STAT 345/445 Lecture 5

Section 1.4: Random Variables

1. A Random Variables is a numerical representation of outcomes.

Random Variable

- Recall a probability model: $(S, \mathcal{B}, P)$
  - Sample space S, σ-algebra $\mathcal{B}$
    and a probability function $P : \mathcal{B} \to [0, 1]$

Definition: Random variable

A **random variable**, denoted X or $X(\cdot)$ is a function with domain S and range in the real line:

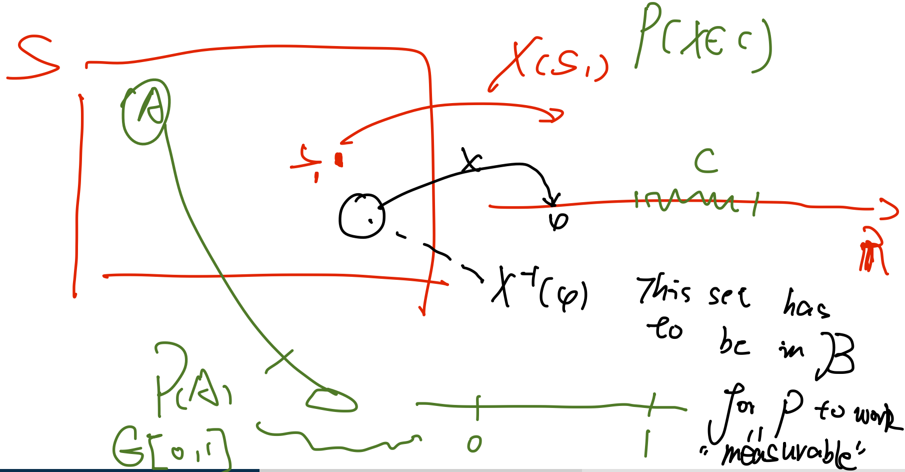
$$X : S \to \mathbb{R}$$

- Technically, X has to satisfy that $\forall r \in \mathbb{R}$ the set
  $\underbrace{A_r = \{s : X(s) \leq r\} \text{ is a set in } \mathcal{B}}$
- Note: To really dive into the knots and bolts of probability theory we need measure-theory. Only get a peak in this course.

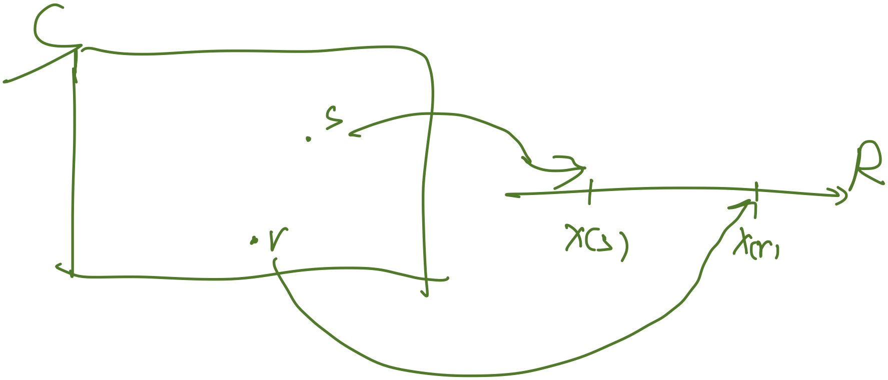
Random Variables

X and P have to play together

- We can think of random variables as a numerical representation of the outcomes in S



$$P(X=\varphi) = P(\underbrace{\{s \in S : X(s)=\varphi\}}_{\text{inverse image of } \varphi})$$



C
s
r
R
x(s)
x(r)

More on random variables

Definition

Range of a random variable X is

$$\mathcal{X} : \{r \in \mathbb{R} : \exists s \in S \text{ such that } X(s) = r\} \subset \mathbb{R}$$

there exists an $s \in S$

- If $\mathcal{X}$ is finite or countable, we say that X is a **discrete random variable**
- If $\mathcal{X}$ is uncountable, we say that X is a **continuous random variable**

e.g. $[0, 10]$

Examples of Random Variables

Experiment: Rolling 2 dice

- $S = \{(s_1, s_2) : \ s_1, s_2 \in \{1, 2, 3, 4, 5, 6\}\}$ $= \{(1,1), (1,2), \cdots, (6,6)\}$
- $X(s_1, s_2) = s_1 + s_2$ Range: $\{2, 3, \cdots, 12\}$
- $Y(s_1, s_2) = \max\{s_1, s_2\}$ Range: $\{1, 2, 3 \cdots 6\}$

Experiment: Lifetime of a light bulb

- $S = (0, \infty)$
- $\underbrace{X(s) = s}_{\text{Identify function}}$ Range: $\mathcal{X} = (0, \infty) = S$

Connecting $P(\cdot)$ and random variables

- Probability functions P are functions of events (in $\mathcal{B}$)
- Events are subsets of S
- Can define events that are sets of outcomes that correspond to some value(s) of the random variable.
  - Examples:

$$A = \{s \in S : \ X(s) = 5\} = X^{-1}(5)$$
$$B = \{s \in S : \ X(s) > 25\} = X^{-1}((25, \infty))$$

  If $A \in \mathcal{B}$ we can find the probability that $X(s) = 5$
  If $B \in \mathcal{B}$ we can find the probability that $X(s) > 25$

- Get an *Induced probability function*

Induced probability function

Definition: Induced probability function

Let $(S, \mathcal{B}, P)$ be a probability model. A probability function induced by X is defined as follows.
If X is a discrete random variable:

$$P_X(X = x) = P(\{s \in S : \ X(s) = x\})$$

If X is a continuous random variable:

$$P_X(X \in R) = P(\{s \in S : \ X(s) \in R\})$$

Sketch of P, S, $\mathcal{B}$, X etc

$$\{s\in S : X(s)=5\} = X^{-1}(5)$$

$$P(X=5) = P(\{s\in S : X(s)=5\})$$

This is only defined if $\{s\in S : X(s)=5\}\in \mathcal{B}$