Computation of Tangent, Euler, and Bernoulli Numbers*

Abstract

Some elementary methods are described which may be used to calculate tangent numbers, Euler numbers, and Bernoulli numbers much more easily and rapidly on electronic computers than the traditional recurrence relations which have been used for over a century. These methods have been used to prepare an accompanying table which extends the existing tables of these numbers. Some theorems about the periodicity of the tangent numbers, which were suggested by the tables, are also proved.

Introduction

The tangent numbers T_n, Euler numbers E_n, and Bernoulli numbers B_n, are defined to be the coefficients in the following power series:

$$(1)\ \tan z = T_0/0! + T_1 z/1! + T_2 z^2/2! + \cdots = \sum_{n\geq 0} T_n z^n/n!\,,$$

$$(2)\ \sec z = E_0/0! + E_1 z/1! + E_2 z^2/2! + \cdots = \sum_{n\geq 0} E_n z^n/n!\,,$$

$$(3)\ z/(e^z - 1) = B_0/0! + B_1 z/1! + B_2 z^2/2! + \cdots = \sum_{n\geq 0} B_n z^n/n!\,.$$

Formulas for Computation.

$$B_m = 1 - \sum_{k=0}^{m-1} \binom{m}{k} \frac{B_k}{m-k+1}$$

$$T_n = \frac{2^{2n}(2^{2n}-1)\,|\,B_{2n}\,|}{2n}$$

$$E_n = 1 - \sum_{k=0}^{n-1} \left[E_k \cdot 2^{(n-1-k)} \cdot \binom{n}{k} \right]$$

Details of the Computation

6	.	1	6	,	4	0	,	2	4	.	,	8	,

↑ P ↑ Q

Here *P* and *Q* represent variables in the program that point to the current places of interest in the memory; *P* points to the number that will be accessed next, and *Q* points to the place where the next value is to be written. Only locations from *P* to *Q* contain information that will be used subsequently by the program. The symbols "." and "," represent special negative codes in the table which delimit the numbers in an obvious fashion. As we begin the calculation for *n = 5*, we set area *A* to zero and a variable *k* to 1. The basic cycle is then

- **(a)** Set area *B* to k times the next value indicated by *P*, and move *P* to the right.
- **(b)** Store the value of *A + B* into the locations indicated by *Q*, and move *Q* to the right.
- **(c)** Transfer the contents of *B* to area *A*.
- **(d)** Increase fc by *2*.

6	.	1	6	,	4	0	,	2	4	.	1	6	,

↑ Q ↑ P

k = 3 *A = 16* *B = 16*

Value table Bernoulli Numbers Tangent, Euler

Tangent Number		Euler Numbers		Bernoulli Numbers
n	T_n	n	E_n	$B_0 = 1$
1	1.	0	1.	$B_1 = -1/2$
3	2.	2	1.	$B_2 = 1/6$
5	16.	4	5.	$B_4 = -1/30$
7	272.	6	61.	$B_6 = 1/42$
9	7936.	8	1385.	$B_8 = -1/30$
11	353792.			
13	22368256.			

Addendum

If I were writing this paper today, I would mention the convergent sum

$$T_n + E_n = \frac{2^{n+2}n!}{\pi^{n+1}}\left(1 + \left(-\frac{1}{3}\right)^{n+1} + \left(\frac{1}{5}\right)^{n+1} + \left(-\frac{1}{7}\right)^{n+1} + \cdots\right)$$

instead of merely stating the first term in T_n's asymptotics. And I'd cite D. H. Lehmer's paper "Lacunary recurrence formulas for the numbers of Bernoulli and Euler," *Annals of Mathematics* (2) **36** (1935), 637–649.

References

1. Thomas Clausen, "Theorem," Asir. Nachrichten, v. 17, 1840, cols. 351-352.
2. S. A. Joffe, "Calculation of the first thirty-two Eulerian numbers from central differences of zero," Quart.J. Math., v. 47, 1916,pp. 103-126.
3. S. A. Joffe, "Calculation of eighteen more, fifty in all, Eulerian numbers from central differencesof zero," Quart. J. Math., v. 48, 1917-1920,pp. 193-271
4. D. H. Lehmee, "An extension of the table of Bernoulli numbers," Duke Math. J., v. 2, 1936,pp. 460-464.
5. Niels Nielsen, Traité Élémentaire des Nombres de Bernoulli, Paris, 1923.
6. J. Peters & J. Stein, Zehnstellige Logarithmentafel, Berlin, 1922
7. S. Z. Serebrennikoff, "Tables des premiers quatre vingt dix nombres de Bernoulli," Mém. Acad. St. Petersbourg 8, v. 16, 1905, no. 10, pp. 1-8.
8. K. G. C. von Staudt, "Beweis eines Lehrsatzes die Bernoullischen Zahlen betreffend," J.für Math., v. 21, 1840,pp. 372-374.