Stat 346/446: Theoretical Statistics II: Homework 2 Solutions

Textbook Exercises

5.32 (346 & 446: 3 pts.)Let $X_1, X_2, \ldots$ be a sequence of random variables that converges in probability to a constant a. Assume that $P(X_i > 0) = 1$ for all i.

(a) (346 & 446: 2 pts.) Verify that the sequences defined by $Y_i = \sqrt{X_i}$ and $Y_i' = a/X_i$ converge in probability.

For any $\epsilon > 0$,

$$\begin{aligned} P(|\sqrt{X_n} - \sqrt{a}| > \epsilon) &= P\left(|\sqrt{X_n} - \sqrt{a}||\sqrt{X_n} + \sqrt{a}| > \epsilon|\sqrt{X_n} + \sqrt{a}|\right) \\ &= P\left(|X_n - a| > \epsilon|\sqrt{X_n} + \sqrt{a}|\right) \\ &\leq P\left(|X_n - a| > \epsilon\sqrt{a}\right) \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Thus, $\lim_{n\to\infty} P(|\sqrt{X_n} - \sqrt{a}| > \epsilon) = 0 \implies \sqrt{X_n} \xrightarrow{p} \sqrt{a}$.
For any $\epsilon > 0$,

$$\begin{aligned} P\left(\left|\frac{a}{X_n} - 1\right| \leq \epsilon\right) &= P\left(\frac{a}{1+\epsilon} \leq X_n \leq \frac{a}{1-\epsilon}\right) \\ &= P\left(a - \frac{a\epsilon}{1+\epsilon} \leq X_n \leq a + \frac{a\epsilon}{1-\epsilon}\right) \\ &= P\left(|X_n - a| \leq \frac{a\epsilon}{1+\epsilon}\right) \to 1, \quad \text{as } n \to \infty. \end{aligned}$$

Thus, $\lim_{n\to\infty} P\left(\left|\frac{a}{X_n} - 1\right| \leq \epsilon\right) = 1 \implies \frac{a}{X_n} \xrightarrow{p} 1$.

(b) (346 & 446: 1 pts.) Use the results in part (a) to prove the fact used in Example 5.5.18, that $\frac{\sigma}{S_n}$ converges in probability to 1.

$$S_n^2 \xrightarrow{p} \sigma^2 \quad \text{in probability.}$$

By $\sqrt{X_n} \xrightarrow{p} \sqrt{a}$, it follows that

$$S_n = \sqrt{X_n} \xrightarrow{p} \sqrt{a} = \sigma \quad \text{in probability.}$$

By $\frac{a}{X_n} \xrightarrow{p} 1$, it implies that

$$\frac{\sigma}{S_n} \xrightarrow{p} 1 \quad \text{in probability.}$$

5.44 (346 & 446 : 4 pts.) Let $X_i, i = 1, 2, \ldots,$ be independent Bernoulli(p) random variables and let $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

(a) (346 : 2pts, 446: 1 pts.) Show that $\sqrt{n}(Y_n - p) \xrightarrow{d} N[0, p(1-p)]$ in distribution.

We have $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(p)$ and $Y_n = \frac{1}{n}\sum_{i=1}^n X_i$.

$$\mathbb{E}[X_i] = p, \quad \text{Var}[X_i] = p(1-p), \quad i = 1, \ldots, n.$$

By the Central Limit Theorem, for n large enough:

$$\sqrt{n}\frac{\left(\frac{1}{n}\sum_{i=1}^n X_i - \mathbb{E}[X_i]\right)}{\sqrt{\text{Var}[X_i]}} \xrightarrow{D} Z, \quad Z \sim N(0,1).$$

Therefore:

$$\sqrt{n}(Y_n - p) \xrightarrow{D} Z \quad \text{and} \quad \sqrt{p(1-p)}Z \sim N(0, p(1-p)).$$

Thus:

$$\sqrt{n}(Y_n - p) \xrightarrow{D} \sqrt{p(1-p)}Z \quad \text{and} \quad \sqrt{n}(Y_n - p) \sim N(0, p(1-p)).$$

(b) (346 : 2 pts, 446 : 1 pts.) Show that for $p \neq \frac{1}{2}$, the estimate of variance $Y_n(1-Y_n)$ satisfies $\sqrt{n}[Y_n(1-Y_n) - p(1-p)] \xrightarrow{d} n\left[0, (1-2p)^2 p(1-p)\right]$ in distribution.

Let $g(X) = X(1-X)$. Then $g'(X) = 1 - 2X$. The derivative $g'(X)$ exists and is not zero if $X \neq \frac{1}{2}$. As shown above:

$$\sqrt{n}(Y_n - p) \xrightarrow{D} N(0, p(1-p)).$$

Using the Delta Method:

$$\sqrt{n}\left[g(Y_n) - g(p)\right] \xrightarrow{D} W, \quad W \sim N(0, [g'(p)]^2\sigma^2),$$

where $\sigma^2 = p(1-p)$. Since $p \neq \frac{1}{2}$, we know $g'(p) \neq 0$. Thus:

$$\sqrt{n}\left[Y_n(1-Y_n) - p(1-p)\right] \xrightarrow{D} W, \quad W \sim N\left(0, (1-2p)^2 p(1-p)\right).$$

(c) (446: 2 pts.) Show that for $p = \frac{1}{2}$, $n\left[Y_n(1-Y_n) - \frac{1}{4}\right] \xrightarrow{d} -\frac{1}{4}\chi_1^2$ in distribution. (If this appears strange, note that $Y_n(1-Y_n) \leq \frac{1}{4}$, so the left-hand side is always negative. An equivalent form is $2n\left[\frac{1}{4} - Y_n(1-Y_n)\right] \xrightarrow{d} \chi_1^2$.)

$g''(X) = -2 \neq 0.$

Using the second-order Delta Method:

$$n\left[Y_n(1-Y_n) - p(1-p)\right] \xrightarrow{D} p(1-p) \cdot \frac{1}{2}g''(p) \cdot \chi_1^2.$$

$$\implies n\left[Y_n(1-Y_n) - p(1-p)\right] \xrightarrow{D} -\frac{1}{4}\chi_1^2.$$

Extra Problems

1. DS Exercise 13 from Section 6.3 (346 & 446 : 1 pts.) Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution with unknown mean θ and variance σ^2. Assuming that $\theta \neq 0$, determine the asymptotic distribution of $\overline{X}_n^3$.

We know that the sample mean $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ follows normal distribution:

$$\overline{X}_n \sim N\left(\theta, \frac{\sigma^2}{n}\right).$$

Using the Delta Method, we let the distribution of $g(\overline{X}_n)$ be $g(x) = x^3$. And we have:

$$g'(x) = 3x^2.$$

Thus the asymptotic variance of $g(\overline{X}_n)$ is:

$$\text{Var}(g(\overline{X}_n)) \approx [g'(\theta)]^2 \cdot \text{Var}(\overline{X}_n).$$

Substituting $g'(\theta) = 3\theta^2$ and $\text{Var}(\overline{X}_n) = \frac{\sigma^2}{n}$, we get:

$$\text{Var}(g(\overline{X}_n)) = (3\theta^2)^2 \cdot \frac{\sigma^2}{n} = \frac{9\theta^4\sigma^2}{n}.$$

and the mean of $g(\theta)$ is:

$$\text{mean } g(\theta) = \theta^3.$$

The asymptotic distribution of $\overline{X}_n^3$ is:

$$\overline{X}_n^3 \xrightarrow{D} N\left(\theta^3, \frac{9\theta^4\sigma^2}{n}\right).$$

2. DS Exercise 14a from Section 6.3 (346 & 446 : 2 pts.) Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution with mean 0 and unknown variance σ^2.

(a) Determine the asymptotic distribution of the statistic

$$\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right)^{-1}.$$

Let $S_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. Know that, S_n^2 is an unbiased estimator for the variance σ^2:

$$S_n^2 \sim \frac{\sigma^2}{n} \cdot \chi_n^2,$$

where χ_n^2 is a chi-squared random variable with n degrees of freedom.
As $n \to \infty$, by the Central Limit Theorem :

$$\sqrt{n}\left(S_n^2 - \sigma^2\right) \xrightarrow{D} N(0, 2\sigma^4).$$

Let $g(x) = x^{-1}$. We have:

$$g'(x) = -\frac{1}{x^2}.$$

Using the Delta Method, The mean of $g(S_n^2)$ is:

$$\text{mean } g(S_n^2) = g(\sigma^2) = (\sigma^2)^{-1}.$$

And the variance of $g(S_n^2)$ is:

$$\text{Var}\left(g(S_n^2)\right) \approx \left[g'(\sigma^2)\right]^2 \cdot \text{Var}(S_n^2),$$

where $g'(\sigma^2) = -\frac{1}{\sigma^4}$ and $\text{Var}(S_n^2) = \frac{2\sigma^4}{n}$. Thus:

$$\text{Var}\left(g(S_n^2)\right) = \frac{1}{\sigma^8} \cdot \frac{2\sigma^4}{n} = \frac{2}{n\sigma^4}.$$

Thus, the asymptotic distribution of $\left(S_n^2\right)^{-1}$ is:

$$\left(S_n^2\right)^{-1} \xrightarrow{D} N\left(\frac{1}{\sigma^2}, \frac{2}{n\sigma^4}\right).$$