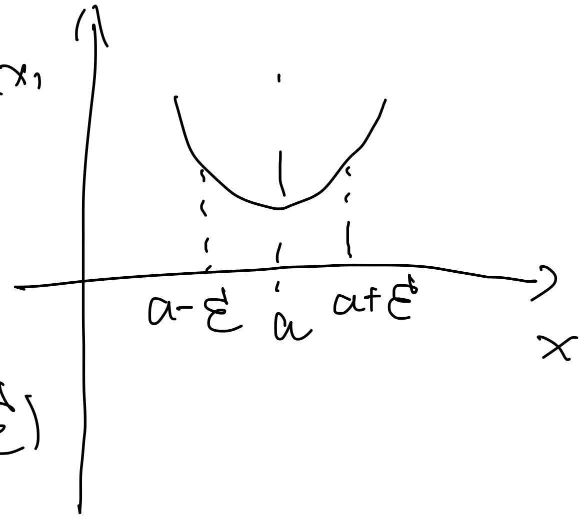
2.26 Let $f(x)$ be a pdf and let a be a number such that, for all $\epsilon > 0$, $f(a+\epsilon) = f(a-\epsilon)$. Such a pdf is said to be *symmetric* about the point a.

(b) Show that if $X \sim f(x)$, symmetric, then the median of X (see Exercise 2.17) is the number a.

Let $x = a + \varepsilon$, $x - a = \varepsilon$



$$\int_a^\infty f(x)\,dx = \int_0^\infty f(a+\varepsilon)\,d(a+\varepsilon)$$

$$= \int_0^\infty f(a+\varepsilon)\,d\varepsilon \quad \cdots \text{①}$$

Let $x = a - \varepsilon$, $a - x = \varepsilon$

$$\int_{-\infty}^a f(x)\,dx = \int_{-\infty}^0 f(a-\varepsilon)\,d(a-\varepsilon)$$

$$= \int_0^\infty f(a-\varepsilon)\,d\varepsilon \quad \cdots \text{②}$$

$\because f(a+\varepsilon) = f(a-\varepsilon)$

$\therefore$ ① = ②

$$① + ② = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = 1$$

Therefore, $\int_{-\infty}^{a} f(x)\,dx = \int_{a}^{\infty} f(x)\,dx = \frac{1}{2}$

a is the median of X.

3.28 Show that each of the following families is an exponential family.

(b) gamma family with either parameter α or β known or both unknown

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, \quad x>0,\ \alpha>0,\ \beta>0$$

①: If α known,

$$h(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} \qquad c(\beta) = \frac{1}{\beta^{\alpha}} \qquad k=1$$

$$w_1(\beta) = -\frac{1}{\beta} \qquad t_1(x) = x.$$

Therefore, in this case, gamma family is an exponential family.

② : β known.

$$f(x|\alpha) = e^{-x/\beta} \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{\log x^{\alpha-1}}$$

$$= e^{-x/\beta} \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{(\alpha-1)\cdot\log x}$$

$h(x) = e^{-x/\beta}$ $\quad c(\alpha) = \frac{1}{\Gamma(\alpha)\beta^\alpha}$ $\quad k = 1$

$w_1(\alpha) = \alpha - 1$ $\quad t_1(x) = \log x$

Therefore, in this case, gamma family is an exponential family, too.

③: both α and β unknown

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{\frac{-x}{\beta}}$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{(\alpha-1)\log x} \cdot e^{\frac{-x}{\beta}}$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{(\alpha-1)\log x + \left(\frac{-x}{\beta}\right)}$$

$h(x) = I_{(x>0)}(x)$, $c(\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}$, $k=2$

$w_1(\alpha) = \alpha - 1$ $\quad t_1(x) = \log x$

$w_2(\beta) = -\frac{1}{\beta}$ $\quad t_2(x) = x$

Therefore, in this case, gamma family is an exponential family, too.

(d) **Poisson family**

$$f(x|\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad x = 0, 1, 2, \cdots$$

$$= \frac{1}{x!} \cdot e^{-\lambda} \cdot e^{\log \lambda^{x}}$$

$$= \frac{1}{x!} \cdot e^{-\lambda} \cdot e^{x \log \lambda}$$

$h(x) = \frac{1}{x!}$ $\quad c(\lambda) = e^{-\lambda}$ $\quad k = 1$

$w_1(\lambda) = \log \lambda$ $\quad t_1(x) = x$.

Therefore, Poisson family is exponential family.

3.37 Show that if $f(x)$ is a pdf, symmetric about 0, then μ is the median of the location–scale pdf $(1/\sigma)f((x-\mu)/\sigma)$, $-\infty < x < \infty$.

Let $x = \mu + \varepsilon$

$$\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma} f\left(\frac{(\mu+\varepsilon)-\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma} f\left(\frac{\varepsilon}{\sigma}\right) \quad \cdots \quad (*)$$

$\because$ $f(x)$ is a pdf, symmetric about 0

$\therefore$ $f(x) = f(-x)$, $x \in R$

$$(*) = \frac{1}{\sigma} f\left(\frac{-\varepsilon}{\sigma}\right) = \frac{1}{\sigma} f\left(\frac{(\mu-\varepsilon)-\mu}{\sigma}\right)$$

Therefore, μ is the median.

3.39 Consider the Cauchy family defined in Section 3.3. This family can be extended to a location–scale family yielding pdfs of the form

$$f(x|\mu,\sigma) = \frac{1}{\sigma\pi\left(1+\left(\frac{x-\mu}{\sigma}\right)^2\right)}, \quad -\infty < x < \infty.$$

The mean and variance do not exist for the Cauchy distribution. So the parameters μ and σ^2 are not the mean and variance. But they do have important meaning. Show that if X is a random variable with a Cauchy distribution with parameters μ and σ, then:

(a) μ is the median of the distribution of X, that is, $P(X \geq \mu) = P(X \leq \mu) = \frac{1}{2}$.

$$Z = \frac{X-\mu}{\sigma}$$

the standard Cauchy distribution is that $\frac{1}{\pi(1+Z^2)}$,

$$P(Z \geq 0) = \int_0^\infty \frac{1}{\pi(1+Z^2)}\, dZ$$

$$\int \frac{1}{1+Z^2}\, dZ = \tan^{-1}(Z)$$

Therefore, $P(Z \geq 0) = \frac{1}{\pi} \tan^{-1}(Z)\Big|_0^\infty$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - 0\right) = \frac{1}{2}$$

and $Z=0$ is indeed the median for a standard Cauchy distribution, so μ is the median of the distribution of X, $P(X \geq \mu) = P(X \leq \mu) = \frac{1}{2}$

3.42 Refer to Exercise 3.41 for the definition of a stochastically increasing family.

(a) Show that a location family is stochastically increasing in its location parameter.

$$\left.\begin{array}{l} X_1 \sim f(x-\theta_1) \\ X_2 \sim f(x-\theta_2) \\ \theta_1 > \theta_2 \\ Z \sim f(z) \end{array}\right\} \quad \text{cdf}: \begin{cases} F(x|\theta_1) \\ F(x|\theta_2) \end{cases}$$

$$F(x|\theta_1) = P(X_1 \leq x) = P(Z+\theta_1 \leq x)$$

$$= P(Z \leq x-\theta_1) = F(x-\theta_1) \cdots ①$$

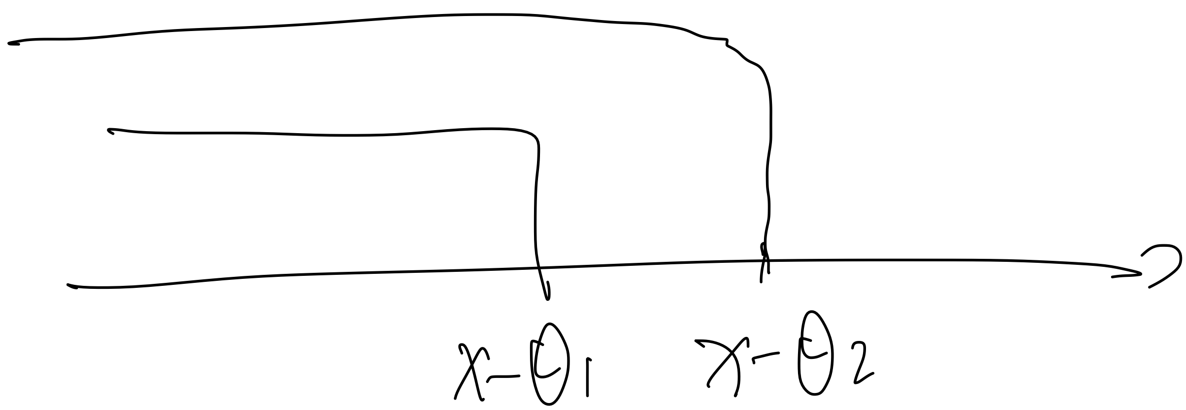
$$F(x|\theta_2) = \cdots\cdots = F(x-\theta_2) \cdots ②$$

According to ①, ②, Since $\theta_1 > \theta_2$.

$\Rightarrow x-\theta_2 > x-\theta_1$. besides, F is nondecreasing function.

$$① - ② = F(x|\theta_1) - F(x|\theta_2)$$

$$= F(x-\theta_1) - F(x-\theta_2) < 0$$



$$F(x-\theta_1) < F(x-\theta_2)$$

$$\Updownarrow$$

$$F(x|\theta_1) < F(x|\theta_2)$$

3.46 Calculate $P(|X - \mu_X| \geq k\sigma_X)$ for $X \sim \text{uniform}(0,1)$ and $X \sim \text{exponential}(\lambda)$, and compare your answers to the bound from Chebychev's Inequality.

① : $X \sim \text{uniform}(0,1)$

$$\mu = \frac{a+b}{2} = \frac{0+1}{2} = \frac{1}{2}$$

$$\sigma = \sqrt{\frac{(b-a)^2}{12}} = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}} \quad \sigma^2 = \frac{1}{12}$$

$$P(|X-\mu| > k\sigma) = P(X-\mu > k\sigma) + P(X-\mu < -k\sigma)$$

$$= P(X > \mu + k\sigma) + P(X < -k\sigma + \mu)$$

$$= 1 - P(\mu - k\sigma \leq X \leq \mu + \sigma k)$$

$$= 1 - P\left(\frac{1}{2} - \frac{k}{2\sqrt{3}} \leq X \leq \frac{1}{2} + \frac{k}{2\sqrt{3}}\right)$$

$$= (*)$$

$$\begin{cases} \frac{1}{2} - \frac{k}{2\sqrt{3}} > 0 \\ \frac{1}{2} + \frac{k}{2\sqrt{3}} < 1 \end{cases} \Rightarrow k < \sqrt{3}$$

$$(*) = \begin{cases} 1 - \frac{2k}{\sqrt{12}}, & k < \sqrt{3} \\ 0, & \text{otherwise.} \end{cases}$$

		$\frac{1}{k^2}$
$k = 0.5$	$(*) = 1 - \frac{1}{\sqrt{12}} = 0.711$	4
$k = 1$	$(*) = 1 - \frac{2}{\sqrt{12}} = 0.423$	1
$k = 1.5$	$(*) = 1 - \frac{3}{\sqrt{12}} = 0.134$	0.44
$k = 2$	$(*) = 0$, $k = 2 > \sqrt{3}$	0.25
$k = 3$	$(*) = 0$, $k = 3 > \sqrt{3}$.	0.11

② $X \sim \text{exponential}(\lambda)$, $\mu = \lambda$

$\sigma^2 = \lambda^2$

$$P(|X-\mu| > k\sigma) = 1 - P(\lambda - \lambda k \le X \le \lambda + \lambda k)$$

$$= (*)$$

$\because \lambda - \lambda k \geq 0$

$\therefore k \le 1$

$$\therefore (*) = \begin{cases} 1 + e^{-(k+1)} - e^{k-1}, & k \le 1 \\ e^{-(k+1)}, & k > 1 \end{cases}$$

		$\frac{1}{k^2}$
$k = 0.5 \leq 1$	0.617	φ
$k = 1 \leq 1$	0.135	1
$k = 1.5 > 1$	0.0821	0.44
$k = 2 > 1$	0.0498	0.25
$k = 3 > 1$	0.0183	0.11