STAT 345/445 Lecture 7

Section 2.1: Distributions of Functions of a Random Variable

Functions of random variables

Sometimes we want to transform a random variable.
For example:

- If X is the temperature in Fahrenheit, what is the distribution of the temperature in Celsius?

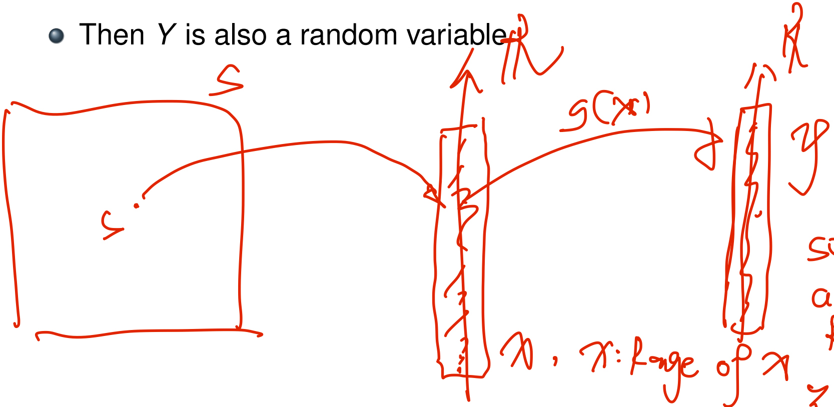
$$Y = (X - 32)\frac{5}{9}$$

- If X and Y denote height and weight, what is the distribution of the BMI?

$$B = \frac{X}{Y^2}$$

Functions of random variables

- Let X be a random variable and let $g(\cdot)$ be a function.
- Then Y is also a random variable



so Y is also a R.V.

$Y(s) = g(X(s))$

i.e. $\{x \in \mathbb{R}, \exists s \in S, \text{ s.t. } X(s) = x\}$

Functions of random variables

cdf or pdf/pmf of y

- What is the distribution of the random variable $Y = g(X)$?
- Have the cdf $F_X(x)$ or pmf/pdf $f_X(x)$ of X
  - want to find the cdf $F_Y(y)$ or pmf/pdf $f_Y(y)$ of Y.

notice that $g^{-1}(y)$ may not be just one value

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(y))$$

Let's review *inverse mappings...*

Inverse mapping

- For a function $g(x) : \mathcal{X} \to \mathcal{Y}$ we define an **inverse mapping** as

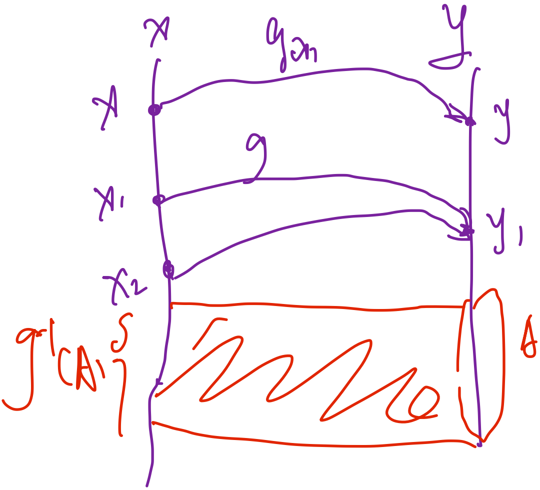
$$g^{-1}(A) = \{x \in \mathcal{X} : \; g(x) \in A\}$$

  for any set $A \subset \mathcal{Y}$
  - Note that $g^{-1}(A) \subset \mathcal{X}$

- In particular:

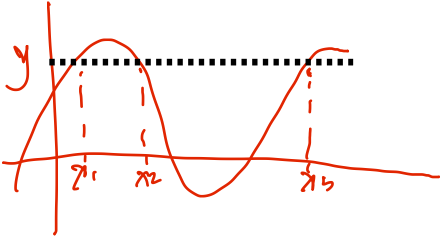
$$g^{-1}(\{y\}) = \{x \in \mathcal{X} : \; g(x) = y\}$$

  - Can still be a set in $\mathcal{X}$ rather than just one number
  - Usually just write $g^{-1}(y)$

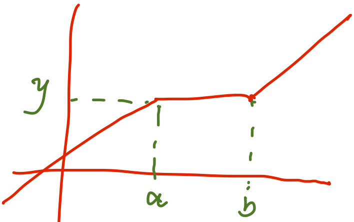


$$g^{-1}(\{y_1\}) = g^{-1}(y_1)$$

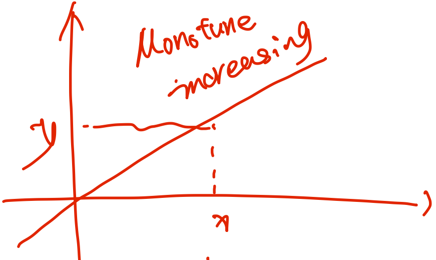
$$= \{x_1, x_2\}$$



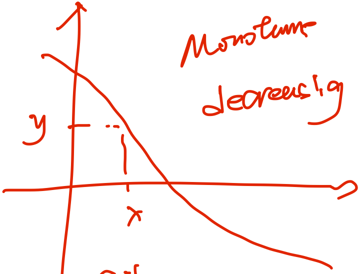
$g^{-1}(y) = \{x_1, x_2, x_3\}$



$g^{-1}(y) = [a, b]$



$g^{-1}(y) = x$



$g^{-1}(y) = x$

More on inverse mapping

- A function $g(x) : \mathcal{X} \to \mathcal{Y}$ is a **one-to-one** function if and only if $\forall\, y \in \mathcal{Y}$ we have

$$g^{-1}(\{y\}) = \{x\}$$

  - Can write $g^{-1}(y) = x$

- *Strictly monotone* functions are one-to-one

Probability of a transformation

- Let X be a random variable in $(S, \mathcal{B}, P)$ and let $Y = g(X)$.
- Probabilities for Y can be obtained from probabilities of X and the inverse mapping $g^{-1}(\cdot)$
- In general

$A \subset \mathcal{Y}$

$$P(Y \in A) = P(X \in g^{-1}(A))$$

In particular:

$$F_Y(y) = P(\underbrace{Y \le y}_{Y \in (-\infty, y]}) = P(X \in g^{-1}(-\infty, y])$$

- Will look at discrete and continuous variables separately

Discrete random variables

- Let X be a discrete random variable and let $Y = g(X)$ for some function $g(\cdot)$.
- Then Y is a discrete random variable
- Then

$$f_Y(y) = P(Y=y) = P(g(X)=y) = P(X \in g^{-1}(y))$$

$$= P(X \in \{x : g(x) = y\})$$

$$= \sum_{x \in g^{-1}(y)} P(X=x) = \sum_{x \in g^{-1}(y)} f_X(x)$$

and

$$F_Y(y) = \sum_{v \le y} f_Y(v)$$

Discrete example 1

- Let $X \sim \text{Binom}(n, p)$, i.e. X has pmf

$$f_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x \in \{0, 1, 2, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}$$

- X can be thought of as the number of successes in n independent Bernoulli trials
- What is the distribution of $Y = n - X$? (number of failiares instead number of success)

Example 1:

$g(x) = n - x$

$x = \{0, 1, 2, \cdots n\}$ $\quad y = \{0, 1, \cdots, n\}$ $\quad g^{-1}(y) = \{x : \underbrace{n - x}_{g(x)} = y\} = \{n - y\}$

i.e. a set of one value.

$$f_Y(y) = \sum_{x \in g^{-1}(y)} f_X(x) = f_X(n-y) = \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)}$$

could stop here

$$= \binom{n}{y} p^{n-y} (1-p)^{y}$$

$$\binom{n}{n-y} = \frac{n!}{(n-y)!\,y!}$$

$\vdots$

put for Binomial $(n, 1-p)$

$$\frac{n!}{y!\,(n-y)!}$$

Discrete example 2

- Let $X \sim \text{Binom}(10, p)$. What is the distribution of $Y = |X - 5|$?

$\mathcal{X} = \{0, 1, 2, \ldots 10.\}$

$g(x) = |x-5|$

$\mathcal{Y} = \{0, 1, 2, 3, 4, 5\}$

$y = 0$, only one x : $x = 5$

If $y = |x-5|$ then

either $y = x-y$ or $y = -x+5 \Rightarrow x = y+5$ or $x = 5-y$

For $y = 0$: $g^{-1}(0) = \{5\}$, For $y = 1, 2, 3, 4, 5$; $g^{-1}(y) = \{y+5, 5-y\}$

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$$f_y(0) = f_x(5) = \binom{10}{5} p^5 (1-p)^5$$

for $y = 1, \cdots 5$,

$$f_y(y) = f_x(y+5) + f_x(5-y) = \binom{10}{y+5} p^{y+5} (1-p)^{10-y-5}$$

$$+ \binom{10}{5-y} p^{5-y} (1-p)^{10-5+y}$$

~~Continuous random variables~~ Monotone transf

- It's easiest to deal with *monotone* functions g:

$$\text{Increasing: } u > v \Rightarrow g(u) > g(v)$$
$$\text{Decreasing: } u > v \Rightarrow g(u) < g(v)$$

- The **support** of a distribution (or random variable) is defined as

$$\mathcal{X} = \{x : f_X(x) > 0\} \tag{1}$$
$$\text{and let } \mathcal{Y} = \{y : \exists x \in \mathcal{X} \text{ such that } g(x) = y\} \tag{2}$$

Support of y support of X

- If g is monotone on $\mathcal{X}$ then it is *one-to-one* and *onto* from $\mathcal{X}$ to $\mathcal{Y}$.
  - Uniquely pairs an x to one y
  - Get an inverse function: $g^{-1}(y) = x$

cdf – method

Theorem ("cdf-method")

Let X be a random variable with cdf $F_X(x)$ and let $Y = g(X)$. Then

(a) If g is an increasing function on $\mathcal{X}$ then

$$F_Y(y) = F_X(g^{-1}(y))$$

(b) If g is a decreasing function on $\mathcal{X}$ *and* X is continuous , then

$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

In general: i.e. for both continuous and discrete

$$F_Y(y) = 1 - F_X(g^{-1}(y)) + P(X = g^{-1}(y))$$

cdf – method, proof

(a) $F_Y(y) = F_X(g^{-1}(y))$

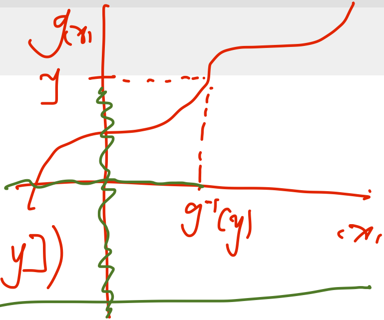
$F_Y(y) = P(Y \le y) = P(Y \in (-\infty, y])$

$Y \le y \Rightarrow X \le g^{-1}(y)$

$= P(X \in (-\infty, g^{-1}(y)])$

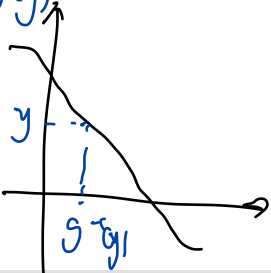
$= P(X \le g^{-1}(y))$

$= F_X(g^{-1}(y))$



(b)

$Y \le y \Rightarrow X \ge g^{-1}(y)$



$$F_Y(y) = P(Y \leq y) = P(X \geq g^{-1}(y))$$

$$= 1 - P(X < g^{-1}(y)) = 1 - P(X \leq g^{-1}(y)) = 1 - F_X(g^{-1}(y))$$

if X is continous

$$= 1 - P(X < g^{-1}(y)) - P(X = g^{-1}(y)) + P(X = g^{-1}(y))$$

$$= 1 - P(X \leq g^{-1}(y)) + P(X = g^{-1}(y)) = 1 - F_X(g^{-1}(y)) + P(X = g^{-1}(y))$$

Example: Exponential and Weibull

Let $X \sim \text{Expo}(1)$, i.e.

$$F_X(x) = \begin{cases} 0 & , \ x < 0 \\ 1 - e^{-x} & , \ x \geq 0 \end{cases}$$

Let $Y = X^\alpha$ for $\alpha > 0$. What is the distribution of Y?

$g(x) = x^\alpha$, $\alpha > 0$ (constant)

if $x_1 < x_2$, then $x_1^\alpha < x_2^\alpha$

so g is monotone increasing. (only need increasing on X)

Or if $\frac{d}{dx} g(x) = \frac{d}{dx} x^\alpha > 0$ for all $x > 0$ $x \in [0, \infty)$

Suppore of Y? $F_X(0) = 1 - e^0 = 0$, $\lim_{x\to\infty} F_X(x) = 1 - 0 = 1$

$F_Y(y) = F_X(g^{-1}(y))$

$y = g(x) = x^\alpha \quad x \in [0,\infty) \Rightarrow y \in [0,\infty) = \mathcal{Y}$

$y^{1/\alpha} = x$

$\Rightarrow F_Y(y) = F_X(y^{1/\alpha}) = 1 - e^{-y^{\frac{1}{\alpha}}}$ for $y \in [0,\infty)$.

$F_Y(y) = 0$ for $y < 0$

pdf – method

Theorem ("pdf method")

Let X be a continuous random variable with pdf $f_X(s)$ and let $Y = g(X)$ where g is a *monotone* function.

Suppose that $f_X(x)$ is continuous on $\mathcal{X}$ and that $g^{-1}(y)$ has a continuous derivative on $\mathcal{Y}$.

Then the pdf of Y is given by

$$f_Y(y) = \frac{d}{dy} F_X(g^{-1}(y))$$

$$= F_X'(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$$

$$= \underbrace{f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)}_{\text{always Positive. cuz increasing}}$$

Pdf method:

If $g(x)$ is increasing,

$$f_Y(y) = \frac{d}{dy} F_X(y) = \frac{d}{dy} F_X(g^{-1}(y))$$

Chain Rule.

$$\frac{d}{dx} h(g(x)) = h'(g(x)) g'(x)$$

If g is decreasing, $f_Y(y) = \frac{d}{dy}(1 - F_X(g^{-1}(y))) = -f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)$

Note: $\frac{d}{dy}g^{-1}(y)$ is negative.

Therefore:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \text{ for } y \in \mathcal{Y}$$

0 otherwise.

Example: Exponential and Uniform

Let $X \sim \text{Expo}(1)$, i.e.

$$f_X(x) = \begin{cases} e^{-x} & , \ x \geq 0 \\ 0 & , \ x < 0 \end{cases}$$

Let $Y = 1 - e^{-X}$. What is the distribution of Y?

$y = g(x) = 1 - e^{-x} \Rightarrow \mathcal{Y} = [0, 1)$

$\mathcal{X} = [0, \infty)$

$g(0) = 1 - 1 = 0.$

$\lim_{x \to \infty} g(x) = 1 - 0 = 1$

$e^{-x} = 1 - y \Rightarrow -x = \log(1-y)$

$\Rightarrow x = -\log(1-y)$

So $g^{-1}(y) = -\log(1-y)$

$\frac{d}{dy} g^{-1}(y) = \frac{-1}{1-y}(-1)$

$= \frac{1}{1-y}$

$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$

$= e^{-(-\log(1-y))} \frac{1}{1-y}$

$= (1-y) \frac{1}{1-y} = 1$

$$\Rightarrow f_Y(y) = \begin{cases} 1 & \text{for } y \in [0,1) \\ 0 & \text{otherwise.} \end{cases}$$

Note: $F_X(x) = \begin{cases} 1 - e^{-x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$

i.e. $g(x) = F_X(x)$

Probability integral transformation

Theorem

Let X have a continuous cdf $F_X(x)$ and let $Y = F_X(X)$. Then Y is uniformly distributed on $(0, 1)$, i.e.

$$F_Y(y) = \begin{cases} 0 & , \ y \leq 0 \\ y & , \ 0 < y < 1 \\ 1 & , \ y \geq 1 \end{cases}$$

$X \sim F_X(x) \Rightarrow Y = F_X(X) \sim \text{Uniform}(0,1)$

and $Y \sim \text{Uniform} \Rightarrow X = F_X^{-1}(Y) \sim F_X$

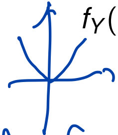
When g is monotone only on certain intervals

- See Theorem 2.1.8 for more detail

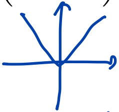
Support of X

- If $\mathcal{X}$ can be split into sets $A_1, \ldots, A_k$ and g can be split into $g_1(x), \ldots, g_k(x)$ such that
  - $g(x) = g_i(x)$ for $x \in A_i$
  - g_i is a monotone function from A_i *onto* $\mathcal{Y}$

  then



$$f_Y(y) = \begin{cases} \sum_{i=1}^{k} f_X\left(g_i^{-1}(y)\right)\left|\frac{d}{dy} g_i^{-1}(y)\right| & , \ y \in \mathcal{Y} \\ 0 & , \ \text{otherwise} \end{cases}$$



E.g. $g(x) = x^2$ or $g(x) = |x|$

$g(x) = x^2 \cdot x \in N(0,1)$

$g_1(x) = x^2$, for $x \in [0, \infty) = A_1$

$g_2(x) = x^2$, for $x \in (-\infty, 0) = A_2$

$y = x^2 \Rightarrow x = \sqrt{y}$ if $x \in A_1$, $x = -\sqrt{y}$ if $x \in A_2$

$g_1(y) = \sqrt{y}$, $\frac{d}{dy} = \frac{1}{2\sqrt{y}}$

$g_2(y) = -\sqrt{y}$, $\frac{d}{dy} = -\frac{1}{2\sqrt{y}}$

Example: Standard normal and χ^2 distribution

Let $X \sim N(0,1)$, i.e.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad \text{for } x \in \mathbb{R}$$

Let $Y = X^2$. What is the distribution of Y?

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{y})^2/2} \frac{1}{2\sqrt{y}} \quad \text{for } y \geq 0$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y/2} \frac{1}{\sqrt{y}}, \quad \text{for } y \geq 0$$

Chi-square

= Pdf for Chi-square distribution with 1 d.f. χ^2