STAT 345/445 Lecture 3

Section 1.2: Basics of Probability Theory

Subsections 1.2.3 and 1.2.4: Counting methods

1. Counting
   - Four cases: Ordered/unordered, with/without replacement
   - Counting and Probabilities for finite samples spaces
   - Examples

Counting methods

Bridge hand : 13 cards

- Probability of getting more than 7 cards of the same suit in bridge? E.g. 8 or more diamonds ...
  - Define a sample space with all possible outcomes so that all are equally likely
  - A = the event of getting more than 7 cards of the same suit . Then

$$P(A) = \frac{|A|}{n}$$

$|A|$ = number of elements in A

  - $|A|$ = how many randomly dealt ~~suits or~~ hands of 13 cards have ~~7~~ 8 or more of the same suit
  - n = total number of possible hands of 13 cards

- Can you solve it?
  - I haven't ... yet

Basics of counting

like dealing a Bridge hand

Multiplication rule

Suppose a project can be described as follows

- The project has $\boxed{k \text{ tasks}}$ that all need to be completed
- Task i can be done in $\underline{n_i}$ ways, $i = 1, 2, 3, \ldots, k$

Then the number ow ways the project can be performed is

$$n_1 \times n_2 \times \cdots \times n_k = \prod_{i=1}^{k} n_i$$

Example: Coffee shop

- When ordering coffee at a café we need to choose size (small, medium, large) and type (regular, latte, cappuccino, americano). In how many ways can we order coffee?

$k = 2$ tasks $n_1 = 3$ $n_2 = 4$

small → regular, latte, cappuccino, americano

medium

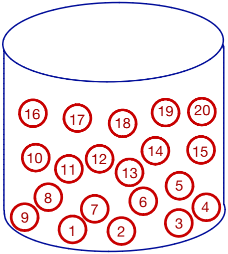
large

$3 \times 4 = 12$

possible orders

Four different cases of counting

- Want to count how many ways we can pick r items out of n items.
- Think:
  - n balls in an urn, numbered from 1 to n
  - We pick r balls at random
- Four cases
  - Case 1: Ordered, without replacement
  - Case 2: Ordered, with replacement
  - Case 3: Unordered, without replacement
  - Case 4: Unordered, with replacement



In each case it is helpful to define the number of tasks that need to be completed and in how many ways each task can be done. Then we can apply the multiplication rule.

- Four cases
  - Case 1: Ordered, without replacement
  - Case 2: Ordered, with replacement
  - Case 3: Unordered, without replacement
  - Case 4: Unordered, with replacement

Say $r = 3$ $n = 20$

if: picking up 3 balls

Ordered: $(17, 9, 5)$ and $(9, 5, 17)$

are different outcomes.

Unordered: $(17, 9, 5)$ and $(9, 5, 17)$ are same outcome.

With replacement: Put the ball back in the urn befor picking the next one. So a ball can be picked more than once: (5, 5, 12)

without replacement

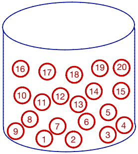
(5, 17, 12)

Case 1: Ordered, without replacement

Example: $n = 20$, $r = 4$.

Number of tasks : r

Task	1	2	3	4
Ways	20	19	18	17



If $r = n$ this is called permutation: $20!$

| 20 | 19 | 18 | 17 |

Total of $20 \times 19 \times 18 \times 17$ outcomes

In general: $\underbrace{n(n-1)(n-2)\times\cdots(n-r+1)}_{r \text{ terms}} = \frac{n!}{(n-r)!}$ outcomes

Case 2: Ordered, with replacement

Today 9/5

Counting methods

Conditional prob

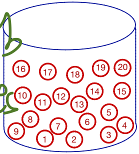
Independent events

Example: $n = 20$, $r = 4$.

Tasks: Pick 1^{st}, 2^{nd}, 3^{rd} and 4^{th}

Task	1	2	3	4
Ways	20	20	20	20

Total 20^4



In general: n^r

Case 3: Unordered, without replacement

Example: $n = 20$, $r = 4$.

Know that if ordered: $20 \times 19 \times 18 \times 17$ possibilities

but $(6, 19, 7, 3)$ is the same as $(3, 7, 6, 19)$

How many permutation? $4! = 4 \times 3 \times 2 = 24$.

So, each outcome has been counted 24 times in $(*)$.

$\Rightarrow$ There are $\dfrac{20 \times 19 \times 18 \times 17}{4 \times 3 \times 2} = \dfrac{n!}{(n-r)!\, r!}$ ← factorial

In general:

$$= \binom{20}{4} = \binom{n}{r}$$

Case 4: Unordered, with replacement

Balls 1, 2, 3, 4

Smaller example: $n = 4$, $r = 2$.

Possible outcomes:

10

(1, 1) (1, 2) (1, 3), (1, 4)

(2, 2) (2, 3) (2, 4)

(3, 3) (3, 4)

(4, 4)

These outcome only have one not two permutations.

So dividing by $r!$ takes out too many outcomes.

In general:

Try this schematic:

$n = 4$ balls and pick 2 with replacement.

outcomes:

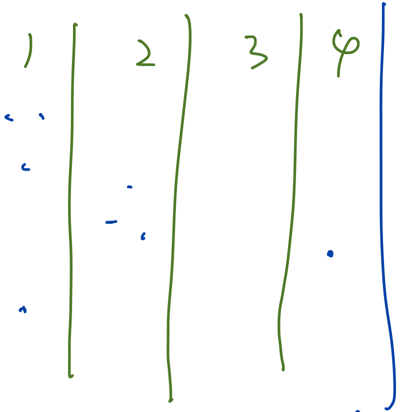
(1, 1)

(1, 2)

(2, 2)

(1, 4)

⋮



··||| 5 items

·|·|| 2 dots "·"

|··|| 3 dividers

·|||·   "|"

⋮

$$\binom{n-1+r}{r} = \binom{5}{2} \text{ possibilities} = \frac{5 \times 4}{2} = 10 \text{ ways}$$

Counting and Probabilities for finite samples spaces

Suppose

- $S = \{s_1, s_2, \ldots, s_n\}$
- All outcomes are equally likely, i.e.

$$P(\{s_i\}) = \frac{1}{n} \qquad \text{for } i = 1, 2, \ldots, n$$

Then

$$P(A) = \sum_{\{i:\, s_i \in A\}} P(\{s_i\}) = \sum_{\{i:\, s_i \in A\}} \frac{1}{n} = \frac{|A|}{n}$$

where $|A|$ is the number of elements in the set A

- Read examples in Chapter 1.2.4 carefully
- Make sure that $n = |S|$ and $|A|$ are counted the same way (both ordered or both unordered)

Counting and Probabilities for finite samples spaces

- Read examples in Chapter1.2.4 carefully
- Note: When sampling *without* replacement and we want to calculate the probability of an event that does not depend on order it does not matter whether we use ordered or unordered sample space
  - Ordered sample space: each outcome equally likely
  - Unordered sample space: each outcome equally likely - because there are exactly $r!$ replicates of each
  - So make sure that $n = |S|$ and $|A|$ are counted the same way (both ordered or both unordered)
- Sampling *with* replacement is a bit trickier - see example 1.2.19 in the textbook

Example: Choosing student representatives

Have a class of 40 people. 25 stat majors, 22 international students, and 10 of the international students are math majors. (no double majors)

	International	Domestic	Total
Stat majors	12	13	25
Math majors	10	5	15
Total	22	18	40

1. On person is chose at random. What is the probability that it is a Stat major?

$\frac{25}{40} = 0.625$

Example: Choosing student representatives

2. A student counsel is chosen with three positions: President, Treasurer, and Scapegoat. If the counsel is chosen at random, what is the probability that there is a Stat major in all positions?

How many possible counsels? $\leftarrow 60 \times 39 \times 38$

Ordered without replacement.

How many stat only cousels?

$25 \times 24 \times 23$

$$\Rightarrow P(\text{all stat counsel}) = \frac{25 \times 24 \times 23}{60 \times 39 \times 38} = 0.233$$

3. Same setting, but same person can hold more than one office

Ordered with replacement

Total ~~st~~ councels: 60^3 Total ~~stat~~ ~~only stat councels~~ only stat only cousel 25^3.

$$P(\text{all stat cousels}) = \frac{25^3}{60^3}$$

Example: Choosing student representatives

4. A task force of 3 is needed for a student paper (no positions). If the task force is chosen at random, what is the probability that they are all Stat majors? Unordered, without replacement

task forces: $\binom{60}{3}$ # stat only task forces $\binom{25}{3}$

$P(\text{stat only forces}) = \dfrac{\binom{25}{3}}{\binom{60}{3}}$

5. Same setting, but 1 editor and 2 members.

2 tasks: 1) Pick editor 2) Pick members

Total # : 1) 60 $\quad$ 2) $\binom{39}{2}$

Z) $60 \times \binom{39}{2}$

$\binom{60}{2} \cdot 38$

Same.

Total stat only : 1) 25 $\quad$ 2) $\binom{24}{2}$

$$P(\text{stat only}) = \frac{25 \times 24 \times 23/2}{60 \times 39 \times 38/2} = 0.233$$

Example: Poker player

Dealer gives you 5 cards at random. What is the probability that you get a full house?

= Triple and a pair

Total # of poker hands: $\binom{52}{5}$

Total # of a full house : $13 \times \binom{4}{3} \times 12 \times \binom{4}{2}$

$$P(\text{full house}) = \frac{13 \times \binom{4}{3} \times 12 \times \binom{4}{2}}{\binom{52}{5}}$$