Discrete Space Markov Chains

Matthew Wascher

February 18, 2025

Exit Times

Example: At a local two year college, 60% of fresh men become sophomores, 25% remain freshmen, and 15% drop out. 70% of sophomores graduate and transfer to a four year college, 20% remain sophomores and 10% drop out. What fraction of new students eventually graduate?

Let's recall with the transition matrix:

$$\begin{array}{c|cccc} & F & S & G & D \\ \hline F & .25 & .6 & 0 & .15 \\ S & 0 & .2 & .7 & .1 \\ G & 0 & 0 & 1 & 0 \\ D & 0 & 0 & 0 & 1 \end{array}$$

Exit Times

Previous we asked what fraction of students will eventually graduate. Next we will consider the average time until a student graduates or drops out.

$$g(1) = 1 + .25\, g(1) + .6\, g(2)$$

$$g(2) = 1 + .2\, g(2)$$

$$.8\, g(2) = 1 \implies g(2) = \frac{5}{4}$$

$$.75\, g(1) = 1 + .6\left(\frac{5}{4}\right) \implies g(1) = \frac{1.75}{.75} = 2.33$$

Exit Time

Given a Markov chain X_n on a state space S with one or more absorbing states, we might be interested in the time until X_n eventually reaches some absorbing state a.

We will call this the **exit time**. For now we will consider another

example.

Exit Times

Example: In a game of tennis, the first player to 4 points wins, with the stipulation that one must win by at least 2 point. Suppose a game of tennis is tied 3-3, the serving player wins each point, and successive points are independent. What is the expected time until the game ends?

$p = .6$

$$
\begin{array}{c|ccccc}
 & -2 & -1 & 0 & 1 & 2 \\
\hline
-2 & 1 & 0 & 0 & 0 & 0 \\
-1 & .4 & 0 & .6 & 0 & 0 \\
0 & 0 & .4 & 0 & .6 & 0 \\
1 & 0 & 0 & .4 & 0 & .6 \\
2 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

Exit Times

What is the expected time until the game ends?

$s(2) = s(-2) = 0$

$s(-1) = 1 + .6\, s(0)$

$s(0) = 1 + .4\, s(-1) + .6\, s(1)$

$s(1) = 1 + .4\, s(0)$

$\vdots$

$s(0) = 3.846$

Exit Times

Theorem 1.28: Let X_n be a Markov chain on finite state space S. Let $A \subset S$ and $V_a = \inf\{n \geq \cancel{1}0 : X_n \in A\}$. Suppose $C = S - A$ is finite and $P_x(V_a < \infty) > 0$ for all $x \in C$. Let $g(a) = 0$ for all $a \in A$ and for all x in C

$$g(x) = 1 + \sum_y p(x, y) g(y) \qquad (**)$$

Then $g(x) = E_x(V_A)$.

Exit Times

Proof: By lemma 1.3 $E_x(V_A) < \infty$ for all $x \in C$.

(**) implies $g(x) = 1 + E_x\, g(X_1)$ when $x \in A$

By the Markov property $(T = V_A)$

$$g(x) = E_x[\min(T, n)] + E_x[g(X_{\min(T,n)})]$$

Claim: $E_x[g(X_{\min(T,n)})] \to 0$

$P(T < \infty) = 1$ so eventually $n > T$

$E_x[g(X_T)] = 0$ b/c $X_T \notin A$

so $g(X_T) = 0$

Exit Times

Example: Watiting time for TT - Suppose flip a fair coin repeatedly. let T_{TT} be the (random) number of flips needed to observe two tails in a row. We want to find $E(T_{TT})$.

X_n = current # consecutive tails

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1/2 & 1/2 & 0 \\ 1 & 1/2 & 0 & 1/2 \\ 2 & 0 & 0 & 1 \end{array}$$

$$g(0) = 1 + .5\,g(0) + .5\,g(1)$$

$$g(1) = 1 + .5\,g(0)$$

$$\cancel{g(0)} + g(1) = 2 + \cancel{g(0)} + .5\,g(1)$$

$$.5\,g(1) = 2 \qquad g(1) = 4 \qquad g(0) = 6$$

Exit Times

We want to find $E(T_{TT})$.

Exit Times

Example: Bob, who has 15 pennies, and Charlie, who has 10 pennies, decide to play a game. They each flip a coin. If the two coins match, Bob gets the two pennies (for a profit of 1). If the two coins are different, then Charlie gets the two pennies. They quit when someone has all the pennies. What is the expected time until someone has all the pennies? X_n: Bob's pennies

N: total pennies

Let $\tau = \min\{n : X_n \in \{0, N\}\}$ the first time someone has all the pennies. We claim

$$E_x(\tau) = x(N - x)$$

We can see this in two ways.

Exit Times

Guess and check: $g(x) = x(N-x)$

$$g(0) = g(N) = 0$$

$$g(x) = 1 + \frac{1}{2} g(x-1) + \frac{1}{2} g(x+1)$$

$$x(N-x) = 1 + \frac{1}{2}(x-1)(N-(x-1)) + \frac{1}{2}(x+1)(N-(x+1))$$

$$= 1 + \frac{1}{2}\Big[\big(x(N-x) - \cancel{x} + \cancel{N} - x - 1\big) + \big(x(N-x) + \cancel{x} - (\cancel{N} - x + 1)\big)\Big]$$

$$= 1 + \frac{1}{2}\big[2x(N-x) - 2\big] = x(N-x) = g(x)$$

Exit Times

Derive the solution:

$$(**) \qquad g(x) = 1 + \sum_y g(y)\, p(x,y)$$

$$g(x) = 1 + \tfrac{1}{2} g(x+1) + \tfrac{1}{2} g(x-1)$$

$$g(x+1) - g(x) = -2 + g(x) - g(x-1)$$

Let $g(1) - g(0) = c \implies g(2) - g(1) = c - 2$

$$g(3) - g(2) = c - 4 \quad \dots$$

$$0 = g(0) \qquad g(k) - g(k-1) = c - 2(k-1)$$

$$0 = g(N) = \sum_{k=1}^{N} g(k) - g(k-1) = \sum_{k=1}^{N} c - 2(k-1) = cN - 2\frac{N(N-1)}{2}$$

$$0 = cN - N(N-1)$$

$$c = N-1$$

$$g(x) = \sum_{k=1}^{x} N-1 - 2(k-1) = x(N-1) - x(x-1)$$

$$= x(N-x)$$

Exit Times

For an example of an unfair game we return to our Gambler's Ruin. Suppose we have a Gambler's Ruin with N and $p \neq 1/2$ and let $q = 1 - p$. We can consider expected time that starting from \$$x$ until the gambler reaches \$$N$ or goes bankrupt. We will let

$$\tau = \min\{n : X_n \in \{0, N\}\}$$

We claim that

$$E_x(\tau) = \frac{\cancel{1}\ x}{q-p} - \left(\frac{N}{q-p}\right)\left(\frac{1-(q/p)^x}{1-(q/p)^N}\right)$$

We can verify using Theorem 1.28 by showing that

$$E_x(\tau) := g(x) = 1 + pg(x+1) + qg(x-1)$$

Exit Times

Claim: $g(x) = p\,g(x+1) + q\,g(x-1)$

$$g(x) = 1 + p\frac{x+1}{q-p} + q\frac{x-1}{q-p} - \frac{N}{q-p}\left[p\frac{1-(q/p)^{x+1}}{1-(q/p)^N} + q\frac{1-(q/p)^{x-1}}{1-(q/p)^N} \right]$$

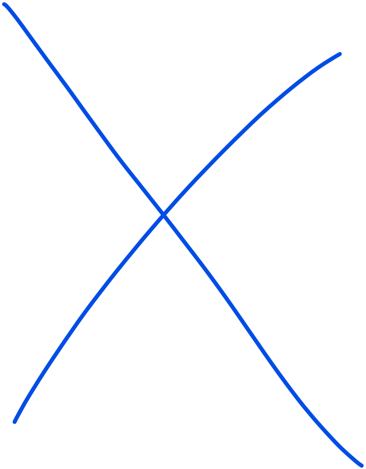
$$= 1 + \frac{x}{q-p} + \frac{p-q}{q-p} - \frac{N}{q-p}\left[\frac{p+q-(q/p)^x\,(q+p)}{1-(q/p)^N} \right]$$

b/c $q+p=1$

$$= \cancel{1} + \frac{x}{q-p} + \cancel{\frac{p-q}{q-p}} - \frac{N}{q-p}\left[\frac{1-(q/p)^x}{1-(q/p)^N} \right]$$

$$\frac{x}{q-p} - \frac{N}{q-p}\left[\frac{1-(q/p)^x}{1-(q/p)^N} \right] = E_x[\tau]$$

Exit Times



Exit Times

For a concrete example of this we can consider roulette. Suppose we bring \$50 to a casino with the goal of winning \$100 before we go bankrupt. If we bet \$1 on red each spin, then $p = 18/38$ since there are

$q = 20/38$

- 18 red spaces
- 18 black spaces
- 2 green spaces (0 and 00)

How long do we expect to play until we reach \$100 or go bankrupt?

Exit Times

$$E_{50}[\tau] = \frac{50}{\frac{20}{38} - \frac{18}{38}} - \frac{100}{\frac{20}{38} - \frac{18}{38}} \left[\frac{1 - (20/18)^{50}}{1 - (20/18)^{100}} \right]$$