

5. Berechnen Sie (mit Herleitung/Begründung, keine fertigen Ergebnisse aus Nachschlagewerken/Integralrechnern) die folgenden bestimmten oder unbestimmten Integrale:

(a) $\int \sin^2(x) \cos(x) \, dx$

$$u = \sin(x)$$

$$\frac{du}{dx} = \cos(x)$$

$$du = \cos(x) \, dx$$

$$\int \sin^2(x) \cos(x) \, dx = \int u^2 \cos(x) \, dx$$

$$= \int u^2 \, du$$

$$= \frac{u^3}{3}$$

$$= \frac{\sin^3(x)}{3} + C$$

(b) $\int_0^{\pi} \sin^3(x) \, dx$

$$\int_0^{\pi} \sin^3(x) \, dx = \int_0^{\pi} \sin^2(x) \sin(x) \, dx$$

$$= \int_0^{\pi} (1 - \cos^2(x)) \sin(x) \, dx$$

$$u = \cos(x)$$

$$\frac{du}{dx} = -\sin(x)$$

$$du = -\sin(x) \, dx$$

$$\int_0^{\pi} (1 - \cos^2(x)) \sin(x) \, dx = - \int_0^{\pi} (1 - \cos^2(x)) (-\sin(x)) \, dx$$

$$= - \int_1^{-1} (1 - u^2) \, du$$

$$= - \left(\int_1^{-1} 1 \, du - \int_1^{-1} u^2 \, du \right)$$

$$= - \left([u]_1^{-1} - \left[\frac{u^3}{3} \right]_1^{-1} \right)$$

$$= - \left((-1 - 1) - \left(\frac{-1}{3} - \frac{1}{3} \right) \right)$$

$$= \frac{4}{3}$$

$$(c) \int_0^{2\pi} \sin^4(x) \, dx$$

$$\int \sin^4(x) \, dx = \int \sin(x) \sin^3(x) \, dx$$

$$f' = \sin(x)$$

$$g = \sin^3(x)$$

$$\int \sin^4(x) \, dx = -\cos(x) \sin^3(x) - \int -\cos(x) \frac{d(\sin^3(x))}{dx} \, dx$$

$$= -\cos(x) \sin^3(x) + 3 \int \cos(x) \sin^2(x) \cos(x) \, dx$$

$$= -\cos(x) \sin^3(x) + 3 \int \cos^2(x) \sin^2(x) \, dx$$

$$= -\cos(x) \sin^3(x) + 3 \int (1 - \sin^2(x)) \sin^2(x) \, dx$$

$$= -\cos(x) \sin^3(x) + 3 \int \sin^2(x) - \sin^4(x) \, dx$$

$$= -\cos(x) \sin^3(x) + 3 \left(\int \sin^2(x) \, dx - \int \sin^4(x) \, dx \right)$$

$$= -\cos(x) \sin^3(x) + 3 \int \sin^2(x) \, dx - 3 \int \sin^4(x) \, dx$$

$$4 \int \sin^4(x) \, dx = -\cos(x) \sin^3(x) + 3 \int \sin^2(x) \, dx$$

$$f'(x) = \sin(x)$$

$$g(x) = \sin(x)$$

$$\int \sin^2(x) \, dx = -\cos(x) \sin(x) - \int -\cos(x) \cos(x) \, dx$$

$$= -\cos(x) \sin(x) + \int (1 - \sin^2(x)) \, dx$$

$$= -\cos(x) \sin(x) + \int 1 \, dx - \int \sin^2(x) \, dx$$

$$2 \int \sin^2(x) \, dx = -\cos(x) \sin(x) + \int 1 \, dx$$

$$\int \sin^2(x) \, dx = \frac{-\cos(x) \sin(x)}{2} + \frac{x}{2}$$

$$4 \int \sin^4(x) \, dx = -\cos(x) \sin^3(x) - \frac{3 \cos(x) \sin(x)}{2} + \frac{3x}{2}$$

$$\int \sin^4(x) \, dx = \frac{-\cos(x) \sin^3(x)}{4} - \frac{3 \cos(x) \sin(x)}{8} + \frac{3x}{8}$$

$$\int_0^{2\pi} \sin^4(x) \, dx = \left[\frac{-\cos(x) \sin^3(x)}{4} - \frac{3 \cos(x) \sin(x)}{8} + \frac{3x}{8} \right]_0^{2\pi}$$

$$= \left(\frac{-1 \cdot 0}{4} - \frac{3 \cdot 1 \cdot 0}{8} + \frac{3 \cdot 2\pi}{8} \right) - \left(\frac{-1 \cdot 0}{4} - \frac{3 \cdot 1 \cdot 0}{8} + \frac{3 \cdot 0}{8} \right)$$

$$= \frac{3\pi}{4}$$

(d) $\int e^{-2x} \sin(x) \, dx$

$$f' = e^{-2x}$$

$$g = \sin(x)$$

$$\begin{aligned} \int e^{-2x} \sin(x) \, dx &= -\frac{e^{-2x} \sin(x)}{2} - \int -\frac{e^{-2x} \cos(x)}{2} \, dx \\ &= -\frac{e^{-2x} \sin(x)}{2} + \frac{1}{2} \int e^{-2x} \cos(x) \, dx \end{aligned}$$

$$f' = e^{-2x}$$

$$g = \cos(x)$$

$$\begin{aligned} \int e^{-2x} \cos(x) \, dx &= -\frac{e^{-2x} \cos(x)}{2} - \int -\frac{e^{-2x} (-\sin(x))}{2} \, dx \\ &= -\frac{e^{-2x} \cos(x)}{2} - \frac{1}{2} \int e^{-2x} \sin(x) \, dx \end{aligned}$$

$$\begin{aligned} \int e^{-2x} \sin(x) \, dx &= -\frac{e^{-2x} \cos(x)}{2} + \frac{1}{2} \left(-\frac{e^{-2x} \cos(x)}{2} - \frac{1}{2} \int e^{-2x} \sin(x) \, dx \right) \\ &= -\frac{e^{-2x} \sin(x)}{2} - \frac{e^{-2x} \cos(x)}{4} - \frac{1}{4} \int e^{-2x} \sin(x) \, dx \end{aligned}$$

$$\begin{aligned} \frac{5 \int e^{-2x} \sin(x) \, dx}{4} &= -\frac{e^{-2x} \sin(x)}{2} - \frac{e^{-2x} \cos(x)}{4} \\ \int e^{-2x} \sin(x) \, dx &= -\frac{2e^{-2x} \sin(x) - e^{-2x} \cos(x)}{5} + C \\ &= -\frac{e^{-2x} (2 \sin(x) + \cos(x))}{5} + C \end{aligned}$$

(e) $\int_0^1 \frac{4x^3 - 3x^2 + 6x - 1}{x^4 - x^3 + 3x^2 - x + 2} \, dx$

$$u = x^4 - x^3 + 3x^2 - x + 2$$

$$\frac{du}{dx} = 4x^3 - 3x^2 + 6x - 1$$

$$du = (4x^3 - 3x^2 + 6x - 1) \, dx$$

$$\begin{aligned} \int_0^1 \frac{4x^3 - 3x^2 + 6x - 1}{x^4 - x^3 + 3x^2 - x + 2} \, dx &= \int_0^1 \frac{4x^3 - 3x^2 + 6x - 1}{u} \, dx \\ &= \int_2^4 \frac{du}{u} \\ &= [\ln |u|]_2^4 \\ &= \ln(4) - \ln(2) \\ &= \ln(2) \end{aligned}$$

$$(f) \int_0^{\frac{e-\frac{1}{e}}{2}} \frac{1}{\sqrt{x^2+1}} dx$$

$$x = \sinh(u)$$

$$u = \operatorname{arsinh}(x)$$

$$\frac{dx}{du} = \cosh(u)$$

$$dx = \cosh(u) du$$

$$\int \frac{1}{\sqrt{x^2+1}} dx = \int \frac{\cosh(u)}{\sqrt{\sinh^2(u)+1}} du$$

$$= \int \frac{\cosh(u)}{\sqrt{\cosh^2(u)}} du$$

$$= \int du$$

$$= u$$

$$= \operatorname{arsinh}(x) + C$$

$$\int_0^{\frac{e-\frac{1}{e}}{2}} \frac{1}{\sqrt{x^2+1}} dx = [\operatorname{arsinh}(x)]_0^{\frac{e-\frac{1}{e}}{2}}$$

$$\frac{e-\frac{1}{e}}{2} = \sinh(y)$$

$$\frac{e-\frac{1}{e}}{2} = \frac{e^y - e^{-y}}{2}$$

$$\frac{e-\frac{1}{e}}{2} = \frac{e^1 - e^{-1}}{2}$$

$$\operatorname{arsinh}\left(\frac{e-\frac{1}{e}}{2}\right) = 1$$

$$\operatorname{arsinh}(0) = 0, \text{ da } \sinh(0) \text{ ist}$$

$$[\operatorname{arsinh}(x)]_0^{\frac{e-\frac{1}{e}}{2}} = 1$$

$$\int_0^{\frac{e-\frac{1}{e}}{2}} \frac{1}{\sqrt{x^2+1}} dx = 1$$

$$(g) \int \frac{4x \arctan(x^2)}{1+x^4} dx$$

$$\begin{aligned} u &= \arctan(x^2) \\ \frac{d[\arctan(f(x))]}{dx} &= \frac{1}{1+(f(x))^2} \cdot f'(x) \\ \frac{du}{dx} &= \frac{1}{1+x^4} 2x \\ du &= \frac{2x}{1+x^4} dx \\ \int \frac{4x \arctan(x^2)}{1+x^4} dx &= 2 \int \frac{2x \cdot u}{1+x^4} dx \\ &= 2 \int u du \\ &= 2 \frac{u^2}{2} \\ &= u^2 \\ &= \arctan^2(x^2) + C \end{aligned}$$

$$(h) \int_1^2 x \sqrt{1+x^2} dx$$

$$\begin{aligned} u &= 1+x^2 \\ \frac{du}{dx} &= 2x \\ du &= 2x dx \\ \int_1^2 x \sqrt{1+x^2} dx &= \frac{1}{2} \int_1^2 2x \sqrt{u} dx \\ &= \frac{1}{2} \int_2^5 \sqrt{u} du \\ &= \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_2^5 \\ &= \frac{5^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2^{\frac{3}{2}}}{\frac{3}{2}} \\ &= \frac{5^{\frac{3}{2}} - 2^{\frac{3}{2}}}{\frac{3}{2}} \end{aligned}$$

$$(i) \int_2^3 4x \ln(2x) \, dx$$

$$\int_2^3 4x \ln(2x) \, dx = 4 \int_2^3 x \ln(2x) \, dx$$

$$f' = x$$

$$g = \ln(2x)$$

$$\int x \ln(2x) \, dx = \frac{1}{2} x^2 \ln(2x) - \int \frac{1}{2} x^2 \frac{1}{x} \, dx$$

$$= \frac{1}{2} x^2 \ln(2x) - \frac{1}{2} \int x \, dx$$

$$= \frac{1}{2} x^2 \ln(2x) - \frac{1}{2} \frac{1}{2} x^2$$

$$= \frac{x^2 \ln(2x)}{2} - \frac{x^2}{4}$$

$$\int_2^3 4x \ln(2x) \, dx = 4 \left[\frac{x^2 \ln(2x)}{2} - \frac{x^2}{4} \right]_2^3$$

$$= [2x^2 \ln(2x) - x^2]_2^3$$

$$= (2 \cdot 3^2 \cdot \ln(2 \cdot 3) - 3^2) - (2 \cdot 2^2 \cdot \ln(2 \cdot 2) - 2^2)$$

$$= (18 \ln(6) - 9) - (8 \ln(4) - 4)$$

$$= 18 \ln(6) - 8 \ln(4) - 5$$

$$(j) \int 3e^x \sqrt{e^x + 1} \, dx$$

$$u = e^x + 1$$

$$\frac{du}{dx} = e^x$$

$$du = e^x \, dx$$

$$\int 3e^x \sqrt{e^x + 1} \, dx = 3 \int e^x \sqrt{u} \, dx$$

$$= 3 \int \sqrt{u} \, du$$

$$= 3 \frac{2u^{\frac{3}{2}}}{\frac{3}{2}}$$

$$= 2u^{\frac{3}{2}}$$

$$= 2(e^x + 1)^{\frac{3}{2}} + C$$

(k) $\int \tan(x) \, dx$

$$\begin{aligned}\int \tan(x) \, dx &= \int \frac{\sin(x)}{\cos(x)} \, dx \\ u &= \cos(x) \\ \frac{du}{dx} &= -\sin(x) \\ du &= -\sin(x) \, dx \\ \int \frac{\sin(x)}{\cos(x)} \, dx &= - \int \frac{-\sin(x)}{\cos(x)} \, dx \\ &= - \int \frac{du}{u} \\ &= -\ln(|u|) \\ &= -\ln(|\cos(x)|) + C\end{aligned}$$

(l) $\int \frac{\sin(x)}{1+\cos^2(x)} \, dx$

$$\begin{aligned}u &= \cos(x) \\ \frac{du}{dx} &= -\sin(x) \\ du &= -\sin(x) \, dx \\ \int \frac{\sin(x)}{1+\cos^2(x)} \, dx &= - \int \frac{-\sin(x)}{1+\cos^2(x)} \, dx \\ &= - \int \frac{1}{1+u^2} \, du \\ &= -\arctan(u) \\ &= -\arctan(\cos(x)) + C\end{aligned}$$

(m) $\int \left(\frac{(2\sqrt{x}+1)^2}{x^2} - \frac{1}{x\sqrt{x}} \right) \, dx$

$$\begin{aligned}\int \left(\frac{(2\sqrt{x}+1)^2}{x^2} - \frac{1}{x\sqrt{x}} \right) \, dx &= \int \frac{(2\sqrt{x}+1)^2}{x^2} \, dx - \int \frac{1}{x\sqrt{x}} \, dx \\ \int \frac{(2\sqrt{x}+1)^2}{x^2} \, dx &= \int \frac{4x+4\sqrt{x}+1}{x^2} \, dx \\ &= \int \frac{4x}{x^2} \, dx + \int \frac{4\sqrt{x}}{x^2} \, dx + \int \frac{1}{x^2} \, dx \\ &= 4 \int \frac{1}{x} \, dx + 4 \int \frac{1}{x^{\frac{3}{2}}} \, dx + \int \frac{1}{x^2} \, dx \\ &= 4 \ln(|x|) - \frac{8}{\sqrt{x}} - \frac{1}{x} \\ \int \frac{1}{x\sqrt{x}} \, dx &= \int \frac{1}{x^{\frac{3}{2}}} \, dx \\ &= -\frac{2}{\sqrt{x}} \\ \int \frac{(2\sqrt{x}+1)^2}{x^2} \, dx - \int \frac{1}{x\sqrt{x}} \, dx &= 4 \ln(|x|) - \frac{8}{\sqrt{x}} - \frac{1}{x} - \left(-\frac{2}{\sqrt{x}} \right) \\ \int \left(\frac{(2\sqrt{x}+1)^2}{x^2} - \frac{1}{x\sqrt{x}} \right) \, dx &= 4 \ln(|x|) - \frac{6}{\sqrt{x}} - \frac{1}{x} + C\end{aligned}$$

(n) $\int x^2 \sin(x) \, dx$

$$f' = \sin(x)$$

$$g = x^2$$

$$\int x^2 \sin(x) \, dx = -x^2 \cos(x) - \int -2x \cos(x) \, dx$$

$$= -x^2 \cos(x) + 2 \int x \cos(x) \, dx$$

$$f' = \cos(x)$$

$$g = x$$

$$\int x^2 \sin(x) \, dx = -x^2 \cos(x) + 2(x \sin(x) - \int \sin(x) \cdot 1 \, dx)$$

$$= -x^2 \cos(x) + 2(x \sin(x) - (-\cos(x)))$$

$$= -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) + C$$

$$= 2x \sin(x) + \cos(x)(2 - x^2) + C$$

(o) $\int e^{ax} \sin(bx) \, dx$, für $a, b \in \mathbb{R} \setminus \{0\}$

$$f' = e^{ax}$$

$$g = \sin(bx)$$

$$\int e^{ax} \sin(bx) \, dx = \frac{e^{ax} \sin(bx)}{a} - \int \frac{e^{ax} b \cos(bx)}{a} \, dx$$

$$= \frac{e^{ax} \sin(bx)}{a} - \frac{b}{a} \int e^{ax} \cos(bx) \, dx$$

$$f' = e^{ax}$$

$$g = \cos(bx)$$

$$\int e^{ax} \sin(bx) \, dx = \frac{e^{ax} \sin(bx)}{a} - \frac{b}{a} \left(\frac{e^{ax} \cos(bx)}{a} - \int \frac{-e^{ax} b \sin(bx)}{a} \, dx \right)$$

$$= \frac{e^{ax} \sin(bx)}{a} - \frac{b}{a} \left(\frac{e^{ax} \cos(bx)}{a} + \frac{b}{a} \int e^{ax} \sin(bx) \, dx \right)$$

$$= \frac{e^{ax} \sin(bx)}{a} - \frac{be^{ax} \cos(bx)}{a^2} - \frac{b^2}{a^2} \int e^{ax} \sin(bx) \, dx$$

$$\int e^{ax} \sin(bx) \, dx + \frac{b^2}{a^2} \int e^{ax} \sin(bx) \, dx = \frac{e^{ax} \sin(bx)}{a} - \frac{be^{ax} \cos(bx)}{a^2}$$

$$\frac{a^2 \int e^{ax} \sin(bx) \, dx + b^2 \int e^{ax} \sin(bx) \, dx}{a^2} = \frac{e^{ax} \sin(bx)}{a} - \frac{be^{ax} \cos(bx)}{a^2}$$

$$a^2 \int e^{ax} \sin(bx) \, dx + b^2 \int e^{ax} \sin(bx) \, dx = ae^{ax} \sin(bx) - be^{ax} \cos(bx)$$

$$(a^2 + b^2) \int e^{ax} \sin(bx) \, dx = ae^{ax} \sin(bx) - be^{ax} \cos(bx)$$

$$\int e^{ax} \sin(bx) \, dx = \frac{ae^{ax} \sin(bx)}{a^2 + b^2} - \frac{be^{ax} \cos(bx)}{a^2 + b^2}$$

$$= \frac{ae^{ax} \sin(bx) - be^{ax} \cos(bx)}{a^2 + b^2} + C$$

(p) $\int (x^2 + 2x) \sinh\left(\frac{x}{2}\right) dx$

$$f' = \sinh\left(\frac{x}{2}\right)$$

$$g = x^2 + 2x$$

$$\begin{aligned} \int (x^2 + 2x) \sinh\left(\frac{x}{2}\right) dx &= 2 \cosh\left(\frac{x}{2}\right) (x^2 + 2x) - \int 2 \cosh\left(\frac{x}{2}\right) (2x + 2) dx \\ &= 2 \cosh\left(\frac{x}{2}\right) (x^2 + 2x) - 2 \int \cosh\left(\frac{x}{2}\right) (2x + 2) dx \end{aligned}$$

$$f' = \cosh\left(\frac{x}{2}\right)$$

$$g = 2x + 2$$

$$\begin{aligned} \int \cosh\left(\frac{x}{2}\right) (2x + 2) dx &= 2 \sinh\left(\frac{x}{2}\right) (2x + 2) - \int 2 \sinh\left(\frac{x}{2}\right) 2 dx \\ &= 2 \sinh\left(\frac{x}{2}\right) (2x + 2) - 4 \int \sinh\left(\frac{x}{2}\right) dx \\ &= 2 \sinh\left(\frac{x}{2}\right) (2x + 2) - 8 \cosh\left(\frac{x}{2}\right) \end{aligned}$$

$$\begin{aligned} \int (x^2 + 2x) \sinh\left(\frac{x}{2}\right) dx &= 2 \cosh\left(\frac{x}{2}\right) (x^2 + 2x) - 2 \int \cosh\left(\frac{x}{2}\right) (2x + 2) dx \\ &= 2 \cosh\left(\frac{x}{2}\right) (x^2 + 2x) - 2(2 \sinh\left(\frac{x}{2}\right) (2x + 2) - 8 \cosh\left(\frac{x}{2}\right)) \\ &= 2 \cosh\left(\frac{x}{2}\right) (x^2 + 2x) - 8 \sinh\left(\frac{x}{2}\right) (x + 1) + 16 \cosh\left(\frac{x}{2}\right) + C \end{aligned}$$

(q) $\int \sin(ax) \cos(bx) \, dx$, für $a, b \in \mathbb{R} \setminus \{0\}$ mit $|a| \neq |b|$

$$\sin(\alpha) \cos(\beta) = \frac{1}{2}(\sin(\beta + \alpha) - \sin(\beta - \alpha))$$

$$\alpha = ax$$

$$\beta = bx$$

$$\begin{aligned} \int \sin(ax) \cos(bx) \, dx &= \int \left(\frac{1}{2}(\sin(bx + ax) - \sin(bx - ax)) \right) \, dx \\ &= \frac{1}{2} \left(\int \sin(x(b + a)) \, dx - \int \sin(x(b - a)) \, dx \right) \end{aligned}$$

$$u_1 = x(b + a)$$

$$\frac{du_1}{dx} = b + a$$

$$du_1 = (b + a) \, dx$$

$$\begin{aligned} \int \sin(x(b + a)) \, dx &= \frac{1}{b + a} \int (b + a) \sin(x(b + a)) \, dx \\ &= \frac{1}{b + a} \int \sin(u_1) \, du_1 \\ &= \frac{-\cos(u_1)}{b + a} \\ &= \frac{-\cos(x(b + a))}{b + a} \end{aligned}$$

$$u_2 = x(b - a)$$

$$\frac{du_2}{dx} = b - a$$

$$du_2 = (b - a) \, dx$$

$$\begin{aligned} \int \sin(x(b - a)) \, dx &= \frac{1}{b - a} \int (b - a) \sin(x(b - a)) \, dx \\ &= \frac{1}{b - a} \int \sin(u_2) \, du_2 \\ &= \frac{-\cos(u_2)}{b - a} \\ &= \frac{-\cos(x(b - a))}{b - a} \end{aligned}$$

$$\begin{aligned} \int \sin(ax) \cos(bx) \, dx &= \frac{1}{2} \left(\frac{-\cos(x(b + a))}{b + a} + \frac{\cos(x(b - a))}{b - a} \right) \\ &= \frac{\cos(x(b - a))}{2(b - a)} - \frac{\cos(x(b + a))}{2(b + a)} + C \end{aligned}$$

(r) $\int \sin(ax) \cos(bx) \, dx$, für $a, b \in \mathbb{R} \setminus \{0\}$ mit $|a| = |b|$

(1) $a = b$

$$\int \sin(ax) \cos(ax) \, dx$$

$$u_1 = \sin(ax)$$

$$\frac{du_1}{dx} = a \cos(ax)$$

$$du_1 = a \cos(ax) dx$$

$$\int \sin(ax) \cos(ax) \, dx = \frac{1}{a} \int \sin(ax) a \cos(ax) \, dx$$

$$= \frac{1}{a} \int u_1 \, du_1$$

$$= \frac{u^2}{2a}$$

$$= \frac{\sin^2(ax)}{2a} + C$$

(2) $a = -b$

$$\int \sin(-ax) \cos(ax) \, dx$$

$$u_2 = \cos(ax)$$

$$\frac{du_2}{dx} = -a \sin(ax)$$

$$du_2 = -a \sin(ax) dx$$

$$\int \sin(ax) \cos(ax) \, dx = \frac{-1}{a} \int -a \sin(ax) \cos(ax) \, dx$$

$$= \frac{-1}{a} \int u_2 \, du_2$$

$$= \frac{-u^2}{2a}$$

$$= \frac{-\sin^2(ax)}{2a} + C$$