STAT 345/445 Lecture 11

Families of Continuous Distributions – Section 3.3

Families of Continuous Distributions

We will learn about some of the most commonly used continuous distributions, including their

- $f(x)$ (usually $F(x)$ is not available in closed form)
  - Notation for pdf that emphasizes the parameters:

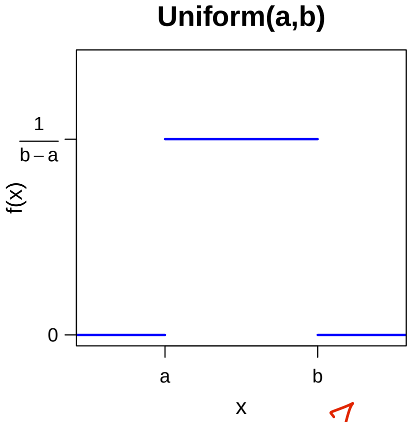
$$f(x \mid \theta)$$

- parameter space Θ and support $\mathcal{X} = \{x : f(x) > 0\}$
- $E(X)$, $\mathrm{Var}(X)$, $M(t)$
- special features and connections between distributions

See tables p. 621-627 in the Textbook

Uniform Distributions

- Probability mass is evenly spread over an interval $[a, b]$



$$\int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}\,x\Big|_a^b$$

$$= \frac{b-a}{b-a} = 1$$

$$F(x) = \int_a^x \frac{1}{b-a}\,du = \frac{1}{b-a}\,u\Big|_{u=a}^{x}$$

$$= \frac{x-a}{b-a} \quad x \in [a,b]$$

$$F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & x \in [a,b] \\ 1, & x > b \end{cases}$$

Uniform Distributions – $\mathrm{Uniform}(a, b)$

Probability density function

$$f(x \mid a, b) = \frac{1}{b - a} \qquad \text{for } x \in [a, b]$$

- Parameter space: $-\infty < a \leq b < \infty$
- cdf: $F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$

Mean and Variance

$$E(X) = \frac{a+b}{2} \qquad \mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

Moment generating function

$$M_X(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}$$

Beta Distributions – $\mathrm{Beta}(\alpha, \beta)$

- Flexible family of distributions with bounded support
- Defined on $X \in [0, 1]$
  - Often used to model proportions
- Can be transformed to have support on a bounded interval $[a, b]$:

$$Y = a + bX$$

- Recall the Gamma function, for any $\alpha > 0$: $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$
  - $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$
  - $\Gamma(n) = (n - 1)!$ for a positive integer n
  - $\Gamma(0.5) = \sqrt{\pi}$

Beta Distributions – $\mathrm{Beta}(\alpha, \beta)$

Probability density function

$$f(x \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1}$$
$$= \frac{1}{B(\alpha,\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1} \qquad \text{for } x \in [0,1]$$

- Parameter space: $\alpha > 0$, $\beta > 0$
- Special case: $\mathrm{Beta}(1,1) = \mathrm{Uniform}(0,1)$
- **Beta function:**

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}\,(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Beta special case: $\alpha = 1$, $\beta = 1$

$$f(x) = \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} x^0 (1-x)^0 = \frac{1!}{0!\,!} = 1, \text{ for } x \in [0,1]$$

$= \text{pdf for Uniform } (0,1)$

Known integral: For $\alpha > 0$, $\beta > 0$

$$\int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx = 1$$

$$E(X^n) = \int_0^1 x^n \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\overbrace{\alpha+n-1}^{=\tilde{\alpha}}} (1-x)^{\beta-1} \, dx \qquad \tilde{\alpha} = n+\alpha .$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\tilde{\alpha})}{\Gamma(\tilde{\alpha}+\beta)} \underbrace{\int_0^1 \frac{\Gamma(\tilde{\alpha}+\beta)}{\Gamma(\tilde{\alpha})\Gamma(\beta)} x^{\tilde{\alpha}-1} (1-x)^{\beta-1} \, dx}_{\text{pdf of Beta}(\alpha+n,\,\beta)}$$

$$\Rightarrow E(X^n) = \frac{\Gamma(\alpha+\beta)\,\Gamma(\alpha+n)}{\Gamma(\alpha)\,\Gamma(\alpha+n+\beta)}$$

For $n = 1$

$$Z(x) = \frac{\alpha}{\alpha + \beta}$$

Beta Distributions – $\mathrm{Beta}(\alpha, \beta)$

Mean and Variance

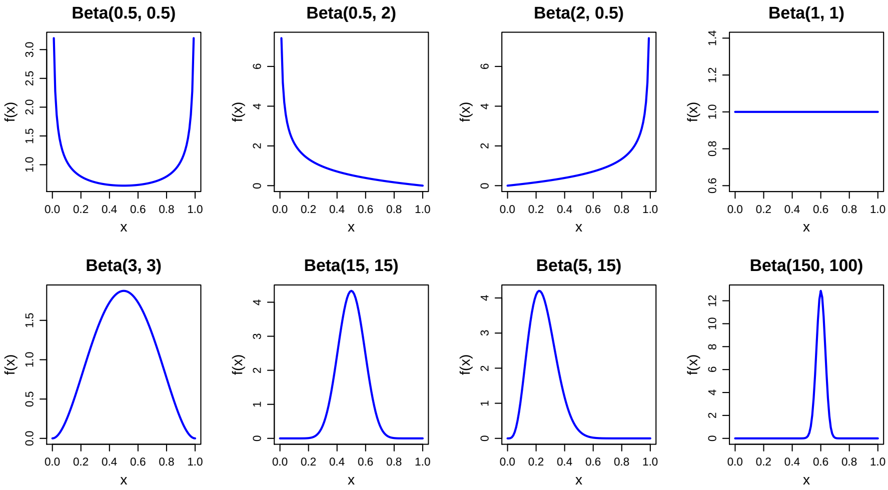
$$E(X) = \frac{\alpha}{\alpha + \beta} \qquad \mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2\,(\alpha+\beta+1)}$$

Moment generating function

$$M_X(t) = \text{ugly (see book) but:}$$

$$E(X^n) = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(\alpha+\beta+n)}$$

Beta pdfs


Beta(0.5, 0.5)
Beta(0.5, 2)
Beta(2, 0.5)
Beta(1, 1)
Beta(3, 3)
Beta(15, 15)
Beta(5, 15)
Beta(150, 100)
f(x)
x
0.0 0.2 0.4 0.6 0.8 1.0

Gamma distributions – $\mathrm{Gamma}(\alpha, \beta)$

Probability density function

$$f(x \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\, x^{\alpha-1}\, e^{-x/\beta} \qquad \text{for } x > 0$$

- Parameter space: $\alpha > 0$, $\beta > 0$
- Several special cases ...

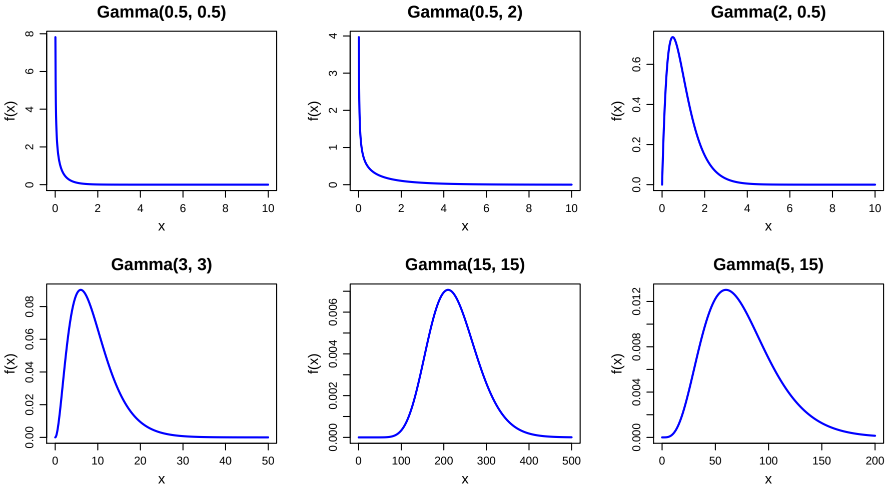
Mean and Variance

$$E(X) = \alpha\beta \qquad \mathrm{Var}(X) = \alpha\beta^2$$

Moment generating function

$$M_X(t) = \frac{1}{(1 - t\beta)^\alpha} \qquad \text{for } t < \frac{1}{\beta}$$

Gamma pdfs


Gamma(0.5, 0.5)
Gamma(0.5, 2)
Gamma(2, 0.5)
Gamma(3, 3)
Gamma(15, 15)
Gamma(5, 15)
f(x)
x

Chi-square distributions – χ^2_p

Special case of Gamma distributions

- $\mathrm{Gamma}(p/2, 2)$ for $p = 1, 2, 3, \ldots$ is called the **Chi-square distribution with p degrees of freedom**

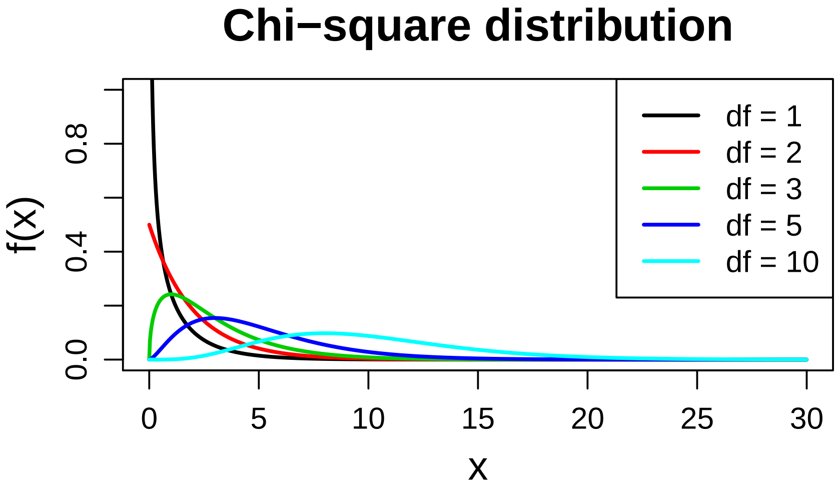
- pdf:

$$f(x) = \frac{1}{\Gamma\left(\frac{p}{2}\right)\ 2^{p/2}}\ x^{\frac{p}{2}-1}\ e^{-x/2} \qquad \text{for } x > 0$$

- If $X \sim \chi^2_p$ then $E(X) = p$ and $\mathrm{Var}(X) = 2p$

- Very important distribution for statistical inference

Chi-square pdfs



Exponential distributions – $\mathrm{Expo}(\beta)$

Special case of Gamma distributions

- $\mathrm{Gamma}(1, \beta)$ for $\beta > 0$ is called the **Exponential distribution**
- pdf:

$$f(x \mid \beta) = \frac{1}{\beta}\, e^{-x/\beta} \qquad \text{for } x > 0$$

Support : $(0, \infty)$

- cdf:

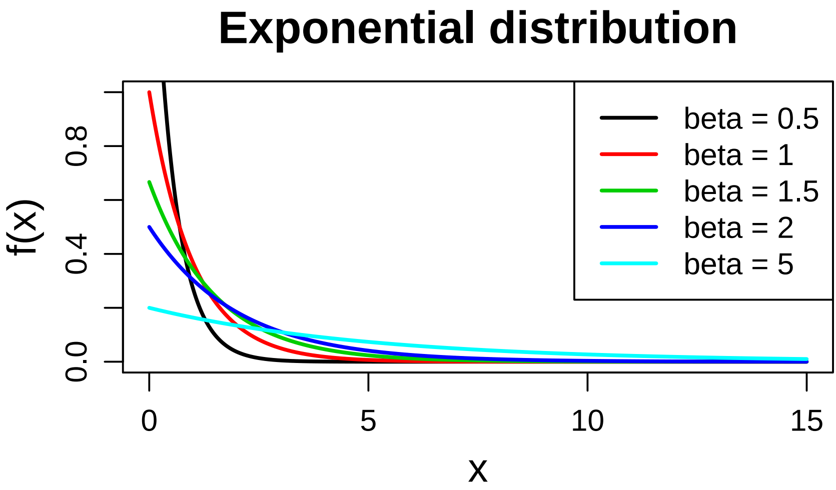
$$F(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-x/\beta} & x > 0 \end{cases}$$

- Memoryless property: If $X \sim \mathrm{Expo}(\beta)$, $t > 0$, and $h > 0$ then

Homework $\longrightarrow$

$$P(X > t + h \mid X > t) = P(X > h)$$

Exponential pdfs



Relationship between Gamma and Poisson

- A **Poisson process** describes events that happen at random times (or places)
  - See Poisson postulates in Section 3.8.1
- In a Poisson process
  - the number of events in an interval has a Poisson distribution
  - the time until the next event has an Exponential distribution
  - the time until the r^{th} event has a Gamma distribution
- Let $X \sim \mathrm{Gamma}(r, \beta)$ where r is an integer. Then for any x

$$P(X \leq x) = P(Y \geq r)$$

where $Y \sim \mathrm{Poisson}(x/\beta)$

Easy to see for $r = 1$

$$P(X \le x) = 1 - e^{-x/\beta}$$

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - \frac{e^{-x/\beta}(x/\beta)^0}{0!}$$

$$Y \sim \text{Poisson}(x/\beta)$$

Double exponential distributions – $\mathrm{DExpo}(\mu, \sigma)$

Probability density function

$$f(x \mid \mu, \sigma) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma} \qquad \text{for } -\infty < x < \infty$$

if $x \geq \mu$, then $|x-\mu| = x-\mu$.

$x < \mu$, then $|x-\mu| = \mu - x$

- Parameter space: $\mu \in \mathbb{R}, \sigma > 0$
- Also called the **Laplace distributions**

$$\int_{-\infty}^{\infty} f(x|\mu,\sigma) = \int_{-\infty}^{\mu} \frac{1}{2\sigma} e^{(x-\mu)/\sigma} dx + \int_{\mu}^{\infty} \frac{1}{2\sigma} e^{-(x-\mu)/\sigma} dx$$

Mean and Variance

$$E(X) = \mu \qquad \mathrm{Var}(X) = 2\sigma^2$$

$$= \frac{1}{2\sigma} e^{-(x-\mu)/\sigma} \cdot (-\sigma) \Big|_{x=\mu}^{\infty} + \frac{1}{2\sigma} e^{(x-\mu)/\sigma} \cdot \sigma \Big|_{x=-\infty}^{\mu}$$

⋮

$(*)$

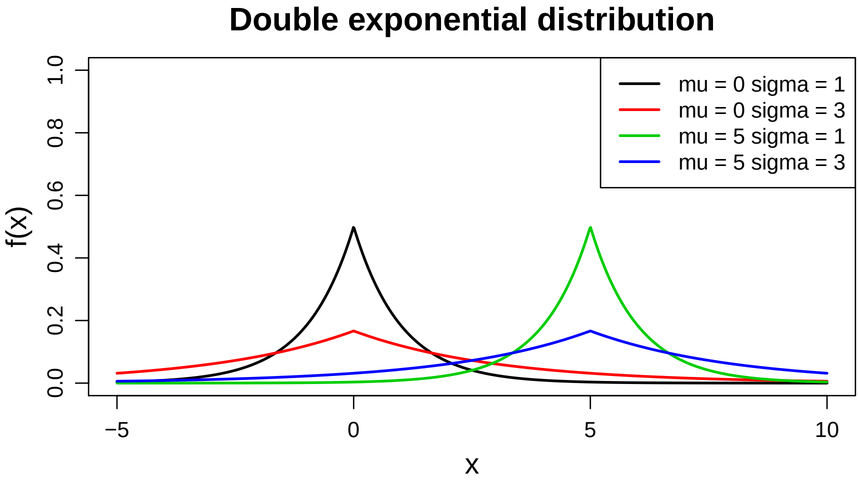
Moment generating function

$$M_X(t) = \frac{e^{\mu t}}{1-(\sigma t)^2} \qquad \text{for } |t| < \frac{1}{\sigma}$$

$$(*) = \frac{-1}{2}\left(\lim_{x\to\infty} e^{-(x-\mu)/s} - e^{-(\mu-\mu)/s}\right) + \frac{1}{2}\left(e^{(\mu-\mu)/s} - \lim_{x\to-\infty} e^{(x-\mu)/s}\right)$$

$$= -\frac{1}{2}(0-1) + \frac{1}{2}(1-0) = \frac{1}{2} + \frac{1}{2} = 1$$

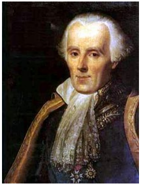
Double exponential pdfs



Normal Distributions – $\mathrm{N}(\mu, \sigma^2)$

- Works well in practice. Many physical experiments have distributions that are approximately normal
- Central Limit Theorem: Sum of many independent random variables (with the same distribution) are approximately normally distributed
- Mathematically convenient
  - especially the multivariate normal distribution.
- Developed by Gauss and then Laplace in the early 1800s
- Also known at the **Gaussian distribution**

Gauss



Laplace

Normal Distributions – $\mathrm{N}(\mu, \sigma^2)$

Probability density function

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} \qquad \text{for } -\infty < x < \infty$$

- Parameter space: $\mu \in \mathbb{R}, \sigma^2 > 0$
- Also called the **Gaussian distributions**

Mean and Variance

$$E(X) = \mu \qquad \mathrm{Var}(X) = \sigma^2$$

Moment generating function

$$M_X(t) = e^{\mu t + \sigma^2 t^2/2}$$

Show : if $X \sim N(\mu, \sigma^2)$ then.

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

$$F(z) = P(Z \le z) = P\left(\frac{X-\mu}{\sigma} \le z\right)$$

$\longrightarrow$ pdf method.

$$g(z) = \frac{x-\mu}{\sigma} = z$$

$$x = \sigma z + \mu = g^{-1}(z)$$

$$\frac{d}{dz} g^{-1}(z) = \underset{>0}{\sigma}, \quad -\infty < x < \infty, \quad -\infty < z < \infty$$

$$f(z) = f_x(g^{-1}(z)) \left| \frac{d}{dz} g^{-1}(z) \right| = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(\mu+\sigma z-\mu)^2/2\sigma^2} \cdot \sigma$$

$$= \frac{1}{\sqrt{2\pi}} e^{-(\sigma z)^2/2\sigma^2} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} = \text{pdf for } N(0,1)$$

Same steps for $Z \sim N(0,1) \Rightarrow X = \mu + \sigma Z \sim N(\mu, \sigma^2)$

Let $Z \sim N(0,1)$

$$M(t) = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2 + tz} dz$$

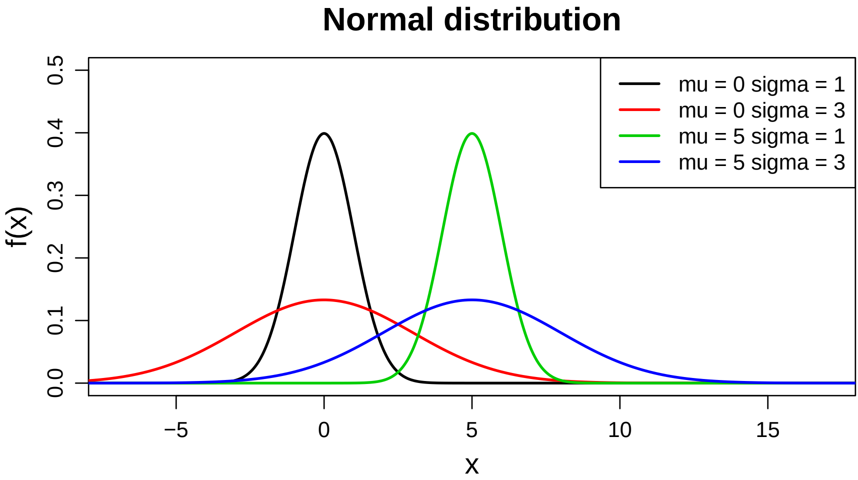
pdf for $N(\mu, \sigma^2)$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2 - 2zt + t^2 - t^2)} dz = \cdots (*)$$

$$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x^2-2x\mu+\mu^2)}$$

$$(A) = e^{t^2/2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2-2zt+t^2)}\, dz}_{\text{pdf of } N(t,1)} = e^{t^2/2}$$

Normal pdfs


Normal distribution
mu = 0 sigma = 1
mu = 0 sigma = 3
mu = 5 sigma = 1
mu = 5 sigma = 3
f(x)
0.0
0.1
0.2
0.3
0.4
0.5
−5
0
5
10
15
x

Standard Normal Distribution – $\mathrm{N}(0, 1)$

- $N(0, 1)$ is called the **standard normal distribution**
- Tradition: Use Z for a $N(0, 1)$ random variable
- Tradition: Use $\phi(\cdot)$ and $\Phi(\cdot)$ for pdf and cdf instead of $f(\cdot)$ and $F(\cdot)$

Theorem

- If $X \sim N(\mu, \sigma^2)$ then $Z = \dfrac{X - \mu}{\sigma} \sim N(0, 1)$
- If $Z \sim N(0, 1)$ then $X = \mu + \sigma Z \sim N(\mu, \sigma^2)$

Derive the mgf, mean, and variance for the normal

- Can start by finding the mgf for $N(0,1)$ and then use the fact that

$$M_{aX+b}(t) = e^{bt} M_X(at)$$

$Z \sim N(0,1), \quad M_Z(t) = e^{t^2/2}$

$\Rightarrow M_{\mu+\sigma Z}(t) = e^{\mu t + \sigma^2 t^2/2}$

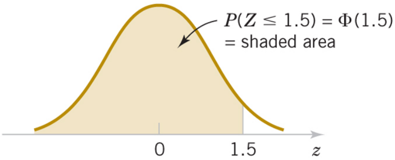
The normal cdf

- The cdf for a normal distribution:

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(u-\mu)^2}{2\sigma^2}} du$$

- Cannot be expressed in closed for and is evaluated using numerical approximations
- Use computer (e.g. R), calculator, or a standard normal probability tables

Standard normal table



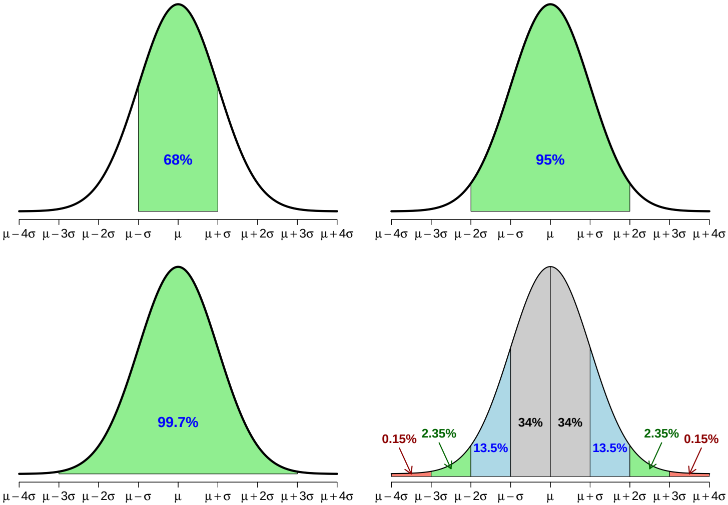
z	0.00	0.01	0.02	0.03
0	0.50000	0.50399	0.50398	0.51197
$\vdots$		$\vdots$		
1.5	0.93319	0.93448	0.93574	0.93699

- Look up z in the table to find $\Phi(z) = P(Z \le z)$

  up to 2 digits

- Examples:
  - $P(Z \le 1.50) = \Phi(1.50) = 0.93319$
  - $P(Z \le 1.51) = \Phi(1.51) = 0.93448$
  - $P(Z \le 1.52) = \Phi(1.52) = 0.93574$

Empirical Rule



Cauchy Distributions – $\mathrm{Cauchy}(\theta, \sigma)$

Probability density function

$$f(x \mid \theta, \sigma) = \frac{1}{\pi\,\sigma}\,\frac{1}{1 + (x-\theta)^2/\sigma^2} \qquad \text{for } -\infty < x < \infty$$

- Parameter space: $\mu \in \mathbb{R}, \sigma^2 > 0$
- A go to extreme case of a distribution without moments or an mgf
- $\theta =$ median = mode

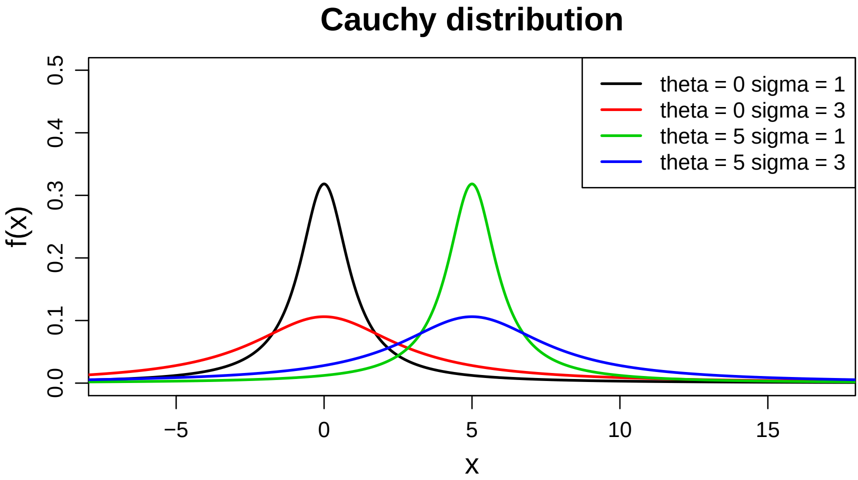
Mean and Variance

$$E(X) = \text{does not exist} \qquad \mathrm{Var}(X) = \text{does not exist}$$

Moment generating function

$$M_X(t) = \text{does not exist}$$

Cauchy pdfs


Cauchy distribution
theta = 0 sigma = 1
theta = 0 sigma = 3
theta = 5 sigma = 1
theta = 5 sigma = 3
f(x)
0.0
0.1
0.2
0.3
0.4
0.5
−5
0
5
10
15
x

LogNormal Distributions – $\mathrm{LogNormal}(\mu, \sigma)$

Probability density function

$0 \le x < \infty$

$$f(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\, x\, \sigma} e^{-(\ln(x)-\mu)^2/2\sigma^2} \qquad \text{for } -\infty < x < \infty$$

- Parameter space: $\mu \in \mathbb{R}, \sigma^2 > 0$
- If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $Y = e^X \sim \mathrm{LogNormal}(\mu, \sigma^2)$
- If $X \sim \mathrm{LogNormal}(\mu, \sigma^2)$ then $Y = \ln(X) \sim \mathrm{N}(\mu, \sigma^2)$

Mean and Variance

$$E(X) = e^{\mu + \sigma^2/2} \qquad \mathrm{Var}(X) = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}$$

Moment generating function

$$M_X(t) = \text{does not exist, but:} \quad E(X^n) = e^{n\mu + n^2\sigma^2/2}$$

Can use Normal Distribution tool to calculate prob for a LogNormal.

E.g. $X \sim \text{LogNormal}(\mu, \sigma^2)$ then

$$P(X \le x) = P(\ln(X) \le \ln(x))$$

$$= F(\ln(x)) \quad F \text{ is the cdf of } N(\mu, \sigma^2)$$

$$= \Phi\left(\frac{\ln(x) - \mu}{\sigma}\right)$$

Let $X \sim N(\mu, \sigma^2)$, $Y = e^X$

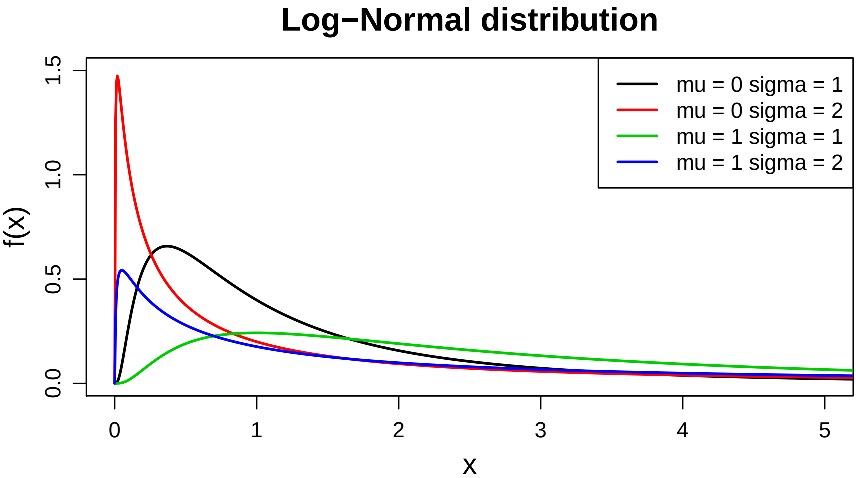
$g(x) = e^x = y \Rightarrow x = \ln(y) = g^{-1}(y)$

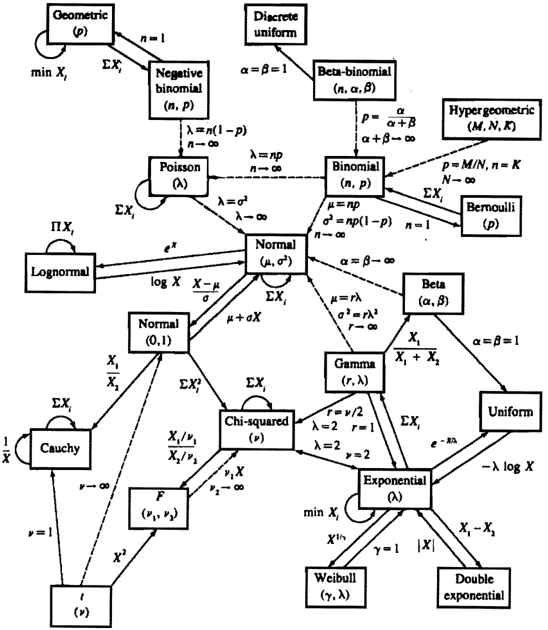
$\frac{d}{dy} g^{-1}(y) = \frac{1}{y}$, $-\infty < x < \infty$, $y > 0$

$\Rightarrow f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(\ln(y) - \mu)^2 / 2\sigma^2} \cdot \frac{1}{y}$,

for $y > 0$

LogNormal pdfs





Relationships among common distributions. Solid lines represent transformations and