5.1 Exercises

1. Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? Why or why not?

2. Is $\lambda = -2$ an eigenvalue of $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$? Why or why not?

3. Is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}$? If so, find the eigenvalue.

4. Is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$? If so, find the eigenvalue.

5. Is $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$? If so, find the eigenvalue.

6. Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 2 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}$? If so, find the eigenvalue.

7. Is $\lambda = 4$ an eigenvalue of $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find one corresponding eigenvector.

8. Is $\lambda = 3$ an eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$? If so, find one corresponding eigenvector.

In Exercises 9–16, find a basis for the eigenspace corresponding to each listed eigenvalue.

9. $A = \begin{bmatrix} 9 & 0 \\ 2 & 3 \end{bmatrix}, \lambda = 3, 9$

10. $A = \begin{bmatrix} 14 & -4 \\ 16 & -2 \end{bmatrix}, \lambda = 6$

11. $A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}, \lambda = 10$

12. $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \lambda = -2, 5$

13. $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \lambda = 1, 2, 3$

14. $A = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 3 & 3 \\ 6 & 6 & 2 \end{bmatrix}, \lambda = -4$

15. $A = \begin{bmatrix} 8 & 3 & -4 \\ -1 & 4 & 4 \\ 2 & 6 & -1 \end{bmatrix}, \lambda = 7$

16. $A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \lambda = 4$

Find the eigenvalues of the matrices in Exercises 17 and 18.

17. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$

18. $\begin{bmatrix} 8 & 0 & 0 \\ -7 & 0 & 0 \\ 6 & -5 & -4 \end{bmatrix}$

19. For $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, find one eigenvalue, with no calculation. Justify your answer.

20. Without calculation, find one eigenvalue and two linearly independent eigenvectors of $A = \begin{bmatrix} 4 & 4 & -4 \\ 4 & 4 & -4 \\ 4 & 4 & -4 \end{bmatrix}$. Justify your answer.

In Exercises 21–30, A is an $n \times n$ matrix. Mark each statement True or False **(T/F)**. Justify each answer.

21. **(T/F)** If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, then λ is an eigenvalue of A.

22. **(T/F)** If $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ, then $\mathbf{x}$ is an eigenvector of A.

23. **(T/F)** A matrix A is invertible if and only if 0 is an eigenvalue of A.

24. **(T/F)** A number c is an eigenvalue of A if and only if the equation $(A - cI)\mathbf{x} = 0$ has a nontrivial solution.

25. **(T/F)** Finding an eigenvector of A may be difficult, but checking whether a given vector is in fact an eigenvector is easy.

26. **(T/F)** To find the eigenvalues of A, reduce A to echelon form.

27. **(T/F)** If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

28. **(T/F)** The eigenvalues of a matrix are on its main diagonal.

29. **(T/F)** If $\mathbf{v}$ is an eigenvector with eigenvalue 2, then $2\mathbf{v}$ is an eigenvector with eigenvalue 4.

30. **(T/F)** An eigenspace of A is a null space of a certain matrix.

31. Explain why a 2×2 matrix can have at most two distinct eigenvalues. Explain why an $n \times n$ matrix can have at most n distinct eigenvalues.

32. Construct an example of a 2×2 matrix with only one distinct eigenvalue.

33. Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}. [*Hint:* Suppose a nonzero $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$.]

34. Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0.

35. Show that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T. [*Hint:* Find out how $A - \lambda I$ and $A^T - \lambda I$ are related.]

36. Use Exercise 35 to complete the proof of Theorem 1 for the case when A is lower triangular.

37. Consider an $n \times n$ matrix A with the property that the row sums all equal the same number s. Show that s is an eigenvalue of A. [*Hint:* Find an eigenvector.]

38. Consider an $n \times n$ matrix A with the property that the column sums all equal the same number s. Show that s is an eigenvalue of A. [*Hint:* Use Exercises 35 and 37.]

In Exercises 39 and 40, let A be the matrix of the linear transformation T. Without writing A, find an eigenvalue of A and describe the eigenspace.

39. T is the transformation on $\mathbb{R}^2$ that reflects points across some line through the origin.

40. T is the transformation on $\mathbb{R}^3$ that rotates points about some line through the origin.

41. Let $\mathbf{u}$ and $\mathbf{v}$ be eigenvectors of a matrix A, with corresponding eigenvalues λ and μ, and let c_1 and c_2 be scalars. Define

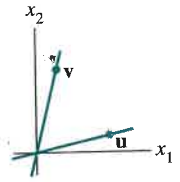
$$\mathbf{x}_k = c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v} \quad (k = 0, 1, 2, \ldots)$$

a. What is $\mathbf{x}_{k+1}$, by definition?

b. Compute $A\mathbf{x}_k$ from the formula for $\mathbf{x}_k$, and show that $A\mathbf{x}_k = \mathbf{x}_{k+1}$. This calculation will prove that the sequence $\{\mathbf{x}_k\}$ defined above satisfies the difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ $(k = 0, 1, 2, \ldots)$.

42. Describe how you might try to build a solution of a difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ $(k = 0, 1, 2, \ldots)$ if you were given the initial $\mathbf{x}_0$ and this vector did not happen to be an eigenvector of A. [*Hint:* How might you relate $\mathbf{x}_0$ to eigenvectors of A?]

43. Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors shown in the figure, and suppose $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of a 2×2 matrix A that correspond to eigenvalues 2 and 3, respectively. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x}$ in $\mathbb{R}^2$, and let $\mathbf{w} = \mathbf{u} + \mathbf{v}$. Make a copy of the figure, and on the same coordinate system, carefully plot the vectors $T(\mathbf{u})$, $T(\mathbf{v})$, and $T(\mathbf{w})$.



44. Repeat Exercise 43, assuming $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of A that correspond to eigenvalues -1 and 3, respectively.

[T] In Exercises 45–48, use a matrix program to find the eigenvalues of the matrix. Then use the method of Example 4 with a row reduction routine to produce a basis for each eigenspace.

45. $\begin{bmatrix} 8 & -10 & -5 \\ 2 & 17 & 2 \\ -9 & -18 & 4 \end{bmatrix}$

46. $\begin{bmatrix} 9 & -4 & -2 & -4 \\ -56 & 32 & -28 & 44 \\ -14 & -14 & 6 & -14 \\ 42 & -33 & 21 & -45 \end{bmatrix}$

47. $\begin{bmatrix} 4 & -9 & -7 & 8 & 2 \\ -7 & -9 & 0 & 7 & 14 \\ 5 & 10 & 5 & -5 & -10 \\ -2 & 3 & 7 & 0 & 4 \\ -3 & -13 & -7 & 10 & 11 \end{bmatrix}$

48. $\begin{bmatrix} -4 & -4 & 20 & -8 & -1 \\ 14 & 12 & 46 & 18 & 2 \\ 6 & 4 & -18 & 8 & 1 \\ 11 & 7 & -37 & 17 & 2 \\ 18 & 12 & -60 & 24 & 5 \end{bmatrix}$

Solutions to Practice Problems

1. The number 5 is an eigenvalue of A if and only if the equation $(A - 5I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution. Form

$$A - 5I = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \\ 3 & -5 & 5 \\ 2 & 2 & 1 \end{bmatrix}$$

and row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 3 & -5 & 5 & 0 \\ 2 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 8 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & -5 & 0 \end{bmatrix}$$

Numerical Notes (Continued)

2. The best numerical methods for finding eigenvalues avoid the characteristic polynomial entirely. In fact, MATLAB finds the characteristic polynomial of a matrix A by first computing the eigenvalues $\lambda_1, \dots, \lambda_n$ of A and then expanding the product $(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$.
3. Several common algorithms for estimating the eigenvalues of a matrix A are based on Theorem 4. The powerful *QR algorithm* is discussed in the exercises. Another technique, called *Jacobi's method*, works when $A = A^T$ and computes a sequence of matrices of the form

$$A_1 = A \quad \text{and} \quad A_{k+1} = P_k^{-1} A_k P_k \quad (k = 1, 2, \dots)$$

Each matrix in the sequence is similar to A and so has the same eigenvalues as A. The nondiagonal entries of A_{k+1} tend to zero as k increases, and the diagonal entries tend to approach the eigenvalues of A.
4. Other methods of estimating eigenvalues are discussed in Section 5.8.

Practice Problem

Find the characteristic equation and eigenvalues of $A = \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$.

5.2 Exercises

Find the characteristic polynomial and the eigenvalues of the matrices in Exercises 1–8.

1. $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}$

3. $\begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}$

4. $\begin{bmatrix} 5 & -5 \\ -2 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -4 \\ 4 & 6 \end{bmatrix}$

7. $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$

8. $\begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}$

Exercises 9–14 require techniques from Section 3.1. Find the characteristic polynomial of each matrix using expansion across a row or down a column. [*Note:* Finding the characteristic polynomial of a 3×3 matrix is not easy to do with just row operations, because the variable λ is involved.]

9. $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 6 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 3 \\ 1 & 0 & 2 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 1 \\ -3 & 6 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 3 & -2 & 3 \\ 0 & -1 & 0 \\ 6 & 7 & -4 \end{bmatrix}$

For the matrices in Exercises 15–17, list the eigenvalues, repeated according to their multiplicities.

15. $\begin{bmatrix} 7 & -5 & 3 & 0 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

16. $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}$

17. $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 0 & 0 \\ 0 & -7 & 2 & 1 & 0 \\ -4 & 1 & 9 & -2 & 3 \end{bmatrix}$

18. It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in the matrix A below such that the eigenspace for $\lambda = 6$ is two-dimensional:

$$A = \begin{bmatrix} 6 & 3 & 9 & -5 \\ 0 & 9 & h & 2 \\ 0 & 0 & 6 & 8 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

19. Let A be an $n \times n$ matrix, and suppose A has n real eigenvalues, $\lambda_1, \dots, \lambda_n$, repeated according to multiplicities, so that
$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$

Explain why $\det A$ is the product of the n eigenvalues of A. (This result is true for any square matrix when complex eigenvalues are considered.)

20. Use a property of determinants to show that A and A^T have the same characteristic polynomial.

In Exercises 21–30, A and B are $n \times n$ matrices. Mark each statement True or False **(T/F)**. Justify each answer.

21. **(T/F)** If 0 is an eigenvalue of A, then A is invertible.

22. **(T/F)** The zero vector is in the eigenspace of A associated with an eigenvalue λ.

23. **(T/F)** The matrix A and its transpose, A^T, have different sets of eigenvalues.

24. **(T/F)** The matrices A and $B^{-1}AB$ have the same sets of eigenvalues for every invertible matrix B.

25. **(T/F)** If 2 is an eigenvalue of A, then $A - 2I$ is not invertible.

26. **(T/F)** If two matrices have the same set of eigenvalues, then they are similar.

27. **(T/F)** If $\lambda + 5$ is a factor of the characteristic polynomial of A, then 5 is an eigenvalue of A.

28. **(T/F)** The multiplicity of a root r of the characteristic equation of A is called the algebraic multiplicity of r as an eigenvalue of A.

29. **(T/F)** The eigenvalue of the $n \times n$ identity matrix is 1 with algebraic multiplicity n.

30. **(T/F)** The matrix A can have more than n eigenvalues.

A widely used method for estimating eigenvalues of a general matrix A is the *QR algorithm*. Under suitable conditions, this algorithm produces a sequence of matrices, all similar to A, that become almost upper triangular, with diagonal entries that approach the eigenvalues of A. The main idea is to factor A (or another matrix similar to A) in the form $A = Q_1R_1$, where $Q_1^T = Q_1^{-1}$ and R_1 is upper triangular. The factors are interchanged to form $A_1 = R_1Q_1$, which is again factored as $A_1 = Q_2R_2$; then to form $A_2 = R_2Q_2$, and so on. The similarity of $A, A_1, \ldots$ follows from the more general result in Exercise 31.

31. Show that if $A = QR$ with Q invertible, then A is similar to $A_1 = RQ$.

32. Show that if A and B are similar, then $\det A = \det B$.

T 33. Construct a random integer-valued 4×4 matrix A, and verify that A and A^T have the same characteristic polynomial (the same eigenvalues with the same multiplicities). Do A and A^T have the same eigenvectors? Make the same analysis of a 5×5 matrix. Report the matrices and your conclusions.

T 34. Construct a random integer-valued 4×4 matrix A.

a. Reduce A to echelon form U with no row scaling, and compute $\det A$. (If A happens to be singular, start over with a new random matrix.)

b. Compute the eigenvalues of A and the product of these eigenvalues (as accurately as possible).

c. List the matrix A, and, to four decimal places, list the pivots in U and the eigenvalues of A. Compute $\det A$ with your matrix program, and compare it with the products you found in (a) and (b).

T 35. Let $A = \begin{bmatrix} -6 & 28 & 21 \\ 4 & -15 & -12 \\ -8 & a & 25 \end{bmatrix}$. For each value of a in the set $\{32, 31.9, 31.8, 32.1, 32.2\}$, compute the characteristic polynomial of A and the eigenvalues. In each case, create a graph of the characteristic polynomial $p(t) = \det(A - tI)$ for $0 \le t \le 3$. If possible, construct all graphs on one coordinate system. Describe how the graphs reveal the changes in the eigenvalues as a changes.

Solution to Practice Problem

The characteristic equation is

$$0 = \det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & -4 \\ 4 & 2-\lambda \end{bmatrix}$$
$$= (1-\lambda)(2-\lambda) - (-4)(4) = \lambda^2 - 3\lambda + 18$$

From the quadratic formula,

$$\lambda = \frac{3 \pm \sqrt{(-3)^2 - 4(18)}}{2} = \frac{3 \pm \sqrt{-63}}{2}$$

It is clear that the characteristic equation has no real solutions, so A has no real eigenvalues. The matrix A is acting on the real vector space $\mathbb{R}^2$, and there is no nonzero vector $\mathbf{v}$ in $\mathbb{R}^2$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ.

Practice Problems

1. Compute A^8, where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

2. Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are told that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A. Use this information to diagonalize A.

3. Let A be a 4×4 matrix with eigenvalues 5, 3, and -2, and suppose you know that the eigenspace for $\lambda = 3$ is two-dimensional. Do you have enough information to determine if A is diagonalizable?

5.3 Exercises

In Exercises 1 and 2, let $A = PDP^{-1}$ and compute A^4.

1. $P = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

2. $P = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

In Exercises 3 and 4, use the factorization $A = PDP^{-1}$ to compute A^k, where k represents an arbitrary positive integer.

3. $\begin{bmatrix} a & 0 \\ 3(a-b) & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 15 & -36 \\ 6 & -15 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$

In Exercises 5 and 6, the matrix A is factored in the form PDP^{-1}. Use the Diagonalization Theorem to find the eigenvalues of A and a basis for each eigenspace.

5. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} =$
$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & -3/4 \\ 1/4 & -1/2 & 1/4 \end{bmatrix}$

6. $\begin{bmatrix} 7 & -1 & 1 \\ 6 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix} =$
$\begin{bmatrix} 2 & 1 & 1 \\ 5 & 2 & 3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 3 & -1 & -1 \\ -2 & 1 & -1 \end{bmatrix}$

Diagonalize the matrices in Exercises 7–20, if possible. The eigenvalues for Exercises 11–16 are as follows: (11) $\lambda = 1, 2, 3$; (12) $\lambda = 1, 4$; (13) $\lambda = 5, 1$; (14) $\lambda = 3, 4$; (15) $\lambda = 3, 1$; (16) $\lambda = 2, 1$. For Exercise 18, one eigenvalue is $\lambda = 5$ and one eigenvector is $(-2, 1, 2)$.

7. $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

8. $\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$

10. $\begin{bmatrix} 3 & 6 \\ 4 & 1 \end{bmatrix}$

11. $\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

12. $\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$

14. $\begin{bmatrix} 4 & 0 & 2 \\ 2 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

15. $\begin{bmatrix} -7 & 24 & -16 \\ -2 & 7 & -4 \\ 2 & -6 & 5 \end{bmatrix}$

16. $\begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$

17. $\begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

18. $\begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

20. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

In Exercises 21–28, A, P, and D are $n \times n$ matrices. Mark each statement True or False **(T/F)**. Justify each answer. (Study Theorems 5 and 6 and the examples in this section carefully before you try these exercises.)

21. **(T/F)** A is diagonalizable if $A = PDP^{-1}$ for some matrix D and some invertible matrix P.

22. **(T/F)** If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.

23. **(T/F)** A is diagonalizable if and only if A has n eigenvalues, counting multiplicities.

24. **(T/F)** If A is diagonalizable, then A is invertible.

25. **(T/F)** A is diagonalizable if A has n eigenvectors.

26. **(T/F)** If A is diagonalizable, then A has n distinct eigenvalues.

27. **(T/F)** If $AP = PD$, with D diagonal, then the nonzero columns of P must be eigenvectors of A.

28. **(T/F)** If A is invertible, then A is diagonalizable.

29. A is a 5×5 matrix with two eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two-dimensional. Is A diagonalizable? Why?

30. A is a 3×3 matrix with two eigenvalues. Each eigenspace is one-dimensional. Is A diagonalizable? Why?

31. A is a 4×4 matrix with three eigenvalues. One eigenspace is one-dimensional, and one of the other eigenspaces is two-dimensional. Is it possible that A is *not* diagonalizable? Justify your answer.

32. A is a 7×7 matrix with three eigenvalues. One eigenspace is two-dimensional, and one of the other eigenspaces is three-dimensional. Is it possible that A is *not* diagonalizable? Justify your answer.

33. Show that if A is both diagonalizable and invertible, then so is A^{-1}.

34. Show that if A has n linearly independent eigenvectors, then so does A^T. [*Hint:* Use the Diagonalization Theorem.]

35. A factorization $A = PDP^{-1}$ is not unique. Demonstrate this for the matrix A in Example 2. With $D_1 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$, use the information in Example 2 to find a matrix P_1 such that $A = P_1 D_1 P_1^{-1}$.

36. With A and D as in Example 2, find an invertible P_2 unequal to the P in Example 2, such that $A = P_2 D P_2^{-1}$.

37. Construct a nonzero 2×2 matrix that is invertible but not diagonalizable.

38. Construct a nondiagonal 2×2 matrix that is diagonalizable but not invertible.

Diagonalize the matrices in Exercises 39–42. Use your matrix program's eigenvalue command to find the eigenvalues, and then compute bases for the eigenspaces as in Section 5.1.

T 39. $\begin{bmatrix} -6 & 4 & 0 & 9 \\ -3 & 0 & 1 & 6 \\ -1 & -2 & 1 & 0 \\ -4 & 4 & 0 & 7 \end{bmatrix}$

T 40. $\begin{bmatrix} 0 & 13 & 8 & 4 \\ 4 & 9 & 8 & 4 \\ 8 & 6 & 12 & 8 \\ 0 & 5 & 0 & -4 \end{bmatrix}$

T 41. $\begin{bmatrix} 11 & -6 & 4 & -10 & -4 \\ -3 & 5 & -2 & 4 & 1 \\ -8 & 12 & -3 & 12 & 4 \\ 1 & 6 & -2 & 3 & -1 \\ 8 & -18 & 8 & -14 & -1 \end{bmatrix}$

T 42. $\begin{bmatrix} 4 & 4 & 2 & 3 & -2 \\ 0 & 1 & -2 & -2 & 2 \\ 6 & 12 & 11 & 2 & -4 \\ 9 & 20 & 10 & 10 & -6 \\ 15 & 28 & 14 & 5 & -3 \end{bmatrix}$

Solutions to Practice Problems

1. $\det(A - \lambda I) = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$. The eigenvalues are 2 and 1, and the corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Next, form

$$P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Since $A = PDP^{-1}$,

$$\begin{aligned} A^8 = PD^8P^{-1} &= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^8 & 0 \\ 0 & 1^8 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 256 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix} \end{aligned}$$

2. Compute $A\mathbf{v}_1 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1 \cdot \mathbf{v}_1$, and

$$A\mathbf{v}_2 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \cdot \mathbf{v}_2$$

So, $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for the eigenvalues 1 and 3, respectively. Thus

$$A = PDP^{-1}, \quad \text{where} \quad P = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Numerical Notes

An efficient way to compute a $\mathcal{B}$-matrix $P^{-1}AP$ is to compute AP and then to row reduce the augmented matrix $[\,P \quad AP\,]$ to $[\,I \quad P^{-1}AP\,]$. A separate computation of P^{-1} is unnecessary. See Exercise 22 in Section 2.2.

Practice Problems

1. Find $T(a_0 + a_1t + a_2t^2)$, if T is the linear transformation from $\mathbb{P}_2$ to $\mathbb{P}_2$ whose matrix relative to $\mathcal{B} = \{1, t, t^2\}$ is

$$[\,T\,]_{\mathcal{B}} = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}$$

2. Let A, B, and C be $n \times n$ matrices. The text has shown that if A is similar to B, then B is similar to A. This property, together with the statements below, shows that "similar to" is an *equivalence relation*. (Row equivalence is another example of an equivalence relation.) Verify parts (a) and (b).
   a. A is similar to A.
   b. If A is similar to B and B is similar to C, then A is similar to C.

5.4 Exercises

1. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for the vector space V. Let $T : V \to V$ be a linear transformation with the property that

$$T(\mathbf{b}_1) = 3\mathbf{b}_1 - 5\mathbf{b}_2,\ T(\mathbf{b}_2) = -\mathbf{b}_1 + 6\mathbf{b}_2,\ T(\mathbf{b}_3) = 4\mathbf{b}_2$$

Find $[T]_{\mathcal{B}}$, the matrix for T relative to $\mathcal{B}$.

2. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis for a vector space V. Let $T : V \to V$ be a linear transformation with the property that

$$T(\mathbf{b}_1) = 7\mathbf{b}_1 + 4\mathbf{b}_2,\ T(\mathbf{b}_2) = 6\mathbf{b}_1 - 5\mathbf{b}_2$$

Find $[T]_{\mathcal{B}}$, the matrix for T relative to $\mathcal{B}$.

3. Assume the mapping $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T\left(a_0 + a_1t + a_2t^2\right) = 2a_0 + (3a_1 + 4a_2)t + (5a_0 - 6a_2)t^2$$

is linear. Find the matrix representation of T relative to the basis $\mathcal{B} = \{1, t, t^2\}$.

4. Define $T : \mathbb{P}_2 \to \mathbb{P}_2$ by $T(\mathbf{p}) = \mathbf{p}(0) - \mathbf{p}(1)t + \mathbf{p}(2)t^2$.
   a. Show that T is a linear transformation.
   b. Find $T(\mathbf{p})$ when $\mathbf{p}(t) = -2 + t$. Is $\mathbf{p}$ an eigenvector of T?
   c. Find the matrix for T relative to the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$.

5. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for a vector space V. Find $T(2\mathbf{b}_1 - 5\mathbf{b}_3)$ when T is a linear transformation from V to V whose matrix relative to $\mathcal{B}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 3 \\ 2 & 0 & -1 \end{bmatrix}$$

6. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for a vector space V. Find $T(2\mathbf{b}_1 - \mathbf{b}_2 + 4\mathbf{b}_3)$ when T is a linear transformation from V to V whose matrix relative to $\mathcal{B}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}$$

In Exercises 7 and 8, find the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$, when $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$.

7. $A = \begin{bmatrix} 4 & 9 \\ 1 & 4 \end{bmatrix}, \mathbf{b}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

8. $A = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}, \mathbf{b}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

In Exercises 9–12, define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal.

9. $A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$

10. $A = \begin{bmatrix} 5 & -3 \\ -7 & 1 \end{bmatrix}$

11. $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$

12. $A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}$

13. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, for $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$.

a. Verify that $\mathbf{b}_1$ is an eigenvector of A but A is not diagonalizable.

b. Find the $\mathcal{B}$ matrix for T.

14. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, where A is a 3×3 matrix with eigenvalues 5 and -2. Does there exist a basis $\mathcal{B}$ for $\mathbb{R}^3$ such that the $\mathcal{B}$-matrix for T is a diagonal matrix? Discuss.

15. Define $T : \mathbb{P}_2 \to \mathbb{P}_2$ by $T(\mathbf{p}) = \mathbf{p}(1) + \mathbf{p}(1)t + \mathbf{p}(1)t^2$.

a. Find $T(\mathbf{p})$ when $\mathbf{p}(t) = 1 + t + t^2$. Is $\mathbf{p}$ an eigenvector of T? If $\mathbf{p}$ is an eigenvector, what is its eigenvalue?

b. Find $T(\mathbf{p})$ when $\mathbf{p}(t) = -2 + t$. Is $\mathbf{p}$ an eigenvector of T? If $\mathbf{p}$ is an eigenvector, what is its eigenvalue?

16. Define $T : \mathbb{P}_3 \to \mathbb{P}_3$ by $T(\mathbf{p}) = \mathbf{p}(0) + \mathbf{p}(2)t - \mathbf{p}(0)t^2 - \mathbf{p}(2)t^3$.

a. Find $T(\mathbf{p})$ when $\mathbf{p}(t) = 1 - t^2$. Is $\mathbf{p}$ an eigenvector of T? If $\mathbf{p}$ is an eigenvector, what is its eigenvalue?

b. Find $T(\mathbf{p})$ when $\mathbf{p}(t) = t - t^3$. Is $\mathbf{p}$ an eigenvector of T? If $\mathbf{p}$ is an eigenvector, what is its eigenvalue?

In Exercises 17 through 20, mark each statement True or False (**T/F**). Justify each answer.

17. **(T/F)** Similar matrices have the same eigenvalues.

18. **(T/F)** Similar matrices have the same eigenvectors.

19. **(T/F)** Only linear transformations on finite vectors spaces have eigenvectors.

20. **(T/F)** If there is a nonzero vector in the kernel of a linear transformation T, then 0 is an eigenvalue of T.

Verify the statements in Exercises 21–28 by providing justification for each statement. In each case, the matrices are square.

21. If A is invertible and similar to B, then B is invertible and A^{-1} is similar to B^{-1}. [*Hint:* $P^{-1}AP = B$ for some invertible P. Explain why B is invertible. Then find an invertible Q such that $Q^{-1}A^{-1}Q = B^{-1}$.]

22. If A is similar to B, then A^2 is similar to B^2.

23. If B is similar to A and C is similar to A, then B is similar to C.

24. If A is diagonalizable and B is similar to A, then B is also diagonalizable.

25. If $B = P^{-1}AP$ and $\mathbf{x}$ is an eigenvector of A corresponding to an eigenvalue λ, then $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding also to λ.

26. If A and B are similar, then they have the same rank. [*Hint:* Refer to Supplementary Exercises 31 and 32 for Chapter 4.]

27. The *trace* of a square matrix A is the sum of the diagonal entries in A and is denoted by tr A. It can be verified that $\mathrm{tr}(FG) = \mathrm{tr}(GF)$ for any two $n \times n$ matrices F and G. Show that if A and B are similar, then $\mathrm{tr}\, A = \mathrm{tr}\, B$.

28. It can be shown that the trace of a matrix A equals the sum of the eigenvalues of A. Verify this statement for the case when A is diagonalizable.

Exercises 29–32 refer to the vector space of signals, $\mathbb{S}$, from Section 4.7. The shift transformation, $S(\{y_k\}) = \{y_{k-1}\}$, shifts each entry in the signal one position to the right. The moving average transformation, $M_2(\{y_k\}) = \left\{\dfrac{y_k + y_{k-1}}{2}\right\}$, creates a new signal by averaging two consecutive terms in the given signal. The constant signal of all ones is given by $\chi = \{1^k\}$ and the alternating signal by $\alpha = \{(-1)^k\}$.

29. Show that χ is an eigenvector of the shift transformation S. What is the associated eigenvalue?

30. Show that α is an eigenvector of the shift transformation S. What is the associated eigenvalue?

31. Show that α is an eigenvector of the moving average transformation M_2. What is the associated eigenvalue?

32. Show that χ is an eigenvector of the moving average transformation M_2. What is the associated eigenvalue?

In Exercises 33 and 34, find the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$ when $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

T **33.** $A = \begin{bmatrix} -14 & 4 & -14 \\ -33 & 9 & -31 \\ 11 & -4 & 11 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$

T **34.** $A = \begin{bmatrix} -7 & -48 & -16 \\ 1 & 14 & 6 \\ -3 & -45 & -19 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$

T **35.** Let T be the transformation whose standard matrix is given below. Find a basis for $\mathbb{R}^4$ with the property that $[T]_{\mathcal{B}}$ is diagonal.

$$A = \begin{bmatrix} 15 & -66 & -44 & -33 \\ 0 & 13 & 21 & -15 \\ 1 & -15 & -21 & 12 \\ 2 & -18 & -22 & 8 \end{bmatrix}$$

5.5 Exercises

Let each matrix in Exercises 1–6 act on $\mathbb{C}^2$. Find the eigenvalues and a basis for each eigenspace in $\mathbb{C}^2$.

1. $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

2. $\begin{bmatrix} -1 & -1 \\ 5 & -5 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix}$

4. $\begin{bmatrix} -7 & 1 \\ -5 & -3 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$

In Exercises 7–12, use Example 6 to list the eigenvalues of A. In each case, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the composition of a rotation and a scaling. Give the angle φ of the rotation, where $-\pi < \varphi \le \pi$, and give the scale factor r.

7. $\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

8. $\begin{bmatrix} \sqrt{3} & 3 \\ -3 & \sqrt{3} \end{bmatrix}$

9. $\begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}$

10. $\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$

11. $\begin{bmatrix} .1 & .1 \\ -.1 & .1 \end{bmatrix}$

12. $\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$

In Exercises 13–20, find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that the given matrix has the form $A = PCP^{-1}$. For Exercises 13–16, use information from Exercises 1–4.

13. $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

14. $\begin{bmatrix} -1 & -1 \\ 5 & -5 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix}$

16. $\begin{bmatrix} -7 & 1 \\ -5 & -3 \end{bmatrix}$

17. $\begin{bmatrix} 1 & -.8 \\ 4 & -2.2 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -1 \\ .4 & .6 \end{bmatrix}$

19. $\begin{bmatrix} 1.52 & -.7 \\ .56 & .4 \end{bmatrix}$

20. $\begin{bmatrix} -1.64 & -2.4 \\ 1.92 & 2.2 \end{bmatrix}$

21. In Example 2, solve the first equation in (2) for x_2 in terms of x_1, and from that produce the eigenvector $\mathbf{y} = \begin{bmatrix} 2 \\ -1+2i \end{bmatrix}$ for the matrix A. Show that this $\mathbf{y}$ is a (complex) multiple of the vector $\mathbf{v}_1$ used in Example 2.

22. Let A be a complex (or real) $n \times n$ matrix, and let $\mathbf{x}$ in $\mathbb{C}^n$ be an eigenvector corresponding to an eigenvalue λ in $\mathbb{C}$. Show that for each nonzero complex scalar μ, the vector $\mu\mathbf{x}$ is an eigenvector of A.

In Exercises 23–26, A is a 2×2 matrix with real entries, and $\mathbf{x}$ is a vector in $\mathbb{R}^2$. Mark each statement True or False **(T/F)**. Justify each answer.

23. **(T/F)** The matrix A can have one real and one complex eigenvalue.

24. **(T/F)** The points $A\mathbf{x}, A^2\mathbf{x}, A^3\mathbf{x}, \ldots$ always lie on the same circle.

25. **(T/F)** The matrix A always has two eigenvalues, but sometimes they have algebraic multiplicity 2 or are complex numbers.

26. **(T/F)** If the matrix A has two complex eigenvalues, then it also has two linearly independent real eigenvectors.

Chapter 7 will focus on matrices A with the property that $A^T = A$. Exercises 27 and 28 show that every eigenvalue of such a matrix is necessarily real.

27. Let A be an $n \times n$ real matrix with the property that $A^T = A$, let $\mathbf{x}$ be any vector in $\mathbb{C}^n$, and let $q = \overline{\mathbf{x}}^T A\mathbf{x}$. The equalities below show that q is a real number by verifying that $\overline{q} = q$. Give a reason for each step.

$$\overline{q} = \overline{\overline{\mathbf{x}}^T A\mathbf{x}} \underset{(a)}{=} \mathbf{x}^T \overline{A\mathbf{x}} \underset{(b)}{=} \mathbf{x}^T A\overline{\mathbf{x}} \underset{(c)}{=} (\mathbf{x}^T A\overline{\mathbf{x}})^T \underset{(d)}{=} \overline{\mathbf{x}}^T A^T \mathbf{x} \underset{(e)}{=} q$$

28. Let A be an $n \times n$ real matrix with the property that $A^T = A$. Show that if $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}$ in $\mathbb{C}^n$, then, in fact, λ is real and the real part of $\mathbf{x}$ is an eigenvector of A. [*Hint:* Compute $\overline{\mathbf{x}}^T A\mathbf{x}$, and use Exercise 27. Also, examine the real and imaginary parts of $A\mathbf{x}$.]

29. Let A be a real $n \times n$ matrix, and let $\mathbf{x}$ be a vector in $\mathbb{C}^n$. Show that $\text{Re}(A\mathbf{x}) = A(\text{Re}\,\mathbf{x})$ and $\text{Im}(A\mathbf{x}) = A(\text{Im}\,\mathbf{x})$.

30. Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \ne 0)$ and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$.
    a. Show that $A(\text{Re}\,\mathbf{v}) = a\,\text{Re}\,\mathbf{v} + b\,\text{Im}\,\mathbf{v}$ and $A(\text{Im}\,\mathbf{v}) = -b\,\text{Re}\,\mathbf{v} + a\,\text{Im}\,\mathbf{v}$. [*Hint:* Write $\mathbf{v} = \text{Re}\,\mathbf{v} + i\,\text{Im}\,\mathbf{v}$, and compute $A\mathbf{v}$.]
    b. Verify that if P and C are given as in Theorem 9, then $AP = PC$.

T In Exercises 31 and 32, find a factorization of the given matrix A in the form $A = PCP^{-1}$, where C is a block-diagonal matrix with 2×2 blocks of the form shown in Example 6. (For each conjugate pair of eigenvalues, use the real and imaginary parts of one eigenvector in $\mathbb{C}^4$ to create two columns of P.)

31. $\begin{bmatrix} .7 & 1.1 & 2.0 & 1.7 \\ -2.0 & -4.0 & -8.6 & -7.4 \\ 0 & -.5 & -1.0 & -1.0 \\ 1.0 & 2.8 & 6.0 & 5.3 \end{bmatrix}$

32. $\begin{bmatrix} -1.4 & -2.0 & -2.0 & -2.0 \\ -1.3 & -.8 & -.1 & -.6 \\ .3 & -1.9 & -1.6 & -1.4 \\ 2.0 & 3.3 & 2.3 & 2.6 \end{bmatrix}$

decomposition of $\mathbf{x}_0$ is positive when the entries in $\mathbf{x}_0$ are nonnegative. Thus the owl population will grow slowly, with a long-term growth rate of 1.01. The eigenvector $\mathbf{v}_1$ describes the eventual distribution of the owls by life stages: for every 31 adults, there will be about 10 juveniles and 3 subadults. ■

Practice Problems

1. The matrix A below has eigenvalues 1, $\frac{2}{3}$, and $\frac{1}{3}$, with corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$:

$$A = \frac{1}{9}\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Find the general solution of the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ if $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 11 \\ -2 \end{bmatrix}$.

2. What happens to the sequence $\{\mathbf{x}_k\}$ in Practice Problem 1 as $k \to \infty$?

5.6 Exercises

1. Let A be a 2×2 matrix with eigenvalues 3 and 1/3 and corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Let $\{\mathbf{x}_k\}$ be a solution of the difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$, $\mathbf{x}_0 = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$.
   a. Compute $\mathbf{x}_1 = A\mathbf{x}_0$. [*Hint:* You do not need to know A itself.]
   b. Find a formula for $\mathbf{x}_k$ involving k and the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

2. Suppose the eigenvalues of a 3×3 matrix A are 3, 4/5, and 3/5, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$, and $\begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}$. Let $\mathbf{x}_0 = \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix}$. Find the solution of the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for the specified $\mathbf{x}_0$, and describe what happens as $k \to \infty$.

In Exercises 3–6, assume that any initial vector $\mathbf{x}_0$ has an eigenvector decomposition such that the coefficient c_1 in equation (1) of this section is positive.[3]

3. Determine the evolution of the dynamical system in Example 1 when the predation parameter p is .2 in equation (3). (Give a formula for $\mathbf{x}_k$.) Does the owl population grow or decline? What about the wood rat population?

4. Determine the evolution of the dynamical system in Example 1 when the predation parameter p is .125. (Give a formula for $\mathbf{x}_k$.) As time passes, what happens to the sizes of the owl and wood rat populations? The system tends toward what is sometimes called an unstable equilibrium. What do you think might happen to the system if some aspect of the model (such as birth rates or the predation rate) were to change slightly?

5. The tawny owl is a widespread breeding species in Europe that feeds mostly on mice. Suppose the predator–prey matrix for these two populations is $A = \begin{bmatrix} .5 & .4 \\ -p & 1.2 \end{bmatrix}$. Show that if the predation parameter p is .15, both populations grow. Estimate the long-term growth rate and the eventual ratio of owls to mice.

6. Show that if the predation parameter p in Exercise 5 is .3, both the owls and the mice will eventually perish. Find a value of p for which populations of both owls and mice tend toward constant levels. What are the relative population sizes in this case?

7. Let A have the properties described in Exercise 1.
   a. Is the origin an attractor, a repeller, or a saddle point of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$?
   b. Find the directions of greatest attraction and/or repulsion for this dynamical system.
   c. Make a graphical description of the system, showing the directions of greatest attraction or repulsion. Include a rough sketch of several typical trajectories (without computing specific points).

8. Determine the nature of the origin (attractor, repeller, or saddle point) for the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ if A has

[3] One of the limitations of the model in Example 1 is that there always exist initial population vectors $\mathbf{x}_0$ with positive entries such that the coefficient c_1 is negative. The approximation (7) is still valid, but the entries in $\mathbf{x}_k$ eventually become negative.

the properties described in Exercise 2. Find the directions of greatest attraction or repulsion.

In Exercises 9–14, classify the origin as an attractor, repeller, or saddle point of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$. Find the directions of greatest attraction and/or repulsion.

9. $A = \begin{bmatrix} 1.7 & -.3 \\ -1.2 & .8 \end{bmatrix}$ **10.** $A = \begin{bmatrix} .3 & .4 \\ -.3 & 1.1 \end{bmatrix}$

11. $A = \begin{bmatrix} .4 & .5 \\ -.4 & 1.3 \end{bmatrix}$ **12.** $A = \begin{bmatrix} .5 & .6 \\ -.3 & 1.4 \end{bmatrix}$

13. $A = \begin{bmatrix} .8 & .3 \\ -.4 & 1.5 \end{bmatrix}$ **14.** $A = \begin{bmatrix} 1.7 & .6 \\ -.4 & .7 \end{bmatrix}$

15. Let $A = \begin{bmatrix} .4 & 0 & .2 \\ .3 & .8 & .3 \\ .3 & .2 & .5 \end{bmatrix}$. The vector $\mathbf{v}_1 = \begin{bmatrix} .1 \\ .6 \\ .3 \end{bmatrix}$ is an eigenvector for A, and two eigenvalues are .5 and .2. Construct the solution of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ that satisfies $\mathbf{x}_0 = (0, .3, .7)$. What happens to $\mathbf{x}_k$ as $k \to \infty$?

T 16. Produce the general solution of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ when $A = \begin{bmatrix} .90 & .01 & .09 \\ .01 & .90 & .01 \\ .09 & .09 & .90 \end{bmatrix}$.

17. Construct a stage-matrix model for an animal species that has two life stages: juvenile (up to 1 year old) and adult. Suppose the female adults give birth each year to an average of 1.6 female juveniles. Each year, 30% of the juveniles survive to become adults and 80% of the adults survive. For $k \geq 0$, let $\mathbf{x}_k = (j_k, a_k)$, where the entries in $\mathbf{x}_k$ are the numbers of female juveniles and female adults in year k.

a. Construct the stage-matrix A such that $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for $k \geq 0$.

b. Show that the population is growing, compute the eventual growth rate of the population, and give the eventual ratio of juveniles to adults.

T c. Suppose that initially there are 15 juveniles and 10 adults in the population. Produce four graphs that show how the population changes over eight years: (a) the number of juveniles, (b) the number of adults, (c) the total population, and (d) the ratio of juveniles to adults (each year). When does the ratio in (d) seem to stabilize? Include a listing of the program or keystrokes used to produce the graphs for (c) and (d).

18. Manta ray populations can be modeled by a stage matrix similar to that for the spotted owls. The females can be divided into yearlings (up to 1 year old), juveniles (1 to 9 years), and adults. Suppose an average of 50 female rays are born each year per 100 adult females. (Only adults produce offspring.) Each year, about 63% of the yearlings survive, 86% of the juveniles survive (among which 11% become adults), and 95% of the adults survive. For $k \geq 0$, let $\mathbf{x}_k = (y_k, j_k, a_k)$, where the entries in $\mathbf{x}_k$ are the numbers of females in each life stage at year k.

a. Construct the stage-matrix A for the manta ray population, such that $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for $k \geq 0$.

T b. Show that the manta ray population is growing, determine the expected growth rate after many years, and give the expected numbers of yearlings and juveniles present per 100 adults.

Solutions to Practice Problems

1. The first step is to write $\mathbf{x}_0$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Row reduction of $[\,\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3 \;\; \mathbf{x}_0\,]$ produces the weights $c_1 = 2$, $c_2 = 1$, and $c_3 = 3$, so that

$$\mathbf{x}_0 = 2\mathbf{v}_1 + 1\mathbf{v}_2 + 3\mathbf{v}_3$$

Since the eigenvalues are 1, $\frac{2}{3}$, and $\frac{1}{3}$, the general solution is

$$\begin{aligned} \mathbf{x}_k &= 2 \cdot 1^k \mathbf{v}_1 + 1 \cdot \left(\frac{2}{3}\right)^k \mathbf{v}_2 + 3 \cdot \left(\frac{1}{3}\right)^k \mathbf{v}_3 \\ &= 2\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + \left(\frac{2}{3}\right)^k \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 3 \cdot \left(\frac{1}{3}\right)^k \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \end{aligned} \tag{12}$$

2. As $k \to \infty$, the second and third terms in (12) tend to the zero vector, and

$$\mathbf{x}_k = 2\mathbf{v}_1 + \left(\frac{2}{3}\right)^k \mathbf{v}_2 + 3\left(\frac{1}{3}\right)^k \mathbf{v}_3 \to 2\mathbf{v}_1 = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$$

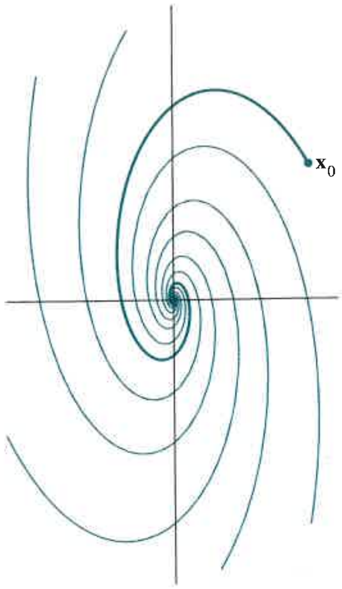


FIGURE 5
The origin as a spiral point.

Since $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent functions, they form a basis for the two-dimensional real vector space of solutions of $\mathbf{x}' = A\mathbf{x}$. Thus the general solution is

$$\mathbf{x}(t) = c_1\begin{bmatrix} -\sin 5t \\ 2\cos 5t \end{bmatrix} e^{-2t} + c_2\begin{bmatrix} \cos 5t \\ 2\sin 5t \end{bmatrix} e^{-2t}$$

To satisfy $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, we need $c_1\begin{bmatrix} 0 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, which leads to $c_1 = 1.5$ and $c_2 = 3$. Thus

$$\mathbf{x}(t) = 1.5\begin{bmatrix} -\sin 5t \\ 2\cos 5t \end{bmatrix} e^{-2t} + 3\begin{bmatrix} \cos 5t \\ 2\sin 5t \end{bmatrix} e^{-2t}$$

or

$$\begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} = \begin{bmatrix} -1.5\sin 5t + 3\cos 5t \\ 3\cos 5t + 6\sin 5t \end{bmatrix} e^{-2t}$$

See Figure 5. ■

In Figure 5, the origin is called a **spiral point** of the dynamical system. The rotation is caused by the sine and cosine functions that arise from a complex eigenvalue. The trajectories spiral inward because the factor e^{-2t} tends to zero. Recall that -2 is the real part of the eigenvalue in Example 3. When A has a complex eigenvalue with positive real part, the trajectories spiral outward. If the real part of the eigenvalue is zero, the trajectories form ellipses around the origin.

Practice Problems

A real 3×3 matrix A has eigenvalues $-.5$, $.2 + .3i$, and $.2 - .3i$, with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1+2i \\ 4i \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1-2i \\ -4i \\ 2 \end{bmatrix}$$

1. Is A diagonalizable as $A = PDP^{-1}$, using complex matrices?
2. Write the general solution of $\mathbf{x}' = A\mathbf{x}$ using complex eigenfunctions, and then find the general real solution.
3. Describe the shapes of typical trajectories.

5.7 Exercises

1. A particle moving in a planar force field has a position vector $\mathbf{x}$ that satisfies $\mathbf{x}' = A\mathbf{x}$. The 2×2 matrix A has eigenvalues 4 and 2, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Find the position of the particle at time t, assuming that $\mathbf{x}(0) = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$.
2. Let A be a 2×2 matrix with eigenvalues -3 and -1 and corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let $\mathbf{x}(t)$ be the position of a particle at time t. Solve the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

In Exercises 3–6, solve the initial value problem $\mathbf{x}'(t) = A\mathbf{x}(t)$ for $t \geq 0$, with $\mathbf{x}(0) = (3, 2)$. Classify the nature of the origin as an attractor, repeller, or saddle point of the dynamical system described by $\mathbf{x}' = A\mathbf{x}$. Find the directions of greatest attraction and/or repulsion. When the origin is a saddle point, sketch typical trajectories.

3. $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$
4. $A = \begin{bmatrix} -2 & -5 \\ 1 & 4 \end{bmatrix}$
5. $A = \begin{bmatrix} 2 & -4 \\ 5 & -7 \end{bmatrix}$
6. $A = \begin{bmatrix} 7 & -3 \\ 5 & -1 \end{bmatrix}$

In Exercises 7 and 8, make a change of variable that decouples the equation $\mathbf{x}' = A\mathbf{x}$. Write the equation $\mathbf{x}(t) = P\mathbf{y}(t)$ and show the calculation that leads to the uncoupled system $\mathbf{y}' = D\mathbf{y}$, specifying P and D.

7. A as in Exercise 5

8. A as in Exercise 6

In Exercises 9–18, construct the general solution of $\mathbf{x}' = A\mathbf{x}$ involving complex eigenfunctions and then obtain the general real solution. Describe the shapes of typical trajectories.

9. $A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$

10. $A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

11. $A = \begin{bmatrix} -3 & -9 \\ 2 & 3 \end{bmatrix}$

12. $A = \begin{bmatrix} -7 & 10 \\ -4 & 5 \end{bmatrix}$

13. $A = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix}$

14. $A = \begin{bmatrix} -3 & 2 \\ -9 & 3 \end{bmatrix}$

T 15. $A = \begin{bmatrix} -8 & -12 & -6 \\ 2 & 1 & 2 \\ 7 & 12 & 5 \end{bmatrix}$

T 16. $A = \begin{bmatrix} -6 & -11 & 16 \\ 2 & 5 & -4 \\ -4 & -5 & 10 \end{bmatrix}$

T 17. $A = \begin{bmatrix} 30 & 64 & 23 \\ -11 & -23 & -9 \\ 6 & 15 & 4 \end{bmatrix}$

T 18. $A = \begin{bmatrix} 53 & -30 & -2 \\ 90 & -52 & -3 \\ 20 & -10 & 2 \end{bmatrix}$

T 19. Find formulas for the voltages v_1 and v_2 (as functions of time t) for the circuit in Example 1, assuming that $R_1 = 1/5$ ohm, $R_2 = 1/3$ ohm, $C_1 = 4$ farads, $C_2 = 3$ farads, and the initial charge on each capacitor is 4 volts.

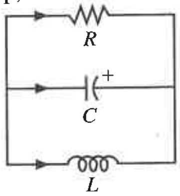
T 20. Find formulas for the voltages v_1 and v_2 for the circuit in Example 1, assuming that $R_1 = 1/15$ ohm, $R_2 = 1/3$ ohm, $C_1 = 9$ farads, $C_2 = 2$ farads, and the initial charge on each capacitor is 3 volts.

T 21. Find formulas for the current i_L and the voltage v_C for the circuit in Example 3, assuming that $R_1 = 1$ ohm, $R_2 = .125$ ohm, $C = .2$ farad, $L = .125$ henry, the initial current is 0 amp, and the initial voltage is 15 volts.

T 22. The circuit in the figure is described by the equation

$$\begin{bmatrix} i_L' \\ v_C' \end{bmatrix} = \begin{bmatrix} 0 & 1/L \\ -1/C & -1/(RC) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix}$$

where i_L is the current through the inductor L and v_C is the voltage drop across the capacitor C. Find formulas for i_L and v_C when $R = .5$ ohm, $C = 2.5$ farads, $L = .5$ henry, the initial current is 0 amp, and the initial voltage is 12 volts.



Solutions to Practice Problems

1. Yes, the 3×3 matrix is diagonalizable because it has three distinct eigenvalues. Theorem 2 in Section 5.1 and Theorem 6 in Section 5.3 are valid when complex scalars are used. (The proofs are essentially the same as for real scalars.)

2. The general solution has the form

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-.5t} + c_2 \begin{bmatrix} 1+2i \\ 4i \\ 2 \end{bmatrix} e^{(.2+.3i)t} + c_3 \begin{bmatrix} 1-2i \\ -4i \\ 2 \end{bmatrix} e^{(.2-.3i)t}$$

The scalars c_1, c_2, and c_3 here can be any complex numbers. The first term in $\mathbf{x}(t)$ is real, provided c_1 is real. Two more real solutions can be produced using the real and imaginary parts of the second term in $\mathbf{x}(t)$:

$$\begin{bmatrix} 1+2i \\ 4i \\ 2 \end{bmatrix} e^{.2t}(\cos .3t + i \sin .3t)$$

The general real solution has the following form, with *real* scalars c_1, c_2, and c_3:

$$c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-.5t} + c_2 \begin{bmatrix} \cos .3t - 2\sin .3t \\ -4\sin .3t \\ 2\cos .3t \end{bmatrix} e^{.2t} + c_3 \begin{bmatrix} \sin .3t + 2\cos .3t \\ 4\cos .3t \\ 2\sin .3t \end{bmatrix} e^{.2t}$$

5.8 Exercises

In Exercises 1–4, the matrix A is followed by a sequence $\{\mathbf{x}_k\}$ produced by the power method. Use these data to estimate the largest eigenvalue of A, and give a corresponding eigenvector.

1. $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$;
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ .2 \end{bmatrix}, \begin{bmatrix} 1 \\ .2414 \end{bmatrix}, \begin{bmatrix} 1 \\ .2486 \end{bmatrix}, \begin{bmatrix} 1 \\ .2498 \end{bmatrix}$

2. $A = \begin{bmatrix} 1.8 & -.8 \\ -3.2 & 4.2 \end{bmatrix}$;
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -.5625 \\ 1 \end{bmatrix}, \begin{bmatrix} -.3021 \\ 1 \end{bmatrix}, \begin{bmatrix} -.2601 \\ 1 \end{bmatrix}, \begin{bmatrix} -.2520 \\ 1 \end{bmatrix}$

3. $A = \begin{bmatrix} .5 & .2 \\ .4 & .7 \end{bmatrix}$;
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ .8 \end{bmatrix}, \begin{bmatrix} .6875 \\ 1 \end{bmatrix}, \begin{bmatrix} .5577 \\ 1 \end{bmatrix}, \begin{bmatrix} .5188 \\ 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 4.1 & -6 \\ 3 & -4.4 \end{bmatrix}$;
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ .7368 \end{bmatrix}, \begin{bmatrix} 1 \\ .7541 \end{bmatrix}, \begin{bmatrix} 1 \\ .7490 \end{bmatrix}, \begin{bmatrix} 1 \\ .7502 \end{bmatrix}$

5. Let $A = \begin{bmatrix} 15 & 16 \\ -20 & -21 \end{bmatrix}$. The vectors $\mathbf{x}, \ldots, A^5\mathbf{x}$ are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$,
$\begin{bmatrix} 31 \\ -41 \end{bmatrix}, \begin{bmatrix} -191 \\ 241 \end{bmatrix}, \begin{bmatrix} 991 \\ -1241 \end{bmatrix}, \begin{bmatrix} -4991 \\ 6241 \end{bmatrix}, \begin{bmatrix} 24991 \\ -31241 \end{bmatrix}$.
Find a vector with a 1 in the second entry that is close to an eigenvector of A. Use four decimal places. Check your estimate, and give an estimate for the dominant eigenvalue of A.

6. Let $A = \begin{bmatrix} -3 & -4 \\ 8 & 9 \end{bmatrix}$. Repeat Exercise 5, using the following sequence $\mathbf{x}, A\mathbf{x}, \ldots, A^5\mathbf{x}$.
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -7 \\ 17 \end{bmatrix}, \begin{bmatrix} -47 \\ 97 \end{bmatrix}, \begin{bmatrix} -247 \\ 497 \end{bmatrix}, \begin{bmatrix} -1247 \\ 2497 \end{bmatrix}, \begin{bmatrix} -6247 \\ 12497 \end{bmatrix}$

Exercises 7–12 require MATLAB or other computational aid. In Exercises 7 and 8, use the power method with the $\mathbf{x}_0$ given. List $\{\mathbf{x}_k\}$ and $\{\mu_k\}$ for $k = 1, \ldots, 5$. In Exercises 9 and 10, list μ_5 and μ_6.

T 7. $A = \begin{bmatrix} 6 & 7 \\ 8 & 5 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

T 8. $A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

T 9. $A = \begin{bmatrix} 8 & 0 & 12 \\ 1 & -2 & 1 \\ 0 & 3 & 0 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

T 10. $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 9 \\ 0 & 1 & 9 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Another estimate can be made for an eigenvalue when an approximate eigenvector is available. Observe that if $A\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T(\lambda\mathbf{x}) = \lambda(\mathbf{x}^T\mathbf{x})$, and the **Rayleigh quotient**

$$R(\mathbf{x}) = \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$$

equals λ. If $\mathbf{x}$ is close to an eigenvector for λ, then this quotient is close to λ. When A is a symmetric matrix ($A^T = A$), the Rayleigh quotient $R(\mathbf{x}_k) = (\mathbf{x}_k^T A\mathbf{x}_k)/(\mathbf{x}_k^T\mathbf{x}_k)$ will have roughly twice as many digits of accuracy as the scaling factor μ_k in the power method. Verify this increased accuracy in Exercises 11 and 12 by computing μ_k and $R(\mathbf{x}_k)$ for $k = 1, \ldots, 4$.

T 11. $A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

T 12. $A = \begin{bmatrix} -3 & 2 \\ 2 & 0 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Exercises 13 and 14 apply to a 3×3 matrix A whose eigenvalues are estimated to be 4, -4, and 3.

13. If the eigenvalues close to 4 and -4 are known to have different absolute values, will the power method work? Is it likely to be useful?

14. Suppose the eigenvalues close to 4 and -4 are known to have exactly the same absolute value. Describe how one might obtain a sequence that estimates the eigenvalue close to 4.

15. Suppose $A\mathbf{x} = \lambda\mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$. Let α be a scalar different from the eigenvalues of A, and let $B = (A - \alpha I)^{-1}$. Subtract $\alpha\mathbf{x}$ from both sides of the equation $A\mathbf{x} = \lambda\mathbf{x}$, and use algebra to show that $1/(\lambda - \alpha)$ is an eigenvalue of B, with $\mathbf{x}$ a corresponding eigenvector.

16. Suppose μ is an eigenvalue of the B in Exercise 15, and that $\mathbf{x}$ is a corresponding eigenvector, so that $(A - \alpha I)^{-1}\mathbf{x} = \mu\mathbf{x}$. Use this equation to find an eigenvalue of A in terms of μ and α. [*Note:* $\mu \neq 0$ because B is invertible.]

T 17. Use the inverse power method to estimate the middle eigenvalue of the A in Example 3, with accuracy to four decimal places. Set $\mathbf{x}_0 = (1, 0, 0)$.

T 18. Let A be as in Exercise 9. Use the inverse power method with $\mathbf{x}_0 = (1, 0, 0)$ to estimate the eigenvalue of A near $\alpha = -1.4$, with an accuracy to four decimal places.

In Exercises 19 and 20, find (a) the largest eigenvalue and (b) the eigenvalue closest to zero. In each case, set $\mathbf{x}_0 = (1, 0, 0, 0)$ and carry out approximations until the approximating sequence seems accurate to four decimal places. Include the approximate eigenvector.

19. $A = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix}$

20. $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 12 & 13 & 11 \\ -2 & 3 & 0 & 2 \\ 4 & 5 & 7 & 2 \end{bmatrix}$

21. A common misconception is that if A has a strictly dominant eigenvalue, then, for any sufficiently large value of k, the vector $A^k\mathbf{x}$ is approximately equal to an eigenvector of A. For the three matrices below, study what happens to $A^k\mathbf{x}$ when $\mathbf{x} = (.5, .5)$, and try to draw general conclusions (for a 2×2 matrix).

a. $A = \begin{bmatrix} .8 & 0 \\ 0 & .2 \end{bmatrix}$ b. $A = \begin{bmatrix} 1 & 0 \\ 0 & .8 \end{bmatrix}$

c. $A = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$

Solution to Practice Problem

For the given A and $\mathbf{x}$,

$$A\mathbf{x} = \begin{bmatrix} 5 & 8 & 4 \\ 8 & 3 & -1 \\ 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1.00 \\ -4.30 \\ 8.10 \end{bmatrix} = \begin{bmatrix} 3.00 \\ -13.00 \\ 24.50 \end{bmatrix}$$

If $A\mathbf{x}$ is nearly a multiple of $\mathbf{x}$, then the ratios of corresponding entries in the two vectors should be nearly constant. So compute:

{entry in $A\mathbf{x}$} $\div$	{entry in $\mathbf{x}$} $=$	{ratio}
3.00	1.00	3.000
−13.00	−4.30	3.023
24.50	8.10	3.025

Each entry in $A\mathbf{x}$ is about 3 times the corresponding entry in $\mathbf{x}$, so $\mathbf{x}$ is close to an eigenvector. Any of the ratios above is an estimate for the eigenvalue. (To five decimal places, the eigenvalue is 3.02409.)

5.9 Applications to Markov Chains

The Markov chains described in this section are used as mathematical models of a wide variety of situations in biology, business, chemistry, engineering, physics, and elsewhere. In each case, the model is used to describe an experiment or measurement that is performed many times in the same way, where the outcome of each trial of the experiment will be one of several specified possible outcomes, and where the outcome of one trial depends only on the immediately preceding trial.

For example, if the population of a city and its suburbs were measured each year, then a vector such as

$$\mathbf{x}_0 = \begin{bmatrix} .60 \\ .40 \end{bmatrix} \tag{1}$$

could indicate that 60% of the population lives in the city and 40% in the suburbs. The decimals in $\mathbf{x}_0$ add up to 1 because they account for the entire population of the region. Percentages are more convenient for our purposes here than population totals.

5.9 Exercises

1. A small remote village receives radio broadcasts from two radio stations, a news station and a music station. Of the listeners who are tuned to the news station, 70% will remain listening to the news after the station break that occurs each half hour, while 30% will switch to the music station at the station break. Of the listeners who are tuned to the music station, 60% will switch to the news station at the station break, while 40% will remain listening to the music. Suppose everyone is listening to the news at 8:15 A.M.
   a. Give the stochastic matrix that describes how the radio listeners tend to change stations at each station break. Label the rows and columns.
   b. Give the initial state vector.
   c. What percentage of the listeners will be listening to the music station at 9:25 A.M. (after the station breaks at 8:30 and 9:00 A.M.)?

2. A laboratory animal may eat any one of three foods each day. Laboratory records show that if the animal chooses one food on one trial, it will choose the same food on the next trial with a probability of 50%, and it will choose the other foods on the next trial with equal probabilities of 25%.
   a. What is the stochastic matrix for this situation?
   b. If the animal chooses food #1 on an initial trial, what is the probability that it will choose food #2 on the second trial after the initial trial?

3. On any given day, a student is either healthy or ill. Of the students who are healthy today, 95% will be healthy tomorrow. Of the students who are ill today, 55% will still be ill tomorrow.
   a. What is the stochastic matrix for this situation?
   b. Suppose 20% of the students are ill on Monday. What fraction or percentage of the students are likely to be ill on Tuesday? On Wednesday?
   c. If a student is well today, what is the probability that he or she will be well two days from now?

4. The weather in Edinburgh is either good, indifferent, or bad on any given day. If the weather is good today, there is a 50% chance the weather will be good tomorrow, a 30% chance the weather will be indifferent, and a 20% chance the weather will be bad. If the weather is indifferent today, it will be good tomorrow with probability .20 and indifferent with probability .70. Finally, if the weather is bad today, it will be good tomorrow with probability .10 and indifferent with probability .30.
   a. What is the stochastic matrix for this situation?
   b. Suppose there is a 30% chance of bad weather today and a 70% chance of indifferent weather. What are the chances of good weather tomorrow?
   c. Suppose the predicted weather for Friday is 50% indifferent weather and 50% good weather. What are the chances for bad weather on Sunday?

In Exercises 5–8, find the steady-state vector.

5. $\begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}$

6. $\begin{bmatrix} .8 & .5 \\ .2 & .5 \end{bmatrix}$

7. $\begin{bmatrix} .7 & .1 & .1 \\ .2 & .8 & .2 \\ .1 & .1 & .7 \end{bmatrix}$

8. $\begin{bmatrix} .7 & .2 & .2 \\ 0 & .2 & .4 \\ .3 & .6 & .4 \end{bmatrix}$

9. Determine if $P = \begin{bmatrix} .7 & 0 \\ .3 & 1 \end{bmatrix}$ is a regular stochastic matrix.

10. Determine if $P = \begin{bmatrix} 0 & .7 \\ 1 & .3 \end{bmatrix}$ is a regular stochastic matrix.

11. a. Find the steady-state vector for the Markov chain in Exercise 1.
    b. At some time late in the day, what fraction of the listeners will be listening to the news?

12. Refer to Exercise 2. Which food will the animal prefer after many trials?

13. a. Find the steady-state vector for the Markov chain in Exercise 3.
    b. What is the probability that after many days a specific student is ill? Does it matter if that person is ill today?

14. Refer to Exercise 4. In the long run, how likely is it for the weather in Edinburgh to be good on a given day?

In Exercises 15–20, P is an $n \times n$ stochastic matrix. Mark each statement True or False **(T/F)**. Justify each answer.

15. **(T/F)** The steady state vector is an eigenvector of P.

16. **(T/F)** Every eigenvector of P is a steady state vector.

17. **(T/F)** The all ones vector is an eigenvector of P^T.

18. **(T/F)** The number 2 can be an eigenvalue of a stochastic matrix.

19. **(T/F)** The number 1/2 can be an eigenvalue of a stochastic matrix.

20. **(T/F)** All stochastic matrices are regular.

21. Is $\mathbf{q} = \begin{bmatrix} .6 \\ .8 \end{bmatrix}$ a steady state vector for $A = \begin{bmatrix} .2 & .6 \\ .8 & .4 \end{bmatrix}$? Justify your answer.

22. Is $\mathbf{q} = \begin{bmatrix} .4 \\ .4 \end{bmatrix}$ a steady state vector for $A = \begin{bmatrix} .2 & .8 \\ .8 & .2 \end{bmatrix}$? Justify your answer.

23. Is $\mathbf{q} = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$ a steady state vector for $A = \begin{bmatrix} .4 & .6 \\ .6 & .4 \end{bmatrix}$? Justify your answer.

24. Is $\mathbf{q} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$ a steady state vector for $A = \begin{bmatrix} .2 & .6 \\ .8 & .4 \end{bmatrix}$? Justify your answer.

T 25. Suppose the following matrix describes the likelihood that an individual will switch between an iOS and an Android smartphone:

$$\begin{array}{cc} \text{From} & \text{To} \\ \text{iOS Android} & \\ \begin{bmatrix} .70 & .15 \\ .30 & .85 \end{bmatrix} & \begin{array}{l} \text{iOS} \\ \text{Android} \end{array} \end{array}$$

In the long run, what percentage of smartphone owners would you expect to have an Android operating system?

T 26. In Rome, Europcar Rent A Car has a fleet of about 2500 cars. The pattern of rental and return locations is given by the fractions in the table below. On a typical day, about how many cars will be rented or ready to rent from the Fiumicino Airport?

$$\begin{array}{cc} \text{Cars Rented from:} & \\ \begin{array}{ccc} \text{Ciampino Airport} & \text{Railway Station} & \text{Fiumicino Airport} \end{array} & \text{Returned to} \\ \begin{bmatrix} .90 & .02 & .08 \\ .02 & .90 & .02 \\ .08 & .08 & .90 \end{bmatrix} & \begin{array}{l} \text{Ciampino Airport} \\ \text{Railway Station} \\ \text{Fiumicino Airport} \end{array} \end{array}$$

27. Let P be an $n \times n$ stochastic matrix. The following argument shows that the equation $P\mathbf{x} = \mathbf{x}$ has a nontrivial solution. (In fact, a steady-state solution exists with nonnegative entries. A proof is given in some advanced texts.) Justify each assertion below. (Mention a theorem when appropriate.)

a. If all the other rows of $P - I$ are added to the bottom row, the result is a row of zeros.

b. The rows of $P - I$ are linearly dependent.

c. The dimension of the row space of $P - I$ is less than n.

d. $P - I$ has a nontrivial null space.

28. Show that every 2×2 stochastic matrix has at least one steady-state vector. Any such matrix can be written in the form $P = \begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix}$, where α and β are constants between 0 and 1. (There are two linearly independent steady-state vectors if $\alpha = \beta = 0$. Otherwise, there is only one.)

29. Let S be the $1 \times n$ row matrix with a 1 in each column,

$$S = [\,1 \quad 1 \quad \cdots \quad 1\,]$$

a. Explain why a vector $\mathbf{x}$ in $\mathbb{R}^n$ is a probability vector if and only if its entries are nonnegative and $S\mathbf{x} = 1$. (A 1×1 matrix such as the product $S\mathbf{x}$ is usually written without the matrix bracket symbols.)

b. Let P be an $n \times n$ stochastic matrix. Explain why $SP = S$.

c. Let P be an $n \times n$ stochastic matrix, and let $\mathbf{x}$ be a probability vector. Show that $P\mathbf{x}$ is also a probability vector.

30. Use Exercise 29 to show that if P is an $n \times n$ stochastic matrix, then so is P^2.

T 31. Examine powers of a regular stochastic matrix.

a. Compute P^k for $k = 2, 3, 4, 5$, when

$$P = \begin{bmatrix} .3355 & .3682 & .3067 & .0389 \\ .2663 & .2723 & .3277 & .5451 \\ .1935 & .1502 & .1589 & .2395 \\ .2047 & .2093 & .2067 & .1765 \end{bmatrix}$$

Display calculations to four decimal places. What happens to the columns of P^k as k increases? Compute the steady-state vector for P.

b. Compute Q^k for $k = 10, 20, \ldots, 80$, when

$$Q = \begin{bmatrix} .97 & .05 & .10 \\ 0 & .90 & .05 \\ .03 & .05 & .85 \end{bmatrix}$$

(Stability for Q^k to four decimal places may require $k = 116$ or more.) Compute the steady-state vector for Q. Conjecture what might be true for any regular stochastic matrix.

c. Use Theorem 11 to explain what you found in parts (a) and (b).

T 32. Compare two methods for finding the steady-state vector $\mathbf{q}$ of a regular stochastic matrix P: (1) computing $\mathbf{q}$ as in Example 5, or (2) computing P^k for some large value of k and using one of the columns of P^k as an approximation for $\mathbf{q}$. [The *Study Guide* describes a program *nulbasis* that almost automates method (1).]

Experiment with the largest random stochastic matrices your matrix program will allow, and use $k = 100$ or some other large value. For each method, describe the time *you* need to enter the keystrokes and run your program. (Some versions of MATLAB have commands `flops` and `tic...toc` that record the number of floating point operations and the total elapsed time MATLAB uses.) Contrast the advantages of each method, and state which you prefer.