STAT 345/445 Lecture 1

Section 1.1: Set Theory

.. and some points about proofs

Sample spaces and events

- **Experiment** (conceptual): A process with *uncertain* outcomes
- **Sample space S:** Set of all possible outcomes of an experiment

Examples:

Experiment	**Sample space S**
Roll a die	$S = \{1, 2, 3, 4, 5, 6\}$
Toss a coin	
Soccer game	
No. of cardinal nests in a city	
Height of a statistics student	

- **finite**, **countable**, and **uncountable** sample spaces

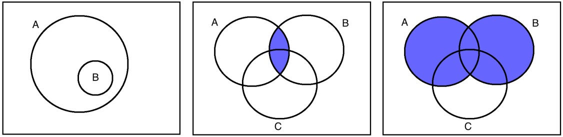
Sample spaces and events

- **Outcome** - observable result of one trial of an experiment
- **Sample Space** S - set of all potential possible outcomes
- **Event** A - a collection of outcomes (subset of S)
  - An event A **occurs** if the outcome of the experiment is the set A

Examples of events:

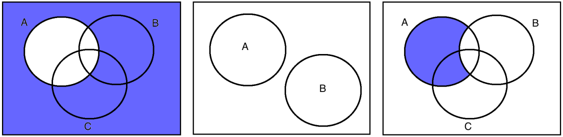
Experiment	Events
Roll a die	Rolled an even number: $A = \{2, 4, 6\}$
No. of cardinal nests in a city	

Set operations



1. **Subset:** $B \subset A$
   - $x \in B \Rightarrow x \in A$
   - Occurrence of B implies occurrence of A
2. **Intersection:** $A \cap B$
   - $A \cap B = \{x : x \in A \text{ and } x \in B\}$
   - Both events A and B occur
3. **Union** $A \cup B$:
   - $A \cup B = \{x : x \in A \text{ or } x \in B\}$
   - At least one of A or B occur

Set operations



1. **Complement:** A^c
    - $A^c = \{x : x \in S, x \notin A\}$
    - The event A does not occur
2. **Empty set** $\emptyset$: contains no elements, but is still treated like a set
3. A and B are **disjoint (mutually exclusive)** if $A \cap B = \emptyset$.
   The two events cannot occur at the same time
4. **Set difference:** $A \setminus B$
    - A happened but B did not
    - $A \setminus B = \{x : x \in A, x \notin B\} = A \cap B^c$

Properties of set operations

Can all be proven using definitions of set operations

Let A, B, and C be events defined on S. Then the following holds

- **Commutative property**

$$A \cup B = B \cup A \quad \text{and} \quad A \cap B = B \cap A$$

- **Associativity property**

$$A \cup (B \cup C) = (A \cup B) \cup C \quad \text{and} \quad A \cap (B \cap C) = (A \cap B) \cap C$$

- **Distributive law**

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

- **DeMorgan's law**

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

Proving a set equality

- Suppose we want to prove a set equality $D = E$
- One Strategy:
  1. Prove that $D \subset E$
  2. Also prove that $E \subset D$

  Since $D \subset E$ and $E \subset D$ we have $D = E$
- To prove $D \subset E$:
  - Prove $x \in D \Rightarrow x \in E$
  - That is: prove that if x is an arbitrary element in D then it follows that x is also an element in E

Proving a set equality - DeMorgan's law

- Prove one of DeMorgan's laws: $(A \cup B)^c = A^c \cap B^c$

Proving a set equality $D = E$

- Another strategy: Prove $D = E$ (or $E = D$) directly
    - Prove $x \in D \Leftrightarrow x \in E$
    - That is: prove that x is an element in D if and only if x is also an element in E
    - Sometimes we realize that all our "$\Rightarrow$" steps are actually also "$\Leftrightarrow$" steps
    - But "$\Rightarrow$" and then "$\Leftarrow$" is generally easier to prove than "$\Leftrightarrow$" directly
- Yet another strategy: Prove $D = ... = E$ by using known set equalities

Proving another set equality

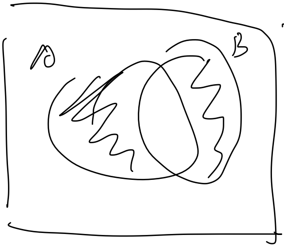
- **Symmetric difference** or **xor** of two sets is defined as

$$
\begin{aligned}
A \triangle B &= (A \setminus B) \cup (B \setminus A) \\
&= \{x : x \text{ is in either } A \text{ or } B \text{ but not both}\}
\end{aligned}
$$

- Show that $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$ drow

Today 8/29

- More on Sets
- Def of Prob
  Law of total prob
  Boole Bonferroni



$= A \triangle B$

$$(A\triangle B)\cap C = (A\cap C)\triangle (B\cap C)$$

Proof: $(A\triangle B)\cap C = ((A\setminus B)\cup(B\setminus A))\cap C$ def of $\triangle$

$= ((A\cap B^c)\cup(B\cap A^c))$ def of "$\setminus$"

$= (A\cap B^c\cap C)\cup(B\cap A^c\cap C)$ distributive law.

$A\cap C \setminus B\cap C$

Let's try from the other end.

$(A\cap C)\triangle(B\cap C) = [A\cap C\cap(B\cap C)^c]\cup[B\cap C\cap(A\cap C)^c]$ def of $\triangle$ and

$= [A\cap C\cap(B^c\cup C^c)]\cup[B\cap C\cap(A^c\cup C^c)]$

$= [(A\cap C\cap B^c)\cup\underbrace{(A\cap C\cap C^c)}_{\emptyset}]\cup[(B\cap C\cap A^c)\cup\underbrace{(B\cap C\cap C^c)}_{\emptyset}] =$

Union and Intersections of many sets

Similar $\sum_{i=1}^{n} a_i = a_1 + \dots + a_n$.

- Notation for union of n sets:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$
$$= \{x : x \in A_i \text{ for at least one } i\}$$

- Notation for union of infinite number of sets:

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cdots$$
$$= \{x : \exists\, i \text{ so that } x \in A_i\}$$

"There exists"

Union and Intersections of many sets

- Notation for intersection of n sets:

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$
$$= \{x : x \in A_1, x \in A_2, \ldots, x \in A_n\}$$

- Notation for intersection of infinite number of sets:

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cdots$$
$$= \{x : x \in A_i, \ \forall i\}$$

for all i

DeMorgan's law for many sets

- DeMorgan's law generalizes to n sets:

$$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c \quad \text{and} \quad \left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c$$

- DeMorgan's law generalizes to infinite number of sets:

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c \quad \text{and} \quad \left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} A_i^c$$

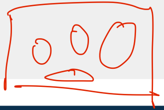
Mutually exclusive sets and a partition

- Recall: Two events are **disjoint** if $A \cap B = \emptyset$

Definition: Mutually exclusive

Events $A_1, A_2, A_3, \ldots$ are called **mutually exclusive** or **pairwise disjoint** if

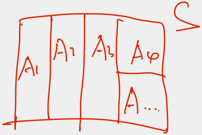
$$A_i \cap A_j = \emptyset \quad \text{for all } i \neq j$$



Definition: Partition

If $A_1, A_2, A_3, \ldots$ are mutually exclusive and

$$\bigcup_{i=1}^{\infty} A_i = S$$



then $A_1, A_2, A_3, \ldots$ is called a **partition** of S

Example

- Example:

open close

$$A_i = \left(\frac{1}{2^i}, \frac{1}{2^{i-1}}\right] \qquad i = 1, 2, 3, \ldots$$

not included included.

$A_1 = [\frac{1}{2}, 1]$ $A_2 = (\frac{1}{4}, \frac{1}{2}]$, $A_3 = (\frac{1}{8}, \frac{1}{4}]$

.... mutually exclusive

not close $\lim_{i \to \infty} \frac{1}{2^i} = 0 \Longrightarrow$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i = (0, 1]$$ So $A_1, A_2, A_3 \ldots$ is a partition of $(0, 1]$