STAT 345/445 Lecture 19

Order Statistics – Section 5.4

Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample. The statistics

- $X_{(1)} = \min\{X_1, X_3, \ldots, X_n\}$
- $X_{(2)} = \text{second smallest}\{X_1, X_3, \ldots, X_n\}$
- $\ldots$
- $X_{(n-1)} = \text{second largest}\{X_1, X_3, \ldots, X_n\}$
- $X_{(n)} = \max\{X_1, X_3, \ldots, X_n\}$

are called **order statistics**.

Get used to the $X_{(j)}$ notation and $X_{(j)} \neq X_j$

Examples:

- Weight of smallest kitten in a litter
- Highest score on an exam

Could have used $U_1 = \min(X_1, \ldots X_n)$

$\vdots$

We want to find the *(sampling) distributions* of order statistics

$U_n = \max(X_1, \ldots, X_n)$

Order Statistics - Example

- Random sample $X_1, X_2, \ldots, X_{10}$
- Order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(10)}$

Observed random sample:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
1.9	3.5	9.0	2.8	3.3	2.1	6.1	1.1	0.7	7.9

Corresponding observed order statistics:

$x_{(1)}$	$x_{(2)}$	$x_{(3)}$	$x_{(4)}$	$x_{(5)}$	$x_{(6)}$	$x_{(7)}$	$x_{(8)}$	$x_{(9)}$	$x_{(10)}$
0.7	1.1	1.9	2.1	2.8	3.3	3.5	6.1	7.9	9.0

Distributions of order statistics

Theorem: distribution of order statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution, with pdf $f(x)$ and cdf $F(x)$. Then the cdf and pdf of the jth order statistic $X_{(j)}$ are

$$F_{(j)}(x) = \sum_{k=j}^{n} \binom{n}{k} F(x)^k (1 - F(x))^{n-k}$$

$$f_{(j)}(x) = \frac{n!}{(j-1)!(n-j)!} f(x)\, F(x)^{j-1}\, (1 - F(x))^{n-j}$$

proof... Lets take $j = 1$ and $j = n$ first

Same as $X_1, \dots X_n$

$$F_{(j)}(x) = \sum_{k=j}^{n} \binom{n}{k} F(x)^k (1-F(x))^{n-k}$$

Suppose?

$$f_{(j)}(x) = \frac{n!}{(j-1)!(n-j)!} f(x)\, F(x)^{j-1}\, (1-F(x))^{n-j}$$

Show last time

$(uv)' = uv' + vu'$

$$f_{(j)}(x) = \frac{d}{dx} \sum_{k=j}^{n} \binom{n}{k} F(x)^k (1-F(x))^{n-k}$$

$$= \sum_{k=j}^{n} \left[\binom{n}{k} k F(x)^{k-1} f(x) (1-F(x))^{n-k} - \binom{n}{k} F(x)^k (n-k) (1-F(x))^{n-k-1} f(x) \right]$$

$= \dots\dots$ Some algebra, term cancel...

$$= \frac{n!}{(j-1)!(n-j)!} f(x) F(x)^{j-1} (1-F(x))^{n-j}$$

Order Statistics as transformations

- Transformations from a random sample to order statistics

$$(X_1, X_2, \ldots, X_n) \longmapsto X_{(j)}$$
$$\text{or } (X_1, X_2, \ldots, X_n) \longmapsto (X_{(1)}, X_{(2)}, \ldots, X_{(n)})$$

  are not one-to-one transformations

- **Example:** $n = 3$ and $(X_1, X_2, X_3) \longmapsto (X_{(1)}, X_{(2)}, X_{(3)})$
  - The following observations of (X_1, X_2, X_3)

$$(3, 5, 8),\ (3, 8, 5),\ (5, 3, 8),\ (5, 8, 3),\ (8, 3, 5),\ (8, 5, 3)$$

    are all mapped to $(X_{(1)}, X_{(2)}, X_{(3)}) = (3, 5, 8)$

- So we can't use out transformation formulas to obtain $f_{(j)}(x)$
  - So use the "cdf-method" to find $F_{(j)}(x)$ ← slide.

Untangling expressions in the Theorem

- Recall that by the Binomial formula

$$\sum_{k=0}^{n} \binom{n}{k} F(x)^k (1 - F(x))^{n-k} = (F(x) + 1 - F(x))^{n} = 1$$

- So the cdf $F_{(1)}(x)$ can be written as

$$\begin{aligned} F_{(1)}(x) &= \sum_{k=1}^{n} \binom{n}{k} F(x)^k (1 - F(x))^{n-k} \qquad n-1 \text{ terms} \\ &= \sum_{k=0}^{n} \binom{n}{k} F(x)^k (1 - F(x))^{n-k} - \binom{n}{0} F(x)^0 (1 - F(x))^{n-0} \\ &= 1 - (1 - F(x))^n \end{aligned}$$

add and substract term for $k=0$

and

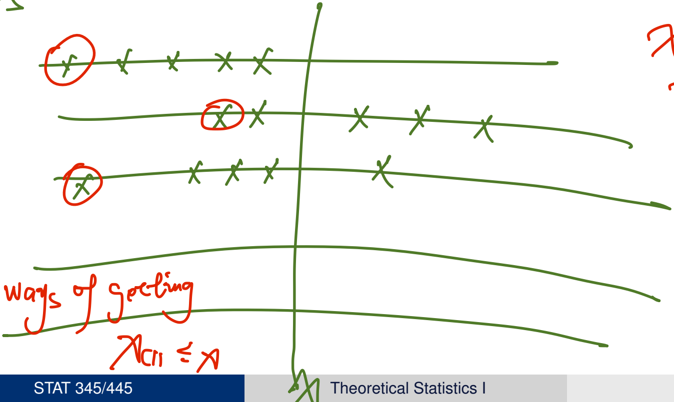
$$\begin{aligned} f_{(1)}(x) &= \frac{n!}{(1-1)!(n-1)!} f(x)\, F(x)^{1-1}\, (1 - F(x))^{n-1} \\ &= n\, f(x)\, (1 - F(x))^{n-1} = \frac{d}{dx} F_{(1)}(x) \end{aligned}$$

Distribution of $X_{(1)}$

- Have the joint pdf of $X_1, X_2, \ldots, X_n$
- How can we relate $X_{(1)}$ to $(X_1, X_2, \ldots, X_n)$?
- "Smallest value is bigger than x" is the same event as "all values are bigger than x"

$x_{(1)} \le x$

$n = 5$



etc

many ways of getting

$x_{(1)} \le x$

$x_{(1)} \ge x$?

Distri of $X_{(1)}$

$$F_{(1)}(x) = P(X_{(1)} \leq x) = 1 - P(X_{(1)} > x)$$

Distribution of $X_{(n)}$

- Theorem for $j = n$:

$$F_{(n)}(x) = \binom{n}{n} F(x)^n (1 - F(x))^{n-n} = F(x)^n$$

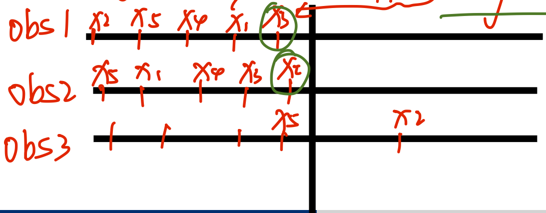
$$f_{(n)}(x) = \frac{n!}{(n-1)!(n-n)!} f(x)\, F(x)^{n-1}\ (1 - F(x))^{n-n} = n\, f(x)\, F(x)^{n-1}$$

$$\frac{d}{dx} F_{(n)}(x) = \frac{d}{dx}\left[F(x)^n\right] = n\left[F(x)\right]^{n-1} \cdot f_{(n)}$$

Chain Rule

- How can we relate $X_{(n)}$ to $(X_1, X_2, \ldots, X_n)$?
- "Largest value is less than x" is the same event as "all values are less than x"

Event $X_{(n)} \le x$, largest



here $X_{(n)} > x$

Distribution of $X_{(n)}$

$$F_{(n)}(x) = P(\underbrace{X_{(n)} \le x}_{\text{largest } X \le x}) = P(\overbrace{X_1 \le x, X_2 \le x, \ldots, X_n \le x}^{\text{all } X\text{'s} \le x})$$

"and"

$$= P(X_1 \le x)\,P(X_2 \le x) \cdots P(X_n \le x), \quad \text{Since } X_1, \ldots X_n \text{ are indep Same distribution}$$

$$= F(x)\,F(x) \cdots F(x) = F(x)^n$$

Distribution of $X_{(j)}$, $j = 2, \ldots, n-1$

- How can we relate $X_{(j)}$ to $(X_1, X_2, \ldots, X_n)$?
- "$X_{(j)} \leq x$" is the same event as
  "all least j of the random variables are less than x"

Example: Order stats for Uniform

- Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be order statistics from a random sample $X_1, X_2, \ldots, X_n$ from the $\mathrm{Uniform}(0, 1)$. What are the (marginal) pdfs of $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$?

$x_1, \ldots, x_n$ iid Uniform$(0,1)$ $\Rightarrow f(x) = 1$ for $x \in (0,1)$

and cdf $F(x) = \begin{cases} x & \text{for } x \in (0,1) \\ = 0 & x \in (-\infty, 0) \\ = 1 & x \in [1, \infty) \end{cases}$

In general: $f_{(j)}(x) = \dfrac{n!}{(n-j)!(j-1)!} f(x) F(x)^{j-1} (1-F(x))^{n-j}$

Here: $f_{(j)}(x) = \dfrac{n!}{(n-j)!(j-1)!} 1 \cdot x^{j-1} (1-x)^{n-j}$, for $x \in (0,1)$

Recall: the pdf of Beta(α, β):

$f(x) = \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$, for $x \in (0,1)$

$\alpha = j$, $\beta = n-j+1$

$\Gamma(\alpha+\beta) = \Gamma(n+1) = n!$ $\Gamma(\alpha) = \Gamma(j) = (j-1)!$

$\Gamma(\beta) = \Gamma(n-j+1) = (n-j)!$

$\Rightarrow X_{(j)} \sim Beta(j, n-j+1)$

Example: $X_1, \ldots, X_9 \overset{ind}{\sim} \text{uniform}(0,1)$

$$X_{(1)} \sim \text{Beta}(1, 9)$$

$$X_{(2)} \sim \text{Beta}(2, 8)$$

$$\vdots$$

$$X_{(9)} \sim \text{Beta}(9, 1)$$

Example: Order stats for Uniform

- Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be order statistics from a random sample $X_1, X_2, \ldots, X_n$ from the $\mathrm{Uniform}(0, 1)$.
- Then

$$f_{(j)}(x) = \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n-j+1)!} x^{j-1} (1-x)^{(n-j+1)-1}$$

So $X_{(j)} \sim \mathrm{Beta}(j, n-j+1)$

- R simulation:

```
x.order <- c()
for(i in 1:10000){
  x <- runif(n=9)
  x.order <- rbind(x.order, x[order(x)])
}
```

makes an empty vector.

Sample

$n = 9$

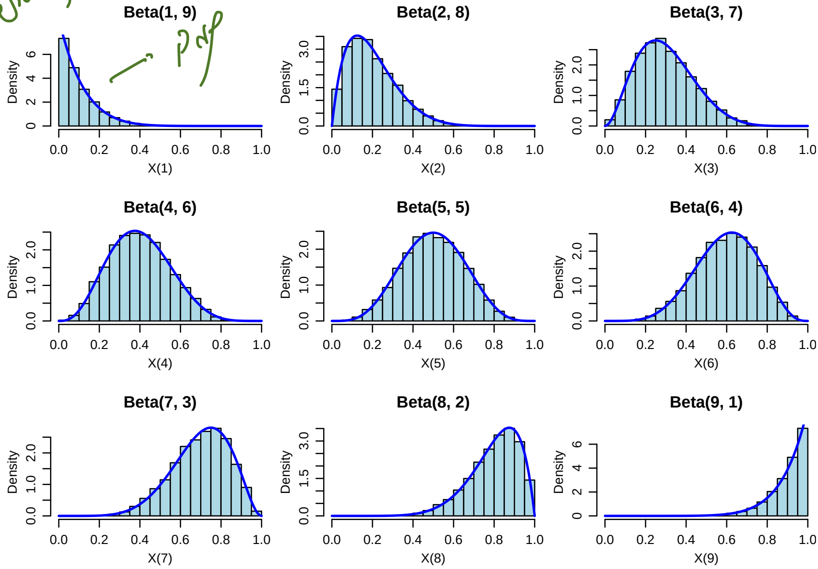
ordering elements of x

1st col of x.order are Sample of x^{1st}

Example

histogram of samples

pdf



Joint distribution of order statistics

range of data: $x_{(n)} - x_{(1)}$

Joint pdf of two order statistics

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be order statistics from a random sample $X_1, X_2, \ldots, X_n$ with pdf $f(x)$ and cdf $F(x)$.
The joint pdf of $\left(X_{(i)}, X_{(j)}\right)$, $1 \leq i < j \leq n$ is

$$f_{(i,j)}(u, v) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(u) f(v) \times F(u)^{i-1} \left(F(v) - F(u)\right)^{j-1-i} \left(1 - F(v)\right)^{n-j}$$

for $-\infty < u < v < \infty$

Think: 3-dimensional Multinomial distribution

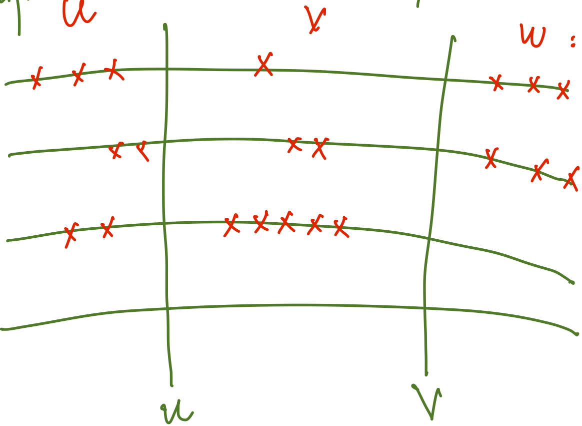
very similar argument but with multinomial instead of Binomial "cdf" method

Joint cdf of $X_{(i)}, X_{(j)}$, $i<j$:

$$f_{(i,j)}(u,v) = P(X_{(i)} \le u,\ X_{(j)} \le v) = P(U \ge i,\ U+V \ge j) \qquad u<v$$

W: can't in the three ranges

e.g. $n=7$, $i=2$, $j=4$



See Ex. 5.26 for details

$X_{(i)} \le u$, and $X_{(j)} \le v$: Same as

at least i of $X_1, \dots, X_n$ are $\le u$ and

at least j of $X_1, \dots, X_n$ are $\le v$

$(u, v, w) \longleftarrow \text{Multinomial}(m = n, p = (F(u), F(v) - F(u), 1 - F(v)))$

Joint distribution of order statistics

Let $X_1, X_2, \dots, X_n$ be a random sample from a continuous distribution, with pdf $f(x)$ and cdf $F(x)$.

Joint pdf for all n order statistics $(X_{(1)}, X_{(2)}, \dots, X_{(n)})$:

$$f_{(1),\dots,(n)}(u_1, \dots, u_n) = n! f(u_1) f(u_2) \cdots f(u_n)$$

for $-\infty < u_1 < u_2 < \cdots < u_n < \infty$ Proof in book...

Can get any marginal and joint marginal pdfs by integrating out other variables.

See book for discrete order stats