Stat 346/446: Theoretical Statistics II: Practice Exercises 6 Solutions

Textbook Exercises

8.5 (446) A random sample, $X_1, \dots, X_n$, is drawn from a Pareto population with pdf

$$f(x \mid \theta, \nu) = \frac{\theta \nu^\theta}{x^{\theta+1}} \mathbf{1}_{[\nu,\infty)}(x), \quad \theta > 0, \quad \nu > 0.$$

(a) Find the MLEs of θ and ν.

$$\begin{aligned} L(\theta, \nu \mid x) &= \prod_{i=1}^n \frac{\theta \nu^\theta}{x_i^{\theta+1}} \mathbf{I}_{[\nu,\infty)}(x_i) \\ &= \theta^n \nu^{n\theta} \left(\prod_{i=1}^n \frac{1}{x_i} \right)^{\theta+1} \mathbf{I}_{[\nu,\infty)}(x_{(1)}). \end{aligned}$$

Taking log-likelihood:

$$\ell(\theta, \nu \mid x) = n \log \theta + n\theta \log \nu - (\theta + 1) \log \left(\prod_{i=1}^n x_i \right)$$

For any given θ, $\ell(\theta, \nu \mid x)$ is an increasing function of ν, so it is maximized at the largest possible value:

$$\hat{\nu} = x_{(1)}$$

Setting $\nu = x_{(1)}$, we have

$$\frac{d}{d\theta} \ell(\theta, x_{(1)} \mid x) = \frac{n}{\theta} + n \log x_{(1)} - \log \left(\prod_{i=1}^n x_i \right) = 0$$

Simplifying:

$$\frac{n}{\theta} = \log \left(\prod_{i=1}^n x_i \right) - n \log x_{(1)} = \log \left(\frac{\prod_{i=1}^n x_i}{(x_{(1)})^n} \right)$$

Thus:

$$\hat{\theta} = \frac{n}{\log \left(\frac{\prod_{i=1}^n x_i}{(x_{(1)})^n} \right)}$$

(b) Show that the LRT of

$$H_0 : \theta = 1, \nu \text{ unknown}, \quad \text{versus} \quad H_1 : \theta \neq 1, \nu \text{ unknown},$$

has critical region of the form $\{x : T(x) \leq c_1 \text{ or } T(x) \geq c_2\}$, where $0 < c_1 < c_2$, and

$$T = \log \left(\frac{\prod_{i=1}^n X_i}{(\min_i X_i)^n} \right).$$

We see that the MLE is $\hat{\theta} = \dfrac{n}{T}$. Under H_0, the likelihood is maximized at $\theta = 1$ and $\nu = x_{(1)}$.

$$\begin{aligned}
&\Rightarrow \lambda(x) = \frac{\sup_{\Theta_0} L(\theta, \nu \mid x)}{\sup_{\Theta} L(\theta, \nu \mid x)} \\
&= \frac{L\left(1, x_{(1)} \mid x\right)}{L\left(\frac{n}{T}, x_{(1)} \mid x\right)} \\
&= \frac{x_{(1)}^n \left(\prod_{i=1}^n \frac{1}{x_i}\right)^2 \mathbf{I}_{[x_{(1)},\infty)}(x_{(1)})}{\left(\frac{n}{T}\right)^n x_{(1)}^{n\frac{n}{T}} \left(\prod_{i=1}^n \frac{1}{x_i}\right)^{\frac{n}{T}+1} \mathbf{I}_{[x_{(1)},\infty)}(x_{(1)})}.
\end{aligned}$$

Note that:

$$T = \log\left(\frac{\prod_{i=1}^n X_i}{(x_{(1)})^n}\right) \quad \Rightarrow \quad \frac{\prod_{i=1}^n X_i}{(x_{(1)})^n} = e^T.$$

Thus:

$$\begin{aligned}
\lambda(x) &= \left(\frac{T}{n}\right)^n \frac{e^{-T}}{e^{-n}} \\
&= \left(\frac{T}{n}\right)^n e^{-T+n} \\
&= \lambda(T).
\end{aligned}$$

The LRT rejects if $\lambda(T) \le c$. Taking the derivative:

$$\frac{d}{dT} \log \lambda(T) = \frac{d}{dT}\left(n \log \frac{T}{n} - T + n\right) = n\frac{1}{T} - 1 = \frac{n}{T} - 1 \equiv \star$$

Thus: - $\star = 0$ if $T = n$, - $\star \le 0$ if $T > n$ (decreasing), - $\star \ge 0$ if $T < n$ (increasing). Hence, $\lambda(T) \le c$ is equivalent to:

$$T \le c_1 \quad \text{or} \quad T \ge c_2$$

for some constants c_1 and c_2.

(c) Show that, under H_0, $2T$ has a chi-squared distribution, and find the number of degrees of freedom. *(Hint: Obtain the joint distribution of the $n-1$ nontrivial terms $X_i/(\min_i X_i)$ conditional on $\min_i X_i$. Put these $n-1$ terms together, and notice that the distribution of T given $\min_i X_i$ does not depend on $\min_i X_i$, so it is the unconditional distribution of T.)*
We will not use the hint, although the problem can be solved that way. Instead, make the following three transformations. First, let $Y_i = \log X_i$, $i = 1, \ldots, n$. Next, make the n-to-1 transformation that sets $Z_1 = \min_i Y_i$ and sets $Z_2, \ldots, Z_n$ equal to the remaining Y_i's, with their order unchanged. Finally, let $W_1 = Z_1$ and $W_i = Z_i - Z_1$, $i = 2, \ldots, n$. Then you find that the W_i's are independent with $W_1 \sim f_{W_1}(w) = n\nu e^{-nw}$, $w > \log \nu$, and $W_i \sim \text{exponential}(1)$, $i = 2, \ldots, n$. Now $T = \sum_{i=2}^n W_i \sim \text{gamma}(n-1, 1)$, and hence

$$2T \sim \text{gamma}(n-1, 2) = \chi^2_{2(n-1)}.$$

8.11 (346 & 446) In Exercise 7.23 the posterior distribution of σ^2, the variance of a normal population, given S^2, the sample variance based on a sample of size n, was found using a conjugate prior for σ^2 (the inverted gamma pdf with parameters α and β). Based on observing S^2, a decision about the hypotheses

$$H_0 : \sigma \le 1 \quad \text{versus} \quad H_1 : \sigma > 1$$

is to be made.

(a) Find the region of the sample space for which $P(\sigma \le 1 \mid s^2) > P(\sigma > 1 \mid s^2)$, the region for which a Bayes test will decide that $\sigma \le 1$.
From Exercise 7.23, the posterior distribution of σ^2 given S^2 is $\mathrm{IG}(\gamma, \delta)$, where $\gamma = \alpha + (n-1)/2$ and $\delta = [(n-1)S^2/2 + 1/\beta]^{-1}$. Let $Y = 2/(\sigma^2\delta)$. Then $Y \mid S^2 \sim \mathrm{gamma}(\gamma, 2)$. (Note: If 2α is an integer, this is a $\chi^2_{2\gamma}$ distribution.) Let M denote the median of a $\mathrm{gamma}(\gamma, 2)$ distribution. Note that M depends on only α and n, not on S^2 or β. Then we have

$$P(Y \ge 2/\delta \mid S^2) = P(\sigma^2 \le 1 \mid S^2) > 1/2$$

if and only if

$$M > \frac{2}{\delta} = (n-1)S^2 + \frac{2}{\beta}, \quad \text{that is,} \quad S^2 < \frac{M - 2/\beta}{n-1}.$$

(b) Compare the region in part (a) with the acceptance region of an LRT. Is there any choice of prior parameters for which the regions agree?
From Example 7.2.11, the unrestricted MLEs are $\hat{\mu} = \bar{X}$ and $\hat{\sigma}^2 = (n-1)S^2/n$. Under H_0, $\hat{\mu}$ is still $\bar{X}$, because this was the maximizing value of μ, regardless of σ^2. Then because $L(\bar{x}, \sigma^2 \mid x)$ is a unimodal function of σ^2, the restricted MLE of σ^2 is $\hat{\sigma}^2$ if $\hat{\sigma}^2 \le 1$, and is 1 if $\hat{\sigma}^2 > 1$. So the LRT statistic is

$$\lambda(x) = \begin{cases} 1, & \text{if } \hat{\sigma}^2 \le 1, \\ (\hat{\sigma}^2)^{n/2} e^{-n(\hat{\sigma}^2 - 1)/2}, & \text{if } \hat{\sigma}^2 > 1. \end{cases}$$

We have that, for $\hat{\sigma}^2 > 1$,

$$\frac{\partial}{\partial(\hat{\sigma}^2)} \log \lambda(x) = \frac{n}{2}\left(\frac{1}{\hat{\sigma}^2} - 1\right) < 0.$$

So $\lambda(x)$ is decreasing in $\hat{\sigma}^2$, and rejecting H_0 for small values of $\lambda(x)$ is equivalent to rejecting for large values of $\hat{\sigma}^2$, that is, large values of S^2. The LRT accepts H_0 if and only if $S^2 < k$, where k is a constant. We can pick the prior parameters so that the acceptance regions match in this way. First, pick α large enough that $M/(n-1) > k$. Then, as β varies between 0 and ∞, $(M - 2/\beta)/(n-1)$ varies between $-\infty$ and $M/(n-1)$. So, for some choice of β, $(M - 2/\beta)/(n-1) = k$ and the acceptance regions match.

8.19 (346 & 446)The random variable X has pdf $f(x) = e^{-x}$, $x > 0$. One observation is obtained on the random variable $Y = X^\theta$, and a test of

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta = 2$$

needs to be constructed. Find the UMP level $\alpha = 0.10$ test and compute the Type II Error probability.
Transformation:

$$Y = g(X) = X^\theta \quad \Rightarrow \quad x = g^{-1}(y) = y^{1/\theta}$$
$$\frac{d}{dy} g^{-1}(y) = \frac{1}{\theta} y^{1/\theta - 1}$$

Thus,

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = e^{-y^{1/\theta}} \frac{1}{\theta} y^{1/\theta - 1}, \quad y > 0$$

Neyman-Pearson: Reject if

$$\frac{f(y \mid 2)}{f(y \mid 1)} > k$$

where

$$\frac{f(y \mid 2)}{f(y \mid 1)} = \frac{e^{-y^{1/2}} \frac{1}{2} y^{1/2 - 1}}{e^{-y} \cdot 1 \cdot y^{1-1}} = e^{y - y^{1/2}} \frac{1}{2} y^{-1/2} \quad \text{for some } k > 0$$

To find the form of the rejection region:
Differentiate:

$$\frac{d}{dy}\left(e^{y - y^{1/2}} \frac{1}{2} y^{-1/2}\right)$$

Applying the product rule:

$$= \frac{1}{2}\left(-e^{y-y^{1/2}}y^{-3/2} + \left(1 - \frac{1}{2}y^{-1/2}\right)e^{y-y^{1/2}}y^{-1/2}\right)$$

$$= \frac{1}{2}e^{y-y^{1/2}}y^{-3/2}\left(-\frac{1}{2} + y - \frac{y^{1/2}}{2}\right)$$

Thus, - derivative $= 0$ at $y = 1$,
- derivative < 0 for $0 < y < 1$ (decreasing),
- derivative > 0 for $y > 1$ (increasing).

Thus, $f(y \mid 2)/f(y \mid 1)$ has a minimum at $y = 1$, and is decreasing for $0 < y < 1$ and increasing for $y > 1$. Thus, the rejection region is of the form:

$$y \leq c_0 \quad \text{or} \quad y \geq c_1$$

for some constants c_0 and c_1.
Finding c_0 and c_1: Size constraint:

$$\alpha = P_{\theta=1}(\text{reject } H_0) = P_{\theta=1}(Y \leq c_0) + P_{\theta=1}(Y \geq c_1)$$

Since under $\theta = 1$, $Y \sim \text{Exp}(1)$, so:

$$P(Y \leq y) = 1 - e^{-y} \quad \text{and} \quad P(Y \geq y) = e^{-y}$$

Thus:

$$\alpha = (1 - e^{-c_0}) + e^{-c_1}$$

i.e.,

$$0.1 = 1 - e^{-c_0} + e^{-c_1}$$

Second constraint: At the cutoffs c_0 and c_1, the likelihood ratio $f(y \mid 2)/f(y \mid 1)$ must be constant (Neyman-Pearson condition), so:

$$e^{c_0 - c_0^{1/2}}\frac{1}{2}c_0^{-1/2} = e^{c_1 - c_1^{1/2}}\frac{1}{2}c_1^{-1/2}$$

Thus, two equations in two unknowns c_0 and c_1. Solve numerically (e.g., using software or iterative methods), and get:

$$c_0 \approx 0.0765, \quad c_1 \approx 3.638$$

Type II Error (under $\theta = 2$): We need:

$$P_{\theta=2}(c_0 < Y < c_1)$$

Under $\theta = 2$, the density is:

$$f(y \mid 2) = \frac{1}{2}y^{-1/2}e^{-y^{1/2}}$$

thus:

$$P_{\theta=2}(c_0 < Y < c_1) = \int_{c_0}^{c_1} \frac{1}{2}y^{-1/2}e^{-y^{1/2}}\,dy$$

Let's compute:

$$\int_{c_0}^{c_1} \frac{1}{2}y^{-1/2}e^{-y^{1/2}}\,dy = \left[-e^{-y^{1/2}}\right]_{c_0}^{c_1} = e^{-c_0^{1/2}} - e^{-c_1^{1/2}}$$

Thus, the Type II error probability is approximately:

$$0.609$$

8.20 (346 & 446)Let X be a random variable whose pmf under H_0 and H_1 is given by

x	1	2	3	4	5	6	7
$f(x \mid H_0)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$f(x \mid H_1)$	0.06	0.05	0.04	0.03	0.02	0.01	0.79

Use the Neyman–Pearson Lemma to find the most powerful test for H_0 versus H_1 with size $\alpha = 0.04$. Compute the probability of Type II Error for this test.
Neyman-Pearson:
A UMP level $\alpha = 0.04$ test rejects H_0 for x where

$$\frac{f(x \mid H_1)}{f(x \mid H_0)} > k$$

for some k, where $\alpha = P_{H_0}(\text{reject})$.

x	1	2	3	4	5	6	7
$\frac{f(x\mid\theta_1)}{f(x\mid\theta_0)}$	$\frac{0.06}{0.01} = 6$	$\frac{0.05}{0.01} = 5$	$\frac{0.04}{0.01} = 4$	$\frac{0.03}{0.01} = 3$	$\frac{0.02}{0.01} = 2$	$\frac{0.01}{0.01} = 1$	$\frac{0.79}{0.94} = 0.84$

The likelihood ratio $f(x \mid H_1)/f(x \mid H_0)$ decreases with x. Thus, rejecting for large $f(x \mid H_1)/f(x \mid H_0)$ is equivalent to rejecting for small x. Thus, reject H_0 if $x \leq c$ for some c.
Finding c:
To achieve size $\alpha = 0.04$:

$$P_{H_0}(X \leq c) = \begin{cases} 0.01 & \text{if } c = 1 \\ 0.02 & \text{if } c = 2 \\ 0.03 & \text{if } c = 3 \\ 0.04 & \text{if } c = 4 \end{cases}$$

Thus, choose $c = 4$ to get $\alpha = 0.04$. Type II Error:
Only have one θ in H_1. Type II error is:

$$P_{H_1}(\text{don't reject}) = P_{H_1}(X > 4) = P_{H_1}(X = 5 \text{ or } 6 \text{ or } 7)$$

$$= 0.02 + 0.01 + 0.79 = 0.82$$