CSDS 452 Causality and Machine Learning

Lecture 6: Unobserved confounders

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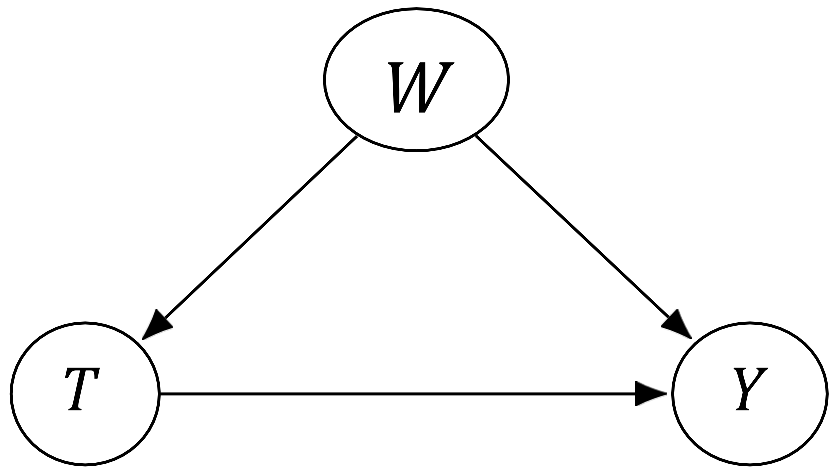
Outline

- Bounds without unconfoundedness
  - No-assumption bound
  - Monotone treatment response
  - Monotone treatment selection
  - Optimal treatment selection
- Sensitivity analysis
  - Linear single confounder
  - More general settings

Recap: Adjusting for Confounders



$$E[Y(1) - Y(0)] = E_W[E[Y|T = 1, W] - E[Y|T = 0, W]]$$

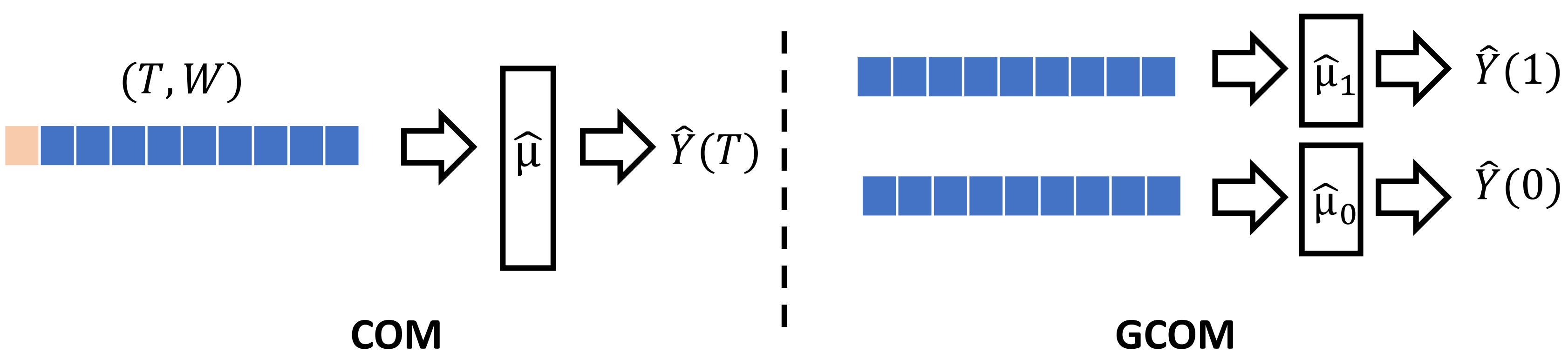
Recap: COM/GCOM estimation

- COM Estimator:

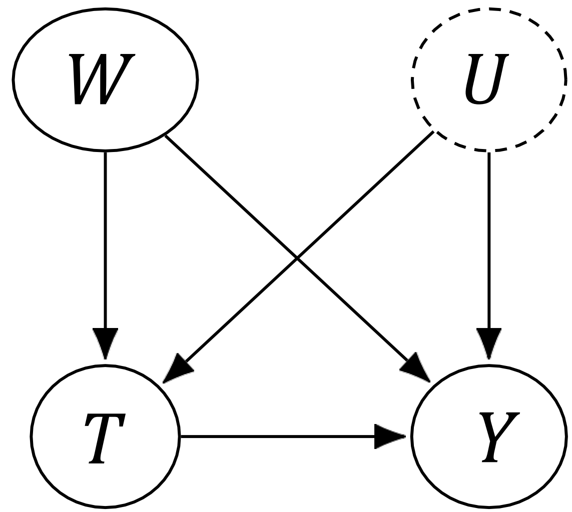
$$\hat{\tau} = \frac{1}{n}\sum_i (\hat{\mu}(1, w_i) - \hat{\mu}(0, w_i))$$

- GCOM Estimator

$$\hat{\tau} = \frac{1}{n}\sum_i (\hat{\mu}_1(w_i) - \hat{\mu}_0(w_i))$$

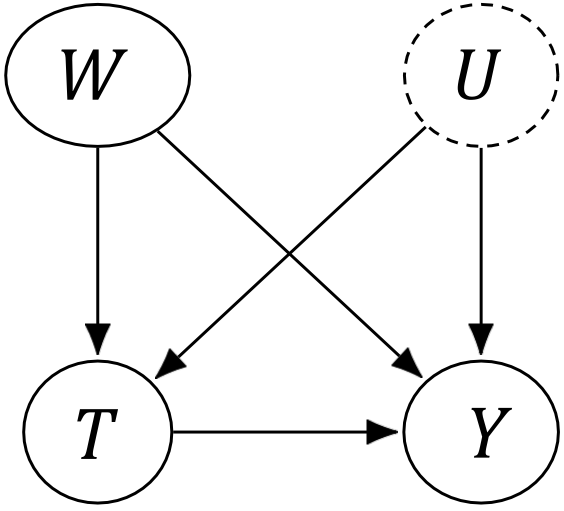


Unobserved Confounders



$$E[Y(1) - Y(0)] = E_{W,U}[E[Y|T=1, W, U] - E[Y|T=0, W, U]]$$

Unobserved Confounders



$$E[Y(1) - Y(0)] = E_{W,U}[E[Y|T=1, W, U] - E[Y|T=0, W, U]]$$

How different is it compared to $E_W[E[Y|T=1, W] - E[Y|T=0, W]]$?

Unconfoundedness assumption is often violated in real world

- Unobserved confounders are ubiquitous!
  - E.g., when study the causal effect of face mask on COVID infection, many confounders (culture background, personality, lifestyles, …) are unmeasured.
- "The Law of Decreasing Credibility: The credibility of inference decreases with the strength of the assumptions maintained" (Manski, 2003).

Can we make weaker assumption?

- Unconfoundedness assumption

$$E[Y(1) - Y(0)] = E_W[E[Y|T = 1, W] - E[Y|T = 0, W]]$$

Identify a "Point"

Can we make weaker assumption?

- Unconfoundedness assumption

$$E[Y(1) - Y(0)] = E_W[E[Y|T = 1,\ W] - E[Y|T = 0,\ W]]$$

Identify a “Point”

- Weaker assumption
  - Allow the existence of some unobserved confounders
  - Instead of a point, identify an interval
  - “Partial identification” or “set identification”

Now, let’s try to throw out the unconfoundedness assumption!

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  - More general settings

Bounded Potential Outcomes

- Example: Suppose that we know the potential outcomes are within a range [0, 1].

$$0 \leq Y(0), Y(1) \leq 1$$

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$$0 - 1 \leq \ Y_i(1) - Y_i(0) \leq 1 - 0$$

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$$\Downarrow$$

$$0 - 1 \leq \ Y_i(1) - Y_i(0) \leq 1 - 0$$

$$\Downarrow$$

$$-1 \leq \ Y_i(1) - Y_i(0) \leq 1$$

An interval with length 2

More general cases

- Example: Suppose that we know the potential outcomes are within a range [a, b].

$$\mathrm{a} \leq Y(0), Y(1) \leq b$$

More general cases

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More general cases

- Example: Suppose that we know the potential outcomes are within a range $[a, b]$.

$$a \leq Y(0), Y(1) \leq b$$

$$\Downarrow$$

$$a - b \leq \; Y_i(1) - Y_i(0) \leq b - a$$

$$\Downarrow$$

$$a - b \leq E[Y(1) - Y(0)] \leq b - a$$

Trivial bound: an interval with length $2(b - a)$

Observational-Counterfactual Decomposition

$E[Y(1) - Y(0)]$

Observational-Counterfactual Decomposition

$$E[Y(1) - Y(0)]$$
$$= E[Y(1)] - E[Y(0)]$$

Observational-Counterfactual Decomposition

$E[Y(1) - Y(0)]$
$= E[Y(1)] - E[Y(0)]$

Conditioning and marginalization

$= P(T = 1)E[Y(1)|T = 1] + P(T = 0)E[Y(1)|T = 0]$
$\quad -P(T = 1)E[Y(0)|T = 1] - P(T = 0)E[Y(0)|T = 0]$

Observational-Counterfactual Decomposition

$E[Y(1) - Y(0)]$

$= E[Y(1)] - E[Y(0)]$

Consistency

$= P(T = 1)E[Y(1)|T = 1] + P(T = 0)E[Y(1)|T = 0]$

$\quad -P(T = 1)E[Y(0)|T = 1] - P(T = 0)E[Y(0)|T = 0]$

$= P(T = 1)E[Y|T = 1] + P(T = 0)E[Y(1)|T = 0]$

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Observational

Observational-Counterfactual Decomposition

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Observational

Counterfactual

No-Assumption Bound

- Observational-Counterfactual Decomposition

$$E[Y(1) - Y(0)] = P(T=1)E[Y|T=1] + P(T=0)E[Y(1)|T=0]$$
$$-P(T=1)E[Y(0)|T=1] - P(T=0)E[Y|T=0]$$

$$= \mathrm{e} \cdot E[Y|T=1] + (1-e) \cdot E[Y(1)|T=0] -$$
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$$\Downarrow \; e = P(T=1)$$

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$$E[Y(1) - Y(0)] \leq ?$$

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Counterfactual

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Counterfactual

$$E[Y(1) - Y(0)] \leq \mathrm{e} \cdot E[Y|T = 1] + (1 - e)b - ea - (1 - e)E[Y|T = 0]$$

No-Assumption Bound

- Observational-Counterfactual Decomposition

$$E[Y(1)-Y(0)] = P(T=1)E[Y|T=1] + P(T=0)E[Y(1)|T=0] \\ -P(T=1)E[Y(0)|T=1] - P(T=0)E[Y|T=0]$$

$e = P(T=1)$

$$= \mathrm{e} \cdot E[Y|T=1] + (1-e) \cdot \boxed{E[Y(1)|T=0]} - \\ \mathrm{e} \cdot \boxed{E[Y(0)|T=1]} - (1-e) \cdot E[Y|T=0]$$

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No-Assumptions Interval Length

- Trivial bound:

$$a - b \leq E[Y(1) - Y(0)] \leq b - a$$

Length: $2(b - a)$

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Length: $(1 - e)b + eb - ea - (1 - e)a = b - a$

Questions

- What kind of bounds can we get on the ATE if the potential outcomes are unbounded?

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Nonnegative Monotone Treatment Response (MTR)

- Assume treatment always helps. Mathematically,

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- Proof:

(Observational-Counterfactual Decomposition)

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consistency

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Example: No-assumption & MTR

- Potential outcome bounded [a=0, b=1]
  $$e = 0.3, E[Y|T = 1] = 0.9, E[Y|T = 0] = 0.2$$

- No-assumption bound: ?
- Nonnegative MTR lower bound: ?

Example: No-assumption & MTR

- Potential outcome bounded [a=0, b=1]
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- No-assumption bound:

$$E[Y(1) - Y(0)] \leq \mathrm{e} \cdot E[Y|T = 1] + (1 - e)b - ea - (1 - e)E[Y|T = 0]$$

$$= 0.3 * 0.9 + 0.7 - 0 - 0.7 * 0.2 = 0.83$$

- Nonnegative MTR lower bound:

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$$E[Y(1) - Y(0)] \geq \mathrm{e} \cdot E[Y|T = 1] + (1 - e)a - eb - (1 - e)E[Y|T = 0]$$

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- Nonnegative MTR lower bound:
  $$E[Y(1) - Y(0)] \geq 0$$

- Combine them together: $0 \leq E[Y(1) - Y(0)] \leq 0.83$

Nonpositive Monotone Treatment Response

- Assume treatment always cannot help. Mathematically,

$$\forall i, Y_i(1) \leq Y_i(0)$$

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Monotone Treatment Selection (MTS)

- Treatment groups' potential outcomes are no worse than control groups':

$$E[Y(1)|T = 1] \geq E[Y(1)|T = 0]$$
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- Under the MTS assumption, ATE is bounded by:

$$E[Y(1)-Y(0)] \leq E[Y|T=1] - E[Y|T=0]$$

(Observational-Counterfactual Decomposition)

Proof:

$$E[Y(1)-Y(0)] = \mathrm{e} \cdot E[Y|T=1] + (1-e) \cdot E[Y(1)|T=0] -$$
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Proof:

$$\begin{aligned} E[Y(1) - Y(0)] &= \mathrm{e} \cdot E[Y|T = 1] + (1 - e) \cdot E[Y(1)|T = 0] - \\ &\quad \mathrm{e} \cdot \underline{E[Y(0)|T = 1]} - (1 - e) \cdot E[Y|T = 0] \\ &\leq \mathrm{e} \cdot E[Y|T = 1] + (1 - e) \cdot E[Y(1)|T = 1] - \\ &\quad \mathrm{e} \cdot \underline{E[Y(0)|T = 0]} - (1 - e) \cdot E[Y|T = 0] \end{aligned}$$

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&\quad \mathrm{e}\cdot E[Y|T=0] - (1-e)\cdot E[Y|T=0] \\
&= E[Y|T=1] - E[Y|T=0]
\end{aligned}$$

Example: MTS and MTR

- Potential outcome bounded [a=0, b=1]
  $e = 0.3, E[Y|T = 1] = 0.9, E[Y|T = 0] = 0.2$
- No-assumption bound:
  $$-0.17 \leq E[Y(1) - Y(0)] \leq 0.83$$
- MTR lower bound:
  $$E[Y(1) - Y(0)] \geq 0$$

Example: MTS and MTR

- Potential outcome bounded [a=0, b=1]
  $e = 0.3, E[Y|T = 1] = 0.9, E[Y|T = 0] = 0.2$
- No-assumption bound:
  $$-0.17 \leq E[Y(1) - Y(0)] \leq 0.83$$
- MTR lower bound:
  $$E[Y(1) - Y(0)] \geq 0$$
- MTS upper bound:
  $$E[Y|T = 1] - E[Y|T = 0] = 0.9 - 0.2 = 0.7$$

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- Bound together:
  $$0 \leq E[Y(1) - Y(0)] \leq 0.7$$

Outline

- Bounds without unconfoundedness
  - No-assumption bound
  - Monotone treatment response
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Optimal Treatment Selection (OTS) Assumption

- Individuals always receive the treatment that is best for them:

$$T_i = 1 \Rightarrow Y_i(1) \geq Y_i(0)$$
$$T_i = 0 \Rightarrow Y_i(0) > Y_i(1)$$

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- Individuals always receive the treatment that is best for them:

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$$T_i = 0 \Rightarrow Y_i(0) > Y_i(1)$$

$$E[Y(1)|T = 0] \leq E[Y(0)|T = 0] = E[Y|T = 0]$$
$$E[Y(0)|T = 1] \leq E[Y(1)|T = 0] = E[Y|T = 1]$$

OTS Upper Bound

- Based on OTS, we have $E[Y(1)|T = 0] \leq E[Y|T = 0]$

OTS Upper Bound

- Based on OTS, we have $E[Y(1)|T=0] \leq E[Y|T=0]$

(Observational-Counterfactual Decomposition)

$$E[Y(1) - Y(0)] = \mathrm{e} \cdot E[Y|T=1] + (1-e) \cdot E[Y(1)|T=0] - \mathrm{e} \cdot E[Y(0)|T=1] - (1-e) \cdot E[Y|T=0]$$

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$$\leq \mathrm{e} \cdot E[Y|T=1] + (1-e) \cdot E[Y|T=0] - e \cdot E[Y(0)|T=1] - (1-e) \cdot E[Y|T=0]$$

OTS Upper Bound

- Based on OTS, we have $E[Y(1)|T=0] \leq E[Y|T=0]$

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$$\leq \mathrm{e}\cdot E[Y|T=1] + \cancel{(1-e)\cdot E[Y|T=0]} - e\cdot E[Y(0)|T=1] - \cancel{(1-e)\cdot E[Y|T=0]}$$

OTS Upper Bound

- Based on OTS, we have $E[Y(1)|T=0] \leq E[Y|T=0]$

(Observational-Counterfactual Decomposition)

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$$\leq \mathrm{e} \cdot E[Y|T=1] + \cancel{(1-e) \cdot E[Y|T=0}] - e \cdot E[Y(0)|T=1] - \cancel{(1-e) \cdot E[Y|T=0}]$$

$$= e \cdot E[Y|T=1] - eE[Y(0)|T=1]$$

$$\leq e \cdot E[Y|T=1] - ea$$

OTS Lower Bound

- Based on OTS, we have $E[Y(0)|T=1] \leq E[Y|T=1]$

(Observational-Counterfactual Decomposition)

$$E[Y(1) - Y(0)] = \mathrm{e} \cdot E[Y|T=1] + (1-e) \cdot E[Y(1)|T=0] - \mathrm{e} \cdot E[Y(0)|T=1] - (1-e) \cdot E[Y|T=0]$$

OTS Lower Bound

- Based on OTS, we have $\underline{E[Y(0)|T=1] \leq E[Y|T=1]}$

(Observational-Counterfactual Decomposition)

$$
\begin{aligned}
E[Y(1)-Y(0)] = \mathrm{e} \cdot E[Y|T=1] + (1-e) \cdot E[Y(1)|T=0] - \\
\mathrm{e} \cdot \underline{E[Y(0)|T=1]} - (1-e) \cdot E[Y|T=0] \\
\geq \mathrm{e} \cdot E[Y|T=1] + (1-e) \cdot E[Y(1)|T=0] - \\
\mathrm{e} \cdot \underline{E[Y|T=1]} - (1-e) \cdot E[Y|T=0]
\end{aligned}
$$

OTS Lower Bound

- Based on OTS, we have $\underline{E[Y(0)|T = 1] \leq E[Y|T = 1]}$

(Observational-Counterfactual Decomposition)

$$
\begin{aligned}
E[Y(1) - Y(0)] = \mathrm{e} \cdot E[Y|T = 1] + (1 - e) \cdot E[Y(1)|T = 0] - \\
\mathrm{e} \cdot E[Y(0)|T = 1] - (1 - e) \cdot E[Y|T = 0] \\
\geq \cancel{\mathrm{e} \cdot E[Y|T = 1]} + (1 - e) \cdot E[Y(1)|T = 0] - \\
\cancel{\mathrm{e} \cdot E[Y|T = 1]} - (1 - e) \cdot E[Y|T = 0]
\end{aligned}
$$

OTS Lower Bound

- Based on OTS, we have $\underline{E[Y(0)|T=1] \leq E[Y|T=1]}$

(Observational-Counterfactual Decomposition)

$$E[Y(1)-Y(0)] = \mathrm{e} \cdot E[Y|T=1] + (1-e) \cdot E[Y(1)|T=0] - \mathrm{e} \cdot E[Y(0)|T=1] - (1-e) \cdot E[Y|T=0]$$

$$\geq \cancel{\mathrm{e} \cdot E[Y|T=1]} + (1-e) \cdot E[Y(1)|T=0] - \cancel{\mathrm{e} \cdot E[Y|T=1]} - (1-e) \cdot E[Y|T=0]$$

$$= (1-e) \cdot E[Y(1)|T=0] - (1-e) \cdot E[Y|T=0]$$

OTS Lower Bound

- Based on OTS, we have $\underline{E[Y(0)|T=1] \leq E[Y|T=1]}$

(Observational-Counterfactual Decomposition)

$$
\begin{aligned}
E[Y(1)-Y(0)] = &\ \mathrm{e} \cdot E[Y|T=1] + (1-e) \cdot E[Y(1)|T=0] - \\
&\ \mathrm{e} \cdot E[Y(0)|T=1] - (1-e) \cdot E[Y|T=0] \\
\geq &\ \cancel{\mathrm{e} \cdot E[Y|T=1]} + (1-e) \cdot E[Y(1)|T=0] - \\
&\ \cancel{\mathrm{e} \cdot E[Y|T=1]} - (1-e) \cdot E[Y|T=0] \\
= &\ (1-e) \cdot E[Y(1)|T=0] - (1-e) \cdot E[Y|T=0] \\
\geq &\ (1-e) \cdot a - (1-e) \cdot E[Y|T=0]
\end{aligned}
$$

OTS Complete Bound

- $E[Y(1) - Y(0)] \geq (1 - e) \cdot a - (1 - e) \cdot E[Y|T = 0]$
- $E[Y(1) - Y(0)] \leq e \cdot E[Y|T = 1] - ea$
- Interval length:
  $$e \cdot E[Y|T = 1] - a + (1 - e) \cdot E[Y|T = 0]$$

Example: OTS

- Potential outcome bounded [a=0, b=1]
  $e = 0.3, E[Y|T = 1] = 0.9, E[Y|T = 0] = 0.2$
- OTS Bound 1:
  $$-0.14 \leq E[Y(1) - Y(0)] \leq 0.27$$

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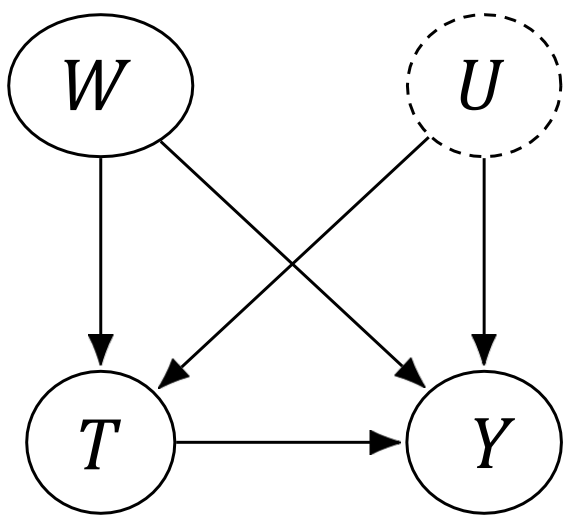
Sensitivity Analysis

- We completely threw out the unconfoundedness assumption just now.
- Now, we assume that there are observed confounders W and unobserved confounders U again:



$$E[Y(1) - Y(0)] = E_{W,U}[E[Y|T=1,\ W, U] - E[Y|T=0,\ W, U]]$$

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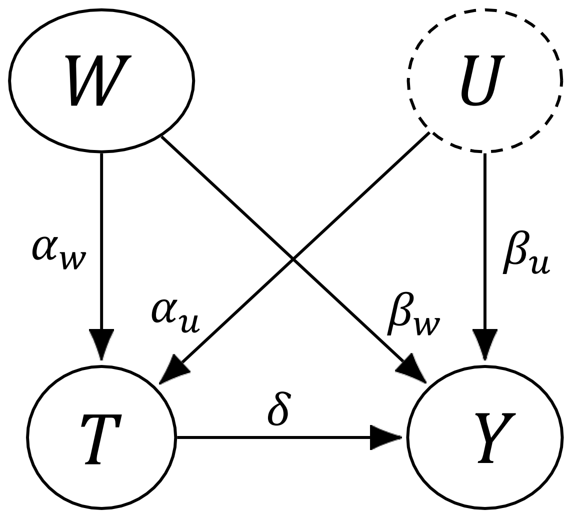
Bias in Simple Linear Setting



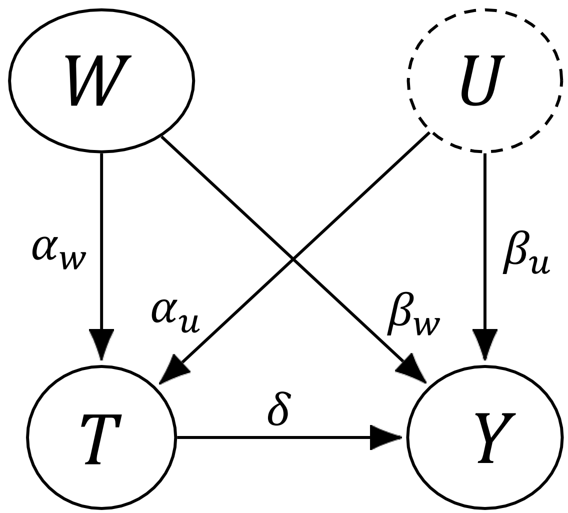
$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

$$E[Y(1) - Y(0)] = E_{W,U}[E[Y|T = 1, W, U] - E[Y|T = 0, W, U]]$$

What is the bias of this estimator?

$$E_W[E[Y|T = 1, W] - E[Y|T = 0, W]]$$

Bias in Simple Linear Setting



$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

$$E[Y(1) - Y(0)] = E_{W,U}[E[Y|T=1, W, U] - E[Y|T=0, W, U]] = \delta$$

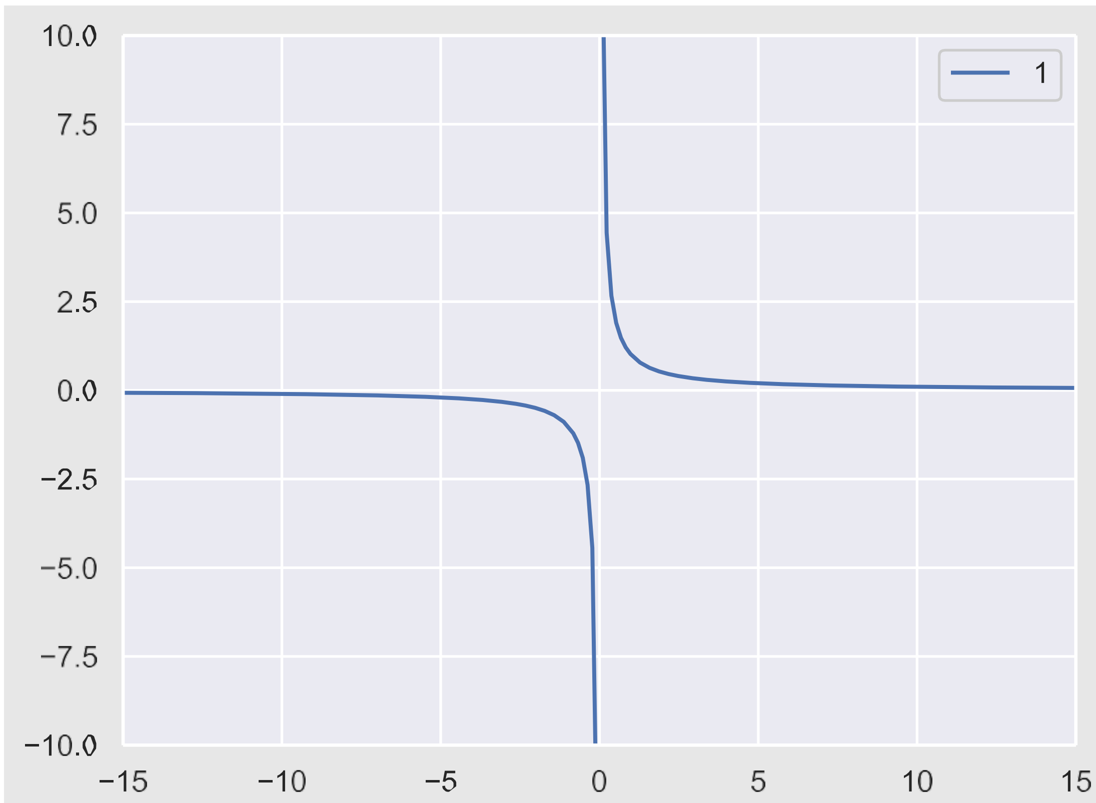
$$E_W[E[Y|T=1, W] - E[Y|T=0, W]] = \delta + \beta_u / \alpha_u$$

Bias of only adjusting for W: $\delta + \frac{\beta_u}{\alpha_u} - \delta = \frac{\beta_u}{\alpha_u}$

Contour Plots for Sensitivity to Confounding

$\frac{\beta_u}{\alpha_u}$

β_u

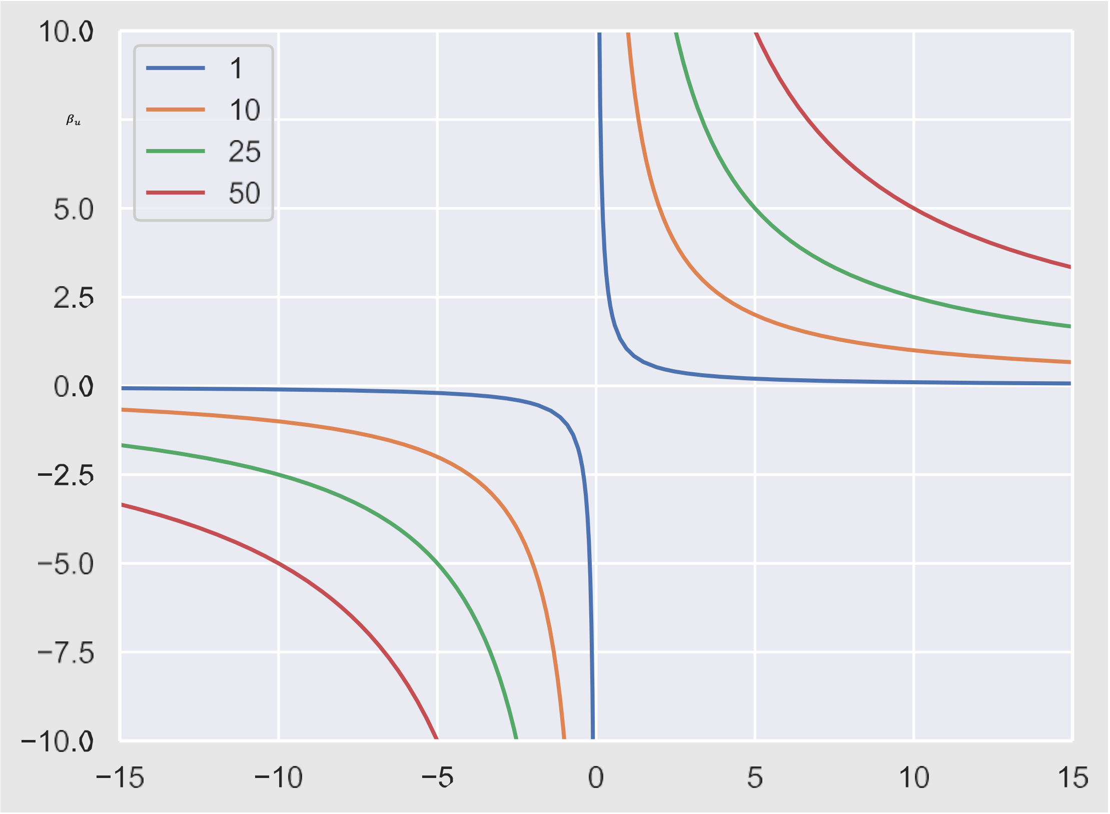


$\frac{1}{\alpha_u}$

Contour Plots for Sensitivity to Confounding

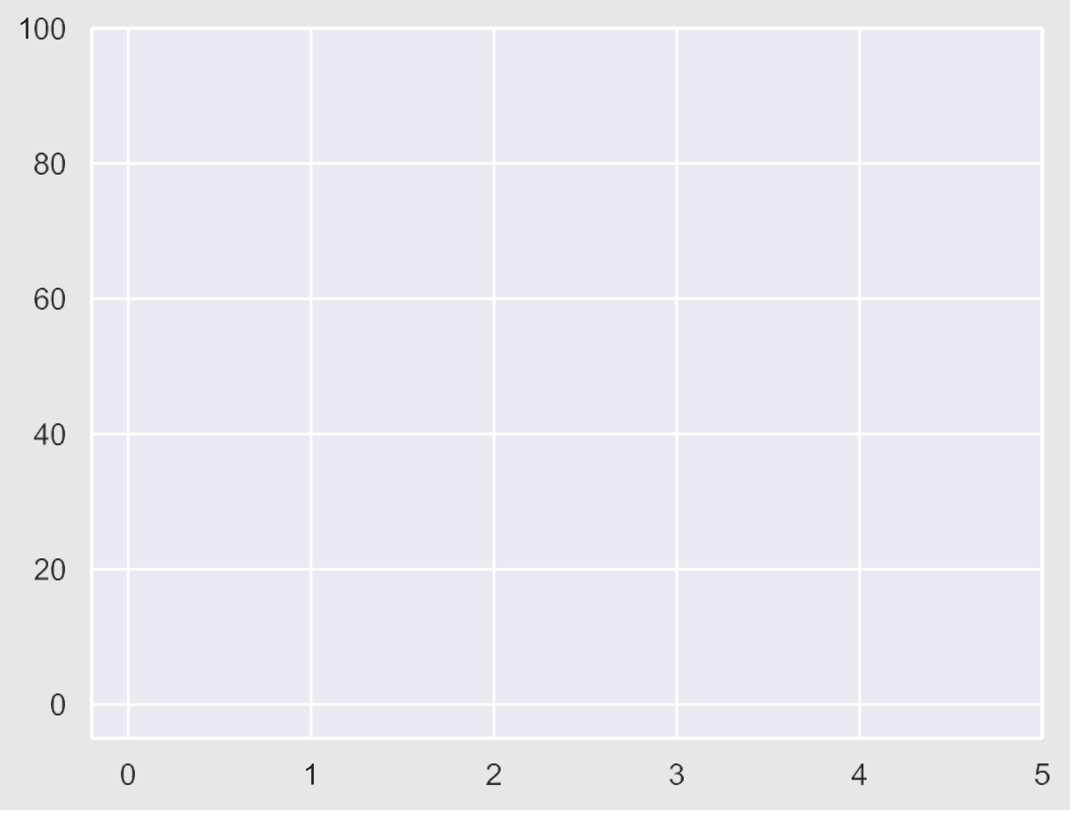
$\frac{\beta_u}{\alpha_u}$

β_u



$\frac{1}{\alpha_u}$

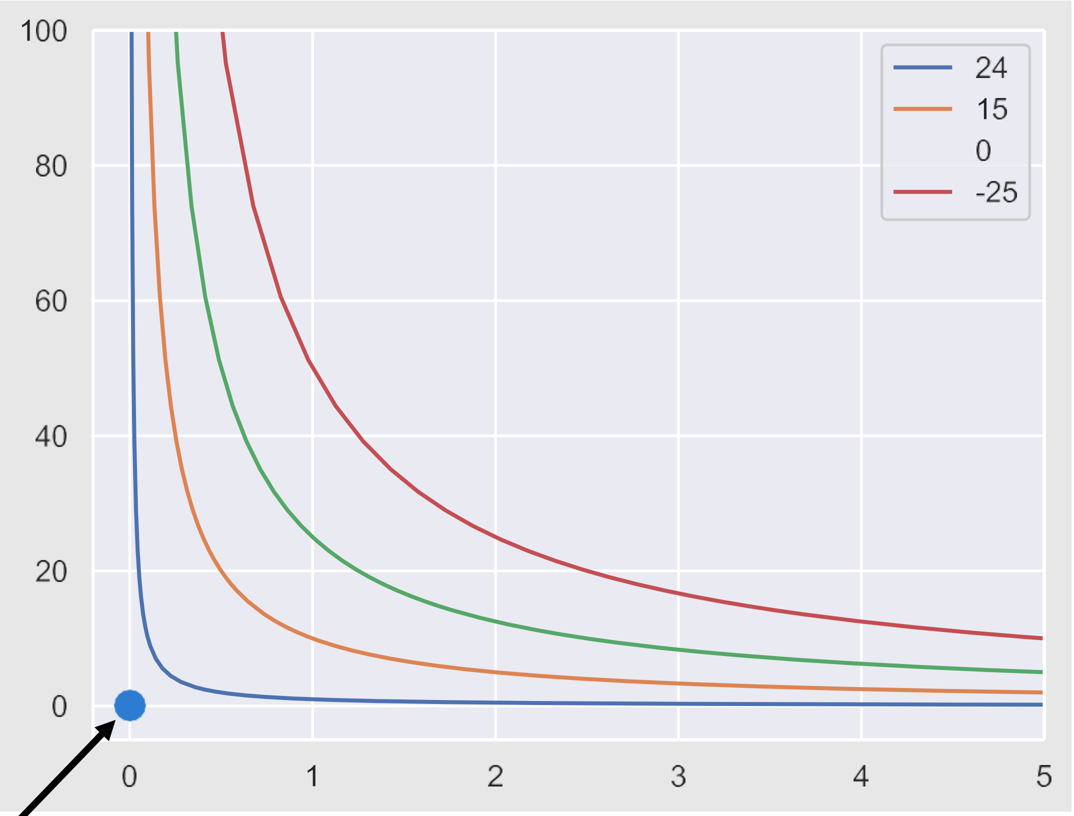
Contour Plots for Sensitivity to Confounding



β_u

$\frac{1}{\alpha_u}$

Contour Plots for Sensitivity to Confounding



β_u

$\delta = 25$

$\frac{1}{\alpha_u}$

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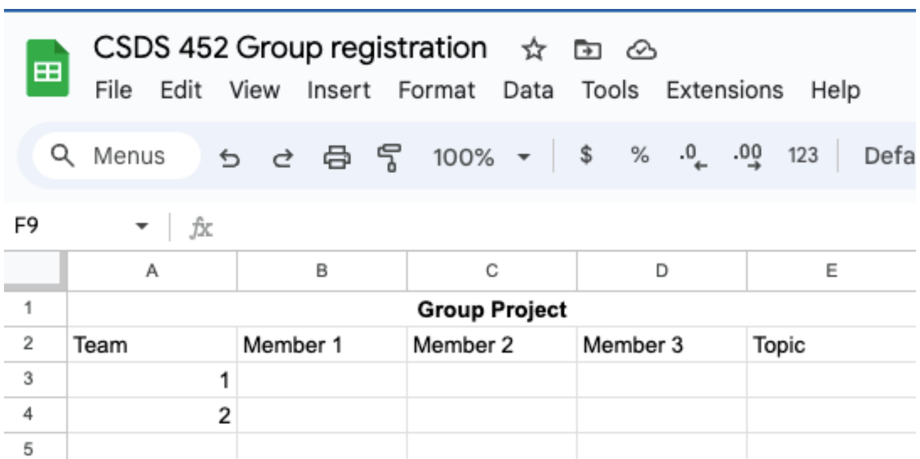
More General Cases

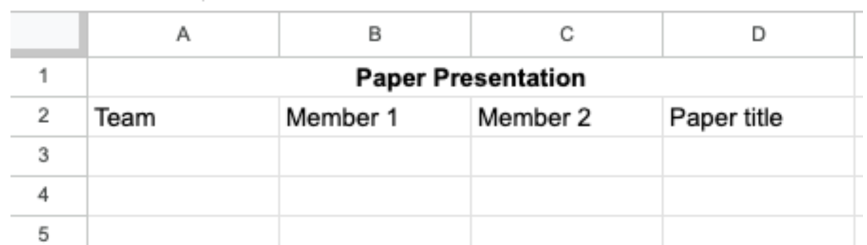
- Arbitrary graphs with linear structural equations
  - Cinelli C, Kumor D, Chen B, et al. Sensitivity analysis of linear structural causal models[C]//International conference on machine learning. PMLR, 2019: 1252-1261.
- Binary cases
  - Rosenbaum P R, Rubin D B. Assessing sensitivity to an unobserved binary covariate in an observational study with binary outcome[J]. Journal of the Royal Statistical Society: Series B (Methodological), 1983, 45(2): 212-218.
  - Imbens G W. Sensitivity to exogeneity assumptions in program evaluation[J]. American Economic Review, 2003, 93(2): 126-132.
- Treatment mechanism and the outcome mechanism can be modeled with arbitrary machine learning models
  - Veitch V, Zaveri A. Sense and sensitivity analysis: Simple post-hoc analysis of bias due to unobserved confounding[J]. Advances in Neural Information Processing Systems, 2020, 33: 10999-11009.

Register your group

- Once you have formed a group, please register in
  - https://docs.google.com/spreadsheets/d/1laxOBG-nUDNzJq8v7RvirW5iT1wJRPdHV9GCTnHvD64/edit?usp=sharing



	A	B	C	D	E
1			**Group Project**		
2	Team	Member 1	Member 2	Member 3	Topic
3	1				
4	2				
5					



	A	B	C	D
1		**Paper Presentation**		
2	Team	Member 1	Member 2	Paper title
3				
4				
5				

Thank you!
Q&A