STAT 345/445, PQHS 481: Theoretical Statistics I

Selected Textbook Exercises from Sections 3.1-3.3 material

1. **Exercise 3.1**
Find expressions for $E(X)$ and $\mathrm{Var}(X)$ if X is a random variable with the general discrete uniform(N_0, N_1) distribution that puts equal probability on each of the values $N_0, N_0+1, \ldots, N_1$. Here $N_0 \leq N_1$ and both are integers.

Since the possible outcomes for X are $N_0, N_0+1, N_0+2, \ldots, N_1$ there are in total $N = N_1 - N_0 + 1$ possible outcomes. The pmf is therefore

$$f(x) = \frac{1}{N_1 - N_0 + 1} \qquad \text{for } x = N_0, N_0+1, N_0+2, \ldots, N_1$$

The easy way: Note that $X = Y + N_0 - 1$ where $Y \sim \mathrm{DiscrUniform}(N = N_1 - N_0 + 1)$. Therefore

$$E(X) = E(Y + N_0 - 1) = E(Y) + N_0 - 1 = \frac{N_1 - N_0 + 1 + 1}{2} + N_0 - 1 = \frac{N_1 + N_0}{2}$$
$$V(X) = V(Y + N_0 - 1) = V(Y) = \frac{(N_1 - N_0 + 2)(N_1 - N_0)}{12}$$

From scratch: First recall that in general

$$\sum_{x=1}^{n} x = \frac{n(n+1)}{2} \qquad \text{and} \qquad \sum_{x=1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}$$

Now, let's find the mean $E(X)$:

$$E(X) = \sum_{x=N_0}^{N_1} x \frac{1}{N_1 - N_0 + 1} = \frac{1}{N_1 - N_0 + 1} \sum_{x=N_0}^{N_1} x \tag{1}$$

Here we can recognize that

$$\sum_{x=N_0}^{N_1} x = \sum_{x=1}^{N_1} x - \sum_{x=1}^{N_0 - 1} x = \frac{N_1(N_1+1)}{2} - \frac{(N_0 - 1)N_0}{2}$$

so

$$E(X) = \frac{N_1^2 + N_1 - N_0^2 + N_0}{2(N_1 - N_0 + 1)} = \frac{(N_1 + N_0)(N_1 - N_0 + 1)}{2(N_1 - N_0 + 1)} = \frac{N_1 + N_0}{2}$$

Or we can use a substitution in (1) and set $t = x - N_0 + 1$ and get

$$\begin{aligned} E(X) &= \frac{1}{N_1 - N_0 + 1} \sum_{t=1}^{N_1 - N_0 + 1} (t + N_0 - 1) \\ &= \frac{1}{N_1 - N_0 + 1} \left(\frac{(N_1 - N_0 + 1)(N_1 - N_0 + 2)}{2} + (N_0 - 1)(N_1 - N_0 + 1) \right) \\ &= \frac{(N_1 - N_0 + 1)(N_1 - N_0 + 2 + 2(N_0 - 1))}{2(N_1 - N_0 + 1)} = \frac{N_1 + N_0}{2} \end{aligned}$$

Next, to find the variance $\text{Var}(X)$ we start with

$$E(X^2) = \sum_{x=N_0}^{N_1} x^2 \frac{1}{N_1 - N_0 + 1} = \frac{1}{N_1 - N_0 + 1}\left(\sum_{x=1}^{N_1} x^2 - \sum_{x=1}^{N_0-1} x^2\right)$$
$$= \frac{1}{N_1 - N_0 + 1}\left(\frac{N_1(N_1+1)(2N_1+1)}{6} - \frac{(N_0-1)N_0(2N_0-1)}{6}\right)$$

and get

$$\begin{aligned}
\text{Var}(X) &= \frac{1}{N_1 - N_0 + 1}\left(\frac{(N_1^2+N_1)(2N_1+1)}{6} - \frac{(N_0^2-N_0)(2N_0-1)}{6}\right) - \frac{(N_1+N_0)^2}{4} \\
&= \frac{2(2N_1^3 + N_1^2 + 2N_1^2 + N_1 - 2N_0^3 + N_0^2 + 2N_0^2 - N_0) - 3(N_1^2 + 2N_1N_0 + N_0^2)(N_1 - N_0 + 1)}{12(N_1 - N_0 + 1)} \\
&= \frac{1}{12(N_1 - N_0 + 1)}\left(4N_1^3 + 6N_1^2 + 2N_1 - 4N_0^3 + 6N_0^2 - 2N_0\right. \\
&\qquad \left.-3N_1^3 + 3N_1^2N_0 - 3N_1^2 - 6N_1^2N_0 + 6N_1N_0^2 - 6N_1N_0 - 3N_1N_0^2 + 3N_0^3 - 3N_0^2\right) \\
&= \frac{N_1^3 - N_0^3 + 3N_1^2 + 3N_0^2 + 2N_1 - 2N_0 - 3N_1^2N_0 + 3N_1N_0^2 - 6N_1N_0}{12(N_1 - N_0 + 1)} \\
&= \frac{(N_1 - N_0 + 2)(N_1 - N_0)}{12}
\end{aligned}$$

The last steps comes from noting that $(N_1 - N_0 + 1)(N_1 - N_0 + 2)(N_1 - N_0)$ is equal to the numerator.

2. **Exersice 3.4**
   A man with n keys wants to open his door and tries the keys at random. Exactly one key will open the door. Find the mean number of trials if

   (a) unsuccessful keys are not eliminated from further selections.
   Since the unsuccessful keys are not eliminated there is $1/n$ chance of success each time. Also the "tries the keys at random" can be interpreted as the trials being independent. Therefore, if X denotes the number of trials until success we have

   $$X \sim \text{Geometric}\left(p = \frac{1}{n}\right) \Rightarrow E(X) = \frac{1}{1/n} = n$$

   (b) unsuccessful keys are eliminated.
   Let X denote the number of trials. The possible values of X are $\{1, 2, 3, \ldots, n\}$ as in part (a) but here the probability of success is not constant. In fact

   $$\begin{aligned}
   f(1) &= P(X=1) = \frac{1}{n} \\
   f(2) &= P(X=2) = \frac{n-1}{n}\frac{1}{n-1} = \frac{1}{n} \\
   f(3) &= P(X=3) = \frac{n-1}{n}\frac{n-2}{n-1}\frac{1}{n-2} = \frac{1}{n} \\
   f(4) &= P(X=4) = \frac{n-1}{n}\frac{n-2}{n-1}\frac{n-3}{n-2}\frac{1}{n-3} = \frac{1}{n} \\
   &\vdots \\
   f(x) &= P(X=x) = \frac{1}{n}
   \end{aligned}$$

   So $X \sim \text{DistreteUniform}(n)$ with $E(X) = \frac{n+1}{2}$

3. **Exersice 3.5**
A standard drug is known to be effective in 80% of the cases in which it is used. A new drug is tested on 100 patients and found to be effective in 85 cases. Is the new drug superior? (Hint: Evaluate the probability of observing 85 or more successes assuming that the new and old drugs are equally effective.)

Assume that X is the number of cases the new drug is effective out of 100 cases. Then $X \sim \text{Binomial}(100, p)$. Assuming that the new drug has the same efficacy as the old one, the probability of getting 85 or more successes is

$$P(X \geq 85) = \sum_{x=85}^{100} \binom{100}{x} 0.8^x 0.2^{100-x} = 0.1285$$

About 13% chance. So, based on these data we can't say with confidence that the new drug is superior to the old one since getting 85 success or more our of 100 samples has a 13% chance for the old drug.

4. **Exersice 3.7**
Let the number of chocolate chips in a certain type of cookie have a Poisson distribution.We want the probability that a randomly chosen cookie has at least two chocolate chips to be greater than .99. Find the smallest value of the mean of the distribution that ensures this probability.

Let $X \sim \text{Poisson}(\lambda)$ be the number of chocolate chips in a cookie. We want

$$0.99 < P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - e^{\lambda} \left(\frac{\lambda^0}{0!} + \frac{\lambda}{1!} \right) = 1 - e^{-\lambda}(1 + \lambda)$$

$$\Rightarrow e^{-\lambda}(1 + \lambda) < 0.01$$

Trial and error yields $\lambda \approx 6.638$

5. **Exersice 3.17**
Establish a formula similar to (3.3.18) for the gamma distribution. If $X \sim \text{Gamma}(\alpha, \beta)$, then for any positive constant ν,

$$E\left(X^\nu\right) = \frac{\beta^\nu \Gamma(\nu + \alpha)}{\Gamma(\alpha)}$$

$$\begin{aligned}
E\left(X^\nu\right) &= \int_0^\infty x^\nu \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} \Gamma(\alpha + \nu)\beta^{\alpha+\nu} \int_0^\infty \frac{1}{\Gamma(\alpha + \nu)\beta^{\alpha+\nu}} x^{\alpha+\nu-1} e^{-x/\beta} dx \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} \Gamma(\alpha + \nu)\beta^{\alpha+\nu} = \frac{\beta^\nu \Gamma(\nu + \alpha)}{\Gamma(\alpha)}
\end{aligned}$$