

Numerical Notes

1. The fastest way to obtain AB on a computer depends on the way in which the computer stores matrices in its memory. The standard high-performance algorithms, such as in LAPACK, calculate AB by columns, as in our definition of the product. (A version of LAPACK written in C++ calculates AB by rows.)
2. The definition of AB lends itself well to parallel processing on a computer. The columns of B are assigned individually or in groups to different processors, which independently and hence simultaneously compute the corresponding columns of AB .

Practice Problems

1. Since vectors in \mathbb{R}^n may be regarded as $n \times 1$ matrices, the properties of transposes in Theorem 3 apply to vectors, too. Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Compute $(A\mathbf{x})^T$, $\mathbf{x}^T A^T$, $\mathbf{x}\mathbf{x}^T$, and $\mathbf{x}^T \mathbf{x}$. Is $A^T \mathbf{x}^T$ defined?

2. Let A be a 4×4 matrix and let \mathbf{x} be a vector in \mathbb{R}^4 . What is the fastest way to compute $A^2 \mathbf{x}$? Count the multiplications.
3. Suppose A is an $m \times n$ matrix, all of whose rows are identical. Suppose B is an $n \times p$ matrix, all of whose columns are identical. What can be said about the entries in AB ?

2.1 Exercises

In Exercises 1 and 2, compute each matrix sum or product if it is defined. If an expression is undefined, explain why. Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

1. $-2A$, $B - 2A$, AC , CD
2. $A + 2B$, $3C - E$, CB , EB

In the rest of this exercise set and in those to follow, you should assume that each matrix expression is defined. That is, the sizes of the matrices (and vectors) involved “match” appropriately.

3. Let $A = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$. Compute $3I_2 - A$ and $(3I_2)A$.
4. Compute $A - 5I_3$ and $(5I_3)A$, when

$$A = \begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -3 \\ -4 & 1 & 8 \end{bmatrix}.$$

In Exercises 5 and 6, compute the product AB in two ways: (a) by the definition, where $A\mathbf{b}_1$ and $A\mathbf{b}_2$ are computed separately, and (b) by the row-column rule for computing AB .

$$5. A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix}$$

$$6. A = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}$$

7. If a matrix A is 5×3 and the product AB is 5×7 , what is the size of B ?
8. How many rows does B have if BC is a 3×4 matrix?
9. Let $A = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -6 \\ 3 & k \end{bmatrix}$. What value(s) of k , if any, will make $AB = BA$?

10. Let $A = \begin{bmatrix} 3 & -6 \\ -4 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 6 \\ 5 & 7 \end{bmatrix}$, $C = \begin{bmatrix} 6 & -2 \\ 4 & 3 \end{bmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.

11. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Compute AD and DA . Explain how the columns or rows of A change when A is multiplied by D on the right or on the left. Find a 3×3 matrix B , not the identity matrix or the zero matrix, such that $AB = BA$.

12. Let $A = \begin{bmatrix} 2 & -8 \\ -1 & 4 \end{bmatrix}$. Construct a 2×2 matrix B such that AB is the zero matrix. Use two different nonzero columns for B .
13. Let $\mathbf{r}_1, \dots, \mathbf{r}_p$ be vectors in \mathbb{R}^n , and let Q be an $m \times n$ matrix. Write the matrix $[Q\mathbf{r}_1 \cdots Q\mathbf{r}_p]$ as a product of two matrices (neither of which is an identity matrix).
14. Let U be the 3×2 cost matrix described in Example 6 of Section 1.8. The first column of U lists the costs per dollar of output for manufacturing product B , and the second column lists the costs per dollar of output for product C . (The costs are categorized as materials, labor, and overhead.) Let \mathbf{q}_1 be a vector in \mathbb{R}^2 that lists the output (measured in dollars) of products B and C manufactured during the first quarter of the year, and let $\mathbf{q}_2, \mathbf{q}_3$, and \mathbf{q}_4 be the analogous vectors that list the amounts of products B and C manufactured in the second, third, and fourth quarters, respectively. Give an economic description of the data in the matrix UQ , where $Q = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3 \quad \mathbf{q}_4]$.

Exercises 15–24 concern arbitrary matrices A , B , and C for which the indicated sums and products are defined. Mark each statement True or False (T/F). Justify each answer.

15. (T/F) If A and B are 2×2 with columns $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{b}_1, \mathbf{b}_2$, respectively, then $AB = [\mathbf{a}_1\mathbf{b}_1 \quad \mathbf{a}_2\mathbf{b}_2]$.
16. (T/F) If A and B are 3×3 and $B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3]$, then $AB = [A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3]$.
17. (T/F) Each column of AB is a linear combination of the columns of B using weights from the corresponding column of A .
18. (T/F) The second row of AB is the second row of A multiplied on the right by B .
19. (T/F) $AB + AC = A(B + C)$
20. (T/F) $A^T + B^T = (A + B)^T$
21. (T/F) $(AB)C = (AC)B$
22. (T/F) $(AB)^T = A^T B^T$
23. (T/F) The transpose of a product of matrices equals the product of their transposes in the same order.
24. (T/F) The transpose of a sum of matrices equals the sum of their transposes.
25. If $A = \begin{bmatrix} 1 & -3 \\ -3 & 8 \end{bmatrix}$ and $AB = \begin{bmatrix} -1 & 3 & -2 \\ 1 & -7 & 3 \end{bmatrix}$, determine the first and second columns of B .
26. Suppose the first two columns, \mathbf{b}_1 and \mathbf{b}_2 , of B are equal. What can you say about the columns of AB (if AB is defined)? Why?
27. Suppose the third column of B is the sum of the first two columns. What can you say about the third column of AB ? Why?
28. Suppose the second column of B is all zeros. What can you say about the second column of AB ?
29. Suppose the last column of AB is all zeros, but B itself has no column of zeros. What can you say about the columns of A ?
30. Show that if the columns of B are linearly dependent, then so are the columns of AB .
31. Suppose $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why A cannot have more columns than rows.
32. Suppose $AD = I_m$ (the $m \times m$ identity matrix). Show that for any \mathbf{b} in \mathbb{R}^m , the equation $A\mathbf{x} = \mathbf{b}$ has a solution. [Hint: Think about the equation $AD\mathbf{b} = \mathbf{b}$.] Explain why A cannot have more rows than columns.
33. Suppose A is an $m \times n$ matrix and there exist $n \times m$ matrices C and D such that $CA = I_n$ and $AD = I_m$. Prove that $m = n$ and $C = D$. [Hint: Think about the product CAD .]
34. Suppose A is a $3 \times n$ matrix whose columns span \mathbb{R}^3 . Explain how to construct an $n \times 3$ matrix D such that $AD = I_3$.

In Exercises 35 and 36, view vectors in \mathbb{R}^n as $n \times 1$ matrices. For \mathbf{u} and \mathbf{v} in \mathbb{R}^n , the matrix product $\mathbf{u}^T \mathbf{v}$ is a 1×1 matrix, called the **scalar product**, or **inner product**, of \mathbf{u} and \mathbf{v} . It is usually written as a single real number without brackets. The matrix product $\mathbf{u}\mathbf{v}^T$ is an $n \times n$ matrix, called the **outer product** of \mathbf{u} and \mathbf{v} . The products $\mathbf{u}^T \mathbf{v}$ and $\mathbf{u}\mathbf{v}^T$ will appear later in the text.

35. Let $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Compute $\mathbf{u}^T \mathbf{v}$, $\mathbf{v}^T \mathbf{u}$, $\mathbf{u}\mathbf{v}^T$, and $\mathbf{v}\mathbf{u}^T$.
36. If \mathbf{u} and \mathbf{v} are in \mathbb{R}^n , how are $\mathbf{u}^T \mathbf{v}$ and $\mathbf{v}^T \mathbf{u}$ related? How are $\mathbf{u}\mathbf{v}^T$ and $\mathbf{v}\mathbf{u}^T$ related?
37. Prove Theorem 2(b) and 2(c). Use the row-column rule. The (i, j) -entry in $A(B + C)$ can be written as $a_{i1}(b_{1j} + c_{1j}) + \cdots + a_{in}(b_{nj} + c_{nj})$ or $\sum_{k=1}^n a_{ik}(b_{kj} + c_{kj})$
38. Prove Theorem 2(d). [Hint: The (i, j) -entry in $(rA)B$ is $(ra_{i1})b_{1j} + \cdots + (ra_{in})b_{nj}$.]
39. Show that $I_m A = A$ when A is an $m \times n$ matrix. You can assume $I_m \mathbf{x} = \mathbf{x}$ for all \mathbf{x} in \mathbb{R}^m .
40. Show that $AI_n = A$ when A is an $m \times n$ matrix. [Hint: Use the (column) definition of AI_n .]
41. Prove Theorem 3(d). [Hint: Consider the j th row of $(AB)^T$.]
42. Give a formula for $(AB\mathbf{x})^T$, where \mathbf{x} is a vector and A and B are matrices of appropriate sizes.
43. Use a web search engine such as Google to find documentation for your matrix program, and write the commands that

will produce the following matrices (without keying in each entry of the matrix).

- A 5×6 matrix of zeros
- A 3×5 matrix of ones
- The 6×6 identity matrix
- A 5×5 diagonal matrix, with diagonal entries 3, 5, 7, 2, 4

A useful way to test new ideas in matrix algebra, or to make conjectures, is to make calculations with matrices selected at random. Checking a property for a few matrices does not prove that the property holds in general, but it makes the property more believable. Also, if the property is actually false, you may discover this when you make a few calculations.

T 44. Write the command(s) that will create a 6×4 matrix with random entries. In what range of numbers do the entries lie? Tell how to create a 3×3 matrix with random integer entries between -9 and 9 . [Hint: If x is a random number such that $0 < x < 1$, then $-9.5 < 19(x - .5) < 9.5$.]

T 45. Construct a random 4×4 matrix A and test whether $(A + I)(A - I) = A^2 - I$. The best way to do this is to compute $(A + I)(A - I) - (A^2 - I)$ and verify that this difference is the zero matrix. Do this for three random matrices. Then test $(A + B)(A - B) = A^2 - B^2$ the same way for three pairs of random 4×4 matrices. Report your conclusions.

T 46. Use at least three pairs of random 4×4 matrices A and B to test the equalities $(A + B)^T = A^T + B^T$ and $(AB)^T = A^T B^T$. (See Exercise 45.) Report your conclusions. [Note: Most matrix programs use A' for A^T .]

T 47. Let

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Compute S^k for $k = 2, \dots, 6$.

T 48. Describe in words what happens when you compute A^5 , A^{10} , A^{20} , and A^{30} for

$$A = \begin{bmatrix} 1/6 & 1/2 & 1/3 \\ 1/2 & 1/4 & 1/4 \\ 1/3 & 1/4 & 5/12 \end{bmatrix}$$

T 49. The matrix M can detect a particular 2×2 colored pattern like in Example 9. Create a nonzero 4×1 vector \mathbf{x} by choosing each entry to be a zero or one. Test to see if \mathbf{x} corresponds

to the right pattern by calculating $\mathbf{x}^T M \mathbf{x}$. If $\mathbf{x}^T M \mathbf{x} = 0$, then \mathbf{x} is the pattern identified by M . If $\mathbf{x}^T M \mathbf{x} \neq 0$, try a different nonzero vector of zeros and ones. You may want to be systematic in the way that you choose each \mathbf{x} in order to avoid testing the same vector twice. You are using “guess and check” to determine which pattern of 2×2 colored squares the matrix M detects.

$$M = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

T 50. Repeat Exercise 49 with the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

T 51. Use the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to switch the first and second rows of the matrix M containing dates of accidents at the Montreal Trudeau Airport.

Montreal:

$$M = \begin{bmatrix} 2 & 3 & 16 & 24 & 25 & 26 & 6 & 7 & 19 & 26 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \end{bmatrix}$$

This data in matrix M has been scrubbed in matrix AM and can be fed into the same machine as the other data from Example 10.

T 52. Use the matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ to remove the last row from the matrix N containing dates of accidents at the New York JFK Airport.

New York:

$$N = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 12 & 21 & 22 & 3 & 20 & 21 \\ 2020 & 2020 & 2020 & 2020 & 2020 & 2020 & 2020 \end{bmatrix}$$

The data in matrix N has been scrubbed in matrix BN and can be fed into the same machine as the other data from Example 10.

Solutions to Practice Problems

1. $A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$. So $(A\mathbf{x})^T = \begin{bmatrix} -4 & 2 \end{bmatrix}$. Also,

$$\mathbf{x}^T A^T = \begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 2 \end{bmatrix}.$$

Numerical Note

In practical work, A^{-1} is seldom computed, unless the entries of A^{-1} are needed. Computing both A^{-1} and $A^{-1}\mathbf{b}$ takes about three times as many arithmetic operations as solving $A\mathbf{x} = \mathbf{b}$ by row reduction, and row reduction may be more accurate.

Practice Problems

1. Use determinants to determine which of the following matrices are invertible.

a. $\begin{bmatrix} 3 & -9 \\ 2 & 6 \end{bmatrix}$ b. $\begin{bmatrix} 4 & -9 \\ 0 & 5 \end{bmatrix}$ c. $\begin{bmatrix} 6 & -9 \\ -4 & 6 \end{bmatrix}$

2. Find the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$, if it exists.

3. If A is an invertible matrix, prove that $5A$ is an invertible matrix.

2.2 Exercises

Find the inverses of the matrices in Exercises 1–4.

1. $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$ 2. $\begin{bmatrix} 5 & 4 \\ 9 & 7 \end{bmatrix}$

3. $\begin{bmatrix} 8 & 3 \\ -7 & -3 \end{bmatrix}$ 4. $\begin{bmatrix} 3 & -2 \\ 7 & -4 \end{bmatrix}$

5. Verify that the inverse you found in Exercise 1 is correct.
 6. Verify that the inverse you found in Exercise 2 is correct.
 7. Use the inverse found in Exercise 1 to solve the system

$$\begin{aligned} 8x_1 + 3x_2 &= 2 \\ 5x_1 + 2x_2 &= -1 \end{aligned}$$

8. Use the inverse found in Exercise 2 to solve the system

$$\begin{aligned} 5x_1 + 4x_2 &= -3 \\ 9x_1 + 7x_2 &= -5 \end{aligned}$$

9. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, and $\mathbf{b}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

- a. Find A^{-1} , and use it to solve the four equations $A\mathbf{x} = \mathbf{b}_1$, $A\mathbf{x} = \mathbf{b}_2$, $A\mathbf{x} = \mathbf{b}_3$, $A\mathbf{x} = \mathbf{b}_4$.
 b. The four equations in part (a) can be solved by the same set of row operations, since the coefficient matrix is the same in each case. Solve the four equations in part (a) by row reducing the augmented matrix $[A \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4]$.
 10. Use matrix algebra to show that if A is invertible and D satisfies $AD = I$, then $D = A^{-1}$.

In Exercises 11–20, mark each statement True or False (T/F). Justify each answer.

11. (T/F) In order for a matrix B to be the inverse of A , both equations $AB = I$ and $BA = I$ must be true.
 12. (T/F) A product of invertible $n \times n$ matrices is invertible, and the inverse of the product is the product of their inverses in the same order.
 13. (T/F) If A and B are $n \times n$ and invertible, then $A^{-1}B^{-1}$ is the inverse of AB .
 14. (T/F) If A is invertible, then the inverse of A^{-1} is A itself.
 15. (T/F) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ab - cd \neq 0$, then A is invertible.
 16. (T/F) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad = bc$, then A is not invertible.
 17. (T/F) If A is an invertible $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for each \mathbf{b} in \mathbb{R}^n .
 18. (T/F) If A can be row reduced to the identity matrix, then A must be invertible.
 19. (T/F) Each elementary matrix is invertible.
 20. (T/F) If A is invertible, then the elementary row operations that reduce A to the identity I_n also reduce A^{-1} to I_n .
 21. Let A be an invertible $n \times n$ matrix, and let B be an $n \times p$ matrix. Show that the equation $AX = B$ has a unique solution $A^{-1}B$.

22. Let A be an invertible $n \times n$ matrix, and let B be an $n \times p$ matrix. Explain why $A^{-1}B$ can be computed by row reduction:

$$\text{If } [A \ B] \sim \cdots \sim [I \ X], \text{ then } X = A^{-1}B.$$

If A is larger than 2×2 , then row reduction of $[A \ B]$ is much faster than computing both A^{-1} and $A^{-1}B$.

23. Suppose $AB = AC$, where B and C are $n \times p$ matrices and A is invertible. Show that $B = C$. Is this true, in general, when A is not invertible?
24. Suppose $(B - C)D = 0$, where B and C are $m \times n$ matrices and D is invertible. Show that $B = C$.
25. Suppose A , B , and C are invertible $n \times n$ matrices. Show that ABC is also invertible by producing a matrix D such that $(ABC)D = I$ and $D(ABC) = I$.
26. Suppose A and B are $n \times n$, B is invertible, and AB is invertible. Show that A is invertible. [Hint: Let $C = AB$, and solve this equation for A .]
27. Solve the equation $AB = BC$ for A , assuming that A , B , and C are square and B is invertible.
28. Suppose P is invertible and $A = PBP^{-1}$. Solve for B in terms of A .
29. If A , B , and C are $n \times n$ invertible matrices, does the equation $C^{-1}(A + X)B^{-1} = I_n$ have a solution, X ? If so, find it.
30. Suppose A , B , and X are $n \times n$ matrices with A , X , and $A - AX$ invertible, and suppose
- $$(A - AX)^{-1} = X^{-1}B \quad (3)$$
- a. Explain why B is invertible.
- b. Solve (3) for X . If you need to invert a matrix, explain why that matrix is invertible.
31. Explain why the columns of an $n \times n$ matrix A are linearly independent when A is invertible.
32. Explain why the columns of an $n \times n$ matrix A span \mathbb{R}^n when A is invertible. [Hint: Review Theorem 4 in Section 1.4.]
33. Suppose A is $n \times n$ and the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why A has n pivot columns and A is row equivalent to I_n . By Theorem 7, this shows that A must be invertible. (This exercise and Exercise 34 will be cited in Section 2.3.)
34. Suppose A is $n \times n$ and the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each \mathbf{b} in \mathbb{R}^n . Explain why A must be invertible. [Hint: Is A row equivalent to I_n ?]

Exercises 35 and 36 prove Theorem 4 for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

35. Show that if $ad - bc = 0$, then the equation $A\mathbf{x} = \mathbf{0}$ has more than one solution. Why does this imply that A is not invertible? [Hint: First, consider $a = b = 0$. Then, if a and b are not both zero, consider the vector $\mathbf{x} = \begin{bmatrix} -b \\ a \end{bmatrix}$.]

36. Show that if $ad - bc \neq 0$, the formula for A^{-1} works.

Exercises 37 and 38 prove special cases of the facts about elementary matrices stated in the box following Example 5. Here A is a 3×3 matrix and $I = I_3$. (A general proof would require slightly more notation.)

37. a. Use equation (1) from Section 2.1 to show that $\text{row}_i(A) = \text{row}_i(I) \cdot A$, for $i = 1, 2, 3$.
- b. Show that if rows 1 and 2 of A are interchanged, then the result may be written as EA , where E is an elementary matrix formed by interchanging rows 1 and 2 of I .
- c. Show that if row 3 of A is multiplied by 5, then the result may be written as EA , where E is formed by multiplying row 3 of I by 5.
38. Show that if row 3 of A is replaced by $\text{row}_3(A) - 4\text{row}_1(A)$, the result is EA , where E is formed from I by replacing $\text{row}_3(I)$ by $\text{row}_3(I) - 4\text{row}_1(I)$.

Find the inverses of the matrices in Exercises 39–42, if they exist. Use the algorithm introduced in this section.

39. $\begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$

40. $\begin{bmatrix} 9 & 7 \\ 8 & 6 \end{bmatrix}$

41. $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

42. $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$

43. Use the algorithm from this section to find the inverses of

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Let A be the corresponding $n \times n$ matrix, and let B be its inverse. Guess the form of B , and then prove that $AB = I$ and $BA = I$.

44. Repeat the strategy of Exercise 43 to guess the inverse of

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & & 0 \\ 1 & 2 & 3 & & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}$$

correct.

45. Let $A = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$. Find the third column of A^{-1} without computing the other columns.

46. Let $A = \begin{bmatrix} -25 & -9 & -27 \\ 546 & 180 & 537 \\ 154 & 50 & 149 \end{bmatrix}$. Find the second and third columns of A^{-1} without computing the first column.

47. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{bmatrix}$. Construct a 2×3 matrix C (by trial and

error) using only 1, -1, and 0 as entries, such that $CA = I_2$. Compute AC and note that $AC \neq I_3$.

48. Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Construct a 4×2 matrix D

using only 1 and 0 as entries, such that $AD = I_2$. Is it possible that $CA = I_4$ for some 4×2 matrix C ? Why or why not?

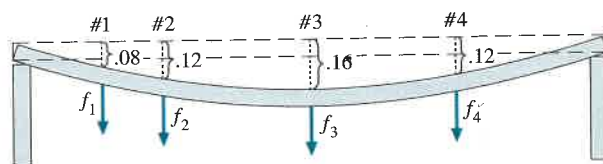
49. Let $D = \begin{bmatrix} .005 & .002 & .001 \\ .002 & .004 & .002 \\ .001 & .002 & .005 \end{bmatrix}$ be a flexibility matrix,

with flexibility measured in inches per pound. Suppose that forces of 30, 50, and 20 lb are applied at points 1, 2, and 3, respectively, in Figure 1 of Example 3. Find the corresponding deflections.

- T 50.** Compute the stiffness matrix D^{-1} for D in Exercise 49. List the forces needed to produce a deflection of .04 in. at point 3, with zero deflections at the other points.

T 51. Let $D = \begin{bmatrix} .0040 & .0030 & .0010 & .0005 \\ .0030 & .0050 & .0030 & .0010 \\ .0010 & .0030 & .0050 & .0030 \\ .0005 & .0010 & .0030 & .0040 \end{bmatrix}$ be a

flexibility matrix for an elastic beam with four points at which force is applied. Units are centimeters per newton of force. Measurements at the four points show deflections of .08, .12, .16, and .12 cm. Determine the forces at the four points.



Deflection of elastic beam in Exercises 51 and 52.

- T 52.** With D as in Exercise 51, determine the forces that produce a deflection of .24 cm at the second point on the beam, with zero deflections at the other three points. How is the answer related to the entries in D^{-1} ? [Hint: First answer the question when the deflection is 1 cm at the second point.]

Solutions to Practice Problems

1. a. $\det \begin{bmatrix} 3 & -9 \\ 2 & 6 \end{bmatrix} = 3 \cdot 6 - (-9) \cdot 2 = 18 + 18 = 36$. The determinant is nonzero, so the matrix is invertible.

b. $\det \begin{bmatrix} 4 & -9 \\ 0 & 5 \end{bmatrix} = 4 \cdot 5 - (-9) \cdot 0 = 20 \neq 0$. The matrix is invertible.

c. $\det \begin{bmatrix} 6 & -9 \\ -4 & 6 \end{bmatrix} = 6 \cdot 6 - (-9)(-4) = 36 - 36 = 0$. The matrix is not invertible.

2. $[A \ I] \sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{bmatrix}$$

So $[A \ I]$ is row equivalent to a matrix of the form $[B \ D]$, where B is square and has a row of zeros. Further row operations will not transform B into I , so we stop. A does not have an inverse.

3. Since A is an invertible matrix, there exists a matrix C such that $AC = I = CA$. The goal is to find a matrix D so that $(5A)D = I = D(5A)$. Set $D = 1/5 C$. Applying Theorem 2 from Section 2.1 establishes that $(5A)(1/5 C) = (5)(1/5)(AC) = 1 I = I$, and $(1/5 C)(5A) = (1/5)(5)(CA) = 1 I = I$. Thus $1/5 C$ is indeed the inverse of A , proving that A is invertible.

2.3 Exercises

Unless otherwise specified, assume that all matrices in these exercises are $n \times n$. Determine which of the matrices in Exercises 1–10 are invertible. Use as few calculations as possible. Justify your answers.

1. $\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$

2. $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$

3. $\begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix}$

4. $\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 4 & 7 \\ 1 & 0 & 5 \\ -5 & 8 & -2 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix}$

7. $\begin{bmatrix} -1 & 0 & 2 & 1 \\ -5 & -3 & 9 & 3 \\ 3 & 0 & 1 & -3 \\ 0 & 3 & 1 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 5 & 9 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

9. $\begin{bmatrix} 4 & 0 & -7 & -7 \\ -6 & 1 & 11 & 9 \\ 7 & -5 & 10 & 19 \\ -1 & 2 & 3 & -1 \end{bmatrix}$

10. $\begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 6 & 4 & 2 & 8 & -8 \\ 7 & 5 & 3 & 10 & 9 \\ 9 & 6 & 4 & -9 & -5 \\ 8 & 5 & 2 & 11 & 4 \end{bmatrix}$

In Exercises 11–20, the matrices are all $n \times n$. Each part of the exercises is an *implication* of the form “If ‘statement 1’, then ‘statement 2’.” Mark an implication as True if the truth of “statement 2” *always* follows whenever “statement 1” happens to be true. An implication is False if there is an instance in which “statement 2” is false but “statement 1” is true. Justify each answer.

11. (T/F) If the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix.
12. (T/F) If there is an $n \times n$ matrix D such that $AD = I$, then there is also an $n \times n$ matrix C such that $CA = I$.
13. (T/F) If the columns of A span \mathbb{R}^n , then the columns are linearly independent.
14. (T/F) If the columns of A are linearly independent, then the columns of A span \mathbb{R}^n .
15. (T/F) If A is an $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbb{R}^n .
16. (T/F) If the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbb{R}^n , then the solution is unique for each \mathbf{b} .
17. (T/F) If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then A has fewer than n pivot positions.

18. (T/F) If the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps \mathbb{R}^n into \mathbb{R}^n , then A has n pivot positions.
19. (T/F) If A^T is not invertible, then A is not invertible.
20. (T/F) If there is a \mathbf{b} in \mathbb{R}^n such that the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not one-to-one.
21. An $m \times n$ **upper triangular matrix** is one whose entries *below* the main diagonal are 0's (as in Exercise 8). When is a square upper triangular matrix invertible? Justify your answer.
22. An $m \times n$ **lower triangular matrix** is one whose entries *above* the main diagonal are 0's (as in Exercise 3). When is a square lower triangular matrix invertible? Justify your answer.
23. Can a square matrix with two identical columns be invertible? Why or why not?
24. Is it possible for a 5×5 matrix to be invertible when its columns do not span \mathbb{R}^5 ? Why or why not?
25. If A is invertible, then the columns of A^{-1} are linearly independent. Explain why.
26. If C is 6×6 and the equation $C\mathbf{x} = \mathbf{v}$ is consistent for every \mathbf{v} in \mathbb{R}^6 , is it possible that for some \mathbf{v} , the equation $C\mathbf{x} = \mathbf{v}$ has more than one solution? Why or why not?
27. If the columns of a 7×7 matrix D are linearly independent, what can you say about solutions of $D\mathbf{x} = \mathbf{b}$? Why?
28. If $n \times n$ matrices E and F have the property that $EF = I$, then E and F commute. Explain why.
29. If the equation $G\mathbf{x} = \mathbf{y}$ has more than one solution for some \mathbf{y} in \mathbb{R}^n , can the columns of G span \mathbb{R}^n ? Why or why not?
30. If the equation $H\mathbf{x} = \mathbf{c}$ is inconsistent for some \mathbf{c} in \mathbb{R}^n , what can you say about the equation $H\mathbf{x} = \mathbf{0}$? Why?
31. If an $n \times n$ matrix K cannot be row reduced to I_n , what can you say about the columns of K ? Why?
32. If L is $n \times n$ and the equation $L\mathbf{x} = \mathbf{0}$ has the trivial solution, do the columns of L span \mathbb{R}^n ? Why?
33. Verify the boxed statement preceding Example 1.
34. Explain why the columns of A^2 span \mathbb{R}^n whenever the columns of A are linearly independent.
35. Show that if AB is invertible, so is A . You cannot use Theorem 6(b), because you cannot *assume* that A and B are invertible. [Hint: There is a matrix W such that $ABW = I$. Why?]
36. Show that if AB is invertible, so is B .
37. If A is an $n \times n$ matrix and the equation $A\mathbf{x} = \mathbf{b}$ has more than one solution for some \mathbf{b} , then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is

not one-to-one. What else can you say about this transformation? Justify your answer.

38. If A is an $n \times n$ matrix and the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one, what else can you say about this transformation? Justify your answer.
39. Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbb{R}^n . Without using Theorems 5 or 8, explain why each equation $A\mathbf{x} = \mathbf{b}$ has in fact exactly one solution.
40. Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Without using the Invertible Matrix Theorem, explain directly why the equation $A\mathbf{x} = \mathbf{b}$ must have a solution for each \mathbf{b} in \mathbb{R}^n .

In Exercises 41 and 42, T is a linear transformation from \mathbb{R}^2 into \mathbb{R}^2 . Show that T is invertible and find a formula for T^{-1} .

41. $T(x_1, x_2) = (-9x_1 + 7x_2, 4x_1 - 3x_2)$
42. $T(x_1, x_2) = (6x_1 - 8x_2, -5x_1 + 7x_2)$
43. Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an invertible linear transformation. Explain why T is both one-to-one and onto \mathbb{R}^n . Use equations (1) and (2). Then give a second explanation using one or more theorems.
44. Let T be a linear transformation that maps \mathbb{R}^n onto \mathbb{R}^n . Show that T^{-1} exists and maps \mathbb{R}^n onto \mathbb{R}^n . Is T^{-1} also one-to-one?
45. Suppose T and U are linear transformations from \mathbb{R}^n to \mathbb{R}^n such that $T(U\mathbf{x}) = \mathbf{x}$ for all \mathbf{x} in \mathbb{R}^n . Is it true that $U(T\mathbf{x}) = \mathbf{x}$ for all \mathbf{x} in \mathbb{R}^n ? Why or why not?
46. Suppose a linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ has the property that $T(\mathbf{u}) = T(\mathbf{v})$ for some pair of distinct vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n . Can T map \mathbb{R}^n onto \mathbb{R}^n ? Why or why not?
47. Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an invertible linear transformation, and let S and U be functions from \mathbb{R}^n into \mathbb{R}^n such that $S(T(\mathbf{x})) = \mathbf{x}$ and $U(T(\mathbf{x})) = \mathbf{x}$ for all \mathbf{x} in \mathbb{R}^n . Show that $U(\mathbf{v}) = S(\mathbf{v})$ for all \mathbf{v} in \mathbb{R}^n . This will show that T has a unique inverse, as asserted in Theorem 9. [Hint: Given any \mathbf{v} in \mathbb{R}^n , we can write $\mathbf{v} = T(\mathbf{x})$ for some \mathbf{x} . Why? Compute $S(\mathbf{v})$ and $U(\mathbf{v})$.]
48. Suppose T and S satisfy the invertibility equations (1) and (2), where T is a linear transformation. Show directly that S is a linear transformation. [Hint: Given \mathbf{u}, \mathbf{v} in \mathbb{R}^n , let $\mathbf{x} = S(\mathbf{u})$, $\mathbf{y} = S(\mathbf{v})$. Then $T(\mathbf{x}) = \mathbf{u}$, $T(\mathbf{y}) = \mathbf{v}$. Why? Apply S to both sides of the equation $T(\mathbf{x}) + T(\mathbf{y}) = T(\mathbf{x} + \mathbf{y})$. Also, consider $T(c\mathbf{x}) = cT(\mathbf{x})$.]

49. Suppose an experiment leads to the following system of equations:

$$\begin{aligned} 4.5x_1 + 3.1x_2 &= 19.249 \\ 1.6x_1 + 1.1x_2 &= 6.843 \end{aligned} \quad (3)$$

- a. Solve system (3), and then solve system (4), below, in which the data on the right have been rounded to two decimal places. In each case, find the *exact* solution.

$$\begin{aligned} 4.5x_1 + 3.1x_2 &= 19.25 \\ 1.6x_1 + 1.1x_2 &= 6.84 \end{aligned} \quad (4)$$

- b. The entries in (4) differ from those in (3) by less than .05%. Find the percentage error when using the solution of (4) as an approximation for the solution of (3).
- c. Use your matrix program to produce the condition number of the coefficient matrix in (3).

Exercises 50–52 show how to use the condition number of a matrix A to estimate the accuracy of a computed solution of $A\mathbf{x} = \mathbf{b}$. If the entries of A and \mathbf{b} are accurate to about r significant digits and if the condition number of A is approximately 10^k (with k a positive integer), then the computed solution of $A\mathbf{x} = \mathbf{b}$ should usually be accurate to at least $r - k$ significant digits.

50. Find the condition number of the matrix A in Exercise 9. Construct a random vector \mathbf{x} in \mathbb{R}^4 and compute $\mathbf{b} = A\mathbf{x}$. Then use your matrix program to compute the solution \mathbf{x}_1 of $A\mathbf{x} = \mathbf{b}$. To how many digits do \mathbf{x} and \mathbf{x}_1 agree? Find out the number of digits your matrix program stores accurately, and report how many digits of accuracy are lost when \mathbf{x}_1 is used in place of the exact solution \mathbf{x} .

51. Repeat Exercise 50 for the matrix in Exercise 10.

52. Solve an equation $A\mathbf{x} = \mathbf{b}$ for a suitable \mathbf{b} to find the last column of the inverse of the *fifth-order Hilbert matrix*.

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{bmatrix}$$

How many digits in each entry of \mathbf{x} do you expect to be correct? Explain. [Note: The exact solution is $(630, -12600, 56700, -88200, 44100)$.]

53. Some matrix programs, such as MATLAB, have a command to create Hilbert matrices of various sizes. If possible, use an inverse command to compute the inverse of a twelfth-order or larger Hilbert matrix, A . Compute AA^{-1} . Report what you find.

STUDY GUIDE offers additional resources for reviewing and reflecting on what you have learned.

Solutions to Practice Problems

1. The columns of A are obviously linearly dependent because columns 2 and 3 are multiples of column 1. Hence, A cannot be invertible (by the Invertible Matrix Theorem).

The exercises that follow give practice with matrix algebra and illustrate typical calculations found in applications.

Practice Problems

1. Show that $\begin{bmatrix} I & 0 \\ A & I \end{bmatrix}$ is invertible and find its inverse.
2. Compute $X^T X$, where X is partitioned as $\begin{bmatrix} X_1 & X_2 \end{bmatrix}$.

2.4 Exercises

In Exercises 1–9, assume that the matrices are partitioned conformably for block multiplication. Compute the products shown in Exercises 1–4.

$$\begin{array}{ll} 1. \begin{bmatrix} I & 0 \\ E & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} & 2. \begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ 3. \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} & 4. \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \end{array}$$

In Exercises 5–8, find formulas for X , Y , and Z in terms of A , B , and C , and justify your calculations. In some cases, you may need to make assumptions about the size of a matrix in order to produce a formula. [Hint: Compute the product on the left, and set it equal to the right side.]

$$\begin{array}{l} 5. \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ X & Y \end{bmatrix} = \begin{bmatrix} 0 & I \\ Z & 0 \end{bmatrix} \\ 6. \begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix} \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ 7. \begin{bmatrix} X & 0 & 0 \\ Y & 0 & I \end{bmatrix} \begin{bmatrix} A & Z \\ 0 & 0 \\ B & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ 8. \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} X & Y & Z \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \end{array}$$

9. Suppose A_{11} is an invertible matrix. Find matrices X and Y such that the product below has the form indicated. Also, compute B_{22} . [Hint: Compute the product on the left, and set it equal to the right side.]

$$\begin{bmatrix} I & 0 & 0 \\ X & I & 0 \\ Y & 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \\ 0 & B_{32} \end{bmatrix}$$

10. The inverse of $\begin{bmatrix} I & 0 & 0 \\ C & I & 0 \\ A & B & I \end{bmatrix}$ is $\begin{bmatrix} I & 0 & 0 \\ Z & I & 0 \\ X & Y & I \end{bmatrix}$.

Find X , Y , and Z .

In Exercises 11–14, mark each statement True or False (T/F). Justify each answer.

11. (T/F) If $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$ and $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, with A_1 and A_2 the same sizes as B_1 and B_2 , respectively, then $A + B = \begin{bmatrix} A_1 + B_1 & A_2 + B_2 \end{bmatrix}$.
12. (T/F) The definition of the matrix–vector product Ax is a special case of block multiplication.
13. (T/F) If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, then the partitions of A and B are conformable for block multiplication.
14. (T/F) If A_1, A_2, B_1 , and B_2 are $n \times n$ matrices, $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, and $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, then the product BA is defined, but AB is not.
15. Let $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$, where B and C are square. Show that A is invertible if and only if both B and C are invertible.
16. Show that the block upper triangular matrix A in Example 5 is invertible if and only if both A_{11} and A_{22} are invertible. [Hint: If A_{11} and A_{22} are invertible, the formula for A^{-1} given in Example 5 actually works as the inverse of A .] This fact about A is an important part of several computer algorithms that estimate eigenvalues of matrices. Eigenvalues are discussed in Chapter 5.

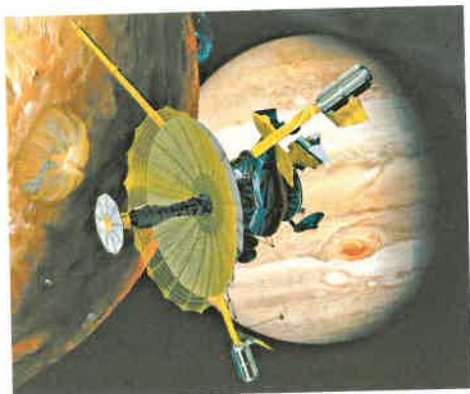
17. Suppose A_{11} is invertible. Find X and Y such that

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} \quad (7)$$

where $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$. The matrix S is called the **Schur complement** of A_{11} . Likewise, if A_{22} is invertible, the matrix $A_{11} - A_{12}A_{22}^{-1}A_{21}$ is called the Schur complement of A_{22} . Such expressions occur frequently in the theory of systems engineering, and elsewhere.

18. Suppose the block matrix A on the left side of (7) is invertible and A_{11} is invertible. Show that the Schur complement S of A_{11} is invertible. [Hint: The outside factors on the right side of (7) are always invertible. Verify this.] When A and A_{11} are both invertible, (7) leads to a formula for A^{-1} , using S^{-1} , A_{11}^{-1} , and the other entries in A .

19. When a deep space probe is launched, corrections may be necessary to place the probe on a precisely calculated trajectory. Radio telemetry provides a stream of vectors, $\mathbf{x}_1, \dots, \mathbf{x}_k$, giving information at different times about how the probe's position compares with its planned trajectory. Let X_k be the matrix $[\mathbf{x}_1 \cdots \mathbf{x}_k]$. The matrix $G_k = X_k X_k^T$ is computed as the radar data are analyzed. When \mathbf{x}_{k+1} arrives, a new G_{k+1} must be computed. Since the data vectors arrive at high speed, the computational burden could be severe. But partitioned matrix multiplication helps tremendously. Compute the column-row expansions of G_k and G_{k+1} , and describe what must be computed in order to update G_k to form G_{k+1} .



The probe Galileo was launched October 18, 1989, and arrived near Jupiter in early December 1995.

20. Let X be an $m \times n$ data matrix such that $X^T X$ is invertible, and let $M = I_m - X(X^T X)^{-1} X^T$. Add a column \mathbf{x}_0 to the data and form

$$W = [X \quad \mathbf{x}_0]$$

Compute $W^T W$. The $(1, 1)$ -entry is $X^T X$. Show that the Schur complement (Exercise 17) of $X^T X$ can be written in the form $\mathbf{x}_0^T M \mathbf{x}_0$. It can be shown that the quantity $(\mathbf{x}_0^T M \mathbf{x}_0)^{-1}$ is the $(2, 2)$ -entry in $(W^T W)^{-1}$. This entry has a useful statistical interpretation, under appropriate hypotheses.

In the study of engineering control of physical systems, a standard set of differential equations is transformed by Laplace transforms into the following system of linear equations:

$$\begin{bmatrix} A - sI_n & B \\ C & I_m \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{y} \end{bmatrix} \quad (8)$$

where A is $n \times n$, B is $n \times m$, C is $m \times n$, and s is a variable. The vector \mathbf{u} in \mathbb{R}^m is the "input" to the system, \mathbf{y} in \mathbb{R}^m is the "output," and \mathbf{x} in \mathbb{R}^n is the "state" vector. (Actually, the vectors \mathbf{x} , \mathbf{u} , and \mathbf{y} are functions of s , but we suppress this fact because it does not affect the algebraic calculations in Exercises 21 and 22.)

21. Assume $A - sI_n$ is invertible and view (8) as a system of two matrix equations. Solve the top equation for \mathbf{x} and substitute

into the bottom equation. The result is an equation of the form $W(s)\mathbf{u} = \mathbf{y}$, where $W(s)$ is a matrix that depends on s . $W(s)$ is called the *transfer function* of the system because it transforms the input \mathbf{u} into the output \mathbf{y} . Find $W(s)$ and describe how it is related to the partitioned *system matrix* on the left side of (8). See Exercise 17.

22. Suppose the transfer function $W(s)$ in Exercise 21 is invertible for some s . It can be shown that the inverse transfer function $W(s)^{-1}$, which transforms outputs into inputs, is the Schur complement of $A - BC - sI_n$ for the matrix below. Find this Schur complement. See Exercise 17.

$$\begin{bmatrix} A - BC - sI_n & B \\ -C & I_m \end{bmatrix}$$

23. a. Verify that $A^2 = I$ when $A = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$.

b. Use partitioned matrices to show that $M^2 = I$ when

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 1 \end{bmatrix}$$

24. Generalize the idea of Exercise 23(a) [not 23(b)] by constructing a 5×5 matrix $M = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$ such that $M^2 = I$. Make C a nonzero 2×3 matrix. Show that your construction works.

25. Use partitioned matrices to prove by induction that the product of two lower triangular matrices is also lower triangular. [Hint: A $(k+1) \times (k+1)$ matrix A_1 can be written in the form below, where a is a scalar, \mathbf{v} is in \mathbb{R}^k , and A is a $k \times k$ lower triangular matrix. See the *Study Guide* for help with induction.]

$$A_1 = \begin{bmatrix} a & \mathbf{0}^T \\ \mathbf{v} & A \end{bmatrix}$$

26. Use partitioned matrices to prove by induction that for $n = 2, 3, \dots$, the $n \times n$ matrix A shown below is invertible and B is its inverse.

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & & 0 \\ 1 & 1 & 1 & & 0 \\ \vdots & & & \ddots & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & & 0 \\ 0 & -1 & 1 & & 0 \\ \vdots & & & \ddots & \\ 0 & & \cdots & -1 & 1 \end{bmatrix}$$

For the induction step, assume A and B are $(k+1) \times (k+1)$ matrices, and partition A and B in a form similar to that displayed in Exercise 25.

27. Without using row reduction, find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 5 & 6 \end{bmatrix}$$

T 28. For block operations, it may be necessary to access or enter submatrices of a large matrix. Describe the functions or commands of your matrix program that accomplish the following tasks. Suppose A is a 20×30 matrix.

- Display the submatrix of A from rows 15 to 20 and columns 5 to 10.
- Insert a 5×10 matrix B into A , beginning at row 10 and column 20.

c. Create a 50×50 matrix of the form $B = \begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}$.

[Note: It may not be necessary to specify the zero blocks in B .]

T 29. Suppose memory or size restrictions prevent your matrix program from working with matrices having more than 32 rows and 32 columns, and suppose some project involves 50×50 matrices A and B . Describe the commands or operations of your matrix program that accomplish the following tasks.

- Compute $A + B$.
- Compute AB .
- Solve $Ax = b$ for some vector b in \mathbb{R}^{50} , assuming that A can be partitioned into a 2×2 block matrix $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, with A_{11} an invertible 20×20 matrix, A_{22} an invertible 30×30 matrix, and A_{12} a zero matrix. [Hint: Describe appropriate smaller systems to solve, without using any matrix inverses.]

Solutions to Practice Problems

1. If $\begin{bmatrix} I & 0 \\ A & I \end{bmatrix}$ is invertible, its inverse has the form $\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$. Verify that

$$\begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} W & X \\ AW + Y & AX + Z \end{bmatrix}$$

So W, X, Y , and Z must satisfy $W = I, X = 0, AW + Y = 0$, and $AX + Z = I$. It follows that $Y = -A$ and $Z = I$. Hence

$$\begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -A & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

The product in the reverse order is also the identity, so the block matrix is invertible, and its inverse is $\begin{bmatrix} I & 0 \\ -A & I \end{bmatrix}$. (You could also appeal to the Invertible Matrix Theorem.)

2. $X^T X = \begin{bmatrix} X_1^T \\ X_2^T \end{bmatrix} \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} X_1^T X_1 & X_1^T X_2 \\ X_2^T X_1 & X_2^T X_2 \end{bmatrix}$. The partitions of X^T and X are automatically conformable for block multiplication because the columns of X^T are the rows of X . This partition of $X^T X$ is used in several computer algorithms for matrix computations.

2.5 Matrix Factorizations

A *factorization* of a matrix A is an equation that expresses A as a product of two or more matrices. Whereas matrix multiplication involves a *synthesis* of data (combining the effects of two or more linear transformations into a single matrix), matrix factorization is an *analysis* of data. In the language of computer science, the expression of A as a product amounts to a *preprocessing* of the data in A , organizing that data into two or more parts whose structures are more useful in some way, perhaps more accessible for computation.