

# MATHEMATICAL RECREATIONS

by Ian Stewart

## Monopoly Revisited

In the April column I described a mathematical model of the board game Monopoly. At the start of the game, when everyone emerges from the GO position by throwing dice, the probability of the first few squares being occupied is high, and the distant squares are unoccupied. Using the concept of Markov chains, I showed that this initial bunching of probabilities ultimately evens out so that the game is fair: everyone has an equal chance to occupy any square and to buy that property. This outcome is true, however, only when certain simplifying assumptions are made. Monopoly enthusiasts were quick to point out that in the real game, the long-term distribution of probabilities is not even.

So what are the true probabilities? The Markov chain method can also be applied to the real game; I have to warn you, however, that the analysis is complex and requires substantial computer assistance. Let me first remind you how Markov chains are used for Monopoly. A player can be in any one of 40 squares on the board, which, for convenience, we number clockwise from zero to 39, starting with GO (which is zero).

Given any two squares A and B, there is a quantity called the transition probability—the probability that a player who starts from A will reach B at the conclusion of his or her turn at throwing the dice. If this move is impossible,

then the transition probability is zero.

There are  $40 \times 40 = 1,600$  transition probabilities in all, and they can conveniently be encoded in a square matrix  $M$  with 40 horizontal rows and 40 vertical columns. For example, the entry in the sixth row and 10th column describes the probability of moving from Reading Railroad to Connecticut Avenue in one turn. The initial probabilities for a player are 1 for position 0 and 0 for all the rest; they can be encoded as a vector  $v = (1, 0, \dots, 0)$ .

The theory of Markov chains tells us that the evolution of this probability distribution is given by the sequence of vectors  $v$ ,  $Mv$ ,  $M^2v$ ,  $M^3v$  and so on: each throw of the dice corresponds to the matrix  $M$  operating on the vector  $v$ . The resulting vectors can be calculated by standard matrix methods, available on any good computer algebra package. Such packages can also calculate the so-called eigenvectors and eigenvalues of  $M$ . A vector  $u$  is an eigenvector with eigenvalue  $c$  if  $Mu = c \times u$ , where  $c$  can be a real or complex number. Markov's key theorem is that the long-term probability distribution is given by the eigenvector whose eigenvalue has the largest absolute value.

So in order to analyze the fairness of Monopoly, all we need to do is compute  $M$  and apply matrix algebra. For my simplified model this was easy, but for the real game we must also take into

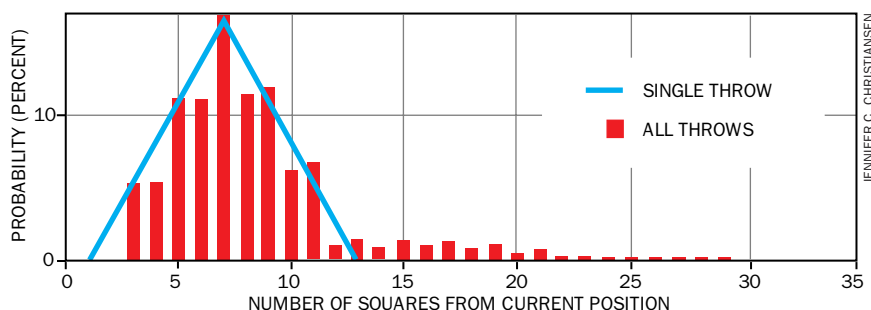


*JAIL,  
and the many ways to land in it,  
makes Monopoly complex.*

account multiple rolls of the dice, special squares such as GO TO JAIL and instructions on cards that players draw when they land on CHANCE and COMMUNITY CHEST.

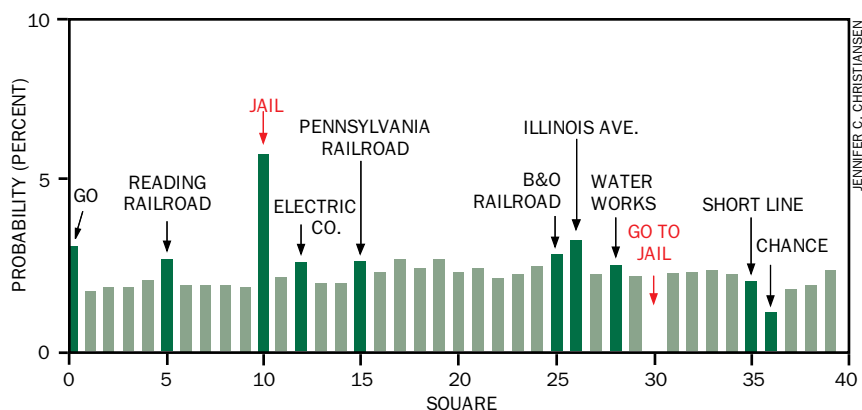
Many readers sent me their analyses of the game. The most extensive were from William J. Butler of Portsmouth, R.I., Thomas H. Friddell, a Boeing engineer from Maple Valley, Wash., and Stephen Abbott of the mathematics department at St. Olaf College in Northfield, Minn., who collaborated with his colleague Matt Richey. Butler wrote a Pascal program, Friddell used Mathcad and Abbott used Maple. The discussion that follows is a synthesis of their results. (All models of Monopoly make assumptions about the degree of detail to be incorporated; there were insignificant differences in the assumptions made by various correspondents.)

The first modification of my original model is to take full account of the rules for the dice. A pair of dice is thrown, and if the result is a double, the player throws again, but three consecutive doubles lands him or her in Jail. The throw of the dice is a tiny Markov chain in its own right and can be solved by the usual method. The result is a graph of the probability of moving any given distance from the current position [see illustration at left]. Notice that the most likely distance is 7, but that it is possible to move up to 35 squares (by throwing 6,6; 6,6; 6,5). Yet the probabilities of



### PROBABILITIES OF MOVING

*a given number of squares in one turn, after accounting for rules for throwing dice, peak at the number seven. If a player throws a double, he or she gets to throw again, but three consecutive doubles mean Jail.*



**LONG-TERM PROBABILITY DISTRIBUTION**  
shows that the Jail square is most likely to be occupied.

moving more than 29 squares are so small that they fail to show up on the graph. These results are incorporated into  $M$  by appropriately changing each individual entry.

Next the effect of the GO TO JAIL square must be included. The Jail rules pose a problem, because players can elect to buy their way out or stay in and try to throw doubles to get out. (Or at later stages, when Jail becomes a refuge from high rents, they can stay in and hope *not* to throw doubles!) The probabilities associated with this choice depend on the player's psychology, so the process is non-Markovian. Most correspondents got around this poser by assuming that the player did not buy his

or her way out. Then Jail becomes not so much a single square as a Markov subprocess—a series of three (virtual) squares where players move from Just in Jail to In Jail One Turn Already to Must Come Out of Jail Next Turn. The GO TO JAIL square itself has probability zero because nobody actually occupies it.

The next step is to modify  $M$  to account for the CHANCE and COMMUNITY CHEST cards, which may send a player to Jail or to some other position on the board. This refinement can be made quite straightforwardly (if laboriously) by counting the proportion of cards that send the player to any given square. The extra probability is then added to the corresponding position in  $M$ .

Having set up an accurate transition matrix, one can work out the steady state probabilities either by numerically computing its eigenvalues and eigenvectors or by calculating the effect of a large number of moves from the powers  $M^2$ ,  $M^3$  and so on. Thanks to Markov's general theorem, these two methods are mathematically equivalent.

The long-term probabilities of occupying different squares are shown in the table [see illustration at left]. The most dramatic feature is that players are almost twice as likely to occupy the Jail square (5.89 percent) as any other. The next most frequented square is Illinois Avenue (3.18 percent). Of the railroads, B&O is occupied most often (3.06 percent) with Reading (2.99 percent) and Pennsylvania (2.91 percent) just behind; however, the probability of occupying Short Line is much less (2.44 percent). The reason for this is that unlike the others, Short Line does not feature a CHANCE card. Among the utilities, Water Works (2.81 percent) wins out, with Electric Company (2.62 percent) being marginally less probable. GO (3.11 percent) is the third most likely square, and the third CHANCE square (0.87 percent) is the least likely—except for GO TO JAIL (0 percent occupation by logical necessity).

Friddell went further and analyzed Monopoly's property market, which is what really makes the game interesting. His aim was to find the break-even point for buying houses—the stage at which income starts to exceed costs—and to determine the best strategies for buying houses and hotels. The exigencies of the property market depend on the number of players and which version of the rules is being adhered to. Assuming that houses can be bought from the start, a number of general principles emerge:

- Although it costs more to buy houses early, the break-even point will be reached more quickly if you do.
- With two houses or fewer, it typically takes around 20 moves or more to break even. Three houses produces a definite improvement.
- Between GO and Indiana Avenue the property square that offers the quickest break-even point for three houses is New York Avenue, which breaks even in about 10 turns.

Properties beyond Indiana Avenue were not evaluated: Friddell says he

## FEEDBACK

**A**lan St. George's sculptures in the May column stimulated a discussion of how to make three-dimensional objects based on regular polyhedra. William J. Sheppard of Columbus, Ohio, sent details of his cunning method for cutting a regular tetrahedron or octahedron from solid wood, pointing out that "sturdy, solid models are more convenient than hollow models made by taping together equilateral triangles." His methods can be found in the *Journal of Chemical Education*, Vol. 44, page 683; November 1967.

Norman Gallatin of Garrison, Iowa, has been working on Platonic solids for a quarter of a century and has developed remarkable sculptures, some made from mirror glass. The picture at the right represents a three-dimensional projection of a four-dimensional hypercube and makes clever use of reflections to create a complex effect from simple components.

Any more mathematical sculptors or modelers out there?  
—I.S.



"Tesseract View of HyperCube III"

NORMAN GALLATIN



stopped there because he never expected to publish his results.

Many other readers contributed interesting observations, and I can mention a few. Simulations by Earl A. Paddon of Maryland Heights, Mo., and calculations by David Weiblen of Reston, Va., confirmed the pattern of probabilities. Weiblen points out that these probabilities do not really affect how "fair" the game is, because all players face the same situation. Developing this point, he notes that "if the rewards for landing on low-probability squares were out of proportion to that lowered probability, then there would be a problem. When out of sheer luck, a player in a game gets a big advantage, the game is unfair." He concludes that Monopoly is not unfair in that manner.

Bruce Moskowitz of East Setauket, N.Y., remarked, "In my youth I played Monopoly many times with my brothers and friends, and it was common knowledge that the tan-colored properties, St. James Place, Tennessee Avenue and New York Avenue, are especially valuable since there is a relatively high probability of landing on one of them when leaving Jail." This suggestion receives confirmation from the calculations, given that all three of these properties figure among the top 12 in the chart of probabilities.

Jonathan D. Simon of Cambridge, Mass., chided me for suggesting that cheap properties were put near the start to help even out the game. "Monopoly was...created during the Great Depression by a *single designer*, Charles Darrow, with lots of presumably unwelcome time on his hands. Under the trappings of wealth, the illustrated fat and rich men, it is (slyly) a poor man's game. In virtually all Monopoly contests...the 'cheap' properties turn out to be the most vital to Monopolize.... The 'lucrative' properties...are expensive to own and prohibitively expensive to build without a source of income provided by ownership of a cheap group with houses." Point taken, though I would still argue that putting a lucrative property on the first half of the board would definitely be unfair, by Weiblen's criterion that no player should gain a big advantage purely by chance. And I'm not convinced that buying up lots of cheap properties and renting them out is a poor man's strategy!

## Learn Another Language on Your Own!



Learn to speak a foreign language fluently on your own and at your own pace with what are considered the finest in-depth courses available. Many were developed by the U.S. Department of State for diplomatic personnel who must learn a language quickly and thoroughly. Emphasis is on learning to speak and to understand the spoken language. A typical course (equivalent to a college semester) includes an album of 10 to 12 audio cassettes (10 to 18 hours), recorded by native-born speakers, plus a 250-page textbook.

Some of our Courses:

<input type="checkbox"/> Arabic, Saudi	\$185	<input type="checkbox"/> German I	\$185	<input type="checkbox"/> Latvian	\$185	<input type="checkbox"/> Slovak	\$185
<input type="checkbox"/> Egyptian	\$185	<input type="checkbox"/> German II	\$155	<input type="checkbox"/> Lithuanian	\$135	<input type="checkbox"/> Spanish I	\$185
<input type="checkbox"/> Bulgarian	\$245	<input type="checkbox"/> Greek	\$185	<input type="checkbox"/> Mandarin	\$185	<input type="checkbox"/> Spanish II	\$165
<input type="checkbox"/> Cantonese	\$185	<input type="checkbox"/> Haitian Creole	\$225	<input type="checkbox"/> Norwegian	\$125	<input type="checkbox"/> Swedish	\$185
<input type="checkbox"/> Catalan	\$185	<input type="checkbox"/> Hebrew	\$255	<input type="checkbox"/> Persian	\$185	<input type="checkbox"/> Swahili	\$225
<input type="checkbox"/> Czech	\$125	<input type="checkbox"/> Hungarian	\$195	<input type="checkbox"/> Polish	\$185	<input type="checkbox"/> Tagalog	\$295
<input type="checkbox"/> Danish	\$185	<input type="checkbox"/> Italian	\$185	<input type="checkbox"/> Portuguese	\$215	<input type="checkbox"/> Thai	\$195
<input type="checkbox"/> Dutch	\$125	<input type="checkbox"/> Japanese	\$185	(Brazilian)		<input type="checkbox"/> Turkish	\$195
<input type="checkbox"/> Estonian	\$295	<input type="checkbox"/> Korean	\$195	<input type="checkbox"/> Romanian	\$115	<input type="checkbox"/> Ukrainian	\$195
<input type="checkbox"/> French I	\$185	<input type="checkbox"/> Lakota	\$185	<input type="checkbox"/> Russian	\$225	<input type="checkbox"/> Urdu	\$185
<input type="checkbox"/> French II	\$215	<input type="checkbox"/> Latin	\$160	<input type="checkbox"/> Serbo-Croatian	\$195	<input type="checkbox"/> Vietnamese	\$225

You can order now with a full 3-week money-back guarantee.

Call toll-free 1-800-243-1234, fax (203) 453-9774, e-mail: 74537.550@compuserve.com

Or clip this ad and send with your name and address and a check or money order – or charge to any major credit card, by enclosing card number, expiration date, and your signature. Our 56-page *Whole World Language Catalog* offers courses in 96 languages. Call or write for your free copy. Our 25th year.

**AUDIO-FORUM**  
THE LANGUAGE SOURCE

Audio-Forum, Room 3715,  
96 Broad Street, Guilford, CT 06437 (203) 453-9794



### Planting Trees to give themselves A Future

Together, American children,  
and children throughout the  
world, are planting trees to  
save our environment.  
If you'd like to help them,  
please contact:



1(800)643-0001

## Bargain Books

America's Biggest Selection  
at up to 80% savings!

- Save up to 80% on recent publishers' overstocks, imports, reprints. Quality hardcover books, starting at \$3.95.
- Save 30% or more on hundreds of current books and best sellers.
- Beautiful art and nature books: rare, unusual, fascinating books for every interest.
- Over 60 subject areas: Science, Biography, Politics, Travel, Nature, Science, Collecting, more.



Free  
Catalog

Please send me your Free Catalog of Bargain Books.

Name \_\_\_\_\_

Address \_\_\_\_\_

City / State / Zip \_\_\_\_\_

**HAMILTON**

5244 Oak, Falls Village, CT 06031-5005

Copyright of Scientific American Archive Online © 1996 is the property of Scientific American Inc. and its content may not be copied or emailed to multiple sites or posted to a listserv without the copyright holder's express written permission. However, users may print, download, or email articles for individual use.

Materials received from the Scientific American Archive Online may only be displayed and printed for your personal, non-commercial use following "fair use" guidelines. Without prior written permission from Scientific American, Inc., materials may not otherwise be reproduced, transmitted or distributed in any form or by any means (including but not limited to, email or other electronic means), via the Internet, or through any other type of technology-currently available or that may be developed in the future.