Stat 345/445: Theoretical Statistics I: Homework 7 Solutions

Textbook Exercises

4.31 (345: 5 pts.) Suppose that the random variable Y has a binomial distribution with n trials and success probability X, where n is a given constant and X is a uniform(0,1) random variable.

(a) Find EY and Var$Y$.

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}(Y|X)] = \mathbb{E}[nX] = n\mathbb{E}[X] = \frac{n}{2}.$$

$$\text{Var}(Y) = \text{Var}(\mathbb{E}(Y|X)) + \mathbb{E}(\text{Var}(Y|X)) = \text{Var}(nX) + \mathbb{E}[nX(1-X)] = \frac{n^2}{12} + \frac{n}{6}.$$

(b) Find the joint distribution of X and Y

$$P(Y = y, X \leq x) = \binom{n}{y} x^y (1-x)^{n-y}, \quad y = 0, 1, \ldots, n, \quad 0 < x < 1.$$

(c) Find the marginal distribution of Y.

$$P(y = y) = \binom{n}{y} \frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)}.$$

(d) Find the conditional distribution of X given Y.

$$P(X|Y) = \frac{P(Y = y, X \leq x)}{P(Y = y)} = \frac{\Gamma(n+2)x^y(1-x)^{x-y}}{\Gamma(y+1)\Gamma(n-y+1)}$$

4.32 (345 & 445: 2pts.)

(a) For the hierarchical model $Y|\Lambda \sim \text{Poisson}(\Lambda)$ and $\Lambda \sim \text{gamma}(\alpha, \beta)$ find the marginal distribution, mean, and variance of Y. Show that the marginal distribution of Y is a negative binomial if α is an integer.

The pmf of Y, for $y = 0, 1, \ldots,$ is

$$\begin{aligned} f_Y(y) &= \int_0^\infty f_Y(y|\lambda) f_\Lambda(\lambda) d\lambda = \int_0^\infty \frac{\lambda^y e^{-\lambda}}{y!} \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta} d\lambda \\ &= \frac{1}{y!\Gamma(\alpha)\beta^\alpha} \int_0^\infty \lambda^{(y+\alpha)-1} \exp\left\{\frac{-\lambda}{(\frac{\beta}{1+\beta})}\right\} d\lambda = \frac{1}{y!\Gamma(\alpha)\beta^\alpha} \Gamma(y+\alpha)(\frac{\beta}{1+\beta})^{y+\alpha} \end{aligned}$$

If α is a positive integer,

$$f_Y(y) = \binom{y+\alpha-1}{y} (\frac{\beta}{1+\beta})^y (\frac{1}{1+\beta})^\alpha, \text{ the negative binomial } (\alpha, \frac{1}{1+\beta}) \text{ pmf.}$$

Then

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|\Lambda]] = \mathbb{E}[\Lambda] = \alpha\beta$$
$$\mathrm{V}[Y] = \mathrm{V}(\mathbb{E}[Y|\Lambda]) + \mathbb{E}(\mathrm{V}[Y|\Lambda]) = \mathrm{V}[\Lambda] + \mathbb{E}[\Lambda] = \alpha\beta^2 + \alpha\beta = \alpha\beta(\beta+1).$$

(b) Show that the three-stage model

$$Y|N \sim \text{binomial}(N,p), \quad N|\Lambda \sim \text{Poisson}(\Lambda), \quad \text{and} \quad \Lambda \sim \text{gamma}(\alpha,\beta)$$

leads to the same marginal (unconditional) distribution of Y.

For $y = 0, 1, \cdots$, we have

$$\begin{aligned}
P(Y=y|\lambda) &= \sum_{n=y}^{\infty} P(Y=y|N=n,\lambda)P(N=n|\lambda) \\
&= \sum_{n=y}^{\infty} \binom{n}{y} p^y (1-p)^{n-y} \frac{e^{-\lambda}\lambda^n}{n!} \\
&= \sum_{n=y}^{\infty} \frac{1}{y!(n-y)!} (\frac{p}{1-p})^y [(1-p)\lambda]^n e^{-\lambda} \\
&= e^{-\lambda} \sum_{m=0}^{\infty} \frac{1}{y!m!} (\frac{p}{1-p})^y [(1-p)\lambda]^{m+y} && m = n-y \\
&= \frac{e^{-\lambda}}{y!} (\frac{p}{1-p})^y [(1-p)\lambda]^y \Big[\sum_{m=0}^{\infty} \frac{[(1-p)\lambda]^m}{m!}\Big] \\
&= e^{-\lambda} (p\lambda)^y e^{(1-p)\lambda} \\
&= \frac{(p\lambda)^y e^{-p\lambda}}{y!}, && \text{the Poisson}(p\lambda) \text{ pmf.}
\end{aligned}$$

Thus $Y|\Lambda \sim \text{Poisson}(p\lambda)$. The pmf of Y is

$$f_Y(y) = \frac{1}{\Gamma(\alpha) y! (p\beta)^\alpha} \Gamma(y+\alpha) (\frac{p\beta}{1+p\beta})^{y+\alpha}.$$

Again, if α is a positive integer, $Y \sim$ negative binomial$(\alpha, \frac{1}{1+p\beta})$

4.36 (445: 3 pts.) One generalization of the Bernouli trials hierarchy in Example 4.4.6 is to allow the success probability to vary from trial to trial, keeping the trials independent. A standard model for this situation is

$$X_i|P_i \sim \text{Bernoulli}(P_i), \quad i = 1, \cdots, n,$$
$$P_i \sim \text{beta}(\alpha,\beta).$$

This model might be appropriate, for example, if we are measuring the success of a drug on n patients and, because the patients are different, we are reluctant to assume that the success probabilities are constant.

A random variable of interest is $Y = \sum^n n_{i=1} X_i$, the total number of successes.

(a) Show that E$Y = n\alpha/(\alpha+\beta)$.

$$E[Y] = E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i] = \sum_{i=1}^n E_p[E_{X_i|p_i}[X_i|P_i]] = \sum_{i=1}^n E_{p_i}[P_i] = \sum_{i=1}^n \frac{\alpha}{\alpha+\beta} = n\frac{\alpha}{\alpha+\beta}$$

(b) Show that Var $Y = n\alpha\beta/(\alpha+\beta)^2$, and hence Y has the same mean and variance as a binomial$(n, \frac{\alpha}{\alpha+\beta})$ random variable. What is the distribution of Y?

$$
\begin{aligned}
V[Y] &= V[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} V[X_i] && X_i\text{'s are independent from each other} \\
&= \sum_{i=1}^{n} (V_{p_i}[E_{x_i|p_i}[X_i|P_i]] + E_{p_i}[V_{x_i|p_i}[x_i|p_i]]) \\
&= \sum_{i=1}^{n} (V_{p_i}[p_i] + E_{p_i}[p_i(1-p_i)]) \\
&= \sum_{i=1}^{n} (V_{p_i}[p_i] + E[p_i] - V_{p_i}[p_i] - (E[p_i])^2 \\
&= \sum_{i=1}^{n} (E_{p_i}[p_i] - (E_{p_i}[p_i])^2) \\
&= \sum_{i=1}^{n} \Big(\frac{\alpha}{\alpha+\beta} - (\frac{\alpha}{\alpha+\beta})^2 \Big) \\
&= \frac{n\alpha\beta}{(\alpha+\beta)^2}
\end{aligned}
$$

$$Y \sim \text{Binomial}(n, \frac{\alpha}{\alpha+\beta})$$

4.43 (345 & 445: 1 pt.) Let X_1, X_2, and X_3 be uncorrelated random variables, each with mean μ and variance σ^2. Find, in terms of μ and σ^2, $\text{Cov}(X_1 + X_2, X_2 + X_3)$ and $\text{Cov}(X_1 + X_2, X_1 - X_2)$.

$$E[X_1] = E[X_2] = E[X_3] = \mu$$

$$V[X_1] = V[X_2] = V[X_3] = \sigma^2$$

$$\text{Cov}(x_i, x_j) = \begin{cases} 0 & \forall i \neq j, ij = 1,2,3 \\ \sigma^2 & i = j \end{cases}$$

$$
\begin{aligned}
\text{Cov}(X_1 + X_2, X_2 + X_3) &= \text{Cov}(X_1X_2) + \text{Cov}(X_1X_3) + V(X_2) + \text{Cov}(X_2X_3) \\
&= 0 + 0 + \sigma^2 + 0 \\
&= \sigma^2
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\text{Cov}(X_1 + X_2, X_1 - X_2) &= V(X_1) - \text{Cov}(X_1X_2) + \text{Cov}(X_1X_2) - V(X_2) \\
&= \sigma^2 - 0 + 0 - \sigma^2 \\
&= 0
\end{aligned}
$$

4.45 Show that if $(X, Y) \sim$ bivariate normal$(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$, then the following are true.

(a) (345: 2 pts & 445: 1 pt.) The marginal distribution of X is $n(\mu_X, \sigma_X^2)$ and the marginal distribution of Y is $n(\mu_Y, \sigma_Y^2)$.

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y} \exp\Big\{ -\frac{1}{2(1-\rho^2)} \{ (\frac{x-\mu_X}{\sigma_X})^2 - 2\rho(\frac{x-\mu_X}{\sigma_X})(\frac{y-\mu_Y}{\sigma_Y}) + (\frac{y-\mu_Y}{\sigma_Y})^2 \} \Big\}$$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{xy}(xy)dy = \int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma_X\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(w^2-2\rho wz+z^2)}\sigma_Y dz$$

$$z = \frac{y-\mu_Y}{\sigma_Y}, \quad dy = \sigma_Y dz, \quad w = \frac{x-\mu_X}{\sigma_X}$$

$$\begin{aligned} f_X(x) &= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} e^{-\frac{w^2}{2(1-\rho^2)}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2(1-\rho^2)}}[(z^2-2\rho wz+\rho^2w^2)-\rho^2w^2]dz \\ &= \frac{e^{-\frac{w^2}{2(1-\rho^2)}}e^{\frac{\rho^2w^2}{2(1-\rho^2)}}}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2(1-\rho^2)}(z-\rho w)^2}dz = \frac{e^{-\frac{1}{2}w^2}}{2\pi\sigma_X\sqrt{1-\rho^2}}\sqrt{2\pi}\sqrt{1-\rho^2} = \frac{1}{\sqrt{2\pi}\sigma_X}e^{-\frac{1}{2}w^2} \\ &= \frac{1}{\sqrt{2\pi}\sigma_X}e^{-\frac{1}{2}(\frac{x-\mu_X}{\sigma_X})^2} \end{aligned}$$

pdf of $N(\mu_X, \sigma_X^2)$

$f_Y(y)$ is obtained similarly.

(b) (445: 1 pt.) The conditional distribution of Y given $X = x$ is

$$n\Big(\mu_Y + \rho(\sigma_Y/\sigma_X)(x-\mu_X), \sigma_Y^2(1-\rho^2)\Big).$$

$$\begin{aligned} f_{y|x}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} &= \frac{\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}[(\frac{x-\mu_X}{\sigma_X})^2-2\rho(\frac{x-\mu_x}{\sigma_X})(\frac{y-\mu_Y}{\sigma_Y})+(\frac{y-\mu_Y}{\sigma_Y})^2]}}{\frac{1}{\sqrt{2\pi}\sigma_X}e^{-\frac{1}{2\sigma_X^2}(x-\mu_X)^2}} \\ &= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}[(\frac{x-\mu_X}{\sigma_X})^2-(1-\rho^2)(\frac{x-\mu_X}{\sigma_X})^2-2\rho(\frac{x-\mu_X}{\sigma_X})(\frac{y-\mu_Y}{\sigma_Y})+(\frac{y-\mu_Y}{\sigma_Y})^2]} \\ &= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}[\rho^2(\frac{x-\mu_X}{\sigma_X})^2-2\rho(\frac{x-\mu_X}{\sigma_X})(\frac{y-\mu_Y}{\sigma_Y})+(\frac{y-\mu_Y}{\sigma_Y})^2]} \\ &= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}}e^{-\frac{1}{2\sigma_Y^2\sqrt{1-\rho^2}}[(y-\mu_Y)-(\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X))]^2} \end{aligned}$$

pdf of $N(\mu_Y - \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X), \sigma_Y\sqrt{1-\rho^2})$

(c) (445: 2 pts.) For any constants a and b, the distribution of $aX + bY$ is

$$n(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y).$$

Mean:

$$\mathbb{E}[aX+bY] = a\mathbb{E}[X] + b\mathbb{E}[Y] = a\mu_X + b\mu_Y$$

Variance:

$$\begin{aligned} V[aX+bY] &= a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(X,Y) \\ &= a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y \end{aligned}$$

As we show in (a), X and Y are normal.

Thus, linear combination of normal distribution is also normal distribution.