**5.4** A generalization of iid random variables is *exchangeable* random variables, an idea due to deFinetti (1972). A discussion of exchangeability can also be found in Feller (1971). The random variables $X_1, \dots, X_n$ are *exchangeable* if any permutation of any subset of them of size $k$ ($k \le n$) has the same distribution. In this exercise we will see an example of random variables that are exchangeable but not iid. Let $X_i|P \sim$ iid Bernoulli$(P)$, $i = 1, \dots, n$, and let $P \sim$ uniform$(0,1)$.

(a) Show that the marginal distribution of any $k$ of the $X$s is the same as

$$P(X_1 = x_1, \dots, X_k = x_k) = \int_0^1 p^t (1-p)^{k-t} dp = \frac{t!(k-t)!}{(k+1)!},$$

where $t = \sum_{i=1}^k x_i$. Hence, the $X$s are exchangeable.

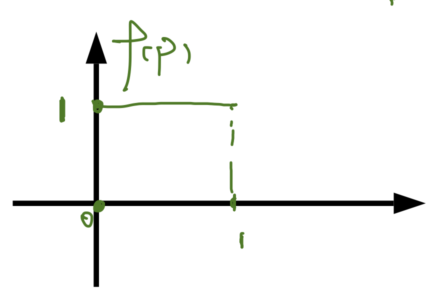
(b) Show that, marginally,

$$P(X_1 = x_1, \dots, X_n = x_n) \ne \Pi_{i=1}^n P(X_i = x_i),$$

so the distribution of the $X$s is exchangeable but not iid.

(a) $X_i | P \sim$ iid Bernoulli$(P)$, $i = 1, \dots, n$

$P \sim$ uniform$(0,1)$




$f(p) = 1$

$$P(X_1 = x_1, \dots, X_k = x_k) = f(x_1, x_2, \dots x_k) = \int f(x_1, x_2, \dots x_k | p) f(p) dp$$

$$= \int_0^1 \prod_{i=1}^k f(x_i|p)\, dp = \int_0^1 \prod_{i=1}^k p^{x_i} \cdot (1-p)^{1-x_i} dp = \int_0^1 p^{\sum_{i=1}^k x_i} \cdot (1-p)^{k - \sum_{i=1}^k x_i} dp \quad \dots\dots \text{①}$$

Let $t = \sum_{i=1}^k x_i$

Beta function

$$\therefore \text{①} = \int_0^1 p^t \cdot (1-p)^{k-t} dp = \frac{\Gamma(t+1)\Gamma(k-t+1)}{\Gamma(k+2)} \underbrace{\int_0^1 \frac{\Gamma(k+1)}{\Gamma(k+1)\Gamma(k-t+1)} p^t \cdot (1-p)^{k-t} dp}_{=1}$$

$$= \frac{t!(k-t)!}{(k+1)!}$$

(2) $P(X_i = x_i) = f(x) = \int_0^1 f(x|p) f(p) dp$

$= \int_0^1 f(x|p) dp \quad \cdots ②$

Since $k = 1$

$② = \frac{x!(1-x)!}{2!} = \frac{x!(1-x)!}{2}$

where $x = 0, 1$

$② = \begin{cases} \frac{1}{2}, x = 0 \\ \frac{1}{2}, x = 1 \end{cases} \Rightarrow = \frac{1}{2}$

$\prod_{i=1}^n P(X_i = x_i) = (\frac{1}{2})^n, \quad \cdots ③$

$t = \sum_{i=1}^n x_i$

However, $③ = (\frac{1}{2})^n \neq \frac{t!(n-t)!}{(n+1)!} = P(X_1 = x_1, \ldots, X_n = x_n)$

So the distribution of $X$s is not i.i.d.

**5.6** If $X$ has pdf $f_X(x)$ and $Y$, independent of $X$, has pdf $f_Y(y)$, establish formulas, similar to (5.2.3), for the random variable $Z$ in each of the following situations.

(a) $Z = X - Y$

(b) $Z = XY$

(a) $Z = X - Y$, $W = Y$, $\quad \frac{\partial x}{\partial z} = 1 \quad \frac{\partial x}{\partial w} = 1$

$X = Z + W = h_1(z, w) \quad Y = W = h_2(z, w) \quad \frac{\partial y}{\partial z} = 0 \quad \frac{\partial y}{\partial w} = 1$

$$|J| = |1 \cdot 1 - 0 \cdot 1| = 1$$

$$f_{ZW}(z, w) = f_{XY}(h_1(z,w), h_2(z,w)) \cdot 1$$

$$= f_X(h_1(z,w)) \cdot f_Y(h_2(z,w)), \text{ since } x, y \text{ indep}$$

$$= f_X(z+w) \cdot f_Y(w)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z, w)\, dw = \int_{-\infty}^{\infty} f_X(z+w) \cdot f_Y(w)\, dw$$

(b) : $Z = XY$, $W = Y$. $\frac{\partial X}{\partial Z} = \frac{1}{W}$ $\frac{\partial X}{\partial W} = -\frac{Z}{W^2}$

$X = \frac{Z}{W}$ $Y = W$ $\frac{\partial Y}{\partial Z} = 0$ $\frac{\partial Y}{\partial W} = 1$

$= h_1(Z, W)$ $= h_2(Z, W)$

$$|J| = \left| \frac{1}{W} \cdot 1 - 0 \cdot \left(-\frac{Z}{W^2}\right) \right| = \left| \frac{1}{W} \right|.$$

$$f_{ZW}(Z, W) = f_{XY}(h_1(Z, W), h_2(Z, W)) \cdot \left| \frac{1}{W} \right|$$

$$= f_X\left(\frac{Z}{W}\right) \cdot f_Y(W) \cdot \left| \frac{1}{W} \right|$$

$$f_Z(Z) = \int_{-\infty}^{\infty} f_X\left(\frac{Z}{W}\right) \cdot f_Y(W) \cdot \left| \frac{1}{W} \right| dW.$$

**5.10** Let $X_1, \ldots, X_n$ be a random sample from a $\mathrm{n}(\mu, \sigma^2)$ population.

(a) Find expressions for $\theta_1, \ldots, \theta_4$, as defined in Exercise 5.8, in terms of $\mu$ and $\sigma^2$.

(b) Use the results of Exercise 5.8, together with the results of part (a), to calculate $\operatorname{Var} S^2$.

(c) Calculate $\operatorname{Var} S^2$ a completely different (and easier) way: Use the fact that $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$.

(a): $\theta_1 = E(X_i) = \mu$, $\theta_2 = E(X_i-\mu)^2 = \sigma^2$.

$$\theta_3 = E(X_i-\mu)^3 = E\underbrace{(X_i-\mu)^2}_{g(x)}\cdot(X_i-\mu) \quad \cdots \text{①}$$

According to Stein's Lemma:

$$\text{①} = E(g(x)\cdot(x-\mu)) = \sigma^2\cdot E(g'(x)) = 2\sigma^2 E(X_i-\mu)$$

$$= 2\sigma^2(E(X_i)-\mu) = 0.$$

$$\theta_4 = E(X_i-\mu)^4 = E\underbrace{(X_i-\mu)^3}_{g(x)}(X_i-\mu)$$

$$= 3\sigma^2 E(X_i-\mu)^2 = 3\sigma^2\cdot\sigma^2 = 3\sigma^4$$

(b):

$$\operatorname{Var}(S^2) = \frac{1}{n}\left(\theta_4 - \frac{n-3}{n-1}\theta_2^2\right)$$

$$= \frac{1}{n}\left(3\sigma^4 - \frac{n-3}{n-1}\sigma^4\right)$$

$$= \frac{1}{n}\sigma^4 \cdot \frac{3n-3-n+3}{n-1}$$

$$= \frac{2\sigma^4}{n-1}$$

(C) According to $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$

$$\mathrm{Var}\,\chi^2_{n-1} = 2n-2.$$

We can get:

$$\mathrm{Var}\left(\frac{(n-1)S^2}{\sigma^2}\right) = 2n-2.$$

$$\frac{(n-1)^2}{\sigma^4}\mathrm{Var}(S^2) = 2n-2.$$

$$\mathrm{Var}(S^2) = \frac{2\sigma^4}{n-1}$$

**5.15** Establish the following recursion relations for means and variances. Let $\bar{X}_n$ and $S_n^2$ be the mean and variance, respectively, of $X_1, \ldots, X_n$. Then suppose another observation, $X_{n+1}$, becomes available. Show that

(a) $\bar{X}_{n+1} = \frac{x_{n+1} + n\bar{X}_n}{n+1}$.

(b) $nS_{n+1}^2 = (n-1)S_n^2 + \left(\frac{n}{n+1}\right)(X_{n+1} - \bar{X}_n)^2$.

(a) $$\bar{X}_{n+1} = \frac{1}{n+1}\sum_{i=1}^{n+1} X_i$$

$$= \frac{\sum_{i=1}^{n} X_i + X_{n+1}}{n+1} \quad \cdots \text{①}$$

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \Rightarrow \sum_{i=1}^{n} X_i = n\bar{X}_n$$

So, ① $= \dfrac{X_{n+1} + n\bar{X}_n}{n+1}$

(b) $$nS_{n+1}^2 = \frac{n}{n}\sum_{i=1}^{n+1}(X_i - \bar{X}_{n+1})^2$$

$$= \sum_{i=1}^{n+1}(X_i - \bar{X}_{n+1})^2 \quad \cdots \quad ②$$

$$\bar{X}_{n+1} = \frac{X_{n+1} + n\bar{X}_n}{n+1}$$

$$② = \sum_{i=1}^{n+1}\left(X_i - \frac{X_{n+1} + n\bar{X}_n}{n+1}\right)^2$$

$$= \sum_{i=1}^{n+1}\left(X_i - \frac{X_{n+1}}{n+1} - \frac{1}{n+1}\bar{X}_n - \bar{X}_n\right)^2$$

$$= \sum_{i=1}^{n+1}\left((X_i - \bar{X}_n) - \left(\frac{X_{n+1}}{n+1} - \frac{\bar{X}_n}{n+1}\right)\right)^2$$

$$= \sum_{i=1}^{n+1}\left((X_i-\bar{X}_n)^2 - 2(X_i - \bar{X}_n)\cdot\frac{(X_{n+1}-\bar{X}_n)}{n+1} + \left(\frac{X_{n+1}-\bar{X}_n}{n+1}\right)^2\right)$$

$$= \sum_{i=1}^{n}\left((X_i-\bar{X}_n)^2 - 2(X_i-\bar{X}_n)\cdot\frac{(X_{n+1}-\bar{X}_n)}{n+1} + \left(\frac{X_{n+1}-\bar{X}_n}{n+1}\right)^2\right) +$$

$$(X_{n+1}-\bar{X}_n)^2 - \frac{2(X_{n+1}-\bar{X}_n)^2}{n+1} + \left(\frac{X_{n+1}-\bar{X}_n}{n+1}\right)^2$$

$$= \sum_{i=1}^{n}(X_i-\bar{X}_n)^2 - \frac{2(X_{n+1}-\bar{X}_n)^2}{n+1} + \frac{\cancel{(n+1)}(X_{n+1}-\bar{X}_n)^2}{(n+1)^{\cancel{2}}} + (X_{n+1}-\bar{X}_n)^2$$

$$= (n-1)S_n^2 + \frac{n}{n+1}(X_{n+1}-\bar{X}_n)^2$$

**5.16** Let $X_i, i = 1, 2, 3$, be independent with $n(i, i^2)$ distributions. For each of the following situations, use the $X_i$s to construct a statistic with the indicated distribution.

(a) chi squared with 3 degrees of freedom

(b) $t$ distribution with 2 degrees of freedom

(c) $F$ distribution with 1 and 2 degrees of freedom

(a) standard each $X_i$,

$$X_i \sim N(i, i^2)$$

$$X_1 \sim N(1, 1^2)$$

$$X_2 \sim N(2, 2^2) \qquad Z_i = \frac{X_i - i}{i}$$

$$X_3 \sim N(3, 3^2)$$

$$\left(\frac{X_1 - 1}{1}\right)^2 + \left(\frac{X_2 - 2}{2}\right)^2 + \left(\frac{X_3 - 3}{3}\right)^2 \sim \chi_3^2$$

$$\sum_{i=1}^{3} \left(\frac{X_i - i}{i}\right)^2 \sim \chi_3^2$$

(b) $Z_i = \dfrac{X_i - 1}{i}$ $Z_2 = \dfrac{X_2 - 2}{2}$ $Z_3 = \dfrac{X_3 - 3}{3}$

$$W = Z_2^2 + Z_3^2 \sim \chi_2^2$$

$$T = \frac{Z}{\sqrt{W/2}}$$

$$= \frac{\dfrac{X_i - 1}{i}}{\sqrt{\dfrac{\left(\frac{X_2-2}{2}\right)^2 + \left(\frac{X_3-3}{3}\right)^2}{2}}}$$

$$= \frac{\left(\dfrac{X_i - 1}{i}\right)}{\sqrt{\dfrac{\sum_{i=2}^{3}\left(\frac{X_i - i}{i}\right)^2}{2}}} \sim t_2$$

(C):

$$F = \frac{Z^2}{\frac{W}{2}}$$

$$= \frac{2\left(\frac{X_i - 1}{i}\right)}{\sum_{i=2}^{3}\left(\frac{X_i - i}{i}\right)^2} \sim F_{1,2}$$