STAT 346

Theoretical Statistics II
Spring Semester 2018

Exam 3

Name: Solution

- You have 75 min to complete this exam
- Justify your answers
- Evaluate expressions as much as you can

Note: There is a table on the last page that lists pdf/pmf, mean, variance and mgf for a few distributions.

Some (possibly) useful results and definitions

- If $X_1, X_2, \ldots, X_n$ are iid Poisson(θ) then $\sum_{i=1}^n X_i \sim$ Poisson($n\theta$)

- MLE for exponential
- $\log y - y$ is a concave function with a maximum at 1

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\theta, \sigma^2)$ where σ^2 is known. We are interested in estimating the parameter θ using squared error loss.Consider the estimator $\delta(\mathbf{X}) = a\overline{X} + b$ where a and b are constants. Show that the risk function for $\delta(\mathbf{X})$ is

$$R(\theta, \delta) = a^2 \frac{\sigma^2}{n} + (b - (1-a)\theta)^2$$

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from exponential(θ). We are interested in testing

$$H_0 : \theta = \theta_0 \qquad \text{versus} \qquad H_1 : \theta \neq \theta_0$$

   (a) Find the likelihood ratio test (LRT) to test these hypotheses, and

   (b) show that it can be expressed in the following form: Reject H_0 if $\overline{x}/\theta_0 \leq c_0$ or $\overline{x}/\theta_0 \leq c_1$. (You do not have to determine the values of c_0 and c_1)

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\theta, \sigma^2)$ where σ^2 is known and let $N(\mu, \tau^2)$ be the prior distribution for θ. Then it is known that the posterior distribution of θ is $N(\tilde{\mu}, \tilde{\tau}^2)$ where

$$\tilde{\mu} = \frac{n\tau^2}{n\tau^2 + \sigma^2}\,\overline{x} + \frac{\sigma^2}{n\tau^2 + \sigma^2}\,\mu \qquad \text{and} \qquad \tilde{\tau}^2 = \frac{\tau^2\sigma^2}{n\tau^2 + \sigma^2}$$

   Find the Bayesian test procedure for the hypotheses $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta < \theta_0$ and show that is can be expressed as: Reject H_0 if $\overline{x} > c$, for some constant c.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from Poisson(θ). Use the Neyman-Pearson lemma to find the most powerful test for $H_0 : \theta = 3$ versus $H_1 : \theta = 5$ of level $\alpha = 0.05$

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from Poisson(θ). Find the uniformly most powerful test for $H_0 : \theta \leq 3$ versus $H_1 : \theta > 3$ of level $\alpha = 0.05$
   Hint: You may assume that the Poisson family has a monotone likelihood ratio (MLR)

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\theta, \sigma^2)$ where σ^2 is known. We are interested in estimating the parameter θ using squared error loss. Consider the estimator $\delta(\mathbf{X}) = a\overline{X} + b$ where a and b are constants. Show that the risk function for $\delta(\mathbf{X})$ is

$$R(\theta, \delta) = a^2 \frac{\sigma^2}{n} + (b - (1-a)\theta)^2$$

Under squared error loss the risk function is the MSE:

$$R(\theta, \delta) = E\left(\left(\theta - \delta(\underline{x})\right)^2\right) = \mathrm{Var}(\delta(\underline{x})) + \mathrm{bias}(\delta(\underline{x}))^2$$

$$= \mathrm{Var}(a\overline{X} + b) + \left(E(a\overline{X}+b) - \theta\right)^2$$

$$= a^2 \frac{\sigma^2}{n} + (a\theta + b - \theta)^2$$

$$= a^2 \frac{\sigma^2}{n} + (b - (1-a)\theta)^2$$

Or: work out $E\left((\theta - a\overline{x} - b)^2\right) = \cdots$

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution exponential(θ). We are interested in testing

$$H_0 : \theta = \theta_0 \qquad \text{versus} \qquad H_1 : \theta \neq \theta_0$$

(a) Find the likelihood ratio test (LRT) to test these hypotheses, and

(b) show that it can be expressed in the following form: Reject H_0 if $\overline{x}/\theta_0 \leq c_0$ or $\overline{x}/\theta_0 \leq c_1$. (You do not have to determine the values of c_0 and c_1)

LRT: Reject H_0 if $\lambda(\underline{x}) \leq c$ where

$$\lambda(\underline{x}) = \frac{\sup_{\theta \in \Theta_0} L(\theta \mid \underline{x})}{\sup_{\theta \in \Theta} L(\theta \mid \underline{x})} = \frac{L(\theta_0 \mid \underline{x})}{L(\hat{\theta} \mid \underline{x})}$$

where $\hat{\theta}$ is the MLE for θ

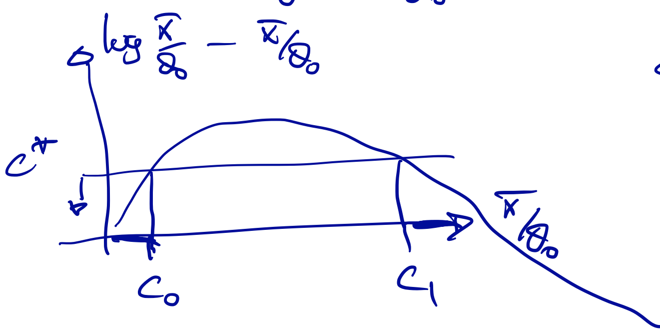
$\to$ We know that $\hat{\theta} = \overline{x}$

$$L(\theta \mid \underline{x}) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x}{\theta}} = \theta^{-n} e^{-\frac{n\overline{x}}{\theta}}$$

$$\Rightarrow \lambda(\underline{x}) = \frac{\theta_0^{-n} e^{-\frac{n\overline{x}}{\theta_0}}}{\overline{x}^{-n} e^{-n\overline{x}/\overline{x}}} = \left(\frac{\overline{x}}{\theta_0}\right)^n e^{n - n\overline{x}/\theta_0} \leq c$$

$$\Leftrightarrow \quad n \log \frac{\overline{x}}{\theta_0} + n - n\frac{\overline{x}}{\theta_0} \leq \log c$$

$$\Leftrightarrow \quad \log \frac{\overline{x}}{\theta_0} - \frac{\overline{x}}{\theta_0} \leq \frac{\log c - n}{n} = c^*$$



equivalent to saying

$\frac{\overline{x}}{\theta_0} < c_0$ or $\frac{\overline{x}}{\theta_0} > c_1$

for some c_0 and c_1

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\theta, \sigma^2)$ where σ^2 is known and let $N(\mu, \tau^2)$ be the prior distribution for θ. Then it is known that the posterior distribution of θ is $N(\tilde{\mu}, \tilde{\tau}^2)$ where

$$\tilde{\mu} = \frac{n\tau^2}{n\tau^2 + \sigma^2}\,\overline{x} + \frac{\sigma^2}{n\tau^2 + \sigma^2}\,\mu \qquad \text{and} \qquad \tilde{\tau}^2 = \frac{\tau^2\sigma^2}{n\tau^2 + \sigma^2}$$

Find the Bayesian test procedure for the hypotheses $H_0 : \theta \le \theta_0$ versus $H_1 : \theta < \theta_0$ and show that is can be expressed as: Reject H_0 if $\overline{x} > c$, for some constant c.

Reject H_0 if $P_{\theta}(\theta \le \theta_0 \mid \underline{x}) < \frac{1}{2}$

$$P(\theta \le \theta_0 / \underline{x}) = \Phi\left(\frac{\theta_0 - \tilde{\mu}}{\tilde{\tau}}\right) < \frac{1}{2}$$

$$\Rightarrow \frac{\theta_0 - \tilde{\mu}}{\tilde{\tau}} < \Phi^{-1}(0.5) = 0$$

$$\Rightarrow \theta_0 < \tilde{\mu} = \frac{n\tau^2}{n\tau^2 + \sigma^2}\,\overline{X} + \frac{\sigma^2}{n\tau^2 + \sigma^2}\,\mu$$

$$\Rightarrow \overline{X}\,\frac{n\tau^2}{n\tau^2 + \sigma^2} > \theta_0 - \frac{\sigma^2 \mu}{n\tau^2 + \sigma^2}$$

$$\Rightarrow \overline{X} > \frac{\left(\theta_0 - \frac{\sigma^2\mu}{n\tau^2 + \sigma^2}\right)(n\tau^2 + \sigma^2)}{n\tau^2} \equiv C$$

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from Poisson(θ). Use the Neyman-Pearson lemma to find the most powerful test for $H_0 : \theta = 3$ versus $H_1 : \theta = 5$ of level α ~~= 0.05~~

UMP : Reject H_0 if

(a)

$$\frac{f(5 \mid \underline{x})}{f(3 \mid \underline{x})} > k$$

$$f(\theta \mid \underline{x}) = \prod_{i=1}^{n} \frac{e^{-\theta} \theta^{x_i}}{x_i!} = e^{-n\theta} \theta^{n\bar{x}} \prod_{i=1}^{n} \frac{1}{x_i!}$$

$$\frac{f(5 \mid \underline{x})}{f(3 \mid \underline{x})} = \frac{e^{-n5}\, 5^{n\bar{x}} \prod_{i=1}^{n} \frac{1}{x_i!}}{e^{-n3}\, 3^{n\bar{x}} \prod_{i=1}^{n} \frac{1}{x_i!}} = e^{-2n} \left(\frac{5}{3}\right)^{n\bar{x}} > k$$

$$\Rightarrow \quad -2n + n\bar{x} \log(5/3) > \log k$$

$$\Rightarrow \quad n\bar{x} > \frac{\log k + 2n}{\log(5/3)} = c$$

Determine c so that we have level α

$$\alpha \geq P\left(\sum_{i=1}^{n} X_i > c\right) = 1 - P\left(\sum X_i \leq c-1\right)$$

$$= 1 - \sum_{x=0}^{c-1} e^{-n\theta} (n\theta)^x \frac{1}{x!} = *$$

$n = 20$

Note: $\sum X_i \sim \text{Poisson}(n\theta)$

(b) Find the smallest c so that

$* \leq 0.05$

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from Poisson(θ). ~~Find~~ Give the uniformly most powerful (UMP) test for $H_0 : \theta \leq 3$ versus $H_1 : \theta > 3$ of level α ~~= 0.05~~

Hint: You may assume that the Poisson family has a monotone likelihood ratio (MLR)

$T = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ

$T \sim \text{Poisson}(n\theta) \rightarrow$ a family that has MLR

$\Rightarrow$ By Karlin-Rubin, the UMP level α test rejects H_0 when $T < t_0$ where $P_{\theta_0}(T < t_0) = \alpha$

Problem	1	2	3	4	5	6	7/8	Total
Missed Score								
out of	5	8	10	5	8	4	10	50

Name	**pdf or pmf**	**Parameters**	**Mean**	**Variance**	**Mgf**
Exponential(β)	$f(x) = \frac{1}{\beta} e^{-x/\beta},\ x \geq 0$	$\beta > 0$	$\mathrm{E}(X) = \beta$	$\mathrm{Var}(X) = \beta^2$	$M_X(t) = \frac{1}{1-\beta t},\ t < \frac{1}{\beta}$
Gamma(α, β)	$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta},\ x \geq 0$	$\alpha, \beta > 0$	$\mathrm{E}(X) = \alpha\beta$	$\mathrm{Var}(X) = \alpha\beta^2$	$M_X(t) = \left(\frac{1}{1-\beta t}\right)^\alpha,\ t < \frac{1}{\beta}$
InvGamma(α, β)	$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{-\alpha-1} e^{-1/x\beta},\ x \geq 0$	$\alpha, \beta > 0$	$\mathrm{E}(X) = \frac{1}{\beta(\alpha-1)}$	$\mathrm{Var}(X) = \frac{1}{\beta^2(\alpha-1)^2(\alpha-2)}$	
			if $\alpha > 1$	if $\alpha > 2$	$M_X(t)$ does not exist
$\mathrm{N}(\mu, \sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/(2\sigma^2)},\ x \in \mathbb{R}$	$\mu \in \mathbb{R}, \sigma > 0$	$\mathrm{E}(X) = \mu$	$\mathrm{Var}(X) = \sigma^2$	$M_X(t) = e^{\mu t + \sigma^2 t^2/2}$
Uniform(a, b)	$f(x) = \frac{1}{b-a},\ a \leq x \leq b$	$a, b \in \mathbb{R},\ a < b$	$\mathrm{E}(X) = \frac{b+a}{2}$	$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$	$M_X(t) = \frac{e^{bt}-e^{at}}{(b-a)t}$
Beta(α, β)	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1},\ 0 \leq x \leq 1$	$\alpha, \beta > 0$	$\mathrm{E}(X) = \frac{\alpha}{\alpha+\beta}$	$\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
					$M_X(t) = 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k} \frac{\alpha+r}{\alpha+\beta+r}\right) \frac{t^k}{k!}$
Binomial(n, p)	$f(x) = \binom{n}{x} p^x (1-p)^{n-x},\ x = 0, 1, \ldots, n$	$n \in \mathbb{N},\ 0 \leq p \leq 1$	$\mathrm{E}(X) = np$	$\mathrm{Var}(X) = np(1-p)$	$M_X(t) = \left(pe^t + (1-p)\right)^n$
Poisson(λ)	$f(x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0, 1, 2, \ldots$	$\lambda \geq 0$	$\mathrm{E}(X) = \lambda$	$\mathrm{Var}(X) = \lambda$	$M_X(t) = e^{\lambda(e^t-1)}$