1.45

Show that the induced probability function defined in (1.4.1) defines a legitimate probability function in that it satisfies the Kolmogorov Axioms.

According to this question, $\mathcal{X} = \{x_1, x_2 \cdots x_m\}$.

Therefore we need to prove 3 points.

① $P_X(A) \geq 0$ ② $P_X(\mathcal{X}) = P(S) = 1$

③ $P_X\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P_X(A_k)$

Prove ①: Since $A \in \mathcal{B}$, P is a probability function, we can get that

$$P_X(A) = P\left(\bigcup_{x_i \in A} \{s_j \in S : X(s_j) = x_i\}\right) \geq 0$$

Therefore, ① was proved.

Prove ②: Since $\mathcal{X} = \{x_1, x_2, \cdots x_m\}$.

and $\mathcal{X}$ is finite, we can get that

$$P_X(\mathcal{X}) = P\left(\bigcup_{i=1}^{m} \{s_j \in S : X(s_j) = x_i\}\right)$$

$$= P(S) = 1$$

Therefore, ② was proved.

Prove ③: For disjoint sets $A_1, A_2 \cdots \in \mathcal{B}$,

$$P_X\left(\bigcup_{k=1}^{\infty} A_k\right) = P\left(\bigcup_{k=1}^{\infty} \left\{\bigcup_{x_i \in A_k} \{s_j \in S : X(s_j) = x_i\}\right\}\right)$$

$$= \sum_{k=1}^{\infty} P\left(\bigcup_{x_i \in A_k} \{s_j \in S : X(s_j) = x_i\}\right) = \sum_{k=1}^{\infty} P_X(A_k)$$

Therefore, ③ was proved.

1.49 A cdf F_X is *stochastically greater* than a cdf F_Y if $F_X(t) \leq F_Y(t)$ for all t and $F_X(t) < F_Y(t)$ for some t. Prove that if $X \sim F_X$ and $Y \sim F_Y$, then

$$P(X > t) \geq P(Y > t) \quad \text{for every } t$$

and

$$P(X > t) > P(Y > t) \quad \text{for some } t,$$

that is, X tends to be bigger than Y.

According to this question, we can have:

$F_X(t) \leq F_Y(t)$ for all t

$F_X(t) < F_Y(t)$ for some t

They show that the cumulative probability up to any t for X is less than or equal to that of Y, and for some t, the cumulative probability for X is strictly less than for Y.

And then, we can use Complementary

Prob:

$$P(X>t) = 1 - F_X(t)$$

$$P(Y>t) = 1 - F_Y(t)$$

Since $F_X(t) \leq F_Y(t)$, for $\forall t$, we can get that

$$1 - F_X(t) \geqslant 1 - F_Y(t).$$

That's to say:

$$P(X>t) \geqslant P(Y>t), \text{ for all } t.$$

Given that $F_X(t') < F_Y(t')$ for some t', we can get

$$P(X > t') = 1 - P(X \le t') = 1 - F_X(t') \cdots ①$$

$$① > 1 - F_Y(t') = P(Y > t')$$

Therefore $P(X > t') > P(Y > t')$,

for some t'.

1.51 An appliance store receives a shipment of 30 microwave ovens, 5 of which are (unknown to the manager) defective. The store manager selects 4 ovens at random, without replacement, and tests to see if they are defective. Let X = number of defectives found. Calculate the pmf and cdf of X and plot the cdf.

According to this question, we have

30 ovens 5 defective ovens

Non-defective ovens = 30 - 5 = 25.

PMF: $r = 4$

$$P(X=0) = \frac{\binom{5}{0}\binom{25}{4}}{\binom{30}{4}} = 0.4616$$

$$P(X=1) = \frac{\binom{5}{1}\binom{25}{3}}{\binom{30}{4}} = 0.4196$$

$$P(X=2) = \frac{\binom{5}{2}\binom{25}{2}}{\binom{30}{4}} = 0.1095$$

$$P(X=3) = \frac{\binom{5}{3}\binom{25}{1}}{\binom{30}{4}} = 0.0091$$

$$P(X=4) = \frac{\binom{5}{4}\binom{25}{0}}{\binom{30}{4}} = 0.0002$$

CDF:

$$P(X\le 0) = P(X=0) = 0.4616$$

$$P(X\le 1) = P(X=0)+P(X=1) = 0.8812$$

$$P(X\le 2) = P(X=0)+P(X=1)+P(X=2) = 0.9907$$

$$P(X\le 3) = P(X\le 2)+P(X=3) = 0.9998$$

$$P(X \leq 4) = P(X \leq 3) + P(X = 4)$$

$$= 1.0$$

1.52 Let X be a continuous random variable with pdf $f(x)$ and cdf $F(x)$. For a fixed number x_0, define the function

$$g(x) = \begin{cases} f(x)/[1 - F(x_0)] & x \geq x_0 \\ 0 & x < x_0. \end{cases}$$

Prove that $g(x)$ is a pdf. (Assume that $F(x_0) < 1$.)

$$g(x) = \begin{cases} \dfrac{f(x)}{1-F(x_0)}, & x \geq x_0 \\ 0, & x < x_0. \end{cases}$$

$\because F(x_0) < 1$

$\therefore \dfrac{f(x)}{1-F(x_0)} \geq 0.$

Therefore, $g(x) \geq 0$ for all x.

and it is non-negative.

$$\int_{x_0}^{\infty} g(x)\,dx = \int_{x_0}^{\infty} \frac{f(x)}{1-F(x_0)}\,dx$$

$$= \frac{1}{1-F(x_0)} \int_{x_0}^{\infty} f(x)\,dx \quad \cdots (*)$$

Since $f(x)$ is pdf, thus

$$\int_{x_0}^{\infty} f(x)\,dx = 1 - F(x_0)$$

$$\text{So, } (*) = \frac{1-F(x_0)}{1-F(x_0)} = 1$$

Therefore, $g(x)$ is a pdf.

1.53 A certain river floods every year. Suppose that the low-water mark is set at 1 and the high-water mark Y has distribution function

$$F_Y(y) = P(Y \le y) = 1 - \frac{1}{y^2}, \quad 1 \le y < \infty.$$

(a) Verify that $F_Y(y)$ is a cdf.
(b) Find $f_Y(y)$, the pdf of Y.
(c) If the low-water mark is reset at 0 and we use a unit of measurement that is $\frac{1}{10}$ of that given previously, the high-water mark becomes $Z = 10(Y - 1)$. Find $F_Z(z)$.

(a):

$$\frac{d F_Y(y)}{dy} = \frac{2}{y^3} > 0, \quad y \in [1, \infty)$$

$F_Y(y)$ is increasing, $y \in [1, \infty)$.

$F_Y(y) = 0, \quad y \in (-\infty, 1)$

$\therefore$ $F_Y(y)$ is non-decreasing, when, $y < 1$, $F_Y(y) = 0$.

$$\lim_{y \to -\infty} F_Y(y) = 0.$$

when $y \geq 1$, then.

$$\lim_{y \to \infty} F_Y(y) = \lim_{y \to \infty} \left(1 - \underbrace{\frac{1}{y^2}}_{0}\right) = 1$$

Therefor, $F_Y(y)$ is cdf.

(b) When $y \in [1, \infty)$,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{2}{y^3}.$$

When $y < 1$

$$f_Y(y) = 0$$

$$f_Y(y) = \begin{cases} \frac{2}{y^3}, & [1, \infty) \\ 0, & (-\infty, 1) \end{cases}$$

1(c):

$$Z = 10(Y-1)$$

$$F_Z(z) = P(Z \le z)$$

$$= P(10(Y-1) \le z)$$

$$= P(Y \le (z/10)+1)$$

$$= F_Y((z/10)+1)$$

$$= \begin{cases} 1 - \dfrac{1}{(\frac{z}{10}+1)^2}, & z \ge 0 \\ 0, & z < 0 \end{cases}$$