

# Advanced Graph Theory

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## Disclaimer

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To separate the contents of the course to actual additions or out of context information, a black band will be added by its side like the globing this comment.

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# Chapter 1

# Fundamental Concepts

## 1.1. Types Of Graphs

**Definition 1.1.1 (Undirected Graph):** An undirected graph  $G$  is an ordered pair  $G = (V, E)$  where  $V$  is a finite set called the vertex set (vertices) and  $E \subset \{\{u, v\} \mid u, v \in V\}$  called the set of edges.

**Definition 1.1.2 (Directed Graph):** A directed graph  $G$  is an ordered pair  $G = (V, A)$  where  $V$  is the set of vertices and  $A \subset V \times V$  called the set of arcs or directed edges.

**Definition 1.1.3 (Unweighted Graph):** An unweighted graph  $G$  is a graph where all edges are considered equal.

**Definition 1.1.4 (Weighted Graph):** A weighted graph  $G = (V, E)$  is a graph where there is a function  $w : E \rightarrow \mathbb{R}$  where for each edge  $e \in E$ , the weight of  $e$  is  $w(e)$ .

## 1.2. Concepts On Vertices & Edges

**Definition 1.2.5 (Adjacent/Incident/Isolated):** Let  $G = (V, E)$  a graph:

1. Two vertices  $u, v \in V$  are adjacent or neighbors if  $\{u, v\} \in E$ .
2. A vertex  $v$  is called incident to an edge  $e$  if  $v$  is an endpoint of  $e$ .

3. A vertex with no incident edges is called isolated.
4. The order of  $G$  is  $\# V$ .
5. The size of  $G$  is  $\# E$ .
6. The open neighborhood of  $v \in V$  is  $\mathcal{N}(v) = \{u \in V \mid \{u, v\} \in E\}$ .
7. The closed neighborhood of  $v \in V$  is  $[v] = \mathcal{N}(v) \cup \{v\}$ .
8. A degree of a vertex  $v$  denoted  $d(v)$  or  $\deg(v)$  is the number of edges incident to  $v$ , that is  $\deg(v) = \# \mathcal{N}(v)$ .
9. If  $G$  is directed then we define the following:
  1. The out neighborhood of  $v \in V$  is  $\mathcal{N}^+(v) = \{u \in V \mid (v, u) \in A\}$ .
  2. The in neighborhood of  $v \in V$  is  $\mathcal{N}^-(v) = \{u \in V \mid (u, v) \in A\}$ .
  3. The in/out degree are  $\deg^-(v) = \# \mathcal{N}^-(v)$ ,  $\deg^+(v) = \# \mathcal{N}^+(v)$ .
10. The number of edges incident to  $v$  with maximum degree of  $G$ ,  $\Delta(G) = \max_{v \in V} d(v)$  and minimum degree is  $\delta(G) = \min_{v \in V} d(v)$ .

**Lemma 1.2.6 (Handshake):** For an undirected graph  $G = (V, E)$  then

$$\sum_{v \in V} \deg(v) = 2 \# E$$

*Proof.* Every edge  $\{u, v\} \in E$  contributes exactly 1 to  $\deg(u)$  and 1 to  $\deg(v)$  thus adding 2 to  $\sum_{v \in V} \deg(v)$ , so all edges counted twice in the sum of all edges.  $\square$

**Theorem 1.2.7:** The number of vertices with odd degree is even.

*Proof.* Let  $V_o = \{v \in V \mid d(v) \text{ odd}\}$  and  $V_e = \{v \in V \mid d(v) \text{ even}\}$ , by the handshake lemma we have

$$2 \# E = \sum_{v \in V} \deg(v) = \sum_{v \in V_o} \deg(v) + \sum_{v \in V_e} \deg(v)$$

given that  $2 \# E$  is even and  $\sum_{v \in V_e} \deg(v)$  is even then necessarily  $\sum_{v \in V_o} \deg(v)$  is even, and since the sum of odd numbers is even if and only if the number of odd numbers is even then  $\# V_o$  is even so there is an even number of odd degree vertices.  $\square$

## 1.3. Simple & Multigraphs

**Definition 1.3.8 (Simple/Multi Graph):** A graph  $G$  is said to be simple if it has no loops and no multiple edges. A multigraph is a graph that may have loops and have multiple edges.

**Theorem 1.3.9:** In any simple graph  $G$  with order  $n \geq 2$ , there exists at least two vertices with same degree.

*Proof.* The possible degrees in a simple graph is  $\{0, \dots, n - 1\}$ , however a graph cannot have a vertex with degree 0 and  $n - 1$  at the same time, so either it has vertices of degree  $\{1, \dots, n - 1\}$  or  $\{0, \dots, n - 2\}$ , and since there are  $n$  vertices and  $n - 1$  choices for degrees, then by the pigeonhole principle, there are at least two vertices with the same degrees.  $\square$

**Proposition 1.3.10:** There are  $\binom{n}{2}$  maximum edges in a simple graph.

**Definition 1.3.11 (Walk/Trail):** Let  $G = (V, E)$  be a graph

1. A walk of length  $k$  is a sequence  $v_1, e_2, v_2, \dots, e_k, v_k$  with  $e_i = \{v_{i-1}, v_i\} \in E$ .
2. A trail is a walk with no repeated edges.
3. A path is a walk with no repeated vertices.
4. A walk is closed if  $v_0 = v_k$ .

**Theorem 1.3.12:** If there is a walk from vertex  $u$  to  $v$  then there is also a path from  $u$  to  $v$ .

*Proof.* You can do it, I suppose  $\square$

**Definition 1.3.13 (Circuit/Cycle):**

1. A circuit is a closed trail.
2. A cycle is a closed path.

3. A cycle of length  $k$  is denoted  $C_k$ , unless otherwise specified.
4. The length of the cycle is the number of edges in it.
5. A graph  $G$  is connected if for every pair of vertices  $u, v \in V$  there exists a path from  $u$  to  $v$ , otherwise it is disconnected.
6. A subgraph  $H$  of  $G$  is a graph such that  $V(H) \subset V(G)$  and  $E(H) \subset E(G)$ .
7. A connected component of  $G$  is a maximum connected subgraph of  $G$ .
8. A spanning subgraph  $H$  of  $G$  is a subgraph such that  $V(H) = V(G)$ .

**Theorem 1.3.14:** In a connected graph, any two longest paths share at least one vertex.