Stat 346/446: Theoretical Statistics II:
Practice Exercises 2 Solutions

Textbook Exercises

7.33 (346 & 446)In Example 7.3.5 the MSE of the Bayes estimator, $\hat{p}_B$, of a success probability was calculated (the estimator was derived in Example 7.2.14). Show that the choice $\alpha = \beta = \sqrt{n}/4$ yields a constant MSE for $\hat{p}_B$.
Given that:

$$\alpha = \beta = \sqrt{\frac{n}{4}}$$

The MSE of the Bayes estimator $\hat{p}_B$ is given by:

$$\text{MSE}(\hat{p}_B) = \frac{np(1-p)}{(\alpha+\beta+n)^2} + \left(\frac{np+\alpha}{\alpha+\beta+n} - p\right)^2.$$

Substituting $\alpha = \beta = \sqrt{\frac{n}{4}}$:

$$\text{MSE}(\hat{p}_B) = \frac{1}{(\sqrt{n}+n)^2}\left(np(1-p) + \left(np + \sqrt{\frac{n}{4}} - p(\sqrt{n}+n)\right)^2\right)..$$

Rewriting the terms:

$$\begin{aligned}
\text{MSE}(\hat{p}_B) &= \frac{1}{(\sqrt{n}+n)^2}\left[np - np^2 + n^2p^2 + n^{3/2}p + \frac{n}{4} - 2p\left(n^{3/2}p + n^2p + \frac{1}{2}n + \frac{n^{3/2}}{2}\right) + (p^2(n + 2n^{3/2} + n^2))\right] \\
&= \frac{1}{(\sqrt{n}+n)^2}\left[np - np^2 + n^2p^2 + n^{3/2}p + \frac{n}{4} - 2n^{3/2}p^2 - 2n^2p^2 - np - n^{3/2}p + np^2 + 2n^{3/2}p^2 + n^2p^2\right] \\
&= \frac{1}{(\sqrt{n}+n)^2}\left[\frac{n}{4}\right] \\
&= \frac{n}{4(\sqrt{n}+n)^2}
\end{aligned}$$

Since this expression does not contain p, the MSE is independent of p.

7.38 For each of the following distributions, let $X_1, \ldots, X_n$ be a random sample. Is there a function of θ, say $g(\theta)$, for which there exists an unbiased estimator whose variance attains the **Cramér-Rao Lower Bound**? If so, find it. If not, show why not.

(a) (346 & 446) $f(x \mid \theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \quad \theta > 0.$
Use Corollary 7.3.15
Given the probability density function:

$$f(x \mid \theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \quad \theta > 0.$$

We differentiate the log-likelihood function:

$$\begin{aligned}
\frac{\partial}{\partial\theta}\ln L(\theta \mid x) &= \frac{\partial}{\partial\theta}\ln\prod_{i=1}^{n} x_i^{\theta-1} \\
&= \frac{\partial}{\partial\theta}\sum_{i=1}^{n}\ln\left(\theta x_i^{\theta-1}\right) \\
&= \frac{\partial}{\partial\theta}\sum_{i=1}^{n}\left[\ln\theta + (\theta-1)\ln x_i\right] \\
&= \sum_{i=1}^{n}\left[\frac{1}{\theta} + \ln x_i\right] \\
&= n\left[\frac{1}{\theta} + \frac{1}{n}\sum_{i=1}^{n}\ln x_i\right].
\end{aligned}$$

Thus, solving for θ:

$$-\sum_{i=1}^{n}\frac{\ln x_i}{n} = \frac{1}{\theta}.$$

This implies that:

$$-\sum_{i=1}^{n}\frac{\ln x_i}{n} \quad \text{is the UMVUE of} \quad g(\theta) = \frac{1}{\theta}.$$

(b) (446) $f(x \mid \theta) = \frac{\log(\theta)}{\theta-1}\theta^x, \quad 0 < x < 1, \quad \theta > 1.$
Use Corollary 7.3.15
Given the probability density function:

$$f(x \mid \theta) = \frac{\ln\theta}{\theta-1}\theta^x, \quad 0 < x < 1, \quad \theta > 1.$$

We compute the derivative of the log-likelihood function:

$$\frac{\partial}{\partial\theta}\ln L(\theta \mid x) = \frac{\partial}{\partial\theta}\ln\prod_{i=1}^{n}\frac{\ln\theta}{\theta-1}\theta^{x_i}.$$

Expanding the logarithm:

$$= \frac{\partial}{\partial\theta}\sum_{i=1}^{n}\left[\ln(\ln\theta) - \ln(\theta-1) + x_i\ln\theta\right].$$

$$\begin{aligned}
&= \sum_{i=1}^{n}\left[\frac{1}{\theta\ln\theta} - \frac{1}{\theta-1} + \frac{x_i}{\theta}\right]. \\
&= n\left[\frac{1}{\theta\ln\theta} - \frac{1}{\theta-1} + \frac{\bar{X}}{\theta}\right]. \\
&= \frac{n}{\theta}\left[\bar{X} - \left(\frac{\theta}{\theta-1} - \frac{1}{\ln\theta}\right)\right].
\end{aligned}$$

Thus, solving for $g(\theta)$:

$$\bar{X} \quad \text{is the UMVUE of} \quad g(\theta) = \frac{\theta}{\theta-1} - \frac{1}{\ln\theta}.$$

7.40 (346 & 446)Let $X_1, \dots, X_n$ be iid Bernoulli(p). Show that the variance of $\bar{X}$ attains the Cramér-Rao Lower Bound, and hence $\bar{X}$ is the best unbiased estimator of p.

Since $X_1, \dots, X_n$ are independent and iid random variables following the Bernoulli distribution with parameter p:

$$X_i \sim \text{Bernoulli}(p),$$

which has the probability mass function:

$$P(X_i = x_i \mid p) = p^{x_i}(1-p)^{1-x_i}, \quad x_i \in \{0,1\}.$$

The likelihood function for the sample is:

$$L(p \mid X) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}.$$

Differentiate the log-likelihood function:

$$\frac{\partial}{\partial p} \log L(p \mid x) = \frac{\partial}{\partial p} \log \prod_i p^{x_i}(1-p)^{1-x_i}.$$

$$= \frac{\partial}{\partial p} \sum_i \left[x_i \log p + (1-x_i)\log(1-p)\right].$$

$$= \sum_i \left[\frac{x_i}{p} - \frac{(1-x_i)}{1-p}\right].$$

$$= \frac{n\bar{X}}{p} - \frac{n-n\bar{X}}{1-p} = \frac{n}{p(1-p)}[\bar{X}-p].$$

By Corollary 7.3.15, $\bar{X}$ is the UMVUE of p and attains the Cramér-Rao lower bound. Alternatively, we could calculate:

$$-nE_0\left(\frac{\partial^2}{\partial p^2}\log f(X \mid \theta)\right).$$

$$= -nE\left(\frac{\partial^2}{\partial p^2}\log\left[p^X(1-p)^{1-X}\right]\right).$$

$$= -nE\left(\frac{\partial^2}{\partial p^2}\left[X\log p + (1-X)\log(1-p)\right]\right).$$

$$= -nE\left(\frac{\partial}{\partial p}\left[\frac{X}{p} - \frac{(1-X)}{1-p}\right]\right).$$

$$= -nE\left(\frac{-X}{p^2} - \frac{(1-X)}{(1-p)^2}\right).$$

$$= -n\left(-\frac{1}{p} - \frac{1}{1-p}\right) = \frac{n}{p(1-p)}.$$

Then using $\tau(\theta) = p$ and $\tau'(\theta) = 1$:

$$\frac{\tau'(\theta)}{-nE_0\left(\frac{\partial^2}{\partial p^2}\log f(X \mid \theta)\right)} = \frac{1}{\frac{n}{p(1-p)}} = \frac{p(1-p)}{n} = \text{Var}(\bar{X}).$$

Since we know that $E\bar{X} = p$, it follows that $\bar{X}$ attains the Cramér-Rao bound.

7.46 (346 & 446) Let X_1, X_2, and X_3 be a random sample of size three from a uniform $(\theta, 2\theta)$ distribution, where $\theta > 0$.

(a) Find the method of moments estimator of θ.
The expectation of X is given by:

$$E(X) = \frac{\theta + 2\theta}{2} = \frac{3}{2}\theta.$$

Solving for θ:

$$\frac{3}{2}\theta = \bar{X} \quad \Rightarrow \quad \hat{\theta}_{\text{MOM}} = \frac{2}{3}\bar{X}.$$

(b) Find the MLE, $\hat{\theta}$, and find a constant k such that $E_\theta(k\hat{\theta}) = \theta$.
Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics. The likelihood function is given by:

$$L(\theta \mid X) = \frac{1}{\theta^n} I(\theta \leq X_{(1)} \leq X_{(n)} \leq 2\theta) = \frac{1}{\theta^n} I\left[\frac{X_{(n)}}{2}, X_{(n)}\right](\theta).$$

Since $\hat{\theta}$ is a decreasing function, the MLE is:

$$\hat{\theta}_{\text{MLE}} = \frac{X_{(n)}}{2}.$$

Using the pdf of $X_{(n)}$, we have:

$$E(X_{(n)}) = \frac{2n+1}{2n+2}\theta.$$

To satisfy $E(k\hat{\theta}_{\text{MLE}}) = \theta$, we solve for k:

$$k = \frac{(2n+2)}{(2n+1)}.$$

(d) Find the method of moments estimate and the MLE of θ based on the data:

$$1.29, \quad 0.86, \quad 1.33,$$

three observations of average berry sizes (in centimeters) of wine grapes.

$$\hat{\theta}_{\text{MOM}} = \frac{2}{3}\bar{X} = 0.7733$$

$$\hat{\theta}_{\text{MLE}} = \frac{1}{2}X_{(3)} = 0.6650$$