Stat 345/445: Theoretical Statistics I
Homework 1 Solutions

Textbook Exercises

1.1 (345 & 445: 2 pts.) For each of the following experiments, describe the sample space.

(a) Toss a coin four times. Each sample point describes the result of the toss (H or T) for each of the four tosses. So, for example THTT denotes T on 1st, H on 2nd, T on 3rd and T on 4th. There are $2^4 = 16$ such sample points.

S = {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH,
THTH, THHT, TTHH, HTTT, THTT, TTHT, TTTH, TTTT}

(b) Count the number of insect-damaged leaves on a plant. The number of damaged leaves is a non-negative integer. So we might use $S = \{0, 1, 2, \dots\}$.

(c) Measure the lifetime (in hours) of a particular brand of light bulb. We might observe fractions of an hour. So we might use $S = \{t : t \geq 0\}$, that is, the half infinite interval $[0, \infty)$.

(d) Record the weights of 10-day-old rats. Suppose we weigh the rats in ounces. The weight must be greater than zero so we might use $S = (0, \infty)$. If we know no 10-day-old rate weighs more than 100oz., we could use $S = (0, 100]$.

(e) Observe the proportion of defectives in a shipment of electronic components. If n is the number of items in the shipment, then $S = \{0/n, 1/n, \dots, 1\}$.

1.4 (345: 2 pts.) For events A and B, find formulas for the probabilities of the following events in terms of the quantities $P(A), P(B)$, and $P(A \cap B)$.

(a) either A or B or both is $A \cup B$, and $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

(b) either A or B but not both is $(A \cap B^c) \cup (B \cap A^c)$, so we have $P[(A \cap B^c) \cup (B \cap A^c)] = P(A \cap B^c) + P(B \cap A^c) = [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)] = P(A) + P(B) - 2P(A \cap B)$.

(c) at least one of A or B is $A \cup B$, so we get the same answer as in part (a).

(d) at most one of A or B is $(A \cap B)^c$, and $P((A \cap B)^c) = 1 - P(A \cap B)$.

1.6 (445: 2 pts.) Two pennies, one with $P(\text{head}) = u$ and one with $P(\text{head}) = w$, are to be tossed together independently. Define $p_0 = P(0 \text{ heads occur})$, $p_1 = P(1 \text{ heads occur})$, $p_2 = P(2 \text{ heads occur})$.

$$p_0 = (1-u)(1-w), \quad p_1 = u(1-w) + w(1-u), \quad p_2 = uw$$

$$p_0 = p_2 \implies u + w = 1$$

$$p_1 = p_2 \implies uw = \frac{1}{3}$$

These two equations imply $u(1-u) = \frac{1}{3}$, which has no solution in the real numbers. We cannot find a real value u and w that makes $p_0 = p_1 = p_2$. Thus, the probability assignment is not legitimate.

Extra Problems

1. (345: 4 pts.) Let $(S, \mathcal{B}, P)$ be a probability model and let $A, B \in \mathcal{B}$. Using only the Kolmogorov axioms and Theorem 1 on slide 9 in Lecture 2 show the following:

(a) $P(A) = P(A \cap B) + P(A \cap B^c)$ First, note that $A = (A \cap B) \cup (A \cap B^c)$.

$$\begin{aligned} P(A) &= P\big((A \cap B) \cup (A \cap B^c)\big) \\ &= P(A \cap B) + P(A \cap B^c) && \text{by axiom(iii)} \end{aligned}$$

since $A \cap B$ and $A \cap B^c$ are disjoint

(b) $P(A \backslash B) = P(A) - P(A \cap B)$ Recall that $A \backslash B = A \cap B^c$
$\implies P(A \backslash B) = P(A \cap B^c) = P(A) - P(A \cap B)$ by part (a).

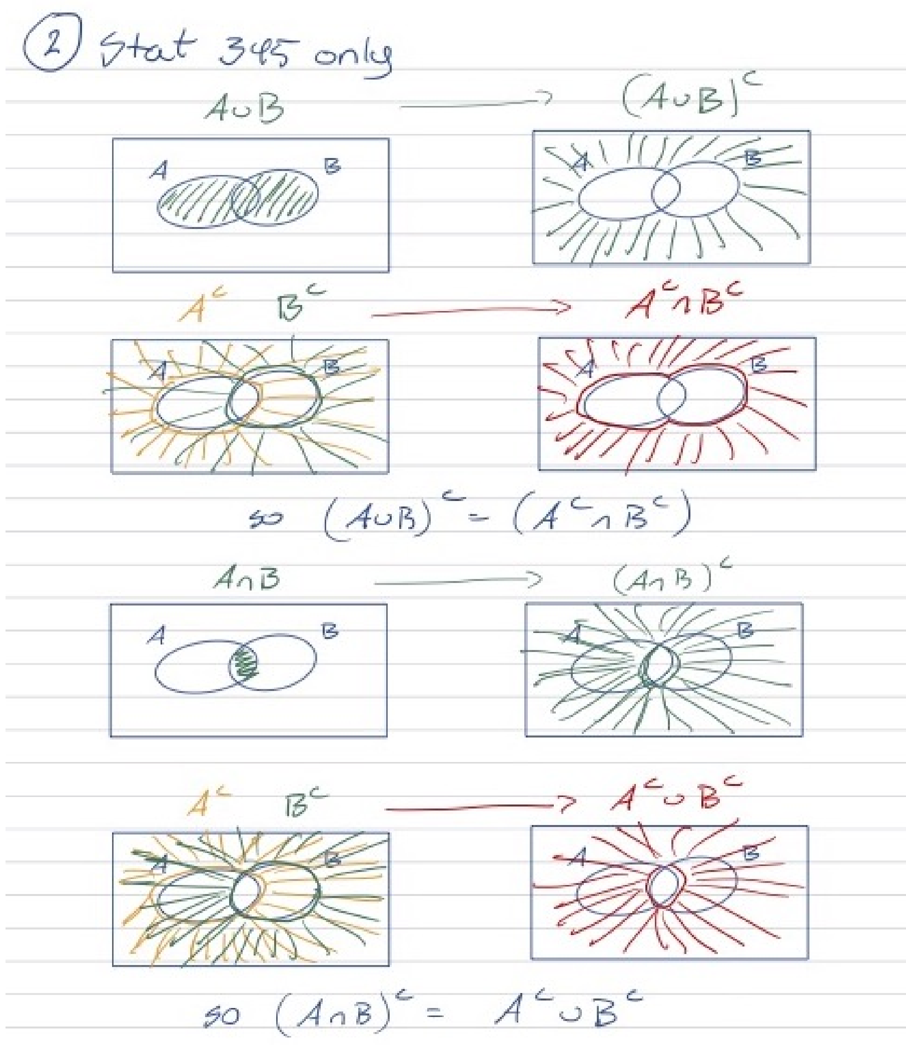
(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ Note that $A \cup B = A \cup (B \backslash A)$ and A and $B \backslash A$ are disjoint
$\implies P(A \cup B) = P(A \cup (B \backslash A)) = P(A) + P(B \backslash A) = P(A) + P(B) - P(B \cap A)$ by (b)

(d) If $A \subseteq B$, then $P(A) \leq P(B)$

$$\begin{aligned} P(B) &= P((B \cap A) \cup (B \cap A^c)) = P(B \cap A) + P(B \cap A^c) && \text{by (a)} \\ &= P(A) + P(B \cap A^c) && \text{since } A \subset B \\ &\geq P(A) && \text{since } P(B \cap A^c) \geq 0 \text{ by axiom(i)} \end{aligned}$$

2. (345: 2 pts.) Draw Venn diagrams to illustrate DeMorgan's laws:

$$(A \cup B)^c = A^c \cap B^c \quad \text{and} \quad (A \cap B)^c = A^c \cup B^c$$



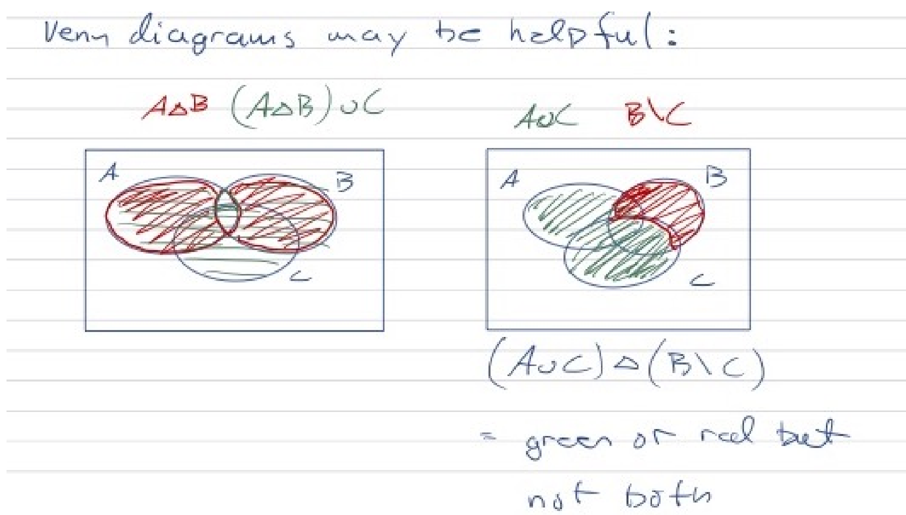
3. (445: 4 pts.) Recall from Lecture 1 the symmetric difference (xor) of two sets:

$$A \triangle B = (A \backslash B) \cup (B \backslash A) = \{x : x \text{ is in either } A \text{ or } B \text{ but not both}\}$$

Show that

(a) $(A \triangle B) \cup C = (A \cup C) \triangle (B \backslash C)$

$$\begin{aligned}
(A \cup C) \triangle (B \backslash C) &= (A \cup C) \triangle (B \cap C^c) && \text{by definition of } \backslash \\
&= [(A \cup C) \cap (B \cap C^c)^c] \cup [(B \cap C^c) \cap (A \cup C)^c] && \text{by definition of } \triangle \text{ and } \backslash \\
&= [(A \cup C) \cap (B^c \cup C)] \cup [(B \cap C^c) \cap (A^c \cap C^c)] && \text{by DeMorgan} \\
&= [(A \cap B^c) \cup C] \cup [B \cap A^c \cap C^c] && \text{by distributive law and associative law} \\
&= A \cap B^c \cup [[C \cup (B \cap A^c)] \cap (C \cup C^c)] && \text{by distributive and associative law} \\
&= (A \cap B^c) \cup C \cup (B \cap A^c) \\
&= (A \triangle B) \cup C && \text{by definition of } \triangle \text{ and } \backslash
\end{aligned}$$



(b) $(A \cup B) \triangle C = (A \triangle C) \triangle (B \backslash A)$

$$\begin{aligned}
&(A \triangle C) \triangle (B \backslash A) \\
&= [(A \cap C^c) \cup (C \cap A^c)] \triangle (B \cap A^c) && \text{def of } \triangle \text{ and } \backslash \\
&= [(A \cap C^c) \cup (C \cap A^c)] \cap (B \cap A^c)^c \cup (B \cap A^c) \cap [(A \cap C^c) \cup (C \cap A^c)]^c && \text{def of } \triangle \text{ and } \backslash \\
&= (A \cap C^c) \cup (C \cap A^c)] \cap (B^c \cup A) \cup (B \cap A^c) \cap [(A \cap C^c)^c \cap (C \cap A^c)^c] && \text{deMorgan (x2)} \\
&= [(A \cap C^c) \cap (B^c \cup A)] \cup (C \cap A^c) \cap (B^c \cup A) && \text{distr. law} \\
&\qquad \cup (B \cap A^c) \cap (A^c \cup C) \cap (C^c \cap A) && \text{deMorgan} \\
&= (A \cap C^c \cap B^c) \cup (A \cap C^c \cap A) \cup (C \cap A^c \cap B^c) \cup (C \cap A^c \cap A) && \text{distr. law} \\
&\qquad \cup [B \cap A^c \cap A^c \cup B \cap A^c \cap C] \cap (C^c \cup A) \\
&= (A \cap C^c \cap B^C) \cup (A \cap C^C) \cup (C \cap A^c \cap B^c) \cup \emptyset \\
&\qquad \cup B \cap A^c \cap (C^c \cap A) \cup B \cap A^c \cap C \cap (C^c \cup A) \\
&= (A \cap C^c \cap B^c) \cup (A \cap C^c) \cup (C \cap A^c \cap B^c) \cup (B \cap A^c \cap C^c) \\
&\qquad \cup (B \cap A^c \cap A) \cup (B \cap A^c \cap C \cap C^c) \cup (B \cap A^c \cap C \cap A) \\
&= (A \cap C^c \cap B^c) \cup (A \cap C^c) \cup (C \cap A^c \cap B^c) \cup (B \cap A^c \cap C) \\
&= (A \cap C^c) \cup (C \cap A^c \cap B^c) \cup (B \cap A^c \cap C^c)
\end{aligned}$$

Since $A \cap C^c \cap B^c \subset A \cap C^c$, their union is just $A \cap C^c$.
So far we have

$$(A \triangle C) \triangle (B \backslash A) = (A \cap C^c) \cup (C \cap A^c \cap B^c) \cup (B \cap A^c \cap C^c) \qquad (\star)$$

Starting from the other end:

$$\begin{aligned}
(A\cup B)\triangle C &= \big((A\cup B)\cap C^c\big)\cup\big(C\cap(A\cup B)^c\big) && \text{def of } \triangle \text{ and } \setminus\\
&= (A\cap C^c)\cup(B\cap C^c)\cup(C\cap A^c\cap B^c) && \text{distr. \& deMorgan}\\
&= (A\cap C^c)\cup(B\cap C^c\cap A)\cup(B\cap C^c\cap A^c)\cup(C\cap A^c\cap B^c) && \text{law of total prop.}
\end{aligned}$$

Again, $B\cap C^c\cap A$ is a subset in $A\cap C^c$, so

$$\begin{aligned}
(A\cup B)\triangle C &= (A\cap C^c)\cup(B\cap C^c\cap A^c)\cup(C\cap A^c\cap B^c)\\
&= (A\triangle C)\triangle(B\setminus A) && \text{by } (\star)
\end{aligned}$$

Venn diagrams of what we wanted to prove:

$(A\cup B)\triangle C$

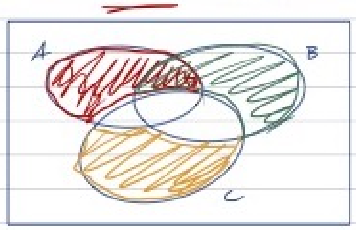
$(A\triangle C)\triangle(B\setminus A)$

= red + blue but not both

When I had trouble connecting the two ends, this helped me along:

$= A\cap C^c \cup B\cap C^c \cup C\cap A^c\cap B^c$

law of tot. prob.



$B\cap C^c = B\cap C^c\cap A \cup B\cap C^c\cap A^c$

$= A\cap C^c \cup B\cap C^c\cap A^c$
$\cup\ B\cap C^c\cap A$ subset in $A\cap C^c$
$\cup\ C\cap A^c\cap B^c$

$= A\cap C^c\cap B^c \cup A\cap C^c \cup C\cap A^c\cap B^c \cup B\cap A^c\cap C^c$

subset

4. (445: 2 pts.) Prove the Bonferroni inequality for infinite number of sets, without using the Boole's inequality for infinite number of sets.

Prove Bonferroni inequality:

$$P\Big(\bigcap_{i=1}^{\infty} A_i\Big) \geq 1 - \sum_{i=1}^{\infty} P(A_i^c) \qquad (*)$$

Strategy. Use DeMorgan to move from $\cap$ to $\cup$ and then rewrite the union as a union of disjoint sets.

$$P\Big(\bigcap_{i=1}^{\infty} A_i\Big) = 1 - \Big(P\Big(\bigcap_{i=1}^{\infty} A_i\Big)^c\Big) = 1 - P\Big(\bigcup_{i=1}^{\infty} A_i^c\Big) \qquad \text{de Morgan}$$

So, if we can show that (Boole's inequality, but we have to prove it)

$$P\Big(\bigcup_{i=1}^{\infty} A_i^c\Big) \leq \sum_{i=1}^{\infty} P(A_i^c) \qquad (**)$$

then $*$ follows.

Using the same trick as in the proof of Boole's inequlity in the textbook, we set

$$\begin{aligned}
A_1^* &= A_1^c \\
A_2^* &= A_2^c \backslash A_1^c \\
A_3^* &= A_3^c \backslash \bigcup_{j=1}^{2} A_j^c \\
&\vdots \\
A_k^* &= A_k^c \backslash \bigcup_{j=1}^{k-1} A_j^c \qquad \text{for } k = 2, 3, 4, \ldots
\end{aligned}$$

Note that

(a) $\bigcup_{i=1}^{\infty} A_i^* = \bigcup_{i=1}^{\infty} A_i^c$ and

(b) $A_1^*, A_2^*, A_3^*, \ldots$ are disjoint

So

$$P\Big(\bigcup_{i=1}^{\infty} A_i^c\Big) = P\Big(\bigcup_{i=1}^{\infty} A_i^*\Big) = \sum_{i=1}^{\infty} P(A_i^*) \qquad \text{by axiom (iii)}$$

And since $A_i^* \subset A_i^c$ for all i we have $P(A_i^*) \leq P(A_i^c)$ for all i. Therefore,

$$P\Big(\bigcup_{i=1}^{\infty} A_i^c\Big) = \sum_{i=1}^{\infty} P(A_i^*) \leq \sum_{i=1}^{\infty} P(A_i^c)$$

which shows $(**)$.

To show that (a) hols:

" $\implies$ " Let $x \in \bigcup_{i=1}^{\infty} A_i^* \implies \exists$ some k such that $x \in A_k^* = A_k^c \backslash \bigcup_{j=1}^{k-1} A_j^c = A_k^c \cap \Big(\bigcup_{j=1}^{k-1} A_j^c\Big)^c$
$\implies x \in A_k^c \implies x \in \bigcup_{i=1}^{\infty} A_i^c$

" $\impliedby$ " Let $x \in_{i=1}^{\infty} A_i^c \implies \exists$ some k such that $x \in A_k^c$. Let k be the lowest number such that $x \in A_k^c$: meaning that $X \notin A_1^c, \ldots, x \notin A_{k-1}^c \implies x \notin \bigcup_{j=1}^{k-1} A_j^c$
$\implies x \in A_k^c \backslash \bigcup_{j=1}^{k=1} A_j^c = A_k^* \implies x \in \bigcup_{i=1}^{\infty} A_i^*$

To show that (b) holds: For any i, k with $i \neq k$

$$
\begin{aligned}
A_i^* \cap A_k^* &= \Big(A_i^c \backslash \bigcup_{j=1}^{i-1} A_j^c\Big) \cap A_k^c \backslash \bigcup_{j=1}^{k-1} A_j \\
&= A_i^c \cap \Big(\bigcup_{j=1}^{i-1} A_j^c\Big)^c \cap A_k^c \cap \Big(\bigcup_{j=1}^{k-1} A_j^c\Big)^c \\
&= A_i^c \cap \Big(\bigcap_{j=1}^{i-1} A_j\Big) \cap A_k^c \cap \Big(\bigcap_{j=1}^{k-1} A_j\Big) \\
& A_i^c \cap A_1 \cap \cdots \cap A_{i-1} \cap A_k^c \cap A_1 \cap \cdots \cap A_{k-1}
\end{aligned}
$$

If $i < k$ then there will be a A_i in the $A_1 \cap \cdots \cap A_{k-1}$ part and the whole intersection is empty. Similarly, if $i > k$ there is an A_k set in the $A_1 \cap \cdots \cap A_{i-1}$ part which together with A_k^c makes the intersection empty.

$$\implies A_i^* \cap A_k^* = \emptyset$$

$$\implies A_1^*, A_2^*, A_3^*, \ldots \text{ are disjoint.}$$