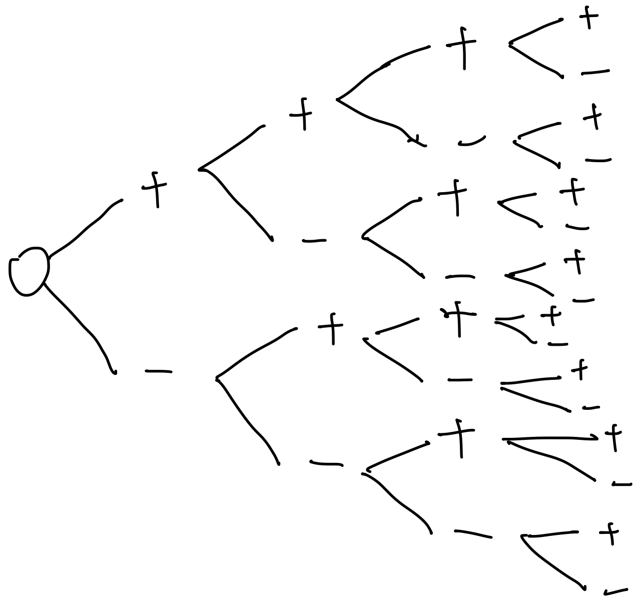
STAT 445: HW1

1.1

(a):



$+ \sim$ Heads $\sim H$

$- \sim$ Tails $\sim T$

$$S = \{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT,\}$$

$$2^4 = 16$$

(b):

$$S = \{0, 1, 2, 3 \cdots\}$$

0 is No Leaves are damaged.

1 is One leave is demaged.

2 is two leaves are demaged.

3 is three leaves are demaged.

⋮

(c):

$$S = \{ t : t \geq 0 \text{ hours} \}$$

(d):

$$S = \{ w : w > 0 \}$$

(e): We set n as the total number of items.

$$S = \{ 0\%, \frac{1}{n}, \frac{2}{n}, \frac{3}{n} \cdots, \frac{n-1}{n}, 100\% \}$$

1.6:

$$\text{Penny A} \begin{cases} P_A(H) = U \\ P_A(T) = 1 - U \end{cases}$$

$$\text{Penny B} \begin{cases} P_B(H) = W \\ P_B(T) = 1 - W \end{cases}$$

$$P_0 = P(\text{0 Heads occur}) = P_A(T) \times P_B(T)$$
$$= (1-U)(1-W)$$
$$= 1 - U - W + WU$$

$$P_1 = P_A(H) \times P_B(T) + P_A(T) \times P_B(H)$$
$$= U \times (1-W) + (1-U) \times W$$
$$= U - UW + W - UW = U + W - 2UW$$

$$P_2 = P_A(H) \times P_B(H) = UW$$

Let $\begin{cases} P_0 = P_2 & \cdots ① \\ P_1 = P_2 & \cdots ② \end{cases}$

①: $1 - U - W + WU = UW$

$U + W = 1$

②: $U + W - 2UW = UW$

$U + W = 3UW$.

$\begin{cases} U + W = 1 & \cdots ① \\ U + W = 3UW & \cdots ② \end{cases}$

According to ①, $U = 1 - W$, we can get

$1 = 3(1-W) \times W$

$3W^2 - 3W + 1 = 0$ $\quad \Delta = b^2 - 4ac = 9 - 4 \times 3 \times 1$

$= -3 < 0$

$\because \Delta < 0$

$\therefore$ no real number solution for w and v.

$\therefore$ v and w can't be chosen to make

$p_0 = p_1 = p_2$.

Extra problem 3:

(a) Prove $\underbrace{(A\triangle B)\cup C}_{①} = \underbrace{(A\cup C)\triangle(B\setminus C)}_{②}$

① $= [(A\setminus B)\cup(B\setminus A)]\cup C = (A\setminus B)\cup(B\setminus A)\cup C$

② $= (A\cup C)\triangle(B\cap C^c)$

$= [(A\cup C)\setminus(B\cap C^c)]\cup[(B\cap C^c)\setminus(A\cup C)]$

$= [(A\cup C)\cap(B^c\cup C)]\cup[(B\cap C^c)\cap(A^c\cap C^c)]$

$= [[(A\cup C)\cap B^c]\cup[(A\cup C)\cap C]]\cup[[(B\cap C^c)\cap A^c]\cap[(B\cap C^c)\cap C^c]]$

$= [[(A\cap B^c)\cup(B^c\cap C)]\cup C]\cup[A^c\cap B\cap C^c]$

$= [(A\cap B^c)\cup[(B^c\cap C)\cup C]]\cup(A^c\cap B\cap C^c)$

$= [(A\cap B^c)\cup C]\cup(A^c\cap B\cap C^c)$

$$= (A \cap B^c) \cup \left[C \cup (A^c \cap B \cap C^c) \right]$$

$$= (A \cap B^c) \cup \left[\left[(A^c \cap B) \cup C \right] \cap (C \cup C^c) \right]$$

$$= (A \cap B^c) \cup \left[\left[(B \cap A^c) \cup C \right] \cap S \right]$$

$$= (A \cap B^c) \cup (B \cap A^c) \cup C$$

$$= (A \setminus B) \cup (B \setminus A) \cup C = 1$$

Therefore (a) was proved.

(b): Prove $\underbrace{(A\cup B)\triangle C}_{(1)} = \underbrace{(A\triangle C)\triangle(B\setminus A)}_{(2)}$

$$(2) = (A\triangle C)\triangle(B\setminus A) = (A\triangle C)\triangle(B\cap A^c)$$

$$= \underbrace{\left[(A\triangle C)\setminus(B\cap A^c)\right]}_{(3)} \cup \underbrace{\left[(B\cap A^c)\setminus(A\triangle C)\right]}_{(4)}$$

$$(3) = \left[\left[(A\setminus C)\cup(C\setminus A)\right]\setminus(B\cap A^c)\right]$$

$$= \left[\left[(A\cap C^c)\cup(C\cap A^c)\right]\cap(B^c\cup A)\right]$$

$$= \left[(A\cap C^c)\cap(B^c\cup A)\right]\cup\left[(C\cap A^c)\cap(B^c\cup A)\right]$$

$$= \left[(A\cap C^c\cap B^c)\cup(A\cap C^c)\right]\cup\left[(C\cap A^c\cap B^c)\cup\underbrace{(C\cap A^c\cap A)}_{\phi}\right]$$

$$\boxed{= (A\cap C^c)\cup(A^c\cap B^c\cap C)}$$

$$(4) = \left[(B\cap A^c)\setminus(A\triangle C)\right]$$

$$= \left[(B\cap A^c)\cap(A\triangle C)^c\right] = \left[(B\cap A^c)\cap\left[(A\cap C^c)\cup(C\cap A^c)\right]^c\right]$$

$$= (B\cap A^c)\cap\left[(A^c\cup C)\cap(C^c\cup A)\right]$$

$$= \left[(B\cap A^c\cap A^c)\cup(B\cap A^c\cap C)\right]\cap(C^c\cup A)$$

$$= \left[(B \cap A^c) \cup (B \cap A^c \cap C)\right] \cap (C^c \cup A)$$

$$= \left[(B \cap A^c) \cap (C^c \cup A)\right] \cup \left[(B \cap A^c \cap C) \cap (C^c \cup A)\right]$$

$$= \left[(B \cap A^c \cap C^c) \cup \phi\right] \cup \left[(B \cap A^c \cap C) \cap (C^c \cup A)\right]$$

$$= (B \cap A^c \cap C^c) \cup \left[(B \cap A^c \cap \underbrace{C \cap C^c}_{\phi}) \cup (B \cap \underbrace{A^c \cap C \cap A}_{\phi})\right]$$

$$\boxed{= (B \cap A^c \cap C^c)}$$

$$\boxed{③ \cup ④ = (A \cap C^c) \cup (A^c \cap B^c \cap C) \cup (B \cap A^c \cap C^c) = ②}$$

$$① = (A \cup B) \triangle C$$

$$= \left[(A \cup B) \setminus C\right] \cup \left[C \setminus (A \cup B)\right]$$

$$= \left[(A \cup B) \cap C^c\right] \cup \left[C \cap (A \cup B)^c\right]$$

$$= \left[(A \cap C^c) \cup (B \cap C^c)\right] \cup \left[C \cap (A^c \cap B^c)\right]$$

$$= (A \cap C^c) \cup (B \cap C^c) \cup (A^c \cap B^c \cap C)$$

$$= (A \cap C^c) \cup (A^c \cap B^c \cap C) \cup ((A \cup A^c) \cap (B \cap C^c))$$

$$= (A \cap C^c) \cup (A^c \cap B^c \cap C) \cup (B \cap A^c \cap C^c) \cup (B \cap A \cap C^c)$$

$$\because (B \cap A \cap C^c) \subseteq (A \cap C^c)$$

$$\therefore (B \cap A \cap C^c) \cup (A \cap C^c) = (A \cap C^c)$$

$$\therefore ① = (A \cap C^c) \cup (A^c \cap B^c \cap C) \cup (B \cap A^c \cap C^c) = ②$$

$\therefore$ (b) was proved.

φ :

Prove: Since $C_1, C_2 \dots$ form a partition,

We can get for $\forall i, j, i \neq j$, $C_i \cap C_j = \emptyset$

and $S = \bigcup_{i=1}^{\infty} C_i$.

$$A^c = A^c \cap S = A^c \cap \left(\bigcup_{i=1}^{\infty} C_i \right) = \bigcup_{i=1}^{\infty} (A^c \cap C_i)$$

According to the Distributive Law, we can have

$$P(A^c) = P\left(\bigcup_{i=1}^{\infty} (A^c \cap C_i) \right)$$

Since C_i are disjoint, thus $A^c \cap C_i$ are also disjoint. We can get

$$P(A^c) = P\left(\bigcup_{i=1}^{\infty} (A^c \cap C_i) \right) = \sum_{i=1}^{\infty} P(A^c \cap C_i)$$

Then, we construct a disjoint collection $A_1^{c*}, A_2^{c*}, \dots$, with the property $\bigcup_{i=1}^{\infty} A_i^{c*} = \bigcup_{i=1}^{\infty} A_i^c$.

We define A_i^{c*} by $A_1^{c*} = A_1^c$,

$$A_i^{c*} = A_i^c \setminus \left(\bigcup_{j=1}^{i-1} A_j^c \right), \quad i = 2, 3, \cdots$$

Thus, $P(A^c) = P\left(\bigcup_{i=1}^{\infty} A_i^c\right) = P\left(\bigcup_{i=1}^{\infty} A_i^{c*}\right)$

$$= \sum_{i=1}^{\infty} P(A_i^{c*}) \leq \sum_{i=1}^{\infty} P(A_i^c)$$

, because $A_i^{c*} \subseteq A_i^c$

$P(A_i^*) \leq P(A_i)$

each A_i^{c*} are disjoint.

Then. $P(A^c) = \underbrace{P\left(\bigcup_{i=1}^{\infty} A_i^c\right) = P\left(\left(\bigcap_{i=1}^{\infty} A_i\right)^c\right)}_{\text{De Morgan's Law}}$ $\boxed{P\left(\left(\bigcap_{i=1}^{\infty} A_i\right)^c\right) \leq \sum_{i=1}^{\infty} P(A_i^c)}$.

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② $\because$ $P\left(\left(\bigcap_{i=1}^{\infty} A_i\right)^c\right) \leq \sum_{i=1}^{\infty} P(A_i^c)$

$\because$ $\left(\bigcap_{i=1}^{\infty} A_i\right)^c \cup \left(\bigcap_{i=1}^{\infty} A_i\right) = S$

$\therefore$ $P\left(\left(\bigcap_{i=1}^{\infty} A_i\right)^c\right) + P\left(\left(\bigcap_{i=1}^{\infty} A_i\right)\right) = P(S) = 1$

$\therefore$ $1 - P\left(\bigcap_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i^c)$

$\therefore$ $P\left(\bigcap_{i=1}^{\infty} A_i\right) \geq 1 - \sum_{i=1}^{\infty} P(A_i^c)$