STAT 445

Theoretical Statistics I

Fall Semester 2017

Quiz 2

Name: Solution

- You have 30 min to complete this quiz
- Justify your answers
- Evaluate expressions as much as you can

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1. (6 points) Show that the family of Beta distributions is an exponential family. Recall that the pdf of a Beta(α, β) distribution is given by

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1}\,\mathrm{I}_{[0,1]}(x) \qquad \forall x \in \mathbb{R}$$

Rewrite $f(x)$ as:

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} I_{[0,1]}(x) \exp\{(\alpha-1)\log x + (\beta-1)\log(1-x)$$

set

$$c(\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$h(x) = I_{[0,1]}(x)$$

$$t_1(x) = \log x$$

$$w_1(\alpha,\beta) = \alpha - 1$$

$$t_2(x) = \log(1-x)$$

$$w_2(\alpha,\beta) = \beta - 1$$

$$k = 2$$

Then

$$f(x) = c(\alpha,\beta)\, h(x) \exp\left\{\sum_{i=1}^{k} t_i(x)\, w_i(\alpha,\beta)\right\} \quad \forall x \in \mathbb{R}$$

$\Rightarrow$ Exponential family

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2. The joint probability mass function (pmf) of two discrete random variables is given by

$$f(x,y) = \begin{cases} \frac{1}{42}(2x+y) & \text{for } x = 0,1,2, \text{ and } y = 0,1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

(a) (8 points) Find the marginal pmfs of X and Y

Hint: Give the value of the pmf for every possible outcome for X, and then the same for Y.

$f(x,y)$:

	0	1	2	3	$f_X(x)$
0	$\frac{0}{42}$	$\frac{1}{42}$	$\frac{2}{42}$	$\frac{3}{42}$	$\frac{6}{42}$
1	$\frac{2}{42}$	$\frac{3}{42}$	$\frac{4}{42}$	$\frac{5}{42}$	$\frac{14}{42}$
2	$\frac{4}{42}$	$\frac{5}{42}$	$\frac{6}{42}$	$\frac{7}{42}$	$\frac{22}{42}$
$f_Y(x)$	$\frac{6}{42}$	$\frac{9}{42}$	$\frac{12}{42}$	$\frac{15}{42}$	

(b) (4 points) Find the conditional pmf of Y, given $X = 1$.

$$f(y|1) = \frac{f(1,y)}{f_X(1)}$$

$$f(0|1) = \frac{2/42}{14/42} = \frac{2}{14} \qquad f(2|1) = \frac{4/42}{14/42} = \frac{4}{14}$$

$$f(1|1) = \frac{3/42}{14/42} = \frac{3}{14} \qquad f(3|1) = \frac{5/42}{14/42} = \frac{5}{14}$$

(c) (4 points) Find $E(X \mid Y = 2)$.

$$f(x|2) = \frac{f(x,2)}{f_Y(2)}$$

$$f(0|2) = \frac{2/42}{12/42} = \frac{2}{12} = \frac{1}{6}$$

$$f(1|2) = \frac{4/42}{12/42} = \frac{4}{12} = \frac{2}{6}$$

$$f(2|2) = \frac{6/42}{12/42} = \frac{6}{12} = \frac{3}{6}$$

$$E(X|Y=2) = 0 \cdot \frac{1}{6} + 1 \cdot \frac{2}{6} + 2 \cdot \frac{3}{6} = \frac{2+6}{6} = \frac{8}{6} = \frac{4}{3}$$

(d) (2 points) Are X and Y independent? Why / why not?

No. $f(x,y)$ is not equal to $f_X(x) f_Y(y) \ \forall x,y \in \mathbb{R}$

For example:

$$f(0,0) = 0$$

but $f_X(0) f_Y(0) = \frac{6}{42} \cdot \frac{6}{42} = \frac{36}{42^2} \neq 0$

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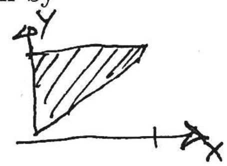
2. The joint probability density function (pdf) of two continuous random variables is given by

$$f(x,y) = \begin{cases} 2 & \text{for } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$



(a) (8 points) Find the marginal pdfs of X and Y.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^1 2\,dy = 2y\Big|_{y=x}^{1} = 2-2x$$

for $x \in (0,1)$

$$f_Y(y) = \int_0^y 2\,dx = 2x\Big|_{x=0}^{y} = 2y$$

for $y \in (0,1)$

(b) (4 points) Find the conditional pdf of X, given $Y = 3/4$.

$$f(x|3/4) = \frac{f(x,3/4)}{f_Y(3/4)} = \frac{2\,I_{(0,3/4)}(x)}{2\cdot 3/4}$$

$$= \frac{4}{3} \text{ for } 0 < x < \frac{3}{4}$$

(c) (4 points) Find $E(Y \mid X = 1/2)$.

$$f(y \mid \tfrac{1}{2}) = \frac{f(\frac{1}{2}, y)}{f_X(\frac{1}{2})} = \frac{2 \; I_{(1/2,1)}(y)}{2 - 2\frac{1}{2}}$$

$$= \frac{2}{1} = 2 \quad \text{for} \quad 0.5 < y < 1$$

$$E(Y \mid X = \tfrac{1}{2}) = \int_{0.5}^{1} y\, 2 \, dy = \frac{2y^2}{2}\Big|_{y=0.5}^{1} = 1 - \frac{1}{4} = \frac{3}{4}$$

(d) (2 points) Are X and Y independent? Why / why not?

No. $f(x,y)$ is not equal to $f_X(x) \cdot f_Y(y) \;\; \forall x,y \in \mathbb{R}$

For example: $f(0.5, 0.1) = 0$, but

$$f_X(0.5) f_Y(0.1) = (2 - 2 \cdot \tfrac{1}{2}) \, 2 \cdot 0.1 = 0.2 \neq 0$$

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3. (6 points) Let X and Y be independent random variables and let $X \sim \text{Gamma}(\alpha_1, \beta)$ and $Y \sim \text{Gamma}(\alpha_2, \beta)$. What is the distribution of $X + Y$?
Hint: The moment generating function (mgf) for $\text{Gamma}(\alpha, \beta)$ is $M(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha}$.

Since X and Y are independent we have

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

$$= \left(\frac{1}{1-\beta t}\right)^{\alpha_1} \left(\frac{1}{1-\beta t}\right)^{\alpha_2}$$

$$= \left(\frac{1}{1-\beta t}\right)^{\alpha_1+\alpha_2}$$

$= \text{mgf of Gamma}(\alpha_1+\alpha_2, \beta)$

$\Rightarrow X+Y \sim \text{Gamma}(\alpha_1+\alpha_2, \beta)$

Problem	1	2	3	Total
Missed Score				
out of	6	18	6	30