FORMAL AND INFORMAL PROOFS

Introduction to proof

- Mathematical proof is an argument we give logically to validate a mathematical statement. In order to validate a statement, we consider two things: A **statement** and **Logical operators**.
- A statement is either true or false but not both. Logical operators are AND, OR, NOT, If then, and If and only if. Coupled with quantifiers like for all and there exists. We apply operators on the statement to check the correctness of it.

What is proof?

- The process of establishing the validity of a statement

- Formal proof
  - Driven by logic
  - Based on deductive reasoning

- Informal Proof
  - Driven by experience, observation and intuition
  - Based on inductive reasoning

Formal Proof Example

- Let us adopt the following abbreviations:
  1. $sunny$ = "*It is sunny*";
  2. $cold$ = "*It is cold*";
  3. $swim$ = "*We will swim*";
  4. $canoe$ = "*We will canoe*";
  5. $early$ = "*We will be home early*".
- Then, the premises can be written as:
  1. $\neg sunny \wedge cold$;
  2. $swim \longrightarrow sunny$;
  3. $\neg swim \longrightarrow canoe$;
  4. $canoe \longrightarrow early$.

Formal Proof Example

Step	Proved by
1. $\neg sunny \wedge cold$	Premise #1.
2. $\neg sunny$	Simplification of 1.
3. $swim \rightarrow sunny$	Premise #2.
4. $\neg swim$	Modustollens on 2,3.
5. $\neg swim \rightarrow canoe$	Premise #3.
6. $canoe$	Modusponens on 4,5.
7. $canoe \rightarrow early$	Premise #4.
8. $early$	Modusponens on 6,7.

We will learn these in the next slides

Indirect proof

- To show $p \to q$ prove its contrapositive $\neg q \to \neg p$
- Why? **$p \to q$ and $\neg q \to \neg p$ are equivalent !!!**
- Assume $\neg q$ is true, show that $\neg p$ is true.

Example: Prove If $3n + 2$ is odd then n is odd.

Proof:

- Assume n is even, that is $n = 2k$, where k is an integer.
- Then: $$\begin{aligned} 3n + 2 &= 3(2k) + 2 \\ &= 6k + 2 \\ &= 2(3k+1) \end{aligned}$$
- Therefore $3n + 2$ is even.
- We proved **¬ "n is odd" → ¬ "3n + 2 is odd".** This is equivalent to **"3n + 2 is odd" → "n is odd".** □

Indirect Proof Example

Theorem

(For all integers n) If $3n + 2$ is odd, then n is odd.

Proof.

Suppose that the conclusion is false, i.e., that n is even. Then $n = 2k$ for some integer k. Then $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$. Thus $3n + 2$ is even, because it equals $2j$ for integer $j = 3k + 1$. So $3n + 2$ is not odd. We have shown that $\neg(n \text{ is odd}) \rightarrow \neg(3n + 2 \text{ is odd})$, thus its contra-positive $(3n + 2 \text{ is odd}) \rightarrow (n \text{ is odd})$ is also true. □

Inference Rule

Definition

An **argument** is a sequence of statements, and an **argument form** is a sequence of statement forms. All statements in an argument and all statement forms in an argument form, except for the final one, are called **premises** (or **assumptions** or **hypotheses**). The final statement or statement form is called the **conclusion**. The symbol $\therefore$, which is read "therefore," is normally placed just before the conclusion.

To say that an *argument form* is **valid** means that no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then the conclusion is also true. To say that an *argument* is **valid** means that its form is valid.

- Inference Rule
  - Pattern establishing that if we know that a set of antecedent statements of certain forms are all true, then a certain related consequent statement is true.

$$\begin{array}{l} \quad \textit{antecedent } 1 \\ \quad \textit{antecedent } 2 \ldots \\ \hline \therefore \textit{ consequent} \end{array}$$

  PS. "$\therefore$" means “therefore”

- Each logical inference rule corresponds to an implication that is a tautology

$$((\textit{ante. } 1) \land (\textit{ante. } 2) \land \ldots) \rightarrow \textit{consequent}$$

How to proof inference rules

- Truth tables.
  - The consequent statement must be true if all antecedent statements are ture.
- Prove the corresponding implication proposition is a tautology.

Testing an Argument Form for Validity

1. Identify the premises and conclusion of the argument form.
2. Construct a truth table showing the truth values of all the premises and the conclusion.
3. A row of the truth table in which all the premises are true is called a **critical row**. If there is a critical row in which the conclusion is false, then it is possible for an argument of the given form to have true premises and a false conclusion, and so the argument form is invalid. If the conclusion in *every* critical row is true, then the argument form is valid.

Determining Validity or Invalidity

Determine whether the following argument form is valid or invalid by drawing a truth table,

$$p \to q \vee \sim r$$

$$q \to p \wedge r$$

$$\therefore p \to r$$

						premises		conclusion
p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	$p \to q \vee \sim r$	$q \to p \wedge r$	$p \to r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	
T	F	T	F	F	T	F	T	
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	
F	T	F	T	T	F	T	F	
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

This row shows that an argument of this form can have true premises and a false conclusion. Hence this form of argument is invalid.

Some Inference Rules

- $\dfrac{p}{\therefore p \vee q}$ Rule of Addition
- $\dfrac{p \wedge q}{\therefore p}$ Rule of Simplification
- $\dfrac{\begin{matrix} p \\ q \end{matrix}}{\therefore p \wedge q}$ Rule of Conjunction

- **Addition**

  $p \rightarrow (p \vee q)$

  $\dfrac{p}{\therefore p \vee q}$

- **Example:** It is below freezing now. Therefore, it is below freezing or raining snow.

- **Simplification**

  $(p \wedge q) \rightarrow p$

  $\dfrac{p \wedge q}{\therefore p}$

- **Example:** It is below freezing and snowing. Therefore it is below freezing.

Two rules

$\boxed{\dfrac{p \wedge q}{\therefore q}}$ $\boxed{\dfrac{p \wedge q}{\therefore p}}$

Modus Ponens

An argument form consisting of two premises and a conclusion is called a **syllogism**.

The first and second premises are called the **major premise** and **minor premise**, respectively.

The most famous form of syllogism in logic is called **modus ponens**

Modus Ponens

$$\begin{array}{rl} & \text{If } p \text{ then } q. \\ & p \\ \therefore & q \end{array}$$

Here is an argument of this form:

If the sum of the digits of 371,487 is divisible by 3, then 371,487 is divisible by 3.

The sum of the digits of 371,487 is divisible by 3.

∴ 371,487 is divisible by 3.

Modus Tollens

Modus tollens is Latin meaning "method of denying" (the conclusion is a denial)

$$\begin{aligned} &\text{If } p \text{ then } q. \\ &\sim q \\ \therefore\ &\sim p \end{aligned}$$

Here is an example of modus tollens:

$$\begin{aligned} &\text{If Zeus is human, then Zeus is mortal.} \\ &\text{Zeus is not mortal.} \\ \therefore\ &\text{Zeus is not human.} \end{aligned}$$

Modus Ponens and Modus Tollens

- $$\begin{array}{c} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Rule of modus ponens
(a.k.a. law of detachment)
"the mode of affirming"

- $$\begin{array}{c} \neg p \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$$

Rule of modus tollens
"the mode of denying"

Generalization

a. p

$\therefore p \vee q$

b. q

$\therefore p \vee q$

Anton is a junior.

∴ (more generally) Anton is a junior or Anton is a senior.

Specialization

a. $p \wedge q$

$\therefore p$

b. $p \wedge q$

$\therefore q$

Ana knows numerical analysis and Ana knows graph algorithms.

$\therefore$ (in particular) Ana knows graph algorithms.

Elimination

a. $p \vee q$

$\sim q$

$\therefore p$

b. $p \vee q$

$\sim p$

$\therefore q$

Transitivity

$$
\begin{aligned}
&p \to q \\
&q \to r \\
\therefore\ &p \to r
\end{aligned}
$$

If n is divisible by 18, then n is divisible by 9.

If n is divisible by 9, then the sum of the digits of n is divisible by 9.

$\therefore$ If n is divisible by 18, then the sum of the digits of n is divisible by 9.

Modus Ponens	$p \to q$ p $\therefore q$	**Elimination**	a. $p \vee q$ $\sim q$ $\therefore p$ b. $p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	a. p $\therefore p \vee q$ b. q $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	a. $p \wedge q$ $\therefore p$ b. $p \wedge q$ $\therefore q$		
Conjunction	p q $\therefore p \wedge q$	**Contradiction Rule**	$\sim p \to \mathbf{c}$ $\therefore p$

Syllogism Inference Rules

Disjunctive Syllogism

If $\neg P$ and $P \vee Q$ are two premises, we can use Disjunctive Syllogism to derive Q.

$$\frac{\begin{array}{c}\neg P \\ P \vee Q\end{array}}{\therefore Q}$$

Example

"The ice cream is not vanilla flavored", $\neg P$

"The ice cream is either vanilla flavored or chocolate flavored", $P \vee Q$

Therefore − "The ice cream is chocolate flavored”

Syllogism Inference Rules

Modus Tollens

If $P \rightarrow Q$ and $\neg Q$ are two premises, we can use Modus Tollens to derive $\neg P$.

$$\begin{array}{c} P \rightarrow Q \\ \neg Q \\ \hline \therefore \neg P \end{array}$$

Example

"If you have a password, then you can log on to facebook", $P \rightarrow Q$

"You cannot log on to facebook", $\neg Q$

Therefore – "You do not have a password "

Syllogism Inference Rules

If $P \rightarrow Q$ and $Q \rightarrow R$ are two premises, we can use Hypothetical Syllogism to derive $P \rightarrow R$

$$\begin{array}{c} P \rightarrow Q \\ Q \rightarrow R \\ \hline \therefore P \rightarrow R \end{array}$$

Example

"If it rains, I shall not go to school", $P \rightarrow Q$

"If I don't go to school, I won't need to do homework", $Q \rightarrow R$

Therefore − "If it rains, I won't need to do homework"

Constructive Dilemma

$$\begin{array}{l} p \to q \\ r \to s \\ p \vee r \\ \hline \therefore q \vee s \end{array}$$

Disjunction of modus ponens

Corresponding Tautology:

$$((p \to q) \wedge (r \to s) \wedge (p \vee r)) \to (q \vee s)$$

Example:

Let *p* be "I will study discrete math."
Let *q* be "I will study computer science."
Let *r* be "I will study protein structures."
Let *s* be "I will study biochemistry."

"If I will study discrete math, then I will study computer science."
"If I will study protein structures, then I will study biochemistry."
"I will study discrete math or I will study protein structures."

"Therefore, I will study computer science or biochemistry."

Destructive Dilemma

Disjunction of modus tollens

$$\begin{array}{l} p \to q \\ r \to s \\ \underline{\neg q \vee \neg s} \\ \therefore \neg p \vee \neg r \end{array}$$

Corresponding Tautology:

$$(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s) \to (\neg p \vee \neg r)$$

Example:

Let *p* be "I will study discrete math."
Let *q* be "I will study computer science."
Let *r* be "I will study protein structures."
Let *s* be "I will study biochemistry."

"If I will study discrete math, then I will study computer science."
"If I will study protein structures, then I will study biochemistry."
"I will not study computer science or I will not study biochemistry."

"Therefore, I will not study discrete math
or I will not study protein structures."

Try it

If the argument is valid, use the rules of inference and the laws of logic to prove the validity of the argument. If the argument is not valid, give truth assignments that demonstrate that it is an invalid argument.

1. (Translate to logic first)

   If I study hard, then I get A's or I get rich.

   I get A's.

   ∴If I don't study hard, then I get rich.

2.

   $p \to q$

   $p \wedge r$

   $r \to u$

   $\therefore q \wedge u$

3.

   $p \to r$

   $q \vee \neg r$

   $q \to r$

   $\therefore \neg p$

Assignments

Use symbols to write the logical form of each argument. If the argument is valid, identify the rule of inference that guarantees its validity.

Given the following information about a computer program, find the mistake in the program.

a. There is an undeclared variable or there is a syntax error in the first five lines.

b. If there is a syntax error in the first five lines, then there is a missing semicolon or a variable name is misspelled.

c. There is not a missing semicolon.

d. There is not a misspelled variable name.

The famous detective Percule Hoirot was called in to solve a baffling murder mystery. He determined the following facts:

a. Lord Hazelton, the murdered man, was killed by a blow on the head with a brass candlestick.
b. Either Lady Hazelton or a maid, Sara, was in the dining room at the time of the murder.
c. If the cook was in the kitchen at the time of the murder, then the butler killed Lord Hazelton with a fatal dose of strychnine.
d. If Lady Hazelton was in the dining room at the time of the murder, then the chauffeur killed Lord Hazelton.
e. If the cook was not in the kitchen at the time of the murder, then Sara was not in the dining room when the murder was committed.
f. If Sara was in the dining room at the time the murder was committed, then the wine steward killed Lord Hazelton.

Is it possible for the detective to deduce the identity of the murderer from these facts? If so, who did murder Lord Hazelton? (Assume there was only one cause of death.)