Stat 345/445: Theoretical Statistics I: Homework 4 Solutions

Textbook Exercises

2.1 (345 & 445: 2 pts.) Find the pdf of Y. Show that the pdf integrates to 1.

(a) $Y = X^3$ and $f_X(x) = 42x^5(1-x),\ \ 0 < x < 1$

$$f_X(x) = 42x^5(1-x),\ \ 0 < x < 1; \quad y = x^3 = g(x),\ \text{monotone, and } \mathcal{Y} = (0,1).$$

Use Theorem 2.1.5. $x = g^{-1}(y) = y^{1/3}$

$$f_Y(y) = f_x(g^{-1}(y))\Big|\frac{d}{dy}g^{-1}(y)\Big| = f_x(y^{\frac{1}{3}})\frac{d}{dy}(y^{\frac{1}{3}}) = 42y^{\frac{5}{3}}(1-y^{\frac{1}{3}})(\frac{1}{3}y^{-\frac{2}{3}}) = 14y(1-y^{\frac{1}{3}})$$

$$f_Y(y) = 14y - 14y^{\frac{4}{3}}, \quad 0 < y < 1.$$

Check integral: $\int_0^1 (14y - 14y^{\frac{4}{3}})dy = 7y^2 - 14\frac{y^{7/3}}{(7/3)}\Big|_0^1 = 7y^2 - 6y^{\frac{7}{3}}\Big|_0^1 = 1$

2.3 (345 & 445: 2 pts.) Suppose X has the geometric pmf $f_X(x) = \frac{1}{3}(\frac{2}{3})^x$, $x = 0, 1, 2, \cdots$ Determine the probability distribution of $Y = X/(X+1)$. Note that here both X and Y are discrete random variables. To specify the probability distribution of Y, specify its pmf.

$$y = \frac{x}{x+1} \implies y(x+1) = x \implies xy + y = x \implies x - xy = y \implies x(1-y) = y \implies x = \frac{y}{1-y}$$

$$P(Y = y) = P(\frac{X}{X+1} = y) = P(X = \frac{y}{1-y}) = (\frac{1}{3})(\frac{2}{3})^{y/(1-y)}, \quad y = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{x}{x+1}, \cdots$$

2.17 (345: 1 pt.) A *median* of a distribution is a value m such that $P(X \le m) \ge \frac{1}{2}$ and $P(X \ge m) \ge \frac{1}{2}$. (If X is continuous, m satisfies $\int_{-\infty}^m f(x)dx = \int_m^\infty f(x)dx = \frac{1}{2}$.) Find the median of the following distribution.

(a) $f(x) = 3x^2,\ \ 0 < x < 1$

$$\int_0^m 3x^2 dx = m^3 = \frac{1}{2} \implies m = (\frac{1}{2})^{1/3} = 0.794.$$

2.18 (445: 2 pts.) Show that if X is a continuous random variable, then

$$\min_a E|X-a| = E|X-m|,$$

where m is the median of X.

$$E|X-a| = \int_{-\infty}^{\infty} |x-a|f(x)dx = \int_{-\infty}^{a} -(x-a)f(x)dx + \int_a^\infty f(x)dx.$$

$$\frac{d}{da}E|X-a| = \int_{-\infty}^{a} f(x)dx - \int_a^\infty f(x)dx = 0.$$

The solution to this equation is $a =$ median. This is a minimum since $d^2/da^2 E|X-a| = 2f(a) > 0$.

2.23 (445: 2 pts.) Let X have the pdf

$$f(x) = \frac{1}{2}(1+x), \quad -1 < x < 1.$$

(a) Find the pdf of $Y = X^2$

Use Theorem 2.1.8 with $A_0\{0\}, A_1 = (-1,0)$ and $A_2 = (0,1)$. Then $g_1(x) = x^2$ on A_1 and $g_2(x) = x^2$ on A_2. Then

$$f_Y(y) = \frac{1}{2}y^{-\frac{1}{2}}, \quad 0 < y < 1.$$

(b) Find EY and Var Y.

$$EY = \int_0^1 y f_Y(y)dy = \frac{1}{3}$$
$$EY^2 = \int_0^1 y^2 f_Y(y)dy = \frac{1}{5}$$
$$VarY = \frac{1}{5} - (\frac{1}{3})^2 = \frac{4}{45}$$

2.24 (345: 3 pts.) Compute EX and Var X for each of the following probability distributions.

(a) $f_x(x) = ax^{a-1}$, $0 < x < 1, \quad a > 0$

$$EX = \int_0^1 x a x^{a-1} dx = \int_0^1 a x^a dx = \frac{ax^{a+1}}{a+1}\Big|_0^1 = \frac{a}{a+1}$$
$$EX^2 = \int_0^1 x^2 a x^{a-1} dx = \int_0^1 a x^{a+1} dx = \frac{ax^{a+2}}{a+2}\Big|_0^1 = \frac{a}{a+2}$$
$$VarX = \frac{a}{a+2} - (\frac{a}{a+1})^2 = \frac{a}{(a+2)(a+1)^2}$$

(b) $F_X(x) = \frac{1}{n}, \quad x = 1, 2, \cdots, n, \quad n > 0$ an integer

$$EX = \sum_{x=1}^{n} \frac{x}{n} = \frac{1}{n}\sum_{x=1}^{n} x = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}$$
$$EX^2 = \sum_{i=1}^{n} \frac{x^2}{n} = \frac{1}{n}\sum_{i=1}^{n} x^2 = \frac{1}{n}\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$
$$VarX = \frac{(n+1)(2n+1)}{6} - (\frac{n+1}{2})^2 = \frac{2n^2+3n+1}{6} - \frac{n^2+2n+1}{4} = \frac{n^2-1}{12}$$

(c) $f_X(x) = \frac{3}{2}(x-1)^2, \quad 0 < x < 2$

$$EX = \int_0^2 x\frac{3}{2}(x-1)^2 dx = \frac{3}{2}\int_0^2 (x^3 - 2x^2 + x)dx = 1$$
$$EX^2 = \int_0^2 x^2\frac{3}{2}(x-1)^2 dx = \frac{3}{2}\int_0^2 (x^4 - 2x^3 + x^2)dx = \frac{8}{5}$$
$$VarX = \frac{8}{5} - 1^2 = \frac{3}{5}$$

2.33 (345 & 445: 2 pts.) Verify the expression given for the moment generating function, and in each case use the mgf to calculate EX and Var$X$.

(c) $f_X(x) = \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sqrt{2\pi}\sigma}$, $M_X(t) = e^{\mu t+\sigma^2 t^2/2}$, $-\infty < x < \infty$; $-\infty < \mu < \infty$, $\sigma > 0$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-(x^2-2\mu x-2\sigma^2 tx+\mu^2)/2\sigma^2} dx.$$

Now, complete the square in the numerator by writing

$$\begin{aligned} x^2 - 2\mu x - 2\sigma^2 tx + \mu^2 &= x^2 - 2(\mu + \sigma^2 t)x \pm (\mu + \sigma^2 t)^2 + \mu^2 \\ &= (x - (\mu + \sigma^2 t))^2 - (\mu + \sigma t)^2 + \mu^2 \\ &= (x - (-\mu + \sigma^2 t))^2 - [2\mu\sigma^2 t + (\sigma^2 t)^2] \end{aligned}$$

Then we have

$$M_x(t) = e^{[2\mu\sigma^2 t+(\sigma^2 t)^2]/2\sigma^2} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2} dx$$

$$M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$EX = \frac{d}{dt} M_x(t)|_{t=0} = (\mu + \sigma^2 t) e^{\mu t+\sigma^2 t^2/2}|_{t=0} = \mu$$

$$EX^2 = \frac{d^2}{dt^2} M_x(t)|_{t=0} = (\mu + \sigma^2 t)^2 e^{\mu t+\sigma^2 t^2/2} + \sigma^2 e^{\mu t+\sigma^2 t/2}|_{t=0} = \mu^2 + \sigma^2$$

$$VarX = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$