Estimation - MLE and MOM

Wednesday, March 19, 2025 11:39 AM

Data $X_1, \ldots, X_n \sim f(x_1, \ldots x_n; \theta)$

Q: How to estimate θ?

① Maximum Likelihood Estimation

The likelihood function $L(\theta | x_1 \ldots x_n)$ is

$$L(\theta | x_1, \ldots, x_n) = f(x_1, \ldots, x_n; \theta)$$

$L(\theta | x_1 \ldots x_n)$ tells us how likely we are to observe the data $X_1 = x_1 \ldots X_n = x_n$ if the parameter is θ

Ex: $X_1, X_2 \overset{iid}{\sim} Exp(\lambda)$

Observe $X_1 = 2$, $X_2 = 3$

$$L(2.5 | x_1 = 2, x_2 = 3) = \frac{1}{\lambda} e^{-\lambda x_1} \frac{1}{\lambda} e^{-\lambda x_2} \Big|_{\lambda = 2, x_1 = 2, x_2 = 3}$$

$$= \frac{1}{4} e^{-4} e^{-6} = \frac{1}{4} e^{-10}$$

MLE says choose θ to maximize $L(\theta | x_1 \ldots x_n)$

$$L(\theta \mid x_1, x_2) = \lambda^2 e^{-(x_1+x_2)\lambda}, \quad \lambda > 0$$

Define log-likelihood $\ell(\theta) = \ln(L(\theta \mid x_1, x_2))$

Because $\ln(x)$ is an increasing function

$\arg\max_\theta L(\theta) = \arg\max_\theta \ell(\theta)$

$$\ell(\theta) = 2\ln(\lambda) - (x_1+x_2)\lambda$$

$$\frac{d}{d\lambda} = \frac{2}{\lambda} - (x_1+x_2) = 0$$

$$\frac{1}{\lambda} = \frac{x_1+x_2}{2}$$

$$\hat{\lambda}_{MLE} = \frac{2}{x_1+x_2} = \frac{1}{\bar{X}}$$

$$\frac{d^2}{d\lambda^2}\ell(\lambda) = \frac{-2}{\lambda^2} < 0 \Rightarrow \max$$

<u>Regression</u> $\quad Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \qquad \varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$

Data: $(x_1, Y_1), \ldots, (x_n, Y_n)$

$$Y_i - (\beta_0 + \beta_1 x_i) \overset{iid}{\sim} N(0, \sigma^2)$$

$$L(\beta_0, \beta_1 \mid x_i, y_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(Y_i - (\beta_0 + \beta_1 x_i) - 0)^2}{2\sigma^2}\right)$$

$$\ell(\beta_0, \beta_1) = -n\ln(\sqrt{2\pi}\ldots) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}[Y_i - (\beta_0 + \beta_1 x_i)]^2$$

$$\ell(\beta_0, \beta_1) = -n \ln(\sqrt{2\pi}\sigma) - \frac{1}{2\sigma^2} \underbrace{\sum_{i=1}^{n} [y_i - (\beta_0 + \beta_1 x_i)]^2}$$

$$-\sum_{i=1}^{n} [y_i - (\beta_0 + \beta_1 x_i)]^2$$

Method of Moments

$X_1 \dots X_n \overset{iid}{\sim} f(\theta)$

The pth moment is for $p = 1, 2, 3, \dots$

$E[X_i^p] = h_p(\theta)$

The pth sample moment is

$\frac{1}{n} \sum_{i=1}^{n} X_i^p$

MOM says find $\hat{\theta}_{MoM}$ by setting

$$\frac{1}{n} \sum_{i=1}^{n} X_i^p = E[X_i^p] = h_p(\theta)$$

and solve for θ

Ex: $X_1 \dots X_n \sim Exp(\lambda)$

$E[X_i] = \frac{1}{\lambda}$

$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

$\hat{\lambda} \quad \frac{1}{n} = \frac{1}{\bar{X}}$

$$\hat{\lambda}_{mom} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} X_i} = \frac{1}{\bar{X}}$$

Ex: $X_1 \dots X_n \overset{iid}{\sim} U[0, \theta]$ $\theta > 0$

$$E[X_i] = \frac{\theta}{2}$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i = \theta/2 \Rightarrow \hat{\theta}_{mom} = 2\bar{X}$$

<u>MLE</u>: $L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} \mathbb{1}_{\{0 \le x_i \le \theta\}}(\theta)$

$$\mathbb{1}_{\{0 \le x_i \le \theta\}}(\theta) = \begin{cases} 1 & \text{if } 0 \le x_i \le \theta \\ 0 & \text{else} \end{cases}$$

$$L(\theta) = \frac{1}{\theta^n} \mathbb{1}_{\{\theta \ge x_i \text{ for all } i = 1, \dots n\}}$$

$$\text{argmax}_{\theta} L(\theta) \Rightarrow \hat{\theta}_{MLE} = \max_{i=1,\dots,n} X_i$$

Data is $X_1 = 13$, $X_2 = 1$, $X_3 = 10$

$$\hat{\theta}_{mom} = 2\bar{X} = 8$$

$\hat{\ }$

$$\hat{\theta}_{MLE} = \max(X_i) = 10$$