2.1 In each of the following find the pdf of Y. Show that the pdf integrates to 1.

(a) $Y = X^3$ and $f_X(x) = 42x^5(1-x)$, $0 < x < 1$

$$\frac{d}{dx} g(x) = 3x^2 > 0, \quad (0 < x < 1)$$

so, g is monotone increasing.

$$f_Y(y) = \frac{d}{dy} F_X(g^{-1}(y)) = F_X'(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

$$= f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

$$Y = X^3, \ (0 < X < 1) \qquad X = Y^{\frac{1}{3}}, \ (Y \in (0,1))$$

$$\frac{d}{dy} g^{-1}(y) = \frac{1}{3} y^{-\frac{2}{3}}.$$

Pdf:

$$f_Y(y) = 42 \cdot y^{\frac{5}{3}} (1 - y^{\frac{1}{3}}) \cdot \frac{1}{3} y^{-\frac{2}{3}}$$

$$= 14 y (1 - y^{\frac{1}{3}})$$

$$= 14y - 14y^{\frac{4}{3}}, \quad y \in (0,1)$$

$$\int_0^1 f_Y(y) = \int_0^1 14y - 14y^{\frac{4}{3}}\,dy$$

$$= 7y^2 - 6y^{\frac{7}{3}} \Big|_0^1$$

$$= (7-6) - 0 = 1$$

Therefore, pdf can integrate to 1.

and $f_Y(y) = 14y - 14y^{\frac{4}{3}}$, $y \in (0, 1)$

2.3 Suppose X has the geometric pmf $f_X(x) = \frac{1}{3}\left(\frac{2}{3}\right)^x$, $x = 0, 1, 2, \ldots$. Determine the probability distribution of $Y = X/(X+1)$. Note that here both X and Y are discrete random variables. To specify the probability distribution of Y, specify its pmf.

$$y = \frac{x}{x+1}, \quad x = 0, 1, 2 \cdots$$

$$g(x) = \frac{x}{x+1}, \quad x = 0, 1, 2 \cdots$$

$$\frac{d}{dx} g(x) = \frac{1}{x+1} + x \cdot \frac{-1}{(x+1)^2} = \frac{x+1}{(x+1)^2} - \frac{x}{(x+1)^2}$$

$$= \frac{1}{(x+1)^2} > 0, \quad x = 0, 1, 2, \cdots$$

$\therefore$ g is monotone increasing.

$$\frac{1}{y} = \frac{x+1}{x} = 1 + \frac{1}{x}$$

$$\frac{1}{x} = 1 - \frac{1}{y}$$

$$x = \frac{1}{1 - \frac{1}{y}} = \frac{y}{y-1}, \quad y \in \left\{0, \frac{1}{2}, \frac{2}{3}, \cdots\right\}$$

Since g is monotone increasing,

Y and X are one to one mapping.

Therefore, we can get pmf;

$$f_Y(y) = f_X(g^{-1}(y))$$

$$= \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{y-1}}, \quad y \in \{0, \frac{1}{2}, \frac{2}{3}, \cdots\}$$

2.18 Show that if X is a continuous random variable, then

$$\min_a \mathrm{E}\,|X-a| = \mathrm{E}\,|X-m|,$$

where m is the median of X (see Exercise 2.17).

$$E|X-a| = \int_{-\infty}^{\infty} |x-a| f_X(x)\,dx$$

$$= \int_{-\infty}^{a} (a-x) f_X(x)\,dx + \int_{a}^{\infty} (x-a) f_X(x)\,dx$$

$$\frac{d}{da} E|X-a| = \frac{d}{da}\left(\int_{-\infty}^{a} (a-x) f_X(x)\,dx + \int_{a}^{\infty} (x-a) f_X(x)\,dx \right)$$

$$= \int_{-\infty}^{a} f_X(x)\,dx - \int_{a}^{\infty} f_X(x)\,dx$$

Since we want to find the minimal point,

we can set $\frac{d}{da} E|X-a| = 0$.

Thus, $\int_{-\infty}^{a} f_X(x)\,dx = \int_{a}^{\infty} f_X(x)\,dx$

Since $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$, We get $\int_{-\infty}^{a} f_X(x)\,dx = \int_{a}^{\infty} f_X(x)\,dx = \frac{1}{2}$

Since m is the median point of X,

We can get :

$$\int_{-\infty}^{m} f_X(x)\,dx = \int_{m}^{\infty} f_X(x)\,dx = \frac{1}{2}$$

Therefore, $a = m$, is median.

and the value of a that minimizes $E|X-a|$ is the median m of the distribution of X. $\min_a E|X-a| = E|X-m|$.

because $\frac{d^2}{da^2} E|X-a| = f_X(a) + f_X(a) = 2f_X(a) > 0$

2.23 Let X have the pdf

$$f(x) = \frac{1}{2}(1+x), \quad -1 < x < 1.$$

(a) Find the pdf of $Y = X^2$. (b) Find $\mathrm{E}\,Y$ and $\mathrm{Var}\,Y$.

(a): $g(x) = x^2, \quad -1 < x < 1$

$$\frac{d}{dx} g(x) = 2x, = \begin{cases} > 0, & x \in (-1, 0) \\ < 0, & x \in (0, 1) \end{cases}$$

so, g is monotone decreasing, $x \in (-1, 0)$

and g is monotone increasing, $x \in (0, 1)$.

$\because Y = X^2$, $\therefore Y \in (0, 1)$, $X = \pm Y^{\frac{1}{2}}$

$$\left|\frac{d}{dy} g^{-1}(y)\right| = \frac{1}{2} Y^{-\frac{1}{2}}$$

$$f_X(g_i^{-1}(y)) = \begin{cases} \frac{1}{2}(1+Y^{\frac{1}{2}}) & \cdots ① \\ \frac{1}{2}(1-Y^{\frac{1}{2}}) & \cdots ② \end{cases}$$

$$f_Y(y) = \begin{cases} \sum_{i=1}^{2} f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right|, & y \in \mathcal{Y} \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} (① + ②) \cdot \frac{1}{2} y^{-\frac{1}{2}} = \frac{1}{2} y^{-\frac{1}{2}}, & y \in (0, 1) \\ 0, & \text{otherwise} \end{cases}$$

(b): $f_Y(y) = \frac{1}{2} Y^{-\frac{1}{2}}, \; y \in [0,1)$

$$E(Y) = \int_0^1 y f_Y(y)\, dy$$

$$= \int_0^1 \frac{1}{2} y^{\frac{1}{2}}\, dy$$

$$= \frac{1}{3} y^{\frac{3}{2}} \Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

$$E(Y^2) = \int_0^1 y^2 f_Y(y)\, dy$$

$$= \int_0^1 \frac{1}{2} y^{\frac{3}{2}}\, dy = \frac{1}{5} y^{\frac{5}{2}} \Big|_0^1$$

$$= \frac{1}{5} - 0 = \frac{1}{5}$$

$$\operatorname{Var} Y = E(Y^2) - [E(Y)]^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}$$

2.33 In each of the following cases verify the expression given for the moment generating function, and in each case use the mgf to calculate E X and Var X.

(c) $f_X(x) = \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sqrt{2\pi}\sigma}$, $M_X(t) = e^{\mu t + \sigma^2 t^2/2}$, $-\infty < x < \infty$; $-\infty < \mu < \infty$, $\sigma > 0$

<1>: verify $M_X(t)$ ...... ①

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x) = \int_{-\infty}^{\infty} \frac{e^{\overbrace{-(x-\mu)^2/(2\sigma^2)+tx}^{①}}}{\sqrt{2\pi}\,\sigma}\,dx$$

⋮

(*)

$$① = -(x-\mu)^2/(2\sigma^2) + tx = \frac{2\sigma^2 tx - (x-\mu)^2}{2\sigma^2}$$

$$= \frac{-(x^2 - 2\mu x + \mu^2) + 2\sigma^2 tx}{2\sigma^2}$$

$$= \frac{-\left(x^2 - (2\mu + 2\sigma^2 t)x + \mu^2\right)}{2\sigma^2}$$

$$= \frac{-\left(x^2 - (2\mu + 2\sigma^2 t)x + (\mu + \sigma^2 t)^2\right) + (\mu + \sigma^2 t)^2 - \mu^2}{2\sigma^2}$$

$$= \frac{-(x - (\mu + \sigma^2 t))^2 + \sigma^4 t^2 + 2\mu\sigma^2 t}{2\sigma^2}$$

$$= \frac{-(x-(\mu+\Delta^2 t))^2}{2\Delta^2} + \frac{1}{2}\Delta^2 t^2 + \mu t$$

$$(*) = \int_{-\infty}^{\infty} \frac{e^{\frac{-(x-(\mu+\Delta^2 t))^2}{2\Delta^2}} \cdot e^{\frac{1}{2}\Delta^2 t^2 + \mu t}}{\sqrt{2\pi}\,\Delta}\, dx$$

$$= e^{\frac{1}{2}\Delta^2 t^2 + \mu t} \underbrace{\int_{-\infty}^{\infty} \frac{e^{-(x-(\mu+\Delta^2 t))^2/2\Delta^2}}{\sqrt{2\pi}\,\Delta}\, dx}_{=1}$$

$$= e^{\frac{1}{2}\Delta^2 t^2 + \mu t} = M_X(t)$$

(2)

$$E(X) = E(X^1), \quad M_X(t) = e^{\mu t + \sigma^2 t^2/2}$$

so $$E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0}$$

$$= \frac{d}{dt} e^{\mu t + \sigma^2 t^2/2} \Big|_{t=0}$$

$$= e^{\mu t + \sigma^2 t^2/2} \cdot (\mu + \sigma^2 t) \Big|_{t=0}$$

$$= e^0 \cdot \mu = \mu$$

$$E(X^2) = \frac{d}{dt} e^{\mu t + \sigma^2 t^2/2} \cdot (\mu + \sigma^2 t) \Big|_{t=0}$$

$$= e^{\mu t + \sigma^2 t^2/2} \cdot (\mu + \sigma^2 t)^2 + e^{\mu t + \sigma^2 t^2/2} \cdot \sigma^2 \Big|_{t=0}$$

$$= e^0 \cdot \mu^2 + e^0 \cdot \Delta^2$$

$$= \mu^2 + \Delta^2$$

$$Var(X) = E(X^2) - E(X)^2$$

$$= \mu^2 + \Delta^2 - \mu^2 = \Delta^2$$