STAT 346/446 Lecture 13

Miscellaneous stuff we did not have time to cover

Chapter 9 and Sections 10.2, 10.3, 10.4

Interval estimation

- Statements about parameters
  - Point estimation: "$\theta = W(\mathbf{x})$" (one value)
  - Hypothesis testing: "$\theta \in \Theta_0$" or "$\theta \in \Theta_0^c$"
    (Θ_0 not a function of $\mathbf{x}$)
  - Interval estimation: "$\theta \in C(\mathbf{x})$" (set or interval)

Interval estimator

An **interval estimate** of θ is any pair of functions $L(\mathbf{x})$ and $U(\mathbf{x})$ that satisfy

$$L(\mathbf{x}) \leq U(\mathbf{x}) \qquad \text{for all } \ \mathbf{x} \in \mathcal{X}$$

The *random interval* $[L(\mathbf{X}),\ U(\mathbf{X})]$ is called an **interval estimator**

- Two-sided interval: $[L(\mathbf{x}),\ U(\mathbf{x})]$
- One-sided intervals: $(-\infty, U(\mathbf{x})]$ or $[L(\mathbf{x}), \infty)$

Coverage probability and Confidence

Coverage probability

The **coverage probability** of an interval estimator $[L(\mathbf{X}),\ U(\mathbf{X})]$ is the probability that it covers the true value of the parameter θ. That is:

$$P_\theta(\theta \in [L(\mathbf{X}),\ U(\mathbf{X})])$$

Confidence coefficient

The **confidence coefficient** of an interval estimator $[L(\mathbf{X}),\ U(\mathbf{x})]$ is the *smallest* coverage probability. That is:

$$1 - \alpha = \inf_\theta P_\theta(\theta \in [L(\mathbf{X}),\ U(\mathbf{X})])$$

- The interval is usually called a **confidence interval**

Example: Normal Model

- Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$
- The usual $1-\alpha$ confidence interval

$$\left[\overline{X} - t_{n-1,\alpha/2}\frac{S}{\sqrt{n}},\ \overline{X} + t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}\right]$$

  is an interval estimator of μ

- Coverage probability:

$$\begin{aligned} &P_\mu\left(\mu \in \left[\overline{X} - t_{n-1,\alpha/2}\frac{S}{\sqrt{n}},\ \overline{X} + t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}\right]\right) \\ =&P_\mu\left(\overline{X} - t_{n-1,\alpha/2}\frac{S}{\sqrt{n}} \le \mu \le \overline{X} + t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}\right) \\ =&P_\mu\left(-t_{n-1,\alpha/2} \le \frac{\overline{X}-\mu}{S/\sqrt{n}} \le t_{n-1,\alpha/2}\right) = 1-\alpha \end{aligned}$$

- Confidence coefficient: $\inf_\mu(1-\alpha) = 1-\alpha$

Example: Uniform Model

- Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{Uniform}(0, \theta)$
- For some constants a and b with $1 \leq a < b$

$$[aX_{(n)},\ bX_{(n)}]$$

  is an interval estimator of θ

- Coverage probability:

$$P_\theta\left(\theta \in [aX_{(n)}, bX_{(n)}]\right) = P_\theta\left(aX_{(n)} \leq \theta \leq bX_{(n)}\right) = P_\theta\left(\frac{1}{a} \leq \frac{X_{(n)}}{\theta} \leq \frac{1}{a}\right)$$

  By deriving the pdf of $T = X_{(n)}/\theta$ (which does not depend on θ) we find that the coverage probability is $\left(\frac{1}{a}\right)^n - \left(\frac{1}{b}\right)^n$

- Confidence coefficient: $\left(\frac{1}{a}\right)^n - \left(\frac{1}{b}\right)^n$

Methods of Finding Interval Estimators

- Inverting a Test Statistic
  - Section 9.2.1
- Pivotal Quantities
  - Sections 9.2.2 and 9.2.3
- Bayesian Interval = **credible interval**
  - Section 9.2.4

Inverting a Test Statistic

Theorem - Inverting a Test Statistic

- From test to interval:
  - For any $\theta_0 \in \Theta$ let $A(\theta_0)$ be the *acceptance region* of a level α test of $H_0 : \theta = \theta_0$.
  - For each $\mathbf{x} \in \mathcal{X}$ define $C(\mathbf{x}) = \{\theta : \mathbf{x} \in A(\theta)\}$

  Then $C(\mathbf{X})$ is a $1 - \alpha$ confidence set

- From interval to test:
  - Let $C(\mathbf{X})$ be a $1 - \alpha$ confidence set.
  - For any $\theta_0 \in \Theta$ let $A(\theta_0) = \{\mathbf{x} : \theta_0 \in C(\mathbf{x})\}$

  Then $A(\theta_0)$ is the acceptance region of a level α test of $H_0 : \theta = \theta_0$.

Example: Normal model

- Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$
- The two-sided t-test has acceptance region

$$A(\mu_0) = \left\{ \mathbf{x} \in \mathcal{X} : -t_{n-1,\alpha/2} \leq \frac{\overline{x} - \mu_0}{s/\sqrt{n}} \leq t_{n-1,\alpha/2} \right\}$$

- Set

$$\begin{aligned} C(\mathbf{x}) &= \{\mu_0 : \mathbf{x} \in A(\mu_0)\} \\ &= \left\{ \mu_0 : -t_{n-1,\alpha/2} \leq \frac{\overline{x} - \mu_0}{s/\sqrt{n}} \leq t_{n-1,\alpha/2} \right\} \\ &= \{\mu_0 : \overline{x} - t_{n-1,\alpha/2} s/\sqrt{n} \leq \mu_0 \leq \overline{x} + t_{n-1,\alpha/2} s/\sqrt{n}\} \end{aligned}$$

- By theorem

$$C(\mathbf{X}) = \left[\overline{X} - t_{n-1,\alpha/2} \frac{S}{\sqrt{n}},\ \overline{X} + t_{n-1,\alpha/2} \frac{S}{\sqrt{n}} \right]$$

  is a $1 - \alpha$ confidence set

Inverting a Test Statistic

- Inverting a two-sided test gives a two-sided interval
- Inverting a one-sided test gives a one-sided interval
- Converting a test statistic can in some cases be quite involved - see examples 9.2.3 and 9.2.5

Pivotal Quantities

Pivotal Quantity

A random variable $Q(\mathbf{X}, \theta)$ is a **pivotal quantity (pivot)** if the distribution of $Q(\mathbf{X}, \theta)$ is independent of all parameters.

- For a random sample $X_1, \ldots, X_n$ from $N(\mu, \sigma^2)$ both

$$Q(\mathbf{X}, \mu) = \frac{\overline{X} - \mu}{S/\sqrt{n}} \qquad \text{and} \qquad Q(\mathbf{X}, \sigma^2) = \frac{(n-1)S^2}{\sigma^2}$$

  are pivotal quantities

Pivot

- Find a and b such that

$$P(a \leq Q(\mathbf{X}, \theta) \leq b) \geq 1 - \alpha$$

Note that a and b will *not* depend on θ since $Q(\mathbf{X}, \theta)$ is a pivot

- Then the acceptance region for a level α test of $H_0 : \theta = \theta_0$ is

$$A(\theta_0) = \{\mathbf{x} : a \leq Q(\mathbf{x}, \theta_0) \leq b\}$$

- Then set

$$C(\mathbf{x}) = \{\theta_0 : a \leq Q(\mathbf{x}, \theta_0) \leq b\}$$

Then $C(\mathbf{X})$ is a $1 - \alpha$ confidence set for θ

Evaluating Interval Estimators

- Want large coverage probability
  - Control by setting the confidence coefficient
- Want small sets, i.e. *short* intervals

Bayesian Intervals

Credible Set

Let $\pi(\theta \mid \mathbf{x})$ be the posterior distribution for θ. A set $A \subset \Theta$ for which

$$P(\theta \in A \mid \mathbf{x}) = 1 - \alpha$$

is a $1 - \alpha$ **credible set** for θ

- Very easy to obtain (if you have the posterior distribution)
- Different interpretation from confidence intervals

Approximate tests and confidence intervals

Sections 10.3 and 10.4

- Can use CLT (+ Slutsky, etc) to come up with approximate tests based on a normal approximation
- Remember that MLEs are (usually) approximately normal
- Find asymptotic variance using Fisher information (as in Cramer-Rao Lower bound)

CLT based

- **Wald test** for either one or two-sided hypotheses

$$\begin{aligned} H_0 &: \theta = \theta_0 & H_1 &: \theta \neq \theta_0 \\ \text{or } H_0 &: \theta \leq \theta_0 & H_1 &: \theta > \theta_0 \\ \text{or } H_0 &: \theta \geq \theta_0 & H_1 &: \theta < \theta_0 \end{aligned}$$

  is based on a test statistic of the form

$$Z_n = \frac{W_n - \theta_0}{S_n}$$

  where W_n is an estimator of θ and S_n is the standard error of W_n

- Example: Tests for proportions in intro stats

Approximate LRTs

- Can usually easily construct and evaluate the test statistic

$$\lambda(\mathbf{x}) = \frac{\sup_{\Theta_0} L(\theta \mid \mathbf{x})}{\sup_{\Theta} L(\theta \mid \mathbf{x})}$$

  even if the (constrained) maximization is via numerical methods

- Problem: Detemining a sampling distribution so that we can choose c such that

$$\sup_{\Theta_0} P_\theta(\lambda(\mathbf{X}) \leq c) \leq \alpha$$

- Under some regularity assumptions we have

$$-2\log(\lambda(\mathbf{X})) \xrightarrow{D} \chi^2_\nu$$


That's a
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