Discrete Mathematics

Sets and set operations

Set Theory

Set - Definition

- Definition: A set is a (unordered) collection of objects.
  - The students in this class
  - The chairs in this room
- These objects are sometimes called elements or members of the set. A set is said to contain its elements
- The notation $\boldsymbol{a} \in \mathbf{A}$ denotes that a is an element of the set A
- If a is not a member of A, write $\boldsymbol{a} \notin \mathbf{A}$

Describing a Set

- This is a simple set : $S = \{a, b, c, d\}$
- Order is not important: $S = \{a, b, c, d\} = \{b, c, a, d\}$
- Each distinct object is either a member or not; listing more than once does not change the set
- $S = \{a, b, c, d\} = \{a, b, c, b, c, d\}$
- Ellipses (...) may be used to describe a set without listing all of the members when the pattern is clear: $S = \{a, b, c, \dots, z\}$
- Sets can be elements of sets: $\{\{1,2,3\}, a, \{b, c\}\}$

Describing a Set

- **Can use set-builder notation**
- The set is defined by specifying a property that elements of the set have in common.
- The set is described as $A = \{x : p(x)\}$

Example 1 – The set $\{a, e, i, o, u\}$ is written as –

$$A = \{x : \text{x is a vowel in English alphabet}\}$$

Example 2 – The set $\{1, 3, 5, 7, 9\}$ is written as –

$$B = \{x : 1 \leq x < 10 \; and \; (x\%2) \neq 0\}$$

Some Important Sets

N – the set of all natural numbers = $\{1, 2, 3, 4, ...\}$

Z – the set of all integers = $\{....., -3, -2, -1, 0, 1, 2, 3, ...\}$

$\mathbf{Z}^+$ – the set of all positive integers

Q – the set of all rational numbers

R – the set of all real numbers

W – the set of all whole numbers

Cardinality of a Set

- Cardinality of a set S, denoted by $|S|$, is the number of elements of the set. The number is also referred as the cardinal number. If a set has an infinite number of elements, its cardinality is ∞
- **Example** – $|\{1,4,3,5\}| = 4, |\{1,2,3,4,5,\ldots\}| = \infty$

Cardinality of the Cartesian product

- $|S \times T| = |S| * |T|$.

Example:

- A= {John, Peter, Mike}
- B ={Jane, Ann, Laura}
- A x B= {(John, Jane),(John, Ann) , (John, Laura), (Peter, Jane), (Peter, Ann) , (Peter, Laura) , (Mike, Jane) , (Mike, Ann) , (Mike, Laura)}
- $|A \times B| = 9$
- $|A|=3$, $|B|=3$ → $|A|\,|B|= 9$

Definition: A subset of the Cartesian product A x B is called a relation from the set A to the set B.

Properties of Cartesian Product

- 1. The Cartesian Product is non-commutative: A × B ≠ B × A

Example:

$A = \{1, 2\}$, $B = \{a, b\}$

$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$

$B \times A = \{(a, 1), (b, 1), (b, 1), (b, 2)\}$

Therefore as $A \neq B$ we have $A \times B \neq B \times A$

Properties of Cartesian Product

2. A × B = B × A, only if A = B

Proof:

Let $A \times B = B \times A$ *then we have*

$A \subseteq B$ *and* $B \subseteq A$*, it follows that* $A = B$

Properties of Cartesian Product

3. The cardinality of the Cartesian Product is defined as the number of elements in $A \times B$ and is equal to the product of cardinality of both sets: $|A \times B| = |A| * |B|$

Proof:

Let $a \in A$ then the number of ordered pair (a, b) such that $b \in B$ is $|B|$.

Therefore we have $|B|$ choices for b for each a where $a \in A$ therefore the number of element in $A \times B$ is $|A| * |B|$.

Properties of Cartesian Product

4. A × B = {∅}, if either A = {∅} or B = {∅}

Proof:

We know $|\{\emptyset\}| = 0$.

Now we have $|A \times B| = |\{\emptyset\}| = 0$

As $|A \times B| = |A| * |B|$, we get $|A| * |B| = 0$

Thus atleast one of $|A|$ or $|B|$ should be equal to 0

Hence either $A = \{\emptyset\}$ or $B = \{\emptyset\}$

- If there are two sets X and Y,

  - $|X| = |Y|$ denotes two sets X and Y having same cardinality. It occurs when the number of elements in X is exactly equal to the number of elements in Y. In this case, there exists a bijective function 'f' from X to Y.
  - $|X| \leq |Y|$ denotes that set X's cardinality is less than or equal to set Y's cardinality. It occurs when number of elements in X is less than or equal to that of Y. Here, there exists an injective function 'f' from X to Y.
  - $|X| < |Y|$ denotes that set X's cardinality is less than set Y's cardinality. It occurs when number of elements in X is less than that of Y. Here, the function 'f' from X to Y is injective function but not bijective.
  - $If\ |X| \leq |Y|$ and $|X| \geq |Y|$ then $|X| = |Y|$. The sets X and Y are commonly referred as equivalent sets.

Types of Sets

Sets can be classified into many types. Some of which are finite, infinite, subset, universal, proper, singleton set, etc.

- Finite Set

A set which contains a definite number of elements is called a finite set.

Example – $S = \{x \,|\, x \in N \text{ and } 70 > x > 50\}$

Types of Sets

Infinite Set

- A set which contains infinite number of elements is called an infinite set.
- If a set is not finite, it is called an infinite set because the number of elements in that set is not countable, and also we cannot represent it in Roster form. Thus, infinite sets are also known as **uncountable sets**.
- Examples of Infinite Sets
  - A set of all whole numbers, W= {0, 1, 2, 3, 4,…}
  - A set of all points on a line
  - The set of all integers

Types of Sets

- Subset

A set X is a subset of set Y (Written as X⊆Y) if every element of X is an element of set Y.

Example 1 – Let, $X = \{1,2,3,4,5,6\}$ and $Y = \{1,2\}$. Here set Y is a subset of set X as all the elements of set Y is in set X. Hence, we can write $Y \subseteq X$.

Example 2 – Let, $X = \{1,2,3\}$ and $Y = \{1,2,3\}$. Here set Y is a subset (Not a proper subset) of set X as all the elements of set Y is in set X. Hence, we can write $Y \subseteq X$.

Types of Sets

- Proper Subset

The term “proper subset” can be defined as **“subset of but not equal to”**. A Set X is a proper subset of set Y (Written as X⊂Y) if every element of X is an element of set Y and |X|<|Y|

Example – Let, $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{1, 2\}$. Here set $Y \subset X$ since all elements in Y are contained in X too and X has at least one element is more than set Y.

Types of Sets

Universal Set

- It is a collection of all elements in a particular context or application. All the sets in that context or application are essentially subsets of this universal set. Universal sets are represented as U

Example − We may define U as the set of all animals on earth. In this case, set of all mammals is a subset of U, set of all fishes is a subset of U, set of all insects is a subset of U, and so on.

Types of Sets

Empty Set or Null Set

- An empty set contains no elements. It is denoted by ∅
- As the number of elements in an empty set is finite, empty set is a finite set. The cardinality of empty set or null set is zero.

Example – $S = \{x \mid x \in N \text{ and } 7 < x < 8\} = \emptyset$

Types of Sets

Singleton Set or Unit Set

- Singleton set or unit set contains only one element. A singleton set is denoted by {s}

Example − $S = \{x \mid x \in N,\ 7 < x < 9\} = \{8\}$

Types of Sets

Equivalent Set

- If the cardinalities of two sets are same, they are called equivalent sets.
- Example – If A={1,2,6} and B={16,17,22}

they are equivalent as cardinality of A is equal to the cardinality of B. i.e. |A|=|B|=3

Types of Sets

Disjoint Set

- Two sets A and B are called disjoint sets if they do not have even one element in common. Therefore, disjoint sets have the following properties –
- Example – Let, A={1,2,6} and B={7,9,14}

there is not a single common element, hence these sets are overlapping sets.

Types of Sets

Equal Set

- If two sets contain the same elements they are said to be equal.
- Example – If A={1,2,6} and B={6,1,2}

they are equal as every element of set A is an element of set B and every element of set B is an element of set A.

Subset

- Suppose every element in a set A is also an element of a set B, that is, suppose a ∈ A implies a ∈ B. Then A is called a subset of B. We also say that A is contained in B or that B contains A. This relationship is written

$$A \subseteq B \quad \text{or} \quad B \supseteq A$$

Subset Properties

Theorem: $S \subseteq S$

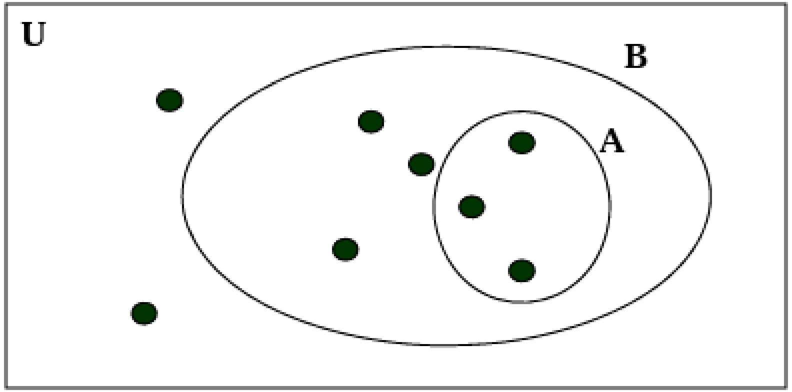
- **Any set S is a subset of itself**

Proof:

- the definition of a subset says: all elements of a set A must be also elements of B: $\forall x\ (x \in A \rightarrow x \in B)$.
- Applying this to S we get:
- $\forall x\ (x \in S \rightarrow x \in S)$ which is trivially **True**
- End of proof

A Proper Subset

Definition: A **set** **A** is said to be a **proper subset** of B if and only if $A \subseteq B$ and $A \neq B$. We denote that A is a proper subset of B with the notation $A \subset B$.



Example: A={1,2,3} B ={1,2,3,4,5}

Is: $A \subset B$? Yes.

Power set

Definition: Given a set S, the **power set** of S is the set of all subsets of S. The power set is denoted by **P(S).**

Examples:

- Assume an empty set $\varnothing$
- What is the power set of $\varnothing$? $P(\varnothing) = \{ \varnothing \}$
- What is the cardinality of $P(\varnothing)$? $|P(\varnothing)| = 1$.

- Assume set $\{1\}$
- $P(\{1\}) = \{ \varnothing, \{1\} \}$
- $|P(\{1\})| = 2$

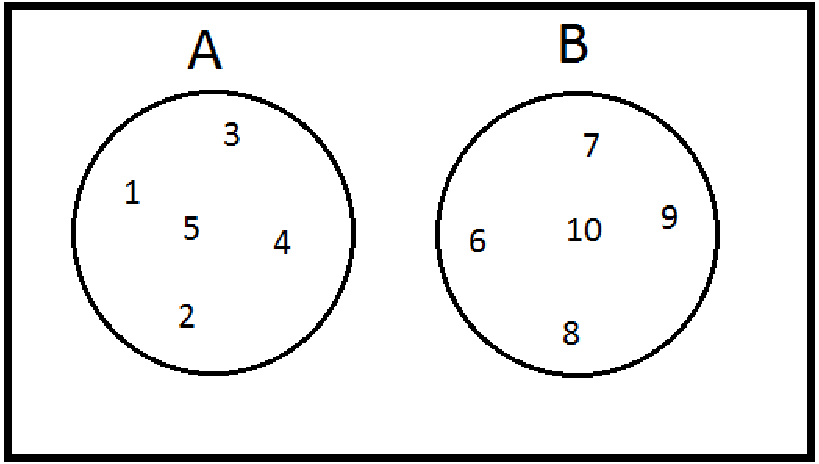
VENN Diagram

What is a Venn Diagram?

- A diagram used to represent all possible relations of different sets. A Venn diagram can be represented by any closed figure, whether it be a Circle or a Polygon (square, hexagon, etc.). But usually, we use circles to represent each set.
- To draw a Venn diagram we first draw a rectangle which will contain every item we want to consider. Since it contains every item, we can refer to it as "the universe."

Venn Diagram

- Suppose now we wanted a set A which is a list of numbers containing 1 up to 5, and a set B which is a list of numbers containing 6 to 10. To represent each set we use circles:

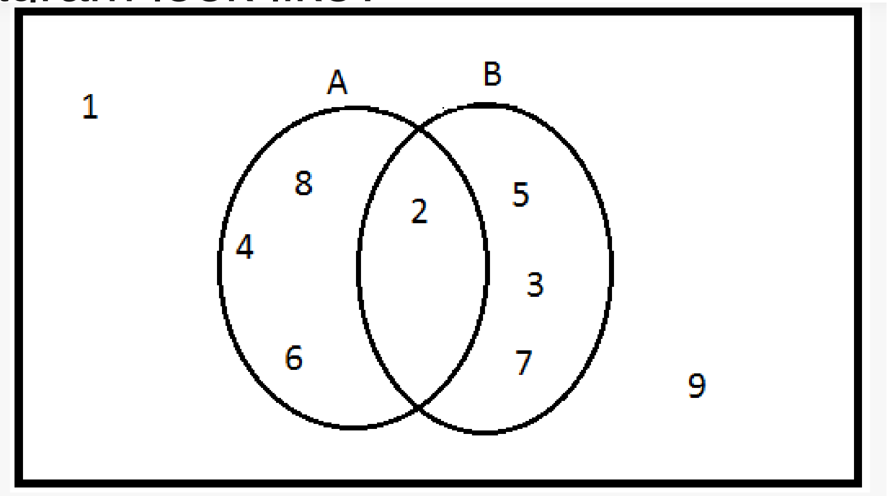


Venn Diagram

- How about if sets A and B have something in common? We can't simply draw two separate circles, as that won't form any logical relationship between the two. As you can see below, the way to show that relationship does indeed exist, where we merge the two circles partially.

Venn Diagram

- In a universal set of all positive integers less than 10, let *A* be the set of all positive even integers less than 10, and *B* the set of all positive prime integers less than 10. Then what will the Venn diagram look like?



Set Operations

Definition

Let A and B be subsets of a universal set U.

1. The **union** of A and B, denoted $A \cup B$, is the set of all elements that are in at least one of A or B.
2. The **intersection** of A and B, denoted $A \cap B$, is the set of all elements that are common to both A and B.
3. The **difference** of B minus A (or **relative complement** of A in B), denoted $B - A$, is the set of all elements that are in B and not A.
4. The **complement** of A, denoted A^c, is the set of all elements in U that are not in A.

Symbolically:

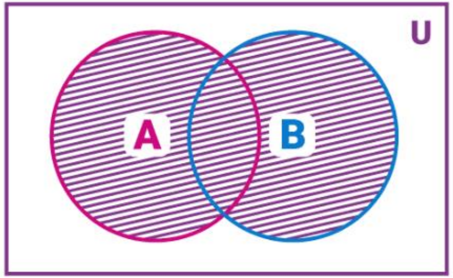
$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$
$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\}$$
$$A^c = \{x \in U \mid x \notin A\}.$$

Set Operations: Union

- Formal definition for the union of two sets: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Properties of the union operation

‣ $A \cup \varnothing = A$	Identity law
‣ $A \cup \boldsymbol{U} = \boldsymbol{U}$	Domination law
‣ $A \cup A = A$	Idempotent law
‣ $A \cup B = B \cup A$	Commutative law
‣ $A \cup (B \cup C) = (A \cup B) \cup C$	Associative law

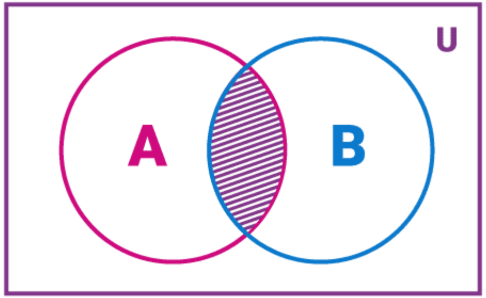


Set Operations: Intersection

- A intersection B is given by: $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Properties of the intersection of sets operation:

- $A \cap B = B \cap A$
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $\varphi \cap A = \varphi$; $U \cap A = A$
- $A \cap A = A$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$



Set Operations: Disjoint

Definition

Two sets are called **disjoint** if, and only if, they have no elements in common.
Symbolically:

$$A \text{ and } B \text{ are disjoint} \quad \Leftrightarrow \quad A \cap B = \varnothing.$$

Further examples

- $\{1, 2, 3\}$ and $\{3, 4, 5\}$ are not disjoint
- $\{a, b\}$ and $\{3, 4\}$ are disjoint
- $\{1, 2\}$ and $\varnothing$ are disjoint
  - Their intersection is the empty set
- $\varnothing$ and $\varnothing$ are disjoint!
  - Their intersection is the empty set

Set Operations: Power Sets

Definition

Given a set A, the **power set** of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

Find the power set of the set $\{x, y\}$. That is, find $\mathcal{P}(\{x, y\})$.

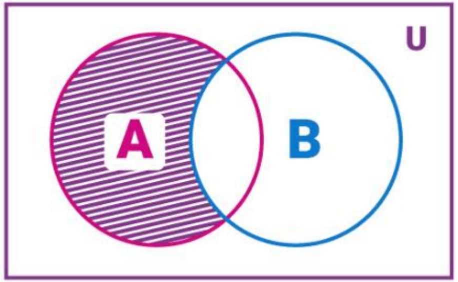
Solution $\mathcal{P}(\{x, y\})$ is the set of all subsets of $\{x, y\}$. In Section 6.2 we will show tha
$\emptyset$ is a subset of every set, and so $\emptyset \in \mathcal{P}(\{x, y\})$. Also any set is a subset of itself, sc
$\{x, y\} \in \mathcal{P}(\{x, y\})$. The only other subsets of $\{x, y\}$ are $\{x\}$ and $\{y\}$, so

$$\mathcal{P}(\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$$

Set Operations: Difference

- **Definition:** let A and B be sets. The difference of A and B, denoted by A – B, is the set containing the elements of A that are not in B.
- The difference of A and B is also called the complement of B with respect to A

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$



Set Operations: Symmetric Difference

Definition: the **symmetric difference** of **A** and **B**, denoted by

$$A \oplus B$$

is the set

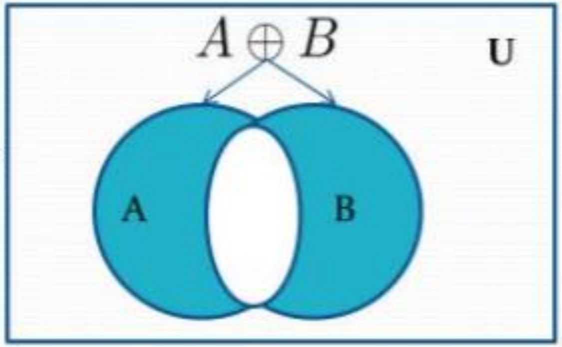
$$(A - B) \cup (B - A)$$

Example:

$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 2, 3, 4, 5\}$ $B = \{4, 5, 6, 7, 8\}$

Solution: $\{1, 2, 3, 6, 7, 8\}$



Venn Diagram

Set Operations: Complement

Definition: if A is a set, then the **complement** of the A (with respect to U), denoted by $\bar{A}$ is the set $U - A$

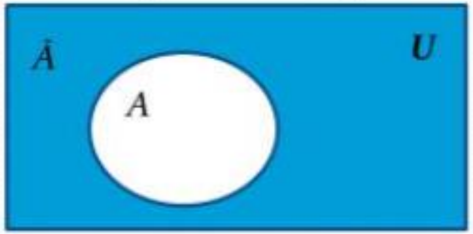
$$\bar{A} = \{x \in U \mid x \notin A\}$$

The complement of A is sometimes denoted by A^c.

Example: if U is the positive integers less than 100 (i.e., 1, 2, ..., 99), what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

Venn Diagram for Complement



Set Operations: Complement

Further examples (assuming $\boldsymbol{U} = \mathbf{Z}$)

- $\{1, 2, 3\}^c = \{ \ldots, -2, -1, 0, 4, 5, 6, \ldots \}$
- $\{a, b\}^c = \mathbf{Z}$

Properties of complement sets

$(A^c)^c = A$	Complementation law
$A \cup A^c = \boldsymbol{U}$	Complement law
$A \cap A^c = \varnothing$	Complement law

Try it

Let $A = \{a, b, c\}$, $B = \{b, c, d\}$, and $C = \{b, c, e\}$.

a. Find $A \cup (B \cap C)$, $(A \cup B) \cap C$, and $(A \cup B) \cap (A \cup C)$. Which of these sets are equal?

b. Find $A \cap (B \cup C)$, $(A \cap B) \cup C$, and $(A \cap B) \cup (A \cap C)$. Which of these sets are equal?

c. Find $(A - B) - C$ and $A - (B - C)$. Are these sets equal?

Sets Identity

- Identity laws

$$A \cup \emptyset = A \qquad A \cap U = A$$

- Domination laws

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

- Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

- Complementation law

$$\overline{(\overline{A})} = A$$

Sets Identity

- Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

- Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Sets Identity

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

- Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \emptyset$$

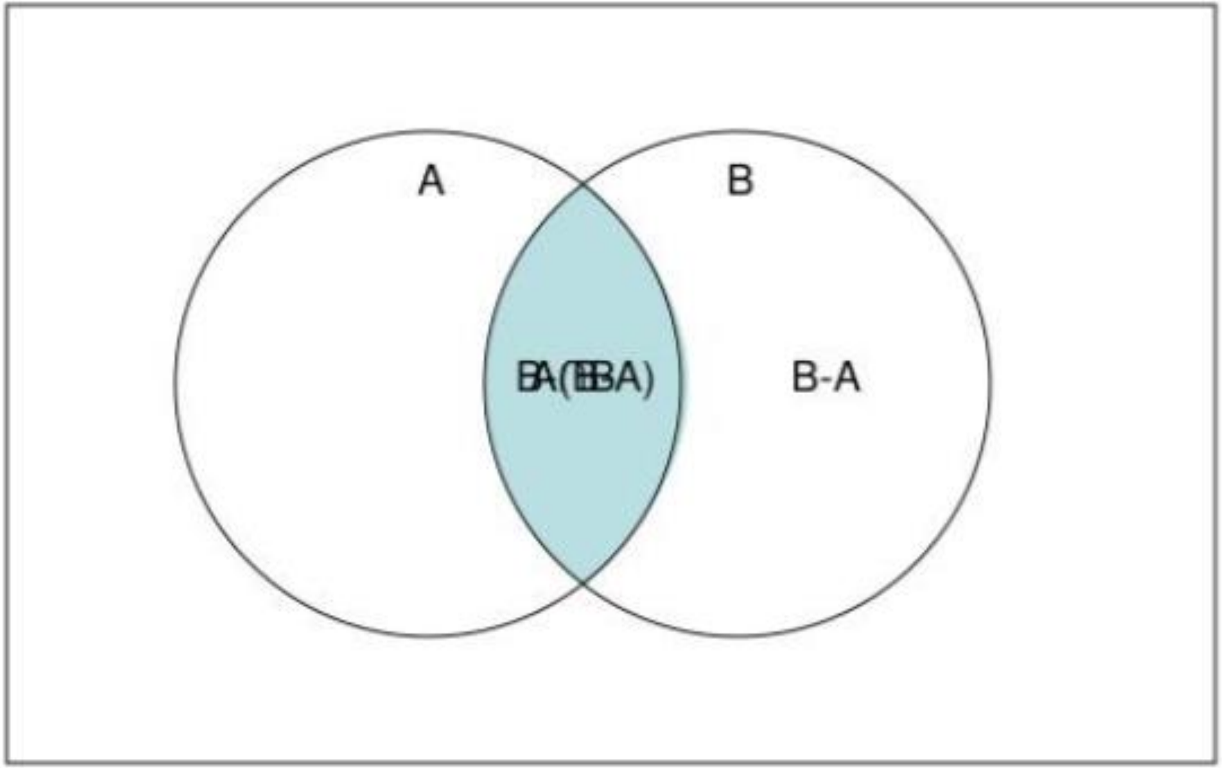
How to prove a set identity

For example: $A \cap B = B - (B - A)$

Four methods:

- Use the basic set identities
- Use membership tables
- Prove each set is a subset of each other
- Use set builder notation and logical equivalences

$A \cap B = B - (B - A)$



♈ $A \cap B = A - (A - B)$

$$\begin{aligned}
\text{Proof) } A - (A - B) &= A - (A \cap B^c) \\
&= A \cap (A \cap B^c)^c \\
&= A \cap (A^c \cup B) \\
&= (A \cap A^c) \cup (A \cap B) \\
&= \varnothing \cup (A \cap B) \\
&= A \cap B
\end{aligned}$$

Example: construct a membership table to show that the distributive law holds.

Solution: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0