SOLUTION This question is Practice Problem 2 in Section 1.3, written here with the term *subspace* rather than Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. The solution there shows that $\mathbf{y}$ is in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if $h = 5$. That solution is worth reviewing now, along with Exercises 11–16 and 19–21 in Section 1.3. ■

Although many vector spaces in this chapter will be subspaces of $\mathbb{R}^n$, it is important to keep in mind that the abstract theory applies to other vector spaces as well. Vector spaces of functions arise in many applications, and they will receive more attention later.

Practice Problems

1. Show that the set H of all points in $\mathbb{R}^2$ of the form $(3s, 2 + 5s)$ is not a vector space, by showing that it is not closed under scalar multiplication. (Find a specific vector $\mathbf{u}$ in H and a scalar c such that $c\mathbf{u}$ is not in H.)
2. Let $W = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$, where $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space V. Show that $\mathbf{v}_k$ is in W for $1 \le k \le p$. [*Hint:* First write an equation that shows that $\mathbf{v}_1$ is in W. Then adjust your notation for the general case.]
3. An $n \times n$ matrix A is said to be symmetric if $A^T = A$. Let S be the set of all 3×3 symmetric matrices. Show that S is a subspace of $M_{3\times3}$, the vector space of 3×3 matrices.

4.1 Exercises

1. Let V be the first quadrant in the xy-plane; that is, let
$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \ge 0, y \ge 0 \right\}$$
   a. If $\mathbf{u}$ and $\mathbf{v}$ are in V, is $\mathbf{u} + \mathbf{v}$ in V? Why?
   b. Find a specific vector $\mathbf{u}$ in V and a specific scalar c such that $c\mathbf{u}$ is *not* in V. (This is enough to show that V is *not* a vector space.)
2. Let W be the union of the first and third quadrants in the xy-plane. That is, let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \ge 0 \right\}$.
   a. If $\mathbf{u}$ is in W and c is any scalar, is $c\mathbf{u}$ in W? Why?
   b. Find specific vectors $\mathbf{u}$ and $\mathbf{v}$ in W such that $\mathbf{u} + \mathbf{v}$ is not in W. (This is enough to show that W is *not* a vector space.)
3. Let H be the set of points inside and on the unit circle in the xy-plane. That is, let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \le 1 \right\}$. Find a specific example—two vectors or a vector and a scalar—to show that H is not a subspace of $\mathbb{R}^2$.
4. Construct a geometric figure that illustrates why a line in $\mathbb{R}^2$ *not* through the origin is not closed under vector addition.

In Exercises 5–8, determine if the given set is a subspace of $\mathbb{P}_n$ for an appropriate value of n. Justify your answers.

5. All polynomials of the form $\mathbf{p}(t) = at^2$, where a is in $\mathbb{R}$.
6. All polynomials of the form $\mathbf{p}(t) = a + t^2$, where a is in $\mathbb{R}$.
7. All polynomials of degree at most 3, with integers as coefficients.
8. All polynomials in $\mathbb{P}_n$ such that $\mathbf{p}(0) = 0$.
9. Let H be the set of all vectors of the form $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix}$. Find a vector $\mathbf{v}$ in $\mathbb{R}^3$ such that $H = \text{Span}\{\mathbf{v}\}$. Why does this show that H is a subspace of $\mathbb{R}^3$?
10. Let H be the set of all vectors of the form $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$. Show that H is a subspace of $\mathbb{R}^3$. (Use the method of Exercise 9.)
11. Let W be the set of all vectors of the form $\begin{bmatrix} 6b + 7c \\ b \\ c \end{bmatrix}$, where b and c are arbitrary. Find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $W = \text{Span}\ \mathbf{u}, \mathbf{v}$. Why does this show that W is a subspace of $\mathbb{R}^3$?
12. Let W be the set of all vectors of the form $\begin{bmatrix} s + 3t \\ s - t \\ 2s - t \\ 4t \end{bmatrix}$. Show that W is a subspace of $\mathbb{R}^4$. (Use the method of Exercise 11.)
13. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

a. Is $\mathbf{w}$ in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

b. How many vectors are in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

c. Is $\mathbf{w}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why?

14. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be as in Exercise 13, and let $\mathbf{w} = \begin{bmatrix} 8 \\ 4 \\ 7 \end{bmatrix}$. Is $\mathbf{w}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why?

In Exercises 15–18, let W be the set of all vectors of the form shown, where a, b, and c represent arbitrary real numbers. In each case, either find a set S of vectors that spans W or give an example to show that W is *not* a vector space.

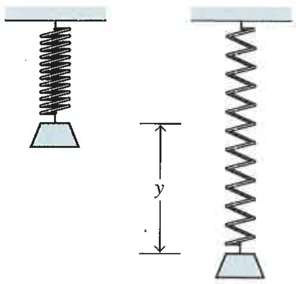
15. $\begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}$ **16.** $\begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$

17. $\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix}$ **18.** $\begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix}$

19. If a mass m is placed at the end of a spring, and if the mass is pulled downward and released, the mass–spring system will begin to oscillate. The displacement y of the mass from its resting position is given by a function of the form

$$y(t) = c_1 \cos \omega t + c_2 \sin \omega t \qquad (5)$$

where ω is a constant that depends on the spring and the mass. (See the figure below.) Show that the set of all functions described in (5) (with ω fixed and c_1, c_2 arbitrary) is a vector space.



20. The set of all continuous real-valued functions defined on a closed interval $[a, b]$ in $\mathbb{R}$ is denoted by $C[a, b]$. This set is a subspace of the vector space of all real-valued functions defined on $[a, b]$.

a. What facts about continuous functions should be proved in order to demonstrate that $C[a, b]$ is indeed a subspace as claimed? (These facts are usually discussed in a calculus class.)

b. Show that $\{\mathbf{f} \text{ in } C[a, b] : \mathbf{f}(a) = \mathbf{f}(b)\}$ is a subspace of $C[a, b]$.

For fixed positive integers m and n, the set $M_{m\times n}$ of all $m \times n$ matrices is a vector space, under the usual operations of addition of matrices and multiplication by real scalars.

21. Determine if the set H of all matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is a subspace of $M_{2\times 2}$.

22. Let F be a fixed 3×2 matrix, and let H be the set of all matrices A in $M_{2\times 4}$ with the property that $FA = 0$ (the zero matrix in $M_{3\times 4}$). Determine if H is a subspace of $M_{2\times 4}$.

In Exercises 23–32, mark each statement True or False **(T/F)**. Justify each answer.

23. **(T/F)** If $\mathbf{f}$ is a function in the vector space V of all real-valued functions on $\mathbb{R}$ and if $\mathbf{f}(t) = 0$ for some t, then $\mathbf{f}$ is the zero vector in V.

24. **(T/F)** A vector is any element of a vector space.

25. **(T/F)** An arrow in three-dimensional space can be considered to be a vector.

26. **(T/F)** If $\mathbf{u}$ is a vector in a vector space V, then $(-1)\,\mathbf{u}$ is the same as the negative of $\mathbf{u}$.

27. **(T/F)** A subset H of a vector space V is a subspace of V if the zero vector is in H.

28. **(T/F)** A vector space is also a subspace.

29. **(T/F)** A subspace is also a vector space.

30. **(T/F)** $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

31. **(T/F)** The polynomials of degree two or less are a subspace of the polynomials of degree three or less.

32. **(T/F)** A subset H of a vector space V is a subspace of V if the following conditions are satisfied: (i) the zero vector of V is in H, (ii) $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ are in H, and (iii) c is a scalar and $c\mathbf{u}$ is in H.

Exercises 33–36 show how the axioms for a vector space V can be used to prove the elementary properties described after the definition of a vector space. Fill in the blanks with the appropriate axiom numbers. Because of Axiom 2, Axioms 4 and 5 imply, respectively, that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ and $-\mathbf{u} + \mathbf{u} = \mathbf{0}$ for all $\mathbf{u}$.

33. Complete the following proof that the zero vector is unique. Suppose that $\mathbf{w}$ in V has the property that $\mathbf{u} + \mathbf{w} = \mathbf{w} + \mathbf{u} = \mathbf{u}$ for all $\mathbf{u}$ in V. In particular, $\mathbf{0} + \mathbf{w} = \mathbf{0}$. But $\mathbf{0} + \mathbf{w} = \mathbf{w}$, by Axiom ______. Hence $\mathbf{w} = \mathbf{0} + \mathbf{w} = \mathbf{0}$.

34. Complete the following proof that $-\mathbf{u}$ is the *unique vector* in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. Suppose that $\mathbf{w}$ satisfies $\mathbf{u} + \mathbf{w} = \mathbf{0}$. Adding $-\mathbf{u}$ to both sides, we have

$(-\mathbf{u}) + [\mathbf{u} + \mathbf{w}] = (-\mathbf{u}) + \mathbf{0}$

$[(-\mathbf{u}) + \mathbf{u}] + \mathbf{w} = (-\mathbf{u}) + \mathbf{0}$ by Axiom ______ (a)

$\mathbf{0} + \mathbf{w} = (-\mathbf{u}) + \mathbf{0}$ by Axiom ______ (b)

$\mathbf{w} = -\mathbf{u}$ by Axiom ______ (c)

35. Fill in the missing axiom numbers in the following proof that $0\mathbf{u} = \mathbf{0}$ for every $\mathbf{u}$ in V.

$0\mathbf{u} = (0 + 0)\mathbf{u} = 0\mathbf{u} + 0\mathbf{u}$ by Axiom ______ (a)

Add the negative of $0\mathbf{u}$ to both sides:

$$0\mathbf{u} + (-0\mathbf{u}) = [0\mathbf{u} + 0\mathbf{u}] + (-0\mathbf{u})$$

$$0\mathbf{u} + (-0\mathbf{u}) = 0\mathbf{u} + [0\mathbf{u} + (-0\mathbf{u})] \qquad \text{by Axiom \_\_\_\_ (b)}$$

$$\mathbf{0} = 0\mathbf{u} + \mathbf{0} \qquad \text{by Axiom \_\_\_\_ (c)}$$

$$\mathbf{0} = 0\mathbf{u} \qquad \text{by Axiom \_\_\_\_ (d)}$$

36. Fill in the missing axiom numbers in the following proof that $c\mathbf{0} = \mathbf{0}$ for every scalar c.

$$c\mathbf{0} = c(\mathbf{0} + \mathbf{0}) \qquad \text{by Axiom \_\_\_\_ (a)}$$

$$= c\mathbf{0} + c\mathbf{0} \qquad \text{by Axiom \_\_\_\_ (b)}$$

Add the negative of $c\mathbf{0}$ to both sides:

$$c\mathbf{0} + (-c\mathbf{0}) = [c\mathbf{0} + c\mathbf{0}] + (-c\mathbf{0})$$

$$c\mathbf{0} + (-c\mathbf{0}) = c\mathbf{0} + [c\mathbf{0} + (-c\mathbf{0})] \qquad \text{by Axiom \_\_\_\_ (c)}$$

$$\mathbf{0} = c\mathbf{0} + \mathbf{0} \qquad \text{by Axiom \_\_\_\_ (d)}$$

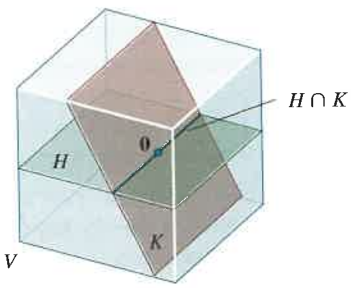
$$\mathbf{0} = c\mathbf{0} \qquad \text{by Axiom \_\_\_\_ (e)}$$

37. Prove that $(-1)\mathbf{u} = -\mathbf{u}$. [*Hint:* Show that $\mathbf{u} + (-1)\mathbf{u} = \mathbf{0}$. Use some axioms and the results of Exercises 34 and 35.]

38. Suppose $c\mathbf{u} = \mathbf{0}$ for some nonzero scalar c. Show that $\mathbf{u} = \mathbf{0}$. Mention the axioms or properties you use.

39. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in a vector space V, and let H be any subspace of V that contains both $\mathbf{u}$ and $\mathbf{v}$. Explain why H also contains Span $\{\mathbf{u}, \mathbf{v}\}$. This shows that Span $\{\mathbf{u}, \mathbf{v}\}$ is the smallest subspace of V that contains both $\mathbf{u}$ and $\mathbf{v}$.

40. Let H and K be subspaces of a vector space V. The **intersection** of H and K, written as $H \cap K$, is the set of $\mathbf{v}$ in V that belong to both H and K. Show that $H \cap K$ is a subspace of V. (See the figure.) Give an example in $\mathbb{R}^2$ to show that the union of two subspaces is not, in general, a subspace.



41. Given subspaces H and K of a vector space V, the **sum** of H and K, written as $H + K$, is the set of all vectors in V that can be written as the sum of two vectors, one in H and the other in K; that is,

$$H + K = \{\mathbf{w} : \mathbf{w} = \mathbf{u} + \mathbf{v} \text{ for some } \mathbf{u} \text{ in } H \text{ and some } \mathbf{v} \text{ in } K\}$$

a. Show that $H + K$ is a subspace of V.

b. Show that H is a subspace of $H + K$ and K is a subspace of $H + K$.

42. Suppose $\mathbf{u}_1, \dots, \mathbf{u}_p$ and $\mathbf{v}_1, \dots, \mathbf{v}_q$ are vectors in a vector space V, and let

$H = \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ and $K = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_q\}$

Show that $H + K = \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$.

43. Show that $\mathbf{w}$ is in the subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, where

$$\mathbf{w} = \begin{bmatrix} 6 \\ -7 \\ 8 \\ -9 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 7 \\ -6 \\ -5 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 2 \\ -1 \\ -4 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 2 \\ -5 \end{bmatrix}$$

44. Determine if $\mathbf{y}$ is in the subspace of $\mathbb{R}^4$ spanned by the columns of A, where

$$\mathbf{y} = \begin{bmatrix} -4 \\ -8 \\ 6 \\ -5 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & -5 & -9 \\ 8 & 7 & -6 \\ -5 & -8 & 3 \\ 2 & -2 & -9 \end{bmatrix}$$

45. The vector space $H = \text{Span}\{1, \cos^2 t, \cos^4 t, \cos^6 t\}$ contains at least two interesting functions that will be used in a later exercise:

$$\mathbf{f}(t) = 1 - 8\cos^2 t + 8\cos^4 t$$

$$\mathbf{g}(t) = -1 + 18\cos^2 t - 48\cos^4 t + 32\cos^6 t$$

Study the graph of $\mathbf{f}$ for $0 \le t \le 2\pi$, and guess a simple formula for $\mathbf{f}(t)$. Verify your conjecture by graphing the difference between $1 + \mathbf{f}(t)$ and your formula for $\mathbf{f}(t)$. (Hopefully, you will see the constant function 1.) Repeat for $\mathbf{g}$.

46. Repeat Exercise 45 for the functions

$$\mathbf{f}(t) = 3\sin t - 4\sin^3 t$$

$$\mathbf{g}(t) = 1 - 8\sin^2 t + 8\sin^4 t$$

$$\mathbf{h}(t) = 5\sin t - 20\sin^3 t + 16\sin^5 t$$

in the vector space Span $\{1, \sin t, \sin^2 t, \dots, \sin^5 t\}$.

Solutions to Practice Problems

1. Take any $\mathbf{u}$ in H—say, $\mathbf{u} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$—and take any $c \ne 1$—say, $c = 2$. Then $c\mathbf{u} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$. If this is in H, then there is some s such that

$$\begin{bmatrix} 3s \\ 2 + 5s \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

$$M_2(\{p_k\}+\{q_k\}) = M_2(\{p_k+q_k\}) = \left\{\frac{p_k+q_k+p_{k-1}+q_{k-1}}{2}\right\}$$
$$= \left\{\frac{p_k+p_{k-1}}{2}\right\} + \left\{\frac{q_k+q_{k-1}}{2}\right\}$$
$$= M_2(\{p_k\}) + M_2(\{q_k\})$$

and

$$M_2(c\{p_k\}) = M_2(\{cp_k\}) = \left\{\frac{cp_k+cp_{k-1}}{2}\right\} = c\left\{\frac{p_k+p_{k-1}}{2}\right\} = cM_2(\{p_k\})$$

thus M_2 is a linear transformation.

To find the kernel of M_2, notice that $\{p_k\}$ is in the kernel if and only if $\frac{p_k+p_{k-1}}{2} = 0$ for all k, and hence $p_k = -p_{k-1}$. Since this relationship is true for all integers k, it can be applied recursively resulting in $p_k = -p_{k-1} = (-1)^2 p_{k-2} = (-1)^3 p_{k-3} \dots$. Working out from $k = 0$, any signal in the kernel can be written as $p_k = p_0(-1)^k$, a multiple of the alternating signal described by $\{(-1)^k\}$. Since the kernel of the two-day moving average function consists of all multiples of the alternating sequence, it smooths out daily fluctuations, without leveling out overall trends. (See Figure 3.)

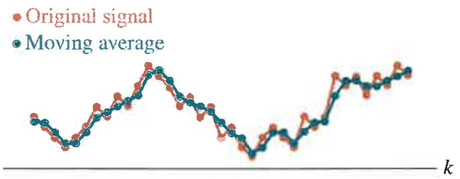


FIGURE 3

Practice Problems

1. Let $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 3b - c = 0 \right\}$. Show in two different ways that W is a subspace of $\mathbb{R}^3$. (Use two theorems.)
2. Let $A = \begin{bmatrix} 7 & -3 & 5 \\ -4 & 1 & -5 \\ -5 & 2 & -4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 7 \\ 6 \\ -3 \end{bmatrix}$. Suppose you know that the equations $A\mathbf{x} = \mathbf{v}$ and $A\mathbf{x} = \mathbf{w}$ are both consistent. What can you say about the equation $A\mathbf{x} = \mathbf{v} + \mathbf{w}$?
3. Let A be an $n \times n$ matrix. If Col A = Nul A, show that Nul $A^2 = \mathbb{R}^n$.

4.2 Exercises

1. Determine if $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ is in Nul A, where
$$A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}.$$

2. Determine if $\mathbf{w} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$ is in Nul A, where
$$A = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix}.$$

In Exercises 3–6, find an explicit description of Nul A by listing vectors that span the null space.

3. $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 7–14, either use an appropriate theorem to show that the given set, W, is a vector space, or find a specific example to the contrary.

7. $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b + c = 2 \right\}$

8. $\left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : 5r - 1 = s + 2t \right\}$

9. $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{matrix} a - 2b = 4c \\ 2a = c + 3d \end{matrix} \right\}$

10. $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{matrix} a + 3b = c \\ b + c + a = d \end{matrix} \right\}$

11. $\left\{ \begin{bmatrix} b - 2d \\ 5 + d \\ b + 3d \\ d \end{bmatrix} : b, d \text{ real} \right\}$

12. $\left\{ \begin{bmatrix} b - 5d \\ 2b \\ 2d + 1 \\ d \end{bmatrix} : b, d \text{ real} \right\}$

13. $\left\{ \begin{bmatrix} c - 6d \\ d \\ c \end{bmatrix} : c, d \text{ real} \right\}$

14. $\left\{ \begin{bmatrix} -a + 2b \\ a - 2b \\ 3a - 6b \end{bmatrix} : a, b \text{ real} \right\}$

In Exercises 15 and 16, find A such that the given set is Col A.

15. $\left\{ \begin{bmatrix} 2s + 3t \\ r + s - 2t \\ 4r + s \\ 3r - s - t \end{bmatrix} : r, s, t \text{ real} \right\}$

16. $\left\{ \begin{bmatrix} b - c \\ 2b + c + d \\ 5c - 4d \\ d \end{bmatrix} : b, c, d \text{ real} \right\}$

For the matrices in Exercises 17–20, (a) find k such that Nul A is a subspace of $\mathbb{R}^k$, and (b) find k such that Col A is a subspace of $\mathbb{R}^k$.

17. $A = \begin{bmatrix} 2 & -8 \\ -1 & 4 \\ 1 & -4 \end{bmatrix}$

18. $A = \begin{bmatrix} 8 & -3 & 0 & -1 \\ -3 & 0 & -1 & 8 \\ 0 & -1 & 8 & -3 \end{bmatrix}$

19. $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$

20. $A = \begin{bmatrix} 1 & -3 & 9 & 0 & -5 \end{bmatrix}$

21. With A as in Exercise 17, find a nonzero vector in Nul A, a nonzero vector in Col A, and a nonzero vector in Row A.

22. With A as in Exercise 3, find a nonzero vector in Nul A, a nonzero vector in Col A, and a nonzero vector in Row A.

23. Let $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Determine if $\mathbf{w}$ is in Col A. Is $\mathbf{w}$ in Nul A?

24. Let $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. Determine if $\mathbf{w}$ is in Col A. Is $\mathbf{w}$ in Nul A?

In Exercises 25–38, A denotes an $m \times n$ matrix. Mark each statement True or False **(T/F)**. Justify each answer.

25. **(T/F)** The null space of A is the solution set of the equation $A\mathbf{x} = \mathbf{0}$.

26. **(T/F)** A null space is a vector space.

27. **(T/F)** The null space of an $m \times n$ matrix is in $\mathbb{R}^m$.

28. **(T/F)** The column space of an $m \times n$ matrix is in $\mathbb{R}^m$.

29. **(T/F)** The column space of A is the range of the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

30. **(T/F)** Col A is the set of all solutions of $A\mathbf{x} = \mathbf{b}$.

31. **(T/F)** If the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then Col $A = \mathbb{R}^m$.

32. **(T/F)** Nul A is the kernel of the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

33. **(T/F)** The kernel of a linear transformation is a vector space.

34. **(T/F)** The range of a linear transformation is a vector space.

35. **(T/F)** Col A is the set of all vectors that can be written as $A\mathbf{x}$ for some $\mathbf{x}$.

36. **(T/F)** The set of all solutions of a homogeneous linear differential equation is the kernel of a linear transformation.

37. **(T/F)** The row space of A is the same as the column space of A^T.

38. **(T/F)** The null space of A is the same as the row space of A^T.

39. It can be shown that a solution of the system below is $x_1 = 3$, $x_2 = 2$, and $x_3 = -1$. Use this fact and the theory from this section to explain why another solution is $x_1 = 30$, $x_2 = 20$, and $x_3 = -10$. (Observe how the solutions are related, but make no other calculations.)

$$\begin{aligned} x_1 - 3x_2 - 3x_3 &= 0 \\ -2x_1 + 4x_2 + 2x_3 &= 0 \\ -x_1 + 5x_2 + 7x_3 &= 0 \end{aligned}$$

40. Consider the following two systems of equations:

$$\begin{aligned} 5x_1 + x_2 - 3x_3 &= 0 \\ -9x_1 + 2x_2 + 5x_3 &= 1 \\ 4x_1 + x_2 - 6x_3 &= 9 \end{aligned} \qquad \begin{aligned} 5x_1 + x_2 - 3x_3 &= 0 \\ -9x_1 + 2x_2 + 5x_3 &= 5 \\ 4x_1 + x_2 - 6x_3 &= 45 \end{aligned}$$

It can be shown that the first system has a solution. Use this fact and the theory from this section to explain why the second system must also have a solution. (Make no row operations.)

41. Prove Theorem 3 as follows: Given an $m \times n$ matrix A, an element in Col A has the form $A\mathbf{x}$ for some $\mathbf{x}$ in $\mathbb{R}^n$. Let $A\mathbf{x}$ and $A\mathbf{w}$ represent any two vectors in Col A.

a. Explain why the zero vector is in Col A.

b. Show that the vector $A\mathbf{x} + A\mathbf{w}$ is in Col A.

c. Given a scalar c, show that $c(A\mathbf{x})$ is in Col A.

42. Let $T : V \to W$ be a linear transformation from a vector space V into a vector space W. Prove that the range of T is a subspace of W. [*Hint:* Typical elements of the range have the form $T(\mathbf{x})$ and $T(\mathbf{w})$ for some $\mathbf{x}$, $\mathbf{w}$ in V.]

43. Define $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$. For instance, if $\mathbf{p}(t) = 3 + 5t + 7t^2$, then $T(\mathbf{p}) = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$.

a. Show that T is a linear transformation. [*Hint:* For arbitrary polynomials $\mathbf{p}$, $\mathbf{q}$ in $\mathbb{P}_2$, compute $T(\mathbf{p} + \mathbf{q})$ and $T(c\mathbf{p})$.]

b. Find a polynomial $\mathbf{p}$ in $\mathbb{P}_2$ that spans the kernel of T, and describe the range of T.

44. Define a linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(0) \end{bmatrix}$. Find polynomials $\mathbf{p}_1$ and $\mathbf{p}_2$ in $\mathbb{P}_2$ that span the kernel of T, and describe the range of T.

45. Let $M_{2\times 2}$ be the vector space of all 2×2 matrices, and define $T : M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A + A^T$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

a. Show that T is a linear transformation.

b. Let B be any element of $M_{2\times 2}$ such that $B^T = B$. Find an A in $M_{2\times 2}$ such that $T(A) = B$.

c. Show that the range of T is the set of B in $M_{2\times 2}$ with the property that $B^T = B$.

d. Describe the kernel of T.

STUDY GUIDE offers additional resources for mastering vector spaces, subspaces, and column row, and null spaces.

46. (*Calculus required*) Define $T : C[0, 1] \to C[0, 1]$ as follows: For $\mathbf{f}$ in $C[0, 1]$, let $T(\mathbf{f})$ be the antiderivative $\mathbf{F}$ of $\mathbf{f}$ such that $\mathbf{F}(0) = 0$. Show that T is a linear transformation, and describe the kernel of T. (See the notation in Exercise 20 of Section 4.1.)

47. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Given a subspace U of V, let $T(U)$ denote the set of all images of the form $T(\mathbf{x})$, where $\mathbf{x}$ is in U. Show that $T(U)$ is a subspace of W.

48. Given $T : V \to W$ as in Exercise 47, and given a subspace Z of W, let U be the set of all $\mathbf{x}$ in V such that $T(\mathbf{x})$ is in Z. Show that U is a subspace of V.

T 49. Determine whether $\mathbf{w}$ is in the column space of A, the null space of A, or both, where

$$\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix}, \quad A = \begin{bmatrix} 7 & 6 & -4 & 1 \\ -5 & -1 & 0 & -2 \\ 9 & -11 & 7 & -3 \\ 19 & -9 & 7 & 1 \end{bmatrix}$$

T 50. Determine whether $\mathbf{w}$ is in the column space of A, the null space of A, or both, where

$$\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} -8 & 5 & -2 & 0 \\ -5 & 2 & 1 & -2 \\ 10 & -8 & 6 & -3 \\ 3 & -2 & 1 & 0 \end{bmatrix}$$

T 51. Let $\mathbf{a}_1, \dots, \mathbf{a}_5$ denote the columns of the matrix A, where

$$A = \begin{bmatrix} 5 & 1 & 2 & 2 & 0 \\ 3 & 3 & 2 & -1 & -12 \\ 8 & 4 & 4 & -5 & 12 \\ 2 & 1 & 1 & 0 & -2 \end{bmatrix}, \quad B = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_4\,]$$

a. Explain why $\mathbf{a}_3$ and $\mathbf{a}_5$ are in the column space of B.

b. Find a set of vectors that spans Nul A.

c. Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$. Explain why T is neither one-to-one nor onto.

T 52. Let $H = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$ and $K = \text{Span}\,\{\mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ -12 \\ -28 \end{bmatrix}.$$

Then H and K are subspaces of $\mathbb{R}^3$. In fact, H and K are planes in $\mathbb{R}^3$ through the origin, and they intersect in a line through $\mathbf{0}$. Find a nonzero vector $\mathbf{w}$ that generates that line. [*Hint:* $\mathbf{w}$ can be written as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ and also as $c_3\mathbf{v}_3 + c_4\mathbf{v}_4$. To build $\mathbf{w}$, solve the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_3\mathbf{v}_3 + c_4\mathbf{v}_4$ for the unknown c_j's.]

it will not be a linear combination of the remaining vectors, and hence the smaller set will no longer span V. Thus a basis is a spanning set that is as small as possible.

A basis is also a linearly independent set that is as large as possible. If S is a basis for V, and if S is enlarged by one vector—say, $\mathbf{w}$—from V, then the new set cannot be linearly independent, because S spans V, and $\mathbf{w}$ is therefore a linear combination of the elements in S.

EXAMPLE 11 The following three sets in $\mathbb{R}^3$ show how a linearly independent set can be enlarged to a basis and how further enlargement destroys the linear independence of the set. Also, a spanning set can be shrunk to a basis, but further shrinking destroys the spanning property.

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}\right\}$$

Linearly independent but does not span $\mathbb{R}^3$ — A basis for $\mathbb{R}^3$ — Spans $\mathbb{R}^3$ but is linearly dependent ■

Practice Problems

1. Let $\mathbf{v}_1 = \begin{bmatrix}1\\-2\\3\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}-2\\7\\-9\end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^3$. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for $\mathbb{R}^2$?

2. Let $\mathbf{v}_1 = \begin{bmatrix}1\\-3\\4\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}6\\2\\-1\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}2\\-2\\3\end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix}-4\\-8\\9\end{bmatrix}$. Find a basis for the subspace W spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

3. Let $\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $H = \left\{\begin{bmatrix}s\\s\\0\end{bmatrix} : s \text{ in } \mathbb{R}\right\}$. Then every vector in H is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ because
$$\begin{bmatrix}s\\s\\0\end{bmatrix} = s\begin{bmatrix}1\\0\\0\end{bmatrix} + s\begin{bmatrix}0\\1\\0\end{bmatrix}$$
Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for H?

4. Let V and W be vector spaces, let $T : V \to W$ and $U : V \to W$ be linear transformations, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a basis for V. If $T(\mathbf{v}_j) = U(\mathbf{v}_j)$ for every value of j between 1 and p, show that $T(\mathbf{x}) = U(\mathbf{x})$ for every vector $\mathbf{x}$ in V.

STUDY GUIDE offers additional resources for mastering the concept of basis.

4.3 Exercises

Determine which sets in Exercises 1–8 are bases for $\mathbb{R}^3$. Of the sets that are *not* bases, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answers.

1. $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$

2. $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$

3. $\begin{bmatrix}1\\0\\-2\end{bmatrix}, \begin{bmatrix}3\\2\\-4\end{bmatrix}, \begin{bmatrix}-3\\-5\\1\end{bmatrix}$

4. $\begin{bmatrix}2\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\-3\\2\end{bmatrix}, \begin{bmatrix}-7\\5\\4\end{bmatrix}$

5. $\begin{bmatrix}1\\-3\\0\end{bmatrix}, \begin{bmatrix}-2\\9\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-3\\5\end{bmatrix}$

6. $\begin{bmatrix}1\\2\\-3\end{bmatrix}, \begin{bmatrix}-4\\-5\\6\end{bmatrix}$

7. $\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$

Find bases for the null spaces of the matrices given in Exercises 9 and 10. Refer to the remarks that follow Example 3 in Section 4.2.

9. $\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix}$

11. Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x + 4y - 5z = 0$. [Hint: Think of the equation as a "system" of homogeneous equations.]

12. Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y = 5x$.

In Exercises 13 and 14, assume that A is row equivalent to B. Find bases for Nul A, Col A, and Row A.

13. $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}$,

$B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 15–18, find a basis for the space spanned by the given vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_5$.

15. $\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -6 \\ 9 \end{bmatrix}$

16. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}$

T **17.** $\begin{bmatrix} 8 \\ 9 \\ -3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ -4 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -9 \\ 6 \\ -7 \end{bmatrix}, \begin{bmatrix} 6 \\ 8 \\ 4 \\ -7 \\ 10 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 11 \\ -8 \\ -7 \end{bmatrix}$

T **18.** $\begin{bmatrix} -8 \\ 7 \\ 6 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 8 \\ -7 \\ -9 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} -8 \\ 7 \\ 4 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 9 \\ 6 \\ -7 \end{bmatrix}, \begin{bmatrix} -9 \\ 3 \\ -4 \\ -1 \\ 0 \end{bmatrix}$

19. Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$, and $H =$ Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. It can be verified that $4\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$. Use this information to find a basis for H. There is more than one answer.

20. Let $\mathbf{v}_1 = \begin{bmatrix} 7 \\ 4 \\ -9 \\ -5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ -7 \\ 2 \\ 5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -5 \\ 3 \\ 4 \end{bmatrix}$. It can be verified that $\mathbf{v}_1 - 3\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$. Use this information to find a basis for $H =$ Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

In Exercises 21–32, mark each statement True or False **(T/F)**. Justify each answer.

21. **(T/F)** A single vector by itself is linearly dependent.

22. **(T/F)** A linearly independent set in a subspace H is a basis for H.

23. **(T/F)** If $H =$ Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$, then $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for H.

24. **(T/F)** If a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis for V.

25. **(T/F)** The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

26. **(T/F)** A basis is a linearly independent set that is as large as possible.

27. **(T/F)** A basis is a spanning set that is as large as possible.

28. **(T/F)** The standard method for producing a spanning set for Nul A, described in Section 4.2, sometimes fails to produce a basis for Nul A.

29. **(T/F)** In some cases, the linear dependence relations among the columns of a matrix can be affected by certain elementary row operations on the matrix.

30. **(T/F)** If B is an echelon form of a matrix A, then the pivot columns of B form a basis for Col A.

31. **(T/F)** Row operations preserve the linear dependence relations among the rows of A.

32. **(T/F)** If A and B are row equivalent, then their row spaces are the same.

33. Suppose $\mathbb{R}^4 =$ Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$. Explain why $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$.

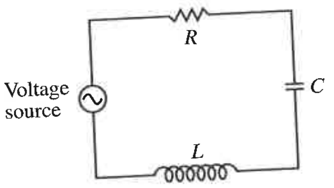
34. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a linearly independent set in $\mathbb{R}^n$. Explain why $\mathcal{B}$ must be a basis for $\mathbb{R}^n$.

35. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and let H be the set of vectors in $\mathbb{R}^3$ whose second and third entries are equal. Then every vector in H has a unique expansion as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, because
$$\begin{bmatrix} s \\ t \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (t-s)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
for any s and t. Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis for H? Why or why not?

36. In the vector space of all real-valued functions, find a basis for the subspace spanned by $\{\sin t, \sin 2t, \sin t \cos t\}$.

37. Let V be the vector space of functions that describe the vibration of a mass–spring system. (Refer to Exercise 19 in Section 4.1.) Find a basis for V.

38. (*RLC circuit*) The circuit in the figure consists of a resistor (R ohms), an inductor (L henrys), a capacitor (C farads), and an initial voltage source. Let $b = R/(2L)$, and suppose R, L, and C have been selected so that b also equals $1/\sqrt{LC}$. (This is done, for instance, when the circuit is used in a voltmeter.) Let $v(t)$ be the voltage (in volts) at time t, measured across the capacitor. It can be shown that v is in the null space H of the linear transformation that maps $v(t)$ into $Lv''(t) + Rv'(t) + (1/C)v(t)$, and H consists of all functions of the form $v(t) = e^{-bt}(c_1 + c_2 t)$. Find a basis for H.



Exercises 39 and 40 show that every basis for $\mathbb{R}^n$ must contain exactly n vectors.

39. Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$, with $k < n$. Use a theorem from Section 1.4 to explain why S cannot be a basis for $\mathbb{R}^n$.

40. Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$, with $k > n$. Use a theorem from Chapter 1 to explain why S cannot be a basis for $\mathbb{R}^n$.

Exercises 41 and 42 reveal an important connection between linear independence and linear transformations and provide practice using the definition of linear dependence. Let V and W be vector spaces, let $T : V \to W$ be a linear transformation, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a subset of V.

41. Show that if $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly dependent in V, then the set of images, $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$, is linearly dependent in W. This fact shows that if a linear transformation maps a set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ onto a linearly *independent* set $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$, then the original set is linearly independent, too (because it cannot be linearly dependent).

42. Suppose that T is a one-to-one transformation, so that an equation $T(\mathbf{u}) = T(\mathbf{v})$ always implies $\mathbf{u} = \mathbf{v}$. Show that if the set of images $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$ is linearly dependent, then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly dependent. This fact shows that *a one-to-one linear transformation maps a linearly independent set onto a linearly independent set* (because in this case the set of images cannot be linearly dependent).

43. Consider the polynomials $\mathbf{p}_1(t) = 1 + t^2$ and $\mathbf{p}_2(t) = 1 - t^2$. Is $\{\mathbf{p}_1, \mathbf{p}_2\}$ a linearly independent set in $\mathbb{P}_3$? Why or why not?

44. Consider the polynomials $\mathbf{p}_1(t) = 1 + t$, $\mathbf{p}_2(t) = 1 - t$, and $\mathbf{p}_3(t) = 2$ (for all t). By inspection, write a linear dependence relation among $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. Then find a basis for Span $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.

45. Let V be a vector space that contains a linearly independent set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$. Describe how to construct a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ in V such that $\{\mathbf{v}_1, \mathbf{v}_3\}$ is a basis for Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

T 46. Let $H = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $K = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where
$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 3 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 2 \\ -3 \\ 6 \\ -5 \end{bmatrix},$$
$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -4 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 7 \\ 6 \\ 5 \end{bmatrix}$$
Find bases for H, K, and $H + K$. (See Exercises 41 and 42 in Section 4.1.)

T 47. Show that $\{t, \sin t, \cos 2t, \sin t \cos t\}$ is a linearly independent set of functions defined on $\mathbb{R}$. Start by assuming that
$$c_1 t + c_2 \sin t + c_3 \cos 2t + c_4 \sin t \cos t = 0 \qquad (5)$$
Equation (5) must hold for all real t, so choose several specific values of t (say, $t = 0, .1, .2$) until you get a system of enough equations to determine that all the c_j must be zero.

T 48. Show that $\{1, \cos t, \cos^2 t, \dots, \cos^6 t\}$ is a linearly independent set of functions defined on $\mathbb{R}$. Use the method of Exercise 47. (This result will be needed in Exercise 54 in Section 4.5.)

If a different basis for H were chosen, would the associated coordinate system also make H isomorphic to $\mathbb{R}^2$? Surely, this must be true. We shall prove it in the next section.

Practice Problems

1. Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$.
   a. Show that the set $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis of $\mathbb{R}^3$.
   b. Find the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis.
   c. Write the equation that relates $\mathbf{x}$ in $\mathbb{R}^3$ to $[\mathbf{x}]_{\mathcal{B}}$.
   d. Find $[\mathbf{x}]_{\mathcal{B}}$, for the $\mathbf{x}$ given above.
2. The set $\mathcal{B} = \{1 + t, 1 + t^2, t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 6 + 3t - t^2$ relative to $\mathcal{B}$.

4.4 Exercises

In Exercises 1–4, find the vector $\mathbf{x}$ determined by the given coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ and the given basis $\mathcal{B}$.

1. $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

2. $\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$

3. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -8 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ -4 \end{bmatrix} \right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$

4. $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} \right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix}$

In Exercises 5–8, find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ relative to the given basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$.

5. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

6. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 5 \\ -6 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

7. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}$

8. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

In Exercises 9 and 10, find the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

9. $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \end{bmatrix} \right\}$

10. $\mathcal{B} = \left\{ \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 8 \end{bmatrix} \right\}$

In Exercises 11 and 12, use an inverse matrix to find $[\mathbf{x}]_{\mathcal{B}}$ for the given $\mathbf{x}$ and $\mathcal{B}$.

11. $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}, \mathbf{x} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

12. $\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}, \mathbf{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

13. The set $\mathcal{B} = \{1 + t^2, t + t^2, 1 + 2t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 1 + 4t + 7t^2$ relative to $\mathcal{B}$.

14. The set $\mathcal{B} = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 3 + t - 6t^2$ relative to $\mathcal{B}$.

In Exercises 15–20, mark each statement True or False (**T/F**). Justify each answer. Unless stated otherwise, $\mathcal{B}$ is a basis for a vector space V.

15. **(T/F)** If $\mathbf{x}$ is in V and if $\mathcal{B}$ contains n vectors, then the $\mathcal{B}$-coordinate vector of $\mathbf{x}$ is in $\mathbb{R}^n$.

16. **(T/F)** If $\mathcal{B}$ is the standard basis for $\mathbb{R}^n$, then the $\mathcal{B}$-coordinate vector of an $\mathbf{x}$ in $\mathbb{R}^n$ is $\mathbf{x}$ itself.

17. **(T/F)** If $P_{\mathcal{B}}$ is the change-of-coordinates matrix, then $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}\,\mathbf{x}$, for $\mathbf{x}$ in V.

18. **(T/F)** The correspondence $[\mathbf{x}]_{\mathcal{B}} \mapsto \mathbf{x}$ is called the coordinate mapping.

19. **(T/F)** The vector spaces $\mathbb{P}_3$ and $\mathbb{R}^3$ are isomorphic.

20. **(T/F)** In some cases, a plane in $\mathbb{R}^3$ can be isomorphic to $\mathbb{R}^2$.

21. The vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$ span $\mathbb{R}^2$ but do not form a basis. Find two different ways to express $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

22. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Explain why the $\mathcal{B}$-coordinate vectors of $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are the columns $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of the $n \times n$ identity matrix.

23. Let S be a finite set in a vector space V with the property that every $\mathbf{x}$ in V has a unique representation as a linear combination of elements of S. Show that S is a basis of V.

24. Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ is a linearly dependent spanning set for a vector space V. Show that each $\mathbf{w}$ in V can be expressed in more than one way as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_4$. [*Hint:* Let $\mathbf{w} = k_1\mathbf{v}_1 + \cdots + k_4\mathbf{v}_4$ be an arbitrary vector in V. Use the linear dependence of $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ to produce another representation of $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_4$.]

25. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix} \right\}$. Since the coordinate mapping determined by $\mathcal{B}$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$, this mapping must be implemented by some 2×2 matrix A. Find it. [*Hint:* Multiplication by A should transform a vector $\mathbf{x}$ into its coordinate vector $[\,\mathbf{x}\,]_{\mathcal{B}}$.]

26. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for $\mathbb{R}^n$. Produce a description of an $n \times n$ matrix A that implements the coordinate mapping $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$. (See Exercise 25.)

Exercises 27–30 concern a vector space V, a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, and the coordinate mapping $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$.

27. Show that the coordinate mapping is one-to-one. [*Hint:* Suppose $[\,\mathbf{u}\,]_{\mathcal{B}} = [\,\mathbf{w}\,]_{\mathcal{B}}$ for some $\mathbf{u}$ and $\mathbf{w}$ in V, and show that $\mathbf{u} = \mathbf{w}$.]

28. Show that the coordinate mapping is *onto* $\mathbb{R}^n$. That is, given any $\mathbf{y}$ in $\mathbb{R}^n$, with entries $y_1, \ldots, y_n$, produce $\mathbf{u}$ in V such that $[\,\mathbf{u}\,]_{\mathcal{B}} = \mathbf{y}$.

29. Show that a subset $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in V is linearly independent if and only if the set of coordinate vectors $\{[\,\mathbf{u}_1\,]_{\mathcal{B}}, \ldots, [\,\mathbf{u}_p\,]_{\mathcal{B}}\}$ is linearly independent in $\mathbb{R}^n$. [*Hint:* Since the coordinate mapping is one-to-one, the following equations have the same solutions, $c_1, \ldots, c_p$.]

$$c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p = \mathbf{0} \quad \text{The zero vector in } V$$
$$[\,c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p\,]_{\mathcal{B}} = [\,\mathbf{0}\,]_{\mathcal{B}} \quad \text{The zero vector in } \mathbb{R}^n$$

30. Given vectors $\mathbf{u}_1, \ldots, \mathbf{u}_p$, and $\mathbf{w}$ in V, show that $\mathbf{w}$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_p$ if and only if $[\,\mathbf{w}\,]_{\mathcal{B}}$ is a linear combination of the coordinate vectors $[\,\mathbf{u}_1\,]_{\mathcal{B}}, \ldots, [\,\mathbf{u}_p\,]_{\mathcal{B}}$.

In Exercises 31–34, use coordinate vectors to test the linear independence of the sets of polynomials. Explain your work.

31. $\{1 + 2t^3,\ 2 + t - 3t^2,\ -t + 2t^2 - t^3\}$

32. $\{1 - 2t^2 - t^3,\ t + 2t^3,\ 1 + t - 2t^2\}$

33. $\{(1 - t)^2,\ t - 2t^2 + t^3,\ (1 - t)^3\}$

34. $\{(2 - t)^3,\ (3 - t)^2,\ 1 + 6t - 5t^2 + t^3\}$

35. Use coordinate vectors to test whether the following sets of polynomials span $\mathbb{P}_2$. Justify your conclusions.

a. $\{1 - 3t + 5t^2, -3 + 5t - 7t^2, -4 + 5t - 6t^2, 1 - t^2\}$

b. $\{5t + t^2, 1 - 8t - 2t^2, -3 + 4t + 2t^2, 2 - 3t\}$

36. Let $\mathbf{p}_1(t) = 1 + t^2$, $\mathbf{p}_2(t) = t - 3t^2$, $\mathbf{p}_3(t) = 1 + t - 3t^2$.

a. Use coordinate vectors to show that these polynomials form a basis for $\mathbb{P}_2$.

b. Consider the basis $\mathcal{B} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for $\mathbb{P}_2$. Find $\mathbf{q}$ in $\mathbb{P}_2$, given that $[\mathbf{q}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

In Exercises 37 and 38, determine whether the sets of polynomials form a basis for $\mathbb{P}_3$. Justify your conclusions.

T 37. $3 + 7t, 5 + t - 2t^3, t - 2t^2, 1 + 16t - 6t^2 + 2t^3$

T 38. $5 - 3t + 4t^2 + 2t^3, 9 + t + 8t^2 - 6t^3, 6 - 2t + 5t^2, t^3$

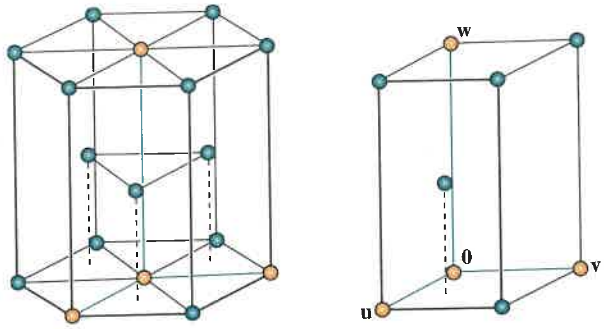
T 39. Let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Show that $\mathbf{x}$ is in H and find the $\mathcal{B}$-coordinate vector of $\mathbf{x}$, for

$$\mathbf{v}_1 = \begin{bmatrix} 11 \\ -5 \\ 10 \\ 7 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 14 \\ -8 \\ 13 \\ 10 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 19 \\ -13 \\ 18 \\ 15 \end{bmatrix}$$

T 40. Let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Show that $\mathcal{B}$ is a basis for H and $\mathbf{x}$ is in H, and find the $\mathcal{B}$-coordinate vector of $\mathbf{x}$, for

$$\mathbf{v}_1 = \begin{bmatrix} -6 \\ 4 \\ -9 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 8 \\ -3 \\ 7 \\ -3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -9 \\ 5 \\ -8 \\ 3 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 7 \\ -8 \\ 3 \end{bmatrix}$$

Exercises 41 and 42 concern the crystal lattice for titanium, which has the hexagonal structure shown on the left in the accompanying figure. The vectors $\begin{bmatrix} 2.6 \\ -1.5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4.8 \end{bmatrix}$ in $\mathbb{R}^3$ form a basis for the unit cell shown on the right. The numbers here are Ångstrom units (1 Å $= 10^{-8}$ cm). In alloys of titanium, some additional atoms may be in the unit cell at the *octahedral* and *tetrahedral* sites (so named because of the geometric objects formed by atoms at these locations).



The hexagonal close-packed lattice and its unit cell.

41. One of the octahedral sites is $\begin{bmatrix} 1/2 \\ 1/4 \\ 1/6 \end{bmatrix}$, relative to the lattice basis. Determine the coordinates of this site relative to the standard basis of $\mathbb{R}^3$.

42. One of the tetrahedral sites is $\begin{bmatrix} 1/2 \\ 1/2 \\ 1/3 \end{bmatrix}$. Determine the coordinates of this site relative to the standard basis of $\mathbb{R}^3$.

Solutions to Practice Problems

1. a. It is evident that the matrix $P_{\mathcal{B}} = [\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3\,]$ is row-equivalent to the identity matrix. By the Invertible Matrix Theorem, $P_{\mathcal{B}}$ is invertible and its columns form a basis for $\mathbb{R}^3$.

b. From part (a), the change-of-coordinates matrix is $P_{\mathcal{B}} = \begin{bmatrix} 1 & -3 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 3 \end{bmatrix}$.

c. $\mathbf{x} = P_{\mathcal{B}}[\,\mathbf{x}\,]_{\mathcal{B}}$

d. To solve the equation in (c), it is probably easier to row reduce an augmented matrix than to compute $P_{\mathcal{B}}^{-1}$:

$$\underbrace{\begin{bmatrix} 1 & -3 & 3 & -8 \\ 0 & 4 & -6 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix}}_{P_{\mathcal{B}} \qquad \mathbf{x}} \sim \underbrace{\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{I \qquad [\,\mathbf{x}\,]_{\mathcal{B}}}$$

Hence

$$[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$$

2. The coordinates of $\mathbf{p}(t) = 6 + 3t - t^2$ with respect to $\mathcal{B}$ satisfy

$$c_1(1+t) + c_2(1+t^2) + c_3(t+t^2) = 6 + 3t - t^2$$

Equating coefficients of like powers of t, we have

$$\begin{aligned} c_1 + c_2 \qquad &= 6 \\ c_1 \qquad + c_3 &= 3 \\ c_2 + c_3 &= -1 \end{aligned}$$

Solving, we find that $c_1 = 5$, $c_2 = 1$, $c_3 = -2$, and $[\,\mathbf{p}\,]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$.

We have refrained from adding to the Invertible Matrix Theorem obvious statements about the row space of A, because the row space is the column space of A^T. Recall from statement (l) of the Invertible Matrix Theorem that A is invertible if and only if A^T is invertible. Hence every statement in the Invertible Matrix Theorem can also be stated for A^T. To do so would double the length of the theorem and produce a list of more than 30 statements!

Numerical Notes

Many algorithms discussed in this text are useful for understanding concepts and making simple computations by hand. However, the algorithms are often unsuitable for large-scale problems in real life.

Rank determination is a good example. It would seem easy to reduce a matrix to echelon form and count the pivots. But unless exact arithmetic is performed on a matrix whose entries are specified exactly, row operations can change the apparent rank of a matrix. For instance, if the value of x in the matrix $\begin{bmatrix} 5 & 7 \\ 5 & x \end{bmatrix}$ is not stored exactly as 7 in a computer, then the rank may be 1 or 2, depending on whether the computer treats $x - 7$ as zero.

In practical applications, the effective rank of a matrix A is often determined from the singular value decomposition of A, to be discussed in Section 7.4. This decomposition is also a reliable source of bases for Col A, Row A, Nul A, and Nul A^T.

Practice Problems

1. Decide whether each statement is True or False, and give a reason for each answer. Here V is a nonzero finite-dimensional vector space.
   a. If $\dim V = p$ and if S is a linearly dependent subset of V, then S contains more than p vectors.
   b. If S spans V and if T is a subset of V that contains more vectors than S, then T is linearly dependent.
2. Let H and K be subspaces of a vector space V. In Section 4.1, Exercise 40, it is established that $H \cap K$ is also a subspace of V. Prove $\dim (H \cap K) \leq \dim H$.

4.5 Exercises

For each subspace in Exercises 1–8, (a) find a basis, and (b) state the dimension.

1. $\left\{ \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} : s, t \text{ in } \mathbb{R} \right\}$

2. $\left\{ \begin{bmatrix} 5s \\ -t \\ -7s \end{bmatrix} : s, t \in \mathbb{R} \right\}$

3. $\left\{ \begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$

4. $\left\{ \begin{bmatrix} a+b \\ 2a \\ 3a-b \\ -b \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$

5. $\left\{ \begin{bmatrix} a-4b-2c \\ 2a+5b-4c \\ -a+2c \\ -3a+7b+6c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$

6. $\left\{ \begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$

7. $\{(a, b, c) : a - 3b + c = 0, b - 2c = 0, 2b - c = 0\}$

8. $\{(a, b, c, d) : a - 3b + c = 0\}$

In Exercises 9 and 10, find the dimension of the subspace spanned by the given vectors.

9. $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 1 \end{bmatrix}$

10. $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}$

Determine the dimensions of Nul A, Col A, and Row A for the matrices shown in Exercises 11–16.

11. $A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

13. $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$

14. $A = \begin{bmatrix} 3 & 4 \\ -6 & 10 \end{bmatrix}$

15. $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 7 \\ 0 & 0 & 5 \end{bmatrix}$ 16. $A = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

In Exercises 17–26, V is a vector space and A is an $m \times n$ matrix. Mark each statement True or False **(T/F)**. Justify each answer.

17. **(T/F)** The number of pivot columns of a matrix equals the dimension of its column space.

18. **(T/F)** The number of variables in the equation $A\mathbf{x} = \mathbf{0}$ equals the nullity A.

19. **(T/F)** A plane in $\mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$.

20. **(T/F)** The dimension of the vector space $\mathbb{P}_4$ is 4.

21. **(T/F)** The dimension of the vector space of signals, $\mathbb{S}$, is 10.

22. **(T/F)** The dimensions of the row space and the column space of A are the same, even if A is not square.

23. **(T/F)** If B is any echelon form of A, then the pivot columns of B form a basis for the column space of A.

24. **(T/F)** The nullity of A is the number of columns of A that are not pivot columns.

25. **(T/F)** If a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ spans a finite-dimensional vector space V and if T is a set of more than p vectors in V, then T is linearly dependent.

26. **(T/F)** A vector space is infinite-dimensional if it is spanned by an infinite set.

27. The first four Hermite polynomials are $1, 2t, -2 + 4t^2$, and $-12t + 8t^3$. These polynomials arise naturally in the study of certain important differential equations in mathematical physics.[2] Show that the first four Hermite polynomials form a basis of $\mathbb{P}_3$.

28. The first four Laguerre polynomials are $1, 1 - t, 2 - 4t + t^2$, and $6 - 18t + 9t^2 - t^3$. Show that these polynomials form a basis of $\mathbb{P}_3$.

29. Let $\mathcal{B}$ be the basis of $\mathbb{P}_3$ consisting of the Hermite polynomials in Exercise 27, and let $\mathbf{p}(t) = 7 - 12t - 8t^2 + 12t^3$. Find the coordinate vector of $\mathbf{p}$ relative to $\mathcal{B}$.

30. Let $\mathcal{B}$ be the basis of $\mathbb{P}_2$ consisting of the first three Laguerre polynomials listed in Exercise 28, and let $\mathbf{p}(t) = 7 - 8t + 3t^2$. Find the coordinate vector of $\mathbf{p}$ relative to $\mathcal{B}$.

31. Let S be a subset of an n-dimensional vector space V, and suppose S contains fewer than n vectors. Explain why S cannot span V.

32. Let H be an n-dimensional subspace of an n-dimensional vector space V. Show that $H = V$.

33. If a 4×7 matrix A has rank 4, find nullity A, rank A, and rank A^T.

34. If a 6×3 matrix A has rank 3, find nullity A, rank A, and rank A^T.

35. Suppose a 5×9 matrix A has four pivot columns. Is Col $A = \mathbb{R}^5$? Is Nul $A = \mathbb{R}^4$? Explain your answers.

36. Suppose a 5×6 matrix A has four pivot columns. What is nullity A? Is Col $A = \mathbb{R}^4$? Why or why not?

37. If the nullity of a 5×6 matrix A is 4, what are the dimensions of the column and row spaces of A?

38. If the nullity of a 7×6 matrix A is 5, what are the dimensions of the column and row spaces of A?

39. If A is a 7×5 matrix, what is the largest possible rank of A? If A is a 5×7 matrix, what is the largest possible rank of A? Explain your answers.

40. If A is a 4×3 matrix, what is the largest possible dimension of the row space of A? If A is a 3×4 matrix, what is the largest possible dimension of the row space of A? Explain.

41. Explain why the space $\mathbb{P}$ of all polynomials is an infinite-dimensional space.

42. Show that the space $C(\mathbb{R})$ of all continuous functions defined on the real line is an infinite-dimensional space.

In Exercises 43–48, V is a nonzero finite-dimensional vector space, and the vectors listed belong to V. Mark each statement True or False **(T/F)**. Justify each answer. (These questions are more difficult than those in Exercises 17–26.)

[2] See *Introduction to Functional Analysis*, 2nd ed., by A. E. Taylor and David C. Lay (New York: John Wiley & Sons, 1980), pp. 92–93. Other sets of polynomials are discussed there, too.

43. **(T/F)** If there exists a set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ that spans V, then $\dim V \leq p$.

44. **(T/F)** If there exists a linearly dependent set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V, then $\dim V \leq p$.

45. **(T/F)** If there exists a linearly independent set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V, then $\dim V \geq p$.

46. **(T/F)** If $\dim V = p$, then there exists a spanning set of $p + 1$ vectors in V.

47. **(T/F)** If every set of p elements in V fails to span V, then $\dim V > p$.

48. **(T/F)** If $p \geq 2$ and $\dim V = p$, then every set of $p - 1$ nonzero vectors is linearly independent.

49. Justify the following equality: $\dim \operatorname{Row} A + \operatorname{nullity} A = n$, the number of columns of A

50. Justify the following equality: $\dim \operatorname{Row} A + \operatorname{nullity} A^T = m$, the number of rows of A

Exercises 51 and 52 concern finite-dimensional vector spaces V and W and a linear transformation $T : V \to W$.

51. Let H be a nonzero subspace of V, and let $T(H)$ be the set of images of vectors in H. Then $T(H)$ is a subspace of W, by Exercise 47 in Section 4.2. Prove that $\dim T(H) \leq \dim H$.

52. Let H be a nonzero subspace of V, and suppose T is a one-to-one (linear) mapping of V into W. Prove that $\dim T(H) = \dim H$. If T happens to be a one-to-one mapping of V *onto* W, then $\dim V = \dim W$. Isomorphic finite-dimensional vector spaces have the same dimension.

T 53. According to Theorem 12, a linearly independent set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ in $\mathbb{R}^n$ can be expanded to a basis for $\mathbb{R}^n$. One way to do this is to create $A = [\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_k \ \ \mathbf{e}_1 \ \cdots \ \mathbf{e}_n\,]$, with $\mathbf{e}_1, \dots, \mathbf{e}_n$ the columns of the identity matrix; the pivot columns of A form a basis for $\mathbb{R}^n$.

a. Use the method described to extend the following vectors to a basis for $\mathbb{R}^5$:

$$\mathbf{v}_1 = \begin{bmatrix} -9 \\ -7 \\ 8 \\ -5 \\ 7 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 9 \\ 4 \\ 1 \\ 6 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 7 \\ -8 \\ 5 \\ -7 \end{bmatrix}$$

b. Explain why the method works in general: Why are the original vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ included in the basis found for $\operatorname{Col} A$? Why is $\operatorname{Col} A = \mathbb{R}^n$?

T 54. Let $\mathcal{B} = \{1, \cos t, \cos^2 t, \dots, \cos^6 t\}$ and $\mathcal{C} = \{1, \cos t, \cos 2t, \dots, \cos 6t\}$. Assume the following trigonometric identities (see Exercise 45 in Section 4.1).

$$\begin{aligned}
\cos 2t &= -1 + 2\cos^2 t \\
\cos 3t &= -3\cos t + 4\cos^3 t \\
\cos 4t &= 1 - 8\cos^2 t + 8\cos^4 t \\
\cos 5t &= 5\cos t - 20\cos^3 t + 16\cos^5 t \\
\cos 6t &= -1 + 18\cos^2 t - 48\cos^4 t + 32\cos^6 t
\end{aligned}$$

Let H be the subspace of functions spanned by the functions in $\mathcal{B}$. Then $\mathcal{B}$ is a basis for H, by Exercise 48 in Section 4.3.

a. Write the $\mathcal{B}$-coordinate vectors of the vectors in $\mathcal{C}$, and use them to show that $\mathcal{C}$ is a linearly independent set in H.

b. Explain why $\mathcal{C}$ is a basis for H.

Solutions to Practice Problems

1. a. False. Consider the set $\{\mathbf{0}\}$.

   b. True. By the Spanning Set Theorem, S contains a basis for V; call that basis S'. Then T will contain more vectors than S'. By Theorem 10, T is linearly dependent.

2. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a basis for $H \cap K$. Notice $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a linearly independent subset of H, hence by Theorem 12, $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ can be expanded, if necessary, to a basis for H. Since the dimension of a subspace is just the number of vectors in a basis, it follows that $\dim (H \cap K) = p \leq \dim H$.

4.6 Change of Basis

When a basis $\mathcal{B}$ is chosen for an n-dimensional vector space V, the associated coordinate mapping onto $\mathbb{R}^n$ provides a coordinate system for V. Each $\mathbf{x}$ in V is identified uniquely by its $\mathcal{B}$-coordinate vector $[\,\mathbf{x}\,]_{\mathcal{B}}$.[1]

[1] Think of $[\,\mathbf{x}\,]_{\mathcal{B}}$ as a name for $\mathbf{x}$ that lists the weights used to build $\mathbf{x}$ as a linear combination of the basis vectors in $\mathcal{B}$.

So

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$$

b. By part (a) and property (6) (with $\mathcal{B}$ and $\mathcal{C}$ interchanged),

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = (\underset{\mathcal{B}\leftarrow\mathcal{C}}{P})^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix} \quad \blacksquare$$

Another description of the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ uses the change-of-coordinate matrices $P_{\mathcal{B}}$ and $P_{\mathcal{C}}$ that convert $\mathcal{B}$-coordinates and $\mathcal{C}$-coordinates, respectively, into standard coordinates. Recall that for each $\mathbf{x}$ in $\mathbb{R}^n$,

$$P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}, \quad P_{\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = \mathbf{x}, \quad \text{and} \quad [\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\mathbf{x}$$

Thus

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\mathbf{x} = P_{\mathcal{C}}^{-1}P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

In $\mathbb{R}^n$, the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ may be computed as $P_{\mathcal{C}}^{-1}P_{\mathcal{B}}$. Actually, for matrices larger than 2×2, an algorithm analogous to the one in Example 3 is faster than computing $P_{\mathcal{C}}^{-1}$ and then $P_{\mathcal{C}}^{-1}P_{\mathcal{B}}$. See Exercise 22 in Section 2.2.

Practice Problems

1. Let $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2\}$ and $\mathcal{G} = \{\mathbf{g}_1, \mathbf{g}_2\}$ be bases for a vector space V, and let P be a matrix whose columns are $[\,\mathbf{f}_1\,]_{\mathcal{G}}$ and $[\,\mathbf{f}_2\,]_{\mathcal{G}}$. Which of the following equations is satisfied by P for all $\mathbf{v}$ in V?
   (i) $[\,\mathbf{v}\,]_{\mathcal{F}} = P[\,\mathbf{v}\,]_{\mathcal{G}}$ (ii) $[\,\mathbf{v}\,]_{\mathcal{G}} = P[\,\mathbf{v}\,]_{\mathcal{F}}$
2. Let $\mathcal{B}$ and $\mathcal{C}$ be as in Example 1. Use the results of that example to find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

4.6 Exercises

1. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V, and suppose $\mathbf{b}_1 = 6\mathbf{c}_1 - 2\mathbf{c}_2$ and $\mathbf{b}_2 = 9\mathbf{c}_1 - 4\mathbf{c}_2$.
   a. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.
   b. Find $[\,\mathbf{x}\,]_{\mathcal{C}}$ for $\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2$. Use part (a).

2. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V, and suppose $\mathbf{b}_1 = -\mathbf{c}_1 + 4\mathbf{c}_2$ and $\mathbf{b}_2 = 5\mathbf{c}_1 - 3\mathbf{c}_2$.
   a. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.
   b. Find $[\,\mathbf{x}\,]_{\mathcal{C}}$ for $\mathbf{x} = 5\mathbf{b}_1 + 3\mathbf{b}_2$.

3. Let $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $\mathcal{W} = \{\mathbf{w}_1, \mathbf{w}_2\}$ be bases for V, and let P be a matrix whose columns are $[\mathbf{u}_1]_{\mathcal{W}}$ and $[\mathbf{u}_2]_{\mathcal{W}}$. Which of the following equations is satisfied by P for all $\mathbf{x}$ in V?
   (i) $[\,\mathbf{x}\,]_{\mathcal{U}} = P[\,\mathbf{x}\,]_{\mathcal{W}}$ (ii) $[\,\mathbf{x}\,]_{\mathcal{W}} = P[\,\mathbf{x}\,]_{\mathcal{U}}$

4. Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ be bases for V, and let $P = [\,[\mathbf{d}_1]_{\mathcal{A}} \;\; [\mathbf{d}_2]_{\mathcal{A}} \;\; [\mathbf{d}_3]_{\mathcal{A}}\,]$. Which of the following equations is satisfied by P for all $\mathbf{x}$ in V?
   (i) $[\,\mathbf{x}\,]_{\mathcal{A}} = P[\,\mathbf{x}\,]_{\mathcal{D}}$ (ii) $[\,\mathbf{x}\,]_{\mathcal{D}} = P[\,\mathbf{x}\,]_{\mathcal{A}}$

5. Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for a vector space V, and suppose $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2$, $\mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, and $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$.
   a. Find the change-of-coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$.
   b. Find $[\,\mathbf{x}\,]_{\mathcal{B}}$ for $\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$.

6. Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be bases for a vector space V, and suppose $\mathbf{f}_1 = 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3$, $\mathbf{f}_2 = 3\mathbf{d}_2 + \mathbf{d}_3$, and $\mathbf{f}_3 = -3\mathbf{d}_1 + 2\mathbf{d}_3$.
   a. Find the change-of-coordinates matrix from $\mathcal{F}$ to $\mathcal{D}$.
   b. Find $[\,\mathbf{x}\,]_{\mathcal{D}}$ for $\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$.

In Exercises 7–10, let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for $\mathbb{R}^2$. In each exercise, find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

7. $\mathbf{b}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

8. $\mathbf{b}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

9. $\mathbf{b}_1 = \begin{bmatrix} -6 \\ -1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

10. $\mathbf{b}_1 = \begin{bmatrix} 8 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

In Exercises 11–14, $\mathcal{B}$ and $\mathcal{C}$ are bases for a vector space V. Mark each statement True or False **(T/F)**. Justify each answer.

11. **(T/F)** The columns of the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are $\mathcal{B}$-coordinate vectors of the vectors in $\mathcal{C}$.

12. **(T/F)** The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are linearly independent.

13. **(T/F)** If $V = \mathbb{R}^n$ and $\mathcal{C}$ is the *standard* basis for V, then $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is the same as the change-of-coordinates matrix $P_{\mathcal{B}}$ introduced in Section 4.4.

14. **(T/F)** If $V = \mathbb{R}^2$, $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, then row reduction of $[\,\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{b}_1 \quad \mathbf{b}_2\,]$ to $[\,I \quad P\,]$ produces a matrix P that satisfies $[\,\mathbf{x}\,]_{\mathcal{B}} = P[\,\mathbf{x}\,]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.

15. In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$. Then find the $\mathcal{B}$-coordinate vector for $-1 + 2t$.

16. In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$ to the standard basis. Then write t^2 as a linear combination of the polynomials in $\mathcal{B}$.

Exercises 17 and 18 provide a proof of Theorem 15. Fill in a justification for each step.

17. Given $\mathbf{v}$ in V, there exist scalars $x_1, \dots, x_n$, such that

$$\mathbf{v} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n$$

because (a) _____. Apply the coordinate mapping determined by the basis $\mathcal{C}$, and obtain

$$[\mathbf{v}]_{\mathcal{C}} = x_1[\mathbf{b}_1]_{\mathcal{C}} + x_2[\mathbf{b}_2]_{\mathcal{C}} + \cdots + x_n[\mathbf{b}_n]_{\mathcal{C}}$$

because (b) _____. This equation may be written in the form

$$[\,\mathbf{v}\,]_{\mathcal{C}} = \begin{bmatrix} [\,\mathbf{b}_1\,]_{\mathcal{C}} & [\,\mathbf{b}_2\,]_{\mathcal{C}} & \cdots & [\,\mathbf{b}_n\,]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{8}$$

by the definition of (c) _____. This shows that the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ shown in (5) satisfies $[\mathbf{v}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{v}]_{\mathcal{B}}$ for each $\mathbf{v}$ in V, because the vector on the right side of (8) is (d) _____.

18. Suppose Q is any matrix such that

$$[\mathbf{v}]_{\mathcal{C}} = Q[\mathbf{v}]_{\mathcal{B}} \quad \text{for each } \mathbf{v} \text{ in } V \tag{9}$$

Set $\mathbf{v} = \mathbf{b}_1$ in (9). Then (9) shows that $[\mathbf{b}_1]_{\mathcal{C}}$ is the first column of Q because (a) _____. Similarly, for $k = 2, \dots, n$, the kth column of Q is (b) _____ because (c) _____. This shows that the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ defined by (5) in Theorem 15 is the only matrix that satisfies condition (4).

T 19. Let $\mathcal{B} = \{\mathbf{x}_0, \dots, \mathbf{x}_6\}$ and $\mathcal{C} = \{\mathbf{y}_0, \dots, \mathbf{y}_6\}$, where $\mathbf{x}_k$ is the function $\cos^k t$ and $\mathbf{y}_k$ is the function $\cos kt$. Exercise 54 in Section 4.5 showed that both $\mathcal{B}$ and $\mathcal{C}$ are bases for the vector space $H = \text{Span}\{\mathbf{x}_0, \dots, \mathbf{x}_6\}$.

a. Set $P = \begin{bmatrix} [\,\mathbf{y}_0\,]_{\mathcal{B}} & \cdots & [\,\mathbf{y}_6\,]_{\mathcal{B}} \end{bmatrix}$, and calculate P^{-1}.

b. Explain why the columns of P^{-1} are the $\mathcal{C}$-coordinate vectors of $\mathbf{x}_0, \dots, \mathbf{x}_6$. Then use these coordinate vectors to write trigonometric identities that express powers of $\cos t$ in terms of the functions in $\mathcal{C}$.

See the *Study Guide*.

T 20. (*Calculus required*)[3] Recall from calculus that integrals such as

$$\int (5\cos^3 t - 6\cos^4 t + 5\cos^5 t - 12\cos^6 t)\, dt \tag{10}$$

are tedious to compute. (The usual method is to apply integration by parts repeatedly and use the half-angle formula.) Use the matrix P or P^{-1} from Exercise 19 to transform (10); then compute the integral.

T 21. Let

$$P = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -5 & 0 \\ 4 & 6 & 1 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -7 \\ 2 \\ 6 \end{bmatrix}$$

a. Find a basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbb{R}^3$ such that P is the change-of-coordinates matrix from $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. [*Hint:* What do the columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ represent?]

b. Find a basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for $\mathbb{R}^3$ such that P is the change-of-coordinates matrix from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

T 22. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ be bases for a two-dimensional vector space.

a. Write an equation that relates the matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, $\underset{\mathcal{D}\leftarrow\mathcal{C}}{P}$, and $\underset{\mathcal{D}\leftarrow\mathcal{B}}{P}$. Justify your result.

b. Use a matrix program either to help you find the equation or to check the equation you write. Work with three bases for $\mathbb{R}^2$. (See Exercises 7–10.)

[3] The idea for Exercises 19 and 20 and five related exercises in earlier sections came from a paper by Jack W. Rogers, Jr., of Auburn University, presented at a meeting of the International Linear Algebra Society, August 1995. See "Applications of Linear Algebra in Calculus," *American Mathematical Monthly* **104** (1), 1997.

THEOREM 18

The set $\mathcal{B}_f = \{S^j(\delta) : \text{where } j \in \mathbb{Z}\}$ is a basis for the infinite dimensional vector space $\mathbb{S}_f$.

PROOF Let $\{y_k\}$ be any signal in $\mathbb{S}_f$. Since only finitely many entries in $\{y_k\}$ are nonzero, there exist integers p and q such that $y_k = 0$ for all $k < p$ and and $k > q$. Thus

$$\{y_k\} = \sum_{j=p}^{q} y_j S^j(\delta),$$

so $\mathcal{B}_f$ is a spanning set for $\mathbb{S}_n$. Moreover, if a linear combination of signals with scalars $c_p, c_{p+1}, \ldots, c_q$ add to zero,

$$\sum_{j=p}^{q} c_j S^j(\delta), = \{0\},$$

then $c_p = c_{p+1} = \cdots = c_q = 0$, and thus the vectors in $\mathcal{B}_f$ form a linearly independent set. This establishes that $\mathcal{B}_f$ is a basis for $\mathbb{S}_f$. Since $\mathcal{B}_f$ contains infinitely many signals, $\mathbb{S}_f$ is an infinite dimensional vector space. ∎

The creative power of the shift LTI transformation falls short of being able to create a basis for $\mathbb{S}$ itself. The definition of linear combination requires that only finitely many vectors and scalars are used in a sum. Consider the unit step signal, υ, from Table 1. Although $\upsilon = \sum_{j=0}^{\infty} S^j(\delta)$, this is an infinite sum of vectors and hence not technically considered a *linear combination* of the basis elements from $\mathcal{B}_f$.

In calculus, sums with infinitely many terms are studied in detail. Although it can be shown that every vector space has a basis (using a finite number of terms in each linear combination), the proof relies on the Axiom of Choice and hence establishing that $\mathbb{S}$ has a basis is a topic you may see in higher level math classes. The sinusoidal and exponential signals, which have infinite support, are explored in detail in Section 4.8

Practice Problems

1. Find $\upsilon + \chi$ from Table 1. Express the answer as a vector and give its formal description.
2. Show that $T(\{x_k\}) = \{3x_k - 2x_{k-1}\}$ is a linear time invariant transformation.
3. Find a nonzero vector in the kernel of T for the linear time invariant transformation given in Practice Problem 2.

4.7 Exercises

For Exercises 1–4, find the indicated sums of the signals in Table 1.

1. $\chi + \alpha$
2. $\chi - \alpha$
3. $\upsilon + 2\alpha$
4. $\upsilon - 3\alpha$

For Exercises 5–8, recall that $I(\{x_k\}) = \{x_k\}$ and $S(\{x_k\}) = \{x_{k-1}\}$.

5. Which signals from Table 1 are in the kernel of $I + S$?
6. Which signals from Table 1 are in the kernel of $I - S$?

7. Which signals from Table 1 are in the kernel of $I - cS$ for a fixed nonzero scalar $c \neq 1$?

8. Which signals from Table 1 are in the kernel of $I - S - S^2$?

9. Show that $T(\{x_k\}) = \{x_k - x_{k-1}\}$ is a linear time invariant transformation.

10. Show that $M_3(\{x_k\}) = \left\{\frac{1}{3}(x_{k-2} + x_{k-1} + x_k)\right\}$ is a linear time invariant transformation.

11. Find a nonzero signal in the kernel of T from Exercise 9.

12. Find a nonzero signal in the kernel of M_3 from Exercise 10.

13. Find a nonzero signal in the range of T from Exercise 9.

14. Find a nonzero signal in the range of M_3 from Exercise 10.

In Exercises 15–22, V is a vector space and A is an $m \times n$ matrix. Mark each statement True or False **(T/F)**. Justify each answer.

15. **(T/F)** The set of signals of length n, $\mathbb{S}_n$, has a basis with $n + 1$ signals.

16. **(T/F)** The set of signals, $\mathbb{S}$, has a finite basis.

17. **(T/F)** Every subspace of the set of signals $\mathbb{S}$ is infinite dimensional.

18. **(T/F)** The vector space $\mathbb{R}^{n+1}$ is a subspace of $\mathbb{S}$.

19. **(T/F)** Every linear time invariant transformation is a linear transformation.

20. **(T/F)** The moving average function is a linear time invariant transformation.

21. **(T/F)** If you scale a signal by a fixed constant, the result is not a signal.

22. **(T/F)** If you scale a linear time invariant transformation by a fixed constant, the result is no longer a linear transformation.

Guess and check or working backwards through the solution to Practice Problem 3 are two good ways to find solutions to Exercises 23 and 24.

23. Construct a linear time invariant transformation that has the signal $\{x_k\} = \left\{\left(\frac{4}{5}\right)^k\right\}$ in its kernel.

24. Construct a linear time invariant transformation that has the signal $\{x_k\} = \left\{\left(\frac{-3}{4}\right)^k\right\}$ in its kernel.

25. Let $W = \left\{\{x_k\} \mid x_k = \begin{cases} 0 & \text{if } k \text{ is a multiple of 2} \\ r & \text{if } k \text{ is not a multiple of 2} \end{cases}\right.$ where r can be any real number$\Big\}$. A typical signal in W looks like

$$(\ldots,\ r,\ 0,\ r,\ \underset{\substack{\uparrow \\ k=0}}{0},\ r,\ 0,\ r,\ \ldots)$$

Show that W is a subspace of $\mathbb{S}$.

26. Let $W = \left\{\{x_k\} \mid x_k = \begin{cases} 0 & \text{if } k < 0 \\ r & \text{if } k \geq 0 \end{cases}\right.$ where r can be any real number.$\Big\}$ A typical signal in W looks like

$$(\ldots,\ 0,\ 0,\ 0,\ \underset{\substack{\uparrow \\ k=0}}{r},\ r,\ r,\ r,\ \ldots)$$

Show that W is a subspace of $\mathbb{S}$.

27. Find a basis for the subspace W in Exercise 25. What is the dimension of this subspace?

28. Find a basis for the subspace W in Exercise 26. What is the dimension of this subspace?

29. Let $W = \left\{\{x_k\} \mid x_k = \begin{cases} 0 & \text{if } k \text{ is a multiple of 2} \\ r_k & \text{if } k \text{ is not a multiple of 2} \end{cases}\right.$ where each r_k can be any real number$\Big\}$. A typical signal in W looks like

$$(\ldots,\ r_{-3},\ 0,\ r_{-1},\ \underset{\substack{\uparrow \\ k=0}}{0},\ r_1,\ 0,\ r_3,\ \ldots)$$

Show that W is a subspace of $\mathbb{S}$.

30. Let $W = \left\{\{x_k\} \mid x_k = \begin{cases} 0 & \text{if } k < 0 \\ r_k & \text{if } k \geq 0 \end{cases}\right.$ where each r_k can be any real number$\Big\}$. A typical signal in W looks like

$$(\ldots,\ 0,\ 0,\ 0,\ \underset{\substack{\uparrow \\ k=0}}{r_0},\ r_1,\ r_2,\ r_3,\ \ldots)$$

Show that W is a subspace of $\mathbb{S}$.

31. Describe an infinite linearly independent subset of the subspace W in Exercise 29. Does this establish that W is infinite dimensional? Justify your answer.

32. Describe an infinite linearly independent subset of the subspace W in Exercise 30. Does this establish that W is infinite dimensional? Justify your.

EXAMPLE 6 Write the following difference equation as a first-order system:

$$y_{k+3} - 2y_{k+2} - 5y_{k+1} + 6y_k = 0 \quad \text{for all } k$$

SOLUTION For each k, set

$$\mathbf{x}_k = \begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \end{bmatrix}$$

The difference equation says that $y_{k+3} = -6y_k + 5y_{k+1} + 2y_{k+2}$, so

$$\mathbf{x}_{k+1} = \begin{bmatrix} y_{k+1} \\ y_{k+2} \\ y_{k+3} \end{bmatrix} = \begin{bmatrix} 0 + y_{k+1} + 0 \\ 0 + 0 + y_{k+2} \\ -6y_k + 5y_{k+1} + 2y_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{bmatrix} \begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \end{bmatrix}$$

That is,

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \quad \text{for all } k, \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{bmatrix}$$

■

In general, the equation

$$y_{k+n} + a_1 y_{k+n-1} + \cdots + a_{n-1} y_{k+1} + a_n y_k = 0 \quad \text{for all } k$$

can be rewritten as $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for all k, where

$$\mathbf{x}_k = \begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+n-1} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

Practice Problem

It can be shown that the signals 2^k, $3^k \sin \frac{k\pi}{2}$, and $3^k \cos \frac{k\pi}{2}$ are solutions of

$$y_{k+3} - 2y_{k+2} + 9y_{k+1} - 18y_k = 0$$

Show that these signals form a basis for the set of all solutions of the difference equation.

4.8 Exercises

Verify that the signals in Exercises 1 and 2 are solutions of the accompanying difference equation.

1. $2^k, (-4)^k$; $y_{k+2} + 2y_{k+1} - 8y_k = 0$

2. $4^k, (-4)^k$; $y_{k+2} - 16y_k = 0$

Show that the signals in Exercises 3–6 form a basis for the solution set of the accompanying difference equation.

3. The signals and equation in Exercise 1

4. The signals and equation in Exercise 2

5. $(-3)^k$, $k(-3)^k$; $y_{k+2} + 6y_{k+1} + 9y_k = 0$

6. $5^k \cos \frac{k\pi}{2}, 5^k \sin \frac{k\pi}{2}$; $y_{k+2} + 25y_k = 0$

In Exercises 7–12, assume the signals listed are solutions of the given difference equation. Determine if the signals form a basis for the solution space of the equation. Justify your answers using appropriate theorems.

7. $1^k, 3^k, (-3)^k$; $y_{k+3} - y_{k+2} - 9y_{k+1} + 9y_k = 0$

8. $2^k, 4^k, (-5)^k$; $y_{k+3} - y_{k+2} - 22y_{k+1} + 40y_k = 0$

9. $1^k, 3^k \cos \frac{k\pi}{2}, 3^k \sin \frac{k\pi}{2}$; $y_{k+3} - y_{k+2} + 9y_{k+1} - 9y_k = 0$

10. $(-1)^k, k(-1)^k, 5^k;\ y_{k+3} - 3y_{k+2} - 9y_{k+1} - 5y_k = 0$

11. $(-1)^k, 3^k;\ y_{k+3} + y_{k+2} - 9y_{k+1} - 9y_k = 0$

12. $1^k, (-1)^k;\ y_{k+4} - 2y_{k+2} + y_k = 0$

In Exercises 13–16, find a basis for the solution space of the difference equation. Prove that the solutions you find span the solution set.

13. $y_{k+2} - y_{k+1} + \frac{2}{9}y_k = 0$ 14. $y_{k+2} - 9y_{k+1} + 14y_k = 0$

15. $y_{k+2} - 25y_k = 0$ 16. $16y_{k+2} + 8y_{k+1} - 3y_k = 0$

17. The Fibonacci Sequence is listed in Table 1 of Section 4.7. It can be viewed as the sequence of numbers where each number is the sum of the two numbers before it. It can be described as the homogeneous difference equation

$$y_{k+2} - y_{k+1} - y_k = 0$$

with the initial conditions $y_0 = 0$ and $y_1 = 1$. Find the general solution of the Fibonacci sequence.

18. If the initial conditions are changed to $y_0 = 1$ and $y_1 = 2$ for the Fibonacci sequence in Exercise 17, list the terms of the sequence for $k = 2, 3, 4$ and 5. Find the solution to the difference equation from 17 with these new initial conditions.

Exercises 19 and 20 concern a simple model of the national economy described by the difference equation

$$Y_{k+2} - a(1+b)Y_{k+1} + abY_k = 1 \tag{14}$$

Here Y_k is the total national income during year k, a is a constant less than 1, called the *marginal propensity to consume*, and b is a positive *constant of adjustment* that describes how changes in consumer spending affect the annual rate of private investment.[1]

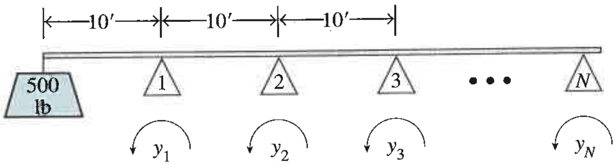
19. Find the general solution of equation (14) when $a = .9$ and $b = \frac{4}{9}$. What happens to Y_k as k increases? [*Hint:* First find a particular solution of the form $Y_k = T$, where T is a constant, called the equilibrium level of national income.]

20. Find the general solution of equation (14) when $a = .9$ and $b = .5$.

A lightweight cantilevered beam is supported at N points spaced 10 ft apart, and a weight of 500 lb is placed at the end of the beam, 10 ft from the first support, as in the figure. Let y_k be the bending moment at the kth support. Then $y_1 = 5000$ ft-lb. Suppose the beam is rigidly attached at the Nth support and the bending moment there is zero. In between, the moments satisfy the *three-moment equation*

$$y_{k+2} + 4y_{k+1} + y_k = 0 \quad \text{for } k = 1, 2, \ldots, N-2 \tag{15}$$

[1] For example, see *Discrete Dynamical Systems*, by James T. Sandefur (Oxford: Clarendon Press, 1990), pp. 267–276. The original *accelerator-multiplier model* is attributed to the economist P. A. Samuelson.



Bending moments on a cantilevered beam.

21. Find the general solution of difference equation (15). Justify your answer.

22. Find the particular solution of (15) that satisfies the *boundary conditions* $y_1 = 5000$ and $y_N = 0$. (The answer involves N.)

23. When a signal is produced from a sequence of measurements made on a process (a chemical reaction, a flow of heat through a tube, a moving robot arm, etc.), the signal usually contains random *noise* produced by measurement errors. A standard method of preprocessing the data to reduce the noise is to smooth or filter the data. One simple filter is a *moving average* that replaces each y_k by its average with the two adjacent values:

$$\tfrac{1}{3}y_{k+1} + \tfrac{1}{3}y_k + \tfrac{1}{3}y_{k-1} = z_k \quad \text{for } k = 1, 2, \ldots$$

Suppose a signal y_k, for $k = 0, \ldots, 14$, is

9, 5, 7, 3, 2, 4, 6, 5, 7, 6, 8, 10, 9, 5, 7

Use the filter to compute $z_1, \ldots, z_{13}$. Make a broken-line graph that superimposes the original signal and the smoothed signal.

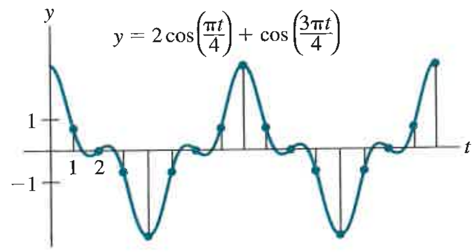
24. Let $\{y_k\}$ be the sequence produced by sampling the continuous signal $2\cos\frac{\pi t}{4} + \cos\frac{3\pi t}{4}$ at $t = 0, 1, 2, \ldots$, as shown in the figure. The values of y_k, beginning with $k = 0$, are

3, .7, 0, −.7, −3, −.7, 0, .7, 3, .7, 0, ...

where .7 is an abbreviation for $\sqrt{2}/2$.

a. Compute the output signal $\{z_k\}$ when $\{y_k\}$ is fed into the filter in Example 2.

b. Explain how and why the output in part (a) is related to the calculations in Example 2.



Sampled data from $2\cos\frac{\pi t}{4} + \cos\frac{3\pi t}{4}$.

Exercises 25 and 26 refer to a difference equation of the form $y_{k+1} - ay_k = b$, for suitable constants a and b.

25. A loan of \$10,000 has an interest rate of 1% per month and a monthly payment of \$450. The loan is made at month $k = 0$, and the first payment is made one month later, at $k = 1$. For

$k = 0, 1, 2, \ldots$, let y_k be the unpaid balance of the loan just after the kth monthly payment. Thus

$$\underset{\text{New balance}}{y_1} = \underset{\text{Balance due}}{10{,}000} + \underset{\text{Interest added}}{(.01)10{,}000} - \underset{\text{Payment}}{450}$$

a. Write a difference equation satisfied by $\{y_k\}$.

T b. Create a table showing k and the balance y_k at month k. List the program or the keystrokes you used to create the table.

T c. What will k be when the last payment is made? How much will the last payment be? How much money did the borrower pay in total?

26. At time $k = 0$, an initial investment of \$1000 is made into a savings account that pays 6% interest per year compounded monthly. (The interest rate per month is .005.) Each month after the initial investment, an additional \$200 is added to the account. For $k = 0, 1, 2, \ldots$, let y_k be the amount in the account at time k, just after a deposit has been made.

a. Write a difference equation satisfied by $\{y_k\}$.

T b. Create a table showing k and the total amount in the savings account at month k, for $k = 0$ through 60. List your program or the keystrokes you used to create the table.

T c. How much will be in the account after two years (that is, 24 months), four years, and five years? How much of the five-year total is interest?

In Exercises 27–30, show that the given signal is a solution of the difference equation. Then find the general solution of that difference equation.

27. $y_k = k^2$; $y_{k+2} + 4y_{k+1} - 5y_k = 8 + 12k$

28. $y_k = 1 + k$; $y_{k+2} - 8y_{k+1} + 15y_k = 2 + 8k$

29. $y_k = 2 - 2k$; $y_{k+2} - \frac{9}{2}y_{k+1} + 2y_k = 2 + 3k$

30. $y_k = 2k - 4$; $y_{k+2} + \frac{3}{2}y_{k+1} - y_k = 1 + 3k$

Write the difference equations in Exercises 31 and 32 as first-order systems, $\mathbf{x}_{k+1} = A\mathbf{x}_k$, for all k.

31. $y_{k+4} - 2y_{k+3} - 3y_{k+2} + 8y_{k+1} - 4y_k = 0$

32. $y_{k+3} - \frac{3}{4}y_{k+2} + \frac{1}{16}y_k = 0$

33. Is the following difference equation of order 3? Explain.

$$y_{k+3} + 5y_{k+2} + 6y_{k+1} = 0$$

34. What is the order of the following difference equation? Explain your answer.

$$y_{k+3} + a_1 y_{k+2} + a_2 y_{k+1} + a_3 y_k = 0$$

35. Let $y_k = k^2$ and $z_k = 2k|k|$. Are the signals $\{y_k\}$ and $\{z_k\}$ linearly independent? Evaluate the associated Casorati matrix $C(k)$ for $k = 0$, $k = -1$, and $k = -2$, and discuss your results.

36. Let f, g, and h be linearly independent functions defined for all real numbers, and construct three signals by sampling the values of the functions at the integers:

$$u_k = f(k), \qquad v_k = g(k), \qquad w_k = h(k)$$

Must the signals be linearly independent in $\mathbb{S}$? Discuss.

Solution to Practice Problem

Examine the Casorati matrix:

$$C(k) = \begin{bmatrix} 2^k & 3^k \sin\frac{k\pi}{2} & 3^k \cos\frac{k\pi}{2} \\ 2^{k+1} & 3^{k+1} \sin\frac{(k+1)\pi}{2} & 3^{k+1} \cos\frac{(k+1)\pi}{2} \\ 2^{k+2} & 3^{k+2} \sin\frac{(k+2)\pi}{2} & 3^{k+2} \cos\frac{(k+2)\pi}{2} \end{bmatrix}$$

Set $k = 0$ and row reduce the matrix to verify that it has three pivot positions and hence is invertible:

$$C(0) = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 4 & 0 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & -13 \end{bmatrix}$$

The Casorati matrix is invertible at $k = 0$, so the signals are linearly independent. Since there are three signals, and the solution space H of the difference equation has dimension 3 (Theorem 20), the signals form a basis for H, by the Basis Theorem.