4.32 (a) For the hierarchical model

$$Y|\Lambda \sim \text{Poisson}(\Lambda) \quad \text{and} \quad \Lambda \sim \text{gamma}(\alpha, \beta)$$

find the marginal distribution, mean, and variance of Y. Show that the marginal distribution of Y is a negative binomial if α is an integer.

(b) Show that the three-stage model

$$Y|N \sim \text{binomial}(N, p), \quad N|\Lambda \sim \text{Poisson}(\Lambda), \quad \text{and} \quad \Lambda \sim \text{gamma}(\alpha, \beta)$$

(a): $f(Y|\Lambda) = \dfrac{e^{-\Lambda}\Lambda^{y}}{y!}, \begin{cases} \Lambda > 0 \\ y = 0, 1, 2, 3 \cdots \end{cases}$

$f(\Lambda) = \dfrac{1}{\Gamma(\alpha)\beta^{\alpha}} \Lambda^{\alpha-1} e^{-\Lambda/\beta}, \begin{cases} \alpha > 0 \\ \beta > 0 \\ \Lambda > 0 \end{cases}$

$f(Y, \Lambda) = f(Y|\Lambda) \cdot f(\Lambda)$

$= \dfrac{1}{\Gamma(\alpha)\beta^{\alpha} y!} \Lambda^{y+\alpha-1} e^{-\Lambda(1+\frac{1}{\beta})}$

$f_Y(y) = \displaystyle\int_{-\infty}^{\infty} f(Y, \Lambda)\, d\Lambda$

$= \displaystyle\int_{0}^{\infty} \dfrac{1}{\Gamma(\alpha)\beta^{\alpha} y!} \Lambda^{y+\alpha-1} e^{-\Lambda(1+\frac{1}{\beta})}\, d\Lambda$

$$= \frac{1}{y!} \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{y+\alpha-1} e^{-\lambda(1+\frac{1}{\beta})} d\lambda$$

$$\cdots (*)$$

$$\int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta} d\lambda = 1$$

$$\begin{cases} \alpha'-1 = y+\alpha-1 \\ \frac{1}{\beta'} = 1+\frac{1}{\beta} \end{cases} \Rightarrow \begin{cases} \alpha = \alpha'-y \\ \beta = \frac{\beta'}{1-\beta'} \end{cases} \quad \begin{cases} \alpha' = \alpha+y \\ \beta' = \frac{\beta}{1+\beta} \end{cases}$$

$$(*) = \frac{1}{y!} \int_0^\infty \frac{1}{\Gamma(\alpha'-y)\left(\frac{\beta'}{1-\beta'}\right)^{\alpha'-y}} \lambda^{\alpha'-1} e^{-\lambda/\beta'} d\lambda$$

$$= \frac{\Gamma(\alpha')\,\beta'^{\alpha'}}{y!\,\Gamma(\alpha'-y)\left(\frac{\beta'}{1-\beta'}\right)^{\alpha'-y}} \underbrace{\int \frac{1}{\Gamma(\alpha')\,\beta'^{\alpha'}}\,\lambda^{\alpha'-1} e^{-\lambda/\beta'}\,d\lambda}_{=1}$$

$$= \frac{\Gamma(\alpha+y)\left(\frac{\beta}{(1+\beta)}\right)^{\alpha+y}}{y!\,\Gamma(\alpha)\,\beta^{\alpha}}$$

$$= \frac{\Gamma(\alpha+y)}{y!\,\Gamma(\alpha)} \cdot \beta^{(\alpha+y)} \cdot \beta^{-\alpha} \cdot \left[\frac{1}{(1+\beta)}\right]^{\alpha} \cdot \left[\frac{1}{(1+\beta)}\right]^{y}$$

$$= \frac{(\alpha+y-1)!}{y!\,(\alpha-1)!} \cdot \left(\frac{\beta}{1+\beta}\right)^{y} \cdot \left(\frac{1}{1+\beta}\right)^{\alpha}$$

$$= \binom{\alpha+y-1}{y} \cdot \left(\frac{\beta}{1+\beta}\right)^{y} \cdot \left(\frac{1}{1+\beta}\right)^{\alpha}, \quad y = 0, 1, 2 \cdots$$

$= \text{negative biomial}, \quad \begin{cases} \alpha > 0 \\ \beta > 0 \end{cases}$

$$f\left(y \mid \underset{\substack{\uparrow \\ r}}{\alpha}, \underset{\substack{\uparrow \\ p}}{\frac{1}{1+\beta}}\right)$$

$$\mathbb{E}(y) = \frac{r(1-p)}{p} = \alpha \cdot \left(1 - \frac{1}{1+\beta}\right) \cdot (1+\beta)$$

$$= \alpha \cdot (1+\beta-1) = \alpha\beta$$

$$\mathrm{Var}(y) = \frac{r(1-p)}{p^2} = \frac{\mathbb{E}(x)}{p} = \alpha\beta(1+\beta)$$

(b) $f(Y|N) = \binom{N}{y} p^{y} (1-p)^{N-y}$, $y \in \{0, 1, 2, \cdots N\}$
$p \in [0, 1]$

$f(N|\Lambda) = \frac{e^{-\Lambda} \Lambda^N}{N!}$, $N \in \{0, 1, 2, \cdots\}$
$\Lambda > 0$

$f(\Lambda) = \frac{1}{\Gamma(\alpha) \beta^\alpha} \Lambda^{\alpha-1} e^{-\Lambda/\beta}$, $\begin{cases} \Lambda > 0 \\ \alpha > 0 \\ \beta > 0 \end{cases}$

$f(N, \Lambda) = f(N|\Lambda) \cdot f(\Lambda)$

According to (a)

$f(N) \sim$ negative binomal

$$f(N) = \binom{\alpha+N-1}{N} \cdot \left(\frac{\beta}{1+\beta}\right)^{N} \cdot \left(\frac{1}{1+\beta}\right)^{\alpha}, \quad N = 0, 1, 2, \ldots$$

$$\alpha > 0$$

$$\beta > 0$$

$$f(N, y) = f(y|N) \cdot f(N)$$

$$= \binom{N}{y} p^{y} \cdot (1-p)^{N-y} \cdot \binom{\alpha+N-1}{N} \cdot \left(\frac{\beta}{1+\beta}\right)^{N} \cdot \left(\frac{1}{1+\beta}\right)^{\alpha}$$

$$= \frac{N!}{y!(N-y)!} \cdot \frac{(\alpha+N-1)!}{N!(\alpha-1)!} p^{y} \cdot (1-p)^{N-y} \cdot \left(\frac{\beta}{1+\beta}\right)^{y} \cdot$$

$$\left(\frac{\beta}{1+\beta}\right)^{N-y} \cdot \left(\frac{1}{1+\beta}\right)^{\alpha}$$

$$= \frac{(\alpha+N-1)!}{y!\,(N-y)!\,(\alpha-1)!} \cdot \left(\frac{p\beta}{1+\beta}\right)^{y} \cdot \left(\frac{(1-p)\beta}{1+\beta}\right)^{N-y} \cdot \left(\frac{1}{1+\beta}\right)^{\alpha}$$

$$f_Y(y) = \sum_{N=0}^{\infty} \frac{(\alpha+N-1)!}{y!\,(N-y)!\,(\alpha-1)!} \cdot \left(\frac{p\beta}{1+\beta}\right)^{y} \cdot \left(\frac{(1-p)\beta}{1+\beta}\right)^{N-y} \cdot \left(\frac{1}{1+\beta}\right)^{\alpha}$$

$$= \left(\frac{p\beta}{1+\beta}\right)^{y} \left(\frac{1}{1+\beta}\right)^{\alpha} \cdot \frac{1}{(\alpha-1)!\,y!} \underbrace{\sum_{N=0}^{\infty} \frac{(\alpha+N-1)!}{(N-y)!} \left(\frac{(1-p)\beta}{1+\beta}\right)^{N-y} \cdots \cancel{\times}}$$

$$\sum_{N=0}^{\infty} \frac{(\alpha+y-1+N-y)!}{(N-y)! \cdot (\alpha+y-1)!} \cdot (\alpha+y-1)! \cdot \left(\frac{\beta-p\beta}{1+\beta}\right)^{N-y} \cdot \left(\frac{p\beta+1}{1+\beta}\right)^{\alpha+y-1} \cdot \left(\frac{1+\beta}{p\beta+1}\right)^{\alpha+y-1}$$

$$= \left(\frac{1+\beta}{p\beta+1}\right)^{\alpha+y} \cdot (\alpha+y-1)! \cdot \underbrace{\sum_{N=0}^{\infty} \binom{\alpha+N-1}{N-y} \left(\frac{\beta-p\beta}{1+\beta}\right)^{N-y} \left(\frac{p\beta+1}{1+\beta}\right)^{\alpha+y}}_{=1}$$

$$\therefore (X) = \left(\frac{p\beta}{1+\beta}\right)^{y} \cdot \left(\frac{1}{1+\beta}\right)^{\alpha} \cdot \underline{\frac{1}{(\alpha-1)!\,y!}} \cdot \left(\frac{1+\beta}{p\beta+1}\right)^{\alpha+y} \cdot \underbrace{(\alpha+y-1)!}$$

$$= \binom{\alpha+y-1}{y} (p\beta)^{y} \cdot \left(\frac{1}{1+\beta}\right)^{\alpha+y} \cdot (1+\beta)^{\alpha+y} \cdot \left(\frac{1}{p\beta+1}\right)^{\alpha+y}$$

$$= \binom{\alpha+y-1}{y} (p\beta)^{y} \cdot \left(\frac{1}{p\beta+1}\right)^{\alpha+y}$$

$$= \binom{\alpha+y-1}{y} \left(\frac{p\beta}{p\beta+1}\right)^{y} \cdot \left(\frac{1}{p\beta+1}\right)^{\alpha}$$

It is negative $y = 0, 1, 2, 3 \ldots$

binomial ? $p \in [0, 1]$ $\beta, \alpha > 0$

$f(y \mid \alpha, \frac{1}{p\beta+1})$

$$\Sigma(x) = \frac{\alpha\left(1 - \frac{1}{p\beta+1}\right)}{\frac{1}{p\beta+1}} = \alpha(p\beta+1-1)$$

$$= \alpha p \beta$$

$$\mathrm{Var}(x) = \frac{\Sigma(x)}{\frac{1}{p\beta+1}} = \alpha p \beta (p\beta+1)$$

4.36 One generalization of the Bernoulli trials hierarchy in Example 4.4.6 is to allow the success probability to vary from trial to trial, keeping the trials independent. A standard model for this situation is

$$X_i|P_i \sim \text{Bernoulli}(P_i), \quad i = 1, \dots, n,$$
$$P_i \sim \text{beta}(\alpha, \beta).$$

This model might be appropriate, for example, if we are measuring the success of a drug on n patients and, because the patients are different, we are reluctant to assume that the success probabilities are constant. (This can be thought of as an *empirical Bayes model*; see Miscellanea 7.5.6.)
A random variable of interest is $Y = \sum_{i=1}^n X_i$, the total number of successes.

(a) Show that $\mathrm{E}Y = n\alpha/(\alpha + \beta)$.
(b) Show that $\mathrm{Var}\, Y = n\alpha\beta/(\alpha+\beta)^2$, and hence Y has the same mean and variance as a binomial$(n, \frac{\alpha}{\alpha+\beta})$ random variable. What is the distribution of Y?
(c) Suppose now that the model is

$$X_i|P_i \sim \text{binomial}(n_i, P_i), \quad i = 1, \dots, k,$$
$$P_i \sim \text{beta}(\alpha, \beta).$$

Show that for $Y = \sum_{i=1}^k X_i$, $\mathrm{E}Y = \frac{\alpha}{\alpha+\beta}\sum_{i=1}^k n_i$ and $\mathrm{Var}\, Y = \sum_{i=1}^k \mathrm{Var}\, X_i$, where

$$\mathrm{Var}\, X_i = n_i \frac{\alpha\beta(\alpha+\beta+n_i)}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

(a): $E(Y) = E\left(\sum_{i=1}^{n} X_i\right) = E(X_1 + X_2 + \cdots X_n) = E(X_1) + E(X_2) \cdots + E(X_n) = \sum_{i=1}^{n} E(X_i) \quad \cdots \quad ①$

$$E_p(X_i) = E_p\big(\underbrace{E(X_i|P_i)}_{\text{mean of binomial } = P_i}\big) = \underbrace{E_p(P_i)}_{\text{mean of Beta } = \frac{\alpha}{\alpha+\beta}}$$

$\therefore ① = \sum_{i=1}^{n} \frac{\alpha}{\alpha+\beta} = \frac{n\alpha}{\alpha+\beta}$

(b) : $Var(Y) = Var\left(\sum_{i=1}^{n} X_i\right) = Var(X_1 + X_2 + X_3 \cdots X_n) = Var(X_1) + Var(X_2) \cdots Var(X_n)$

$$= \sum_{i=1}^{n} Var(X_i) \quad \cdots \quad ②$$

$$Var(X_i) = E_{P_i}\big(\underbrace{V(X|P_i)}_{= P_i(1-P_i)}\big) + V_{P_i}\big(\underbrace{E(X|P_i)}_{= P_i}\big)$$

$$= E(P_i) - E(P_i^2) + V(P_i)$$

$$= E(P_i) - E(P_i^2) + E(P_i^2) - E(P_i)^2$$

$$= E(P_i) - E(P_i)^2$$

$$= \frac{\alpha}{\alpha+\beta} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{\alpha(\alpha+\beta) - \alpha^2}{(\alpha+\beta)^2} = \frac{\alpha\beta}{(\alpha+\beta)^2}$$

$$\therefore ② = \sum_{i=1}^{n} \frac{\alpha\beta}{(\alpha+\beta)^2} = \frac{n\alpha\beta}{(\alpha+\beta)^2} = Var(Y)$$

$$Y \sim Binomial\left(n, \frac{\alpha}{\alpha+\beta}\right)$$

$$f(y) = \binom{n}{y} \left(\frac{\alpha}{\alpha+\beta}\right)^{y} \left(\frac{\beta}{\alpha+\beta}\right)^{n-y}$$

4.43 Let X_1, X_2, and X_3 be uncorrelated random variables, each with mean μ and variance σ^2. Find, in terms of μ and σ^2, $\mathrm{Cov}(X_1 + X_2, X_2 + X_3)$ and $\mathrm{Cov}(X_1 + X_2, X_1 - X_2)$.

$$\mathrm{Cov}(X_1+X_2, X_2+X_3) = E((X_1+X_2)(X_2+X_3)) - E(X_1+X_2)E(X_2+X_3)$$

$$= E(X_1X_2 + X_1X_3 + X_2^2 + X_2X_3) - (\mu_1+\mu_2)\cdot(\mu_2+\mu_3)$$

$$= E(X_1X_2) + E(X_1X_3) + E(X_2^2) + E(X_2X_3) - (\mu_1+\mu_2)(\mu_2+\mu_3)$$

$\because X_1, X_2, X_3$ are independent

$$= E(X_1)\cdot E(X_2) + E(X_1)E(X_3) + E(X_2^2) + E(X_2)E(X_3) -$$

$$(\mu_1\mu_2 + \mu_1\mu_3 + \mu_2^2 + \mu_2\mu_3)$$

$$= \underline{\mu_1\mu_2} + \underline{\mu_1\mu_3} + \sigma_2^2 + \underline{\mu_2^2} + \underline{\mu_2\mu_3} - \underline{\mu_1\mu_2} - \underline{\mu_1\mu_3} -$$

$$\underline{\mu_2^2} - \underline{\mu_2\mu_3} = \sigma^2$$

$$\mathrm{Cov}(X_1+X_2,\, X_1-X_2)$$

$$= E((X_1+X_2)(X_1-X_2)) - E(X_1+X_2)\cdot E(X_1-X_2)$$

$$= E(X_1^2 - X_2^2) - (\mu+\mu)\cdot(\mu-\mu)$$

$$= E(X_1^2 - X_2^2) = E(X_1^2) - E(X_2^2)$$

$$= \sigma^2+\mu^2 - (\sigma^2+\mu^2) = 0$$

4.45 Show that if $(X, Y) \sim$ bivariate normal$(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$, then the following are true.

(a) The marginal distribution of X is $\mathrm{n}(\mu_X, \sigma_X^2)$ and the marginal distribution of Y is $\mathrm{n}(\mu_Y, \sigma_Y^2)$.

(b) The conditional distribution of Y given $X = x$ is

$$\mathrm{n}(\mu_Y + \rho(\sigma_Y/\sigma_X)(x - \mu_X), \sigma_Y^2(1 - \rho^2)).$$

(c) For any constants a and b, the distribution of $aX + bY$ is

$$\mathrm{n}(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y).$$

(a): $f(x,y) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho \left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right)\right]$

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right)\right]$

use $p = \frac{y-\mu_Y}{\sigma_y}$ $\therefore$ $dy = \sigma_y\, dp$

$q = \frac{x-\mu_x}{\sigma_x}$ $\therefore$ $dx = \sigma_x\, dq$.

$$f_X(x) = \frac{1}{2\pi \Delta x \sqrt{1-\rho^2}} e^{-\frac{w^2}{2(1-\rho^2)}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left((p^2 - 2\rho p q + \rho^2 q^2) - \rho^2 q^2\right)} dp$$

$$= \frac{1}{2\pi \Delta x \sqrt{1-\rho^2}} \cdot e^{-\frac{w^2}{2(1-\rho^2)}} \cdot e^{\rho^2 q^2 / 2(1-\rho^2)} \underline{\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}(p-\rho q)^2} dp}$$

$$\cdot \underline{\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}} \cdot \sqrt{2\pi}\sqrt{1-\rho^2}$$

$$||$$

red part = 1

$$= \frac{e^{-q^2/2}\sqrt{2\pi}\sqrt{1-\rho^2}}{2\pi \Delta x \sqrt{1-\rho^2}} = \frac{1}{\sqrt{2\pi}\,\Delta x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\Delta x}\right)^2}$$

$X \sim n(\mu_x, \Delta_x^2)$.

Using the same method for y

$y \sim n(\mu_y, \Delta_y^2)$.

(b).

$$f(Y \mid X=x) = \frac{f(X=x, y)}{f_X(x)}$$

$$= \frac{\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right)}}{\frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\rho^2\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right.}$$

$$\left.+\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right)$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2\sigma_y^2\sqrt{(1-\rho^2)}}\left((y-\mu_y) - \left(\rho\frac{\sigma_y}{\sigma_x}(x-\mu_x)\right)\right)^2}$$

Thus, $f_{Y|X}(y|x) \sim$

Normal distribution

$$N\left(\mu_Y - \rho\left(\frac{\sigma_Y}{\sigma_X}\right)(x-\mu_X),\ \sigma_Y\sqrt{1-\rho^2}\right)$$

(C)

$$E(aX+bY) = aE(X) + bE(Y)$$

$$= a\mu_X + b\mu_Y$$

$$Var(aX+bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\,Cov(X,Y) \quad \cdots\cdots \quad ①$$

$$\frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \rho \qquad Cov(X,Y) = \rho\,\underline{\sigma_X \sigma_Y}$$

$$① = Var(aX+bY) = a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\sigma_x\sigma_y\rho$$

$$m = aX+bY, \quad n = Y$$

$$\frac{\partial X}{\partial m} = \frac{1}{a} \qquad \frac{\partial X}{\partial n} = \frac{-b}{a}$$

$$\frac{\partial Y}{\partial m} = 0 \qquad \frac{\partial y}{\partial n} = 1$$

$$|J| = \left|\frac{1}{a} \times 1 - 0 \cdot \left(-\frac{b}{a}\right)\right| = \frac{1}{a}$$

$$f_{mn}(m,n) = \frac{1}{2\pi a\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\left(\frac{1}{a}(m-bn)\right)^2 - 2\frac{\rho}{a}(m-bn) + n^2\right)} \cdots\cdots 1$$

$$\mu = a\mu_x + b\mu_y \qquad \Delta^2 = a^2\Delta_x^2 + b^2\Delta_y^2 + 2ab\rho\Delta_x\Delta_y$$

$$\Delta = \sqrt{a^2\Delta_x^2 + b^2\Delta_y^2 + 2ab\rho\Delta_x\Delta_y}$$

$$\rho' = \frac{Cov(m,n)}{\Delta m \Delta n} = \frac{a\rho + b}{\sqrt{a^2 + b^2 + ab\rho}} \quad \cdots \text{②}$$

$$\Delta n^2 = 1 \qquad \mu_m = 0$$

$$\Delta m^2 = a^2 + 2ab\rho + b^2 \qquad \mu_n = 0$$

$$1 - \rho'^2 = 1 - \text{②}^2 = \frac{(1-\rho^2)a^2}{a^2 + b^2 + 2ab\rho}$$

$$= \frac{(1-\rho^2)a^2}{\Delta m^2}$$

① =

$$f_{mn}(m,n) = \frac{1}{2\pi \sigma_m \sigma_n \sqrt{1-\rho'^2}} e^{\left(-\frac{1}{2\sqrt{1-\rho'^2}}\left(\frac{m^2}{\sigma_m^2} - 2\rho\frac{mn}{\sigma_m\sigma_n} + \frac{y^2}{\sigma_n^2}\right)\right)}$$

In part (a), I have proved the relationship between $f_m(m)$ and $f_{mn}(m,n)$

$$f_m(m) \sim N(\mu_m, \sigma_m^2)$$

$$f(ax+by) = N(a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\rho\sigma_x\sigma_y)$$