Stat 346/446: Theoretical Statistics II: Practice Exercises 4 Solutions

Textbook Exercises

8.13 (346 & 446)Let X_1, X_2 be iid uniform$(\theta, \theta+1)$. For testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, we have two competing tests:

$$\phi_1(X_1) : \text{Reject } H_0 \text{ if } X_1 > 0.95,$$

$$\phi_2(X_1, X_2) : \text{Reject } H_0 \text{ if } X_1 + X_2 > C.$$

(a) Find the value of C so that ϕ_2 has the same size as ϕ_1.

Size of test ϕ_1:

$$\alpha_1 = P_{\theta=0}(X_1 > 0.95) = 0.05 \quad \text{since } X_1 \sim \text{Uniform}(0,1)$$

Size of test ϕ_2: The CDF of $Y = X_1 + X_2$ for X_1, X_2 i.i.d. Uniform$(\theta, \theta+1)$ is:

$$F(y) = \begin{cases} 0 & y < 2\theta \\ \frac{1}{2}(y-2\theta)^2 & 2\theta \le y < 2\theta+1 \\ 1 - \frac{1}{2}(2\theta+2-y)^2 & 2\theta+1 \le y < 2\theta+2 \\ 1 & y > 2\theta+2 \end{cases}$$

So under $H_0 : \theta = 0$,

$$\alpha_2 = P_{\theta=0}(X_1 + X_2 > c) = 1 - F(c)$$

If $c > 1$, then using the appropriate piece of the CDF:

$$F(c) = 1 - \frac{1}{2}(2-c)^2 \Rightarrow \alpha_2 = 1 - \left(1 - \frac{1}{2}(2-c)^2\right) = \frac{1}{2}(2-c)^2$$

Set $\alpha_2 = \alpha_1 = 0.05$ and solve:

$$\frac{1}{2}(2-c)^2 = 0.05 \Rightarrow (2-c)^2 = 0.1 \Rightarrow c = 2 - \sqrt{0.1} \approx 1.68$$

(b) Calculate the power function of each test. Draw a well-labeled graph of each power function.

Power Function for Test 1:

$$\begin{aligned} \beta(\theta) &= P_\theta(X_1 > 0.95) \\ &= \begin{cases} 1 & \text{if } 0.95 < \theta \\ 0 & \text{if } 0.95 > \theta + 1 \\ \theta - 0.05 & \text{if } -0.05 < \theta < 0.95 \end{cases} \end{aligned}$$

This follows from:

$$P_\theta(X_1 > 0.95) = 1 - F(0.95) = 1 - \frac{0.95 - \theta}{1} = \theta - 0.05$$

$$\beta(\theta) = \begin{cases} 0 & \text{if } \theta \le -0.05 \\ \theta + 0.05 & \text{if } -0.05 < \theta < 0.95 \\ 1 & \text{if } \theta \ge 0.95 \end{cases}$$

Test 2:

$$\beta(\theta) = P_\theta(X_1 + X_2 > c)$$

Let $Y = X_1 + X_2$, and recall the known CDF of the sum of two i.i.d. Uniform$(\theta, \theta+1)$ variables. Then:

$$\beta(\theta) = \begin{cases} 0 & \text{if } \theta < \frac{c-2}{2} \\ \frac{1}{2}(2\theta + 2 - c)^2 & \text{if } \frac{c-2}{2} \le \theta < \frac{c-1}{2} \\ 1 - \frac{1}{2}(c - 2\theta)^2 & \text{if } \frac{c-1}{2} \le \theta < \frac{c}{2} \\ 1 & \text{if } \theta \ge \frac{c}{2} \end{cases}$$

(c) Prove or disprove: ϕ_2 is a more powerful test than ϕ_1.

From the graph it is clear that ϕ_1 is more powerful for θ near 0, but ϕ_2 is more powerful for larger θs. ϕ_2 is not uniformly more powerful than ϕ_1.

8.15 (346 & 446) Show that for a random sample $X_1, \ldots, X_n$ from a $\mathcal{N}(0, \sigma^2)$ population, the most powerful test of

$$H_0 : \sigma = \sigma_0 \quad \text{versus} \quad H_1 : \sigma = \sigma_1, \quad \text{where } \sigma_0 < \sigma_1,$$

is given by

$$\phi\left(\sum X_i^2\right) = \begin{cases} 1 & \text{if } \sum X_i^2 > c \\ 0 & \text{if } \sum X_i^2 \le c \end{cases}$$

For a given value of α, the size of the Type I Error, show how the value of c is explicitly determined.

$$\text{We reject if } \sum_{i=1}^n X_i^2 > c$$

The density function for $X = (X_1, \ldots, X_n)$ under σ^2 is:

$$f(x \mid \sigma^2) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n X_i^2\right)$$

By the Neyman-Pearson Lemma, the UMP level-α test rejects H_0 if:

$$\frac{f(x \mid \sigma_0^2)}{f(x \mid \sigma_1^2)} = \left(\frac{\sigma_1^2}{\sigma_0^2}\right)^{n/2} \exp\left\{\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) \sum_{i=1}^n X_i^2\right\} > k \quad \text{for some } k \ge 0$$

Taking logarithms and solving:

$$\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) \sum_{i=1}^n X_i^2 > \log\left(k\left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{n/2}\right)$$

$$\Rightarrow \sum_{i=1}^n X_i^2 > \frac{2\log\left(k(\sigma_0^2/\sigma_1^2)^{n/2}\right)}{\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)} = c \quad \text{since } \frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0$$

To determine c, set the size of the test to α:

$$\alpha = P_{\sigma_0}(\text{reject } H_0) = P_{\sigma_0}\left(\sum_{i=1}^n X_i^2 > c\right) = P\left(\frac{1}{\sigma_0^2}\sum_{i=1}^n X_i^2 > \frac{c}{\sigma_0^2}\right)$$

Recall:

$$\frac{1}{\sigma_0^2}\sum_{i=1}^n X_i^2 \sim \chi_n^2 \Rightarrow \frac{c}{\sigma_0^2} = \chi_{n,\alpha}^2 \Rightarrow c = \sigma_0^2 \chi_{n,\alpha}^2$$

where $\chi_{n,\alpha}^2$ is the upper α quantile of the χ_n^2 distribution.

8.20 (446)Let X be a random variable whose pmf under H_0 and H_1 is given by

x	1	2	3	4	5	6	7
$f(x \mid H_0)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$f(x \mid H_1)$	0.06	0.05	0.04	0.03	0.02	0.01	0.79

Use the Neyman–Pearson Lemma to find the most powerful test for H_0 versus H_1 with size $\alpha = 0.04$. Compute the probability of Type II Error for this test.

By the Neyman-Pearson Lemma, a UMP level-$\alpha = 0.04$ test rejects H_0 for x where

$$\frac{f(x \mid H_1)}{f(x \mid H_0)} > k \quad \text{for some } k,$$

where $\alpha = P_{H_0}(\text{reject})$.

x	1	2	3	4	5	6	7
$\frac{f(x\mid H_1)}{f(x\mid H_0)}$	6	5	4	3	2	1	0.84

This likelihood ratio decreases with x, so we should reject for small x, i.e., reject if $x \le c$ for some c. To find c such that the test has size $\alpha = 0.04$:

$$P_{H_0}(X \le c) = \begin{cases} 0.01 & \text{if } c = 1 \\ 0.02 & \text{if } c = 2 \\ 0.03 & \text{if } c = 3 \\ 0.04 & \text{if } c = 4 \end{cases} \Rightarrow \text{Choose } c = 4 \text{ to get } \alpha = 0.04$$

Type II error: Only one θ in H_1 is considered, so:

$$P_{H_1}(\text{don't reject}) = P_{H_1}(X > 4) = P_{H_1}(X = 5, 6, 7) = 0.02 + 0.01 + 0.79 = 0.82$$

DS Section 9.1 Exercises

1 (346 & 446) Let X have the exponential distribution with parameter β. Suppose that we wish to test the hypotheses

$$H_0 : \beta \ge 1 \quad \text{versus} \quad H_1 : \beta < 1.$$

Consider the test procedure δ that rejects H_0 if $X \le 1$.

(a) Determine the power function of the test.

Let δ be the test that rejects H_0 when $X \ge 1$. The power function of δ is

$$\pi(\beta \mid \delta) = \Pr(X \ge 1 \mid \beta) = \exp(-\beta),$$

for $\beta > 0$.

(b) Compute the size of the test.

The size of the test δ is $\sup_{\beta \ge 1} \pi(\beta \mid \delta)$. Using the answer to part (a), we see that $\pi(\beta \mid \delta)$ is a decreasing function of β, hence the size of the test is $\pi(1 \mid \delta) = \exp(-1)$.

2 (346 & 446) Suppose that $X_1, \ldots, X_n$ form a random sample from the uniform distribution on the interval $[0, \theta]$, and that the following hypotheses are to be tested:

$$H_0 : \theta \ge 2, \quad H_1 : \theta < 2.$$

Let $Y_n = \max\{X_1, \ldots, X_n\}$, and consider a test procedure such that the critical region contains all the outcomes for which $Y_n \le 1.5$.

(a) Determine the power function of the test.

We want the power function:

$$\pi(\theta) = P_\theta(Y_n \le 1.5)$$

Since the X_i are independent and uniformly distributed on $[0, \theta]$, we have:

$$P(Y_n \le y) = P(X_1 \le y, \ldots, X_n \le y) = \left(\frac{y}{\theta}\right)^n \quad \text{for } 0 \le y \le \theta$$

So, for our test:

$$\pi(\theta) = \begin{cases} \left(\frac{1.5}{\theta}\right)^n & \text{if } \theta \ge 1.5 \\ 1 & \text{if } \theta < 1.5 \end{cases}$$

(b) Determine the size of the test.

The size of the test is the maximum probability of rejecting H_0 when H_0 is true (i.e., when $\theta \ge 2$):

$$\text{Size} = \sup_{\theta \ge 2} \pi(\theta)$$

Since $\pi(\theta) = \left(\frac{1.5}{\theta}\right)^n$ is decreasing in θ for $\theta \ge 1.5$, the maximum occurs at the smallest value under H_0, which is $\theta = 2$:

$$\text{Size} = \left(\frac{1.5}{2}\right)^n = \left(\frac{3}{4}\right)^n$$

8 (346 & 446) Assume that $X_1, \ldots, X_n$ are i.i.d. with the normal distribution that has mean μ and variance 1. Suppose that we wish to test the hypotheses

$$H_0 : \mu \le \mu_0, \quad H_1 : \mu > \mu_0.$$

Consider the test that rejects H_0 if $Z \ge c$, where Z is defined in Eq. (9.1.10).

(a) Show that $\Pr(Z \ge c \mid \mu)$ is an increasing function of μ.

The distribution of Z given μ is the normal distribution with mean $n^{1/2}(\mu - \mu_0)$ and variance 1. We can write:

$$\Pr(Z \ge c \mid \mu) = 1 - \Phi\left(c - n^{1/2}(\mu - \mu_0)\right) = \Phi\left(n^{1/2}\mu - n^{1/2}\mu_0 - c\right).$$

Since Φ is an increasing function and $n^{1/2}\mu - n^{1/2}\mu_0 - c$ is an increasing function of μ, the power function is an increasing function of μ.

(b) Find c to make the test have size α_0.

The size of the test will be the power function at $\mu = \mu_0$, since μ_0 is the largest value in Ω_0 and the power function is increasing. Hence, the size is $\Phi(-c)$. If we set this equal to α_0, we can solve for c:

$$c = -\Phi^{-1}(\alpha_0).$$

13 (346 & 446) Let X have the Poisson distribution with mean θ. Suppose that we wish to test the hypotheses

$$H_0 : \theta \leq 1.0, \quad H_1 : \theta > 1.0.$$

Let δ_c be the test that rejects H_0 if $X \geq c$. Find c to make the size of δ_c as close as possible to 0.1 without being larger than 0.1.

Let δ_c be the test that rejects H_0 if $X \geq c$, for some integer c. We aim to choose c such that the size of the test,

$$\alpha = P_{\theta=1}(X \geq c),$$

is as close as possible to 0.1 but not greater than 0.1. We use the fact that for a Poisson random variable with mean 1,

$$P(X \geq c) = \sum_{x=c}^{\infty} \frac{e^{-1}1^x}{x!}$$

Using values from a Poisson distribution table:

c	$P_{\theta=1}(X \geq c)$
2	0.264
3	0.080
4	0.019

We observe that:

$$P_{\theta=1}(X \geq 2) = 0.264 > 0.1$$
$$P_{\theta=1}(X \geq 3) = 0.080 \leq 0.1$$
$$P_{\theta=1}(X \geq 4) = 0.019 \ll 0.1$$

The best choice is: $c = 3$
which gives a test size of $\alpha = 0.080$
This is the closest possible to 0.1 without exceeding it.