STAT 345

Theoretical Statistics I

Fall Semester 2017

Midterm

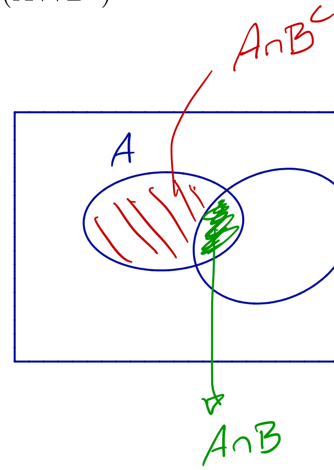
Name: Solution

- You have 75 min to complete this exam
- This exam has 9 questions worth a total of 105 points, but will be graded out of 100 points (so you have the opportunity to earn 5 extra points on this exam).
- Notice that many questions ask for more than just one thing $\Rightarrow$ make sure you read each question to the end.
- Justify your answers

1. (10 points) Let P be a probability function, defined on a sample space S with some σ-algebra $\mathcal{B}$. Using only Kolmogorov's three axioms (and rules for set operations), show that for all $A \in \mathcal{B}$ and $B \in \mathcal{B}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Hint: It might be helpful to first show that $P(A) = P(A \cap B) + P(A \cap B^c)$



We have

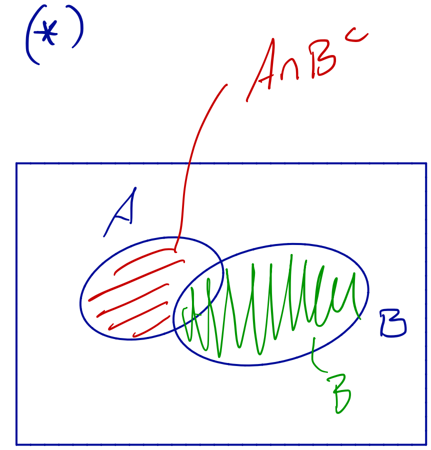
$$A = (A \cap B) \cup (A \cap B^c)$$

where $A \cap B$ and $A \cap B^c$ are disjoint

$$\Rightarrow P(A) = P\left((A \cap B) \cup (A \cap B^c)\right)$$

$$= P(A \cap B) + P(A \cap B^c) \quad \text{by axiom (iii)}$$

$$\Rightarrow P(A \cap B^c) = P(A) - P(A \cap B) \quad (*)$$



We also have

$$A \cup B = B \cup (A \cap B^c)$$

and since B and $A \cap B^c$ are disjoint we get

$$P(A \cup B) = P(B) + P(A \cap B^c) \quad \text{by axiom (iii)}$$

$$= P(B) + P(A) - P(A \cap B) \quad \text{from } (*)$$

$\square$

2. (a) (5 points) A box contains 24 light bulbs, of which two are defective. If a person selects 10 bulbs at random, without replacement, what is the probability that both defective bulbs will be selected?

Sample space: All samples of 10 bulbs

$$|S| = \binom{24}{10}$$

Event A: Pick 2 defective and 8 non-defective

$$|A| = \binom{2}{2}\binom{22}{8}$$

$$\Rightarrow P(A) = \frac{\binom{2}{2}\binom{22}{8}}{\binom{24}{10}} = \frac{1 \cdot \frac{22!}{8!\,14!}}{\frac{24!}{10!\,14!}}$$

$$= \frac{22!\,10!\,14!}{24!\,8!\,14!} = \frac{10 \cdot 9}{24 \cdot 23} = 0.163$$

(b) (5 points) If $P(A) = \frac{1}{3}$ and $P(B^c) = \frac{1}{4}$, can the events A and B be disjoint? Justify your answer.

$$P(B) = 1 - P(B^c) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(A) + P(B) = \frac{1}{3} + \frac{3}{4} = \frac{13}{12} > 1$$

$\Rightarrow$ A and B cannot be disjoint

3. (10 points) Suppose you are given two coins. The first coin is fair and the second coin comes up Heads 75% of the time. (You can't tell from appearances which one is which.) You choose one of the coins at random (each coin has equal chance of being selected) and flip it three times, yielding Heads, Heads, and Tails (H,H,T). What is the probability that the coin you've been flipping is the unfair one?

Define events:

F: Coin is fair

H, T: Get heads, tails

Given:

$$P(H \mid F) = 0.5$$
$$P(H \mid F^c) = 0.75$$
$$P(F) = 0.5$$

Tosses are independent so

$$P(H,H,T \mid F) = 0.5^3$$
$$P(H,H,T \mid F^c) = 0.75^2 * 0.25$$

Also: $$P(H,H,T) = P(H,H,T \mid F)P(F) + P(H,H,T \mid F^c)P(F^c)$$
$$= 0.5^3 * 0.5 + 0.75^2 * 0.25 * 0.5$$
$$= 0.1328$$

Want

$$P(F^c \mid H,H,T) = \frac{P(H,H,T \mid F^c)P(F^c)}{P(H,H,T)}$$
$$= \frac{0.75^2 * 0.25 * 0.5}{0.1328}$$
$$= 0.5294$$

4. (15 points) Determine the distribution for the random variables defined below. Choices include: Binomial(n, p), Geometric(p), NegativeBinomial(r, p), Hypergeometric(N, M, K), and Poisson(λ). Also give the values of the parameters in the distributions. (No need to give the pmf's)

(a) On a desk there are 10 math books, 12 chemistry books and 8 history books. Joe randomly takes 6 of the books. Let Y denote the number of math books Joe takes.

20 books, 10 of them are math books

$\Rightarrow Y \sim \text{Hypergeometric}(N=20, M=10, K=6)$

(b) There is a 25% chance that Jeff will make a three-pointer in a basketball shot. Assume that Jeff's shots are independent. Let Y be the random variable denoting the number of shots until Jeff makes a three pointer.

$Y \sim \text{Geometric}(0.25)$

(c) There is a 25% chance that Jeff will make a three-pointer in basketball. Assume that Jeff's shots are independent. Let Y be the random variable denoting the number of three-pointers Jeff makes in 20 attempts.

$Y \sim \text{Binomial}(n=20, p=0.25)$

(d) There is a 25% chance that Jeff will make a three-pointer in basketball. Assume that Jeff's shots are independent. Let Y be the random variable denoting the number of missed shots before Jeff makes five three-pointers.

$Y \sim \text{NegativeBinomial}(r=5, p=0.25)$

(e) Let Y denote the number of cars that pass through a particular McDonalds drive-through in an hour. Assume that the cars are independent and that average number of cars that go through this drive-through in an hour is 50 cars.

$Y \sim \text{Poisson}(\lambda=50)$

5. (15 points) Let X be a random variable that follows the Poisson distribution with mean λ. Recall that the pmf is $f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$.

Derive the moment generating function (mgf) of X and use it to derive $E(X)$ and $V(X)$.

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=0}^{\infty} \frac{e^{-\lambda}(e^t\lambda)^x}{x!}$$

$$= e^{-\lambda} e^{e^t\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{e^{-e^t\lambda}(e^t\lambda)^x}{x!}}_{=1}$$

$$= e^{-\lambda + e^t\lambda}$$

$$= e^{\lambda(e^t-1)}$$

$$E(X) = \frac{d}{dt} M_X(t)\Big|_{t=0} = \lambda e^t e^{\lambda(e^t-1)}\Big|_{t=0} = \lambda e^0 e^{\lambda(1-1)}$$

$$= \lambda$$

$$E(X^2) = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = \frac{d}{dt} \lambda e^{t+\lambda(e^t-1)}\Big|_{t=0}$$

$$= \lambda(1+\lambda e^t) e^{t+\lambda(e^t-1)}\Big|_{t=0} = \lambda(1+\lambda)$$

$$\Rightarrow Var(X) = \lambda(1+\lambda) - \lambda^2 = \lambda + \lambda^2 - \lambda^2$$

$$= \lambda$$

6. (10 points) Let X be a random variable with pdf

$$f_X(x) = \begin{cases} \frac{3}{8}(x+1)^2 & , -1 < x < 1 \\ 0 & , \text{otherwise} \end{cases}$$

Let $Y = 1 - X^2$. Find the pdf of Y.

$g(x) = 1 - x^2$ is piece-wise monotone on the intervals $(-1, 0]$ and $(0, 1)$

support for Y: $[0, 1)$

$g_1(x) = 1 - x^2$ for $x \in (-1, 0]$

$\Rightarrow \quad x^2 = 1 - y \quad \Rightarrow \quad x = -\sqrt{1-y}$

$g_2(x) = 1 - x^2$ for $x \in (0, 1)$

$\Rightarrow \quad x^2 = 1 - y \quad \Rightarrow \quad x = +\sqrt{1-y}$

$$\frac{d}{dy} g_1^{-1}(y) = +\frac{1}{2\sqrt{1-y}}, \quad \frac{d}{dy} g_2^{-1}(y) = -\frac{1}{2\sqrt{1-y}}$$

$$\Rightarrow f_Y(y) = \sum_{i=1}^{2} f_X\left(g_i^{-1}(y)\right) \left| \frac{d}{dt} g_i^{-1}(y) \right|$$

$$= \frac{3}{8}\left(-\sqrt{1-y}+1\right)^2 \frac{1}{2\sqrt{1-y}} + \frac{3}{8}\left(\sqrt{1-y}+1\right)^2 \frac{1}{2\sqrt{1-y}}$$

$$= \frac{3}{16\sqrt{1-y}}\left(\left((1-y) - 2\sqrt{1-y} + 1 + (1-y) + 2\sqrt{1-y} + 1\right)\right.$$

$$= \frac{3}{16\sqrt{1-y}}(4 - 2y) = \frac{3}{8\sqrt{1-y}}(2-y)$$

for $0 \le y < 1$

7. (10 points) Let X be a random variable with pdf

$$f_X(x) = \begin{cases} \frac{1}{\beta}e^{-x/\beta} & , 0 \leq x \\ 0 & , \text{otherwise} \end{cases}$$

$$\int (uv)' = \int u'v + \int uv'$$

$$uv = \int u'v + \int uv'$$

$$\int uv' = uv - \int u'v$$

Find the mean, median, and mode of X, and then order these three measures of center from smallest to largest.

$$E(x) = \int_0^\infty \frac{x}{\beta} e^{-x/\beta}\,dx = \frac{x}{\beta}(-\beta)e^{-x/\beta}\Big|_{x=0}^{\infty} - \int_0^\infty \frac{1}{\beta}(-\beta)e^{-x/\beta}\,dx$$

$$= \lim_{x\to\infty} \frac{x}{e^{x/\beta}} - 0 + \beta\int_0^\infty \frac{1}{\beta}e^{-x/\beta}\,dx$$

$$= \lim_{x\to\infty} \frac{1}{\frac{1}{\beta}e^{x/\beta}} - 0 + \beta = 0 - 0 + \beta$$

$$= \beta$$

Median: solve for m:

$$\frac{1}{2} = \int_0^m \frac{1}{\beta}e^{-x/\beta}\,dx = -e^{-x/\beta}\Big|_{x=0}^{m} = -e^{-m/\beta} + e^0$$

$$= 1 - e^{-m/\beta}$$

$$\Rightarrow e^{-m/\beta} = \frac{1}{2} \Rightarrow -m/\beta = \log(0.5)$$

$$\Rightarrow m = -\beta\log(0.5)$$

Mode: $\frac{1}{\beta}e^{-x/\beta}$ is a decreasing function of x

$\Rightarrow f(0) = \frac{1}{\beta} \geq f(x) \ \forall x \Rightarrow$ mode $= 0$

Order: mode < median < mean

8. (10 points) State and prove the memoryless property of the Exponential distribution.

If $X \sim Expo(\beta)$ then

$$P(X > t+s \mid X > t) = P(X > s)$$

Proof:

$$P(X > t+s \mid X > t) = \frac{P(X > t+s, X > t)}{P(X > t)}$$

$$= \frac{P(X > t+s)}{P(X > t)} = \frac{e^{-(t+s)/\beta}}{e^{-t/\beta}}$$

since $F(x) = P(X \le x) = 1 - e^{-x/\beta}$

$$= \frac{e^{-t/\beta} e^{-s/\beta}}{e^{-t/\beta}} = e^{-s/\beta} = P(X > s)$$

9. (15 points) Let $X \sim \text{Gamma}(\alpha, \beta)$. The pdf is

$$f(x \mid \alpha, \beta) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} & , 0 < x \\ 0 & , \text{otherwise.} \end{cases}$$

Show that

$$E(X^n) = \frac{\beta^n \Gamma(n+\alpha)}{\Gamma(\alpha)}$$

and then use this expression to find $E(X)$ and $V(X)$.

$$E(X^n) = \int_{-\infty}^{\infty} x^n f(x)\, dx = \int_0^{\infty} x^n \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \frac{\beta^n \Gamma(n+\alpha)}{\Gamma(\alpha)} \int_0^{\infty} \underbrace{\frac{1}{\Gamma(n+\alpha)\beta^{n+\alpha}} x^{n+\alpha-1} e^{-x/\beta}}_{\text{pdf of Gamma}(n+\alpha,\beta)} dx$$

$$= \frac{\beta^n \Gamma(n+\alpha)}{\Gamma(\alpha)}$$

$$\Rightarrow E(X) = \frac{\beta\, \Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\beta\, \alpha\, \Gamma(\alpha)}{\Gamma(\alpha)} = \beta\alpha$$

$$E(X^2) = \frac{\beta^2 \Gamma(\alpha+2)}{\Gamma(\alpha)} = \frac{\beta^2 (\alpha-1)\alpha\Gamma(\alpha)}{\Gamma(\alpha)}$$

$$= \beta^2(\alpha-1)\alpha$$

$$\Rightarrow V(X) = \beta^2\alpha(\alpha-1) - \beta^2\alpha^2 = \beta^2\alpha(\alpha-1-\alpha) = \beta^2\alpha$$

Problem	1	2	3	4	5	6	7	8	9	Total
Missed Score										
out of	10	10	10	15	15	10	10	10	15	100