STAT 346

Theoretical Statistics II
Spring Semester 2018

Exam 1

Name: ______________________________

- You have 75 min to complete this exam
- Justify your answers
- Evaluate expressions as much as you can

Note: There is a table on the last page that lists pdf/pmf, mean, variance and mgf for a few distributions.

1. (6 points) Let $X_1, X_2, \ldots, X_n$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$. Show that $\overline{X} \sim \mathrm{N}(\mu, \sigma^2/n)$. Hint: consider using mgf's.

2. (8 points) Let $X_1, X_2, \ldots, X_9$ be a random sample from Uniform$(0, 1)$. Derive the pdf for the 4th order statistic, $X_{(4)}$, and identify the name and parameter values of that distribution.

3. Let X_1, X_2, X_3, be a random sample from $\mathrm{N}(0, 4)$ and let Y_1, Y_2, Y_3, Y_4, be a random sample from $\mathrm{N}(2, 9)$. Also assume that $\{X_1, X_2, X_3\}$ are independent of $\{Y_1, Y_2, Y_3, Y_4\}$. Determine the distribution of the following random variables. Remember to justify your answers.

   (a) (5 points) $\qquad U_1 = \dfrac{3}{4}\overline{X}^2 + \dfrac{4}{9}(\overline{Y} - 2)^2$

(b) (5 points) $$U_2 = \frac{4(\overline{Y} - 2)}{\sqrt{3\sum_{i=1}^{3} X_i^2}}$$

(c) (5 points) $$U_3 = \frac{3\sum_{i=1}^{3} X_i^2}{\sum_{i=1}^{4}(Y_i - 2)^2}$$

4. (6 points) Let $X_1, X_2, \ldots, X_n$ be a random sample from Gamma$(\theta, 2)$. Use the Central Limit Theorem and other theorems or corollaries to find the asymptotic (limiting) distribution of the following statistic

$$\frac{\sqrt{n}(\overline{X}_n - 2\theta)}{\sqrt{\overline{X}_n}}$$

5. (6 points) Again, let $X_1, X_2, \ldots, X_n$ be a random sample from Gamma$(\theta, 2)$. Use the Central Limit Theorem and other theorems or corollaries to find the asymptotic (limiting) distribution of the following statistic

$$\sqrt{n}(\overline{X}_n^2 - 4\theta^2)$$

6. (9 points) Let X be a random variable and $X_1, X_2, X_3, \ldots$ be a sequence of random variables. Define in mathematical notation what the following statements mean.

   (a) $X_n \xrightarrow{D} X$, i.e. X_n converges to X in distribution as $n \to \infty$

   (b) $X_n \xrightarrow{P} X$, i.e. X_n converges to X in probability as $n \to \infty$

   (c) $X_n \xrightarrow{\text{a.s.}} X$, i.e. X_n converges to X almost surely as $n \to \infty$

Problem	1	2	3	4	5	6	Total
Missed Score							
out of	6	8	15	6	6	9	50

Name	pdf	Parameters	Mean	Variance	Mgf
Exponential(β)	$f(x) = \frac{1}{\beta} e^{-x/\beta},\ x \geq 0$	$\beta > 0$	$\mathrm{E}(X) = \beta$	$\mathrm{Var}(X) = \beta^2$	$M_X(t) = \frac{1}{1-\beta t},\ t < \frac{1}{\beta}$
Gamma(α, β)	$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta},\ x \geq 0$	$\alpha, \beta > 0$	$\mathrm{E}(X) = \alpha\beta$	$\mathrm{Var}(X) = \alpha\beta^2$	$M_X(t) = \left(\frac{1}{1-\beta t}\right)^\alpha,\ t < \frac{1}{\beta}$
$\mathrm{N}(\mu, \sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/(2\sigma^2)},\ x \in \mathbb{R}$	$\mu \in \mathbb{R}, \sigma > 0$	$\mathrm{E}(X) = \mu$	$\mathrm{Var}(X) = \sigma^2$	$M_X(t) = e^{\mu t + \sigma^2 t^2/2}$
Uniform(a, b)	$f(x) = \frac{1}{b-a},\ a \leq x \leq b$	$a, b \in \mathbb{R},\ a < b$	$\mathrm{E}(X) = \frac{b+a}{2}$	$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$	$M_X(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$
Beta(α, β)	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1},\ 0 \leq x \leq 1$	$\alpha, \beta > 0$	$\mathrm{E}(X) = \frac{\alpha}{\alpha+\beta}$	$\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$M_X(t) = 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k} \frac{\alpha+r}{\alpha+\beta+r}\right) \frac{t^k}{k!}$
Binomial(n, p)	$f(x) = \binom{n}{x} p^x (1-p)^{n-x},\ x = 0, 1, \ldots, n$	$n \in \mathbb{N},\ 0 \leq p \leq 1$	$\mathrm{E}(X) = np$	$\mathrm{Var}(X) = np(1-p)$	$M_X(t) = \left(pe^t + (1-p)\right)^n$
Poisson(λ)	$f(x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0, 1, 2, \ldots$	$\lambda \geq 0$	$\mathrm{E}(X) = \lambda$	$\mathrm{Var}(X) = \lambda$	$M_X(t) = e^{\lambda(e^t - 1)}$