STAT 346

Theoretical Statistics II
Spring Semester 2018

Exam 1

Name: Solution

- You have 75 min to complete this exam
- Justify your answers
- Evaluate expressions as much as you can

Note: There is a table on the last page that lists pdf/pmf, mean, variance and mgf for a few distributions.

1. (6 points) Let $X_1, X_2, \ldots, X_n$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$. Show that $\overline{X} \sim \mathrm{N}(\mu, \sigma^2/n)$.
Hint: consider using mgf's.

We know that

$$M_{\overline{X}}(t) = \left(M_X\left(\frac{t}{n}\right)\right)^n$$

and mgf for $N(\mu, \sigma^2)$ is

$$M_X(t) = e^{\mu t + \sigma^2 t^2/2}$$

$$\Rightarrow M_{\overline{X}}(t) = \left(e^{\mu t/n + \sigma^2 t^2/2n^2}\right)^n$$

$$= e^{n\mu t/n + n\sigma^2 t^2/2n^2}$$

$$= e^{\mu t + \frac{\sigma^2}{n} t^2/2}$$

$$= \text{mgf of } N(\mu, \sigma^2/n)$$

2. (8 points) Let $X_1, X_2, \ldots, X_9$ be a random sample from Uniform$(0,1)$. Derive the pdf for the 4th order statistic, $X_{(4)}$, and identify the name and parameter values of that distribution.

Have $f(x) = 1$ and $F(x) = x$ for $x \in (0,1)$

In general:

$$f_{(j)}(x) = \frac{n!}{(j-1)!\,(n-j)!} f(x) \left(F(x)\right)^{j-1} \left(1 - F(x)\right)^{n-j}$$

Have $n = 9$ and $j = 4$

$$f_{(4)}(x) = \frac{9!}{3!\,5!} * 1 * x^3 (1-x)^5 \quad \text{for } x \in (0,1)$$

$$= \frac{\Gamma(6+4)}{\Gamma(4)\Gamma(6)} x^{4-1} (1-x)^{6-1}$$

= pdf of the Beta(4, 6) distribution

3. Let X_1, X_2, X_3, be a random sample from N(0, 4) and let Y_1, Y_2, Y_3, Y_4, be a random sample from N(2, 9). Also assume that $\{X_1, X_2, X_3\}$ are independent of $\{Y_1, Y_2, Y_3, Y_4\}$. Determine the distribution of the following random variables. Remember to justify your answers.

(a) (5 points) $\quad U_1 = \frac{3}{4}\overline{X}^2 + \frac{4}{9}(\overline{Y} - 2)^2$

$$\bar{X} \sim N(0, 4/3) \Rightarrow \frac{\bar{X}}{\sqrt{4/3}} \sim N(0,1)$$

$$\Rightarrow \frac{X^2}{4/3} = \frac{3}{4}\bar{X}^2 \sim \chi^2_1$$

$$\bar{Y} \sim N(2, 9/4) \Rightarrow \frac{\bar{Y}-2}{\sqrt{9/4}} \sim N(0,1)$$

$$\Rightarrow \frac{(\bar{Y}-2)^2}{9/4} = \frac{4}{9}(\bar{Y}-2)^2 \sim \chi^2_1$$

$\bar{X}$ and $\bar{Y}$ independent

$$\Rightarrow \frac{3}{4}\bar{X}^2 + \frac{4}{9}(\bar{Y}-2)^2 \sim \chi^2_2$$

$$\Rightarrow U_1 \sim \chi^2_2$$

(b) (5 points) $$U_2 = \frac{4(\overline{Y}-2)}{\sqrt{3\sum_{i=1}^{3} X_i^2}}$$

$$\frac{2}{3}(\overline{Y}-2) \sim N(0,1)$$

$$X_i \sim N(0,4) \Rightarrow \frac{X_i}{2} \sim N(0,1) \Rightarrow \frac{X_i}{4} \sim \chi_1^2$$

$$\Rightarrow \sum_{i=1}^{3} \frac{X_i^2}{4} \sim \chi_3^2$$

$$\Rightarrow \frac{\frac{2}{3}(\overline{Y}-2)}{\sqrt{\frac{1}{4}\sum_{i=1}^{3} X_i^2 / 3}} \sim t_3$$

$$= \frac{2(\overline{Y}-2)}{\frac{1}{2}\sqrt{\frac{1}{3}\sum_{i=1}^{3} X_i^2}} = \frac{4(\overline{Y}-2)}{\sqrt{\frac{1}{3}\sum_{i=1}^{3} X_i^2}}$$

$$\Rightarrow U_2 \sim t_3$$

(c) (5 points) $U_3 = \dfrac{3\sum_{i=1}^{3} X_i^2}{\sum_{i=1}^{4}(Y_i - 2)^2}$

$$\frac{1}{4}\sum_{i=1}^{3} X_i \sim \chi^2_3$$

$$\frac{Y_i - 2}{3} \sim N(0,1) \Rightarrow \frac{(Y_i - 2)^2}{9} \sim \chi^2_1$$

$$\Rightarrow \frac{1}{9}\sum_{i=1}^{4}(Y_i - 2)^2 \sim \chi^2_4$$

$$\Rightarrow \frac{\frac{1}{4}\sum_{i=1}^{3} X_i \,/3}{\frac{1}{9}\sum_{i=1}^{4}(Y_i-2)^2/4} \sim F_{3,4}$$

$$= \frac{\frac{1}{4\cdot 3}\sum_{i=1}^{3} X_i}{\frac{1}{9\cdot 4}\sum_{i=1}^{4}(Y_i-2)^2} = \frac{3\sum_{i=1}^{3} X_i}{\sum_{i=1}^{4}(Y_i-2)^2}$$

$$\Rightarrow U_3 \sim F_{3,4}$$

4. (6 points) Let $X_1, X_2, \ldots, X_n$ be a random sample from Gamma$(\theta, 2)$. Use the Central Limit Theorem and other theorems or corollaries to find the asymptotic (limiting) distribution of the following statistic

$$\frac{\sqrt{n}(\overline{X}_n - 2\theta)}{\sqrt{\overline{X}_n}}$$

$E(X_i) = \theta 2 \qquad Var(X_i) = \theta 2^2$

By Central Limit Theorem:

$$\frac{\sqrt{n}(\overline{X}_n - 2\theta)}{\sqrt{4\theta}} \xrightarrow{d} Z \sim N(0,1)$$

$$\Rightarrow \sqrt{n}(\overline{X}_n - 2\theta) \xrightarrow{d} \sqrt{4\theta}\, Z \sim N(0, 4\theta)$$

By law of large numbers $\overline{X}_n \xrightarrow{P} 2\theta$

$$\Rightarrow \sqrt{\overline{X}} \xrightarrow{P} \sqrt{2\theta} \Rightarrow \frac{1}{\sqrt{\overline{X}}} \xrightarrow{P} \frac{1}{\sqrt{2\theta}}$$

By Slutsky:

$$\frac{\sqrt{n}(\overline{X}_n - 2\theta)}{\sqrt{\overline{X}}} \xrightarrow{d} \frac{1}{\sqrt{2\theta}}\sqrt{4\theta}\, Z = \sqrt{2}\, Z \sim N(0,2)$$

$\Rightarrow$ Limiting distribution is $N(0,2)$

5. (6 points) Again, let $X_1, X_2, \ldots, X_n$ be a random sample from Gamma$(\theta, 2)$. Use the Central Limit Theorem and other theorems or corollaries to find the asymptotic (limiting) distribution of the following statistic

$\sqrt{n}(\overline{X}_n^2 - 4\theta^2)$

CLT: $\sqrt{n}(\overline{X}_n - 2\theta) \xrightarrow{d} \sqrt{4\theta}\, Z \sim N(0, 4\theta)$

set $g(x) = x^2$. By delta method we get

$$\sqrt{n}\,(g(\overline{X}_n) - g(2\theta)) \xrightarrow{d} N(0, 4\theta\, g'(2\theta)^2)$$

$$g'(x) = 2x \Rightarrow g'(2\theta) = 2 \cdot 2\theta = 4\theta$$

$$\Rightarrow \sqrt{n}\,(\overline{X}_n^2 - 4\theta^2) \xrightarrow{d} N(0, 4\theta \cdot 4^2\theta^2)$$

$$= N(0, 4^3\theta^3)$$

6. (9 points) Let X be a random variable and $X_1, X_2, X_3, \ldots$ be a sequence of random variables. Define in mathematical notation what the following statements mean.

(a) $X_n \xrightarrow{D} X$, i.e. X_n converges to X in distribution as $n \to \infty$

$$\lim_{n \to \infty} F_{X_n}(x) \to F_X(x) \quad \text{for all } x \text{ where}$$
$$F(x) \text{ is continuous}$$

(b) $X_n \xrightarrow{P} X$, i.e. X_n converges to X in probability as $n \to \infty$

$$\lim_{n \to \infty} P(|X_n - X| \overset{<}{\cancel{\geq}} \varepsilon) = 1 \quad \text{for all } \varepsilon > 0$$

$$\text{or } \lim_{n \to \infty} P(|X_n - X| \geq \varepsilon) = 0 \quad \forall \varepsilon > 0$$

(c) $X_n \xrightarrow{\text{a.s.}} X$, i.e. X_n converges to X almost surely as $n \to \infty$

$$P(\lim_{n \to \infty} |X_n - X| \overset{<}{\cancel{\geq}} \varepsilon) = 1 \quad \text{for all } \varepsilon > 0$$

Problem	1	2	3	4	5	6	Total
Missed Score							
out of	6	8	15	6	6	9	50

Name	pdf	Parameters	Mean	Variance	Mgf
Exponential(β)	$f(x) = \frac{1}{\beta}e^{-x/\beta},\ x \geq 0$	$\beta > 0$	$\mathrm{E}(X) = \beta$	$\mathrm{Var}(X) = \beta^2$	$M_X(t) = \frac{1}{1-\beta t},\ t < \frac{1}{\beta}$
Gamma(α, β)	$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta},\ x \geq 0$	$\alpha, \beta > 0$	$\mathrm{E}(X) = \alpha\beta$	$\mathrm{Var}(X) = \alpha\beta^2$	$M_X(t) = \left(\frac{1}{1-\beta t}\right)^\alpha,\ t < \frac{1}{\beta}$
N(μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}e^{-(x-\mu)^2/(2\sigma^2)},\ x \in \mathbb{R}$	$\mu \in \mathbb{R}, \sigma > 0$	$\mathrm{E}(X) = \mu$	$\mathrm{Var}(X) = \sigma^2$	$M_X(t) = e^{\mu t + \sigma^2 t^2/2}$
Uniform(a, b)	$f(x) = \frac{1}{b-a},\ a \leq x \leq b$	$a, b \in \mathbb{R},\ a < b$	$\mathrm{E}(X) = \frac{b+a}{2}$	$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$	$M_X(t) = \frac{e^{bt}-e^{at}}{(b-a)t}$
Beta(α, β)	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1},\ 0 \leq x \leq 1$	$\alpha, \beta > 0$	$\mathrm{E}(X) = \frac{\alpha}{\alpha+\beta}$	$\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
				$M_X(t) = 1 + \sum_{k=1}^{\infty}\left(\prod_{r=0}^{k}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$	
Binomial(n, p)	$f(x) = \binom{n}{x}p^x(1-p)^{n-x},\ x = 0, 1, \ldots, n$	$n \in \mathbb{N},\ 0 \leq p \leq 1$	$\mathrm{E}(X) = np$	$\mathrm{Var}(X) = np(1-p)$	$M_X(t) = \left(pe^t + (1-p)\right)^n$
Poisson(λ)	$f(x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0, 1, 2, \ldots$	$\lambda \geq 0$	$\mathrm{E}(X) = \lambda$	$\mathrm{Var}(X) = \lambda$	$M_X(t) = e^{\lambda(e^t-1)}$