Predicate logic

Predicates in grammar

- In grammar, the word *predicate* refers to the part of a sentence that **gives information about the subject**.
- In the sentence "**James is a student at Bedford College**," the word *James* is the subject and the phrase *is a student at Bedford College* is the predicate.
- The predicate is the part of the sentence from which the subject has been removed.

Predicates in grammar

- The proposition

  "The dog is sleeping"

  has two parts:
  - "the dog" denotes the *subject* – the *object or entity* that the sentence is about.
  - "is sleeping" denotes the *predicate*– a property that the subject can have.

Predicate Logic - Definition

Definition

A predicate is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

Predicate Logic

A predicate is a statement involving variables that can be true or false based on the values of the variables.

Let $P(x)$ be a predicate that represents the statement: "x is greater than 5“

Here, P(x) is a predicate because it contains the variable x, and the truth value of P(x) depends on what value of x is substituted.

Predicate Logic

The domain of the predicate variable x is the set of all values that can be substituted in place of x. For example, let the domain be all integers ($\mathbb{Z}$).

For example:

$P(3)$ denotes "3 is greater than 5.

"This is false because 3 is not greater than 5.

$P(7)$ denotes "7 is greater than 5.

"This is true because 7 is greater than 5.

Predicate Logic with a Single Variable

Predicate logic with a single variable refers to a logical statement that contains one variable and becomes a statement (true or false) when a specific value is assigned to that variable.

The general form is a predicate P(x) where x is the variable, and P(x) is a statement about x.

Example:
Let D(x) represent "x is divisible by 5."
Domain: x is an integer.
Predicate: "x is divisible by 5."
- For D(10): This becomes "10 is divisible by 5." (True)
- For D(7): This becomes "7 is divisible by 5." (False)

Predicate logic with multiple variables

Predicate logic with multiple variables extends the concept of predicate logic to statements involving more than one variable. This type of logic is used to describe relationships between different entities or to express properties involving several variables.

Example:

- Predicate: Let G(x,y) represent "x is greater than y”
- Domain: x and y are integers.
  - G(7,3) denotes "7 is greater than 3." (True)
  - G(2,5) denotes "2 is greater than 5." (False)

What are quantifiers?

Definition

- Quantifiers are words that refer to quantities such as "all" or "some" and they tell for how many elements a given predicate is true

- Introduced into logic by logicians Charles Sanders Pierce and Gottlob Frege during late 19th century
- Two types of quantifiers:
  1. Universal quantifier ($\forall$)
  2. Existential quantifier ($\exists$)

The universe of discourse

- The **universe of discourse** (also known as the domain of discourse) is **the set of all possible** objects or elements that are considered in a particular discussion or logical analysis.
- It defines the scope within which variables, quantifiers, and propositions are interpreted.

Quantifier Expressions

- ***Quantifiers*** allow us to *quantify* (count) *how many* objects in the universe of discourse satisfy a given predicate:
  - "$\forall$" is the FOR$\forall$LL or *universal* quantifier.

    $\forall x\ P(x)$ means *for all* x in the u.d., P holds.

  - "$\exists$" is the $\exists$XISTS or *existential* quantifier.

    $\exists x\ P(x)$ means there *exists* an x in the u.d. (that is, one or more) such that $P(x)$ is true.

Universal Quantifier

- **The universal quantifier ∀ is used to denote that a particular property or condition holds true for every element in a specified domain.**
- **It is commonly read as "for all," "for every," or "for any.“**
- **The universal quantifier is typically written as:**

$$\forall x \, P(x)$$

- **The statement $\forall x P(x)$ means that the proposition $P(x)$ is true for every possible value of x within the domain of discourse.**

Universal Quantifier ∀: Example

- Let $P(x)$ be the *predicate* "x is full."
- Let the u.d. of x be parking spaces at PresUniv.
- The *universal quantification of* $P(x)$,

  $\forall x\ P(x)$, is read as:
    - "All parking spaces at PresUniv are full." or
    - "Every parking space at PresUniv is full." or
    - "For each parking space at PresUniv, that space is full."

Universal Quantifier ∀: Example

Let D {1, 2, 3, 4, 5}, and consider the statement:

$$\forall x \in D, x^2 \geq x.$$

To show that statement is true, we need to verify that $x^2 \geq x$ for every element x in the set D

The Existential Quantifier

The existential quantifier is a fundamental concept in logic and mathematics that asserts the existence of **at least one element** in a given domain that satisfies a certain property or condition.
It is commonly denoted by the symbol $\exists$.

Existential Quantifier ∃ Example

- Let $P(x)$ be the *predicate* "x is full."
- Let the u.d. of x be parking spaces at PresUniv.
- The *universal quantification of* $P(x)$, $\exists x\, P(x)$, is read as:
  - "Some parking space at PresUniv is full." or
  - "There is a parking space at PresUniv that is full." or
  - "At least one parking space at PresUniv is full."

Existential Quantifier ∃ Example

Let E {5, 6, 7, 8} and consider the statement

$$\exists m \in E \text{ such that } m^2 = m.$$

Show that this statement is false.

Existential Quantifier ∃ Example

Let E {5, 6, 7, 8} and consider the statement

$$\exists m \in E \text{ such that } m^2 = m.$$

Show that this statement is false.

Note that $m^2 = m$ is not true for any integers m from 5 through 8:

$$5^2 = 25 \neq 5, \quad 6^2 = 36 \neq 6, \quad 7^2 = 49 \neq 7, \quad 8^2 = 64 \neq 8.$$

Thus "$\exists m \in E$ such that $m^2 = m$" is false.

Formal vs. Informal Language

It is important to be able to translate from formal to informal language when trying to make sense of mathematical concepts that are new to you. It is equally important to be able to translate from informal to formal language when thinking out a complicated problem

Translating from Formal to Informal Language

Rewrite the following formal statements in a variety of equivalent but more informal ways. Do not use the symbol $\forall$ or $\exists$.

a. $\forall x \in \mathbf{R}, x^2 \geq 0.$

b. $\forall x \in \mathbf{R}, x^2 \neq -1.$

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a. Every real number has a nonnegative square.
Or: All real numbers have nonnegative squares.
Or: Any real number has a nonnegative square.
Or: The square of each real number is nonnegative.

b. All real numbers have squares that do not equal -1.
Or: No real numbers have squares equal to -1.
(The words *none are* or *no … are* are equivalent to the words *all are not*.)

Universal Conditional Statements

Universal conditional statements combine two fundamental concepts in logic: **the universal quantifier and conditional statements**.
They assert that a specific condition holds universally across a particular domain.

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

Summary of quantified statements

- When **$\forall x\ P(x)$** and **$\exists x\ P(x)$** are true and false?

Statement	When true?	When false?
$\forall x\ P(x)$	P(x) true for all x	There is an x where P(x) is false.
$\exists x\ P(x)$	There is some x for which P(x) is true.	P(x) is false for all x.

Suppose the elements in the universe of discourse can be enumerated as x1, x2, ..., xN then:

- **$\forall x\ P(x)$ is true whenever $P(x1) \wedge P(x2) \wedge \ldots \wedge P(xN)$ is true**
- **$\exists x\ P(x)$ is true whenever $P(x1) \vee P(x2) \vee \ldots \vee P(xN)$ is true.**

Examples

- Let *L(x, y)* be the predicate "*x* likes y," and let the universe of discourse be the set of all people. Use quantifiers to express each of the following statements.
- a) Everyone likes everyone.

a) $\forall x \ \forall y \ L(x, y)$

b) Everyone likes someone.

b) $\forall x \ \exists y \ L(x, y)$

c) Someone does not like anyone. c) $\exists x \ \forall y \ \neg L(x, y)$

Your Turn

Let *L(x, y)* be the predicate "*x* likes y," and let the universe of discourse be the set of all people. Use quantifiers to express each of the following statements.

d) Everyone likes Budi.
e) There is someone whom everyone likes.
f) There is no one whom everyone likes.
g) Everyone does not like someone.

Write your answer on paper