$X_1, \dots, X_n \sim Beta(\theta, 1)$

pdf of Beta(α, β):

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

$\alpha = \theta$, $\beta = 1$ we get

$$f(x) = \frac{\Gamma(\theta+1)}{\Gamma(\theta)\,\Gamma(1)} x^{\theta-1}(1-x)^{1-1}$$

$$= \frac{\theta\,\Gamma(\theta)}{\Gamma(\theta)} x^{\theta-1} = \theta\, x^{\theta-1}$$

1) Joint pdf:

$$f(x_1, \dots, x_n) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \theta\, x_i^{\theta-1}$$

$$= \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$$

a) set $g(t_1 | \theta) = \theta^n (t_1)^{\theta-1}$ and $h(x) = 1$

$\Rightarrow T_1 = \prod_{i=1}^{n} X_i$ is a sufficient statistic for θ

b) $T_2 = \underbrace{\ln\left(\prod_{i=1}^{n} X_i\right)}_{= \ln(T_1)} = \sum_{i=1}^{n} \ln(X_i)$ a one-to-one function of $T_1 \Rightarrow$ sufficient

or set $g(t_2 | \theta) = \theta^n (e^{t_2})^{\theta-1}$

and $h(x) = 1$

(2) $E(X) = \frac{\theta}{\theta+1}$ $\qquad m_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$

Set $m_1 = \frac{\theta}{\theta+1} \iff m_1\theta + m_1 = \theta$

$\iff \theta(m_1 - 1) = -m_1 \iff \theta = -\frac{m_1}{m_1 - 1}$

$\Rightarrow \hat{\theta}^{MOM} = -\frac{\bar{X}}{\bar{X}-1}$ (note: $\bar{X} < 1$)

(3) Likelihood function

$L(\theta) = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$

$\ell(\theta) = n \log\theta + (\theta-1)\sum_{i=1}^{n} \log x_i$

$\frac{d\,\ell(\theta)}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log x_i = 0$

$\iff \frac{n}{\theta} = -\sum_{i=1}^{n} \log x_i$

$\iff \theta = -\frac{n}{\sum_{i=1}^{n} \log x_i} = -\frac{1}{\frac{1}{n}\sum_{i=1}^{n} \log x_i}$

$\frac{d^2\ell(\theta)}{d\theta^2} = -\frac{n}{\theta^2} < 0 \quad \forall\, \theta \Rightarrow$ maximum

$\Rightarrow \hat{\theta}^{MLE} = -\frac{1}{\frac{1}{n}\sum \log X_i}$

(4) $X_i \sim \text{Beta}(\theta, 1)$

$$\Rightarrow \quad -\log(X_i) \sim \text{Expo}\left(\frac{1}{\theta}\right)$$

$$\Rightarrow \quad -\sum_{i=1}^{n} \log(X_i) \sim \text{Gamma}(n, 1/\theta)$$

$$\Rightarrow -\frac{1}{\sum_{i=1}^{n} \log(X_i)} \sim \text{Inv Gamma}(n, 1/\theta)$$

$$\Rightarrow E(\hat{\theta}^{MLE}) = E\left(\frac{-n}{\sum_{i=1}^{n} \log(X_i)}\right)$$

$$= \quad n\, E\left(-\frac{1}{\sum \log(X_i)}\right) = n \frac{1}{(n-1)\, 1/\theta}$$

$$= \theta \frac{n}{n-1} \neq \theta$$

but $E\left(\frac{n-1}{n} \hat{\theta}^{MLE}\right) = \frac{n-1}{n} \theta \frac{n}{n-1} = \theta$

5 Need Fisher Information

$$I(\theta) = -n\, E\left(\frac{d^2}{d\theta^2} \log(f(X|\theta)) \right)$$

$$f(x|\theta) = \theta\, x^{\theta-1}$$

$$\log f(x|\theta) = \log\theta + (\theta-1)\log x$$

$$\frac{d}{d\theta} \log f(x|\theta) = \frac{1}{\theta} + \log x$$

$$\frac{d^2}{d\theta^2} \log f(x|\theta) = -\frac{1}{\theta^2}$$

$$-n\, E\left(\frac{d^2}{d\theta^2} \log f(X|\theta) \right) = -n\, E\left(-\frac{1}{\theta^2} \right)$$

$$= \frac{n}{\theta^2}$$

$\Rightarrow$ Cramer-Rao lower bound is

$$\frac{\theta^2}{n} \frac{d}{d\theta} \tau(\theta)$$

(6) $\mathrm{Var}(\hat{\theta}^{MLE}) = \mathrm{Var}\left(-\frac{n}{\Sigma \ln X}\right)$

$= n^2\, \mathrm{Var}\left(-\frac{1}{\Sigma \ln(X)}\right)$ InvGamma$(n, 1/\theta)$

$= n^2 \frac{\theta^2}{(n-1)^2 (n-2)}$

$\mathrm{Var}\left(\frac{n-1}{n}\hat{\theta}^{MLE}\right) = \frac{(n-1)^2}{n^2} \frac{n^2\theta^2}{(n-1)^2(n-2)}$

$= \frac{\theta^2}{n-2} > \frac{\theta^2}{n}$

$\Rightarrow$ not efficient

(7)

$f(x|\theta) = \theta x^{\theta-1}$

$= \theta e^{\log x^{\theta-1}}$

$= \theta e^{(\theta-1)\log x}$

expo-family with $h(x) = 1$, $c(\theta) = \theta$

$w(\theta) = \theta - 1$ and $t(x) = \log x$

We know that $T = \sum_{i=1}^{n} t(X_i) = \sum_{i=1}^{n} \log(X_i)$

is a complete sufficient statistic

(not a "curved" expo family)

(8) W depends only on a complete sufficient statistic and $E(W) = \theta$

$\Rightarrow$ W is the best unbiased estimator of θ