# STAT 345/445 Lecture 9

## Differentiating under an integral sign – Section 2.4

- A technical section on when we can switch the order of limits, integrals, and sums

# Moment Generating functions

- Proof of

$$\frac{d}{dt}M_X(t)\Big|_{t=0} = E(X)$$

$e^{tx}f(x)$ is function of both $t$ and $x$

goes like this:

$$\frac{d}{dt}M_X(t) = \frac{d}{dt}\overbrace{\int_{-\infty}^{\infty} e^{tx}f(x)dx}^{M_X(t)} = \int_{-\infty}^{\infty}\left(\frac{\partial}{\partial t}e^{tx}f(x)\right)dx$$

if we can *differentiate under the integral sign*

$$\Rightarrow \frac{d}{dt}M_X(t) = \int_{-\infty}^{\infty} xe^{tx}f(x)dx = E\left(Xe^{tX}\right)$$

$$\Rightarrow \frac{d}{dt}M_X(t)\Big|_{t=0} = E\left(Xe^{0\cdot X}\right) = E(X)$$

# Differentiating under an integral sign

## Theorem: (Simplified Leibnitz's Rule)

If $f(x, \theta)$ is differentiable with respect to $\theta$ and $a$ and $b$ are constants then

$$\frac{d}{d\theta} \underbrace{\int_a^b f(x,\theta)dx}_{\text{function of } \theta} = \int_a^b \frac{\partial}{\partial\theta} f(x,\theta)dx$$

- Finite range integral $\Rightarrow$ can switch the order of $\frac{d}{d\theta}$ and $\int dx$
- If $a = -\infty$ and/or $b = \infty$ we have to be careful
  - Recall: A derivative is a limit
  - $\Rightarrow$ The issue is actually: When can we change the order of limits and integration?

# What is it we want to do?

$\infty + \infty = \infty$

$-\infty - \infty = -\infty$

$\infty - \infty$ not defined

- Recall:

$$\frac{\partial}{\partial\theta} f(x,\theta) = \lim_{\delta\to 0} \frac{f(x,\theta+\delta) - f(x,\theta)}{\delta}$$

Therefore:

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial\theta} f(x,\theta)dx = \int_{-\infty}^{\infty} \lim_{\delta\to 0} \frac{f(x,\theta+\delta) - f(x,\theta)}{\delta} dx$$

$$\frac{d}{d\theta} \int_{-\infty}^{\infty} f(x,\theta)dx = \lim_{\delta\to 0} \frac{\int_{-\infty}^{\infty} f(x,\theta+\delta)dx - \int_{-\infty}^{\infty} f(x,\theta)dx}{\delta}$$

$$= \lim_{\delta\to 0} \int_{-\infty}^{\infty} \frac{f(x,\theta+\delta) - f(x+\theta)}{\delta}$$

# When can we differentiate under the integral sign?

## Theorem

If $f(x, \theta)$ is differentiable with respect to $\theta$ and there exists a constant $\delta_0 > 0$ and a function $g(x, \theta)$ that satisfies

(i) $\left| \frac{\partial}{\partial\theta} f(x, \theta)\Big|_{\theta=\theta'} \right| \leq g(x, \theta)$

for all $\theta'$ such that $|\theta' - \theta| \leq \delta_0$

(ii) $\int_{-\infty}^{\infty} g(x, \theta) dx < \infty$

then

$$\frac{d}{d\theta} \int_{-\infty}^{\infty} f(x, \theta) dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial\theta} f(x, \theta) dx$$

derivative of $f(x, \theta)$ was to behave!
need to be bounded by
Some function

# When can we differentiate under the integral sign?

- Grad: Understand that conditions (i) and (ii) basically mean:
  - The (partial) derivative $\frac{\partial}{\partial\theta} f(x, \theta)$ has to behave!
  - It has to be dominated by a function $g(x, \theta)$ that has a finite integral (w.r.t. $x$)
    - at least at some $\theta'$ close to $\theta$
- UG: Just remember that changing the order of a derivative and an integral with an infinite range can't always be done

# Example

- Let $X \sim \mathrm{Expo}(\lambda)$. The pdf for $X$ is

$$f(x) = \frac{1}{\lambda} e^{-x/\lambda} \quad \text{for } x > 0$$

Show that for an integer $n \geq 1$

$$E\left(X^{n+1}\right) = \lambda E\left(X^n\right) + \lambda^2 \frac{d}{d\lambda} E\left(X^n\right)$$

# Derivatives and infinite sums

- Finite sums are no problem:

$$\frac{d}{d\theta}\sum_{x=0}^{n} f(x,\theta) = \sum_{x=0}^{n} \frac{\partial}{\partial\theta} f(x,\theta)$$

- When does the following hold?

$$\frac{d}{d\theta}\sum_{x=0}^{\infty} f(x,\theta) = \sum_{x=0}^{\infty} \frac{\partial}{\partial\theta} f(x,\theta)$$

  - Basically: Both series have to converge
    See Theorem 2.4.8 in the textbook

# Example: Geometric distribution

- Let $X \sim \mathrm{Geometric}(\theta)$. The pmf for $X$ is

$$f(x) = \theta(1-\theta)^x \quad \text{for } x = 0, 1, 2, \dots \text{ and } 0 < \theta < 1$$

is a pmf since.

$$\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} \theta(1-\theta)^x = \theta \sum_{x=0}^{\infty} (1-\theta)^x = \frac{\theta}{1-(1-\theta)} = 1$$

Geometric Sum

- Lets see where this takes us:

$$\frac{d}{d\theta} \sum_{x=0}^{\infty} f(x)$$

- Convenient facts about the **geometric series**

$$\sum_{x=0}^{\infty} r^x = \frac{1}{1-r} \quad \text{for } |r| < 1$$

$$\sum_{x=0}^{n} r^x = \frac{1-r^{n+1}}{1-r} \quad \text{for } r \neq 1$$

First. $\frac{d}{d\theta}\sum_{x=0}^{\infty} f(x) = \frac{d}{d\theta} 1 = 0$ $(UV)' = U'V + V'U$

Second, $\frac{d}{d\theta}\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} \frac{d}{d\theta} f(x) = \sum_{x=0}^{\infty} \frac{d}{d\theta} \overset{U}{\theta} \overset{V}{(1-\theta)^x}$

See textbook

$= \sum_{x=0}^{\infty} \left[(1-\theta)^x + \theta(1-\theta)^{x-1}(-1)\right]$

$= \sum_{x=0}^{\infty} (1-\theta)^x - \frac{(1-\theta)}{(1-\theta)} \sum_{x=0}^{\infty} x\theta(1-\theta)^{x-1}$

$= \frac{1}{\theta} - \frac{1}{1-\theta} \underbrace{\sum_{x=0}^{\infty} x\theta(1-\theta)^x}_{=E(X)}$

Take this together we get

$0 = \frac{1}{\theta} - \frac{1}{1-\theta} E(X)$

$\Rightarrow E(X) = \frac{1-\theta}{\theta}.$