STAT 345/445 Lecture 3

Section 1.2: Basics of Probability Theory

Subsections 1.2.3 and 1.2.4: Counting methods

1. Counting
   - Four cases: Ordered/unordered, with/without replacement
   - Counting and Probabilities for finite samples spaces
   - Examples

Counting methods

Bridge hand: 13 cards

- Probability of getting more than 7 cards of the same suit in bridge? E.g. 8 or more diamonds, ...
  - Define a sample space with all possible outcomes so that all are equally likely
  - A = the event of getting more than 7 cards of the same suit . Then

$$P(A) = \frac{|A|}{n}$$

$|A|$ = number of elements in A

  - $|A|$ = how many randomly dealt ~~suits of~~ hands of 13 cards have ~~7~~ 8 or more of the same suit
  - n = total number of possible hands of 13 cards
- Can you solve it?
  - I haven't ... yet

Basics of counting

like dealing a Bridge hand

Multiplication rule

Suppose a project can be described as follows

- The project has k tasks that all need to be completed
- Task i can be done in n_i ways, $i = 1, 2, 3, \ldots, k$

Then the number of ways the project can be performed is

$$n_1 \times n_2 \times \cdots \times n_k = \prod_{i=1}^{k} n_i$$

Example: Coffee shop

- When ordering coffee at a café we need to choose size (small, medium, large) and type (regular, latte, cappuccino, americano). In how many ways can we order coffee?

$k=2$ tasks $\quad n_1 = 3 \quad n_2 = 4$

small — regular, latte, cappuccino, americano

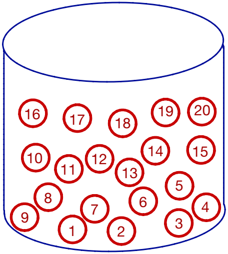
medium — regular, latte, cappuccino, americano

large — regular, latte, cappuccino, americano

$3 \times 4 = 12$ possible orders.

Four different cases of counting

- Want to count how many ways we can pick r items out of n items.
- Think:
  - n balls in an urn, numbered from 1 to n
  - We pick r balls at random
- Four cases
  - Case 1: Ordered, without replacement
  - Case 2: Ordered, with replacement
  - Case 3: Unordered, without replacement
  - Case 4: Unordered, with replacement



In each case it is helpful to define the number of tasks that need to be completed and in how many ways each task can be done. Then we can apply the multiplication rule.

- Four cases
  - Case 1: Ordered, without replacement
  - Case 2: Ordered, with replacement
  - Case 3: Unordered, without replacement
  - Case 4: Unordered, with replacement

Say $r = 3$ $n = 20$
i.e. picking up 3 balls.

Ordered: (17, 9, 5) and (9, 5, 17)
are different outcomes

Un-Ordered: (17, 9, 5) and (9, 5, 17)
are considered the
same outcome

With replacement: Put the ball back
in the urn before picking the next
one. So a ball can be picked more
than once: (5, 5, 12)

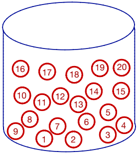
Case 1: Ordered, without replacement

Example: $n = 20$, $r = 4$.

Pick 1st ball, Pick 2nd, then 3rd and 4th

number of tasks: r

Task	1	2	3	4
Ways	20	19	18	17



Total of $20 \times 19 \times 18 \times 17$ outcomes $= 116280$

If $r = n$ this is called permutation.

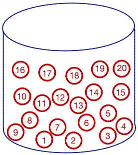
permutation : $20 \times 19 \times 18 \times \cdots \times 1 = 20!$

In general: $\underbrace{n(n-1)(n-2) \times \cdots \times (n-r+1)}_{r \text{ terms}} = \dfrac{n!}{(n-r)!}$

Case 2: Ordered, with replacement

Example: $n = 20$, $r = 4$.

Tasks: Pick 1^{st}, 2^{nd}, 3^{rd} and then 4^{th}



Task	1	2	3	4
Ways	20	20	20	20

Total $20^4 = 160{,}000$

In general: n^r

Case 3: Unordered, without replacement

Example: $n = 20$, $r = 4$.

Know that if ordered: $20 \times 19 \times 18 \times 17$ "(*)" possibilities

but $(6, 19, 7, 3)$ is the same as $(3, 7, 19, 6)$

How many permutations? $4! = 24$

so, each outcome has been counted 24 times in (*)

$\Rightarrow$ There are $\dfrac{20 \times 19 \times 18 \times 17}{4!} = \dfrac{20!}{16!\ 4!} = \binom{20}{4}$

total unordered outcomes

In general: $\dfrac{n!}{(n-r)!\, r!} = \binom{n}{r}$

Case 4: Unordered, with replacement

Balls 1,2,3,4

Smaller example: $n = 4$, $r = 2$.

$\frac{n^r}{r!}$? $\quad \frac{4^2}{2!} = \frac{16}{2} = 8$

Possible outcomes:

(1,1) (1,2) (1,3) (1,4)
(2,2) (2,3) (2,4)
(3,3) (3,4)
(4,4)

10 outcomes.

These outcomes only have 1 (not 2) permutations. So dividing by $r!$ takes out too many outcomes

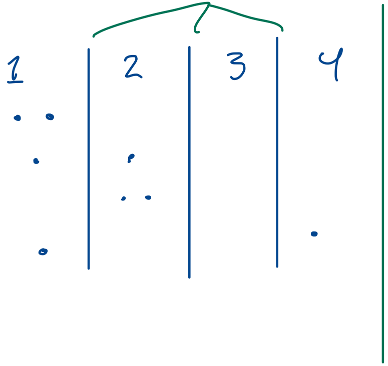
So, how?

In general: $\binom{n+r-1}{r}$

Try this schematic:

$n=4$ balls and pick 2 with repl.

3 dividers, $n-1$



Outcomes:

(1,1)

(1,2)

(2,2)

(1,4)

⋮

etc.

••|||

•|•||

|••||

•|||•

⋮

5 items
2 dots "•"
3 dividers "|"

$\binom{5}{2} = \frac{5!}{3!\,2!} = \frac{5\times 4}{2} = 10$ possible ways of choosing 2 dots $\Leftarrow$ placement of

$\binom{n-1+r}{r}$

Counting and Probabilities for finite samples spaces

Suppose

- $S = \{s_1, s_2, \dots, s_n\}$
- All outcomes are equally likely, i.e.

$$P(\{s_i\}) = \frac{1}{n} \qquad \text{for } i = 1, 2, \dots, n$$

Then

$$P(A) = \sum_{\{i:\, s_i \in A\}} P(\{s_i\}) = \sum_{\{i:\, s_i \in A\}} \frac{1}{n} = \frac{|A|}{n}$$

where $|A|$ is the number of elements in the set A

- Read examples in Chapter 1.2.4 carefully
- Make sure that $n = |S|$ and $|A|$ are counted the same way (both ordered or both unordered)

Counting and Probabilities for finite samples spaces

- Read examples in Chapter1.2.4 carefully
- Note: When sampling *without* replacement and we want to calculate the probability of an event that does not depend on order it does not matter whether we use ordered or unordered sample space
  - Ordered sample space: each outcome equally likely
  - Unordered sample space: each outcome equally likely - because there are exactly $r!$ replicates of each
  - So make sure that $n = |S|$ and $|A|$ are counted the same way (both ordered or both unordered)
- Sampling *with* replacement is a bit trickier - see example 1.2.19 in the textbook

Example: Choosing student representatives

Have a class of 40 people. 25 stat majors, 22 international students, and 10 of the international students are math majors. (no double majors)

	International	Domestic	Total
Stat majors	12	13	25
Math majors	10	5	15
Total	22	18	40

1. One person is chose at random. What is the probability that it is a Stat major?

$$\frac{25}{40} = \frac{5}{8} = 0.625$$

Example: Choosing student representatives

2. A student counsel is chosen with three positions: President, Treasurer, and Scapegoat. If the counsel is chosen at random, what is the probability that there is a Stat major in all positions?

Ordered without replacement

How many possible counsels? $40 \times 39 \times 38$

How many stat only counsels? $25 \times 24 \times 23$

$\Rightarrow P(\text{all stat counsel}) = \dfrac{25 \times 24 \times 23}{40 \times 39 \times 38} = 0.233$

3. Same setting, but same person can hold more than one office

Ordered, with replacement

Total councels: 40^3 Total stat only counsels: 25^3

$\Rightarrow P(\text{all stat councel}) \; \dfrac{25^3}{40^3} = 0.244$

Example: Choosing student representatives

4. A task force of 3 is needed for a student paper (no positions). If the task force is chosen at random, what is the probability that they are all Stat majors?

Unordered, without replacement

task forces : $\binom{40}{3}$ # stat only task forces $\binom{25}{3}$

$$P(\text{stat only forces}) = \frac{\binom{25}{3}}{\binom{40}{3}} = \frac{25 \cdot 24 \cdot 23/3!}{40 \cdot 39 \cdot 38/3!}$$

$= 0.233$ same as in 1. (see slide 11)

5. Same setting, but 1 editor and 2 members.

2 tasks: 1) Pick editor 2) Pick members

5. Same setting, but 1 editor and 2 members. in 1

2 tasks: 1) Pick editor 2) Pick members

Total #: 1) 40 2) $\binom{39}{2}$

$\Rightarrow 40 \cdot \binom{39}{2} = 40 \, \frac{39 \cdot 38}{2}$

$\binom{40}{2} \cdot 38 = \frac{40 \cdot 39}{2} \cdot 38$ same

Total stat only: 1) 25 2) $\binom{24}{2}$

$P(\text{stat only}) = \frac{25 \cdot 24 \cdot 23/2}{40 \cdot 39 \cdot 38/2} = 0.233$

Example: Poker player

Dealer gives you 5 cards at random. What is the probability that you get a full house?

Full house = Triple and a pair

Total number of poker hands: $\binom{52}{5}$

Total number of poker hands that are a full house:

Task 1: Pick a triple
pick a number (1-13) : 13
pick 3 out of 4 suits: $\binom{4}{3}$

Task 2: Pick a pair
pick a number : 12
pick 2 out 4 suits: $\binom{4}{2}$

$$P(\text{Full House}) = \frac{13 \times \binom{4}{3} \times 12 \times \binom{4}{2}}{\binom{52}{5}} = 0.00144 = 0.14\%$$