Signal Processing

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Disclaimer

This document contains the lectures that were supposed to be given by Dr. HOCINE, but instead he chose to teach us about electricity, anyway, hope it helps.

> To separate the contents of the course to actual additions or out of context information, a black band will be added by its side like the one on this comment.

Contents

Chapter 1

Fundamental Concepts

Given the source for reading, the definitions can be somewhat improperly explained or vague as the nature of definition depends highly on context.

Definition 1.1 (Signal/Noise): *A signal is a physical representation of information that is sent from source to destination. Noise is any interference happening in the process of reading or interpreting the signal.*

Definition 1.2 (Signal-Noise Ratio): *Let P_s and P_n represent the power of the signals of signal and noise respectively, then the signal-noise ratio is defined as $\xi = P_s / P_n$.*

This ratio represents the noise affects the signal, as in the higher ξ is, the more powerful is the signal to the noise. In general, we indicate it with a logarithmic measure $\xi_{dB} = 10 \log_{10} \xi$.

1.1. Fourier Transform

Definition 1.1.3 (Fourier Transform): *Let $x : \mathbb{R} \to \mathbb{R}$ a function representing a signal dependent of time t. The Fourier transform of the function x is a function of frequency f, $X(f)$ such that*

$$\mathcal{F}(x)(f) = X(f) = \int_{-\infty}^{\infty} x(t) e^{-i 2\pi f t} \,\mathrm{d}t$$

where i is the imaginary unit.

The Fourier transform is used to transform a function from the time domain to frequency/phase domain. This operation is not one way only, in many cases, we can go back to the time domain just from the frequency/phase domain, which is done by the Inverse Fourier Transform.

Definition 1.1.4 (Inverse Fourier Transform): *Let $X : \mathbb{R} \to \mathbb{R}$ a function representing the strength of frequencies of a signal with parameter f. The inverse Fourier transform of the function X is a function of time $x(t)$ such that*

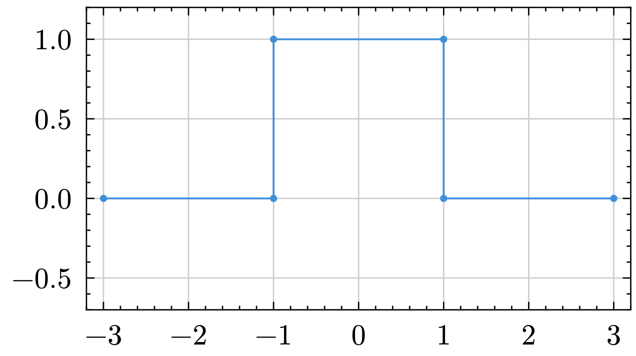
$$\mathcal{F}^{-1}(X)(t) = x(t) = \int_{-\infty}^{\infty} X(f) e^{i 2\pi f t} \,\mathrm{d}f$$

We prove that the inverse Fourier transform is indeed an inverse function.

Theorem 1.1.5 (Fourier Inversion):

An example here will be given with using finite frequencies to explain what does this transform do.

Example: Consider the following function



$$x(t) = \begin{cases} 1 \text{ if } |t| < 1 \\ 0 \text{ otherwise} \end{cases}$$

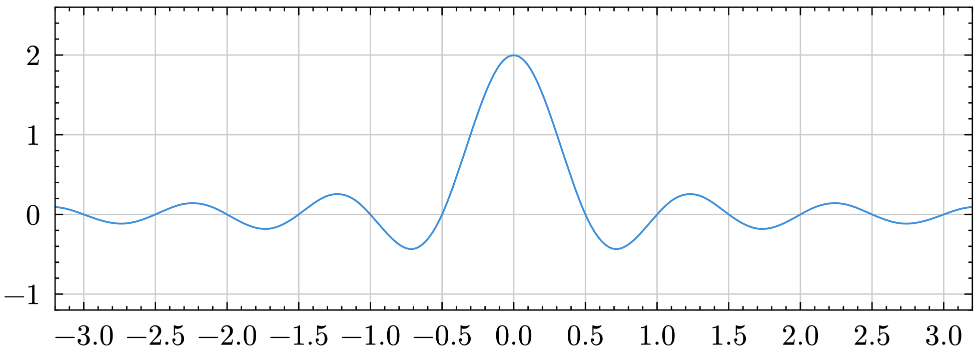
By calculating the Fourier transform we get

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i 2\pi f t} \,\mathrm{d}t = \int_{-1}^{1} e^{-i 2\pi f t} \,\mathrm{d}t$$

For $f = 0$ we get 2, thus we consider $f \neq 0$

$$X(f) = \int_{-1}^{1} e^{-i2\pi f t} \,\mathrm{d}t = \left[\frac{e^{-i2\pi f t}}{-i2\pi f}\right]_{t=-1}^{t=1} = \frac{e^{-(2\pi f)i} - e^{(2\pi f)i}}{-2i\pi f} = \frac{\sin(2\pi f)}{\pi f}$$

Now we plot this function for some values of f



What the previous graph represents in this case is the strength of each frequency in the signal. For example, the frequency 0 is the strongest frequency in the function $x(t)$ which is clearly true since most of the function is constant, the remaining frequencies come from the rising-edge and falling-edge at -1 and 1.

A useful operator in the usage of Fourier transforms is the convolution.

Definition 1.1.6 (Convolution): *Let $x, y : \mathbb{R} \to \mathbb{R}$ two integrable functions. We define the convolution of x and y as $x * y$ as follows*

$$\begin{aligned} x * y &: \mathbb{R} \to \mathbb{R} \\ t &\mapsto \int_{-\infty}^{\infty} x(s) y(t-s) \,\mathrm{d}s \end{aligned}$$

And an important result we get in this case is the convolution theorem stating that $\mathcal{F}(x * y) = \mathcal{F}(x) \cdot \mathcal{F}(y)$. More useful results will be explained later.

1.2. Some Useful Functions

Sign function	$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases} = \begin{cases} \frac{x}{\lvert x \rvert} & \text{if } x \neq 0 \\ x & \text{if } x = 0 \end{cases}$
Jump/Echelon function	$\varepsilon(x) = \frac{1 + \operatorname{sgn}(x)}{2} = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \\ 1/2 & \text{if } x = 0 \end{cases}$
Ramp function	$r(x) = \int_{-\infty}^{x} \varepsilon(t) \,\mathrm{d}t = x \cdot \varepsilon(x)$
Rectangular function	$\operatorname{rect}(x) = \varepsilon\left(x + \frac{1}{2}\right) - \varepsilon\left(x - \frac{1}{2}\right) = \begin{cases} 1 & \text{if } \lvert x \rvert < \frac{1}{2} \\ 0 & \text{if } \lvert x \rvert > \frac{1}{2} \\ \frac{1}{2} & \text{if } \lvert x \rvert = \frac{1}{2} \end{cases}$

1.3. Operations On Signals

Given a signal $x(t)$ we define the following quantities on an interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$:

1. Average value:

$$\overline{x}(T) = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \,\mathrm{d}t$$

2. Quadratic value (Normalized energy):

$$W_x(T) = \int_{-T/2}^{T/2} x^2(t) \,\mathrm{d}t$$

3. Average quadratic value (Normalized power):

$$P_x(T) = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) \,\mathrm{d}t$$