Stat 346/446: Theoretical Statistics II: Homework 4 Solutions

Textbook Exercises

6.2 (446: 1 pts.)Let $X_1, \ldots, X_n$ be independent random variables with densities

$$f_{X_i}(x|\theta) = \begin{cases} e^{i\theta} e^{-x}, & x \geq i\theta, \\ 0, & x < i\theta. \end{cases}$$

Prove that $T = \min_i(X_i/i)$ is a sufficient statistic for θ.

By the Factorization Theorem, $T(X) = \min_i(X_i/i)$ is sufficient because the joint pdf is

$$f(x_1, \ldots, x_n|\theta) = \prod_{i=1}^{n} e^{i\theta - x_i} I_{(i\theta,+\infty)}(x_i) = \underbrace{e^{in\theta} I_{(\theta,+\infty)}(T(X))}_{g(T(X)|\theta)} \cdot \underbrace{e^{-\sum_i x_i}}_{h(x)}.$$

Where $i > 0$, and all x_is satisfy $x_i > i\theta$ if and only if $\min_i(x_i/i) > \theta$.

6.6 (346 & 446 : 1 pts.) Let $X_1, \ldots, X_n$ be a random sample from a gamma(α, β) population. Find a two-dimensional sufficient statistic for (α, β).

The joint pdf is given by

$$f(x_1, \ldots, x_n|\alpha, \beta) = \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\beta^\alpha} x_i^{\alpha-1} e^{-x_i/\beta} = \left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\sum_{i=1}^n x_i/\beta}.$$

From the joint pdf, we can identify: $g(T(X)|\alpha, \beta)$ as:

$$g(T(X)|\alpha, \beta) = \left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\sum_{i=1}^n x_i/\beta},$$

which depends on (α, β) only through: $S_1 = \sum_{i=1}^n X_i$, which appears in the exponential term, $S_2 = \prod_{i=1}^n X_i$, which appears in the power term. Thus, by the Factorization Theorem, the statistic

$$T(X) = \left(\prod_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i\right)$$

is a sufficient statistic for (α, β).

6.9 (346 & 446 : 1 pts.) For each of the following distributions let $X_1, \ldots, X_n$ be a random sample. Find a minimal sufficient statistic for θ.

(b) $f(x|\theta) = e^{-(x-\theta)}, \quad \theta < x < \infty, \quad -\infty < \theta < \infty$ (location exponential)

Note, for $X \sim$ location exponential(θ), the range depends on the parameter. Now

$$\frac{f(\mathbf{x}|\theta)}{f(\mathbf{y}|\theta)} = \frac{\prod_{i=1}^n \left(e^{-(x_i-\theta)} I_{(\theta,\infty)}(x_i)\right)}{\prod_{i=1}^n \left(e^{-(y_i-\theta)} I_{(\theta,\infty)}(y_i)\right)}$$

$$= \frac{e^{n\theta} e^{-\sum_i x_i} \prod_{i=1}^n I_{(\theta,\infty)}(x_i)}{e^{n\theta} e^{-\sum_i y_i} \prod_{i=1}^n I_{(\theta,\infty)}(y_i)}$$

$$= \frac{e^{-\sum_i x_i} I_{(\theta,\infty)}(\min x_i)}{e^{-\sum_i y_i} I_{(\theta,\infty)}(\min y_i)}.$$

To make the ratio independent of θ, we need the ratio of indicator functions independent of θ. This will be the case if and only if $\min\{x_1, \ldots, x_n\} = \min\{y_1, \ldots, y_n\}$. So $T(\mathbf{X}) = \min\{X_1, \ldots, X_n\}$ is a minimal sufficient statistic.

7.48 Suppose that $X_i, i = 1, \ldots, n$, are iid Bernoulli(p).

(a) (346 : 2 pts, 446 : 1 pts.)Show that the variance of the MLE of p attains the Cramér-Rao Lower Bound.

Let $X_1, X_2, \ldots, X_n$ be iid Bernoulli(p), meaning each X_i follows:

$$P(X_i = 1) = p, \quad P(X_i = 0) = 1 - p.$$

The likelihood function is:

$$L(p) = \prod_{i=1}^n p^{X_i} (1-p)^{1-X_i}.$$

Taking the log-likelihood:

$$\ell(p) = \sum_{i=1}^n X_i \log p + \sum_{i=1}^n (1 - X_i) \log(1-p).$$

Differentiating with respect to p:

$$\frac{d\ell}{dp} = \sum_{i=1}^n \frac{X_i}{p} - \sum_{i=1}^n \frac{1 - X_i}{1-p}.$$

Setting this to zero and solving for p:

$$\sum_{i=1}^n \frac{X_i}{p} = \sum_{i=1}^n \frac{1 - X_i}{1-p}.$$

$$\frac{\sum X_i}{p} = \frac{n - \sum X_i}{1-p}.$$

Solving for p:

$$\hat{p} = \frac{\sum X_i}{n} = \bar{X}.$$

Thus, the MLE of p is:

$$\hat{p} = \bar{X}.$$

The Fisher information is given by:

$$I(p) = -E\left[\frac{d^2\ell}{dp^2}\right].$$

From the log-likelihood:

$$\frac{d^2\ell}{dp^2} = -\sum_{i=1}^n \frac{X_i}{p^2} - \sum_{i=1}^n \frac{1 - X_i}{(1-p)^2}.$$

Taking expectation:

$$E\left[\frac{d^2\ell}{dp^2}\right] = -n\left(\frac{p}{p^2} + \frac{1-p}{(1-p)^2}\right).$$

$$= -n\left(\frac{1}{p} + \frac{1}{1-p}\right).$$

$$= -n\left(\frac{1-p+p}{p(1-p)}\right) = -n\left(\frac{1}{p(1-p)}\right).$$

Thus:

$$I(p) = \frac{n}{p(1-p)}.$$

The Cramér-Rao Lower Bound :

$$\frac{d}{dp}p = 1$$

$$\mathrm{Var}(\hat{p}) \geq \frac{1}{I(p)} = \frac{p(1-p)}{n}.$$

Since:

$$\mathrm{Var}(\hat{p}) = \mathrm{Var}(\bar{X}) = \frac{p(1-p)}{n},$$

We see that the variance of the MLE attains the CRLB.

(b) (346 & 446 : 2 pts.)For $n \geq 4$, show that the product $X_1X_2X_3X_4$ is an unbiased estimator of p^4, and use this fact to find the best unbiased estimator of p^4.

By independence,

$$E(X_1X_2X_3X_4) = \prod_i EX_i = p^4,$$

so the estimator is unbiased. Because $\sum_i X_i$ is a complete sufficient statistic, Theorems 7.3.17 and 7.3.23 imply that

$$E(X_1X_2X_3X_4 | \sum_i X_i = t)$$

is the best unbiased estimator of p^4. Evaluating this yields

$$E\left(X_1X_2X_3X_4 \middle| \sum_i X_i = t\right) = \frac{P(X_1 = X_2 = X_3 = X_4 = 1, \sum_{i=5}^n X_i = t-4)}{P(\sum_i X_i = t)}$$

$$= \frac{p^4\binom{n-4}{t-4}p^{t-4}(1-p)^{n-t}}{\binom{n}{t}p^t(1-p)^{n-t}} = \frac{\binom{n-4}{t-4}}{\binom{n}{t}},$$

for $t \geq 4$. For $t < 4$, one of the X_is must be zero, so the estimator is

$$E(X_1X_2X_3X_4 | \sum_i X_i = t) = 0.$$

Thus, the best unbiased estimator of p^4 is:

$$\frac{\binom{n-4}{t-4}}{\binom{n}{t}}, \quad \text{for } t \geq 4.$$

7.52 (346 & 446 : 1 pts.) Let $X_1, \ldots, X_n$ be iid Poisson(λ), and let $\bar{X}$ and S^2 denote the sample mean and variance, respectively. We now complete Example 7.3.8 in a different way. There we used the Cramér-Rao Bound; now we use completeness.

(a) Prove that $\bar{X}$ is the best unbiased estimator of λ without using the Cramér-Rao Theorem.
Let $X_1, X_2, \ldots, X_n \sim$ Poisson(λ). We have:

$$f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$f(X)$ is an exponential family:

$$f(x|\lambda) = e^{x\log\lambda - \lambda - \log x!}$$

where $h(x) = \frac{1}{x!}$, $c(\lambda) = e^{-\lambda}$, $w(\lambda) = \log\lambda$, $T(x) = x$. Thus, $T(X) = \sum X_i$ is a complete sufficient statistic.

$$\bar{X} = \frac{1}{n}\sum X_i = \varphi(T)$$

is a function of a complete sufficient statistic for λ. And $E(\bar{X}) = \lambda$, so $\bar{X}$ is an unbiased estimator of λ. By Lehmann-Scheffé Theorem, $\bar{X}$ is the best unbiased estimator of λ.

7.57 Let $X_1, \ldots, X_{n+1}$ be iid Bernoulli(p), and define the function $h(p)$ by

$$h(p) = P\left(\sum_{i=1}^{n} X_i > X_{n+1} \middle| p\right),$$

the probability that the first n observations exceed the $(n+1)$st.

(a) (346 & 446 : 1 pts.)Show that

$$T(X_1, \ldots, X_{n+1}) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} X_i > X_{n+1}, \\ 0, & \text{otherwise} \end{cases}$$

is an unbiased estimator of $h(p)$.

$$T(X_1, \ldots, X_{n+1}) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} X_i > X_{n+1}, \\ 0, & \text{otherwise.} \end{cases}$$

T is a Bernoulli random variable, so

$$\begin{aligned} E(T) &= 0 \cdot P(T = 0) + 1 \cdot P(T = 1) = P(T = 1) \\ &= P\left(\sum_{i=1}^{n} X_i > X_{n+1}\right) = h(p). \end{aligned}$$

Thus, T is an unbiased estimator of $h(p)$.

(b) (346 & 446 : 2 pts.)Find the best unbiased estimator of $h(p)$.
Bernoulli(p) is an exponential family:

$$f(x) = p^x(1-p)^{1-x} = (1-p)\left(\frac{p}{1-p}\right)^x = (1-p)\exp\left(x\log\frac{p}{1-p}\right).$$

Thus,

$$\sum_i t(x_i) = \sum_i X_i$$

is complete, so

$$T' = \sum_{i=1}^{n+1} X_i$$

is a complete statistic (note: use all $n+1$ observations). The estimator T is a Bernoulli random variable, so

$$\begin{aligned} \varphi(T') &= E(T|T') = P(T = 1|T') \\ &= P\left(\sum_{i=1}^{n} X_i > X_{n+1} \middle| T'\right). \end{aligned}$$

This becomes complicated because the functional form of $\varphi(T')$ depends on the value T' takes. For $T' = t$ $(t = 0, 1, \ldots, n+1)$, we have

$$\varphi(t) = P\left(\sum_{i=1}^{n} X_i > X_{n+1} \middle| T' = t\right)$$

$$= \frac{P\left(\sum_{i=1}^{n} X_i > X_{n+1}, T' = t\right)}{P(T' = t)} = *.$$

Since $T' = \sum_{i=1}^{n+1} X_i \sim \text{Binomial}(n+1, p)$, we get

$$P(T' = t) = \binom{n+1}{t} p^t (1-p)^{n+1-t}.$$

The numerator of *:

$$P\left(\sum_{i=1}^{n} X_i > X_{n+1}, \sum_{i=1}^{n+1} X_i = t\right) = **.$$

For $t = 0$, we have $\sum_{i=1}^{n} X_i = 0$ only if $X_1 = \cdots = X_n = 0$, but then X_{n+1} must also be 0, so $\sum_{i=1}^{n} X_i > X_{n+1}$ is not possible. Hence,

$$** = 0.$$

For $t > 0$, split into cases $X_{n+1} = 0$ and $X_{n+1} = 1$:

$$** = P\left(\sum_{i=1}^{n} X_i > X_{n+1}, T' = t, X_{n+1} = 0\right)$$

$$+ P\left(\sum_{i=1}^{n} X_i > X_{n+1}, T' = t, X_{n+1} = 1\right).$$

Using independence,

$$= P\left(\sum_{i=1}^{n} X_i > 0, \sum_{i=1}^{n} X_i = t, X_{n+1} = 0\right)$$

$$+ P\left(\sum_{i=1}^{n} X_i > 1, \sum_{i=1}^{n} X_i = t-1, X_{n+1} = 1\right).$$

Since $t > 0$,

$$= P\left(\sum_{i=1}^{n} X_i = t\right)(1-p) + P\left(\sum_{i=1}^{n} X_i = t-1\right)p.$$

$$= \binom{n}{t} p^t (1-p)^{n-t}(1-p) + \binom{n}{t-1} p^{t-1}(1-p)^{n-t+1} p.$$

If $t = 1$ or $t = 2$, then

$$\varphi(t) = \frac{\binom{n}{t} p^t (1-p)^{n-t+1}}{\binom{n+1}{t} p^t (1-p)^{n+1-t}} = \frac{\binom{n}{t}}{\binom{n+1}{t}}.$$

$$\binom{n}{t} = \frac{n!}{t!(n-t)!}, \quad \binom{n+1}{t} = \frac{(n+1)!}{t!(n+1-t)!},$$

so

$$\frac{\binom{n}{t}}{\binom{n+1}{t}} = \frac{n!}{t!(n-t)!} \times \frac{t!(n+1-t)!}{(n+1)!}.$$

Simplifying,

$$= \frac{(n+1-t)}{(n+1)}.$$

If $t \geq 2$, then

$$\varphi(t) = \frac{\binom{n}{t} p^t (1-p)^{n-t+1} + \binom{n}{t-1} p^t (1-p)^{n-t+1}}{\binom{n+1}{t} p^t (1-p)^{n-t+1}}.$$

$$= \frac{\frac{n!}{t!(n-t)!} + \frac{n!}{(t-1)!(n-t+1)!}}{\frac{(n+1)!}{t!(n-t)!}}.$$

$$
\begin{aligned}
&= \frac{n!}{t!(n-t)!} + \frac{n!}{(t-1)!(n-t+1)!}. \\
&= \frac{n!(n-t+1) + n!t}{t!(n-t+1)!}. \\
&= \frac{(n+1)!}{t!(n-t+1)!} = \binom{n+1}{t}.
\end{aligned}
$$

Thus,

$$
\frac{\binom{n+1}{t}}{\binom{n+1}{t}} = 1.
$$

$$
\varphi(t) = \begin{cases} 0, & \text{if } t = 0, \\ \frac{n}{n+1}, & \text{if } t = 1, \\ \frac{n-1}{n+1}, & \text{if } t = 2, \\ 1, & \text{if } t \geq 2. \end{cases}
$$