STAT 346

Theoretical Statistics II
Spring Semester 2018

Exam 2

Name: ______________________________

- You have 75 min to complete this exam
- Justify your answers
- Evaluate expressions as much as you can

Note: There is a table on the last page that lists pdf/pmf, mean, variance and mgf for a few distributions.

Some (possibly) useful results and definitions

- $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$, $n > 0$. Also $\Gamma(n) = (n-1)\Gamma(n-1)$, $\Gamma(0.5) = \sqrt{\pi}$ and $\Gamma(n) = (n-1)!$ if n is an integer
- A pmf/pdf $f(x|\boldsymbol{\theta})$ belongs to an exponential family if it can be written as

$$f(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta})\exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\right)$$

- If $X \sim \text{Beta}(\alpha, 1)$ then $Y = -\ln(X) \sim \text{Exponential}(1/\alpha)$
- If $X_1, X_2, \ldots, X_n$ are iid Exponential(α) then $\sum_{i=1}^n X_i \sim \text{Gamma}(n, \alpha)$
- If $X \sim \text{Gamma}(\alpha, \beta)$ then $Y = 1/X \sim \text{InvGamma}(\alpha, \beta)$ - see pdf on last page

For all questions on this exam you will work with a special case of the Beta distribution:

Let $X_1, X_2, \ldots, X_n$ be a random sample from Beta$(\theta, 1)$.

Note that in addition to the table in the back with information for several different distributions, the front page also contains many results that may be helpful on this exam.

1. (5 points) Use the factorization theorem to show that

   (a) $T_1 = \prod_{i=1}^{n} X_i$ is a sufficient statistic for θ.

   (b) $T_2 = \sum_{i=1}^{n} \ln(X_i)$ is a sufficient statistic for θ.

2. (8 points) Find the method of moments (MOM) estimator of θ.

3. (10 points) Show that the maximum likelihood estimator (MLE) of θ is

$$\hat{\theta}^{MLE} = -\frac{n}{\sum_{i=1}^{n} \ln(X_i)}$$

4. (5 points) Show that $W = \frac{n-1}{n}\hat{\theta}^{MLE}$ is an unbiased estimator of θ.
   Hint: To avoid some lengthy derivations, check out the front and back pages of this exam.

5. (8 points) Calculate the Cramer-Rao lower bound for W.

6. (4 points) Calculate the variance of W. Is it equal to the Cramer-Rao lower bound?
   Hint: To avoid some lengthy derivations, check out the front and back pages of this exam.

7. (5 points) Find a complete sufficient statistic for θ.
   (Hint: check whether Beta$(\theta, 1)$ is an exponential family)

8. (5 points) Is W the best unbiased estimator of θ? Use results from earlier parts to justify your answer.

Problem	1	2	3	4	5	6	7/8	Total
Missed Score								
out of	5	8	10	5	8	4	10	50

Name	pdf or pmf	Parameters	Mean	Variance	Mgf
Exponential(β)	$f(x) = \frac{1}{\beta} e^{-x/\beta},\ x \geq 0$	$\beta > 0$	$\mathrm{E}(X) = \beta$	$\mathrm{Var}(X) = \beta^2$	$M_X(t) = \frac{1}{1-\beta t},\ t < \frac{1}{\beta}$
Gamma(α, β)	$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta},\ x \geq 0$	$\alpha, \beta > 0$	$\mathrm{E}(X) = \alpha\beta$	$\mathrm{Var}(X) = \alpha\beta^2$	$M_X(t) = \left(\frac{1}{1-\beta t}\right)^\alpha,\ t < \frac{1}{\beta}$
InvGamma(α, β)	$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{-\alpha-1} e^{-1/x\beta},\ x \geq 0$	$\alpha, \beta > 0$	$\mathrm{E}(X) = \frac{1}{\beta(\alpha-1)}$	$\mathrm{Var}(X) = \frac{1}{\beta^2(\alpha-1)^2(\alpha-2)}$	
			if $\alpha > 1$	if $\alpha > 2$	$M_X(t)$ does not exist
$\mathrm{N}(\mu, \sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/(2\sigma^2)},\ x \in \mathbb{R}$	$\mu \in \mathbb{R}, \sigma > 0$	$\mathrm{E}(X) = \mu$	$\mathrm{Var}(X) = \sigma^2$	$M_X(t) = e^{\mu t + \sigma^2 t^2/2}$
Uniform(a, b)	$f(x) = \frac{1}{b-a},\ a \leq x \leq b$	$a, b \in \mathbb{R},\ a < b$	$\mathrm{E}(X) = \frac{b+a}{2}$	$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$	$M_X(t) = \frac{e^{bt}-e^{at}}{(b-a)t}$
Beta(α, β)	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1},\ 0 \leq x \leq 1$	$\alpha, \beta > 0$	$\mathrm{E}(X) = \frac{\alpha}{\alpha+\beta}$	$\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
				$M_X(t) = 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k} \frac{\alpha+r}{\alpha+\beta+r}\right) \frac{t^k}{k!}$	
Binomial(n, p)	$f(x) = \binom{n}{x} p^x (1-p)^{n-x},\ x = 0, 1, \ldots, n$	$n \in \mathbb{N},\ 0 \leq p \leq 1$	$\mathrm{E}(X) = np$	$\mathrm{Var}(X) = np(1-p)$	$M_X(t) = \left(pe^t + (1-p)\right)^n$
Poisson(λ)	$f(x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0, 1, 2, \ldots$	$\lambda \geq 0$	$\mathrm{E}(X) = \lambda$	$\mathrm{Var}(X) = \lambda$	$M_X(t) = e^{\lambda(e^t-1)}$