

Practice Problems (Continued)

$$\begin{array}{ll} \text{a. } x_1 + 4x_2 - 2x_3 + 8x_4 = 12 & \text{b. } x_1 - 3x_2 + 5x_3 - 2x_4 = 0 \\ x_2 - 7x_3 + 2x_4 = -4 & x_2 + 8x_3 = -4 \\ 5x_3 - x_4 = 7 & 2x_3 = 3 \\ x_3 + 3x_4 = -5 & x_4 = 1 \end{array}$$

2. The augmented matrix of a linear system has been transformed by row operations into the form below. Determine if the system is consistent.

$$\left[\begin{array}{cccc} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{array} \right]$$

3. Is $(3, 4, -2)$ a solution of the following system?

$$\begin{array}{rcl} 5x_1 - x_2 + 2x_3 & = & 7 \\ -2x_1 + 6x_2 + 9x_3 & = & 0 \\ -7x_1 + 5x_2 - 3x_3 & = & -7 \end{array}$$

4. For what values of h and k is the following system consistent?

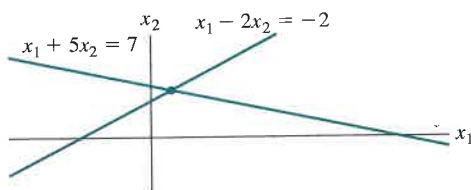
$$\begin{array}{rcl} 2x_1 - x_2 & = & h \\ -6x_1 + 3x_2 & = & k \end{array}$$

1.1 Exercises

Solve each system in Exercises 1–4 by using elementary row operations on the equations or on the augmented matrix. Follow the systematic elimination procedure described in this section.

$$\begin{array}{ll} 1. \quad x_1 + 5x_2 = 7 & 2. \quad 2x_1 + 4x_2 = -4 \\ -2x_1 - 7x_2 = -5 & 5x_1 + 7x_2 = 11 \end{array}$$

3. Find the point (x_1, x_2) that lies on the line $x_1 + 5x_2 = 7$ and on the line $x_1 - 2x_2 = -2$. See the figure.



4. Find the point of intersection of the lines $x_1 - 5x_2 = 1$ and $3x_1 - 7x_2 = 5$.

Consider each matrix in Exercises 5 and 6 as the augmented matrix of a linear system. State in words the next two elementary row operations that should be performed in the process of solving the system.

$$5. \quad \left[\begin{array}{ccccc} 1 & 3 & -4 & 0 & 9 \\ 1 & 1 & 5 & 0 & -8 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -6 \end{array} \right]$$

$$6. \quad \left[\begin{array}{ccccc} 1 & -6 & 4 & 0 & -1 \\ 0 & 2 & -7 & 0 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 3 & 1 & 6 \end{array} \right]$$

In Exercises 7–10, the augmented matrix of a linear system has been reduced by row operations to the form shown. In each case, continue the appropriate row operations and describe the solution set of the original system.

$$7. \quad \left[\begin{array}{cccc} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

$$8. \quad \left[\begin{array}{cccc} 1 & 1 & 5 & 0 \\ 0 & 1 & 9 & 0 \\ 0 & 0 & 7 & -7 \end{array} \right]$$

$$9. \quad \left[\begin{array}{ccccc} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 4 \end{array} \right]$$

$$10. \begin{bmatrix} 1 & -2 & 0 & 3 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Solve the systems in Exercises 11–14.

$$11. \quad x_2 + 4x_3 = -4$$

$$x_1 + 3x_2 + 3x_3 = -2$$

$$3x_1 + 7x_2 + 5x_3 = 6$$

$$12. \quad x_1 - 3x_2 + 4x_3 = -4$$

$$3x_1 - 7x_2 + 7x_3 = -8$$

$$-4x_1 + 6x_2 + 2x_3 = 4$$

$$13. \quad x_1 - 3x_3 = 8$$

$$2x_1 + 2x_2 + 9x_3 = 7$$

$$x_2 + 5x_3 = -2$$

$$14. \quad x_1 - 3x_2 = 5$$

$$-x_1 + x_2 + 5x_3 = 2$$

$$x_2 + x_3 = 0$$

15. Verify that the solution you found to Exercise 11 is correct by substituting the values you obtained back into the original equations.

16. Verify that the solution you found to Exercise 12 is correct by substituting the values you obtained back into the original equations.

17. Verify that the solution you found to Exercise 13 is correct by substituting the values you obtained back into the original equations.

18. Verify that the solution you found to Exercise 14 is correct by substituting the values you obtained back into the original equations.

Determine if the systems in Exercises 19 and 20 are consistent. Do not completely solve the systems.

$$19. \quad x_1 + 3x_3 = 2$$

$$x_2 - 3x_4 = 3$$

$$-2x_2 + 3x_3 + 2x_4 = 1$$

$$3x_1 + 7x_4 = -5$$

$$20. \quad x_1 - 2x_4 = -3$$

$$2x_2 + 2x_3 = 0$$

$$x_3 + 3x_4 = 1$$

$$-2x_1 + 3x_2 + 2x_3 + x_4 = 5$$

21. Do the three lines $x_1 - 4x_2 = 1$, $2x_1 - x_2 = -3$, and $-x_1 - 3x_2 = 4$ have a common point of intersection? Explain.

22. Do the three planes $x_1 + 2x_2 + x_3 = 4$, $x_2 - x_3 = 1$, and $x_1 + 3x_2 = 0$ have at least one common point of intersection? Explain.

In Exercises 23–26, determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

$$23. \begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix}$$

$$24. \begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix}$$

$$25. \begin{bmatrix} 1 & 3 & -2 \\ -4 & h & 8 \end{bmatrix}$$

$$26. \begin{bmatrix} 3 & -4 & h \\ -6 & 8 & 9 \end{bmatrix}$$

In Exercises 27–34, key statements from this section are either quoted directly, restated slightly (but still true), or altered in some way that makes them false in some cases. Mark each statement True or False, and *justify* your answer. (If true, give the approximate location where a similar statement appears, or refer to a definition or theorem. If false, give the location of a statement that has been quoted or used incorrectly, or cite an example that shows the statement is not true in all cases.) Similar true/false questions will appear in many sections of the text and will be flagged with a (T/F) at the beginning of the question.

27. (T/F) Every elementary row operation is reversible.

28. (T/F) Elementary row operations on an augmented matrix never change the solution set of the associated linear system.

29. (T/F) A 5×6 matrix has six rows.

30. (T/F) Two matrices are row equivalent if they have the same number of rows.

31. (T/F) The solution set of a linear system involving variables x_1, \dots, x_n is a list of numbers (s_1, \dots, s_n) that makes each equation in the system a true statement when the values s_1, \dots, s_n are substituted for x_1, \dots, x_n , respectively.

32. (T/F) An inconsistent system has more than one solution.

33. (T/F) Two fundamental questions about a linear system involve existence and uniqueness.

34. (T/F) Two linear systems are equivalent if they have the same solution set.

35. Find an equation involving g , h , and k that makes this augmented matrix correspond to a consistent system:

$$\begin{bmatrix} 1 & -3 & 5 & g \\ 0 & 2 & -3 & h \\ -3 & 5 & -9 & k \end{bmatrix}$$

36. Construct three different augmented matrices for linear systems whose solution set is $x_1 = -2$, $x_2 = 1$, $x_3 = 0$.

37. Suppose the system below is consistent for all possible values of f and g . What can you say about the coefficients c and d ? Justify your answer.

$$x_1 + 5x_2 = f$$

$$cx_1 + dx_2 = g$$

38. Suppose a , b , c , and d are constants such that a is not zero and the system below is consistent for all possible values of f and g . What can you say about the numbers a , b , c , and d ? Justify your answer.

$$ax_1 + bx_2 = f$$

$$cx_1 + dx_2 = g$$

In Exercises 39–42, find the elementary row operation that transforms the first matrix into the second, and then find the reverse row operation that transforms the second matrix into the first.

39. $\begin{bmatrix} 0 & -2 & 5 \\ 1 & 4 & -7 \\ 3 & -1 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 4 & -7 \\ 0 & -2 & 5 \\ 3 & -1 & 6 \end{bmatrix}$

40. $\begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & -5 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & -3 \\ 0 & -5 & 9 \end{bmatrix}$

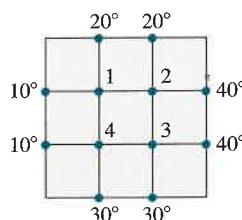
41. $\begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 4 & -5 & 6 \\ 5 & -7 & 8 & -9 \end{bmatrix}, \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 4 & -5 & 6 \\ 0 & 8 & -2 & -9 \end{bmatrix}$

42. $\begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & -3 & 9 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

An important concern in the study of heat transfer is to determine the steady-state temperature distribution of a thin plate when the

temperature around the boundary is known. Assume the plate shown in the figure represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate. Let T_1, \dots, T_4 denote the temperatures at the four interior nodes of the mesh in the figure. The temperature at a node is approximately equal to the average of the four nearest nodes—to the left, above, to the right, and below.² For instance,

$$T_1 = (10 + 20 + T_2 + T_4)/4, \quad \text{or} \quad 4T_1 - T_2 - T_4 = 30$$



43. Write a system of four equations whose solution gives estimates for the temperatures T_1, \dots, T_4 .
44. Solve the system of equations from Exercise 43. [Hint: To speed up the calculations, interchange rows 1 and 4 before starting “replace” operations.]

² See Frank M. White, *Heat and Mass Transfer* (Reading, MA: Addison-Wesley Publishing, 1991), pp. 145–149.

Solutions to Practice Problems

1. a. For “hand computation,” the best choice is to interchange equations 3 and 4. Another possibility is to multiply equation 3 by $1/5$. Or, replace equation 4 by its sum with $-1/5$ times row 3. (In any case, do not use the x_2 in equation 2 to eliminate the $4x_2$ in equation 1. Wait until a triangular form has been reached and the x_3 terms and x_4 terms have been eliminated from the first two equations.)
- b. The system is in triangular form. Further simplification begins with the x_4 in the fourth equation. Use the x_4 to eliminate all x_4 terms above it. The appropriate step now is to add 2 times equation 4 to equation 1. (After that, move to equation 3, multiply it by $1/2$, and then use the equation to eliminate the x_3 terms above it.)
2. The system corresponding to the augmented matrix is

$$x_1 + 5x_2 + 2x_3 = -6$$

$$4x_2 - 7x_3 = 2$$

$$5x_3 = 0$$

The third equation makes $x_3 = 0$, which is certainly an allowable value for x_3 . After eliminating the x_3 terms in equations 1 and 2, you could go on to solve for unique values for x_2 and x_1 . Hence a solution exists, and it is unique. Contrast this situation with that in Example 3.

$$\begin{aligned}
 (8 - 3x_3) - 2(3 - x_3) + (x_3) &= 8 - 3x_3 - 6 + 2x_3 + x_3 = 2 \\
 (8 - 3x_3) - (3 - x_3) + 2(x_3) &= 8 - 3x_3 - 3 + x_3 + 2x_3 = 5 \\
 (3 - x_3) + (x_3) &= 3 - x_3 + x_3 = 3
 \end{aligned}$$

You can now be confident you have a correct solution to the system of equations represented by the augmented matrix.

Practice Problems

1. Find the general solution of the linear system whose augmented matrix is

$$\begin{bmatrix} 1 & -3 & -5 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

2. Find the general solution of the system

$$\begin{aligned}
 x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\
 -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\
 3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2
 \end{aligned}$$

3. Suppose a 4×7 coefficient matrix for a system of equations has 4 pivots. Is the system consistent? If the system is consistent, how many solutions are there?

1.2 Exercises

In Exercises 1 and 2, determine which matrices are in reduced echelon form and which others are only in echelon form.

1. a. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$
2. a. $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
- c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Row reduce the matrices in Exercises 3 and 4 to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

3. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$ 4. $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$
5. Describe the possible echelon forms of a nonzero 2×2 matrix. Use the symbols \blacksquare , $*$, and 0, as in the first part of Example 1.
6. Repeat Exercise 5 for a nonzero 3×2 matrix.
- Find the general solutions of the systems whose augmented matrices are given in Exercises 7–14.
7. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 8 & 9 & 4 \end{bmatrix}$ 8. $\begin{bmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 11 \end{bmatrix}$
9. $\begin{bmatrix} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -4 \end{bmatrix}$ 10. $\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -2 & 2 \end{bmatrix}$
11. $\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$ 12. $\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$
13. $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

14.
$$\begin{bmatrix} 1 & 2 & -5 & -4 & 0 & -5 \\ 0 & 1 & -6 & -4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

You may find it helpful to review the information in the Reasonable Answers box from this section before answering Exercises 15–18.

15. Write down the equations corresponding to the augmented matrix in Exercise 9 and verify your answer to Exercise 9 is correct by substituting the solutions you obtained back into the original equations.
16. Write down the equations corresponding to the augmented matrix in Exercise 10 and verify your answer to Exercise 10 is correct by substituting the solutions you obtained back into the original equations.
17. Write down the equations corresponding to the augmented matrix in Exercise 11 and verify your answer to Exercise 11 is correct by substituting the solutions you obtained back into the original equations.
18. Write down the equations corresponding to the augmented matrix in Exercise 12 and verify your answer to Exercise 12 is correct by substituting the solutions you obtained back into the original equations.

Exercises 19 and 20 use the notation of Example 1 for matrices in echelon form. Suppose each matrix represents the augmented matrix for a system of linear equations. In each case, determine if the system is consistent. If the system is consistent, determine if the solution is unique.

19. a.
$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 0 & \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

20. a.
$$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

In Exercises 21 and 22, determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

21.
$$\begin{bmatrix} 2 & 3 & h \\ 4 & 6 & 7 \end{bmatrix}$$

22.
$$\begin{bmatrix} 1 & -4 & -3 \\ 6 & h & -9 \end{bmatrix}$$

In Exercises 23 and 24, choose h and k such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part.

23. $x_1 + hx_2 = 2$
 $4x_1 + 8x_2 = k$

24. $x_1 + 4x_2 = 5$
 $2x_1 + hx_2 = k$

In Exercises 25–34, mark each statement True or False (T/F). Justify each answer.⁴

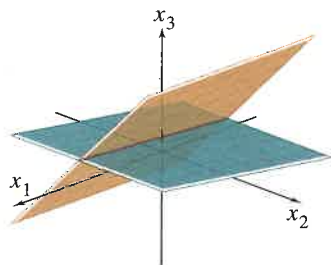
25. (T/F) In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations.
26. (T/F) The echelon form of a matrix is unique.
27. (T/F) The row reduction algorithm applies only to augmented matrices for a linear system.
28. (T/F) The pivot positions in a matrix depend on whether row interchanges are used in the row reduction process.
29. (T/F) A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.
30. (T/F) Reducing a matrix to echelon form is called the *forward phase* of the row reduction process.
31. (T/F) Finding a parametric description of the solution set of a linear system is the same as *solving* the system.
32. (T/F) Whenever a system has free variables, the solution set contains a unique solution.
33. (T/F) If one row in an echelon form of an augmented matrix is $[0 \ 0 \ 0 \ 0 \ 5]$, then the associated linear system is inconsistent.
34. (T/F) A general solution of a system is an explicit description of all solutions of the system.
35. Suppose a 3×5 coefficient matrix for a system has three pivot columns. Is the system consistent? Why or why not?
36. Suppose a system of linear equations has a 3×5 augmented matrix whose fifth column is a pivot column. Is the system consistent? Why (or why not)?
37. Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system is consistent.
38. Suppose the coefficient matrix of a linear system of three equations in three variables has a pivot in each column. Explain why the system has a unique solution.
39. Restate the last sentence in Theorem 2 using the concept of pivot columns: “If a linear system is consistent, then the solution is unique if and only if _____.”
40. What would you have to know about the pivot columns in an augmented matrix in order to know that the linear system is consistent and has a unique solution?
41. A system of linear equations with fewer equations than unknowns is sometimes called an *underdetermined system*.

⁴ True/false questions of this type will appear in many sections. Methods for justifying your answers were described before the True or False exercises in Section 1.1.

Suppose that such a system happens to be consistent. Explain why there must be an infinite number of solutions.

42. Give an example of an inconsistent underdetermined system of two equations in three unknowns.
43. A system of linear equations with more equations than unknowns is sometimes called an *overdetermined system*. Can such a system be consistent? Illustrate your answer with a specific system of three equations in two unknowns.
44. Suppose an $n \times (n + 1)$ matrix is row reduced to reduced echelon form. Approximately what fraction of the total number of operations (flops) is involved in the backward phase of the reduction when $n = 30$? when $n = 300$?

Suppose experimental data are represented by a set of points in the plane. An **interpolating polynomial** for the data is a polynomial whose graph passes through every point. In scientific work, such a polynomial can be used, for example, to estimate values between the known data points. Another use is to create curves for graphical images on a computer screen. One method for finding an interpolating polynomial is to solve a system of linear equations.



The general solution of the system of equations is the line of intersection of the two planes.

45. Find the interpolating polynomial $p(t) = a_0 + a_1t + a_2t^2$ for the data (1, 11), (2, 16), (3, 19). That is, find a_0 , a_1 , and a_2 such that

$$a_0 + a_1(1) + a_2(1)^2 = 11$$

$$a_0 + a_1(2) + a_2(2)^2 = 16$$

$$a_0 + a_1(3) + a_2(3)^2 = 19$$

46. In a wind tunnel experiment, the force on a projectile due to air resistance was measured at different velocities:

Velocity (100 ft/sec)	0	2	4	6	8	10
Force (100 lb)	0	2.90	14.8	39.6	74.3	119

Find an interpolating polynomial for these data and estimate the force on the projectile when the projectile is traveling at 750 ft/sec. Use $p(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5$. What happens if you try to use a polynomial of degree less than 5? (Try a cubic polynomial, for instance.)⁵

⁵ Exercises marked with the symbol **T** are designed to be worked with the aid of a "Matrix program" (a computer program, such as MATLAB, Maple, Mathematica, MathCad, Octave, or Derive, or a programmable calculator with matrix capabilities, such as those manufactured by Texas Instruments or Hewlett-Packard).

Solutions to Practice Problems

1. The reduced echelon form of the augmented matrix and the corresponding system are

$$\begin{bmatrix} 1 & 0 & -8 & -3 \\ 0 & 1 & -1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{cases} x_1 - 8x_3 = -3 \\ x_2 - x_3 = -1 \end{cases}$$

The basic variables are x_1 and x_2 , and the general solution is

$$\begin{cases} x_1 = -3 + 8x_3 \\ x_2 = -1 + x_3 \\ x_3 \text{ is free} \end{cases}$$

Note: It is essential that the general solution describe each variable, with any parameters clearly identified. The following statement does *not* describe the solution:

$$\begin{cases} x_1 = -3 + 8x_3 \\ x_2 = -1 + x_3 \\ x_3 = 1 + x_2 \end{cases} \quad \text{Incorrect solution}$$

This description implies that x_2 and x_3 are *both* free, which certainly is not the case.

2. Row reduce the system's augmented matrix:

$$\begin{aligned} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{bmatrix} &\sim \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \end{aligned}$$

Practice Problems

1. Prove that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for any \mathbf{u} and \mathbf{v} in \mathbb{R}^n .
2. For what value(s) of h will \mathbf{y} be in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

3. Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{u}$, and \mathbf{v} be vectors in \mathbb{R}^n . Suppose the vectors \mathbf{u} and \mathbf{v} are in $\text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Show that $\mathbf{u} + \mathbf{v}$ is also in $\text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. [Hint: The solution requires the use of the definition of the span of a set of vectors. It is useful to review this definition before starting this exercise.]

1.3 Exercises

In Exercises 1 and 2, compute $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - 2\mathbf{v}$.

1. $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$

2. $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

In Exercises 3 and 4, display the following vectors using arrows on an xy -graph: $\mathbf{u}, \mathbf{v}, -\mathbf{v}, -2\mathbf{v}, \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}$, and $\mathbf{u} - 2\mathbf{v}$. Notice that $\mathbf{u} - \mathbf{v}$ is the vertex of a parallelogram whose other vertices are $\mathbf{u}, \mathbf{0}$, and $-\mathbf{v}$.

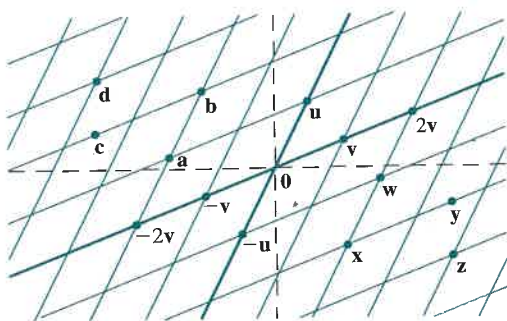
3. \mathbf{u} and \mathbf{v} as in Exercise 1 4. \mathbf{u} and \mathbf{v} as in Exercise 2

In Exercises 5 and 6, write a system of equations that is equivalent to the given vector equation.

5. $x_1 \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -8 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ -5 \end{bmatrix}$

6. $x_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Use the accompanying figure to write each vector listed in Exercises 7 and 8 as a linear combination of \mathbf{u} and \mathbf{v} . Is every vector in \mathbb{R}^2 a linear combination of \mathbf{u} and \mathbf{v} ?



7. Vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and \mathbf{d}
8. Vectors $\mathbf{w}, \mathbf{x}, \mathbf{y}$, and \mathbf{z}

In Exercises 9 and 10, write a vector equation that is equivalent to the given system of equations.

9. $x_2 + 5x_3 = 0$ 10. $4x_1 + x_2 + 3x_3 = 9$
 $4x_1 + 6x_2 - x_3 = 0$ $x_1 - 7x_2 - 2x_3 = 2$
 $-x_1 + 3x_2 - 8x_3 = 0$ $8x_1 + 6x_2 - 5x_3 = 15$

In Exercises 11 and 12, determine if \mathbf{b} is a linear combination of $\mathbf{a}_1, \mathbf{a}_2$, and \mathbf{a}_3 .

11. $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

12. $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$

In Exercises 13 and 14, determine if \mathbf{b} is a linear combination of the vectors formed from the columns of the matrix A .

13. $A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$

In Exercises 15 and 16, list five vectors in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. For each vector, show the weights on \mathbf{v}_1 and \mathbf{v}_2 used to generate the vector and list the three entries of the vector. Do not make a sketch.

15. $\mathbf{v}_1 = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$

16. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$

17. Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$. For what value(s) of h is \mathbf{b} in the plane spanned by \mathbf{a}_1 and \mathbf{a}_2 ?
18. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -5 \\ 1 \\ 7 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} h \\ -1 \\ -5 \end{bmatrix}$. For what value(s) of h is \mathbf{y} in the plane generated by \mathbf{v}_1 and \mathbf{v}_2 ?
19. Give a geometric description of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for the vectors $\mathbf{v}_1 = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 12 \\ 3 \\ -9 \end{bmatrix}$.
20. Give a geometric description of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for the vectors in Exercise 16.
21. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ for all h and k .
22. Construct a 3×3 matrix A , with nonzero entries, and a vector \mathbf{b} in \mathbb{R}^3 such that \mathbf{b} is *not* in the set spanned by the columns of A .
- In Exercises 23–32, mark each statement True or False (T/F). Justify each answer.
23. (T/F) Another notation for the vector $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is $[-4 \ 3]$.
24. (T/F) Any list of five real numbers is a vector in \mathbb{R}^5 .
25. (T/F) The points in the plane corresponding to $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ lie on a line through the origin.
26. (T/F) The vector \mathbf{u} results when a vector $\mathbf{u} - \mathbf{v}$ is added to the vector \mathbf{v} .
27. (T/F) An example of a linear combination of vectors \mathbf{v}_1 and \mathbf{v}_2 is the vector $\frac{1}{2}\mathbf{v}_1$.
28. (T/F) The weights c_1, \dots, c_p in a linear combination $c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$ cannot all be zero.
29. (T/F) The solution set of the linear system whose augmented matrix is $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{b}]$ is the same as the solution set of the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$.
30. (T/F) When \mathbf{u} and \mathbf{v} are nonzero vectors, $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ contains the line through \mathbf{u} and the origin.
31. (T/F) The set $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is always visualized as a plane through the origin.
32. (T/F) Asking whether the linear system corresponding to an augmented matrix $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{b}]$ has a solution amounts to asking whether \mathbf{b} is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.
33. Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$. Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, and let $W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.
- a. Is \mathbf{b} in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$? How many vectors are in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$?
- b. Is \mathbf{b} in W ? How many vectors are in W ?
- c. Show that \mathbf{a}_1 is in W . [Hint: Row operations are unnecessary.]
34. Let $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$, let $\mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$, and let W be the set of all linear combinations of the columns of A .
- a. Is \mathbf{b} in W ?
- b. Show that the third column of A is in W .
35. A mining company has two mines. One day's operation at mine 1 produces ore that contains 20 metric tons of copper and 550 kilograms of silver, while one day's operation at mine 2 produces ore that contains 30 metric tons of copper and 500 kilograms of silver. Let $\mathbf{v}_1 = \begin{bmatrix} 20 \\ 550 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 30 \\ 500 \end{bmatrix}$. Then \mathbf{v}_1 and \mathbf{v}_2 represent the "output per day" of mine 1 and mine 2, respectively.
- a. What physical interpretation can be given to the vector $5\mathbf{v}_1$?
- b. Suppose the company operates mine 1 for x_1 days and mine 2 for x_2 days. Write a vector equation whose solution gives the number of days each mine should operate in order to produce 150 tons of copper and 2825 kilograms of silver. Do not solve the equation.
- T**c. Solve the equation in (b).
36. A steam plant burns two types of coal: anthracite (A) and bituminous (B). For each ton of A burned, the plant produces 27.6 million Btu of heat, 3100 grams (g) of sulfur dioxide, and 250 g of particulate matter (solid-particle pollutants). For each ton of B burned, the plant produces 30.2 million Btu, 6400 g of sulfur dioxide, and 360 g of particulate matter.
- a. How much heat does the steam plant produce when it burns x_1 tons of A and x_2 tons of B?
- b. Suppose the output of the steam plant is described by a vector that lists the amounts of heat, sulfur dioxide, and particulate matter. Express this output as a linear combination of two vectors, assuming that the plant burns x_1 tons of A and x_2 tons of B.
- T**c. Over a certain time period, the steam plant produced 162 million Btu of heat, 23,610 g of sulfur dioxide, and 1623 g of particulate matter. Determine how many tons of each type of coal the steam plant must have burned. Include a vector equation as part of your solution.

37. Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be points in \mathbb{R}^3 and suppose that for $j = 1, \dots, k$ an object with mass m_j is located at point \mathbf{v}_j . Physicists call such objects *point masses*. The total mass of the system of point masses is

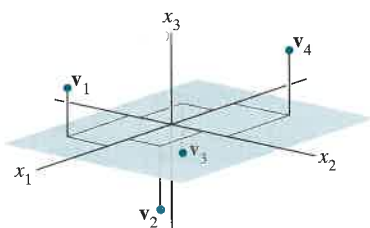
$$m = m_1 + \dots + m_k.$$

The *center of mass* (or *center of gravity*) of the system is

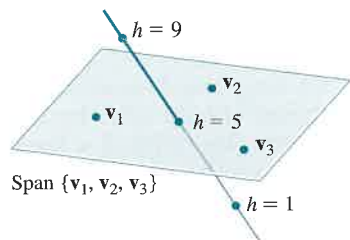
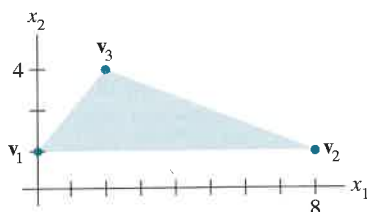
$$\bar{\mathbf{v}} = \frac{1}{m} [m_1 \mathbf{v}_1 + \dots + m_k \mathbf{v}_k]$$

Compute the center of gravity of the system consisting of the following point masses (see the figure):

Point	Mass
$\mathbf{v}_1 = (5, -4, 3)$	2 g
$\mathbf{v}_2 = (4, 3, -2)$	5 g
$\mathbf{v}_3 = (-4, -3, -1)$	2 g
$\mathbf{v}_4 = (-9, 8, 6)$	1 g



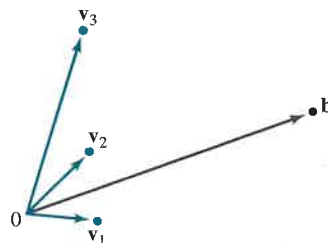
38. Let \mathbf{v} be the center of mass of a system of point masses located at $\mathbf{v}_1, \dots, \mathbf{v}_k$ as in Exercise 37. Is \mathbf{v} in $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$? Explain.
39. A thin triangular plate of uniform density and thickness has vertices at $\mathbf{v}_1 = (0, 1)$, $\mathbf{v}_2 = (8, 1)$, and $\mathbf{v}_3 = (2, 4)$, as in the figure below, and the mass of the plate is 3 g.



The points $\begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$ lie on a line that intersects the plane when $h = 5$.

- a. Find the (x, y) -coordinates of the center of mass of the plate. This “balance point” of the plate coincides with the center of mass of a system consisting of three 1-gram point masses located at the vertices of the plate.
- b. Determine how to distribute an additional mass of 6 g at the three vertices of the plate to move the balance point of the plate to $(2, 2)$. [Hint: Let w_1 , w_2 , and w_3 denote the masses added at the three vertices, so that $w_1 + w_2 + w_3 = 6$.]

40. Consider the vectors \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{b} in \mathbb{R}^2 , shown in the figure. Does the equation $x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + x_3 \mathbf{v}_3 = \mathbf{b}$ have a solution? Is the solution unique? Use the figure to explain your answers.



41. Use the vectors $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n)$, and $\mathbf{w} = (w_1, \dots, w_n)$ to verify the following algebraic properties of \mathbb{R}^n .
- a. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- b. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ for each scalar c
42. Use the vector $\mathbf{u} = (u_1, \dots, u_n)$ to verify the following algebraic properties of \mathbb{R}^n .
- a. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$
- b. $c(d\mathbf{u}) = (cd)\mathbf{u}$ for all scalars c and d

Solutions to Practice Problems

1. Take arbitrary vectors $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ in \mathbb{R}^n , and compute

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (u_1 + v_1, \dots, u_n + v_n) && \text{Definition of vector addition} \\ &= (v_1 + u_1, \dots, v_n + u_n) && \text{Commutativity of addition in } \mathbb{R} \\ &= \mathbf{v} + \mathbf{u} && \text{Definition of vector addition} \end{aligned}$$

2. The vector \mathbf{y} belongs to $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if there exist scalars x_1, x_2, x_3 such that

$$x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

Numerical Note

To optimize a computer algorithm to compute $A\mathbf{x}$, the sequence of calculations should involve data stored in contiguous memory locations. The most widely used professional algorithms for matrix computations are written in Fortran, a language that stores a matrix as a set of columns. Such algorithms compute $A\mathbf{x}$ as a linear combination of the columns of A . In contrast, if a program is written in the popular language C, which stores matrices by rows, $A\mathbf{x}$ should be computed via the alternative rule that uses the rows of A .

PROOF OF THEOREM 4 As was pointed out after Theorem 4, statements (a), (b), and (c) are logically equivalent. So, it suffices to show (for an arbitrary matrix A) that (a) and (d) are either both true or both false. This will tie all four statements together.

Let U be an echelon form of A . Given \mathbf{b} in \mathbb{R}^m , we can row reduce the augmented matrix $[A \ \mathbf{b}]$ to an augmented matrix $[U \ \mathbf{d}]$ for some \mathbf{d} in \mathbb{R}^m :

$$[A \ \mathbf{b}] \sim \cdots \sim [U \ \mathbf{d}]$$

If statement (d) is true, then each row of U contains a pivot position and there can be no pivot in the augmented column. So $A\mathbf{x} = \mathbf{b}$ has a solution for any \mathbf{b} , and (a) is true. If (d) is false, the last row of U is all zeros. Let \mathbf{d} be any vector with a 1 in its last entry. Then $[U \ \mathbf{d}]$ represents an *inconsistent* system. Since row operations are reversible, $[U \ \mathbf{d}]$ can be transformed into the form $[A \ \mathbf{b}]$. The new system $A\mathbf{x} = \mathbf{b}$ is also inconsistent, and (a) is false. ■

Practice Problems

1. Let $A = \begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$. It can be shown

that \mathbf{p} is a solution of $A\mathbf{x} = \mathbf{b}$. Use this fact to exhibit \mathbf{b} as a specific linear combination of the columns of A .

2. Let $A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$. Verify Theorem 5(a) in this case by computing $A(\mathbf{u} + \mathbf{v})$ and $A\mathbf{u} + A\mathbf{v}$.

3. Construct a 3×3 matrix A and vectors \mathbf{b} and \mathbf{c} in \mathbb{R}^3 so that $A\mathbf{x} = \mathbf{b}$ has a solution, but $A\mathbf{x} = \mathbf{c}$ does not.

1.4 Exercises

Compute the products in Exercises 1–4 using (a) the definition, as in Example 1, and (b) the row–vector rule for computing $A\mathbf{x}$. If a product is undefined, explain why.

1. $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$

2. $\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

3. $\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

4. $\begin{bmatrix} 8 & 3 & 1 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 5–8, use the definition of $A\mathbf{x}$ to write the matrix equation as a vector equation, or vice versa.

5. $\begin{bmatrix} 7 & 2 & -9 & 3 \\ -4 & -5 & 7 & -2 \end{bmatrix} \begin{bmatrix} 6 \\ -9 \\ 1 \\ -8 \end{bmatrix} = \begin{bmatrix} -9 \\ 44 \end{bmatrix}$

6. $\begin{bmatrix} 7 & -3 \\ 2 & 1 \\ 9 & -6 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$

$$7. x_1 \begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

$$8. z_1 \begin{bmatrix} 4 \\ -2 \end{bmatrix} + z_2 \begin{bmatrix} -4 \\ 5 \end{bmatrix} + z_3 \begin{bmatrix} -5 \\ 4 \end{bmatrix} + z_4 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}$$

In Exercises 9 and 10, write the system first as a vector equation and then as a matrix equation.

$$\begin{array}{ll} 9. 4x_1 + x_2 - 7x_3 = 8 & 10. 8x_1 - x_2 = 4 \\ x_2 + 6x_3 = 0 & 5x_1 + 4x_2 = 1 \\ & x_1 - 3x_2 = 2 \end{array}$$

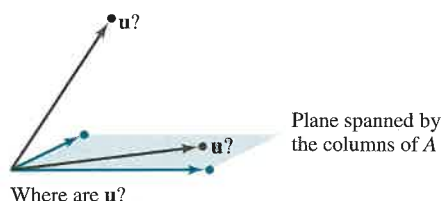
Given A and \mathbf{b} in Exercises 11 and 12, write the augmented matrix for the linear system that corresponds to the matrix equation $A\mathbf{x} = \mathbf{b}$. Then solve the system and write the solution as a vector.

$$11. A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ 2 \\ 9 \end{bmatrix}$$

$$12. A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$13. \text{ Let } \mathbf{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} \text{ and } A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}. \text{ Is } \mathbf{u} \text{ in the plane in } \mathbb{R}^3$$

spanned by the columns of A ? (See the figure.) Why or why not?



$$14. \text{ Let } \mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} \text{ and } A = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}. \text{ Is } \mathbf{u} \text{ in the subset}$$

of \mathbb{R}^3 spanned by the columns of A ? Why or why not?

$$15. \text{ Let } A = \begin{bmatrix} 3 & -4 \\ -6 & 8 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \text{ Show that the equation } A\mathbf{x} = \mathbf{b} \text{ does not have a solution for all possible } \mathbf{b}, \text{ and describe the set of all } \mathbf{b} \text{ for which } A\mathbf{x} = \mathbf{b} \text{ does have a solution.}$$

$$16. \text{ Repeat Exercise 15: } A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Exercises 17–20 refer to the matrices A and B below. Make appropriate calculations that justify your answers and mention an appropriate theorem.

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$$

17. How many rows of A contain a pivot position? Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for each \mathbf{b} in \mathbb{R}^4 ?

18. Do the columns of B span \mathbb{R}^4 ? Does the equation $B\mathbf{x} = \mathbf{y}$ have a solution for each \mathbf{y} in \mathbb{R}^4 ?

19. Can each vector in \mathbb{R}^4 be written as a linear combination of the columns of the matrix A above? Do the columns of A span \mathbb{R}^4 ?

20. Can every vector in \mathbb{R}^4 be written as a linear combination of the columns of the matrix B above? Do the columns of B span \mathbb{R}^4 ?

$$21. \text{ Let } \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span \mathbb{R}^4 ? Why or why not?

$$22. \text{ Let } \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ -1 \\ -5 \end{bmatrix}.$$

Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span \mathbb{R}^3 ? Why or why not?

In Exercises 23–34, mark each statement True or False (T/F). Justify each answer.

23. (T/F) The equation $A\mathbf{x} = \mathbf{b}$ is referred to as a *vector equation*.

24. (T/F) Every matrix equation $A\mathbf{x} = \mathbf{b}$ corresponds to a vector equation with the same solution set.

25. (T/F) If the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then \mathbf{b} is not in the set spanned by the columns of A .

26. (T/F) A vector \mathbf{b} is a linear combination of the columns of a matrix A if and only if the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

27. (T/F) The equation $A\mathbf{x} = \mathbf{b}$ is consistent if the augmented matrix $[A \ \mathbf{b}]$ has a pivot position in every row.

28. (T/F) If A is an $m \times n$ matrix whose columns do not span \mathbb{R}^m , then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some \mathbf{b} in \mathbb{R}^m .

29. (T/F) The first entry in the product $A\mathbf{x}$ is a sum of products.

30. (T/F) Any linear combination vectors can always be written in the form $A\mathbf{x}$ for a suitable matrix A and vector \mathbf{x} .

31. (T/F) If the columns of an $m \times n$ matrix A span \mathbb{R}^m , then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for each \mathbf{b} in \mathbb{R}^m .

32. (T/F) The solution set of a linear system whose augmented matrix is $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{b}]$ is the same as the solution set of $A\mathbf{x} = \mathbf{b}$, if $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$.

33. (T/F) If A is an $m \times n$ matrix and if the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some \mathbf{b} in \mathbb{R}^m , then A cannot have a pivot position in every row.

34. (T/F) If the augmented matrix $[A \ \mathbf{b}]$ has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent.

35. Note that $\begin{bmatrix} 3 & -4 & 2 \\ 6 & -3 & 4 \\ -8 & 9 & -5 \end{bmatrix} \begin{bmatrix} -4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -9 \\ 8 \end{bmatrix}$. Use this fact

(and no row operations) to find scalars c_1, c_2, c_3 such that

$$\begin{bmatrix} -2 \\ -9 \\ 8 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 6 \\ -8 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ -3 \\ 9 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 4 \\ -5 \end{bmatrix}$$

36. Let $\mathbf{u} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$.

It can be shown that $3\mathbf{u} - 5\mathbf{v} - \mathbf{w} = \mathbf{0}$. Use this fact (and no row operations) to find x_1 and x_2 that satisfy the equation

$$\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$$

37. Let $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$, and \mathbf{v} represent vectors in \mathbb{R}^5 , and let x_1, x_2 , and x_3 denote scalars. Write the following vector equation as a matrix equation. Identify any symbols you choose to use.

$$x_1\mathbf{q}_1 + x_2\mathbf{q}_2 + x_3\mathbf{q}_3 = \mathbf{v}$$

38. Rewrite the (numerical) matrix equation below in symbolic form as a vector equation, using symbols $\mathbf{v}_1, \mathbf{v}_2, \dots$ for the vectors and c_1, c_2, \dots for scalars. Define what each symbol represents, using the data given in the matrix equation.

$$\begin{bmatrix} -3 & 5 & -4 & 9 & 7 \\ 5 & 8 & 1 & -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

39. Construct a 3×3 matrix, not in echelon form, whose columns span \mathbb{R}^3 . Show that the matrix you construct has the desired property.

40. Construct a 3×3 matrix, not in echelon form, whose columns do not span \mathbb{R}^3 . Show that the matrix you construct has the desired property.

41. Let A be a 3×2 matrix. Explain why the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all \mathbf{b} in \mathbb{R}^3 . Generalize your argument to the case of an arbitrary A with more rows than columns.

42. Could a set of three vectors in \mathbb{R}^4 span all of \mathbb{R}^4 ? Explain. What about n vectors in \mathbb{R}^m when n is less than m ?

43. Suppose A is a 4×3 matrix and \mathbf{b} is a vector in \mathbb{R}^4 with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. What can you say about the reduced echelon form of A ? Justify your answer.

44. Suppose A is a 3×3 matrix and \mathbf{b} is a vector in \mathbb{R}^3 with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Explain why the columns of A must span \mathbb{R}^3 .

45. Let A be a 3×4 matrix, let \mathbf{y}_1 and \mathbf{y}_2 be vectors in \mathbb{R}^3 , and let $\mathbf{w} = \mathbf{y}_1 + \mathbf{y}_2$. Suppose $\mathbf{y}_1 = A\mathbf{x}_1$ and $\mathbf{y}_2 = A\mathbf{x}_2$ for some vectors \mathbf{x}_1 and \mathbf{x}_2 in \mathbb{R}^4 . What fact allows you to conclude that the system $A\mathbf{x} = \mathbf{w}$ is consistent? (Note: \mathbf{x}_1 and \mathbf{x}_2 denote vectors, not scalar entries in vectors.)

46. Let A be a 5×3 matrix, let \mathbf{y} be a vector in \mathbb{R}^3 , and let \mathbf{z} be a vector in \mathbb{R}^5 . Suppose $A\mathbf{y} = \mathbf{z}$. What fact allows you to conclude that the system $A\mathbf{x} = 4\mathbf{z}$ is consistent?

T In Exercises 47–50, determine if the columns of the matrix span \mathbb{R}^4 .

47. $\begin{bmatrix} 7 & 2 & -5 & 8 \\ -5 & -3 & 4 & -9 \\ 6 & 10 & -2 & 7 \\ -7 & 9 & 2 & 15 \end{bmatrix}$ 48. $\begin{bmatrix} 5 & -7 & -4 & 9 \\ 6 & -8 & -7 & 5 \\ 4 & -4 & -9 & -9 \\ -9 & 11 & 16 & 7 \end{bmatrix}$

49. $\begin{bmatrix} 12 & -7 & 11 & -9 & 5 \\ -9 & 4 & -8 & 7 & -3 \\ -6 & 11 & -7 & 3 & -9 \\ 4 & -6 & 10 & -5 & 12 \end{bmatrix}$

50. $\begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ -7 & -8 & 5 & 6 & -9 \\ 11 & 7 & -7 & -9 & -6 \\ -3 & 4 & 1 & 8 & 7 \end{bmatrix}$

T 51. Find a column of the matrix in Exercise 49 that can be deleted and yet have the remaining matrix columns still span \mathbb{R}^4 .

T 52. Find a column of the matrix in Exercise 50 that can be deleted and yet have the remaining matrix columns still span \mathbb{R}^4 . Can you delete more than one column?

STUDY GUIDE offers additional resources for mastering the concept of span.

Solutions to Practice Problems

1. The matrix equation

$$\begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$$