Course 02402 Introduction to Statistics Lecture 4:

Confidence interval for mean (and standard deviation)

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Agenda

Oversigt

Example - heights:

Sample, $n = 10$:

168	161	167	179	184	166	198	187	191	179

Sample mean and standard deviation:

$$\bar{x} = 178$$
$$s = 12.21$$

Estimate population mean and standard deviation:

$$\hat{\mu} = 178$$
$$\hat{\sigma} = 12.21$$

NEW:**Confidence interval, μ**:

$$178 \pm 2.26 \cdot \frac{12.21}{\sqrt{10}} \Leftrightarrow [169.3;\ 186.7]$$

NEW:**Confidence interval, σ**:

$$[8.4;\ 22.3]$$

Oversigt

Let's simulate the key challenge of statistics!

```r
## Mean
mu <- 178
## Standard deviation
sigma <- 12
## Sample size
n <- 10
## Simulate  normally distributed  X_i
x <- rnorm(n=n, mean=mu, sd=sigma)
## See the realizations
x
## Empirical density
hist(x, prob=TRUE, col='blue')
## Find the sample mean
mean(x)
##  Find the sample variance
var(x)
## Repeat the simulated sampling many times
mat <- replicate(100, rnorm(n=n, mean=mu, sd=sigma))
## Find the sample mean for each of them
xbar <- apply(mat, 2, mean)
## Now we have many realizations of the sample mean
xbar
## See their distribution
hist(xbar, prob=TRUE, col='blue')
## There mean
mean(xbar)
## and sample variance
var(xbar)
```


Theorem 3.2: The distribution of the mean of normal random variables

(Sample-) Distribution/ The (sampling) distribution for $\bar{X}$

Assume that $X_1,\ldots,X_n$ are independent and identically normally distributed random variables, $X_i \sim N(\mu,\sigma^2), i=1,\ldots,n$, then:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Mean and variance follow from 'rules': (Theorem 2.56)

The Mean of $\bar{X}$

$$E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n} \mu = \frac{1}{n} n\mu = \mu$$

The variance of $\bar{X}$

$$\text{Var}(\bar{X}) = \frac{1}{n^2}\sum_{i=1}^{n} \text{Var}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

We now know the distribution of the error we make:

(When using $\bar{x}$ as an estimate of μ)

The standard deviation of $\bar{X}$

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

The standard deviation of $(\bar{X} - \mu)$

$$\sigma_{(\bar{X}-\mu)} = \frac{\sigma}{\sqrt{n}}$$

Standardized version of the same thing, Theorem 3.3:

Distribution for the standardized error we make:

Assume that $X_1, \ldots, X_n$ are independent and identically normally distributed random variables, $X_i \sim N\left(\mu, \sigma^2\right)$ where $i = 1, \ldots, n$, then:

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N\left(0, 1^2\right)$$

That is, the standardized sample mean Z follows a standard normal distribution.

Practical problem in all this, so far:

How to transform this into a specific interval for μ?

When the populations standard deviation σ is in all the formulas?

Obvious solution:

Use the estimtate s in stead of σ in formulas!

BUT BUT:

The given theory then breaks down!!

Luckily:

We have en extended theory to handle it for us!!

Theorem 3.4: More applicable extension of the same stuff: (copy of Theorem 2.89)

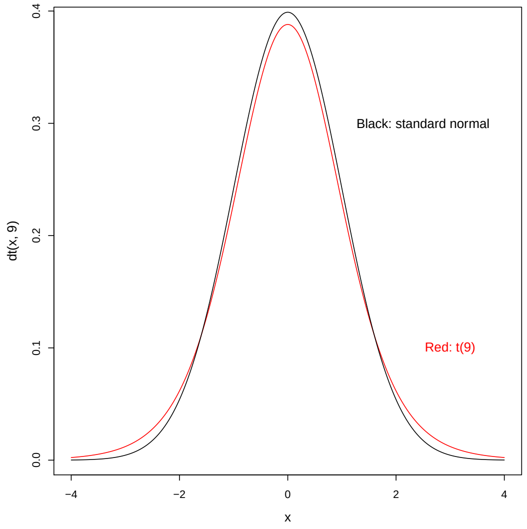
The t-Distribution takes the uncertainty of s into account:

Assume that $X_1, \ldots, X_n$ are independent and identically normally distributed random variables, where $X_i \sim N\left(\mu, \sigma^2\right)$ and $i = 1, \ldots, n$, then:

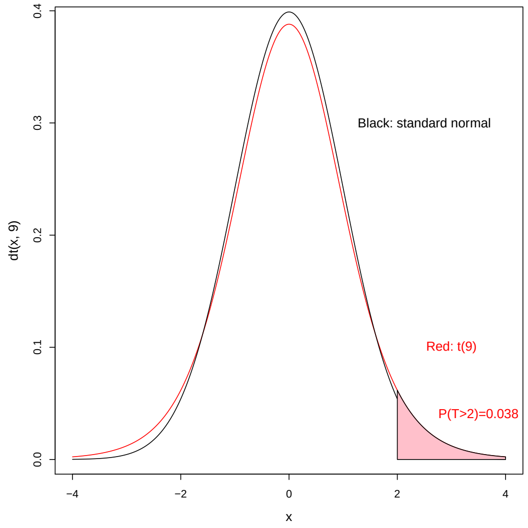
$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t$$

where t is the t-distribution with $n - 1$ degrees of freedom.

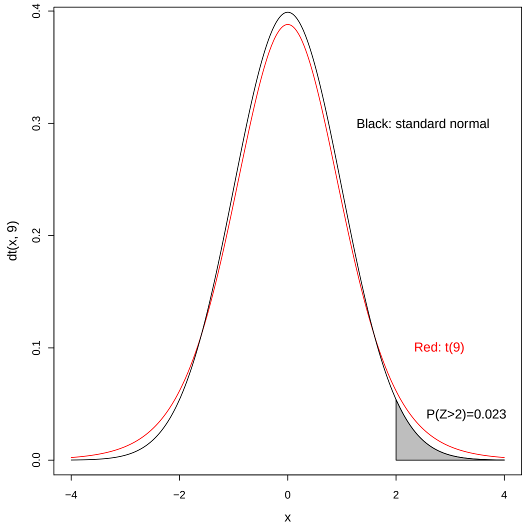
t-Distribution with 9 degrees of freedom $(n = 10)$:



t-Distribution with 9 degrees of freedom and standard normal distribution:



t-Distribution with 9 degrees of freedom and standard normal distribution:



Oversigt

Method box 3.8: One-sample Confidence interval for μ

Use the right t-distribution to make the confidence interval:

For a sample $x_1,\ldots,x_n$ the $100(1-\alpha)\%$ confidence interval is given by:

$$\bar{x} \pm t_{1-\alpha/2} \cdot \frac{s}{\sqrt{n}}$$

where $t_{1-\alpha/2}$ is the $100(1-\alpha)\%$ quantile from the t-distribution with $n-1$ degrees of freedom.

Most commonly using $\alpha = 0.05$:

The most commonly used is the 95%-confidence interval:

$$\bar{x} \pm t_{0.975} \cdot \frac{s}{\sqrt{n}}$$

Student height Example

```
## The t-quantiles for n=10:
qt(0.975,9)

## [1] 2.262
```

and we can recognize the already given result:

$$178 \pm 2.26 \cdot \frac{12.21}{\sqrt{10}}$$

which is:

$$178 \pm 8.74 = [169.3;\ 186.7]$$

Student height example, 99% Confidence interval (CI)

```
qt(0.995,9)

## [1] 3.25
```

$$178 \pm 3.25 \cdot \frac{12.21}{\sqrt{10}}$$

giving

$$178 \pm 12.55 = [165.4;\ 190.6]$$

There is an R-function, that can do it all (and more than that):

```
x <- c(168,161,167,179,184,166,198,187,191,179)
t.test(x,conf.level=0.99)

##
##  One Sample t-test
##
## data:  x
## t = 46, df = 9, p-value = 5e-12
## alternative hypothesis: true mean is not equal to 0
## 99 percent confidence interval:
##  165.5 190.5
## sample estimates:
## mean of x
##       178
```

Oversigt

The formal framework for *statistical inference*

From eNote, Chapter 1:

- An *observational unit* is the single entity/level about which information is sought (e.g. a person) (**Observationsenhed**)
- The *statistical population* consists of all possible "measurements" on each *observational unit* (**Population**)
- The *sample* from a statistical population is the actual set of data collected. (**Sample**)

Language and concepts:

- μ and σ are parameters describing th populationen
- $\bar{x}$ is the *estimate* of μ (specific realization)
- $\bar{X}$ is the *estimator* of μ (now seen as a random variable)
- The word *'statistic(s)'* is used for both

The formal framework for *statistical inference* - Example

From eNote, Chapter 1, heights example

We measure the heights of 10 randomly selected persons in Demark

The sample:

The 10 specific numbers: $x_1,\ldots,x_{10}$

The population:

The heights for all people in Dnemark

Observational unit:

A person

Statistical inference = Learning from data

Learning from data:

Is learning about parameters of distributions that describe populations.

Important for this:

The sample must in a meaningful way represent some well defined population

How to ensure this:

F.ex. by making sure that the sample is taken completely at random

Random Sampling

Definition 3.11

- A random sample from an (infinite) population: A set of observations $X_1, X_2, \ldots, X_n$ constitutes a random sample of size n from the infinite population $f(x)$ if:
  1. Each X_i is a random variable whose distribution is given by $f(x)$
  2. These n random variables are independent

What does that mean????

1. All observations must come from the same population
2. They cannot share any information with each other (e.g. if we sampled entire families)

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Theorem 3.13: The Central Limit Theorem

No matter what, the distribution of the mean becomes a normal distribution:

Let $\bar{X}$ be the mean of a random sample of size n taken from a population with mean μ and variance σ^2, then

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

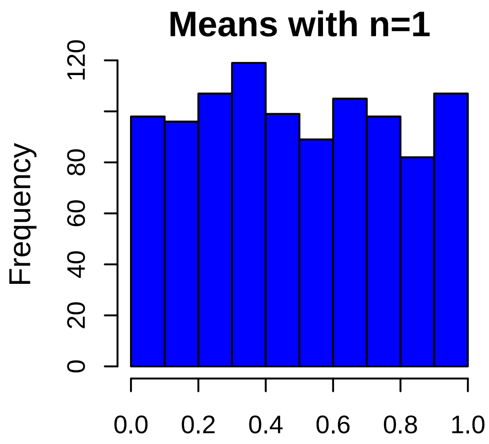
is a random variable whose distribution function approaches that of the standard normal distribution, $N(0, 1^2)$, as $n \to \infty$

Hence, if n is large enough, we can (approximately) assume:

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1^2)$$

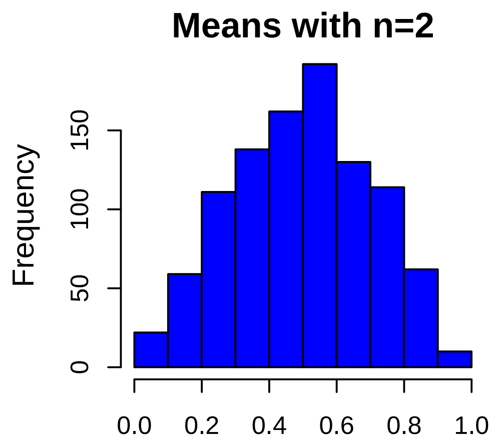
CLT in action - mean of uniformly distributed observations

```
n=1
k=1000
u=matrix(runif(k*n),ncol=n)
hist(apply(u,1,mean), col = "blue", main = "Means with n=1", xlab = "")
```



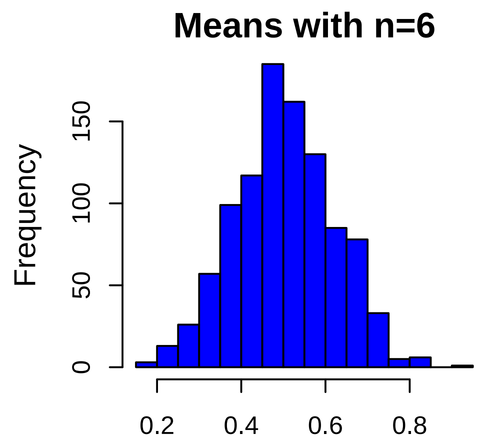
CLT in action - mean of uniformly distributed observations

```r
n=2
k=1000
u=matrix(runif(k*n),ncol=n)
hist(apply(u,1,mean), col = "blue", main = "Means with n=2", xlab = "")
```



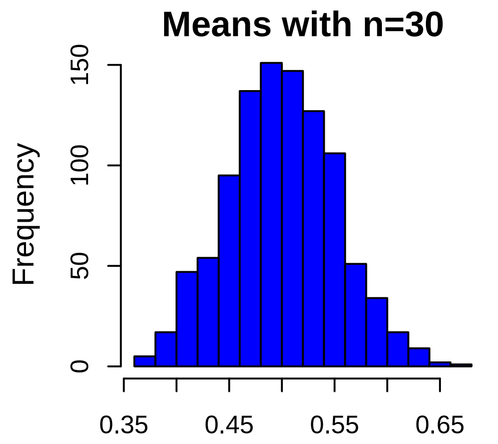
CLT in action - mean of uniformly distributed observations

```r
n=6
k=1000
u=matrix(runif(k*n),ncol=n)
hist(apply(u,1,mean), col = "blue", main = "Means with n=6", xlab = "")
```



CLT in action - mean of uniformly distributed observations

```r
n=30
k=1000
u=matrix(runif(k*n),ncol=n)
hist(apply(u,1,mean), col = "blue", main = "Means with n=30", xlab = "")
```



Consequence of CLT:

Our CI-method also works for non-normal data:

We can use the confidence-interval based on the t-distribution in basically any situation, as long as n is large enough.

What is "large enough"?

Actually difficult to say exactly, BUT:

- Rule of thumb: $n \geq 30$
- Even for smaller n the approach can be (almost) valid for non-normal data.

Oversigt

'Repeated sampling' interpretation

In the long run we catch the true value in 95% of cases:

The confidence interval will vary in both width (s) and position ($\bar{x}$) if the study is repeated.

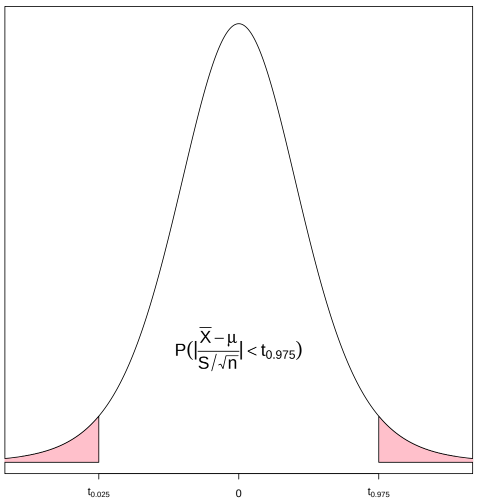
More formally expressed (Theorem 3.4 and 2.89):

$$P\left(\frac{|\bar{X}-\mu|}{S/\sqrt{n}} < t_{0.975}\right) = 0.95$$

Which is equivalent to:

$$P\left(\bar{X} - t_{0.975}\frac{S}{\sqrt{n}} < \mu < \bar{X} + t_{0.975}\frac{S}{\sqrt{n}}\right) = 0.95$$

'Repeated sampling' interpretation



Oversigt

Motivating Example: Production of tablets

In the production of tablets, an active matter is mixed with a powder and then the mixture is formed to tablets. It is important that the mixture is homogenous, so that each tablet has the same strength.

We consider a mixture (of the active matter and powder) from where a large amount of tablets is to be produced.

We seek to produce the mixtures (and the final tablets) so that the mean content of the active matter is 1 mg/g with the smallest variance as possible. A random sample is collected where the amount of active matter is measured. It is assumed that all the measurements follow a normal distribution with the unit mg/g.

The sampling distribution of the variance estimator (Theorem 2.81)

Variance estimators behaves like a χ^2-distribution:

Let

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

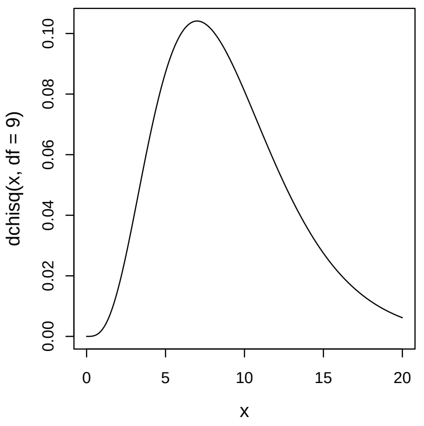
then:

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2}$$

is a stochastic variable following the χ^2-distribution with $v = n-1$ degrees of freedom.

χ^2-distribution with $\nu = 9$ degrees of freedom

```r
x <- seq(0, 20, by = 0.1)
plot(x, dchisq(x, df = 9), type = "l")
```



Method 3.18: Confidence interval for sample variance and standard deviation

The variance:

A $100(1-\alpha)\%$ confidence interval for a sample variance $\hat{\sigma}^2$ is:

$$\left[\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}};\ \frac{(n-1)s^2}{\chi^2_{\alpha/2}}\right]$$

where the quantiles come from a χ^2-distribution with $\nu = n-1$ degrees of freedom.

The standard deviation:

A $100(1-\alpha)\%$ confidence interval for the sample standard deviation $\hat{\sigma}$ is:

$$\left[\sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}};\ \sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2}}}\right]$$

Example

Data:
A random sample with $n = 20$ tablets is taken and from this we get:

$$\hat{\mu} = \bar{x} = 1.01,\ \hat{\sigma}^2 = s^2 = 0.07^2$$

95%-Confidence interval for the variance - we need the χ^2-quantiles:

$$\chi^2_{0.025} = 8.9065,\ \chi^2_{0.975} = 32.8523$$

```
qchisq(c(0.025, 0.975), df = 19)
```

[1] 8.907 32.852

Example

So the confidence interval for the variance σ^2 becomes:

$$\left[\frac{19\cdot 0.7^2}{32.85};\ \frac{19\cdot 0.7^2}{8.907}\right] = [0.002834;\ 0.01045]$$

and the confidence interval for the standard deviation σ becomes:

$$\left[\sqrt{0.002834};\ \sqrt{0.01045}\right] = [0.053;\ 0.102]$$

Heights example

We need the χ^2-quantiles with $\nu = 9$ degrees of freedom:

$$\chi^2_{0.025} = 2.700389,\ \chi^2_{0.975} = 19.022768$$

```
qchisq(c(0.025, 0.975), df = 9)
```

[1] 2.70 19.02

So the confidence interval for the height standard deviation σ becomes:

$$\left[\sqrt{\frac{9\cdot 12.21^2}{19.022768}};\ \sqrt{\frac{9\cdot 12.21^2}{2.700389}}\right] = [8.4;\ 22.3]$$

Example - heights- recap:

Sample, $n = 10$:

168 161 167 179 184 166 198 187 191 179

Sample mean and standard deviation:

$$\bar{x} = 178$$
$$s = 12.21$$

Estimate population mean and standard deviation:

$$\hat{\mu} = 178$$
$$\hat{\sigma} = 12.21$$

NEW:**Confidence interval, μ**:

$$178 \pm 2.26\cdot\frac{12.21}{\sqrt{10}} \Leftrightarrow [169.3;\ 186.7]$$

NEW:**Confidence interval, σ**:

$$[8.4;\ 22.3]$$

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