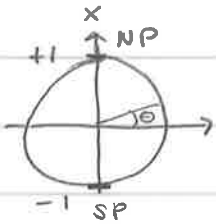
# Numerical Solution of 1-D-energy balance

## Differential equation

$$C \frac{\partial T}{\partial t} = \frac{S}{4}(1-\alpha) - (A + BT) + \frac{\partial}{\partial x}\left(D\,(1-x^2)\frac{\partial T}{\partial x}\right)$$

$= \text{Funktion}(x, t)$

$x = \sin \Theta$




$S = S(x,t) = \frac{Q_0}{4} \underbrace{(1 - 0{,}241 \cdot (3x^2 - 1))}_{f(x)}$ , $Q_0$ = Solar Const.

$T = T(x,t)$

$\alpha = \alpha(x,t)$

$D = D(x)$

$x = -1, \ldots, +1$ $\quad x_i = -1 + (i-1)\cdot \Delta x$ , $i = 1, \ldots, I_{max}$

$t = 0, \ldots, t_{max}$ $\quad t_j = 0 + (j-1)\cdot \Delta t$ , $j = 1, \ldots, J_{max}$.

We use the notation: $T_i^j = T(x_i, t_j)$.

The equation is solved by assuming an initial T $\underline{T}^0 = (T_1^0, \ldots, T_{IMAX}^0)$ and calcutate $\underline{T}(t)$ at each time-step.

## Boundary Conditions:

Set up the difference equation:

Set up the equation to calculate $T_i^{j+1}$

$$\frac{\partial T(x_i, t_j)}{\partial t} = \frac{T_i^{j+1} - T_i^j}{\Delta t}$$

$$\left[\frac{\partial}{\partial x}\left(D(1-x^2)\frac{\partial T}{\partial x}\right)\right]_{i,j} = \frac{D_{i+1}^j (1-x_{i+1}^2)\left(\frac{\partial T}{\partial x}\right)_{i+\frac{1}{2},j} - D_i^j (1-x_i^2)\left(\frac{\partial T}{\partial x}\right)_{i-\frac{1}{2},j}}{\Delta x}$$

$$= \frac{D_{i+1}^j (1-x_{i+1}^2)\frac{T_{i+1}-T_i}{\Delta x} - D_i^j (1-x_i^2)\frac{T_i - T_{i-1}}{\Delta x}}{\Delta x}$$

$$= T_{i+1} \cdot \frac{D_{i+1}^j (1-x_{i+1}^2)}{(\Delta x)^2} + T_i \frac{1}{(\Delta x)^2}\left(-D_{i+1}^j (1-x_{i+1}^2) - D_i^j (1-x_i^2)\right)$$
$$+ T_{i-1} \frac{D_{i-1}^j (1-x_{i-1}^2)}{(\Delta x)^2}$$
$$= T_{i+1}\, lam_{i+1}^j + T_i(-lam_{i+1}^j - lam_i^j) + T_{i-1}\, lam_{i-1}^j$$

We solve the equations by the $\boxed{\text{explicit}}$ method:

$$C\frac{\partial T}{\partial t}(x_i, t_j) = \text{Funktion}(x_i, t_j)$$

$$\Downarrow$$

$$C\frac{T_i^{j+1} - T_i^j}{\Delta t} = \frac{1}{4}Q_0 f_i (1-\alpha_i^j) - (A + BT_i^j) + T_{i+1}\, lam_{i+1}^j$$
$$+ T_i(-lam_{i+1}^j - lam_i^j) + T_{i-1}\, lam_{i-1}^j$$

$$\Updownarrow$$

$$T_i^{j+1} = \Delta t \underbrace{\boxed{\frac{1}{C}\left(\frac{1}{4}Q_0 f_i (1-\alpha_i^j) - A\right)}}_{\text{"} src_i^j}$$
$$+ \Delta t \cdot T_{i+1}^j \cdot \underbrace{\boxed{\frac{1}{C}\, lam_{i+1}^j}}_{\text{"} N_{i,i+1}^j} + \Delta t\, T_{i-1}^j \underbrace{\boxed{\frac{1}{C}\, lam_i^j}}_{\text{"} N_{i,i-1}^j}$$
$$+ \Delta t\, T_i^j \underbrace{\boxed{\frac{1}{C}(-B - lam_{i+1}^j - lam_i^j)}}_{\text{"} N_{i,i}^j} + T_i^j$$

$$\Rightarrow \underline{\underline{T}}^{j+1} = \Delta t \cdot \underbrace{\left(\underline{src}^j + \underline{\underline{N}} \cdot \underline{T}^j\right)}_{\underline{h}^j} + \underline{T}^j$$

Let us now solve the equations by the $\boxed{\text{implicit}}$ method:

$$C \cdot \frac{\partial T(x_i, t_j)}{\partial t} = \frac{1}{2}\left(\text{Funktion}(x_i, t_j) + \text{Funktion}(x_i, t_{j+1})\right)$$

i.e. Funktion is now calculated as an average between the values at current time $t_j$ and the next timestep $t_{j+1}$

$\Downarrow$

Explicit: $\frac{1}{\Delta t}\left(\underline{T}^{j+1} - \underline{T}^{j}\right) = \underline{src}^{j} + \underline{\underline{N}} \cdot \underline{T}^{j}$ (written as vectors)

Implicit: $\frac{1}{\Delta t}\left(\underline{T}^{j+1} - \underline{T}^{j}\right) = \frac{1}{2}\left(src^{j+1} + src^{j}\right) + \frac{1}{2}\underline{\underline{N}}\left(\underline{T}^{j+1} + \underline{T}^{j}\right)$ (written as vectors)

$\Leftrightarrow$

$$-\left(\frac{1}{2}\underline{\underline{N}} - \frac{1}{\Delta t}\cdot\underline{\underline{I}}\right)\underline{T}^{j+1} = \frac{1}{2}\underline{src}^{j+1} + \frac{1}{2}\underline{src}^{j} + \frac{1}{2}\underline{\underline{N}}\cdot\underline{T}^{j} + \frac{1}{\Delta t}\underline{T}^{j}$$
$$= \frac{1}{2}\left(\underline{src}^{j+1} + \underline{h}^{j}\right) + \frac{1}{\Delta t}\underline{T}^{j}$$

matrix with 1 in the diagonal

$\Updownarrow$

$$\underline{T}^{j+1} = -\text{inv}\underbrace{\left(\frac{1}{2}\underline{\underline{N}} - \frac{1}{\Delta t}\underline{\underline{I}}\right)}_{\underline{\underline{M}}} \cdot \left(\frac{1}{2}\left(\underline{src}^{j+1} + \underline{h}^{j}\right) + \frac{1}{\Delta t}\underline{T}^{j}\right)$$

$$\underline{T}^{j+1} = -\text{inv}\left(\underline{\underline{M}}\right) \cdot \left(\frac{1}{2}\left(\underline{src}^{j+1} + \underline{h}^{j}\right) + \frac{1}{\Delta t}\cdot\underline{T}^{j}\right)$$

For the program

D is a constant in time, this implies that

$\underline{\underline{N}}^{j+1} = \underline{\underline{N}}$ the matrix is constant in time.

$\underline{\underline{M}}^{j+1} = \underline{\underline{M}}$ — " —

## Program Flow Chart:

Define parameters jmx, scaleQ, A, B, Dmag, Toffset, Cl, delt, NMAX, delx, Q

Calculate $\underline{x}$, $\underline{phi}$, $\underline{S}$,

Calculate initial parameters $\underline{T}^0$, $\underline{alb}^0$, $\underline{src}^0$, $\underline{\underline{N}}$, $\underline{h}^0 = \underline{\underline{N}} \cdot \underline{T}^0 + \underline{scr}^0$, $Tglob^0$

timestepping loop. $j = 1, NMAX$

Calculate for $j$: $alb^j$, $scr^j$

$\underline{T}^j = -inv(\underline{\underline{M}}) \cdot \left(\frac{1}{2}(h^{j-1} + scr^j) + \frac{T^{j-1}}{\Delta t}\right)$

$h^j = \underline{\underline{N}} \cdot \underline{T}^j + scr^{j-1}$

$Tglob^j = mean(\underline{T}^j)$

Check $Tglob^j - Tglob^{j-1} < 1 \cdot 10^{-12}$

No

yes

$inv(\underline{\underline{M}}) = inv\left(\frac{1}{2}\underline{\underline{N}} - \frac{1}{\Delta t}\underline{\underline{I}}\right)$

$\underline{\underline{N}}$

Are calculated in setupfastM.m