Stat 345/445: Theoretical Statistics I: Homework 6 Solutions

Textbook Exercises

4.7 (345: 2 pts & 445: 1 pt.) A woman leaves for work between 8 AM and 8:30 AM and takes between 40 and 50 minutes to get there. Let the random variable X denote her time of departure, and the random variable Y the travel time. Assuming that these variables are independent and uniformly distributed, find the probability that the woman arrives at work before 9 AM.

We will measure time in minutes past 8 AM. So $X \sim \text{uniform}(0,30)$, $Y \sim \text{uniform }(40,50)$ and the joint pdf is 1/300 on the rectangle $(0,30) \times (40,50)$.

$$P(\text{arrive before 9 AM}) = P(X+Y<60) = \int_{40}^{50}\int_{0}^{60-y} \frac{1}{300} dx dy = \frac{60y - \frac{y^2}{2}}{300}\Big|_{40}^{50}$$
$$= \frac{60(50-40) - \frac{(50^2-40^2)}{2}}{300} = \frac{600-450}{300} = \frac{1}{2}$$

4.10 (345 & 445: 2 pts.) The random pair (X,Y) has the distribution

		X = 1	2	3
Y	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	3	$\frac{1}{6}$	0	$\frac{1}{6}$
	4	0	$\frac{1}{3}$	0

(a) Show that X and Y are dependent.

The marginal distribution of X is $P(X=1) = P(X=3) = \frac{1}{4}$ and $P(X=2) = \frac{1}{2}$. The marginal distribution of Y is $P(Y=2) = P(Y=3) = P(Y=4) = \frac{1}{3}$. But

$$P(X=2, Y=3) = 0 \neq (\frac{1}{2})(\frac{1}{3}) = P(X=2)P(Y=3).$$

Therefore the random variables are not independent.

(b) Give a probability table for random variables U and V that have the same marginals as X and Y but are independent.

The distribution that satisfies $P(U=x, V=y) = P(U=x)P(V=y)$ where $U \sim X$ and $V \sim Y$ is

		U = 1	2	3
V	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	3	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	4	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$

4.19

(a) (345 & 445: 1 pt.) Let X_1 and X_2 be independent $n(0,1)$ random variables. Find the pdf of $(X_1 - X_2)^2/2$.

Since $(X_1 - X_2)/\sqrt{2} \sim n(0,1), (X_1 - X_2)^2/2 \sim \mathcal{X}_1^2$ (see Example 2.1.9).

$$Y = \frac{(X_1 - X_2)^2}{2} \sim \mathcal{X}_1^2, \quad f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}, \quad 0 < y < \infty$$

(b) (445: 2 pts.) If X_i, $i = 1, 2$, are independent gamma$(\alpha_i, 1)$ random variables, find the marginal distributions of $X_1/(X_1 + X_2)$ and $X_2/(X_1 + X_2)$.

Make the transformation $y_1 = \frac{x_1}{x_1+x_2}, y_2 = x_1 + x_2$ then $x_1 = y_1 y_2, x_2 = y_2(1 - y_1)$ and $|J| = y_2$. Then

$$f(y_1, y_2) = \Big[\frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y_1^{\alpha_1 - 1}(1 - y_1)^{\alpha_2 - 1}\Big]\Big[\frac{1}{\Gamma(\alpha_1 + \alpha_2)} y_2^{\alpha_1 + \alpha_2 - 1} e^{-y_2}\Big],$$

thus $Y_1 \sim$ beta$(\alpha_1, \alpha_2), Y_2 \sim$ gamma $(\alpha_1 + \alpha_2, 1)$ and are independent.

4.22 (345: 1 pt.) Let (X, Y) be a bivariate random vector with joint pdf $f(x, y)$. Let $U = aX + b$ and $V = cY + d$, where $a, b, c,$ and d are fixed constants with $a > 0$ and $c > 0$. Show that the joint pdf of (U, V) is

$$f_{U,V}(u, v) = \frac{1}{ac} f(\frac{u - b}{a}, \frac{v - d}{c}).$$

$u = ax + b \implies x = \frac{u-b}{a}, \quad v = cy + d \implies y = \frac{v-d}{c}$

$$J = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{c} \end{vmatrix} = \frac{1}{ac}$$

Thus $J = \frac{1}{ac}$.

$$f_{uv}(uv) = f_{xy}(h_x(u) h_y(v))|J| = \frac{1}{ac} f(\frac{u - b}{a} \ \frac{v - d}{c})$$

4.23 For X and Y as in Example 4.3.3, find the distribution of XY by making the transformations given in (a) and integrating out V.

(a) (345 & 445: 2 pts.) $U = XY$, $V = Y$

Let $y = v, x = u/y = u/v$ then

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}.$$

$$\begin{aligned}
f_{U,V}(u, v) &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} (\frac{u}{v})^{\alpha - 1} (1 - \frac{u}{v})^{\beta - 1} \frac{\Gamma(\alpha + \beta + \gamma)}{\Gamma(\alpha + \beta)\Gamma(\gamma)} v^{\alpha + \beta - 1}(1 - v)^{\gamma - 1}(\frac{1}{v}) \\
&= \frac{\Gamma(\alpha + \beta + \gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma\gamma)} u^{\alpha - 1} v^{1 - \alpha}(1 - \frac{u}{v})^{\beta - 1} v^{\alpha + \beta - 1}(1 - v)^{\gamma - 1} v^{-1} \\
&= \frac{\Gamma(\alpha + \beta + \gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma\gamma)} u^{\alpha - 1} v^{1 - \alpha + \alpha + \beta - 1 - 1}(1 - \frac{u}{v})^{\beta - 1}(1 - v)^{\gamma - 1} \\
&= \frac{\Gamma(\alpha + \beta + \gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma\gamma)} u^{\alpha - 1} v^{\beta - 1}(1 - \frac{u}{v})^{\beta - 1}(1 - v)^{\gamma - 1} \\
&= \frac{\Gamma(\alpha + \beta + \gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma\gamma)} u^{\alpha - 1}(v - u)^{\beta - 1}(1 - v)^{\gamma - 1} \qquad 0 < u < v < 1.
\end{aligned}$$

Then,

$$f_U(u) = \int_u^1 \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma\gamma)} u^{\alpha-1}(v-u)^{\beta-1}(1-v)^{\gamma-1}dv$$

Let $z = \frac{v-u}{1-u} \implies v = (1-u)z + u, \quad dv = (1-u)dz$. For $v = 1, z = 1$ and for $v = u, z = 0$.

$$\begin{aligned} f_U(u) &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1} \int_0^1 \Big((1-u)z + u - u\Big)^{\beta-1}\Big(1-(1-u)z-u\Big)^{\gamma-1}(1-u)dz \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1} \int_0^1 (1-u)^{\beta-1}z^{\beta-1}(1-u)^{\gamma-1}(1-z)^{\gamma-1}(1-u)dz \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta-1+\gamma-1+1}\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} \int_0^1 \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)} z^{\beta-1}(1-z)^{\gamma-1}dz \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \qquad 0 < u < 1. \end{aligned}$$

Thus, $U \sim \text{beta}(\alpha, \beta+\gamma)$.

4.24 (345 & 445: 2 pts.) Let X and Y be independent random variables with $X \sim \text{gamma}(r, 1)$ and $Y \sim \text{gamma}\ (s, 1)$. Show that $Z_1 = X + Y$ and $Z_2 = X/(X + Y)$ are independent, and find the distribution of each. (Z_1 is gamma and Z_2 is beta.)

Let $z_1 = x + y, z_2 = \frac{x}{x+y}$, then $x = z_1 z_2, y = z_1(1 - z_2)$ and

$$J = \begin{vmatrix} \frac{\partial x}{\partial z_1} & \frac{\partial x}{\partial z_2} \\ \frac{\partial y}{\partial z_1} & \frac{\partial y}{\partial z_2} \end{vmatrix} = \begin{vmatrix} z_2 & z_1 \\ 1-z_2 & -z_1 \end{vmatrix} = z_1.$$

The set $\{x > 0, y > 0\}$ is mapped onto the set $\{z_1 > 0, 0 < z_2 < 1\}$.

$$\begin{aligned} f_{Z_1,Z_2}(z_1,z_2) &= \frac{1}{\Gamma(r)}(z_1z_2)^{r-1}e^{-z_1z_2}\frac{1}{\Gamma(s)}(z_1 - z_1z_2)^{s-1}e^{-z_1+z_1z_2}z_1 \\ &= \frac{1}{\Gamma(r+s)} z_1^{r+s-1}e^{-z_1}\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}z_2^{r-1}(1-z_2)^{s-1}, \qquad 0 < z_1, 0 < z_2 < 1. \end{aligned}$$

$f_{Z_1,Z_2}(z_1, z_2)$ can be factorized into two densities. Therefore Z_1 and Z_2 are independent and $Z_1 \sim$ gamma $(r + s, 1)$, $Z_2 \sim \text{beta}(r, s)$.