MTH 497 Midterm Exam

Name____ Date______________

1. Suppose a fire extinguisher can hold up to 3 fuel cannisters. Each day, there is a .4 probability that the fire extinguisher will not be used, a .3 probability that one fuel cannister will be used, a .2 probability that 2 fuel cannisters will be used, and a .1 probability that all 3 fuel cannisters will be used. If the fire extinguisher has less than 2 fuel cannisters at the end of the day, it is refueled to 3 fuel cannisters for the start of the next day. Suppose the fire extinguisher starts with 3 cannisters and let X_n be the number of fuel cannisters remaining at the start of the nth day.

a) Explain why X_n is a Markov chain.

b) Write the transition matrix for the Markov chain X_n.

$$\begin{array}{c|cc} & 2 & 3 \\ \hline 2 & .4 & .6 \\ 3 & .3 & .7 \end{array}$$

c) What is the probability the fire extinguisher has 3 fuel cannisters at the start of day 2?

$X_0 = 3$

$$p(3,3)\,p(3,3) + p(3,2)\,p(2,3)$$
$$(.7)(.7) + (.3)(.6)$$
$$.49 + .18 = .67$$

d) What is the expected number of fuel cannisters at the start of day 2?

$$P(X_2 = 2 \mid X_0 = 3) = p(3,3)\,p(3,2) + p(3,2)\,p(2,2)$$
$$= (.7)(.3) + (.3)(.4) = .33$$

$$E[X_2 \mid X_0 = 3] = \sum_{x=2,3} x P(X_2 = x \mid X_0 = 3) = 2.67$$

2. Suppose that on a given day a faulty machine can be in state 1, 2, 3, or 4 and changes state each day according to a Markov process. If the machine is in state 1, it goes to state 2 with probability .4 and state 3 with probability .6. If the machine is in state 2, it goes to state 4 with probability 1. If the machine is in state 3, it goes to state 1 with probability .7 and state 2 with probability .3. If the machine is in state 4, it goes to state 2 with probability 1. Let X_n track the state of the machine on day n.

a) Show that state 1 is transient.

$\rho_{12} > p(1,2) = .4 > 0 \qquad \rho_{21} = 0$

b) Show that state 2 is recurrent.

$\rho_{22} = 1 \qquad p^2(2,2) = p(2,4)\,p(4,2) = 1$

$\rho_{22} \geq p^2(2,2) = 1$

c) Is this Markov chain irreducible? Justify your answer.

No, for example $4 \not\to 1$

d) Find the period of state 2.

$p^n(2,2) = 1$ if n even

0 else

$\Rightarrow$ period is 2

3. Is a finite state space Markov chain with a stationary distribution that satisfies the detailed balance equations necessarily aperiodic? Consider a Markov chain with states A, B, and C. When in state A, the chain moves to state B with probability 1. When in state B, the chain moves to state A with probability .5 and to state C with probability .5. When in state C, the chain moves to state B with probability 1.

a) Find the stationary distribution π.

$$p = \begin{array}{c|ccc} & A & B & C \\ A & 0 & 1 & 0 \\ B & .5 & 0 & .5 \\ C & 0 & 1 & 0 \end{array}$$

$\pi p = \pi$

$\pi_3 = .5\pi_2$

$\pi_1 + \pi_2 + \pi_3 = 1$

$\pi_1 = .5\pi_2$

$\pi_2 = \pi_1 + \pi_3$

$\Rightarrow \quad \pi = [\frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{4}]$

b) Show that π satisfies the detailed balance equations.

$\pi_i p(i,j) = \pi_j p(j,i)$ for all i,j

$\pi_A p(A,B) = \pi_B p(B,A) \Rightarrow \frac{1}{4}(1) = \frac{1}{2}(\frac{1}{2}) = \frac{1}{4}$ ✓

$\pi_A p(A,C) = \pi_C p(C,A) \Rightarrow 0 = 0$ ✓

$\pi_B p(B,C) = \pi_C p(C,B) \Rightarrow \frac{1}{2}(\frac{1}{2}) = \frac{1}{4}(1)$ ✓

c) Is this chain aperiodic? Justify your answer.

No, $\quad p^n(A,A) > 0$ if n even

$= 0$ if n odd

so the period of A is 2

d) Give an example that shows that

$$p^n(x,y) \to \pi(y)$$

does not hold for some pair of states x,y.

$p^n(A,A) = \frac{1}{2}$ if n even

$= 0$ if n odd

$(0, 1/2, 0, 1/2, 0, 1/2, \ldots$ does not converge at all

4. In the classic children's game "Keep Away", two players throw a ball between them while a third player attempts to intercept the ball. Suppose players A and B throw the ball between them while player C tries to intercept the ball. If player C intercepts the ball, the game ends. Suppose player A starts with the ball. When player A throws the ball, there is a .8 probability it reaches player B and a .2 probability player C intercepts it. When player B throws the ball, there is a .7 probability it reaches player A, a .2 probability player C intercepts it, and a .1 probability player C deflects it back to player B.

$X_n \in \{A, B, C\}$

	A	B	C
A	0	.8	.2
B	.7	.1	.2
C	0	0	1

a) Let T be the time that the game ends. Show that T is a stopping time.

$\{T \le n\} = \{X_k = C \text{ for some } k \le n\}$

b) Let T* be the last time that player A receives the ball. Show that T* is not a stopping time.

$\{T^* = n\} = \{X_n = A,\ X_{n+1} = C\}$ so depends on X_{n+1}

b) What is the expected length of the game?

$s(A) = .8s(B) + 1$

$s(B) = .7s(A) + .1s(B) + 1$

$s(B) = .56s(B) + .7 + .1s(B) + 1$

$.34s(B) = 1.7$

$s(A) = .8s(B) + 1$

$s(B) = \frac{1.7}{.34} = 5$

$s(A) = .8(5) + 1 = 5$

c) What is the probability that the game ends with player B throwing the ball and player C intercepting it? (Hint: it will be helpful to define a Markov chain with two absorbing states.)

	A	B	AC	BC
A	0	.8	.2	0
B	.7	.1	0	.2
AC	0	0	1	0
BC	0	0	0	1

$h(AC) = 0$

$h(BC) = 1$

$h(A) = .8h(B) + \cancel{.2h(AC)}$

$h(B) = .1h(B) + .7h(A) + \cancel{.2h(BC)}\ .2$

$h(B) = .66h(B) + .2$

$h(B) = \frac{.20}{.34}$

$h(A) = \frac{16}{34}$

5. (Bonus Question) Suppose you play a game on an infinite gameboard with spaces labeled {0,1,2, . . .}. Each round you flip a coin and move two spaces if the coin is heads and one space if it is tails. However, if your current coin flip is the same as your previous coin flip, you return to space 0. Let X_n track your position after n rounds.

a) Show that 0 is positive recurrent.

$P(X_n \neq 0, \forall n \geq 1 \mid X_0 = 0) = (1/2)^n \to 0$

b) Let k > 0 be a positive integer. Show that k is positive recurrent. (Hint: you can do this by checking the conditions of the appropriate theorem rather than explicitly calculating $E_k(T_k)$.)

Show X_n is irreducible

c) Let V be the maximum location on the gameboard reached before the first return to 0. Find $E_0(V)$.