3.1 Exercises

Compute the determinants in Exercises 1–8 using a cofactor expansion across the first row. In Exercises 1–4, also compute the determinant by a cofactor expansion down the second column.

1. $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$ **2.** $\begin{vmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}$

3. $\begin{vmatrix} 2 & -2 & 3 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{vmatrix}$ **4.** $\begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 4 & 2 \end{vmatrix}$

5. $\begin{vmatrix} 4 & 5 & -8 \\ 1 & 0 & 2 \\ 7 & 3 & 6 \end{vmatrix}$ **6.** $\begin{vmatrix} 6 & -3 & 2 \\ 0 & 5 & -5 \\ 3 & -7 & 8 \end{vmatrix}$

7. $\begin{vmatrix} 4 & 3 & 0 \\ 6 & 5 & 2 \\ 9 & 7 & 3 \end{vmatrix}$ **8.** $\begin{vmatrix} 4 & 1 & 2 \\ 4 & 0 & 3 \\ 3 & -2 & 5 \end{vmatrix}$

Compute the determinants in Exercises 9–14 by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

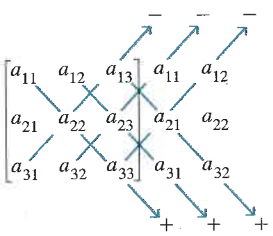
9. $\begin{vmatrix} 7 & 6 & 8 & 4 \\ 0 & 0 & 0 & 6 \\ 8 & 7 & 9 & 3 \\ 0 & 4 & 0 & 5 \end{vmatrix}$ **10.** $\begin{vmatrix} 1 & -2 & 4 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -4 & -3 & 5 \\ 2 & 0 & 3 & 5 \end{vmatrix}$

11. $\begin{vmatrix} 2 & -3 & 4 & 5 \\ 0 & 5 & 3 & -1 \\ 0 & 0 & -2 & 7 \\ 0 & 0 & 0 & 4 \end{vmatrix}$ **12.** $\begin{vmatrix} 3 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -3 \end{vmatrix}$

13. $\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$

14. $\begin{vmatrix} 6 & 0 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix}$

The expansion of a 3×3 determinant can be remembered by the following device. Write a second copy of the first two columns to the right of the matrix, and compute the determinant by multiplying entries on six diagonals:



Add the downward diagonal products and subtract the upward products. Use this method to compute the determinants in Exercises 15–18. ***Warning:*** *This trick does not generalize in any reasonable way to* 4×4 *or larger matrices.*

15. $\begin{vmatrix} 1 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -2 \end{vmatrix}$ **16.** $\begin{vmatrix} 6 & 5 & 0 \\ 4 & 3 & -2 \\ 2 & 0 & 1 \end{vmatrix}$

17. $\begin{vmatrix} 2 & -3 & 3 \\ 3 & 2 & 2 \\ 1 & 3 & -1 \end{vmatrix}$ **18.** $\begin{vmatrix} 1 & 4 & 5 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{vmatrix}$

In Exercises 19–24, explore the effect of an elementary row operation on the determinant of a matrix. In each case, state the row operation and describe how it affects the determinant.

19. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

20. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$

21. $\begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 6 & 5 \\ 3+6k & 4+5k \end{bmatrix}$

22. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$

23. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -4 \\ 3 & -4 & 5 \end{bmatrix}, \begin{bmatrix} k & -2k & 3k \\ 2 & 3 & -4 \\ 3 & -4 & 5 \end{bmatrix}$

24. $\begin{bmatrix} a & b & c \\ 1 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 6 \\ 1 & 4 & 5 \\ a & b & c \end{bmatrix}$

Compute the determinants of the elementary matrices given in Exercises 25–30. (See Section 2.2, Examples 5 and 6.)

25. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$ **26.** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$ **28.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

29. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **30.** $\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Use Exercises 25–30 to answer the questions in Exercises 31 and 32. Give reasons for your answers.

31. What is the determinant of an elementary row replacement matrix?

32. What is the determinant of an elementary scaling matrix with k on the diagonal?

In Exercises 33–36, verify that $\det EA = (\det E)(\det A)$, where E is the elementary matrix shown and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

33. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ **34.** $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

35. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **36.** $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$

37. Let $A = \begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}$. Write $2A$. Is $\det 2A = 2 \det A$?

38. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let k be a scalar. Find a formula that relates $\det kA$ to k and $\det A$.

In Exercises 39 through 42, A is an $n \times n$ matrix. Mark each statement True or False **(T/F)**. Justify each answer.

39. **(T/F)** An $n \times n$ determinant is defined by determinants of $(n-1) \times (n-1)$ submatrices.

40. **(T/F)** The (i, j)-cofactor of a matrix A is the matrix A_{ij} obtained by deleting from A its ith row and jth column.

41. **(T/F)** The cofactor expansion of $\det A$ down a column is equal to the cofactor expansion along a row.

42. **(T/F)** The determinant of a triangular matrix is the sum of the entries on the main diagonal.

43. Let $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute the area of the parallelogram determined by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, and $\mathbf{0}$, and compute the determinant of $[\,\mathbf{u} \;\; \mathbf{v}\,]$. How do they compare? Replace the first entry of $\mathbf{v}$ by an arbitrary number x, and repeat the problem. Draw a picture and explain what you find.

44. Let $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} c \\ 0 \end{bmatrix}$, where a, b, and c are positive (for simplicity). Compute the area of the parallelogram determined by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, and $\mathbf{0}$, and compute the determinants of the matrices $[\,\mathbf{u} \;\; \mathbf{v}\,]$ and $[\,\mathbf{v} \;\; \mathbf{u}\,]$. Draw a picture and explain what you find.

45. Let A be a 2×2 matrix all of whose entries are numbers that are greater than or equal to -10 and less than or equal to 10. Decide if each of the following is a reasonable answer for $\det A$.

a. 0

b. 202

c. -110

d. 555

46. Let A be a 3×3 matrix all of whose entries are numbers that are greater than or equal to -5 and less than or equal to 5. Decide if each of the following is a reasonable answer for $\det A$.

a. 300

b. -220

c. 1000

d. 10

T 47. Construct a random 4×4 matrix A with integer entries between -9 and 9. How is $\det A^{-1}$ related to $\det A$? Experiment with random $n \times n$ integer matrices for $n = 4$, 5, and 6, and make a conjecture. *Note:* In the unlikely event that you encounter a matrix with a zero determinant, reduce it to echelon form and discuss what you find.

T 48. Is it true that $\det AB = (\det A)(\det B)$? To find out, generate random 5×5 matrices A and B, and compute $\det AB - (\det A \det B)$. Repeat the calculations for three other pairs of $n \times n$ matrices, for various values of n. Report your results.

T 49. Is it true that $\det(A + B) = \det A + \det B$? Experiment with four pairs of random matrices as in Exercise 48, and make a conjecture.

T 50. Construct a random 4×4 matrix A with integer entries between -9 and 9, and compare $\det A$ with $\det A^T$, $\det(-A)$, $\det(2A)$, and $\det(10A)$. Repeat with two other random 4×4 integer matrices, and make conjectures about how these determinants are related. (Refer to Exercise 44 in Section 2.1.) Then check your conjectures with several random 5×5 and 6×6 integer matrices. Modify your conjectures, if necessary, and report your results.

T 51. Recall from the introductory section that the larger the determinant of $D^T D$, where D is the design matrix, the better will be the accuracy of the calculated weights for small light objects. Which of the following matrices corresponds to the best design for four weighings of four objects? Describe which of the objects s_1, s_2, s_3, and s_4 you would put in the left and right pans for each weighing corresponding to the best design matrix.

a. $D = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$

b. $D = \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$

c. $D = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

T 52. Repeat Exercise 51 for the case of five weighings of four objects and the following design matrices.

a. $D = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

b. $D = \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$

c. $D = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}$

Solution to Practice Problem

Take advantage of the zeros. Begin with a cofactor expansion down the third column to obtain a 3×3 matrix, which may be evaluated by an expansion down its first column.

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix} = (-1)^{1+3}(2)\begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix}$$

$$= 2\,(-1)^{2+1}(-5)\begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} = 20$$

The $(-1)^{2+1}$ in the next-to-last calculation came from the (2, 1)-position of the -5 in the 3×3 determinant.

3.2 Properties of Determinants

The secret of determinants lies in how they change when row operations are performed. The following theorem generalizes the results of Exercises 19–24 in Section 3.1. The proof is at the end of this section.

THEOREM 3 **Row Operations**

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.
b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.
c. If one row of A is multiplied by k to produce B, then $\det B = k \det A$.

The following examples show how to use Theorem 3 to find determinants efficiently.

EXAMPLE 1 Compute $\det A$, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$.

For brevity, write $|A|$ for det A. Then repeated application of Theorem 3, as rephrased above, shows that

$$\begin{aligned}|AB| &= |E_p \cdots E_1 B| = |E_p||E_{p-1} \cdots E_1 B| = \cdots \\ &= |E_p| \cdots |E_1||B| = \cdots = |E_p \cdots E_1||B| \\ &= |A||B|\end{aligned}$$ ■

Practice Problems

1. Compute $\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix}$ in as few steps as possible.
2. Use a determinant to decide if $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent, when

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$$

3. Let A be an $n \times n$ matrix such that $A^2 = I$. Show that det $A = \pm 1$.

3.2 Exercises

Each equation in Exercises 1–4 illustrates a property of determinants. State the property.

1. $\begin{vmatrix} 0 & 5 & -2 \\ 1 & -3 & 6 \\ 4 & -1 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & -3 & 6 \\ 0 & 5 & -2 \\ 4 & -1 & 8 \end{vmatrix}$

2. $\begin{vmatrix} 3 & -6 & 9 \\ 3 & 5 & -5 \\ 1 & 3 & 3 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 & 3 \\ 3 & 5 & -5 \\ 1 & 3 & 3 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 2 & 2 \\ 0 & 3 & -4 \\ 2 & 7 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 3 & -4 \\ 0 & 3 & 0 \end{vmatrix}$

4. $\begin{vmatrix} 1 & 3 & -4 \\ 2 & 0 & -3 \\ 3 & -5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -4 \\ 0 & -6 & 5 \\ 3 & -5 & 2 \end{vmatrix}$

Find the determinants in Exercises 5–10 by row reduction to echelon form.

5. $\begin{vmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{vmatrix}$

6. $\begin{vmatrix} 3 & -6 & 6 \\ 3 & -5 & 9 \\ 3 & -4 & 8 \end{vmatrix}$

7. $\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix}$

8. $\begin{vmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & 5 & 6 \\ -4 & -9 & 7 & -14 \\ 2 & 5 & 0 & 7 \end{vmatrix}$

9. $\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 0 & 5 & 3 \\ 3 & -3 & -2 & 3 \end{vmatrix}$

10. $\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 1 & -2 & -1 & -3 \\ -2 & -6 & 2 & 3 & 10 \\ 1 & 5 & -6 & 2 & -3 \\ 0 & 2 & -4 & 5 & 9 \end{vmatrix}$

Combine the methods of row reduction and cofactor expansion to compute the determinants in Exercises 11–14.

11. $\begin{vmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 6 & 8 & -4 & -1 \end{vmatrix}$

12. $\begin{vmatrix} -2 & 6 & 0 & 9 \\ 3 & 4 & 8 & 2 \\ 4 & 3 & 0 & 1 \\ 3 & 1 & 2 & -1 \end{vmatrix}$

13. $\begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}$

14. $\begin{vmatrix} 4 & 3 & 2 & 1 \\ 5 & 4 & -3 & 0 \\ 9 & -8 & -7 & 0 \\ 4 & 6 & 2 & 1 \end{vmatrix}$

Find the determinants in Exercises 15–20, where

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$$

15. $\begin{vmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{vmatrix}$

16. $\begin{vmatrix} a & b & c \\ d+3g & e+3h & f+3i \\ g & h & i \end{vmatrix}$

17. $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix}$

18. $\begin{vmatrix} a & b & c \\ 8d & 8e & 8f \\ g & h & i \end{vmatrix}$

19. $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$

20. $\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}$

In Exercises 21–23, use determinants to find out if the matrix is invertible.

21. $\begin{bmatrix} 1 & 3 & 6 \\ 2 & 4 & 7 \\ 0 & 5 & 8 \end{bmatrix}$

22. $\begin{bmatrix} 4 & 5 & 0 \\ 3 & 2 & 1 \\ 1 & -4 & 3 \end{bmatrix}$

23. $\begin{bmatrix} 3 & 0 & 0 & 2 \\ 6 & 8 & 9 & 0 \\ 4 & 5 & 6 & 0 \\ 0 & -8 & -9 & 4 \end{bmatrix}$

In Exercises 24–26, use determinants to decide if the set of vectors is linearly independent.

24. $\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -2 \end{bmatrix}$

25. $\begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$

26. $\begin{bmatrix} 3 \\ 5 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}$

In Exercises 27–34, A and B are $n \times n$ matrices. Mark each statement True or False **(T/F)**. Justify each answer.

27. **(T/F)** A row replacement operation does not affect the determinant of a matrix.

28. **(T/F)** If $\det A$ is zero, then two rows or two columns are the same, or a row or a column is zero.

29. **(T/F)** If the columns of A are linearly dependent, then $\det A = 0$.

30. **(T/F)** The determinant of A is the product of the diagonal entries in A.

31. **(T/F)** If three row interchanges are made in succession, then the new determinant equals the old determinant.

32. **(T/F)** The determinant of A is the product of the pivots in any echelon form U of A, multiplied by $(-1)^r$, where r is the number of row interchanges made during row reduction from A to U.

33. **(T/F)** $\det(A + B) = \det A + \det B$.

34. **(T/F)** $\det A^{-1} = (-1) \det A$.

35. Compute $\det B^4$, where $B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$.

36. Use Theorem 3 (but not Theorem 4) to show that if two rows of a square matrix A are equal, then $\det A = 0$. The same is true for two columns. Why?

In Exercises 37–42, mention an appropriate theorem in your explanation.

37. Show that if A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$.

38. Suppose that A is a square matrix such that $\det A^3 = 0$. Explain why A cannot be invertible.

39. Let A and B be square matrices. Show that even though AB and BA may not be equal, it is always true that $\det AB = \det BA$.

40. Let A and P be square matrices, with P invertible. Show that $\det(PAP^{-1}) = \det A$.

41. Let U be a square matrix such that $U^T U = I$. Show that $\det U = \pm 1$.

42. Find a formula for $\det(rA)$ when A is an $n \times n$ matrix.

Verify that $\det AB = (\det A)(\det B)$ for the matrices in Exercises 43 and 44. (Do not use Theorem 6.)

43. $A = \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 5 & 4 \end{bmatrix}$

44. $A = \begin{bmatrix} 2 & 3 \\ -3 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 \\ -3 & -6 \end{bmatrix}$

45. Let A and B be 3×3 matrices, with $\det A = -2$ and $\det B = 3$. Use properties of determinants (in the text and in the preceding exercises) to compute:

a. $\det AB$ b. $\det 5A$ c. $\det B^T$

d. $\det A^{-1}$ e. $\det A^3$

46. Let A and B be 4×4 matrices, with $\det A = 4$ and $\det B = -5$. Compute:

a. $\det AB$ b. $\det 3A$ c. $\det B^4$

d. $\det BA\,B^T$ e. $\det AB\,A^{-1}$

47. Verify that $\det A = \det B + \det C$, where

$$A = \begin{bmatrix} a+e & b+f \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad C = \begin{bmatrix} e & f \\ c & d \end{bmatrix}$$

48. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that $\det(A + B) = \det A + \det B$ if and only if $a + d = 0$.

49. Verify that $\det A = \det B + \det C$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & u_1 + v_1 \\ a_{21} & a_{22} & u_2 + v_2 \\ a_{31} & a_{32} & u_3 + v_3 \end{bmatrix},$$

$$B = \begin{bmatrix} a_{11} & a_{12} & u_1 \\ a_{21} & a_{22} & u_2 \\ a_{31} & a_{32} & u_3 \end{bmatrix}, \quad C = \begin{bmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ a_{31} & a_{32} & v_3 \end{bmatrix}$$

Note, however, that A is *not* the same as $B + C$.

50. Right-multiplication by an elementary matrix E affects the *columns* of A in the same way that left-multiplication affects the *rows*. Use Theorems 5 and 3 and the obvious fact that E^T is another elementary matrix to show that

$$\det AE = (\det E)(\det A)$$

Do not use Theorem 6.

51. Suppose A is an $n \times n$ matrix and a computer suggests that $\det A = 5$ and $\det(A^{-1}) = 1$. Should you trust these answers? Why or why not?

52. Suppose A and B are $n \times n$ matrices and a computer suggests that $\det A = 5$, $\det B = 2$ and $\det AB = 7$. Should you trust these answers? Why or why not?

T 53. Compute $\det A^T A$ and $\det AA^T$ for several random 4×5 matrices and several random 5×6 matrices. What can you say about $A^T A$ and AA^T when A has more columns than rows?

T 54. If $\det A$ is close to zero, is the matrix A nearly singular? Experiment with the nearly singular 4×4 matrix

$$A = \begin{bmatrix} 4 & 0 & -7 & -7 \\ -6 & 1 & 11 & 9 \\ 7 & -5 & 10 & 19 \\ -1 & 2 & 3 & -1 \end{bmatrix}$$

Compute the determinants of A, $10A$, and $0.1A$. In contrast, compute the condition numbers of these matrices. Repeat these calculations when A is the 4×4 identity matrix. Discuss your results.

Solutions to Practice Problems

1. Perform row replacements to create zeros in the first column, and then create a row of zeros.

$$\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & -4 & 5 & 1 \\ 0 & 1 & -3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & -4 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

2. $$\det[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3] = \begin{vmatrix} 5 & -3 & 2 \\ -7 & 3 & -7 \\ 9 & -5 & 5 \end{vmatrix} = \begin{vmatrix} 5 & -3 & 2 \\ -2 & 0 & -5 \\ 9 & -5 & 5 \end{vmatrix} \qquad \text{Row 1 added to row 2}$$

$$= -(-3)\begin{vmatrix} -2 & -5 \\ 9 & 5 \end{vmatrix} - (-5)\begin{vmatrix} 5 & 2 \\ -2 & -5 \end{vmatrix} \qquad \text{Cofactors of column 2}$$

$$= 3\,(35) + 5\,(-21) = 0$$

By Theorem 4, the matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3]$ is not invertible. The columns are linearly dependent, by the Invertible Matrix Theorem.

3. Recall that $\det I = 1$. By Theorem 6, $\det(AA) = (\det A)(\det A)$. Putting these two observations together results in

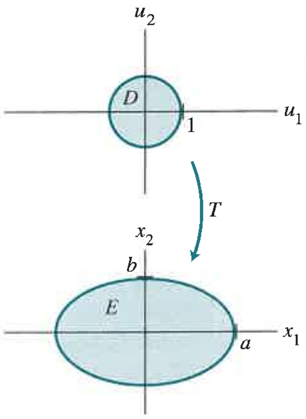
$$1 = \det I = \det A^2 = \det(AA) = (\det A)(\det A) = (\det A)^2$$

Taking the square root of both sides establishes that $\det A = \pm 1$.

The conclusions of Theorem 10 hold whenever S is a region in $\mathbb{R}^2$ with finite area or a region in $\mathbb{R}^3$ with finite volume.

EXAMPLE 5 Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$



SOLUTION We claim that E is the image of the unit disk D under the linear transformation T determined by the matrix $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, because if $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\mathbf{x} = A\mathbf{u}$, then

$$u_1 = \frac{x_1}{a} \quad \text{and} \quad u_2 = \frac{x_2}{b}$$

It follows that $\mathbf{u}$ is in the unit disk, with $u_1^2 + u_2^2 \le 1$, if and only if $\mathbf{x}$ is in E, with $(x_1/a)^2 + (x_2/b)^2 \le 1$. By the generalization of Theorem 10,

$$\begin{aligned} \{\text{area of ellipse}\} &= \{\text{area of } T(D)\} \\ &= |\det A| \cdot \{\text{area of } D\} \\ &= ab\pi(1)^2 = \pi ab \end{aligned}$$

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Practice Problem

Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, and let $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

3.3 Exercises

Use Cramer's rule to compute the solutions of the systems in Exercises 1–6.

1. $\begin{aligned} 5x_1 + 7x_2 &= 3 \\ 2x_1 + 4x_2 &= 1 \end{aligned}$

2. $\begin{aligned} 6x_1 + x_2 &= 3 \\ 5x_1 + 2x_2 &= 4 \end{aligned}$

3. $\begin{aligned} 3x_1 - 2x_2 &= 3 \\ -4x_1 + 6x_2 &= -5 \end{aligned}$

4. $\begin{aligned} -5x_1 + 2x_2 &= 9 \\ 3x_1 - x_2 &= -4 \end{aligned}$

5. $\begin{aligned} x_1 + x_2 \qquad &= 2 \\ -5x_1 \qquad + 4x_3 &= 0 \\ x_2 - x_3 &= -1 \end{aligned}$

6. $\begin{aligned} x_1 + 3x_2 + x_3 &= 8 \\ -x_1 \qquad + 2x_3 &= 4 \\ 3x_1 + x_2 \qquad &= 4 \end{aligned}$

In Exercises 7–10, determine the values of the parameter s for which the system has a unique solution, and describe the solution.

7. $\begin{aligned} 2sx_1 + 5x_2 &= 8 \\ 6x_1 + 3sx_2 &= 4 \end{aligned}$

8. $\begin{aligned} 3sx_1 + 5x_2 &= 3 \\ 12x_1 + 5sx_2 &= 2 \end{aligned}$

9. $\begin{aligned} sx_1 + 2sx_2 &= -1 \\ 3x_1 + 6sx_2 &= 4 \end{aligned}$

10. $\begin{aligned} sx_1 - 2x_2 &= 1 \\ 4sx_1 + 4sx_2 &= 2 \end{aligned}$

In Exercises 11–16, compute the adjugate of the given matrix, and then use Theorem 8 to give the inverse of the matrix.

11. $\begin{bmatrix} 0 & -2 & -1 \\ 5 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 1 & 3 \\ 0 & -1 & 3 \end{bmatrix}$

13. $\begin{bmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 3 & 0 & 6 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 4 & 0 \\ -2 & 3 & -1 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

17. Show that if A is 2×2, then Theorem 8 gives the same formula for A^{-1} as that given by Theorem 4 in Section 2.2.

18. Suppose that all the entries in A are integers and $\det A = 1$. Explain why all the entries in A^{-1} are integers.

In Exercises 19–22, find the area of the parallelogram whose vertices are listed.

19. $(0,0)$, $(5,2)$, $(6,4)$, $(11,6)$

20. $(0,0)$, $(-3,7)$, $(8,-9)$, $(5,-2)$

21. $(-6,0)$, $(0,5)$, $(4,5)$, $(-2,0)$

22. $(0,-2)$, $(5,-2)$, $(-3,1)$, $(2,1)$

23. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1,0,-6)$, $(1,3,5)$, and $(6,7,0)$.

24. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1,5,0)$, $(-3,0,3)$, and $(-1,4,-1)$.

25. Use the concept of volume to explain why the determinant of a 3×3 matrix A is zero if and only if A is not invertible. Do not appeal to Theorem 4 in Section 3.2. [*Hint:* Think about the columns of A.]

26. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation, and let $\mathbf{p}$ be a vector and S a set in $\mathbb{R}^m$. Show that the image of $\mathbf{p} + S$ under T is the translated set $T(\mathbf{p}) + T(S)$ in $\mathbb{R}^n$.

27. Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$, and let $A = \begin{bmatrix} 3 & -4 \\ -4 & 6 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

28. Repeat Exercise 27 with $\mathbf{b}_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$, and $A = \begin{bmatrix} 3 & 4 \\ -2 & -2 \end{bmatrix}$.

29. Find a formula for the area of the triangle whose vertices are $\mathbf{0}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ in $\mathbb{R}^2$.

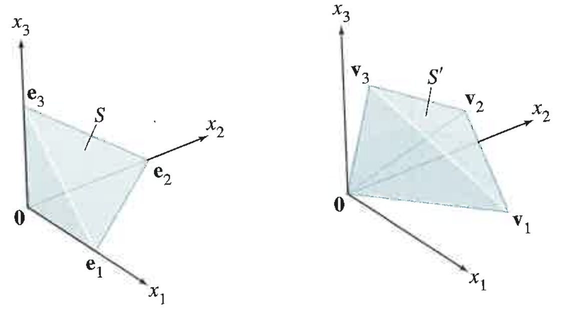
30. Let R be the triangle with vertices at (x_1, y_1), (x_2, y_2), and (x_3, y_3). Show that

$$\{\text{area of triangle}\} = \frac{1}{2}\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

[*Hint:* Translate R to the origin by subtracting one of the vertices, and use Exercise 29.]

31. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by the matrix $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, where a, b, and c are positive numbers. Let S be the unit ball, whose bounding surface has the equation $x_1^2 + x_2^2 + x_3^2 = 1$.
   a. Show that $T(S)$ is bounded by the ellipsoid with the equation $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$.
   b. Use the fact that the volume of the unit ball is $4\pi/3$ to determine the volume of the region bounded by the ellipsoid in part (a).

32. Let S be the tetrahedron in $\mathbb{R}^3$ with vertices at the vectors $\mathbf{0}$, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, and let S' be the tetrahedron with vertices at vectors $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. See the figure.



   a. Describe a linear transformation that maps S onto S'.
   b. Find a formula for the volume of the tetrahedron S' using the fact that

   $\{\text{volume of } S\} = (1/3) \cdot \{\text{area of base}\} \cdot \{\text{height}\}$

33. Let A be an $n \times n$ matrix. If $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$ is computed, what should AA^{-1} be equal to in order to confirm that A^{-1} has been found correctly?

34. If a parallelogram fits inside a circle radius 1 and $\det A = 4$, where A is the matrix whose columns correspond to the edges of the parallelogram, does it seem like A and its determinant have been calculated correctly to correspond to the area of this parallelogram? Explain why or why not.

In Exercises 35–38, mark each statement as True or False (**T/F**). Justify each answer.

35. **(T/F)** Two parallelograms with the same base and height have the same area.

36. **(T/F)** Applying a linear transformation to a region does not change its area.

37. **(T/F)** If A is an invertible $n \times n$ matrix, then $A^{-1} = \operatorname{adj} A$.

38. **(T/F)** Cramer's rule can only be used for invertible matrices.

T 39. Test the inverse formula of Theorem 8 for a random 4×4 matrix A. Use your matrix program to compute the cofactors of the 3×3 submatrices, construct the adjugate, and set $B = (\text{adj } A)/(\det A)$. Then compute $B - \text{inv}(A)$, where $\text{inv}(A)$ is the inverse of A as computed by the matrix program. Use floating point arithmetic with the maximum possible number of decimal places. Report your results.

T 40. Test Cramer's rule for a random 4×4 matrix A and a random 4×1 vector $\mathbf{b}$. Compute each entry in the solution of $A\mathbf{x} = \mathbf{b}$, and compare these entries with the entries in $A^{-1}\mathbf{b}$. Write the command (or keystrokes) for your matrix program that uses Cramer's rule to produce the second entry of $\mathbf{x}$.

T 41. If your version of MATLAB has the `flops` command, use it to count the number of floating point operations to compute A^{-1} for a random 30×30 matrix. Compare this number with the number of flops needed to form $(\text{adj } A)/(\det A)$.

Solution to Practice Problem

The area of S is $\left|\det \begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix}\right| = 14$, and $\det A = 2$. By Theorem 10, the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$ is

$$|\det A| \cdot \{\text{area of } S\} = 2 \cdot 14 = 28$$

CHAPTER 3 PROJECTS

Chapter 3 projects are available online.

A. *Weighing Design*: This project develops the concept of weighing design and their corresponding matrices for use in weighing a few small, light objects.

B. *Jacobians*: This set of exercises examines how a particular determinant called the Jacobian may be used to allow us to change variables in double and triple integrals.

CHAPTER 3 SUPPLEMENTARY EXERCISES

In Exercises 1-15, mark each statement True or False **(T/F)**. Justify each answer. Assume that all matrices here are square.

1. **(T/F)** If A is a 2×2 matrix with a zero determinant, then one column of A is a multiple of the other.

2. **(T/F)** If two rows of a 3×3 matrix A are the same, then $\det A = 0$.

3. **(T/F)** If A is a 3×3 matrix, then $\det 5A = 5 \det A$.

4. **(T/F)** If A and B are $n \times n$ matrices, with $\det A = 2$ and $\det B = 3$, then $\det(A + B) = 5$.

5. **(T/F)** If A is $n \times n$ and $\det A = 2$, then $\det A^3 = 6$.

6. **(T/F)** If B is produced by interchanging two rows of A, then $\det B = \det A$.

7. **(T/F)** If B is produced by multiplying row 3 of A by 5, then $\det B = 5 \det A$.

8. **(T/F)** If B is formed by adding to one row of A a linear combination of the other rows, then $\det B = \det A$.

9. **(T/F)** $\det A^T = -\det A$.

10. **(T/F)** $\det(-A) = -\det A$.

11. **(T/F)** $\det A^TA \geq 0$.

12. **(T/F)** Any system of n linear equations in n variables can be solved by Cramer's rule.

13. **(T/F)** If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^2$ and $\det[\mathbf{u} \;\; \mathbf{v}] = 10$, then the area of the triangle in the plane with vertices at $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$ is 10.

14. **(T/F)** If $A^3 = 0$, then $\det A = 0$.

15. **(T/F)** If A is invertible, then $\det A^{-1} = \det A$.

Use row operations to show that the determinants in Exercises 16–18 are all zero.

16. $\begin{vmatrix} 12 & 13 & 14 \\ 15 & 16 & 17 \\ 18 & 19 & 20 \end{vmatrix}$

17. $\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix}$

18. $\begin{vmatrix} a & b & c \\ a+x & b+x & c+x \\ a+y & b+y & c+y \end{vmatrix}$

Compute the determinants in Exercises 19 and 20.

19. $\begin{vmatrix} 1 & 5 & 4 & 3 & 2 \\ 0 & 8 & 5 & 9 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 3 & 9 & 6 & 5 & 4 \\ 0 & 8 & 0 & 6 & 0 \end{vmatrix}$