Discrete Space Markov Chains - Introduction

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Example: shuffling a deck

Consider a newly opened deck of standard playing cards arranged in order by suit and then by rank within suit (i.e. A-K of clubs, A-K diamonds, A-K hearts, A-K spades). We plan to shuffle the deck and decide to use the following (very inefficient) method:

- Take the top card of the deck and place it a location in the deck chosen uniformly at random from 1-52
- Repeat until the deck is “shuffled”

Example: shuffling a deck

Our example scenario invites several natural questions:

1. Given the current state of the deck, what can we say about the possible states of the deck after the next move?
2. What is the long run result of shuffling cards in this way (i.e. what does "shuffled" mean here)?
3. How long will it take to shuffle the deck this way?

Example: shuffling a deck

Let us first consider the the first question:

Given the current state of the deck, what can we say about the possible states of the deck after the next move?

Example: shuffling a deck

We observed that given the current state of the deck, the distribution of possible states of the deck after the next move does not depend on past states of the deck. This is called the **Markov property**.

A stochastic process satisfying the Markov property is called a **Markov chain**. Markov chains are widely used across many fields to model a wide variety of phenomena because they are simple and thus tractable.

We will define these things mathematically later. For now, let us continue with our example.

Example: shuffling a deck

Our second question:

What is the long run result of shuffling cards in this way (i.e. what does "shuffled" mean here)?

Example: shuffling a deck

Having determined what needs to occur for the deck to be shuffled we can answer our third question:

How long will it take to shuffle the deck this way? The time is random but we can characterize it's distribution, expected value, etc.

Example: Gambler's Ruin

One famous example of a Markov chain is the Gambler's Ruin model. Consider a gambler with initial fortune $\$N_0$ who plays a game such that each round

- The gambler wins $1 with probability p
- The gambler loses $1 with probability $1 - p$

The gamblers plays until they either go bankrupt or reach a target amount of $\$N$.

Example: Gambler's Ruin

Again we can ask several natural questions:

1. What is the long run behavior of this process?
2. How does this behavior depend on N_0, p, and N?