Stat 346/446: Theoretical Statistics II: Practice Exercises 1 Solutions

Textbook Exercises

7.1 (346 & 446)One observation is taken on a discrete random variable X with pmf $f(x|\theta)$, where $\theta \in \{1, 2, 3\}$. Find the MLE of θ.

x	$f(x\|1)$	$f(x\|2)$	$f(x\|3)$
0	$\frac{1}{3}$	$\frac{1}{4}$	0
1	$\frac{1}{3}$	$\frac{1}{4}$	0
2	0	$\frac{1}{4}$	$\frac{1}{4}$
3	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$
4	$\frac{1}{6}$	0	$\frac{1}{4}$

For each value of x, the MLE $\hat{\theta}$ is the value of θ that maximizes $f(x|\theta)$. These values are in the following table:

x	0	1	2	3	4
$\hat{\theta}$	1	1	2 or 3	3	3

The MLE of θ is:

$$\hat{\theta} = \begin{cases} 1, & \text{if } X \in \{0, 1\}, \\ 2 \text{ or } 3, & \text{if } X = 2, \\ 3, & \text{if } X \in \{3, 4\}. \end{cases}$$

7.2a (346 & 446) Let $X_1, \dots, X_n$ be a random sample from a gamma(α, β) population.

(a) Find the MLE of β, assuming α is known.
The likelihood function is:

$$L(\beta|x) = \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)} \beta^{-\alpha} x_i^{\alpha-1} e^{-x_i/\beta} = \frac{1}{\Gamma(\alpha)^n} \beta^{-n\alpha} \left(\prod_{i=1}^{n} x_i \right)^{\alpha-1} e^{-\sum_{i=1}^{n} x_i/\beta}.$$

Taking the log-likelihood:

$$\log L(\beta|x) = -\log \Gamma(\alpha)^n - n\alpha \log \beta + (\alpha - 1) \sum_{i=1}^{n} \log x_i - \frac{\sum_{i=1}^{n} x_i}{\beta}.$$

Differentiating with respect to β:

$$\frac{\partial \log L}{\partial \beta} = -\frac{n\alpha}{\beta} + \frac{\sum_{i=1}^{n} x_i}{\beta^2}.$$

Setting the derivative to zero:

$$-\frac{n\alpha}{\beta} + \frac{\sum_{i=1}^{n} x_i}{\beta^2} = 0.$$

Solve for β:

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i}{n\alpha}.$$

To verify that this is a maximum, calculate the second derivative:

$$\frac{\partial^2 \log L}{\partial \beta^2} = \frac{n\alpha}{\beta^2} - \frac{2\sum_{i=1}^n x_i}{\beta^3}.$$

Substitute $\beta = \hat{\beta}$:

$$\left.\frac{\partial^2 \log L}{\partial \beta^2}\right|_{\beta=\hat{\beta}} = \frac{(n\alpha)^3}{(\sum_{i=1}^n x_i)^2} - \frac{2(n\alpha)^3}{(\sum_{i=1}^n x_i)^2}.$$

Simplify:

$$\left.\frac{\partial^2 \log L}{\partial \beta^2}\right|_{\beta=\hat{\beta}} = -\frac{(n\alpha)^3}{(\sum_{i=1}^n x_i)^2} < 0.$$

Since the second derivative is negative, $\hat{\beta}$ is the unique point where the derivative is zero and is a local maximum. Thus, $\hat{\beta}$ is the global maximum and the MLE.

$$\boxed{\hat{\beta} = \frac{\sum_{i=1}^n x_i}{n\alpha}}$$

7.7 (446)Let $X_1, \ldots, X_n$ be iid with one of two pdfs. If $\theta = 0$, then

$$f(x|\theta) = \begin{cases} 1, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

While if $\theta = 1$, then

$$f(x|\theta) = \begin{cases} \frac{1}{2\sqrt{x}}, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the MLE of θ.

The likelihood function for $\theta = 0$ is:

$$L(0|\mathbf{x}) = \prod_{i=1}^n f(x_i|\theta = 0).$$

Since $f(x_i|\theta = 0) = 1$ for $0 < x_i < 1$, we have:

$$L(0|\mathbf{x}) = 1, \quad \text{if } 0 < x_i < 1 \text{ for all } i.$$

The likelihood function for $\theta = 1$ is:

$$L(1|\mathbf{x}) = \prod_{i=1}^n f(x_i|\theta = 1).$$

Since $f(x_i|\theta = 1) = \frac{1}{2\sqrt{x_i}}$ for $0 < x_i < 1$, we have:

$$L(1|\mathbf{x}) = \prod_{i=1}^n \frac{1}{2\sqrt{x_i}} = \frac{1}{2^n}\prod_{i=1}^n \frac{1}{\sqrt{x_i}}.$$

To find the MLE, compare $L(0|\mathbf{x})$ and $L(1|\mathbf{x})$:

$$L(0|\mathbf{x}) \geq L(1|\mathbf{x}) \quad \text{or} \quad L(0|\mathbf{x}) < L(1|\mathbf{x}).$$

Substitute the likelihoods:

$$1 \geq \frac{1}{2^n}\prod_{i=1}^n \frac{1}{\sqrt{x_i}} \quad \text{or} \quad 1 < \frac{1}{2^n}\prod_{i=1}^n \frac{1}{\sqrt{x_i}}.$$

Simplify the inequality:

$$1 \geq \frac{1}{2^n}\prod_{i=1}^n \frac{1}{\sqrt{x_i}} \implies 2^n \geq \prod_{i=1}^n \frac{1}{\sqrt{x_i}} \implies 1 \geq \prod_{i=1}^n 2\sqrt{x_i}.$$

The MLE is:

$$\text{MLE} = \begin{cases} 0, & \text{if } 1 \geq \prod_{i=1}^n 2\sqrt{x_i}, \\ 1, & \text{if } 1 < \prod_{i=1}^n 2\sqrt{x_i}. \end{cases}$$

7.11 (346 & 446) Let $X_1, \ldots, X_n$ be iid with pdf:

$$f(x|\theta) = \theta x^{\theta-1}, \quad 0 \leq x \leq 1, \quad 0 < \theta < \infty.$$

(a) Find the MLE of θ, and show that its variance $\to 0$ as $n \to \infty$.
The pdf is:

$$f(x|\theta) = \theta x^{\theta-1}.$$

The likelihood function is:

$$L(\theta|x) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}.$$

The log-likelihood is:

$$\log L(\theta|x) = n \log \theta + (\theta - 1) \sum_{i=1}^{n} \log x_i.$$

The derivative of the log-likelihood with respect to θ is:

$$\frac{d}{d\theta} \log L = \frac{n}{\theta} + \sum_{i=1}^{n} \log x_i.$$

Setting $\frac{d}{d\theta} \log L = 0$:

$$\frac{n}{\theta} + \sum_{i=1}^{n} \log x_i = 0 \implies \hat{\theta} = \left(-\frac{1}{n} \sum_{i=1}^{n} \log x_i\right)^{-1}.$$

The second derivative is:

$$\frac{d^2}{d\theta^2} \log L = -\frac{n}{\theta^2} < 0,$$

so $\hat{\theta}$ is the MLE. To calculate the variance of $\hat{\theta}$, note that $Y_i = -\log X_i \sim \text{exponential}(1/\theta)$, so:

$$-\sum_{i=1}^{n} \log X_i \sim \text{gamma}(n, 1/\theta).$$

Thus, $\hat{\theta} = n/T$, where $T \sim \text{gamma}(n, 1/\theta)$. Using the properties of the gamma distribution:

$$\mathbb{E}\left(\frac{1}{T}\right) = \frac{\theta}{n-1}, \quad \mathbb{E}\left(\frac{1}{T^2}\right) = \frac{\theta^2}{(n-1)(n-2)}.$$

Therefore:

$$\mathbb{E}(\hat{\theta}) = \frac{n}{n-1}\theta, \quad \text{Var}(\hat{\theta}) = \frac{n^2}{(n-1)^2(n-2)}\theta^2 \to 0 \quad \text{as } n \to \infty.$$

(b) Find the method of moments estimator of θ.
Since $X \sim \text{beta}(\theta, 1)$, the mean is:

$$\mathbb{E}[X] = \frac{\theta}{\theta + 1}.$$

Equating the sample mean to the population mean:

$$\frac{\theta}{\theta + 1} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Solving for θ:

$$\tilde{\theta} = \frac{\sum_{i=1}^{n} X_i}{n - \sum_{i=1}^{n} X_i}.$$

7.22 (446) This exercise will prove the assertions in Example 7.2.16, and more. Let $X_1, \dots, X_n$ be a random sample from a $n(\theta, \sigma^2)$ population, and suppose that the prior distribution on θ is $n(\mu, \tau^2)$. Here we assume that σ^2, μ, and τ^2 are all known.

(a) Find the joint pdf of $\bar{X}$ and θ.
The joint pdf of $\bar{X}$ and θ is given by:

$$f(\bar{x}, \theta) = f(\bar{x}|\theta)\pi(\theta).$$

The sample mean $\bar{X}$ follows:

$$\bar{X}|\theta \sim n\left(\theta, \frac{\sigma^2}{n}\right),$$

with pdf:

$$f(\bar{x}|\theta) = \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left(-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2\right).$$

The prior distribution is:

$$\pi(\theta) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\left(-\frac{1}{2\tau^2}(\theta-\mu)^2\right).$$

The joint pdf of $\bar{X}$ and θ is:

$$f(\bar{x}, \theta) = f(\bar{x}|\theta)\pi(\theta),$$

substituting the expressions:

$$f(\bar{x}, \theta) = \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left(-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2\right) \cdot \frac{1}{\sqrt{2\pi\tau^2}} \exp\left(-\frac{1}{2\tau^2}(\theta-\mu)^2\right).$$

Combining:

$$f(\bar{x}, \theta) = \frac{1}{\sqrt{2\pi\sigma^2/n}\sqrt{2\pi\tau^2}} \exp\left(-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2 - \frac{1}{2\tau^2}(\theta-\mu)^2\right).$$

(b) Show that $m(\bar{x}|\sigma^2, \mu, \tau^2)$, the marginal distribution of $\bar{X}$, is $n(\mu, (\sigma^2/n) + \tau^2)$.
The joint pdf from part (a) has the exponent:

$$-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2 - \frac{1}{2\tau^2}(\theta-\mu)^2.$$

$$-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2 = -\frac{n}{2\sigma^2}(\theta^2 - 2\theta\bar{x} + \bar{x}^2),$$

$$-\frac{1}{2\tau^2}(\theta-\mu)^2 = -\frac{1}{2\tau^2}(\theta^2 - 2\theta\mu + \mu^2).$$

$$\text{Coefficient of } \theta^2: \quad -\frac{n}{2\sigma^2} - \frac{1}{2\tau^2} = -\frac{1}{2v^2},$$

where $v^2 = \frac{\sigma^2\tau^2/n}{\tau^2+\sigma^2/n}$.

$$\text{Coefficient of } \theta: \quad \frac{n\bar{x}}{\sigma^2} + \frac{\mu}{\tau^2} = \frac{\delta(x)}{v^2},$$

where $\delta(x) = \frac{\tau^2\bar{x}+(\sigma^2/n)\mu}{\tau^2+\sigma^2/n}$.

$$-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2 - \frac{1}{2\tau^2}(\theta-\mu)^2 = -\frac{1}{2v^2}(\theta-\delta(x))^2 - \frac{1}{\tau^2+\sigma^2/n}(\bar{x}-\mu)^2.$$

$$f(\bar{x}, \theta) = n(\theta, \sigma^2/n) \times n(\mu, \tau^2) = n(\delta(x), v^2) \times n(\mu, \tau^2 + \sigma^2/n).$$

- The marginal distribution of $\bar{X}$ is:

$$\bar{X} \sim n\left(\mu, \tau^2 + \frac{\sigma^2}{n}\right).$$

- The posterior distribution of $\theta|\bar{x}$ is:

$$\theta|\bar{x} \sim n\left(\delta(x), v^2\right),$$

where:

$$\delta(x) = \frac{\tau^2\bar{x} + (\sigma^2/n)\mu}{\tau^2 + \sigma^2/n}, \quad v^2 = \frac{\sigma^2\tau^2/n}{\tau^2 + \sigma^2/n}.$$

(c) Show that $\pi(\theta|\bar{x}, \sigma^2, \mu, \tau^2)$, the posterior distribution of θ, is normal with mean and variance given by (7.2.10).
From part (b) we know that:

$$f(\bar{x}, \theta) = n(\theta, \sigma^2/n) \times n(\mu, \tau^2) = n(\delta(x), v^2) \times n(\mu, \tau^2 + \sigma^2/n).$$

The marginal distribution of $\bar{X}$ is:

$$\bar{X} \sim n\left(\mu, \tau^2 + \frac{\sigma^2}{n}\right).$$

Thus the posterior distribution of $\theta|\bar{x}$ is:

$$\theta|\bar{x} \sim n\left(\delta(x), v^2\right),$$

where:

$$\delta(x) = \frac{\tau^2\bar{x} + (\sigma^2/n)\mu}{\tau^2 + \sigma^2/n}, \quad v^2 = \frac{\sigma^2\tau^2/n}{\tau^2 + \sigma^2/n}.$$

7.24 (346 & 446) Let $X_1, \ldots, X_n$ be iid Poisson(λ), and let λ have a gamma(α, β) distribution, the conjugate family for the Poisson.

(a) Find the posterior distribution of λ.
The prior for λ is:

$$\pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}.$$

For n observations, $Y = \sum_i X_i \sim \text{Poisson}(n\lambda)$. The likelihood function is:

$$f(y|\lambda) = \frac{(n\lambda)^y e^{-n\lambda}}{y!}.$$

The marginal distribution of Y is:

$$m(y) = \int_0^\infty f(y|\lambda)\pi(\lambda)d\lambda.$$

Substituting $f(y|\lambda)$ and $\pi(\lambda)$, we get:

$$\begin{aligned}
m(y) &= \int_0^\infty \frac{(n\lambda)^y e^{-n\lambda}}{y!} \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha}\lambda^{\alpha-1}e^{-\lambda/\beta}\, d\lambda \\
&= \frac{n^y}{y!\Gamma(\alpha)\beta^\alpha}\int_0^\infty \lambda^{y+\alpha-1}e^{-\lambda(n+\frac{1}{\beta})}\, d\lambda \\
&= \frac{n^y}{y!\Gamma(\alpha)\beta^\alpha}\Gamma(y+\alpha)\left(\frac{1}{n\beta+1}\right)^{y+\alpha}
\end{aligned}$$

Thus, the posterior distribution is:

$$\pi(\lambda|y) = \frac{f(y|\lambda)\pi(\lambda)}{m(y)} = \frac{\lambda^{y+\alpha-1}e^{-\lambda\frac{\beta}{n\beta+1}}}{\Gamma(y+\alpha)\left(\frac{\beta}{n\beta+1}\right)^{y+\alpha}} \sim \text{Gamma}\left(y+\alpha, \frac{\beta}{n\beta+1}\right).$$

(b) Calculate the posterior mean and variance.

$$\mathbb{E}(\lambda|y) = (y+\alpha)\frac{\beta}{n\beta+1} = \frac{\beta}{n\beta+1}y + \frac{1}{n\beta+1}(\alpha\beta).$$

$$\mathrm{Var}(\lambda|y) = (y+\alpha)\frac{\beta^2}{(n\beta+1)^2}.$$