Discrete Space Markov Chains

Matthew Wascher

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Exit Distributions

Example: At a local two year college, 60% of fresh men become sophomores, 25% remain freshmen, and 15% drop out. 70% of sophomores graduate and transfer to a four year college, 20% remain sophomores and 10% drop out. What fraction of new students eventually graduate?

Let's begin with the transition matrix:

$$\begin{array}{c|cccc} & F & S & G & D \\ \hline F & .25 & .6 & 0 & .15 \\ S & 0 & .2 & .7 & .1 \\ G & 0 & 0 & 1 & 0 \\ D & 0 & 0 & 0 & 1 \end{array}$$

Exit Distributions

What fraction of new students eventually graduate?

$$h(x) = P_x(\text{Ever graduate})$$

$$h(S) = .70 + .2h(S) + .1(0)$$

$$h(F) = .6h(S) + .25h(F) \quad .15(0)$$

$$8h(S) = .7 \Rightarrow h(S) = 7/8$$

$$.75h(F) = .6(7/8) \Rightarrow h(F) = \frac{.6}{.75}(7/8) = .7$$

Exit Distributions

Given a Markov chain X_n on a state space S with one or more absorbing states, we might be interested in the probability that X_n eventually reaches some absorbing state a.

We will also consider how long it takes before the chain either reaches a or we know that it will never reach a.

For now we will consider another example.

Exit Distributions

Example: In a game of tennis, the first player to 4 points wins, with the stipulation that one must win by at least 2 point. Suppose a game of tennis is tied 3-3, the serving player wins each point, and successive points are independent. What is the probability the serving player will eventually win the game?

with $p = .6$ X_n = server - returner

	-2	-1	0	1	2
-2	1	0	0	0	0
-1	.4	0	.6	0	0
0	0	.4	0	.6	0
1	0	0	.4	0	.6
2	0	0	0	0	1

Exit Distributions

What is the probability the serving player will eventually win the game?

$h(2) = 1$

$h(-2) = 0$

$h(1) = .6 + .4\,h(0)$

$h(0) = .6\,h(1) + .4\,h(-1)$

$h(-1) = .6\,h(0) + .4\,(0)$

$\vdots$

$h(1) = .8769$

$h(0) = .6923$

$h(-1) = .4154$

Exit Distributions

General case: What if the serving player wins each point with probability w??

$$h(1) = w + (1-w)h(0)$$

$$h(0) = w\,h(1) + (1-w)\,h(-1)$$

$$h(-1) = w\,h(0)$$

$$\vdots$$

$$h(0) = \frac{w^2}{w^2 + (1-w)^2}$$

Exit Distributions

Theorem 1.27: Suppose X_n is a Markov chain on a finite state space S. Let $a, b \in S$, $C = S - \{a, b\}$, and $V_x = \min\{\overset{n}{\cancel{n \geq 1}}, X_n = x\}$ for each $x \in S$. Suppose $h(a) = 1, h(b) = 0$, and

$$h(x) = \sum_y p(x, y)h(y) \qquad (*)$$

.

Then if $P_x(\min\{V_a, V_b\} < \infty) > 0$ for all $x \in C$, then

$$h(x) = P_x(V_a < V_b)$$

Exit Distributions

Proof: Let $T = \min(V_a, V_b)$. By lemma 1.3

$P_x(T < \infty) = 1$ for all $x \in C$

(*) implies that $h(x) = E_x[h(X_1)]$ as long as $x \neq a, b$. Apply the Markov property

$$h(x) = E_x[h(X_{\min(T,n)})]$$

Need $\min(T, n)$ b/c (*) is only valid for $x \neq a, b$

Since $P_x(T < \infty) \Rightarrow E_x[h(X_{\min(T,n)})] \to P(V_a < V_b)$

Why $E_x[h(X_T)] = h(a)P(V_a < V_b) + h(b)P(V_b < V_a)$

$= P(V_a < V_b)$

Exit Distributions

Example: Bob, who has 15 pennies, and Charlie, who has 10 pennies, decide to play a game. They each flip a coin. If the two coins match, Bob gets the two pennies (for a profit of 1). If the two coins are different, then Charlie gets the two pennies. They quit when someone has all the pennies. What is the probability Bob will eventually have all the pennies?

Let X_n be the number of pennies Bob has after n rounds. Note this is a fair game in the sense that

$$E_x(X_1) = x,$$

that is, Bob's expected number of pennies is constant in time.

Exit Distributions

Let $V_y = \min\{n \geq 0 : X_n = y\}$

B/c expected earnings is constant in time

$$x = N\, P_x(V_N < V_0) + 0\, P_x(V_0 < V_N)$$

$$\Rightarrow \quad P_x(V_N < V_0) = \frac{x}{N} \qquad 0 \leq x \leq N$$

Also $\quad h(x) = \frac{1}{2} h(x+1) + \frac{1}{2} h(x-1)$

$$h(x+1) - h(x) = h(x) - h(x-1)$$

$$h(0) = 0 \qquad h(N) = N$$

Exit Distributions

For an example of an unfair game we return to our Gambler's Ruin. Suppose we have a Gambler's Ruin with N and $p \neq 1/2$. We can consider the probability that starting from \$$x$, the gambler reaches \$$N$ before going bankrupt. We will let

$$h(x) = P(V_N < V_0)$$

and now that $h(N) = 1$ and $h(0) = 0$. We now consider $h(x)$ for $0 < x < N$

$$h(x) = p\,h(x+1) + (1-p)\,h(x-1)$$

$$h(x+1) - h(x) = \left(\frac{1-p}{p}\right)\left(h(x) - h(x-1)\right)$$

$$c = h(1) - h(0)$$

$$\therefore \quad h(x-1) - h(x) = c\left(\frac{1-p}{p}\right)^{x-1}$$

Exit Distributions

$$\sum_{x=1}^{N} h(x) - h(x-1) = h(N) - h(0) = 1$$

$$= \sum_{x=1}^{N} h(x) - h(x-1) = c \sum_{x=1}^{N} \left(\frac{1-p}{p}\right)^{x-1}$$

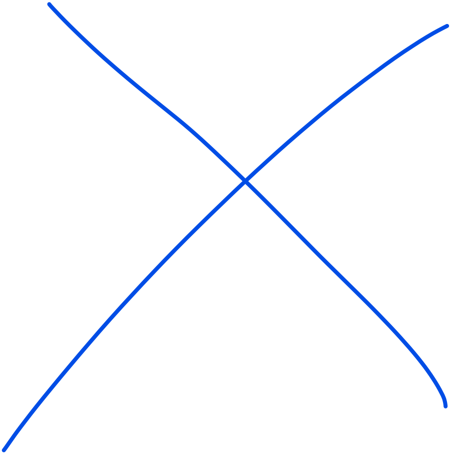
Recall that $\sum_{j=0}^{N-1} \theta^j = \frac{1-\theta^N}{1-\theta}$

Let $\theta = \left(\frac{1-p}{p}\right)$

$$h(x) = h(x) - h(0) = c \sum_{i=0}^{x-1} \theta^i = c \cdot \frac{1-\theta^x}{1-\theta} = \frac{1-\theta^x}{1-\theta^N}$$

$$h(x) = P_x(V_N < V_0) = \frac{\left(\frac{1-p}{p}\right)^x - 1}{\left(\frac{1-p}{p}\right)^N - 1}$$

Exit Distributions



Exit Distributions

For a concrete example of this we can consider roulette. Suppose we bring \$50 to a casino with the goal of winning \$100 before we go bankrupt. If we bet \$1 on red each spin, then $p = 18/38$ since there are

$$\theta = \frac{\left(\frac{20}{38}\right)}{\left(\frac{18}{38}\right)} = \frac{20}{18} = \frac{10}{9}$$

- 18 red spaces
- 18 black spaces
- 2 green spaces (0 and 00)

What is the probability we reach \$100 before we go bankrupt?

$$P(V_{100} < V_0) = \frac{\left(\frac{10}{9}\right)^{50} - 1}{\left(\frac{10}{9}\right)^{100} - 1} = .005128$$

Exit Distributions