

1. Tutorial Series 1: Embeddings

Exercise 1.1: Consider the ring of polynomials $\mathbb{Z}[X]$ with indeterminate X .

Question 1.1.1: Show that $\mathbb{Z}[X]/(X) \cong \mathbb{Z}$.

Take the map $\varphi : \mathbb{Z}[X] \rightarrow \mathbb{Z}$ such that $\varphi(a_0 + a_1X + \dots + a_nX^n) = a_0$, φ is a ring homomorphism with $\text{Ker } \varphi = (X)$ and $\text{Im } \varphi = \mathbb{Z}$ then by the first isomorphism theorem $\mathbb{Z}[X]/(X) \cong \mathbb{Z}$.

Question 1.1.2: Show that $(2) + (x)$ is not generated by a singleton.

Suppose there exists $P \in \mathbb{Z}[X]$ such that $(P) = (2) + (X)$, since $2 \in (2) + (X)$ then $2 \in (P)$ so $2 = PQ$ with $Q \in \mathbb{Z}[X]$ but that means that $\deg(P) + \deg(Q) = 0 \Rightarrow \deg P = 0$ so $P = p \in \mathbb{Z}$, since $2 \in (p)$ then $p \mid 2 \Rightarrow p = \pm 1$ or $p = \pm 2$ which are both impossible since $1 \in \mathbb{Z}[X] \setminus ((2) + (X))$ and $2 + X \in (2) + (X) \setminus (2)$.

Question 1.1.3: Deduce that $\mathbb{Z}[X]$ is not a PID.

- From 1.1.1 we have that $\mathbb{Z}[X]$ is a PID and X is irreducible then (X) is a maximal ideal so $\mathbb{Z}[X]/(X)$ is a field but $\mathbb{Z}[X]/(X) \cong \mathbb{Z}$ which means that \mathbb{Z} is a field, contradiction.
- From 1.1.2 we have that $(2) + (x)$ is an ideal of $\mathbb{Z}[X]$ but it is not a principle ideal.

Question 1.1.4: Is $\mathbb{Z}[X]$ a Euclidean domain ?

$\mathbb{Z}[X]$ is not a Euclidean domain since it is not a PID.

Exercise 1.2: Find embeddings and automorphisms in the following cases.

Question 1.2.1: $K = \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt[4]{2}), \mathbb{Q}(\sqrt[3]{5})$ and $L = \mathbb{C}$.

- $K = \mathbb{Q}(\sqrt{2})$: we have that $\text{Irr}(\sqrt{2}, K, X) = X^2 - 2$ since it is a monic 2-Eisenstein that nullifies $\sqrt{2}$ and we have that $\text{Char } \mathbb{Q} = 0$ and $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ so there are only two embeddings

$$\begin{aligned}\sigma_1 : \sqrt{2} &\mapsto \sqrt{2} \\ \sigma_2 : \sqrt{2} &\mapsto -\sqrt{2}\end{aligned}$$

which are both automorphisms.

- $K = \mathbb{Q}(\sqrt[4]{2})$: we have that $\mathbb{Q}(\sqrt[4]{2}) = \mathbb{Q}(\sqrt{2})(\sqrt[4]{2})$ then $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[4]{2}), \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2 \cdot 2 = 4$ and $X^4 - 2$ nullifies $\sqrt[4]{2}$ then we have that $\text{Irr}(\sqrt[4]{2}, \mathbb{Q}, X) = X^4 - 2$, and we get that the set of conjugates of $\sqrt[4]{2}$ over \mathbb{Q} are $\{\sqrt[4]{2}, -\sqrt[4]{2}, i\sqrt[4]{2}, -i\sqrt[4]{2}\}$ and since

$\text{Char}(\mathbb{Q}) = 0$ then the following 4 embeddings are the only ones

$$\begin{aligned}\sigma_1 : \sqrt[4]{2} &\mapsto \sqrt[4]{2} & \sigma_2 : \sqrt[4]{2} &\mapsto -\sqrt[4]{2} \\ \sigma_3 : \sqrt[4]{2} &\mapsto i\sqrt[4]{2} & \sigma_4 : \sqrt[4]{2} &\mapsto -i\sqrt[4]{2}\end{aligned}$$

and only σ_1, σ_2 are automorphisms.

- $K = \mathbb{Q}(\sqrt[3]{5})$: we have that $X^3 - 5$ is 5-Eisenstein and nullifies $\sqrt[3]{5}$ then $\text{Irr}(\sqrt[3]{5}, \mathbb{Q}, X) = X^3 - 5$ so the conjugates of $\sqrt[3]{5}$ over \mathbb{Q} are $\{\sqrt[3]{5}, j\sqrt[3]{5}, j^2\sqrt[3]{5}\}$ with $j = e^{\frac{2\pi}{3}i}$, thus we get exactly 3 embeddings

$$\begin{aligned}\sigma_1 : \sqrt[3]{5} &\mapsto \sqrt[3]{5} \\ \sigma_2 : \sqrt[3]{5} &\mapsto j\sqrt[3]{5} \\ \sigma_3 : \sqrt[3]{5} &\mapsto j^2\sqrt[3]{5}\end{aligned}$$

and only σ_1 is an automorphism.

Question 1.2.2: Find all $\mathbb{Q}(\sqrt{2})$ -embeddings of $\mathbb{Q}(\sqrt[4]{2})$ into \mathbb{C} .

we have that $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}(\sqrt{2})] = 2$ and its easy to verify that $\text{Irr}(\sqrt[4]{2}, \mathbb{Q}(\sqrt{2}), X) = X^2 - \sqrt{2}$, thus the conjugates of $\sqrt[4]{2}$ over $\mathbb{Q}(\sqrt{2})$ are $\{\sqrt[4]{2}, -\sqrt[4]{2}\}$ thus we get only two embeddings since $\text{Char } \mathbb{Q}(\sqrt{2}) = 0$ which are

$$\begin{aligned}\sigma_1 : \sqrt[4]{2} &\mapsto \sqrt[4]{2} \\ \sigma_2 : \sqrt[4]{2} &\mapsto -\sqrt[4]{2}\end{aligned}$$

Question 1.2.3: Determine all embeddigns of $K = \mathbb{F}_2(\alpha)$ into an algebraic closure \overline{K} and all automorphisms with $\alpha^2 + \alpha + 1 = 0$ then $\alpha^3 + \alpha^2 + 1 = 0$.

- $\alpha^2 + \alpha + 1 = 0$: let $P(X) = X^2 + X + 1$, $P(0) = P(1) = 1 \neq 0$ thus P is irreducible over $\mathbb{F}_2[X]$ and $P(\alpha) = 0$ so $\text{Irr}(\alpha, \mathbb{F}_2, X) = P(X)$, $[\mathbb{F}_2(\alpha) : \mathbb{F}_2] = 2$ thus there are two conjugates of α over \mathbb{F}_2 . $P(\alpha^2) = \alpha^4 + \alpha^2 + 1$, we have $\alpha^2 + \alpha + 1 = 0 \Rightarrow \alpha^2 = \alpha + 1 \Rightarrow \alpha^3 = 1 \Rightarrow \alpha^4 = \alpha$ thus $P(\alpha^2) = \alpha^2 + \alpha + 1 = 0$. So the conjugates are $\{\alpha, \alpha^2\}$ and thus we get the embeddings are

$$\begin{aligned}\sigma_1 : \alpha &\mapsto \alpha \\ \sigma_2 : \alpha &\mapsto \alpha^2\end{aligned}$$

which are both automorphisms.

Question 1.2.4: Determine all embeddigns of $K = \mathbb{F}_3(\beta)$ into an algebraic closure \overline{K} and all automorphisms with $\beta^2 + \beta + 2 = 0$ then $\beta^3 + \beta^2 + 2 = 0$.

- the process is just the same as before.

Exercise 1.3: Let L/K be an algebraic extension and Ω an algebraically closed field.

Question 1.3.1: Let $\theta \in L$, and $\tau : K \rightarrow \Omega$ an embedding, show that τ can be extended to $\sigma : K(\theta) \rightarrow \Omega$.

Question 1.3.2: If $\text{Char } K = 0$ and $[K(\theta) : K] = n$ then there is exactly n extensions to $K(\theta)$.

Question 1.3.3: Apply the above to each embedding $\sigma : \mathbb{Q}(\sqrt{2}) \rightarrow \mathbb{C}$ with $\theta = \sqrt[4]{2}$.

Question 1.3.4: Using the 1.3.1 and Zorn's Lemma, prove that τ can be extended to $\sigma : L \rightarrow \Omega$.

Exercise 1.4: Find the primitive element of the following extensions

1. $\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}$.
2. \mathbb{C}/\mathbb{R} .
3. $\mathbb{Q}(\sqrt{2}, i, \sqrt{3})/\mathbb{Q}(\sqrt{3})$.
4. $\mathbb{Q}(\sqrt{2}, i, \sqrt{3})/\mathbb{Q}$.
5. $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt[4]{2})/\mathbb{Q}$.
6. $\mathbb{F}_2(\alpha, \alpha^2, \alpha + \alpha^2)/\mathbb{F}_2$ with $\alpha^2 + \alpha + 1 = 0$.

1. $\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}$, we consider two methods to find the primitive element

1. Let $\theta = i + \sqrt{2}$, we have that $\theta - i = \sqrt{2} \Rightarrow (\theta - i)^2 = 2$, by distributing the factors, we have $\theta^2 - 2i\theta + 1 = 2 \Rightarrow i = \frac{\theta^2 - 3}{2\theta} \in \mathbb{Q}(\theta)$ and also $\sqrt{2} = \theta - i \in \mathbb{Q}(\theta)$ thus we get that $\mathbb{Q}(i, \sqrt{2}) = \mathbb{Q}(\theta)$.
2. By Eisenstein criterion we have

$$\text{Irr}(\sqrt{2}, \mathbb{Q}, X) = X^2 - 2$$

$$\text{Irr}(i, \mathbb{Q}, X) = X^2 + 1$$

thus the conjugates of $\sqrt{2}$ are $\{\sqrt{2}, -\sqrt{2}\}$ and of i are $\{i, -i\}$, thus by the proof of the primitive element theorem, by taking $k \notin \{0, i\sqrt{2}\}$ thus by taking $k = 1$ we get $\mathbb{Q}(i, \sqrt{2}) = \mathbb{Q}(i + \sqrt{2})$.

2. \mathbb{C}/\mathbb{R} , its clear that $\mathbb{C} = \mathbb{R}(i)$ thus i is a primitive element.
3. $\mathbb{Q}(\sqrt{2}, i, \sqrt{3})/\mathbb{Q}(\sqrt{3})$, we have $\mathbb{Q}(\sqrt{2}, i, \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{i})(\sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{i})(\sqrt{3}) = \mathbb{Q}(\sqrt{3})(\sqrt{2} + i)$ thus $\sqrt{2} + i$ is a primitive element of $\mathbb{Q}(\sqrt{2}, i, \sqrt{3})/\mathbb{Q}(\sqrt{3})$.
4. $\mathbb{Q}(\sqrt{2}, \sqrt{3}, i)/\mathbb{Q}$, we have from before that $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\sqrt{2} + i)$, thus $\mathbb{Q}(\sqrt{2}, \sqrt{3}, i) = \mathbb{Q}(\sqrt{2} + i)(\sqrt{3})$, now consider $\theta = \sqrt{2} + i$ we have then

$$(\theta - i)^2 = 2 \Rightarrow \theta^2 - 3 = 2\theta i$$

$$\Rightarrow (\theta^2 - 3)^2 = -4\theta^2$$

$$\Rightarrow \theta^4 - 2\theta^2 + 9 = 0$$

we can see that θ is a root of $P(X) = X^4 - 2\theta^2 + 9$, notice that if a is a root of P then so is $-a, \bar{a}$ and $-\bar{a}$ thus we get that the conjugates of θ are $\sqrt{2} + i, -\sqrt{2} + i, \sqrt{2} - i, -\sqrt{2} - i$ and we know that the conjugates of $\sqrt{3}$ over \mathbb{Q} are $\sqrt{3}$ and $-\sqrt{3}$, by the proof of the primitive element theorem we have that $k \notin \{0, \sqrt{2/3}, i/\sqrt{3}, (\sqrt{2} + i)/\sqrt{3}\}$, so taking $k = 1$ we get that $\sqrt{2} + \sqrt{3} + i$ is a primitive element.

1. $\mathbb{Q}(\sqrt[4]{2}, \sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{3}, \sqrt[4]{2})$ with the same method.
2. $\mathbb{F}(\alpha, \alpha^2, \alpha^2 + \alpha)/\mathbb{F}$, its easy to notice that $\alpha + \alpha^2 \in \mathbb{F}(\alpha, \alpha^2)$ and from the definition of $\alpha, \alpha^2 = \alpha + 1 \in \mathbb{F}(\alpha)$ thus we get

$$\mathbb{F}(\alpha, \alpha^2, \alpha^2 + \alpha) = \mathbb{F}(\alpha, \alpha^2) = \mathbb{F}(\alpha)(\alpha^2) = \mathbb{F}(\alpha)$$

thus α is a primitive element.

Exercise 1.5: Let K be a field with $\text{Char } K = 0$, L/K an n -degree extension and θ a primitive element of L/K and an algebraically closed field Ω .

Question 1.5.1: Showing that $1, \theta, \dots, \theta^{n-1}$ is a basis of the vector space L over K .

Question 1.5.2: Proving that the embeddings $\sigma_i : L \rightarrow \Omega$ are of the form $\sigma_i(\theta) = \theta_i$ where $\theta_1, \dots, \theta_n$ are distinct conjugates of θ over K .

Question 1.5.3: For any $\eta \in L$, the conjugates of η are contained in $\{\sigma_i(\eta) \mid i \in \llbracket 1, n \rrbracket\}$.

Question 1.5.4: η is a primitive element if and only if $\forall i, j \in \llbracket 1, n \rrbracket, \sigma_i(\eta) = \sigma_j(\eta) \Rightarrow i = j$.

Question 1.5.5: Deduce that for any $(a, b) \in \mathbb{Q}^* \times \mathbb{Q}^*$ we have $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(a\sqrt{2} + b\sqrt{2})$.

Exercise 1.6: Let $\alpha = \sqrt[3]{2}$, $\omega = e^{\frac{2\pi}{3}i}$ and $\beta = \alpha\omega$, prove the following statements

Question 1.6.1: For any $c \in \mathbb{Q}$, $\gamma = \alpha + c\beta$ is a zero of $x^6 + ax^3 + b$ for some $a, b \in \mathbb{Q}$.

Question 1.6.2: the polynomial $\text{Irr}(\alpha + \beta, \mathbb{Q}, X)$ is cubic and $\deg \text{Irr}(\alpha - \beta, \mathbb{Q}, X) = 6$.

Question 1.6.3: $\forall c \in \mathbb{Q}^*, \mathbb{Q}(\alpha, \omega) = \mathbb{Q}(\omega + c\alpha)$.

Question 1.6.4: $\mathbb{Q}(\omega, \sqrt{5}) = \mathbb{Q}(\omega\sqrt{5})$.

2. Tutorial Series 2: Finite Fields

Exercise 2.7: *Decide whether there exists a finite field having the given number of elements.*

$$\begin{array}{ccccccc} 4095 & - & 191 & - & 12345678910 \\ & & 81 & - & 12396 & - & 128 \end{array}$$

We do prime factorization for each of the elements below.

Exercise 2.8: *Determine all finite fields having n elements where $n \leq 15$. Find a basis, a primitive element, a generator for the multiplicative group for every field.*

We will find all the fields of the form \mathbb{F}_{p^n} such that $p^n \leq 15$.

- $p = 2$:
 - $n = 1$:
 - Field: \mathbb{F}_2 .
 - Primitive Element: 1 or 0.
 - Basis Over \mathbb{F}_2 : $\{1\}$.
 - Generator: 1.
 - $n = 2$:
 - Field: $\mathbb{F}_{2^2} = \mathbb{F}_4$.
 - Primitive Element: α with $\alpha^2 + \alpha + 1 = 0$.
 - Basis Over \mathbb{F}_2 : $\{1, \alpha\}$
 - Generator: α .
 - $n = 3$:
 - Field: $\mathbb{F}_{2^3} = \mathbb{F}_8$.
 - Primitive Element: α with $\alpha^3 + \alpha + 1 = 0$.
 - Basis Over \mathbb{F}_2 : $\{1, \alpha, \alpha^2\}$.
 - Generator: α .
- $p = 3$:
 - $n = 1$:
 - Field: \mathbb{F}_3 .
 - Primitive Element: 1 or 0.
 - Basis Over \mathbb{F}_3 : $\{1\}$.
 - Generator: 2.
 - $n = 2$:
 - Field: $\mathbb{F}_{3^2} = \mathbb{F}_9$.
 - Primitive Element: α with $\alpha^2 + 1 = 0$.
 - Basis Over \mathbb{F}_3 : $\{1, \alpha\}$.
 - Generator: .
- for the remaining for any $p \in \{5, 7, 11, 13\}$ we have
 - Field: \mathbb{F}_p .
 - Primitive Element: 1 or 0.
 - Basis Over \mathbb{F}_p : $\{1\}$.
 - Generator: respectively.