STAT 345/445 Lecture 4

Section 1.3: Conditional Probability and Independence

1. Conditional Probability
   - Conditional probability function
   - Examples
   - Bayes Theorem
   - More Examples
   - Independence

STAT 345/445 Lecture 4

Section 1.3: Conditional Probability and Independence

1. Conditional Probability
   - Conditional probability function
   - Examples
   - Bayes Theorem
   - More Examples
   - Independence

Example: Choosing student representatives

	International	Domestic	Total
Stat majors	12	13	25
Math majors	10	5	15
Total	22	18	40

One person is selected at random. What is the probability that

1. a Stat major was selected? $\frac{25}{40} = P(A)$ $P(A \cap B) = \frac{13}{40}$ domestic stat major
2. a domestic person was selected? $\frac{18}{40} = P(B)$

If we know that a domestic person was selected, what is the probability that it was a Stat major? Given B

$\frac{13}{18}$ Now the sample space changed

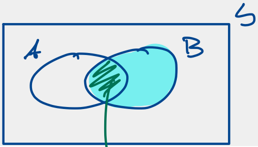
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{13/40}{18/40} = \frac{13}{18}$$

Conditional probability

Definition: Conditional probability

Let A and B be events in S and $P(B) > 0$. Then the **conditional probability of A given B** is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$



normalizing the prob of A to "within B"

know my outcome is in B so this is the only part of A I need.

From the definition we also get:

$$P(A \cap B) = P(A \mid B)P(B)$$
$$\text{and } P(A \cap B) = P(B \mid A)P(A)$$

Cond. probability <u>is</u> a proper probability

use: $P(\cdot \mid B)$ could use $P_B(\cdot)$

still a function from $\mathcal{B}$ to $\mathbb{R}$

Conditional probability function

Theorem

Let B be an event and $P(B) > 0$. Conditional probability $P(\cdot \mid B)$ is a probability function. That is, it satisfies Kolmogorov's axioms:

(i) $P(A \mid B) \geq 0 \quad \forall A \in \mathcal{B}$

(ii) $P(S \mid B) = 1$

(iii) If $A_1, A_2, A_3, \ldots$ are mutually exclusive then

$$P\left(\bigcup_{i=1}^{\infty} A_i \,\middle|\, B \right) = \sum_{i=1}^{\infty} P(A_i \mid B)$$

(i) $P(A) \geq 0$ $\forall A \in \mathcal{B}$

(ii) $P(S) = 1$

(iii) $P(\bigcup A_i)$
$= \sum P(A_i)$

proof .. homework

More on conditional probability

- All properties of probability functions also hold for conditional probability functions
  - Examples:

$P(A) = 1 - P(A^c)$

$$P(A \mid B) = 1 - P(A^c \mid B)$$

$\rightarrow$ – since condition on the same set

$$P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B) - P(A_1 \cap A_2 \mid B)$$

- Can re-write joint probabilities as a series of conditional probabilities
  - Examples:

$\rightarrow P(A \mid B \cap C)\, P(B \cap C)$

"Breakdown" of joint probabilities

$$P(A \cap B) = P(A \mid B)P(B) = P(B \mid A)P(A)$$
$$P(A \cap B \cap C) = P(A \mid B \cap C)P(B \mid C)P(C)$$
$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_3 \cap A_4 \mid A_1 \cap A_2)\, P(A_1 \cap A_2)$$

etc.

Example: Jim plays Bridge

Jim is playing bridge where each player is dealt 13 cards. Jim gets exactly 5 spades. ← Given that this happened

1. What is the probability that 2 of the spades Jim was dealt are the ace and the king?
2. What is the probability that Jim's cards are all black, given that he was dealt exactly 5 spades?

Define: B: Get exactly 5 spades
A: Get ace and king of spades } in a hand 13 rand. dealt cards.

① $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{|A\cap B|/|S|}{|B|/|S|}$

where $|S| = \binom{52}{13}$

$|B|$: Pick 5 spades : $\binom{13}{5}$ $\Rightarrow |B| = \binom{13}{5}\cdot\binom{39}{8}$
Pick 8 other : $\binom{39}{8}$

$|B|$: Pick 5 spades : $\binom{13}{5} \Rightarrow |B| = \binom{13}{5} \cdot \binom{39}{8}$

Pick 8 other : $\binom{39}{8}$

$|A \cap B|$: ♠A, ♠K, 3♠, 8 other |

Pick ace and king of spades : 1

Pick 3 other spaded : $\binom{11}{3}$

Pick remaining 8 : $\binom{39}{8}$

$$\Rightarrow P(A|B) = \frac{1 \cdot \binom{11}{3}\binom{39}{8}}{\binom{13}{5}\binom{39}{8}} = \frac{\binom{11}{3}}{\binom{13}{5}} = \frac{165}{1287} = 0.128$$

② C : All black

$$P(C|B) = \frac{|B \cap C|}{|B|} = \frac{\binom{13}{5} \cdot \binom{13}{8}}{\binom{13}{5}\binom{39}{8}} = 0.0000209$$

Law of total probability:

$$P(A) = \sum_{i=1}^{\infty} P(A \cap B_i) = \sum_{i=1}^{\infty} P(A | B_i) P(B_i)$$

if $B_1, B_2, \ldots$ are a partition of S

Bayes Theorem

Bayes Theorem

Let A and B be events an let $P(B) > 0$. Then

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B)}$$

law of tot prob.

- Bayes Theorem is often written as: Let $\overbrace{A_1, A_2, \ldots}$ be a partition of S and let B be an event in S. Then for each i we have

$$P(A_i \mid B) = \frac{P(B|A_i)\,P(A_i)}{P(B)} = \frac{P(B|A_i)\,P(A_i)}{\sum_{j=1}^{\infty} P(B|A_j)\,P(A_j)}$$

A simple case:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

Example: Computer program

A certain computer program will operate using either of two subroutines, say A and B, depending on the problem. Experience has shown that subroutine A will be used 40 percent of the time and B will be used 60 percent of the time. If A is used, then there is 75 percent probability that the program will run before its time limit is exceeded, and if B is used there is 50 percent chance that it will do so.

1. What is the probability that the program will run without exceeding the time limit? $P(T)$
2. If you know that the program ran without exceeding the time limit, what is the probability that subroutine A was called?

sample space: (subroutine, on time)

subroutine: A or B; on time: yes or no

$$\Omega = \{ \overbrace{(A, no), (A, yes)}^{A}, (B, no), (B, yes) \}$$

Event A: Subroutine A was used

$P(A) = 0.4$

Even T: On time

Given: $P(T|A) = 0.75$

$P(T|B) = 0.5$

1. A and B are a partition of the sample space

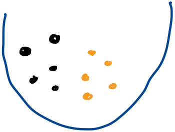
$$P(T) = P(T|A)P(A) + P(T|B)P(B)$$
$$= 0.75 \cdot 0.4 + 0.5 \cdot 0.6 = 0.6$$

2. $$P(A|T) = \frac{P(T|A)P(A)}{P(T)} = \frac{0.75 \cdot 0.4}{0.6}$$
$$= 0.5$$

Example: Balls in an urn

Consider an urn containing 10 balls, 5 of which are black. Choose an integer n at random from the set $\{1, 2, 3, 4, 5, 6\}$, and then choose a random sample of size n without replacement from the urn. Find the probability that all the balls in the sample will be black.

B: All balls in sample are black



Want $P(B)$

Let A_n be the event that integer n was chosen $n = 1, 2, \dots, 6$

Know that $P(A_n) = \frac{1}{6}$

Can figure out $P(B \mid A_n)$ $n = 1, \dots, 6$

and then $P(B) = \sum_{n=1}^{6} P(B \mid A_n) P(A_n)$

$n=1 \quad P(B|A_1) = \frac{5}{10} = \frac{1}{2}$

$n=2 \quad P(B|A_2) = \frac{\binom{5}{2}}{\binom{10}{2}} = \cdots = \frac{2}{9}$

$n=3 \quad P(B|A_3) = \binom{5}{3} \Big/ \binom{10}{3} = \frac{1}{12}$

$n=4 \quad P(B|A_4) = \binom{5}{4} \Big/ \binom{10}{4} = \frac{1}{42}$

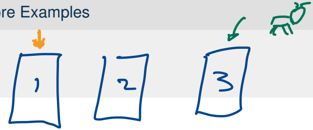
$n=5 \quad P(B|A_5) = \binom{5}{5} \Big/ \binom{10}{5} = \frac{1}{252}$

$n=6 \quad P(B|A_6) = 0$

$\Rightarrow P(B) = \frac{1}{6}\left(\frac{1}{2} + \frac{2}{9} + \frac{1}{12} + \frac{1}{42} + \frac{1}{252}\right) = 0.1389$

$$P(B) = \sum_{n=1}^{6} P(B|A_n)\underbrace{P(A_n)}_{\frac{1}{6}}$$

Example: Monty Hall



You are on a Game Show. There are 3 doors, numbered 1, 2, and 3. You get the price behind the door you pick. Behind two of the doors are goats and behind one door is a car. You want the car. You pick door number 1. Before revealing what is behind door 1, Monty opens door 3 and shows you that there is a goat behind door 3. Monty would never show you where the car is. You now have these options

- Stick with door 1 (and get whatever price behind door 1), or
- Switch to door 2 (and get whatever price behind door 2)

Should you switch, stick with door 1, or does it not matter?

Can be reasoned out with cond. prob.

Similar to the Prisoners dilemma.

One solution (not covered in class)

Define events:

C_1 : Car is behind door 1

C_2 : Car is behind door 2

C_3 : Car is behind door 3

We know that $P(C_1) = P(C_2) = P(C_3) = \frac{1}{3}$ and that C_1, C_2, C_3 is a partition of the sample space. Let

S_3 : Monty opens door 3

We want $P(C_2 | S_3) = \dfrac{P(S_3|C_2)P(C_2)}{P(S_3)}$

and $P(C_1|S_3) = 1 - P(C_2|S_3)$

$$P(S_3) = \underline{P(S_3|C_1)}P(C_1) + \underline{P(S_3|C_2)}P(C_2) + \underline{P(S_3|C_3)}P(C_3)$$

$P(S_3|C_1) = \frac{1}{2}$ Monty knows you have picked the door with the car so he can choose to show you the goat behind door 2 or 3 and chooses one at random

$P(S_3|C_2) = 1$ When the car is behind door 2 Monty has no choice but to show you the goat behind door 3

$P(S_3|C_3) = 0$ When the car is behind door 3 Monty will not open it

$$\Rightarrow P(S_3) = \frac{1}{2}\cdot\frac{1}{3} + 1\cdot\frac{1}{3} + 0\cdot\frac{1}{3} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

$$\Rightarrow P(C_2|S_3) = \frac{1\cdot\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3} \quad \text{and } P(C_1|S_3) = \frac{1}{3}$$

$\Rightarrow$ Always switch!

Independence - two events

Definition: Statistically independent events

Two events A and B are said to be **statistically independent** if

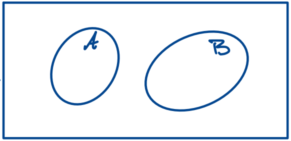
$$P(A \cap B) = P(A)P(B)$$

- If A and B are independent then

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \overset{\text{if indep.}}{=} \frac{P(A)P(B)}{P(B)} = P(A)$$

$$P(B \mid A) = P(B) \quad \text{if } A, B \text{ indep.}$$

disjoint events.



indep? no. $P(A \cap B) = 0$
and $P(A \mid B) = 0 \neq P(A)$

Independence - many events

Definition: Mutually independent

A collection of events $A_1, A_2, \ldots, A_n$ are **mutually independent** if for any sub-collection $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ we have

$$P\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \prod_{j=1}^{k} P(A_{i_j})$$

e.g. $i_1 = 2$, $i_2 = 4$, $i_3 = 5$, $i_4 = 9$

$$P(A_2 \cap A_4 \cap A_5 \cap A_9)$$
$$= P(A_2)P(A_4)P(A_5)P(A_9)$$

- $\longrightarrow$ Read examples 1.3.10, 1.3.11, and 1.3.13 carefully