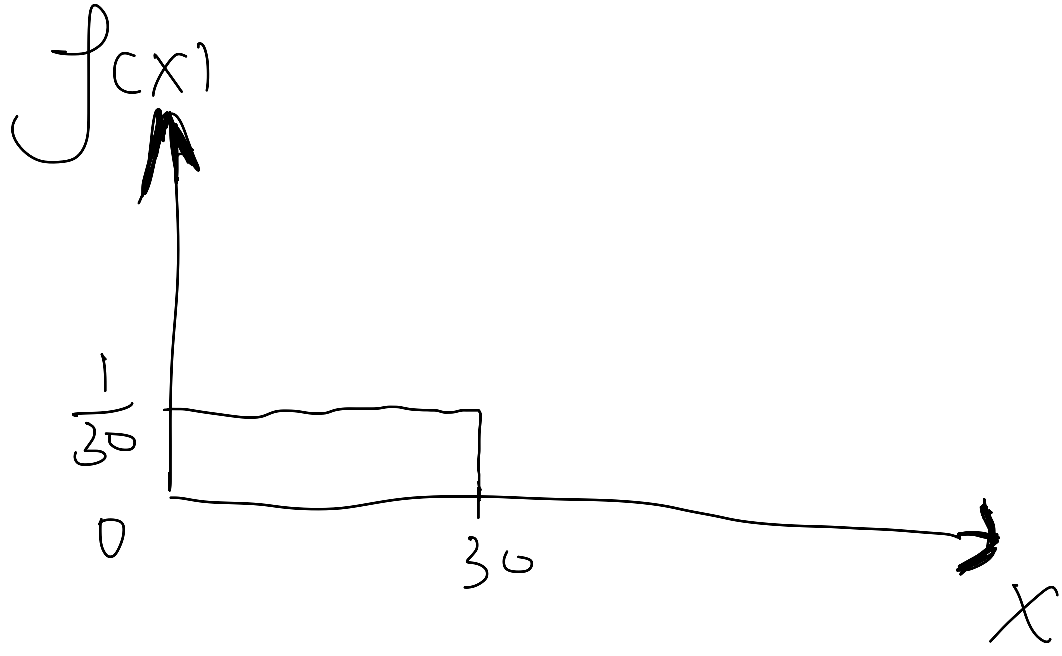
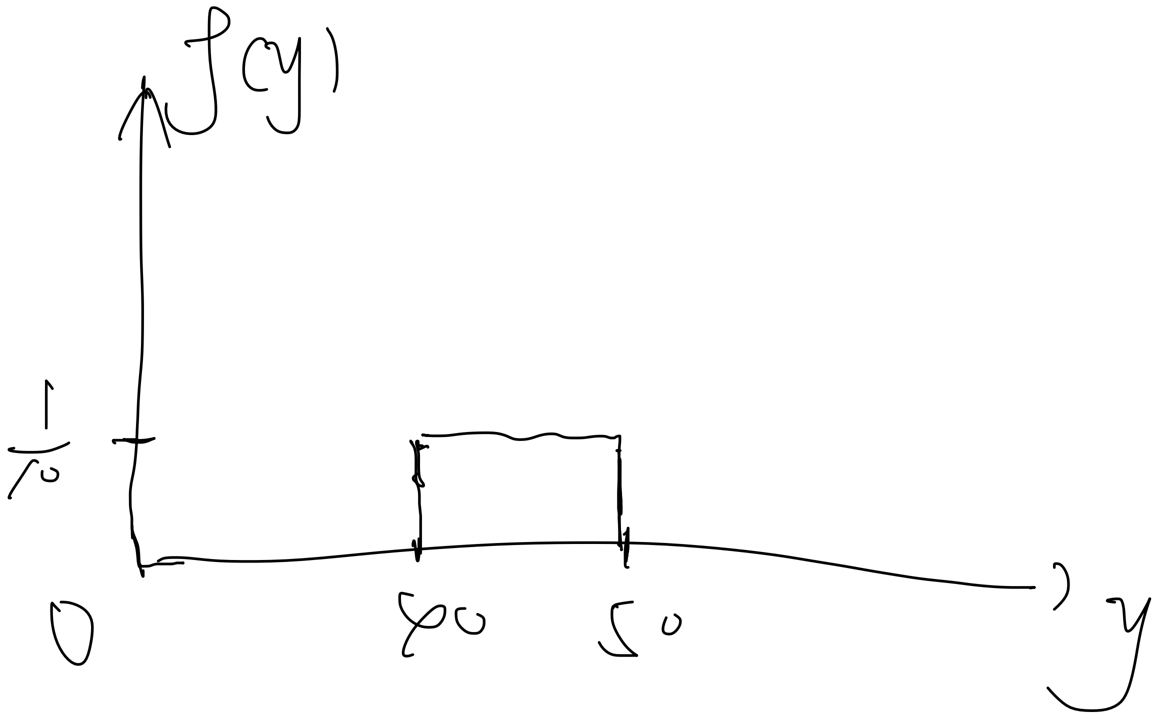
4.7 A woman leaves for work between 8 AM and 8:30 AM and takes between 40 and 50 minutes to get there. Let the random variable X denote her time of departure, and the random variable Y the travel time. Assuming that these variables are independent and uniformly distributed, find the probability that the woman arrives at work before 9 AM.

$$\left.\begin{aligned} X &\in (8{:}00,\ 8{:}30) \\ Y &\in (40,\ 50) \end{aligned}\right\}$$

This question can be transformed that

$$\begin{cases} X \sim \text{Uniform}(0, 30) \\ Y \sim \text{Uniform}(40, 50) \\ P(X+Y<60) = ? \end{cases}$$





$$f_X(x) = \frac{1}{30} \qquad f_Y(y) = \frac{1}{10}$$

Since, X and Y are independent,

$$f(x,y) = f_X(x) \cdot f_Y(y) = \frac{1}{300}.$$

$$\because X + Y < 60 \qquad X \in (0, 30)$$

$$\therefore X \in (0, 60 - Y) \qquad Y \in (40, 50)$$

$$P(X+Y<60) = \int_{40}^{50} \int_{0}^{60-y} f(x,y)\,dx\,dy$$

$$= \int_{40}^{50} \int_{0}^{60-y} \frac{1}{300}\,dx\,dy$$

$$= \int_{40}^{50} \frac{1}{300} x \Big|_{0}^{60-y} dy$$

$$= \int_{40}^{50} \frac{1}{300}(60-y)\,dy$$

$$= \frac{1}{5} y - \frac{1}{600} y^2 \Big|_{60}^{50}$$

$$= \left(\frac{1}{5}\cdot 50 - \frac{1}{600}\cdot 50^2\right) - \left(\frac{1}{5}\cdot 40 - \frac{1}{600}\cdot 40^2\right)$$

$$= 0.5$$

4.10 The random pair (X, Y) has the distribution

		X		
		1	2	3
	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
Y	3	$\frac{1}{6}$	0	$\frac{1}{6}$
	4	0	$\frac{1}{3}$	0

(a) Show that X and Y are dependent.
(b) Give a probability table for random variables U and V that have the same marginals as X and Y but are independent.

(a)

		X			
		1	2	3	$f_Y(y)$
	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{3}$
Y	3	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{3}$
	4	0	$\frac{1}{3}$	0	$\frac{1}{3}$
$f_X(x)$		$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

$$f(1,3) = \frac{1}{6} \neq \frac{1}{3} \cdot \frac{1}{4} = f_X(x=1) \cdot f_Y(y=3)$$

X, Y are not independent.

(b)

		U			
		1	2	3	$p_V(v)$
V	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{3}$
	3	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{3}$
	6	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{3}$
$p_U(u)$		$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	1

4.19 (a) Let X_1 and X_2 be independent n(0,1) random variables. Find the pdf of $(X_1 - X_2)^2/2$.

(b) If $X_i, i = 1, 2$, are independent gamma$(\alpha_i, 1)$ random variables, find the marginal distributions of $X_1/(X_1 + X_2)$ and $X_2/(X_1 + X_2)$.

(a):

$X_1 \sim N(0,1)$ $X_2 \sim N(0,1)$ independent

$$M_{X_1}(t) = E(e^{tX_1}) = e^{\frac{1}{2}t^2}$$

$$M_{X_2}(t) = E(e^{tX_2}) = e^{\frac{1}{2}t^2}$$

$$M_{\frac{X_1}{\sqrt{2}}}(t) = E(e^{t\frac{X_1}{\sqrt{2}}}) = E(e^{\frac{t}{\sqrt{2}}X_1}) = e^{\frac{1}{4}t^2}$$

$$M_{\frac{-X_2}{\sqrt{2}}}(t) = E(e^{t\frac{-X_2}{\sqrt{2}}}) = E(e^{\frac{-t}{\sqrt{2}}X_2}) = e^{\frac{1}{4}t^2}$$

$$M_{\frac{X_1-X_2}{\sqrt{2}}}(t) = e^{\frac{1}{4}t^2} \cdot e^{\frac{1}{4}t^2} = e^{\frac{1}{2}t^2}$$

thus, $\frac{X_1-X_2}{2} \sim N(0,1)$

set $k = \frac{X_1 - X_2}{2}$.

$$f(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}}, \quad k \in R$$

set $g(k) = k^2 = \left(\frac{X_1 - X_2}{2}\right)^2 = X$

$$g'(k) = 2k \begin{cases} 2k > 0 & (0, \infty) \\ 2k < 0 & (\infty, 0) \end{cases}$$

$k = \pm\sqrt{X}$, $X > 0$

$g(k)$ is monotone $\quad \frac{d}{dx} g^{-1}(k) = \pm \frac{1}{2\sqrt{x}}$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{x})^2/2} \left| -\frac{1}{2\sqrt{x}} \right| +$$

$$\frac{1}{\sqrt{2\pi}} e^{-(\sqrt{x})^2/2} \left| \frac{1}{2\sqrt{x}} \right|$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{x}} e^{-x/2}, \quad 0 < x < \infty.$$

$\therefore$ the pdf of $\frac{(X_1 - X_2)^2}{2}$ is

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{x}} e^{-x/2}, \quad 0 < x < \infty.$$

(b) $X_1 \sim gamma(\alpha_1, 1)$, $x_1 > 0$

$X_2 \sim Gamma(\alpha_2, 1)$, $X_2 > 0$

$U = \dfrac{X_1}{X_1 + X_2}$, $U \in (0, 1)$

$V = X_1 + X_2$, $V \in (0, \infty)$

$X_1 = UV \qquad X_2 = V - UV = V(1-U)$

$\dfrac{\partial x_1}{\partial u} = V \quad \dfrac{\partial X_1}{\partial V} = U \quad \dfrac{\partial X_2}{\partial u} = -V \quad \dfrac{\partial X_2}{\partial V} = 1-U$

$$J = \frac{\partial X_1}{\partial u} \cdot \frac{\partial X_2}{\partial V} - \frac{\partial X_2}{\partial u} \cdot \frac{\partial X_1}{\partial V}$$

$$= V(1-U) - U(-V) = V$$

Since X_1, X_2 are independent,

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{\Gamma(\alpha_1)} x_1^{\alpha_1-1} e^{-x_1} \cdot \frac{1}{\Gamma(\alpha_2)} x_2^{\alpha_2-1} e^{-x_2}$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x_1^{\alpha_1-1} x_2^{\alpha_2-1} e^{-(x_1+x_2)}$$

$$f_{U,V}(u,v) = f_{X_1,X_2}(x_1,x_2) \cdot |J|$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (uv)^{\alpha_1-1} (v(1-u))^{\alpha_2-1} e^{-v} \cdot v$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{(\alpha_1-1)} v^{(\alpha_1-1)} \cdot v^{\alpha_2-1} \cdot (1-u)^{\alpha_2-1} \cdot e^{-v} \cdot v$$

$$= \underbrace{\left(\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \cdot u^{\alpha_1-1}(1-u)^{\alpha_2-1}\right)} \cdot \underbrace{\left(\frac{1}{\Gamma(\alpha_1+\alpha_2)} v^{(\alpha_1+\alpha_2-1)} \cdot e^{-v}\right)}$$

$$\frac{X_1}{X_1+X_2} = U \sim \text{Beta}(\alpha_1, \alpha_2) \qquad X_2 = V \sim \text{Gamma}(\alpha_1+\alpha_2, 1)$$

only u function only v function

U and V are independent, because of Lemma.

About $\frac{X_2}{X_1+X_2}$, set $p = \frac{X_2}{X_1+X_2}$, $q = X_1+X_2$

$X_2 = pq$, $X_1 = q - pq = q(1-p)$

$X_1 = q(1-p) > 0$, so $p \in (0,1)$

$\frac{\partial X_1}{\partial p} = -q$ $\frac{\partial X_1}{\partial q} = (1-p)$ $\frac{\partial X_2}{\partial p} = q$ $\frac{\partial X_2}{\partial q} = p$

$J = -q \cdot p - q(1-p) = -q.$

$|J| = q.$

$f_{pq}(p,q) = f_{X_1,X_2}(X_1,X_2) \cdot |J| \cdots\cdots (*)$

$$(*) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \cdot (q(1-p))^{\alpha_1-1} \cdot (pq)^{\alpha_2-1} \cdot e^{-q} \cdot q$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} q^{\alpha_1-1}(1-p)^{\alpha_1-1} \cdot p^{\alpha_2-1} q^{\alpha_2-1} \cdot e^{-q} \cdot q$$

$$= \underbrace{\left(\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} p^{\alpha_2-1}(1-p)^{\alpha_1-1}\right)}_{\text{only } p} \cdot \underbrace{\left(\frac{1}{\Gamma(\alpha_1+\alpha_2)} q^{\alpha_1+\alpha_2-1} e^{-q}\right)}_{\text{only } q}$$

$$p = \frac{X_2}{X_1+X_2} \sim \mathrm{Beta}(\alpha_2, \alpha_1), \quad q = X_1+X_2 \sim \mathrm{Gamma}(\alpha_1+\alpha_2, 1)$$

p, q are independent. according to Lemma.

4.23 For X and Y as in Example 4.3.3, find the distribution of XY by making the transformations given in (a) and (b) and integrating out V.

(a) $U = XY$, $V = Y$

$X \sim Beta(\alpha, \beta)$ $Y \sim Beta(\alpha+\beta, \gamma)$

X, Y are independent.

$$f_{X,Y}(x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} y^{\alpha+\beta-1}(1-y)^{\gamma-1}$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} x^{\alpha-1}(1-x)^{\beta-1} y^{\alpha+\beta-1}(1-y)^{\gamma-1}, \quad x \in (0,1), y \in (0,1)$$

Set $U = XY$, $V = Y$.

$$\frac{U}{V} = X \in (0,1) \quad V \in (0,1)$$

$$0 < \frac{U}{V} < 1, \quad 0 < U < V$$

$$h_1(u,v) = x = \frac{u}{v} \quad h_2(u,v) = y = v$$

$$\frac{\partial x}{\partial u} = \frac{1}{v} \quad \frac{\partial x}{\partial v} = -\frac{u}{v^2} \quad \frac{\partial y}{\partial u} = 0 \quad \frac{\partial y}{\partial v} = 1$$

$$|J| = \left|\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}\right| = \left|\frac{1}{v} - 0\right| = \frac{1}{v}$$

$$f_{U,V}(u,v) = f_{X,Y}(x,y)\,|J|$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\left(\frac{u}{v}\right)^{\alpha-1}\cdot\left(1-\frac{u}{v}\right)^{\beta-1}\cdot v^{\alpha+\beta-1}\cdot(1-v)^{\gamma-1}\cdot\frac{1}{v}$$

where, $0 < u < v < 1$

$$f_U(u) = \int_u^1 \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\, u^{\alpha-1} v^{\beta-1}(1-v)^{\gamma-1}\left(\frac{v-u}{v}\right)^{\beta-1} dv$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\, u^{\alpha-1}(1-u)^{\beta+\gamma-1}\int_0^1 y^{\beta-1}(1-y)^{\gamma-1}\,dy$$

(where $y = \frac{v-u}{1-u}$, $dy = \frac{dv}{1-u}$)

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\, u^{\alpha-1}(1-u)^{\beta+\gamma-1}\cdot\frac{\Gamma(\beta)\cdot\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1}, \quad u \in (0,1)$$

So, $XY = U \sim \text{Gamma}(\alpha, \beta+\gamma)$

4.24 Let X and Y be independent random variables with $X \sim \text{gamma}(r,1)$ and $Y \sim \text{gamma}(s,1)$. Show that $Z_1 = X+Y$ and $Z_2 = X/(X+Y)$ are independent, and find the distribution of each. (Z_1 is gamma and Z_2 is beta.)

$X \sim \text{gamma}(r,1)$, $Y \sim \text{gamma}(s,1)$

X, Y are independent.

$$f_{X,Y}(x,y) = \frac{1}{\Gamma(r)} x^{r-1} e^{-x} \cdot \frac{1}{\Gamma(s)} y^{s-1} e^{-y}, \quad x, y > 0$$

$$Z_1 = X+Y \qquad Z_2 = X/(X+Y)$$

$$X = Z_1 \cdot Z_2 \qquad Y = Z_1 - Z_1 Z_2$$

$$\begin{cases} Z_1 Z_2 > 0 \\ Z_1 - Z_1 Z_2 > 0 \end{cases} \Rightarrow \begin{cases} Z_1 > 0 \\ Z_2 \in (0,1) \end{cases}$$

$$X = h_1(Z_1, Z_2) = Z_1 Z_2 \qquad Y = h_2(Z_1, Z_2) = Z_1(1-Z_2)$$

$$\frac{\partial X}{\partial Z_1} = Z_2 \quad \frac{\partial X}{\partial Z_2} = Z_1, \quad \frac{\partial Y}{\partial Z_1} = (1-Z_2) \quad \frac{\partial Y}{\partial Z_2} = -Z_1$$

$$|J| = \left| Z_2 \cdot (-Z_1) - Z_1(1-Z_2) \right| = Z_1$$

$$f_{Z_1,Z_2}(z_1,z_2) = \frac{1}{\Gamma(r)}(z_1 z_2)^{r-1} e^{-z_1 z_2} \cdot \frac{1}{\Gamma(s)} \left(z_1(1-z_2)\right)^{s-1} e^{z_1(z_2-1)}$$

$$= \frac{1}{\Gamma(r)} z_1^{r-1} \cdot z_2^{r-1} e^{-z_1} \frac{1}{\Gamma(s)} z_1^{s-1} (1-z_2)^{s-1}$$

$$= \underbrace{\frac{1}{\Gamma(r+s)} z_1^{r+s-1} \cdot e^{-z_1}}_{Z_1 \sim \text{gamma}(r+s,1)} \cdot \underbrace{\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} z_2^{r-1} (1-z_2)^{s-1}}_{Z_2 \sim \text{Beta}(r,s)}$$

only z_1 only z_2

Z_1, Z_2 are independent.

$z_1 \in (0,\infty)$, $z_2 \in (0,1)$.