Propositional logic

Week 2 – Discrete Mathematics

Propositional logic: review

- Propositional logic: a formal language for representing knowledge and for making logical inferences
- A proposition is a statement that is either true or false.
- A compound proposition can be created from other propositions using logical connectives
- The truth of a compound proposition is defined by truth values of elementary propositions and the meaning of connectives.
- The truth table for a compound proposition: table with entries (rows) for all possible combinations of truth values of elementary propositions.

Applications of Propositional Logic

Importance of Propositional Logic

- Propositional Logic plays an important role in computer science as well as in a person's daily life.
- The main benefits of studying and using propositional logic are that it prevents us from making inconsistent inferences and incautious decisions. It incorporates reasoning and thinking abilities in one's daily life.

Applications of Propositional Logic

- Translating English Sentence
- System Specifications
- Boolean Searches
- Logic Puzzles
- Logic Circuits

Applications of Propositional Logic:
Translating English Sentence

- Why it is required
  - To remove ambiguity
  - Manipulate expressions (easy in logical sentences)
  - Able to solve puzzles

A variety of English words translate into logic as ∧, ∨, or ~. For instance, the word *but* translates the same as *and* when it links two independent clauses, as in “Jim is tall but he is not heavy.”

p but q	means	p and q
neither p nor q	means	$\sim p$ and $\sim q$.

Applications of Propositional Logic: Translating English Sentence

- **General rule for translation.**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

You can have free coffee if you are senior citizen and it is a Tuesday

Step 1 find logical connectives

Applications of Propositional Logic: Translating English Sentence

- **General rule for translation .**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

You can have free coffee if you are senior citizen and it is a Tuesday

$\underbrace{\text{You can have free coffee}}_{\mathbf{a}}$ if $\underbrace{\text{you are senior citizen}}_{\mathbf{b}}$ and $\underbrace{\text{it is a Tuesday}}_{\mathbf{c}}$

Step 3 rewrite the sentence in propositional logic

$$\mathbf{b \wedge c \rightarrow a}$$

- Assume two elementary statements:
  - **p: you drive over 65 mph ; q: you get a speeding ticket**
- **Translate each of these sentences to logic**
  - you do not drive over 65 mph. $(\neg p)$
  - you drive over 65 mph, but you don't get a speeding ticket. $(p \wedge \neg q)$
  - you will get a speeding ticket if you drive over 65 mph. $(p \rightarrow q)$
  - if you do not drive over 65 mph then you will not get a speeding ticket. $(\neg p \rightarrow \neg q)$
  - driving over 65 mph is sufficient for getting a speeding ticket. $(p \rightarrow q)$
  - you get a speeding ticket, but you do not drive over 65 mph. $\mathbf{(q \wedge \neg p)}$

Applications of Propositional Logic:
Translating English Sentence

Given a sentence “You can purchase this book if you have \$20 or \$10 and a discount coupon.”

Step 1 : Fine the connectives which are connecting two propositions together

Step 2: Rename the propositions

Let: 1:

q: “You can purchase this book ”

r: “you have \$20 ”

s: “you have \$10 ”

t: “you have a discount coupon”

$(r \lor (s \land t) \rightarrow q$

Applications of Propositional Logic:
Translating English Sentence

Example: You are not allowed to drive vehicle if your age is less than 18 years or you have no age proof

Step 1 : Fine the connectives which are connecting two propositions together

Step 2: Rename the propositions

Let: 1:

q: “you are allowed to drive vehicle”

r: “your age is less than 18”

s: “you have age proof”

$$\boxed{(r \: V \: \neg s) \rightarrow \neg q}$$

Try It

- A system sends an alert if a file is larger than 5MB or is not encrypted.
- A backup is initiated if and only if the storage device is available and the system load is below 70%.
- Access to the database is granted only if both the user's credentials are valid and the security check is passed.

Applications of Propositional Logic: System Specifications

When developing/manufacturing a system(Software or Hardware), the developers/manufacturers have to meet certain needs and specifications, which are usually stated in English.

But as English sentences can be ambiguous, developers/engineers translate these system specifications into logical expressions to state specifications rigorously and unambiguously.

Applications of Propositional Logic: System Specifications

Let *a*, *b*, *c*, and *d* represent the sentences ***"The computer is within the local network."***, ***"The computer has a valid login id."***, ***"The computer is under the use of administrator."***, **and** ***"Internet is accessible to the computer."*** So the complex sentence, "***If the computer is within the local network or it is not within the local network but has a valid login id or it is under the use of administrator, then the Internet is accessible to the computer.***"

$$(a \vee (\neg a \wedge b) \vee c) \rightarrow d$$

Applications of Propositional Logic: Logical Puzzles

- Puzzles that are solved using reasoning and logic are called logical puzzles. They can be used for brain exercises, recreational purposes, and for testing a person's reasoning capabilities.
- Solving such puzzles is generally tricky, but it can be done easily using propositional logic.
- Some of the famous logic puzzles are *the muddy children puzzle*, *Smullyan's puzzles about knights and knaves*, etc.

Applications of Propositional Logic: Logical Puzzles

- There is an island that has two kinds of inhabitants, knights, who always tell the truth, and their opposites, knaves, who always lie. You encounter two people A and B. Determine what are A and B if

A says "At least one of us is a knave" and B says nothing

Applications of Propositional Logic: Logical Puzzles

A says “At least one of us is a knave” and B says nothing

Line number	A	B	A says “At least one of us is a knave”
1	Knight	Knight	F
2	Knight	Knave	T
3	Knave	Knight	T
4	Knave	Knave	T

Applications of Propositional Logic: Logical Puzzles

We can eliminate:

Line1, as A would beak knight but he lies

Line 3 and 4, as A would be a knave, but he says the truth

Line2 is valid, and it is the only one. Therefore, A is a knight and B is a knave.

Applications of Propositional Logic: Inference and Decision Making

- Propositional Logic is widely used in the making rules of inference and decision making. These rules of inferences can then be used to build arguments. **When several premises are given, it is hard to tell if a given argument is valid**. Thus, we use these rules of inference to validate an argument and make a decision.

Applications of Propositional Logic: Inference and Decision Making

Assume a sentence:

If you are older than 13 or you are with your parents then you can attend a PG-13 movie.

Parse:

- **If (**you are older than 13** or **you are with your parents**) then (**you can attend a PG-13 movie**)**

Atomic (elementary) propositions:

- A= you are older than 13
- B= you are with your parents
- C=you can attend a PG-13 movie

- **Translation:** $A \lor B \to C$

Propositional Equivalences

Propositional Equivalences

- Two logical expressions are said to be **equivalent if they have the same truth value in all cases**. Sometimes this fact helps in proving a mathematical result by replacing one expression with another equivalent expression, without changing the truth value of the original compound proposition.

Types of propositions based on Truth values

There are three types of propositions when classified according to their truth values

1. Tautology – A proposition which is always true, is called a tautology.
2. Contradiction – A proposition which is always false, is called a contradiction.
3. Contingency – A proposition that is neither a tautology nor a contradiction is called a contingency.

Tautology and Contradiction

- Some propositions are interesting since their values in the truth table are always the same

Definitions:

- A compound proposition that is always true for all possible truth values of the propositions is called a **tautology**.
- A compound proposition that is always false is called a **contradiction**.
- A proposition that is neither a tautology nor contradiction is called a **contingency**.

Example: $p \wedge \neg p$ is a **contradiction.**

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

all T's so $p \vee \sim p$ is a tautology

all F's so $p \wedge \sim p$ is a contradiction

Tautology and Logical Equivalence

Definitions:

- A compound proposition that is always True is called a tautology.
- Two propositions p and q are logically equivalent if their truth tables are the same.
- Namely, p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.
- If p and q are logically equivalent, we write $p \equiv q$.

Example:

Look at the following two compound propositions: $p \rightarrow q$ and $q \vee \neg p$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\neg p$	$q \vee \neg p$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

- The last column of the two truth tables are identical. Therefore $(p \rightarrow q)$ and $(q \vee \neg p)$ are logically equivalent.
- So $(p \rightarrow q) \leftrightarrow (q \vee \neg p)$ is a tautology.
- Thus: $(p \rightarrow q) \equiv (q \vee \neg p)$.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

$\sim(p \wedge q)$ and $\sim p \vee \sim q$ always have the same truth values, so they are logically equivalent

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$		$\sim p \wedge \sim q$
T	T	F	F	T	F		F
T	F	F	T	F	T	$\neq$	F
F	T	T	F	F	T	$\neq$	F
F	F	T	T	F	T		T

$\sim(p \wedge q)$ and $\sim p \wedge \sim q$ have different truth values in rows 2 and 3, so they are not logically equivalent

We have a number of rules for logical equivalence. For example:

De Morgan Law:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \tag{1}$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \tag{2}$$

The following is the truth table proof for (1). The proof for (2) is similar.

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

p	q	$\neg p \vee \neg q$
T	T	F
T	F	T
F	T	T
F	F	T

Distributivity

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \quad (1)$$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \quad (2)$$

The following is the truth table proof of (1). The proof of (2) is similar.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Contrapositives

The proposition $\neg q \to \neg p$ is called the Contrapositive of the proposition $p \to q$. They are logically equivalent.

$$p \to q \quad \equiv \quad \neg q \to \neg p$$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\neg q \to \neg p$
T	T	T
T	F	F
F	T	T
F	F	T

Logic Equivalences

Equivalence	Name
$p \wedge T \equiv p, \quad p \vee F \equiv p$	Identity laws
$p \vee T \equiv T, \quad p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p, \quad p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv T, \quad p \wedge \neg p \equiv F$	Negation laws

Logical Equivalences Involving Conditional Statements

$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical Equivalences Involving Biconditional Statements

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6. *Double negative law:*	$\sim(\sim p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* $\mathbf{t}$ *and* $\mathbf{c}$*:*	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

Prove Equivalence

Example,

Show that $\neg(p \rightarrow q) \equiv p \wedge \neg q$.

Considering LHS,

$$\begin{aligned} \neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) && \textit{Using first equivalence of Conditionals} \\ &\equiv \neg(\neg p) \wedge \neg q && \textit{Using De Morgan's law} \\ &\equiv p \wedge \neg q && \textit{Using Double negation law} \end{aligned}$$

By using these laws, we can prove two propositions are logical equivalent.

Example 1: Prove $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{DeMorgan} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{DeMorgan} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{Distributivity} \\
&\equiv \mathsf{F} \vee (\neg p \wedge \neg q) && \text{Because } \neg p \wedge p \equiv \mathsf{F} \\
&\equiv \neg p \wedge \neg q && \text{Because } \mathsf{F} \vee r \equiv r \text{ for any } r
\end{aligned}
$$

Try it

Use Theorem 2.1.1 to verify the logical equivalences

50. $(p \wedge \sim q) \vee p \equiv p$

51. $p \wedge (\sim q \vee p) \equiv p$

52. $\sim(p \vee \sim q) \vee (\sim p \wedge \sim q) \equiv \sim p$

53. $\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q) \equiv p$

54. $(p \wedge (\sim(\sim p \vee q))) \vee (p \wedge q) \equiv p$

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Assignment (Homework)

1. *Truth tables.* Use a truth table to verify the following equivalence

$$\neg(P \vee (Q \wedge (\neg R))) \leftrightarrow (\neg P) \wedge ((\neg Q) \vee R).$$

2. Translate the following English statements into propositional logic. Define the propositions you will use. Your base propositions should not be negative or compound propositions themselves. Use brackets when necessary to make the order of evaluation clear.
   a) If I study Discrete Math and Linear Algebra then I will get good grades.
   b) The sky is blue or the earth is round.
   c) It is Monday and I am not at work.
   d) I will live to be 100 years old if and only if I exercise everyday and I eat well.
   e) I miss the bus if I sleep in.
3. Use the laws to show that $\neg(p \wedge q) \vee (\neg p \wedge q) \equiv \neg p$