DFAs & NFAs

Exercise 1: *Let $M = (\Sigma, Q, \delta, q_0, F)$ a DFA with*
- $\Sigma = \{a, b, c\}$.
- $Q = \{q_0, q_1, q_2, q_4\}$.
- $F = \{q_0\}$.
- *δ is defined with the table*

$q \in Q$	$c \in \Sigma$	$\delta(q, c)$
q_0	b, c	q_4
q_0	a	q_1
q_1	a, c	q_4
q_1	b	q_2
q_2	a, b	q_4
q_2	c	q_0
q_4	a, b, c	q_4

1. *Draw the diagram of the DFA M.*
2. *Check which of the following strings are in the language of M:*
   - *aaa.*
   - *ababc.*
   - *abcabc.*
3. *Deduce $\mathcal{L}(M)$.*

Exercise 2:
1. *Prove that for $x \in \mathbb{N}, x = \overline{b_n b_{n-1} \ldots b_0}^2$, 2 divides x if and only if $b_0 = 0$.*
2. *Make a DFA taking the alphabet $\Sigma = \{0, 1\}$, and accepts the string $S = s_1 \ldots s_n$ if and only if the number $\overline{s_1 \ldots s_n}^2$ is divisible by 2.*

Exercise 3: *Consider the two deterministic automatons*

$$M_1 = (\Sigma, Q_1, \delta_1, q_0^1, F_1).$$
$$M_2 = (\Sigma, Q_2, \delta_2, q_0^2, F_2).$$

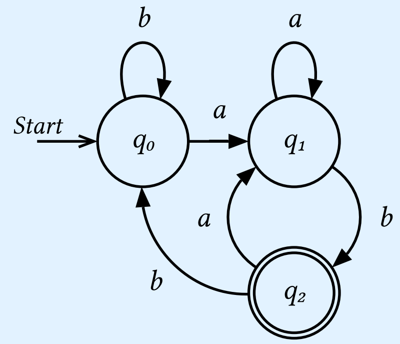
and define the automaton

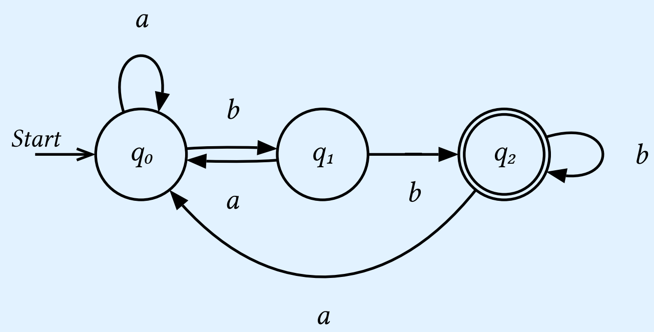
$$M = M_1 \times M_2 = (\Sigma, Q, \delta, q_0, F).$$

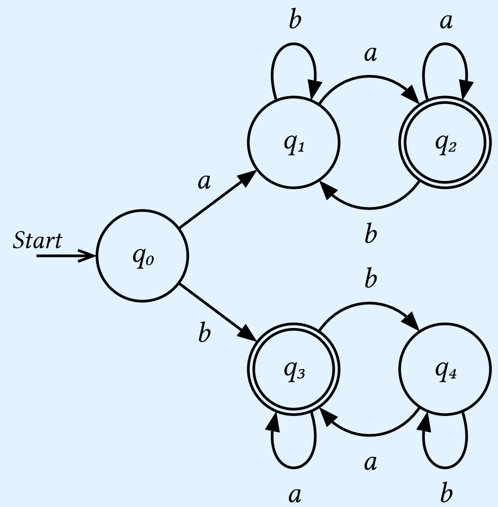
with the components
- $Q = Q_1 \times Q_2$.
- $\delta((q_1, q_2), c) = (\delta(q_1, c), \delta(q_2, c))$.
- $q_0 = (q_0^1, q_0^2)$.
- $F = F_1 \times F_2$.

1. *Prove that $\mathcal{L}(M) = \mathcal{L}(M_1) \cap \mathcal{L}(M_2)$.*
2. *By changing F, find a way to obtain a machine M such that $\mathcal{L}(M) = \mathcal{L}(M_1) \cup \mathcal{L}(M_2)$.*
3. *Try to find a way to also define a machine M such that $\mathcal{L}(M) = \Sigma^* \setminus \mathcal{L}(M_1)$.*
4. *Deduce that the set of regular languages is an algebra of sets.*

Exercise 4: *Find the languages of the following automatons*







Exercise 5: *Construct an NFA for each of the following languages*
1. $\mathcal{L} = \{a^n b^m \mid n, m \in \mathbb{N}\}$.
2. $\mathcal{L} = \{aSa \mid S \in \Sigma^*\}$.
3. *for $x, y \in \mathbb{N}$, $\mathcal{L}_{x,y} = \{a^{xn+y} \mid n \in \mathbb{N}\}$.*

Exercise 6 (Pumping Lemma): *Let M be a DFA with p states and let $S = s_1 s_2 \ldots s_n \in \Sigma^*$ a string.*
1. *Suppose that $S \in \mathcal{L}(M)$ and $n > p$, conclude that there exists at least one loop inside of the DFA, such that S can be divided into 3 sub-words, $x, y, z \in \Sigma^*$ such that $S = xyz$ and y is detected by the loop.*

2. *Deduce that if $S = xyz$ like the previous question then $S = xyz \in \mathcal{L}(M) \Rightarrow \forall i \in \mathbb{N}, xy^i z \in \mathcal{L}(M)$, we call that pumping, that is, we can pump a string and keep it inside of the language which is a property of the regular languages.*
3. *Deduce that if a language does not accept pumping then it is not regular and thus there is no DFA that has that language.*

> **Lemma:** *Let $\mathcal{L}$ be a language. If for all $p \geq 1$ there exists a string $S \in \mathcal{L}$ such that $|S| \geq p$ and for any $x, y, z \in \Sigma^*, |xy| \leq p, |y| > 0$ $S = xyz$ and there exists $i \in \mathbb{N}, xy^i z \notin \mathcal{L}$ then $\mathcal{L}$ is not a regular language thus there is no DFA that has language $\mathcal{L}$.*

4. *Apply the lemma on $\mathcal{L} = \{a^n b^n \mid n \in \mathbb{N}\}$ and deduce that $\mathcal{L}$ is not regular.*