1. Tutorial Series 1: Remainders

Exercise 1.1: *Consider the ring of polynomials* $\mathbb{Z}[X]$ *with indeterminate* X.

Question 1.1.1: *Show that* $\mathbb{Z}[X]/(X) \cong \mathbb{Z}$.

Take the map $\varphi : \mathbb{Z}[X] \to \mathbb{Z}$ such that $\varphi(a_0 + a_1 X + \ldots + a_n X^n) = a_0$, φ is a ring homomorphism with $\operatorname{Ker} \varphi = (X)$ and $\operatorname{Im} \varphi = \mathbb{Z}$ then by the first isomorphism theorem $\mathbb{Z}[X]/(X) \cong \mathbb{Z}$.

Question 1.1.2: *Show that* $(2) + (x)$ *is not generated by a singleton.*

Suppose there exists $P \in \mathbb{Z}[X]$ such that $(P) = (2) + (X)$, since $2 \in (2) + (X)$ then $2 \in (P)$ so $2 = PQ$ with $Q \in \mathbb{Z}[X]$ but that means that $\deg(P) + \deg(Q) = 0 \Rightarrow \deg P = 0$ so $P = p \in \mathbb{Z}$, since $2 \in (p)$ then $p \mid 2 \Rightarrow p = \pm 1$ or $p = \pm 2$ which are both impossible since $1 \in \mathbb{Z}[X] \setminus ((2) + (X))$ and $2 + X \in (2) + (X) \setminus (2)$.

Question 1.1.3: *Deduce that* $\mathbb{Z}[X]$ *is not a PID.*

- From 1.1.1 we have that $\mathbb{Z}[X]$ is a PID and X is irreducible then (X) is a maximal ideal so $\mathbb{Z}[X]/(X)$ is a field but $\mathbb{Z}[X]/(X) \cong \mathbb{Z}$ which means that $\mathbb{Z}$ is a field, contradiction.
- From 1.1.2 we have that $(2) + (x)$ is an ideal of $\mathbb{Z}[X]$ but it is not a principal ideal.

Question 1.1.4: *Is* $\mathbb{Z}[X]$ *a Euclidean domain ?*

$\mathbb{Z}[X]$ is not a Euclidean domain since it is not a PID.

Exercise 1.2: