## Task 5

Determine a fair price for each of the certificates and compute it explicitly for the market parameters from Task 2) with $P_0$ equal to the price quoted on 14.06.2024, barrier level 16, bonus level 21, maturity 1 year and Cap 28.

*Hint: It holds that*

$$\mathbb{E}_{\tilde{Q}}\left[e^{-rT}(P_T - K)\mathbf{1}_{\{\min_{t\in[0,T]} P_t > H\}}\right] = P_0 Q^{(1)} - e^{-rT} K Q^{(2)}, \tag{1}$$

*where $Q^{(1)}$ and $Q^{(2)}$ are given by*

$$\begin{aligned} Q^{(1)} &= N\left(\frac{\log(P_0/H) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\ &\quad - \exp\left(-\frac{2(r + \frac{\sigma^2}{2})\log(P_0/H)}{\sigma^2}\right) N\left(\frac{-\log(P_0/H) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right), \end{aligned} \tag{2}$$

$$\begin{aligned} Q^{(2)} &= N\left(\frac{\log(P_0/H) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\ &\quad - \exp\left(-\frac{2(r - \frac{\sigma^2}{2})\log(P_0/H)}{\sigma^2}\right) N\left(\frac{-\log(P_0/H) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right), \end{aligned} \tag{3}$$

*where $N$ denotes the cumulative distribution function of the standard normal distribution.*

## Proof of the Hint

We need to compute under the risk-neutral measure $\tilde{\mathbb{Q}}$:

$$\mathbb{E}_{\tilde{Q}}\left[e^{-rT}(P_T - K)\mathbf{1}_{\{\min_{t\in[0,T]} P_t > H\}}\right] \tag{4}$$

First, we split this into two parts:

$$= e^{-rT}\left(\mathbb{E}_{\tilde{Q}}\left[P_T\mathbf{1}_{\{\min_{t\in[0,T]} P_t > H\}}\right] - K\mathbb{E}_{\tilde{Q}}\left[\mathbf{1}_{\{\min_{t\in[0,T]} P_t > H\}}\right]\right) \tag{5}$$

$$= e^{-rT}\mathbb{E}_{\tilde{Q}}\left[P_T\mathbf{1}_{\{\min_{t\in[0,T]} P_t > H\}}\right] - e^{-rT}K\,\tilde{\mathbb{Q}}\left(\min_{t\in[0,T]} P_t > H\right) \tag{6}$$

Under the risk-neutral measure $\tilde{\mathbb{Q}}$, we have:

$$P_t = P_0 e^{(r - \frac{\sigma^2}{2})t + \sigma\tilde{W}_t} \tag{7}$$

where $\tilde{W}_t$ is a Brownian motion under $\tilde{\mathbb{Q}}$.

**Computing the second term:**

$$\min_{0\le t\le T} P_t > H \Leftrightarrow \min_{0\le t\le T}\left(r - \frac{\sigma^2}{2}\right)t + \sigma\tilde{W}_t > \ln\left(\frac{H}{P_0}\right) \tag{8}$$

Using the reflection principle with $\mu = r - \frac{\sigma^2}{2}$:

$$\begin{aligned} \tilde{\mathbb{Q}}\left(\min_{t\in[0,T]} P_t > H\right) &= N\left(\frac{\ln(P_0/H) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \\ &\quad - \exp\left(-\frac{2(r - \frac{\sigma^2}{2})\ln(P_0/H)}{\sigma^2}\right) N\left(\frac{-\ln(P_0/H) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \end{aligned} \tag{9}$$

This gives us $Q^{(2)}$.

**Computing the first term using Girsanov's theorem:** We define a new measure $\hat{\mathbb{Q}}$ by:

$$\frac{d\hat{\mathbb{Q}}}{d\tilde{\mathbb{Q}}} = e^{\sigma \tilde{W}_T - \frac{\sigma^2}{2}T} = \frac{P_T}{P_0 e^{rT}} \tag{10}$$

Under $\hat{\mathbb{Q}}$, define $\hat{W}_t = \tilde{W}_t - \sigma t$. By Girsanov's theorem, $\hat{W}_t$ is a Brownian motion under $\hat{\mathbb{Q}}$. We can rewrite:

$$\tilde{W}_t = \hat{W}_t + \sigma t \tag{11}$$

So the price process becomes:

$$P_t = P_0 e^{(r-\frac{\sigma^2}{2})t+\sigma\tilde{W}_t} = P_0 e^{(r-\frac{\sigma^2}{2})t+\sigma(\hat{W}_t+\sigma t)} = P_0 e^{(r+\frac{\sigma^2}{2})t+\sigma\hat{W}_t} \tag{12}$$

Now we compute:

$$\begin{aligned}
\mathbb{E}_{\tilde{Q}}\left[P_T \mathbf{1}_{\{\min_{t\in[0,T]} P_t > H\}}\right] &= \mathbb{E}_{\tilde{Q}}\left[\frac{d\hat{\mathbb{Q}}}{d\tilde{\mathbb{Q}}} P_0 e^{rT} \mathbf{1}_{\{\min_{t\in[0,T]} P_t > H\}}\right] && (13)\\
&= P_0 e^{rT} \mathbb{E}_{\hat{\mathbb{Q}}}\left[\mathbf{1}_{\{\min_{t\in[0,T]} P_t > H\}}\right] && (14)\\
&= P_0 e^{rT} \hat{\mathbb{Q}}\left(\min_{t\in[0,T]} P_t > H\right) && (15)
\end{aligned}$$

Under $\hat{\mathbb{Q}}$, since $P_t = P_0 e^{(r+\frac{\sigma^2}{2})t+\sigma\hat{W}_t}$:

$$\min_{0\le t\le T} P_t > H \Leftrightarrow \min_{0\le t\le T}\left(r+\frac{\sigma^2}{2}\right)t + \sigma\hat{W}_t > \ln\left(\frac{H}{P_0}\right) \tag{16}$$

Using the reflection principle under $\hat{\mathbb{Q}}$ with $\mu = r + \frac{\sigma^2}{2}$:

$$\begin{aligned}
\hat{\mathbb{Q}}\left(\min_{t\in[0,T]} P_t > H\right) &= N\left(\frac{\ln(P_0/H) + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\\
&\quad - \exp\left(-\frac{2(r+\frac{\sigma^2}{2})\ln(P_0/H)}{\sigma^2}\right) N\left(\frac{-\ln(P_0/H) + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) && (17)
\end{aligned}$$

This gives us $Q^{(1)}$.
**Final result:** Substituting back:

$$\begin{aligned}
\mathbb{E}_{\tilde{Q}}\left[e^{-rT}(P_T - K)\mathbf{1}_{\{\min_{t\in[0,T]} P_t > H\}}\right] &= e^{-rT}\cdot P_0 e^{rT} Q^{(1)} - e^{-rT} K Q^{(2)} && (18)\\
&= P_0 Q^{(1)} - e^{-rT} K Q^{(2)} && (19)
\end{aligned}$$