Stat 346/446: Theoretical Statistics II: Homework 6 Solutions

Textbook Exercises

8.25 Show that each of the following families has an MLR.

(a) (346 : 1 pts, 446: 0.5 pts) $n(\theta, \sigma^2)$ family with σ^2 known
For $\theta_2 > \theta_1$,

$$\frac{g(x \mid \theta_2)}{g(x \mid \theta_1)} = \frac{e^{-(x-\theta_2)^2/2\sigma^2}}{e^{-(x-\theta_1)^2/2\sigma^2}} = e^{(\theta_2-\theta_1)x/\sigma^2} \cdot e^{(\theta_1^2-\theta_2^2)/(2\sigma^2)}.$$

Because $\theta_2 - \theta_1 > 0$, the ratio is increasing in x. So the families of $\mathcal{N}(\theta, \sigma^2)$ have MLR.

(b) (346 : 1 pts, 446: 0.5 pts) Poisson(θ) family
For $\theta_2 > \theta_1$,

$$\frac{g(x \mid \theta_2)}{g(x \mid \theta_1)} = \frac{e^{-\theta_2}\theta_2^x/x!}{e^{-\theta_1}\theta_1^x/x!} = \left(\frac{\theta_2}{\theta_1}\right)^x e^{\theta_1-\theta_2},$$

which is increasing in x because $\theta_2/\theta_1 > 1$. Thus the Poisson(θ) family has an MLR.

(c) (346 : 1 pts, 446: 0.5 pts) binomial(n, θ) family with n known
For $\theta_2 > \theta_1$,

$$\frac{g(x \mid \theta_2)}{g(x \mid \theta_1)} = \frac{\binom{n}{x}\theta_2^x(1-\theta_2)^{n-x}}{\binom{n}{x}\theta_1^x(1-\theta_1)^{n-x}} = \left(\frac{\theta_2(1-\theta_1)}{\theta_1(1-\theta_2)}\right)^x \left(\frac{1-\theta_2}{1-\theta_1}\right)^n.$$

Both $\theta_2/\theta_1 > 1$ and $(1-\theta_1)/(1-\theta_2) > 1$. Thus the ratio is increasing in x, and the family has MLR. (Note: You can also use the fact that an exponential family $h(x)c(\theta)\exp(w(\theta)x)$ has MLR if $w(\theta)$ is increasing in θ (Exercise 8.27). For example, the Poisson(θ) pmf is $e^{-\theta}\exp(x \log\theta)/x!$, and the family has MLR because $\log\theta$ is increasing in θ.)

8.28 Let $f(x \mid \theta)$ be the logistic location pdf

$$f(x \mid \theta) = \frac{e^{x-\theta}}{(1+e^{x-\theta})^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

(a) (446 : 0.5 pts.) Show that this family has an MLR.
For $\theta_2 > \theta_1$, the likelihood ratio is

$$\frac{f(x \mid \theta_2)}{f(x \mid \theta_1)} = e^{\theta_1-\theta_2}\left[\frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}}\right]^2.$$

The derivative of the quantity in brackets is

$$\frac{d}{dx}\frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}} = \frac{e^{x-\theta_1}-e^{x-\theta_2}}{(1+e^{x-\theta_2})^2}.$$

Because $\theta_2 > \theta_1$, $e^{x-\theta_1} > e^{x-\theta_2}$, and hence, the ratio is increasing. This family has MLR.

(b) (446 : 1 pts.) Based on one observation, X, find the most powerful size α test of $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. For $\alpha = 0.2$, find the size of the Type II Error.

Because this is a one-observation problem and we have MLR, the Neyman-Pearson Lemma gives that the most powerful test of size α rejects H_0 when:

$$\frac{f(x \mid 1)}{f(x \mid 0)} > k \quad \text{for some } k.$$

Using the ratio:

$$\frac{f(x \mid 1)}{f(x \mid 0)} = e^{-1}\left(\frac{1+e^{x}}{1+e^{x-1}}\right)^2,$$

which increases in x, so the test rejects when $x > x^*$. Let's find the cutoff x^* such that:

$$\alpha = P_{H_0}(\text{Reject } H_0) = P_{\theta=0}(X > x^*) = 0.2.$$

The CDF of logistic($\theta = 0$) is:

$$F(x \mid 0) = \frac{1}{1+e^{-x}}.$$

So:

$$P(X > x^* \mid \theta = 0) = 1 - F(x^*) = \frac{e^{-x^*}}{1+e^{-x^*}} = \frac{1}{1+e^{x^*}} = 0.2.$$

Solving:

$$\frac{1}{1+e^{x^*}} = 0.2 \Rightarrow 1 + e^{x^*} = 5 \Rightarrow e^{x^*} = 4 \Rightarrow x^* = \log 4 \approx 1.386.$$

To find **Type II error** (β) under $\theta = 1$:

$$\beta = P_{H_1}(\text{Fail to reject } H_0) = P_{\theta=1}(X \le x^*) = F(x^* \mid 1) = \frac{1}{1+e^{-(x^*-1)}}.$$

So:

$$\beta = \frac{1}{1+e^{-(\log 4 - 1)}} = \frac{1}{1+e^{-0.386}} \approx \frac{1}{1+0.679} \approx \frac{1}{1.679} \approx 0.595.$$

(c) (446 : 1 pts.) Show that the test in part (b) is UMP size α for testing $H_0 : \theta \le 0$ versus $H_1 : \theta > 0$. What can be said about UMP tests in general for the logistic location family?
Since the logistic location family has a monotone likelihood ratio (MLR) in x, the Karlin–Rubin theorem applies. By that theorem, a most powerful test for a simple null vs simple alternative becomes UMP for composite null

$$H_0 : \theta \le 0 \quad \text{vs} \quad H_1 : \theta > 0$$

when MLR holds.

8.31 Let $X_1, \ldots, X_n$ be iid Poisson(λ).

(a) (346 : 2 pts, 446 : 1 pts.) Find a UMP test of $H_0 : \lambda \le \lambda_0$ versus $H_1 : \lambda > \lambda_0$.
By the Karlin-Rubin Theorem, the UMP test is to reject H_0 if $\sum_i X_i > k$, because $\sum_i X_i$ is sufficient and $\sum_i X_i \sim$ Poisson($n\lambda$) which has MLR. Choose the constant k to satisfy:

$$P\left(\sum_i X_i > k \mid \lambda = \lambda_0\right) = \alpha.$$

(b) (346 & 446 : 1 pts.) Consider the specific case $H_0 : \lambda \le 1$ versus $H_1 : \lambda > 1$. Use the Central Limit Theorem to determine the sample size n so a UMP test satisfies

$$P(\text{reject } H_0 \mid \lambda = 1) = 0.05 \quad \text{and} \quad P(\text{reject } H_0 \mid \lambda = 2) = 0.9.$$

$$P\left(\sum_i X_i > k \,\middle|\, \lambda = 1\right) \approx P\left(Z > \frac{k-n}{\sqrt{n}}\right) \stackrel{\text{set}}{=} 0.05,$$

$$P\left(\sum_i X_i > k \,\middle|\, \lambda = 2\right) \approx P\left(Z > \frac{k-2n}{\sqrt{2n}}\right) \stackrel{\text{set}}{=} 0.90.$$

Thus, solve for k and n in:

$$\frac{k-n}{\sqrt{n}} = 1.645 \quad \text{and} \quad \frac{k-2n}{\sqrt{2n}} = -1.28.$$

Yielding $n = 12$ and $k = 17.70$.

8.37 Let $X_1, \ldots, X_n$ be a random sample from a $\mathcal{N}(\theta, \sigma^2)$ population. Consider testing

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0.$$

(a) (346 & 446 : 1 pts.) If σ^2 is known, show that the test that rejects H_0 when

$$\bar{X} > \theta_0 + z_\alpha \sqrt{\frac{\sigma^2}{n}}$$

is a test of size α. Show that the test can be derived as an LRT.

$$P\left(\bar{X} > \theta_0 + z_\alpha \sigma/\sqrt{n} \,\middle|\, \theta_0\right) = P\left(\frac{(\bar{X}-\theta_0)}{\sigma/\sqrt{n}} > z_\alpha \,\middle|\, \theta_0\right) = P(Z > z_\alpha) = \alpha, \quad Z \sim N(0,1).$$

Because $\bar{x}$ is the unrestricted MLE, and the restricted MLE is θ_0 if $\bar{x} \leq \theta_0$, the LRT statistic is, for $\bar{x} \geq \theta_0$:

$$\lambda(x) = \frac{(2\pi\sigma^2)^{-n/2} e^{-\sum_i (x_i - \theta_0)^2/2\sigma^2}}{(2\pi\sigma^2)^{-n/2} e^{-\sum_i (x_i - \bar{x})^2/2\sigma^2}} = \frac{e^{-[n(\bar{x}-\theta_0)^2 + (n-1)s^2]/2\sigma^2}}{e^{-(n-1)s^2/2\sigma^2}} = e^{-n(\bar{x}-\theta_0)^2/2\sigma^2}.$$

And the LRT statistic is 1 for $\bar{x} < \theta_0$. Thus, rejecting if $\lambda < c$ is equivalent to rejecting if

$$(\bar{x} - \theta_0)/(\sigma/\sqrt{n}) > c' \quad \text{(as long as } c < 1\text{, see Exercise 8.24).}$$

(b) (346 & 446 : 1 pts.) Show that the test in part (a) is a UMP test.
This is a one-parameter exponential family with $\bar{X}$ as a sufficient statistic. The Karlin–Rubin theorem applies, since:

- $\bar{X}$ is sufficient and complete.
- The family has monotone likelihood ratio (MLR) in $\bar{X}$.

So the test is UMP of size α for testing one-sided alternatives.

(c) (346 & 446 : 2 pts.) If σ^2 is unknown, show that the test that rejects H_0 when

$$\bar{X} > \theta_0 + t_{n-1,\alpha} \sqrt{\frac{S^2}{n}}$$

is a test of size α. Show that the test can be derived as an LRT.

$$P\left(\bar{X} > \theta_0 + t_{n-1,\alpha} S/\sqrt{n} \,\middle|\, \theta = \theta_0\right) = P\left(T_{n-1} > t_{n-1,\alpha}\right) = \alpha,$$

where T_{n-1} is a Student's t random variable with $n-1$ degrees of freedom. If we define $\hat{\sigma}^2 = \frac{1}{n}\sum(x_i - \bar{x})^2$ and $\hat{\sigma}_0^2 = \frac{1}{n}\sum(x_i - \theta_0)^2$, then for $\bar{x} \geq \theta_0$ the LRT statistic is

$$\lambda = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}\right)^{n/2}.$$

Writing $\hat{\sigma}^2 = \frac{n-1}{n}s^2$ and $\hat{\sigma}_0^2 = (\bar{x} - \theta_0)^2 + \frac{n-1}{n}s^2$, it is clear that the LRT is equivalent to the t-test because $\lambda < c$ when

$$\frac{n-1}{n}s^2 \Bigg/ \left[(\bar{x} - \theta_0)^2 + \frac{n-1}{n}s^2\right] < c' \quad \text{and} \quad \bar{x} \geq \theta_0,$$

which is the same as rejecting when $(\bar{x} - \theta_0)/(s/\sqrt{n})$ is large.