STAT 345/445 Lecture 1

Section 1.1: Set Theory

.. and some points about proofs

Sample spaces and events

- **Experiment** (conceptual): A process with *uncertain* outcomes
- **Sample space S:** Set of all possible outcomes of an experiment

Examples:

Experiment	Sample space S
Roll a die	$S = \{1, 2, 3, 4, 5, 6\}$ ← set notation
Toss a coin	$S = \{H, T\}$
Soccer game	$S = \{\text{win, loose, tie}\}$
No. of cardinal nests in a city	$S = \mathbb{N}$
Height of a statistics student	$S = [0, \infty)$

- **finite**, **countable**, and **uncountable** sample spaces soccer game:

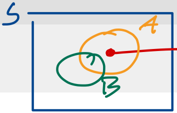
$$S = \{ (x,y) \in \mathbb{N} \times \mathbb{N} : \begin{array}{l} x = \text{goals of team A}, \\ y = \text{goals of team B} \end{array} \}$$

Set notation:

$$\{ \underset{\substack{\uparrow \\ \text{dummy variable}}}{x} : \text{condition} \}$$

$\mathbb{N}$: Natural numbers: $\{0, 1, 2, 3, \dots\}$

Sample spaces and events



outcome ×

A happened but B did not.

- **Outcome** - observable result of one trial of an experiment
- **Sample Space** S - set of all potential possible outcomes
- **Event** A - a collection of outcomes (subset of S)
  - An event A **occurs** if the outcome of the experiment is the set A

Examples of events:

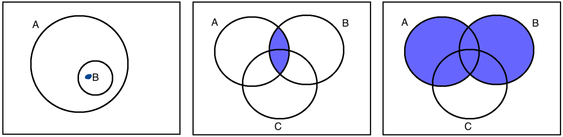
Experiment	Events
Roll a die	Rolled an even number: $A = \{2, 4, 6\}$
No. of cardinal nests in a city :	$A : \{20\}$

or $A : \{x \in \mathbb{N} : x > 19\}$

$= \{20, 21, 22, \dots\}$

Set operations

$B \subset A$



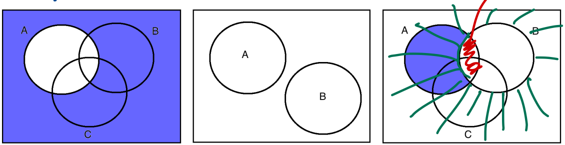
1. **Subset:** $B \subset A$
   - $x \in B \Rightarrow x \in A$
   - Occurrence of B implies occurrence of A
2. **Intersection:** $A \cap B$
   - $A \cap B = \{x : x \in A \text{ and } x \in B\}$
   - Both events A and B occur
3. **Union** $A \cup B$:
   - $A \cup B = \{x : x \in A \text{ or } x \in B\}$
   - At least one of A or B occur

Set operations

Venn diagrams

A^c

$A \cap B$



$A \setminus B = A \cap B^c$

"," : and

1. **Complement:** A^c
   - $A^c = \{x : x \in S, x \notin A\}$
   - The event A does not occur
2. **Empty set** $\emptyset$: contains no elements, but is still treated like a set
3. A and B are **disjoint (mutually exclusive)** if $A \cap B = \emptyset$.
   The two events cannot occur at the same time
4. **Set difference:** $A \setminus B$
   - A happened but B did not
   - $A \setminus B = \{x : x \in A, x \notin B\} = A \cap B^c$

Properties of set operations

Can all be proven using definitions of set operations

Let A, B, and C be events defined on S. Then the following holds

- **Commutative property**

$$A \cup B = B \cup A \quad \text{and} \quad A \cap B = B \cap A$$

- **Associativity property**

$$A \cup (B \cup C) = (A \cup B) \cup C \quad \text{and} \quad A \cap (B \cap C) = (A \cap B) \cap C$$

- **Distributive law**

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

- **DeMorgan's law**

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

Proving a set equality

- Suppose we want to prove a set equality $D = E$
- One Strategy:
  1. Prove that $D \subset E$
  2. Also prove that $E \subset D$

  Since $D \subset E$ and $E \subset D$ we have $D = E$
- To prove $D \subset E$:
  - Prove $x \in D \Rightarrow x \in E$
  - That is: prove that if x is an arbitrary element in D then it follows that x is also an element in E

Proving a set equality - DeMorgan's law

- Prove one of DeMorgan's laws: $(A \cup B)^c = A^c \cap B^c$

1) "$\Rightarrow$" i.e. prove that $(A \cup B)^c \subset A^c \cap B^c$

Suppose $x \in (A \cup B)^c$

$\Rightarrow$ $x \notin A \cup B$

$\Rightarrow$ x is in neither A nor B

i.o.w. $x \notin A$ and $x \notin B$

$\Rightarrow$ $x \in A^c$ and $x \in B^c$

$\Rightarrow$ $x \in A^c \cap B^c$ $\Rightarrow$ $(A \cup B)^c \subset A^c \cap B^c$

2) "$\Leftarrow$" Prove that $A^c \cap B^c \subset (A \cup B)^c$

Suppose $x \in A^c \cap B^c$ $\Rightarrow$ $x \in A^c$ and $x \in B^c$

$\Rightarrow$ $x \notin A$ and $x \notin B$ $\Rightarrow$ $x \notin A \cup B$ $\Rightarrow$ $x \in (A \cup B)^c$

$\Rightarrow$ $A^c \cap B^c \subset (A \cup B)^c$

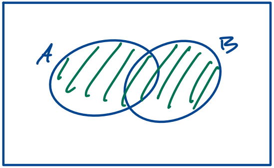
$\Rightarrow$ $(A \cup B)^c = (A^c \cap B^c)$

$(A\cup B)^c = A^c \cap B^c$

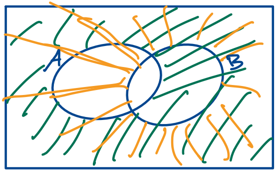
$(A\cup B)^c$

$A^c \cap B^c$

$A\cup B$

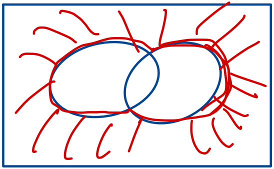


A^c

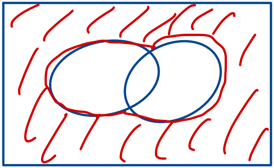


B^c

$(A\cup B)^c$



$A^c\cap B^c$



Proving a set equality $D = E$

- Another strategy: Prove $D = E$ (or $E = D$) directly
  - Prove $x \in D \Leftrightarrow x \in E$
  - That is: prove that x is an element in D if and only if x is also an element in E
  - Sometimes we realize that all our "$\Rightarrow$" steps are actually also "$\Leftrightarrow$" steps
  - But "$\Rightarrow$" and then "$\Leftarrow$" is generally easier to prove than "$\Leftrightarrow$" directly
- Yet another strategy: Prove $D = ... = E$ by using known set equalities

Proving another set equality

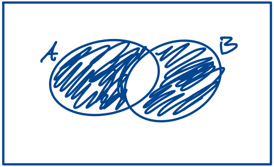
- **Symmetric difference** or **xor** of two sets is defined as

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$
$$= \{x : x \text{ is in either } A \text{ or } B \text{ but not both}\}$$

(= $A \cup B$ if A and B are disjoint)

- Show that $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$

Proof on whiteboard

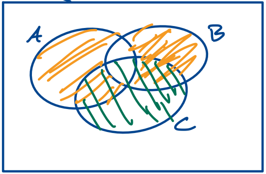


$= A \triangle B$

$(A\triangle B)\cap C = (A\cap C)\triangle(B\cap C)$

$(A\triangle B)\cap C$

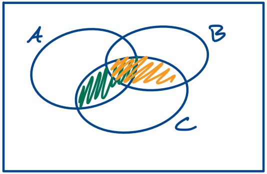
$A\triangle B$



C

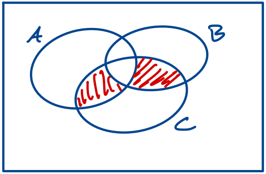
$(A\cap C)\triangle(B\cap C)$

$A\cap C$

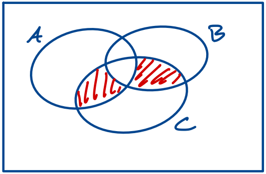


$B\cap C$

$(A\triangle B)\cap C$



$(A\cap C)\triangle(B\cap C)$



Union and Intersections of many sets

$\sum_{i=1}^{n} a_i = a_1 + \cdots + a_n$

- Notation for union of n sets:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$
$$= \{x : x \in A_i \text{ for at least one } i\}$$

- Notation for union of infinite number of sets:

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cdots$$
$$= \{x : \underbrace{\exists}_{\text{"there exists"}} i \text{ so that } x \in A_i\}$$

Union and Intersections of many sets

- Notation for intersection of n sets:

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

"and"

$$= \{x : x \in A_1, x \in A_2, \ldots, x \in A_n\}$$

- Notation for intersection of infinite number of sets:

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cdots$$

$$= \{x : x \in A_i,\ \underline{\forall i}\}$$

"for all"

DeMorgan's law for many sets

- DeMorgan's law generalizes to *n* sets:

$$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c \quad \text{and} \quad \left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c$$

- DeMorgan's law generalizes to infinite number of sets:

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c \quad \text{and} \quad \left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} A_i^c$$

Mutually exclusive sets and a partition

- Recall: Two events are **disjoint** if $A \cap B = \emptyset$

Definition: Mutually exclusive

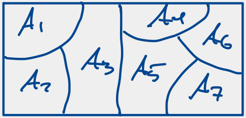
Events $A_1, A_2, A_3, \ldots$ are called **mutually exclusive** or **pairwise disjoint** if

$$A_i \cap A_j = \emptyset \quad \text{for all } i \neq j$$

Definition: Partition

If $A_1, A_2, A_3, \ldots$ are mutually exclusive and

$$\bigcup_{i=1}^{\infty} A_i = S$$



then $A_1, A_2, A_3, \ldots$ is called a **partition** of S

Example of a partition

open closed $\frac{1}{2^i}$ is not included

$\frac{1}{2^{i-1}}$ is included

- Example:

$$A_i = \left(\frac{1}{2^i}, \frac{1}{2^{i-1}}\right] \qquad i = 1, 2, 3, \ldots$$

$A_1 = \left(\frac{1}{2}, 1\right]$ $A_2 = \left(\frac{1}{4}, \frac{1}{2}\right]$, $A_3 = \left(\frac{1}{8}, \frac{1}{4}\right]$

... mutually exclusive

note that $\lim_{i \to \infty} \frac{1}{2^i} = 0$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i = (0, 1]$

So $A_1, A_2, A_3, \ldots$ is a partition of $(0, 1]$