STAT 345/445 Lecture 13

Multiple Random Variables

Joint and Marginal Distributions – Section 4.1

1. Multiple random variables
   - Discrete random vectors
   - Continuous random vectors

Multiple random variables

- Usually we collect data on
  - more than one unit
  - and/or more than one variable per unit
- Need tools to describe the *joint* behavior of multiple random variables
- Joint probability distributions
- Marginal probability distributions
- Conditional probability distributions and Independence
- Covariance, correlation
- Transformations of multiple random variables
- and more...

Random Vectors

Definition: Random vectors

A **Random vector** is a vector of random variables.

- Technically:

$$\mathbf{X}(s) = (X_1(s), X_2(s), \ldots, X_n(s))'$$

  is a multivariate function from the sample space S to $\mathbb{R}^n$

- Usually just write

$$\mathbf{X} = (X_1, X_2, \ldots, X_3)'$$

  or

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

Discrete case

Discrete random vector

A random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)'$ is a **discrete n-dimensional random vector** if it can take only a countable number of values in $\mathbb{R}^n$.

- Will usually work with $n = 2$ here to avoid the endless "$\dots$" notation
- Extensions for $n > 2$ are usually obvious

Joint pmf

Definition: Joint pmf

A **joint probability mass function (joint pmf)** for a discrete random vector (X, Y) is defined as

$$f(x, y) = P(X = x, Y = y)$$

- Note that

$$f(x, y) \geq 0 \qquad \forall x, y$$

- and that

$$\sum_x \sum_y f(x, y) = 1$$

- In general:

$$P\left((X, Y) \in A\right) = \sum_{(x,y) \in A} f(x, y)$$

Example – 3 coins

- A fair coin is tossed 3 times in a row. Let
  - X = number of head on the first toss
  - Y = total number of heads in all 3 tosses
- What is the joint pmf of (X, Y)?

$f(0,0) = P(X=0, Y=0)$

Outcome	X	Y
HHH	1	3
HHT	1	2
HTH	1	2
THH	0	2
HTT	1	1
THT	0	1
TTH	0	1
TTT	0	0

$X \backslash Y$	0	1	2	3	$f_X(x)$
0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{0}{8}$	$\frac{1}{2}$
1	$\frac{0}{8}$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{1}{2}$
$f_Y(y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	

X is $\text{Binom}(1, \frac{1}{2}) \equiv \text{Bernoulli}(\frac{1}{2})$

Y is $\text{Binom}(3, \frac{1}{2})$

Marginal distribution

- a **marginal pmf** is the (joint) pmf of a subset of a random vector
- Can obtain marginal pmf's from the joint pmf

$$f_X(x) = P(X = x) = \sum_y P(X = x, Y = y) = \sum_y f(x, y)$$

and similarly:

$$f_Y(y) = \sum_x f(x, y)$$

- What are the marginal distributions in the 3-coin example?

Marginal distribution

- Can obtain marginal pmf's from the joint pmf, but not the other way around

- Possible:
  - X_1 and X_2 have same distribution
  - Y_1 and Y_2 have same distribution
  - but $f_1(x_1, y_1) \neq f_2(x_2, y_2)$

$X_2 = X$

Outcome	X	Y	Y_2
HHH	1	3	0
HHT	1	2	1
HTH	1	2	1
THH	0	2	1
HTT	1	1	2
THT	0	1	2
TTH	0	1	2
TTT	0	0	3

- Example: 3-coins experiment, set
  - X_2 = number of head on the first toss
  - Y_2 = total number of *tails* in all 3 tosses

then (X, Y) and (X_2, Y_2) do not have the same joint distribution.

X_2 \ Y_2	0	1	2	3	$f_{X_2}(x_2)$
0	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{1}{2}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{1}{2}$
$f_{Y_2}(y_2)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	

$X_2 \sim Bin(1, \frac{1}{2})$

Y_2 are $Bin(3, \frac{1}{2})$

$$\mathcal{E}\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right) = \begin{bmatrix} \mathcal{E}(X) \\ \mathcal{E}(Y) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

$$\mathcal{E}(X+Y) = \sum_{x,y} (x+y) f(x,y) = 0 + (0+1)\left(\frac{2}{8}\right) + (0+2)\left(\frac{1}{8}\right)$$

$$+ 0 + 0 + 0 + (1+1)\left(\frac{1}{8}\right) + (1+2)\left(\frac{2}{8}\right) +$$

$$(1+3)\left(\frac{1}{8}\right) = 2 = \mathcal{E}(X) + \mathcal{E}(Y)$$

Expectation

- Expectation of a function of a random vector:

$$E(g(X, Y)) = \sum_{x,y} g(x,y) f(x,y)$$

- A **mean vector** is defined as

$$E\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right) = \begin{bmatrix} E(X) \\ E(Y) \end{bmatrix}$$

where $E(X)$ and $E(Y)$ are called the **marginal means**

- 3-coins example: Find $E(X + Y)$ and $E\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right)$

$$E(X+Y) = \sum_{x,y} (x+y) f(x,y) = E(X) + E(Y)$$

Joint cdf

$f(x) = P(X \le x)$

Definition

The **joint cumulative distribution function (joint cdf)** is defined as

$$F(x,y) = P(X \le x, Y \le y) = \sum_{\{(t,s) \mid t \le x,\, s \le y\}} f(t,s)$$

	$y<0$	$0 \le y < 1$	$1 \le y < 2$	$2 \le y < 3$	$y \ge 3$
$x<0$	0	0	0	0	0
$0 \le x < 1$	0	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{4}{8}$
$x \ge 1$	0	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	1

Multinomial distribution

from Section 4.6

- Important family of multivariate discrete distributions
- Setup:
  - m independent trials
  - each trial has n possible outcomes
  - outcome i has probability p_i of occurring, $i = 1, 2, \ldots, n$
  - Let
    $$\mathbf{X} = (X_1, X_2, \ldots, X_n)'$$
    where X_i = number of outcomes of type i
- Think: Randomly take m M&M's from a very large bowl that contains an infinite number of M&M's of n different colors.

Pick m M&M's

$X_1 = \#$ y red

$\sum_{i}^{\#} X_i = m$ $\quad X_2 = \#$ y blue

$X_3 = \#$ y green

$X_4 = \#$ y yellow

$X_n = m - \sum_{1}^{n-1} X_i$

$$f(x_1, \ldots, x_{n-1}) = \frac{m!}{\prod_{i=1}^{n} x_i!} \prod_{1}^{n} p_i^{x_i}$$

$$x_n = m - \sum_{1}^{n-1} x_i \qquad \sum_{1}^{x} p_i = 1$$

$$p_n = 1 - (p_1 + \cdots + p_{n-1})$$

$$P(X = x) = \frac{m!}{x!(m-x)!} p^x (1-p)^{m-x}, \quad x = 0, 1, \ldots, m.$$

$M + M$'s

$n = 4$

$$f(x_1) = \sum_{\{x_2, x_3, x_4\}} \frac{m!}{x_1!\, x_2!\, x_3!\, x_4!} p_1^{x_1} p_2^{x_2} p_3^{x_3} p_4^{x_4}$$

$$= \frac{m!\, p_1^{x_1} (1-p_1)^{m-x_1}}{x_1!\,(m-x_1)!} \underbrace{\sum_{\{x_2, x_3, x_4\}} \frac{(m-x_1)!}{x_2!\, x_3!\, x_4!} \left(\frac{p_2}{1-p_2}\right)^{x_2} \left(\frac{p_3}{1-p_3}\right)^{x_3} \left(\frac{p_4}{1-p_4}\right)^{x_4}}_{=1}$$

$$= \frac{m!}{x_1!\,(m-x_1)!} p_1^{x_1} (1-p_1)^{m-x_1}$$

$\mathrm{Bin}(m, p_1)$

Multinomial distribution

Joint probability mass function

$$f(\mathbf{x} \mid m, \mathbf{p}) = \frac{m!}{x_1!x_2!\cdots x_n!} p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n} \qquad \text{for } \mathbf{x} \in \mathbb{N}^n$$

where $x_1 + x_2 + \cdots + x_n = m$

- Parameter space: $m \in \mathbb{N}$, $p_i > 0 \; \forall i$ and $p_1 + p_2 + \cdots + p_n = 1$
- What are the marginal pmf's for this distribution?

Continuous Case

Def: Continuous joint distribution / joint pdf

Two random variables X and Y have a **continuous joint distribution** if there exists a non-negative function $f(x, y)$ such that for every $A \subset \mathbb{R}^2$

$$P((X, Y)' \in A) = \iint_A f(x, y) dxdy$$

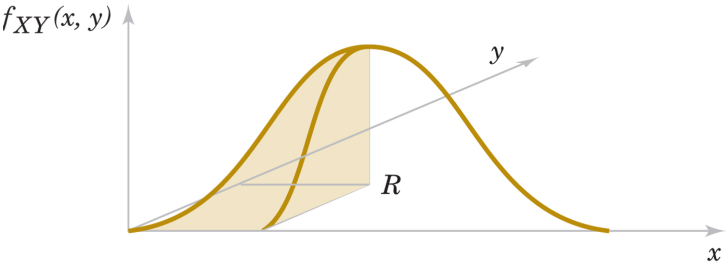
The function $f(x, y)$ is called the **joint probability density function (joint pdf)**.

A joint pdf must satisfy:

$$f(x, y) \geq 0 \quad -\infty < x < \infty, \; -\infty < y < \infty$$

$$\text{and} \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dxdy = 1$$

Calculating probability under a joint pdf



$$P((X, Y) \in R) = \iint_R f(x, y)dxdy$$

More on joint distributions

$f(x,y) \geq 0$ $\iint f(x,y)\,dx\,dy = 1$

- **marginal pdfs**

$$f_X(x) = \int f(x,y)\,dy$$

$$f_Y(y) = \int f(x,y)\,dx$$

- Expectation of a function of a random vector:

$$E(g(X,Y)) = \iint g(x,y) f(x,y)\,dx\,dy$$

- **Joint cdf**

$$F(x,y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt$$

Note that $\frac{\partial^2}{\partial x \partial y} F(x,y) = f(x,y)$

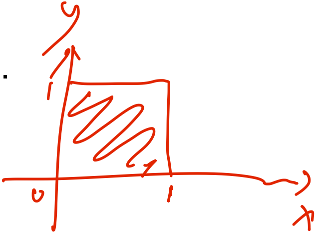
Example 1

Verify that

$$f(x,y) = \begin{cases} 4xy & \text{if } 0 < y < 1 \text{ and } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

is a joint pdf. Find the marginal pdf's and $E(XY)$.

$f(x,y) = 4xy, \quad x \in (0,1)$
$\qquad\qquad\qquad\quad y \in (0,1)$



non-negative

$$\int_0^1 \int_0^1 4xy\,dx\,dy = \int_0^1 \left[2x^2y\right]_0^1 dy = \int_0^1 2y\,dy = y^2\Big|_0^1$$
$$= 1$$

$$f_X(x) = \int_0^1 6xy\,dy = 2xy^2\Big|_0^1 = 2x, \quad 0 < x < 1$$

$$f_Y(y) = \int_0^1 6xy\,dx = 2y, \quad 0 < y < 1$$

X, Y are each Beta$(2,1)$

$$\frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}$$

$$E(XY) = \iint xy\,\varphi xy\,dx\,dy = \int_0^1\int_0^1 \varphi x^2 y^2\,dx\,dy$$

$$= \int_0^1 \left.\frac{\varphi}{3} x^3 y^2\right|_0^1 dy = \int_0^1 \frac{\varphi}{3} y^2\,dy = \left.\frac{\varphi}{9} y^3\right|_0^1 = \frac{\varphi}{9}$$

$$E(X) = \int x f_X(x)\,dx$$

$$E(X) = \iint x f(x,y)\,dx\,dy$$

Example 2

Verify that

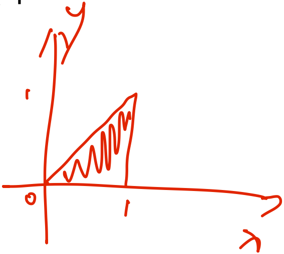
$$f(x,y) = \begin{cases} 8xy & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

is a joint pdf. Find the marginal pdf's and $E(XY)$.

$$\int_0^1\int_0^x 8xy\,dy\,dx = \int_0^1 \left[4xy^2\right]_0^x dx = \int_0^1 4x^3\,dx$$

$$= x^4\Big|_0^1 = 1$$

$$\int_0^1\int_y^1 8xy\,dx\,dy = \cdots = 1$$



$$f_X(x) = \int_0^{x} 8xy\,dy = 4x^2y \Big|_0^x = 4x^3, \quad 0<x<1$$

$$f_Y(y) = \int_y^1 8xy\,dx = 4x^2y \Big|_y^1 = 4y - 4y^3, \quad 0<y<1$$

X is Beta$(4, 1)$