Course 02402 Introduction to Statistics Lecture 10:

Oneway Analysis of Variance, ANOVA

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Agenda

1. Intro: Small example and TV-data from B&O
2. Model and hypothesis
3. Computation - decomposition and the ANOVA table
4. Hypothesis test (F-test)
5. Within-Group variability and the relation to 2-Group t-test
6. Post hoc analysis
7. Model control
8. A complete example - from the book

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Oneway ANOVA - example

Group A	Group B	Group C
2.8	5.5	5.8
3.6	6.3	8.3
3.4	6.1	6.9
2.3	5.7	6.1

Is there a difference (in means) between the groups A, B and C?

Analysis of variance (ANOVA) can be used for the analysis if the observations in eacg group can bve assumed to be normally distributed.

TV set development at Bang & Olufsen

Sound and image quality is measured by th human perceptual instrument:

We developed a tool that is used by B&O to ANOVA (among other things)
PanelCheck (*Show Panelcheck programme with TV data*)

Bang & Olufsen data in R:

```
 # Getting the Bang and Olufsen data from the lmerTest-package:
library(lmerTest) # (Developed by us)
data(TVbo)
head(TVbo)
# Defining the factor identifying the 12 TVset and Picture combs:
TVbo$TVPic <- factor(TVbo$TVset:TVbo$Picture)
# Each of 8 assessors scored each of 12 combinations 2 times
# Averaging the two replicates for each Assessor and TVpic:
library(doBy)
TVbonoise <- summaryBy(Noise ~ Assessor + TVPic, data = TVbo,
                       keep.names = T)
# One-way ANOVA of the Noise: (Not the correct analysis!!)
anova(lm(Noise ~ TVPic, data = TVbonoise))
# Two-way ANOVA of the Noise: (Much better analysis - next week)
anova(lm(Noise ~ Assessor + TVPic, data = TVbonoise))
```

Oneway ANOVA - example

```
######################################
## Input data and plot

## Observations
y <- c(2.8, 3.6, 3.4, 2.3,
       5.5, 6.3, 6.1, 5.7,
       5.8, 8.3, 6.9, 6.1)

## Groups (treatments)
treatm <- factor(c(1, 1, 1, 1,
                   2, 2, 2, 2,
                   3, 3, 3, 3))

## Plot
par(mfrow=c(1,2))
plot(as.numeric(treatm), y, xlab="Treatment", ylab="y")
##
plot(treatm, y, xlab="Treatment", ylab="y")
```

Agenda

Oneway ANOVA, model

- Express the model

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$$

where it is assumed that

$$\varepsilon_{ij} \sim N(0, \sigma^2)$$

- μ is the overall mean
- α_i is the effect of Group (treatment) i
- j indicates the measurements in the groups, from 1 to n_i in each Group

Oneway ANOVA, hypothesis

- We want to compare (more than 2) means $\mu + \alpha_i$ in the model

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim N(0, \sigma^2)$$

- So we can express the hypothesis:

$$\begin{aligned} H_0: &\quad \alpha_i = 0 \quad \text{for all } i \\ H_1: &\quad \alpha_i \neq 0 \quad \text{for at least one } i \end{aligned}$$

Agenda

Oneway ANOVA, decomposition and the ANOVA table

- With the model

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim N(0, \sigma^2)$$

- the total variation in the data can be decomposed:

$$SST = SS(Tr) + SSE$$

- 'Oneway' refers to the fact that there is only one factor in the experiment on k levels

- The method is called analysis of variance, because the testing is carried out by comparing certain variances.

Formulas for sums of squares

- Total sum of squares ("the total variance")

$$SST = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij} - \bar{y})^2$$

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$$SSE = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_i)^2$$

- Sum of squares of treatment ("variance explained by the model")

$$SS(Tr) = \sum_{i=1}^{k} n_i(\bar{y}_i - \bar{y})^2$$

The ANOVA table

Source of variation	Deg. of freedom	Sums of squares	Mean sum of squares	Test-statistic F	p-value
treatment	$k-1$	$SS(Tr)$	$MS(Tr)=\frac{SS(Tr)}{k-1}$	$F_{\text{obs}}=\frac{MS(Tr)}{MSE}$	$P(F>F_{\text{obs}})$
Residual	$n-k$	SSE	$MSE=\frac{SSE}{n-k}$		
Total	$n-1$	SST			

```
anova(lm(y ~ treatm))

## Analysis of Variance Table
##
## Response: y
##           Df Sum Sq Mean Sq F value  Pr(>F)
## treatm     2   30.8   15.40    26.7 0.00017 ***
## Residuals  9    5.2    0.58
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Example

```
## Number of Groups
k <- 3
## Number in each  Group
ni <- 10
## Simulate data from model with 3 means
yModel1 <- rep( c(4, 5, -3), each=ni) + rnorm(ni*k, sd=1)
## Simulate data from model with 3 other means
yModel2 <- rep( c(1, 3, 1), each=ni) + rnorm(ni*k, sd=1)
## 3 Groups
group <- rep(1:k, each=ni)
## Plot them
par(mfrow=c(1,2))
plot(group, yModel1, ylim=range(yModel1,yModel2))
plot(group, yModel2, ylim=range(yModel1,yModel2))

## Compute SST: total variance, which is highest?
(SST1 <- sum( (yModel1 - mean(yModel1))^2 ))
(SST2 <- sum( (yModel2 - mean(yModel2))^2 ))

## Compute SSE: total residual variation, which is highest?
(SSE1 <- sum(tapply(yModel1, group, function(x){ sum((x - mean(x))^2) })))
(SSE2 <- sum(tapply(yModel2, group, function(x){ sum((x - mean(x))^2) })))
```

Agenda

Oneway ANOVA, F-test

- We have: (Theorem 8.2)

$$SST = SS(Tr) + SSE$$

- and can find the test statistic:

$$F = \frac{SS(Tr)/(k-1)}{SSE/(n-k)}$$

  where

  - k is the number of levels of the factor
  - n is the total number of observations

- The significance level α is chosen and the test statistic F is computed

- The test statistic is compared with a quantile in the F distribution

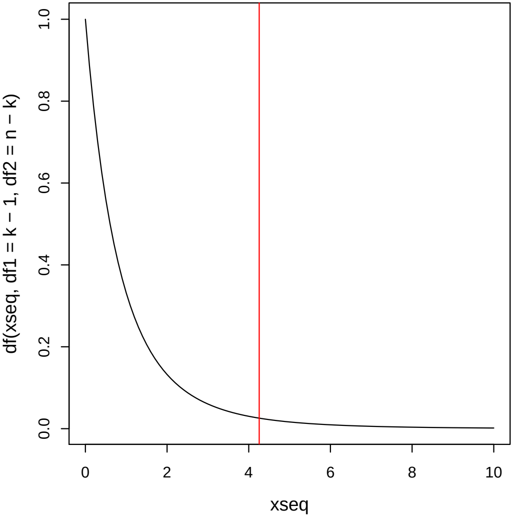
$$F \sim F_{\alpha}(k-1, n-k) \text{ (Theorem 8.6)}$$

The F-distribution

```r
################################
## Plot the F distribution  and see the critical value

## Remember, this is "under H0" (that is we compute as if H0 is true):
## Number of Groups
k <- 3
## number of observations
n <- 12
## Sequence for plot
xseq <- seq(0, 10, by=0.1)
## Plot the density of the  F distribution
plot(xseq, df(xseq, df1=k-1, df2=n-k), type="l")
##The critical value for significance level 5 %
cr <- qf(0.95, df1=k-1, df2=n-k)
## Mark it in the plot
abline(v=cr, col="red")
## The value of the  test statistic
(F <- (SSTr/(k-1)) / (SSE/(n-k)))
## The p-value hence is:
(1 - pf(F, df1=k-1, df2=n-k))
```

The F-distribution



The ANOVA table

Source of variation	Deg. of freedom	Sums of squares	Mean sum of squares	Test-statistic F	p-value
treatment	$k-1$	$SS(Tr)$	$MS(Tr) = \frac{SS(Tr)}{k-1}$	$F_{\text{obs}} = \frac{MS(Tr)}{MSE}$	$P(F > F_{\text{obs}})$
Residual	$n-k$	SSE	$MSE = \frac{SSE}{n-k}$		
Total	$n-1$	SST			

```
anova(lm(y ~ treatm))

## Analysis of Variance Table
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Within-Group variability and the relation to 2-Group t-test (Theorem 8.4)

The residual sum of squares SSE divided by $n-k$, also called Residual mean square $MSE = SSE/(n-k)$ is the average within group variability:

$$MSE = \frac{SSE}{n-k} = \frac{(n_1-1)s_1^2 + \cdots + (n_k-1)s_k^2}{n-k} \tag{1}$$

$$s_i^2 = \frac{1}{n_i-1}\sum_{i=1}^{n_i}(y_{ij}-\bar{y}_i)^2$$

IF $k=2$:(cf. Method 3.51)

$$\text{For } k=2:\ MSE = s_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n-2}$$

$$\text{For } k=2:\ F_{\text{obs}} = t_{\text{obs}}^2$$

where t_{obs} is the pooled version coming from Methods 3.51 and 3.52.

Agenda

Post hoc confidence interval - Method 8.9

- A single pre-planned confidence interval for the difference between treatment i and j is found as:

$$\bar{y}_i - \bar{y}_j \pm t_{1-\alpha/2}\sqrt{\frac{SSE}{n-k}\left(\frac{1}{n_i}+\frac{1}{n_j}\right)} \tag{2}$$

  where $t_{1-\alpha/2}$ is based on the t-distribution with $n-k$ degrees of freedom.

- Note the fewer degrees of freedom as more unknowns are estimated in the computationen of $MSE = SSE/(n-k) = s_p^2$ (i.e. pooled variance estimate)

- If all $M = k(k-1)/2$ combinations of pairwise confidence intervals are found use the formula M times but each time with $\alpha_{\text{Bonferroni}} = \alpha/M$.

Post hoc pairwise hypothesis test- Method 8.10

- A single pre-planned level α hypothesis tests:

$$H_0: \ \mu_i = \mu_j, \ H_1: \ \mu_i \neq \mu_j$$

  is carried out as:

$$t_{\text{obs}} = \frac{\bar{y}_i - \bar{y}_j}{\sqrt{MSE\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}} \tag{3}$$

  and:

$$p - \text{value} = 2P(t > |t_{\text{obs}}|)$$

  where the t-distribution with $n - k$ degrees of freedom is used.

- HIf all $M = k(k-1)/2$ combinations of pairwise hypothesis tests are carried out use the approach M times but each time with test level $\alpha_{\text{Bonferroni}} = \alpha/M$.

Agenda

Variance homogeneity

Look at box-plot to check whether the variability seems different for the groups

```
##################################
## Check assumption of  homogeneous variance

## Box plot
plot(treatm,y)
```

Normal assumption

Look at qq-normal plot

```
##################################
## Check the assumption  of  normality of residuals

## qq-normal plot of residuals
fit1 <- lm(y ~ treatm)
qqnorm(fit1$residuals)
qqline(fit1$residuals)

## Or with a  Wally plot
require(MESS)
qqwrap <- function(x, y, ...) {qqnorm(y, main="",...);
qqline(y)}
## Can we see a deviating qq-norm plot?
wallyplot(fit1$residuals, FUN = qqwrap)
```

Next week: Two-way ANOVA

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A complete example - from the book

Introduction to Statistics

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Dokumentegenskaber...

8.2.5 A complete worked through example: plastic types for lamps

Example 8.17 Plastic types for lamps

On a lamp two plastic screens are to be mounted. It is essential that these plastic screens have a good impact strength. Therefore an experiment is carried out for 5 different types of plastic. 6 samples in each plastic type are tested. The strengths of these items are determined. The following measurement data was found (strength in kJ/m^2):

Type of plastic				
I	II	III	IV	V
44.6	52.8	53.1	51.5	48.2
50.5	58.3	50.0	53.7	40.8
46.3	55.4	54.4	50.5	44.5
48.5	57.4	55.3	54.4	43.9
45.2	58.1	50.6	47.5	45.9
52.3	54.6	53.4	47.8	42.5

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