# Stat 345/445: Theoretical Statistics I: Homework 2 Solutions

## Textbook Exercises

**1.26** (445: 2 pts.) A fair die is cast until a 6 appears. What is the probability that it must be cast more than five times?

For each cast, the probability a 6 appears is $p = \frac{1}{6}$. The probability that it must be cast more than five times until 6 appears is the same as the probability that no 6 appears in the first five times, that is,

$$(1 - \frac{1}{6})^5 = (\frac{5}{6})^5 = 0.4019.$$

Note: If we set $X$ = number of casts until (and including) a 6 appears, then $X$ has a Geometric distribution with $p = \frac{1}{6}$ (see page 97). The pmf is

$$f(x) = p(1-p)^{x-1} \quad \text{for } x = 1, 2, 3, \cdots$$

We want

$$P(X > 5) = 1 - P(X \le 5) = 1 - \sum_{x=1}^{5} f(x) = 1 - \frac{1}{6}(1 + \frac{5}{6} + \frac{5^2}{6^2} + \frac{5^3}{6^3} + \frac{5^4}{6^4}) = 1 - 0.5981 = 0.4019$$

Recall the geometric sum:

$$\sum_{k=1}^{n} r^{k-1} = \frac{1 - r^n}{1 - r} \text{ if } r \neq 1$$

Therefore

$$P(X > 5) = 1 - \sum_{x=1}^{5} \frac{1}{6}(\frac{5}{6})^{x-1} = 1 - \frac{1}{6}(\frac{1 - (\frac{5}{6})^5}{1 - \frac{5}{6}}) = \frac{6(1 - 5/6) - (1 - (5/6)^5)}{6(1 - 5/6)} = \frac{6 - 5 - 1 + (5/6)^5}{6 - 5} = (\frac{5}{6})^5$$

**1.33** (345 & 445: 2 pts.) Suppose that 5% of men and .25% of women are color-blind. A person is chosen at random and that person is color-blind. What is the probability that the person is male? (Assume males and females to be in equal numbers.)

Using Bayes rule

$$P(M|CB) = \frac{P(CB|M)P(M)}{P(CB|M)P(M) + P(CB|F)P(F)} = \frac{0.05(\frac{1}{2})}{0.05(\frac{1}{2}) + 0.0025(\frac{1}{2})} = 0.9524$$

**1.35** (345 & 445: 2 pts.) Prove that $P(\cdot)$ is a legitimate probability function and $B$ is a set with $P(B) > 0$, then $P(\cdot|B)$ also satisfies Kolmogrov's Axioms.

$P(\cdot|B) \ge 0$, and $P(S|B) = 1$.

If $A_1, A_2, \cdots$ are disjoint, then

$$P(\bigcup_{i=1}^{\infty} A_i|B) = \frac{P(\bigcup_{i=1}^{\infty} A_i \cap B)}{P(B)} = \frac{P(\bigcup_{i=1}^{\infty}(A_i \cap B))}{P(B)} = \frac{\sum_{i=1}^{\infty} P(A_i \cap B)}{P(B)} = \sum_{i=1}^{\infty} P(A_i|B)$$

**1.39** (345 & 445: 2 pts.) A pair of events $A$ and $B$ cannot be simultaneously *mutually exclusive* and *independent*. Prove that if $P(A) > 0$ and $P(B) > 0$, then:

(a) If $A$ and $B$ are mutually exclusive, they cannot be independent.

Suppose $A$ and $B$ are mutually exclusive. Then $(A \cap B) = \emptyset$ and $P(A \cap B) = 0$.

If $A$ and $B$ are independent, then $0 = P(A \cap B) = P(A)P(B)$.

But this cannot be since $P(A) > 0$ and $P(B) > 0$. Thus $A$ and $B$ cannot be independent.

(b) If $A$ and $B$ are independent, they cannot be mutually exclusive.

If $A$ and $B$ are independent and $P(A) > 0, P(B) > 0$, then

$$0 < P(A)P(B) = P(A \cap B).$$

This implies $A \cap B \neq \emptyset$, that is $A$ and $B$ are not mutually exclusive.

## Extra Problems

**1.** (345: 2 pts.) A club has 25 members

(a) How many ways are there to choose four members of the club to serve on an executive committee?

Unordered without replacement

$$\binom{25}{4} = \frac{25!}{4!21!} = 12650$$

(b) How many ways are there to choose a president, vice president, secretary, and treasurer of the club?

Ordered without replacement

$$\binom{25}{1}\binom{24}{1}\binom{23}{1}\binom{22}{1} = \frac{25!}{21!} = 303600$$

**2.** (345: 2 pts.) There are three cabinets $A$, $B$ and $C$, each of which has two drawers. Each drawer contains one coin; $A$ has two gold coins, $B$ has two silver coins, and $C$ has one gold and one silver. A cabinet is chosen at random, one drawer is opened, and a silver coin is found. What is the probability that the other drawer in that cabinet contains a silver coin?

Let $A, B, C$ denote the events that cabinet $A, B$, or $C$ were chosen. Let $S$ denote the event that a silver coin is found.

$$P(A) = P(B) = P(C) = \frac{1}{3}$$

$$P(S|A) = 0, \quad P(S|B) = 1, \quad P(S|C) = \frac{1}{2}$$

The problem asks to find $P(B|S)$

Using Bayes Rule

$$P(B|S) = \frac{P(S|B)P(B)}{P(S)} = \frac{P(S|B)P(B)}{P(S|B)(B) + P(S|A)P(A) + P(S|C)P(C)} = \frac{1(\frac{1}{3})}{1(\frac{1}{3}) + 0(\frac{1}{3}) + \frac{1}{2}(\frac{1}{3})} = \frac{2}{3}$$

**3.** (445: 2 pts.) Suppose that a department has 10 mathematicians and 15 statisticians.

(a) How many ways are there to form a committee with 6 members if it must have the same number of mathematicians and statisticians?

For a committee with 6 members having the same number of mathematicians and statisticians, we should have 3 mathematicians and 3 statisticians.

The number of ways to form the committee is $\binom{10}{3}\binom{15}{3} = 120 \times 455 = 54600$

(b) How many ways are there to form a committee with 6 members if it must have more statisticians than mathematicians?

We have the following situations:

| Statisticians | Mathematicians | Number of ways |
|---|---|---|
| 6 | 0 | $\binom{15}{6} = 5005$ |
| 5 | 1 | $\binom{15}{5}\binom{10}{1} = 3003 \times 10 = 30030$ |
| 4 | 2 | $\binom{15}{4}\binom{10}{2} = 1365 \times 45 = 61425$ |

The total number of ways is $5005 + 30030 + 61425 = 96460$