

Mathematical Tools For Cryptography

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Disclaimer

This document contains the lectures given by Dr.ZAIMI.

Some contents were added as remainders and extras for the students.
To separate the contents of the course to actual additions or out of context information, a black band will be added by its side like the one englobing this comment.

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Chapter 0

Remainders

This part will just be a remainder of the important definitions, propositions and theorems of the field extension course that are needed for this course. We assume that rings in this case are commutative rings with unity.

0.1. Rings & Homomorphisms

Definition 0.1.1 (Ring Homomorphism/Kernel/Image): Let R, R' be two rings and $f : R \rightarrow R'$ a map. We say that f is a ring homomorphism if

$$f(a + b) = f(a) + f(b). \quad f(ab) = f(a)f(b).$$

define $\text{Ker } f = f^{-1}(\{0\})$ and $\text{Im } f = f(R)$.

Proposition 0.1.2: Let $f : R \rightarrow R'$ be a ring homomorphism

- $\text{Ker } f$ is an ideal of R and $\text{Im } f$ is a subring of R' .
- $\text{Ker } f = \{0\} \Leftrightarrow f$ injective and $\text{Im } f = R' \Leftrightarrow f$ surjective.
- If R, R' are fields then $f \equiv 0$ or $f(1) = 1$.

Theorem 0.1.3 (First Isomorphism Theorem): Let $f : A \rightarrow B$ be a ring homomorphism, then $\text{Im } f \cong A / \text{Ker } f$ and for $I \subseteq \text{Ker } f$ is an ideal then there exists a unique isomorphism $f_* : A/I \rightarrow B$, $f_*(x + I) = f(x)$ with $f = f_* \circ \pi$ and $\pi(x) = x + I$ the canonical surjection.

$$\begin{array}{ccccc} & f & & & \\ A & \xrightarrow{\pi} & A/I & \xrightarrow{f_*} & B \end{array}$$

0.2. Ideals, UFDs, PIDs & EDs

Definition 0.2.4 (Elements): Let R be a ring, we have the following:

- **Unit:** $x \in R$ is a unit if $\exists y \in R, xy = 1$, denoted R^* .
- **Zero Divisor:** $x \in R \setminus \{0\}$ is a zero divisor if $\exists y \in R, xy = 0$.
- **Irreducible:** $x \in R$ is irreducible if $x = x_1 x_2 \Rightarrow x_1 \in R^* \vee x_2 \in R^*$.

Definition 0.2.5 (Ideals): Let R be a ring, $I \subseteq R$ ideal, we have:

- **Prime Ideal:** I is prime if $\forall a, b \in R, ab \in I \Rightarrow a \in I \vee b \in I$.
- **Principal Ideal:** I is principal if $\exists x \in R, I = (x) = xR$.
- **Maximal Ideal:** I is maximal if $\forall M$ ideal $I \subseteq M \subseteq R \Rightarrow M = I$ or R .

Proposition 0.2.6: Let $f : R \rightarrow R'$ be a ring homomorphism and let I' be an ideal of R' then $I = f^{-1}(I')$ is an ideal of R , if I' is prime then I is prime.

Theorem 0.2.7: Let R be a ring and I, J ideals of R

- I is a prime ideal $\Leftrightarrow R/I$ is an integral domain.
- I is a maximal ideal $\Leftrightarrow R/I$ is a field.
- if $I \subseteq J$ then $(A/I)/(J/I) \cong A/J$.

Definition 0.2.8 (Domains): Let R be a ring, we have the following:

- **Integral Domain:** R is an integral domain if $xy = 0 \Rightarrow x = 0$ or $y = 0$.
- **Principle Ideal Domain:** R is a PID if for any ideal I in R , I is principle.
- **Euclidean Domain:** R is said to be an ED if $\exists \nu : R/\{0\} \rightarrow \mathbb{N}$ a valuation function, $\forall a, b \in R, \exists q, r \in R, a = bq + r, r = 0$ or $\nu(r) < \nu(b)$.
- **Unique Factoriation Domain:** R is a UFD if any element can be decomposed into a unique product of irreducible elements.

Theorem 0.2.9:

- If R is a UFD and $x \in R$ is irreducible, then (x) is prime.
- If R is an integral domain and a PID, every prime ideal is maximal.
- $ED \Rightarrow PID \Rightarrow UFD$.

0.3. Ring Of Polynomials

Definition 0.3.10 (Ring Of Polynomials): Let R be a ring, we define

$$R[X] = \left\{ \sum_{i \in I} a_i X^i \mid I \text{ finite}, \{a_i\}_{i \in I} \subseteq R \right\}.$$

to be the ring of polynomials on R , and for any $P \in R[X]$ we set

$$\deg P = \max\{i \in \mathbb{N} \mid X^i \text{ has a non-zero coefficient in } P\}.$$

Definition 0.3.11 (Polynomials): Let L/K be a field extension and $P \in K[X]$

- **Minimal:** P is the minimal polynomial of $\alpha \in L$ if it is the unique monic polynomial with the smallest degree which vanishes at α denoted $\text{Irr}(\alpha, K, x)$.

Proposition 0.3.12: Let K be a field, $P(X) \in K[X]$ with $\deg P \in \{2, 3\}$ then we have that P is reducible over $K \Leftrightarrow P$ has a zero in K .

Theorem 0.3.13: Let R be a ring

- R is an integral domain $\Rightarrow R[X]$ is an integral domain.
- $P = \sum a_i X^i$ is a unit in $R[X] \Leftrightarrow a_0 \in R^*$ and $\forall i \geq 1, a_i$ nilpotent.
- $P \in R[X]$ irreducible $\Rightarrow R[X]/(P) = \left\{ \sum_{i=0}^{\deg P-1} a_i \alpha^i \mid a_i \in R \right\}$.
- Let α a root of $P \in R[X]$ then $\text{Irr}(\alpha, K, X)$ divides P .
- if R is a field, $R[X]$ is a Euclidean domain with the valuation $\nu(P) = \deg P$.

Proposition 0.3.14 (Eisenstein's Criterions):

- Let $P(x) = a_0 + a_1 x + \dots + a_n x^n \in \mathbb{Z}[X]$, if there is a prime p such that $p \mid a_0, \dots, p \mid a_{n-1}, p^2 \nmid a_0$ and $p \nmid a_n$ then P is irreducible over $\mathbb{Q}[X]$.
- Let $P(X) \in \mathbb{Z}[X]$ and $\varphi : \mathbb{Z}[X] \rightarrow \mathbb{Z}_{n[X]}$ the extension of $k \mapsto k \bmod n$, if $\deg \varphi(P) = \deg P$ and $\varphi(P)$ is irreducible in $\mathbb{Z}_{n[X]}$ then P is irreducible in $\mathbb{Q}[X]$.

0.4. Field Extensions

Definition 0.4.15 (Extension/Degree Of Extension): Let L, K be two fields such that $K \subseteq L$, we call L a field extension of K and we denote it L/K , we define the degree of extension of L on K as $\dim_K L$ if it is finite and $+\infty$ if it is infinite, and we denote it $[L : K] = \dim_K L$

Definition 0.4.16 (Characteristic): we define the characteristic of K to be the smallest n such that $1 + 1 + \dots + 1 = 0$ n times, denoted $\text{Char } K = n$, if n does not exist then we say that $\text{Char } K = 0$.

Definition 0.4.17 (Elements): Let K be a field and $x \in K$.

- **Algebraic:** x is said to be algebraic if $\exists P \in K[X], \deg P > 0, P(x) = 0$.
- **Transcendental:** x is said to be transcendental if it is not algebraic.
- **Conjugate:** α is said to be the conjugate of β if β is a root of $\text{Irr}(\alpha, K, x)$.

Proposition 0.4.18:

- If α, β are conjugates then $\text{Irr}(\alpha, K, X) = \text{Irr}(\beta, K, X)$.
- If L/K then $\text{Char } L = \text{Char } K$.
- if K is a field then $\text{Char } K$ is prime.

Definition 0.4.19 (Fields): Let K be a field

- **Algebraically Closed:** K is said to be algebraically closed if any algebraic extension of K is K .

Definition 0.4.20 (Extensions):

- **L/K Algebraic Extension:** $\forall x \in L$, x is algebraic over K .
- **L/K Transcendental Extension:** if it is not an algebraic extension.
- **Algebraic Closure:** \overline{K} is an algebraic closure of K if \overline{K}/K is an algebraic extension and \overline{K} is algebraically closed.

Theorem 0.4.21: Let L/K be a field extension and $\alpha_1, \dots, \alpha_n \in L$ and set $\alpha = (\alpha_1, \dots, \alpha_n)$ then

- $K[\alpha_1][\alpha_2] \dots [\alpha_n] = K[\alpha_1, \dots, \alpha_n] = \{P(\alpha) \mid P \in K[X_1, \dots, X_n]\}$ is the smallest ring containing K and $\alpha_1, \dots, \alpha_n$.
- $K(\alpha_1)(\alpha_2) \dots (\alpha_n) = K(\alpha_1, \dots, \alpha_n) = \{P(\alpha)/Q(\alpha) \mid P, Q \in K[X_1, \dots, X_n], Q(\alpha) \neq 0\}$ is the smallest field containing K and $\alpha_1, \dots, \alpha_n$.
- any extension of finite degree is algebraic.

Theorem 0.4.22 (Steinitz):

1. Any field is contained inside of an algebraically closed field.
2. Any two algebraic closures of a field are isomorphic.

Chapter 1

Embeddings

Consider in this chapter, K, L, E, Ω denote fields, the lower-case elements are used for elements of fields.

1.1. Embedding

Definition 1.1.1 (Embedding): Let $\sigma : K \rightarrow L$ a homomorphism, if $\sigma \neq 0$ then σ is an embedding from K to L .

Definition 1.1.2 (Extension/Restriction): Suppose E is an extension of K , τ is an embedding of E into L such that $\forall k \in K, \tau(k) = \sigma(k)$, then τ is called an extension σ and σ is called a restriction of τ to K . Moreover if $\sigma = \text{Id}_K$ then τ is called a K -embedding of E into L .

Example:

- The unique embedding $\sigma : \mathbb{Q} \rightarrow \mathbb{C}$ is the identity, we prove $\sigma(1) = 1, \sigma(n) = n, \sigma(-n) = -n, \sigma(a/b) = \sigma(a)/\sigma(b)$ by induction, then $\sigma = \text{Id}$.
- The embeddings $\tau : \mathbb{Q}(\sqrt{2}) \rightarrow \mathbb{C}$ has only two forms, given that $\tau|_{\mathbb{Q}} = \text{Id}_{\mathbb{Q}}$ then $\tau(a + b\sqrt{2}) = a + b\tau(\sqrt{2})$ and since $\tau(\sqrt{2})^2 = \tau(\sqrt{2}^2) = \tau(2) = 2 \Rightarrow \tau(\alpha) = \sqrt{2}$ or $\tau(\alpha) = -\sqrt{2}$.

Proposition 1.1.3: Let τ be a K -embedding of L into E where $K \subseteq L$ and let $\alpha \in L$ be algebraic over K , then $\tau(\alpha)$ is a conjugate of α over K .

Proof. Suppose α algebraic of degree d over K and let $P = \text{Irr}(\alpha, K, x)$, we have $P(\alpha) = 0$ and $\tau(P(\alpha)) = \tau(\sum k_i \alpha^i) = \sum \tau(k_i) \tau(\alpha)^i = \sum k_i \tau(\alpha)^i = P(\tau(\alpha)) = 0$ then $\tau(\alpha)$ is a conjugate of α . \square

We used the fact that τ is a K -embedding in the evaluation $\tau(k_i) = k_i$.

Proposition 1.1.4: Let K be a field, \bar{K} an algebraic closure of K , $\alpha \in \bar{K}$ and let $\beta \in \bar{K}$ be a conjugate of α over K , then there is a K -embedding $\tau : K(\alpha) \rightarrow \bar{K}$ which is a K -isomorphism of $K(\alpha)$ into $K(\beta)$ sending α to β .

Proof Outline:

- $P(X) = \text{Irr}(\alpha, K, X) = \text{Irr}(\beta, K, X)$ since they are conjugates.
- $K[\alpha] = K(\alpha), K[\beta] = K(\beta)$ since they are algebraic.
- $K(\alpha) \cong K[X]/(P(X)) \cong K(\beta)$.
- Construct using those isomorphisms a map that keeps K invariant.

Proof. Let α, β conjugates over K , then we have that $\text{Irr}(\alpha, K, X) = \text{Irr}(\beta, K, X)$. Define $I = (\text{Irr}(\alpha, K, X)), \nu_\alpha : K[X] \rightarrow K[\alpha]$ and $\nu_\beta : K[X] \rightarrow K[\beta]$ such that $\nu_\alpha(P(X)) = P(\alpha)$ and $\nu_\beta(P(X)) = P(\beta)$ which are surjective by definition. We have that $\text{Ker}(\nu_\alpha) = \text{Ker}(\nu_\beta) = (\text{Irr}(\alpha, K, X))$. From the first isomorphism theorem we have that there exists two isomorphisms $(\nu_\alpha)_* : K[X]/I \rightarrow K[\alpha]$ and $(\nu_\beta)_* : K[X]/I \rightarrow K[\beta]$ such that $v_\alpha = (\nu_\alpha)_* \circ \pi$ and $v_\beta = (\nu_\beta)_* \circ \pi$. We also have that $K[\alpha] = K(\alpha)$ and $K[\beta] = K(\beta)$ since α, β are algebraic.

Set $\varphi : K(\alpha) \rightarrow K(\beta), x \mapsto ((v_\beta)_* \circ (v_\alpha)_*)^{-1}(x)$, φ is the composition of isomorphisms then it is an isomorphism, let $x \in K, \varphi(x) = (v_\beta)_*((v_\alpha)_*^{-1}(x)) = (v_\beta)_*(x + I) = x$ so φ is a K -isomorphism and $\varphi(\alpha) = (v_\beta)_*((v_\alpha)_*^{-1}(\alpha)) = (v_\beta)_*(I) = \beta$. \square

Example: Let $\alpha = \sqrt[3]{2}$ we have $\text{Irr}(\alpha, \mathbb{Q}, X) = X^3 - 2$, the conjugates of α over \mathbb{Q} are $\alpha, j\alpha, j^2\alpha$, there are the following embeddings:

- $\tau_1 : \mathbb{Q}(\sqrt[3]{2}) \rightarrow \mathbb{Q}(\sqrt[3]{2})$ which is the identity.
- $\tau_2 : \mathbb{Q}(\sqrt[3]{2}) \rightarrow \mathbb{Q}(j\sqrt[3]{2})$ with $\tau_2(\sqrt[3]{2}) = j\sqrt[3]{2}$ an isomorphism.
- $\tau_3 : \mathbb{Q}(\sqrt[3]{2}) \rightarrow \mathbb{Q}(j^2\sqrt[3]{2})$ with $\tau_3(\sqrt[3]{2}) = j^2\sqrt[3]{2}$ an isomorphism.

so there are exactly three embeddings.

Corollary 1.1.5: Let α be algebraic over K of degree $n, \alpha \in \bar{K}$ an algebraic closure of K and let s be the number of distinct conjugates of α over K , then there are exactly s embeddings of $K(\alpha)$ into \bar{K} sending α to its distinct conjugates.

Proposition 1.1.6: Let L/K be an algebraic extension and σ a K -endomorphism of L , then σ is surjective.

Proof Outline:

1. σ transforms a conjugate to another.
2. the conjugates of an element are finite so σ is a bijection.
3. σ is a permutation thus there is a preimage for any element.

Proof. Let $\sigma : L \rightarrow L$ a K -embedding and $\alpha \in L$. Take $P = \text{Irr}(\alpha, K, X)$ and set $C = \{\beta \in L \mid P(\beta) = 0\}$ so C is the set of conjugates of α over K , $\alpha \in C \neq \emptyset$ and C is finite since P has finite roots. For any $\beta \in C$, $\sigma(\beta) \in C$ since $P(\sigma(\beta)) = \sigma(P(\beta)) = 0$, σ is an injection from a finite set to itself so $\sigma(C) = C$ thus $\exists \beta \in C, \sigma(\beta) = \alpha$ so σ is surjective. \square

1.2. Primitive Element Theorem

Let K be a field and \overline{K} an algebraic closure of K , and let $\alpha \in \overline{K}$ and α is a zero of $P(X) \in K[X] \setminus \{0\}$, recall that α is said to be a zero of P is $P(X) = (X - \alpha)^m Q(X)$ with $Q \in K[X]$, if $m \geq 2$ we say that α is a repeated zero of P , if $m = 1$ we say that α is a simple zero.

Definition 1.2.7 (Derivative/Repeated Factor):

- Let $P \in K[X]$ such that $P(X) = \sum_{i=0}^n k_i X^i$, the formal derivative of $P(X)$ is defined by $P'(X) = \sum_{i=1}^n i k_i X^{i-1}$.
- Let $Q \in K[X]$ of degree ≥ 1 , then Q is said to be a repeated factor of $P \in K[X]$ if $P(X) = Q(X)^m R(X)$ for some $m \geq 2$ and $R \in K[X]$.

Proposition 1.2.8: Let K be a field with $\text{Char } K = 0$, then we have that $\deg P' = \deg P - 1$, thus P does not divide P' .

Proof. Let $P(X) = \sum_{i=0}^n a_i X^i$, with $a_n \neq 0$, its derivative is defined as $P'(X) = \sum_{i=1}^n i a_i X^{i-1}$, the coefficient of the highest degree is $n a_n$, which is not zero since $\text{Char } K = 0$. Thus $\deg P' = n - 1 = \deg P - 1$. \square

Proposition 1.2.9: Let K be a field with $\text{Char } K = 0$ and $P \in K[X]$, then P has a repeated factor in $K[X] \Leftrightarrow P, P'$ have a common factor.

More precisely, if $Q \in K[X]$ a repeated factor of P then it divides both P and P' . Conversely if Q is irreducible and is a common divisor of P and P' then it is a repeated factor of P .

Proof.

- \Rightarrow suppose that $P(X) = Q^m(X)R(X)$, $P'(X) = mQ^{m-1}(X)Q'(X)R(X) + Q^m(X)R'(X)$ then Q divides both P and P' .
- \Leftarrow suppose now that P, P' have a common factor Q irreducible which exists by the fact that $K[X]$ is a UFD, then $P(X) = Q(X)R(X)$ we get then that $P'(X) = Q'(X)R(X) + Q(X)R'(X)$ since Q divides P and P' we have that it divides $Q'(X)R(X) = P'(X) - Q(X)R'(X)$ so Q divides either Q' or R , since $\text{Char } K = 0$ then Q does not divide Q' so it necessarily divides R then $R(X) = Q(X)R_1(X)$, thus $P(X) = Q(X)R(X) = Q(X)Q(X)R_1(X) = Q^2(X)R_1(X)$ so Q is a repeated factor of P . \square

Corollary 1.2.10: Let $P \in K[X]$ irreducible and $\text{Char } K = 0$ then $P(X)$ has no repeated zeros in any algebraic closure \overline{K} .

Proof Outline:

1. Consider the Bezout identity $P(X)U(X) + P'(X)V(X) = 1$.
2. Replace with the roots of $P(X)$ and deduce it doesn't nullify P' .
3. Deduce that the factor $(X - \alpha)$ is not present in P' .
4. Deduce that $(X - \alpha)$ is simple factor of P .

Proof. Let $P \in K[X]$ be irreducible then $P' \neq 0$ and P does not divide P' , if Q is a common factor of P and P' then Q divides P and since P is irreducible then $Q(X) = \lambda P(X)$ thus P divides P' , so $\gcd(P(X), P'(X)) = 1$. By Bezout's theorem we have that $\exists u, v \in K[X]$ such that $P(X)u(X) + P'(X)v(X) = 1$, viewing this identity in $\overline{K}[X]$, let α be a zero of P in \overline{K} , replacing in the previous equation we have that $P(\alpha)u(\alpha) + P'(\alpha)v(\alpha) = P'(\alpha)v(\alpha) = 1$ then $P'(\alpha) \neq$

0 so $(X - \alpha)$ is not a factor of P' so P has no repeated factor $(X - \alpha)$ then P has no repeated zeros in \overline{K} . \square

Corollary 1.2.11: Let K be a field of $\text{Char } K = 0$ and α be algebraic over K of degree d and let $\text{Irr}(\alpha, K, X) = (X - \alpha_1)\dots(X - \alpha_d)$ in $\overline{K}[X]$, then there exists d embeddings of $K(\alpha)$ into \overline{K} of the form $\sigma_i : \alpha \mapsto \alpha_i$.

Proof. Use the previous corollary and the one of the existence of s embeddings in this case to get $s = d$. \square

Theorem 1.2.12 (Primitive Element): Let L be a finite extension of a field K with $\text{Char } K = 0$ then there is $\theta \in L$ such that $L = K(\theta)$.

Proof Outline:

1. Simplify by induction to two elements $L = K(\alpha, \beta)$.
2. Take $P(X) = \text{Irr}(\alpha, K, X)$, $Q(X) = \text{Irr}(\beta, K, X)$.
3. Take $\theta \in K \setminus \{(\alpha - \alpha_i)/(\beta_j - \beta) \mid (i, j) \in [\![1, n]\!] \times [\![2, m]\!]\}$.
4. Consider $\theta = \alpha + k\beta$ and $R(X) = P(\theta - kX)$.
5. Prove that k is the only common factor of R and Q .
6. Deduce that $(X - \beta) \in K(\theta)[X]$ and thus $\beta \in K(\theta)$.

Proof. It is easy to notice that by induction, a proof for the existence of θ when $L = K(\alpha, \beta)$ is sufficient.

Consider $L = K(\alpha, \beta)$. Consider the minimal polynomials

$$P(X) = \text{Irr}(\alpha, K, X) \underset{\overline{K}[X]}{=} (X - \alpha_1)\dots(X - \alpha_n)$$

$$Q(X) = \text{Irr}(\beta, K, X) \underset{\overline{K}[X]}{=} (X - \beta_1)\dots(X - \beta_m)$$

Let $k \in K$ such that $\forall i \in [\![1, n]\!], \forall j \in [\![2, m]\!], k \neq \frac{\alpha - \alpha_i}{\beta_j - \beta}$ which exists since $\text{Char}(K) = 0 \Rightarrow \#K = \infty$ and consider $\theta = \alpha + k\beta$, clearly $K(\theta) \subseteq K(\alpha, \beta)$.

Consider the polynomial $R(X) = P(\theta - kX)$, $\deg R = \deg P = n$ and $R \in K(\theta)[X]$, we have $R(\beta) = P(\theta - k\beta) = P(\alpha) = 0$ so β is a zero of R and also a zero of Q . We want to prove that the only common zero of R and Q is β , let γ be a zero of R and Q in \overline{K} , $R(\gamma) = P(\theta - k\gamma) = 0 \Rightarrow \theta - k\gamma = \alpha_i$ so we get

that γ satisfies $\gamma = \frac{\theta - \alpha_i}{k} = \frac{\alpha - \alpha_i}{k} + \beta$, since γ is a zero of Q then $\gamma = \beta_j$ but this reduces to $k = \frac{\alpha - \alpha_i}{\beta_j - \beta}$ which is not true by choice of k so $\gamma = \beta_1 = \beta$. Thus the unique common zero of Q and R is β then $\text{Irr}(\beta, K, X)$ divides both Q and R and is of degree one so $\text{Irr}(\beta, K, X) = X - \beta \in K(\theta)[X]$ thus we have $\beta \in K(\theta)$.

It is clear that $K(\theta) \subseteq K(\alpha, \beta)$ and we have that $\beta \in K(\theta)$ then $\alpha = \theta - k\beta \in K(\theta)$ since $k, \theta, \beta \in K(\theta)$ and thus $K(\alpha, \beta) = K(\theta)$. \square

Example:

- to find the primitive of $\mathbb{Q}(\sqrt{a}, \sqrt{b})$ with $a, b \in \mathbb{Z}$ non-perfect squares, we have that $\text{Irr}(\sqrt{a}, \mathbb{Q}, K) = X^2 - a$ and $\text{Irr}(\sqrt{b}, \mathbb{Q}, K) = X^2 - b$ thus $\alpha_1 = \sqrt{a}, \alpha_2 = -\sqrt{a}$ and $\beta_1 = \sqrt{b}, \beta_2 = -\sqrt{b}$ and thus we obtain that the set of non-allowed values of k are $\{0, \sqrt{a/b}\}$, given that $a \neq b$ then $k = 1$ is not in that list so we obtain that

$$\mathbb{Q}(\sqrt{a} + \sqrt{b}) = \mathbb{Q}(\sqrt{a} + \sqrt{b})$$

Chapter 2

Extensions Of Finite Fields

The aim of this chapter is to study the structure of the finite fields and how to characterize them using their ground fields. Consider the fields \mathbb{F}_p as the fields \mathbb{Z}_p with the modular addition and multiplication.

2.1. Finite Fields

Proposition 2.1.1: Let F be a finite field with $\# F = q$ and E/F a finite extension of degree n then $\# E = q^n$.

Proof. We have that $[E : F] = \dim_F E = n$, thus $E \cong F^n$ as an F -vector space, so $\# E = \#(F^n) = (\# F)^n = q^n$. \square

Corollary 2.1.2: Let E be a finite field with $\text{Char } E = p$, then $\# E = p^n$ for some $n \in \mathbb{N}^*$.

Proof. We have that $\text{Char } E = p$, then it has a copy of $\mathbb{Z}/p\mathbb{Z}$ by the isomorphism $\varphi : \mathbb{Z}/p\mathbb{Z} \rightarrow E, n \mapsto 1 + 1 + \dots + 1, n$ times, take $F = \varphi(\mathbb{Z}/p\mathbb{Z})$, it is a subfield of E , we get then that $\# E = (\# F)^n = p^n$ where $n = [E : F]$. \square

Theorem 2.1.3: Let E be a finite field with $\# E = p^n$, then the elements of E are precisely the zeros of the polynomial $X^{p^n} - X \in \mathbb{F}_p[X]$ in a certain algebraic closure $\overline{\mathbb{F}_p}$.

Proof. Let $P(X) = X^{p^n} - X$ and $\mathcal{Z}(P) = \{\alpha \in \overline{\mathbb{F}_p} \mid P(\alpha) = 0\}$

- $E \subseteq \mathcal{Z}(P)$: $\# E = p^n$ then $\# E^* = p^n - 1$ thus the multiplicative group E^* is of order $p^n - 1$, by Lagrange's theorem, we have that $\forall a \in E^*, a^{p^n-1} = 1$ thus

we get that $a^{p^n} - a = 0$ so $P(a) = 0$ and notice also that $P(0) = 0$ thus we obtain $E = E^* \cup \{0\} \subseteq \mathcal{Z}(P)$.

- $\mathcal{Z}(P) \subseteq E$: notice that $\# \mathcal{Z}(P) \leq p^n$ since $\deg P = p^n$ and $\# E = p^n$ thus it is clear that $\mathcal{Z}(P) \subseteq E$.

So we conclude that $\mathcal{Z}(P) = E$. \square

Example:

1. $E = \mathbb{F}_2$ with $n = 1$ and $p = 2$ then \mathbb{F}_2 are the zeros of $X^2 - X = X(X - 1)$.
2. $E = \mathbb{F}_2(\alpha)$ with $\alpha^2 + \alpha + 1 = 0$, we have that $E = \{0, 1, \alpha, \alpha^2\}$ and by the theorem the roots of $X^4 - X = X(X - 1)(X^2 + X + 1)$ are the solutions.

Theorem 2.1.4: Let E be a finite field, then the group E^* is cyclic.

Proof. Let E be a finite field and $E^* = E \setminus \{0\}$ the multiplicative group of E , then E^* is abelian and $\# E^* = p^n - 1$ where $\# E = p^n$, assume on the contrary that E^* is not cyclic, so the order of any element of E^* is strictly less than $p^n - 1$. Let $\alpha \in E^*$ of maximal order $s < p^n - 1$. Let $G = \langle \alpha \rangle = \{\alpha, \alpha^2, \dots, \alpha^s\}$ be the cyclic subgroup of E^* generated by α . Any element of G has an order that divides s by Lagrange's theorem, $\forall g \in G, g^s = 1$ thus g is a zero of $X^s - 1 \in E[X]$, and since $\deg(X^s - 1) = s$ then this polynomial has at most s zeros, so G are exactly the roots of $X^s - 1$ in $\overline{\mathbb{F}_p}$, since $\# G = s < p^n - 1 = \# E^*$ then there exists $\beta \in E^*$ with $\beta \notin G$. Let t be the order of β , $\beta^t = 1$ and t does not divide s

$$s = p_1^{l_1} p_2^{l_2} \dots p_j^{l_j} \dots p_r^{l_r} \quad t = p_1^{k_1} p_2^{k_2} \dots p_j^{k_j} \dots p_r^{k_r}$$

with p_i prime numbers and $l_i, k_i \in \mathbb{N}, l_i < k_1, l_j < k_j, l_{j+1} \geq k_{j+1}, \dots, l_r \geq k_r$. j exists since t does not divide s , set $u = p_1^{l_1} \dots p_j^{l_j}, u' = s/u, v = p_1^{k_1} \dots p_j^{k_j}$ and $v' = t/v$. Notice that $u < v$ and $\gcd(u, v') = 1$. Let $\gamma = \alpha^u \beta^{v'}$, $\gamma \in E^*$ since $\alpha, \beta \in E^*$, α^u is of order u' and $\beta^{v'}$ is of order v and since u and v' are coprime then the order of $\alpha^u \beta^{v'}$ is $u'v > u'u = s$, contradiction with the fact that the maximal order is s . \square

Constructive Proof: Let $q = \# E$ and $q - 1 = \prod_{i=1}^k p_i^{l_i}$ by the fundamental theorem of arithmetic. Consider for $i \in \llbracket 1, k \rrbracket$ the polynomials $P_i(X) = X^{\frac{q-1}{p_i}} - 1$, notice that $\# \mathcal{Z}(P_i) \leq \frac{q-1}{q_i} < q - 1 = \# E^*$, thus there exists $y_i \in$

$E^* \setminus \mathcal{Z}(P_i)$. Now for $i \in \llbracket 1, k \rrbracket$ $z_i = y_i^{(q-1)/(p_i^{l_i})}$, notice that z_i has order $p_i^{l_i}$ since $z_i^{p_i} = y_i^{q-1} = 1$ by Lagrange's theorem and $z_i^{p_i^{l_i-1}} = y_i^{(q-1)/p_i} \neq 1$ by the definition of y_i . We chose $z = \prod_{i=1}^n z_i$, since all the orders of z_i are coprime from the fact that they are different primes, then the order of z is $\prod_{i=1}^n p_i^{l_i} = q - 1 = \# E^*$, thus $E^* = \langle z \rangle$ and E^* is cyclic.

Example: Consider \mathbb{F}_7 then $p = 7$, $n = 1$ and $q = 7$

$$\begin{aligned} q-1 = 6 &= 2 \cdot 3 \Rightarrow \begin{cases} p_1 = 2 \\ l_1 = 1 \\ p_2 = 3 \\ l_2 = 1 \end{cases} \Rightarrow \begin{cases} P_1(X) = X^3 - 1 \\ P_2(X) = X^2 - 1 \end{cases} \\ &\Rightarrow \begin{cases} y_1 = 3 \\ y_2 = 4 \end{cases} \Rightarrow \begin{cases} z_1 = 3^3 = 6 \\ z_2 = 4^2 = 2 \end{cases} \Rightarrow z = 6 \cdot 2 = 5 \end{aligned}$$

its easy to verify that $\langle 5 \rangle = \mathbb{F}_7^*$.

Corollary 2.1.5: Let E be a finite extension of a finite field F , then there exists a primitive element $\theta \in E$, $E = F(\theta)$.

Proof. Consider θ the generator of E^* from the previous proposition. $F(\theta) \subseteq E$ since $\theta \in E^* \subseteq E$ and $F \subseteq E$. $E \subseteq F(\theta)$ given that $E^* = \langle \theta \rangle \subseteq F(\theta)$ and $0 \in F \subseteq F(\theta)$ thus we obtain $E = \{0\} \cup E^* \subseteq F(\theta)$ hence $E = F(\theta)$. \square

Theorem 2.1.6: Let $n \in \mathbb{N}^*$ and p a prime number, there exists a unique field of order p^n up to an isomorphism denoted \mathbb{F}_{p^n} .

Proof. Consider $P(X) = X^{p^n} - X$ in $\overline{\mathbb{F}_p}$ and let $K = \{\alpha \in \overline{\mathbb{F}_p} \mid P(\alpha) = 0\}$. We will prove that K is a subfield of $\overline{\mathbb{F}_p}$ containing \mathbb{F}_p .

- $\mathbb{F}_p \subseteq K$: let $\alpha \in \mathbb{F}_p$, we have $\alpha^{p-1} = 1$ by Langrange's theorem, thus $\alpha^p = \alpha$, by induction we have that $\alpha^{p^i} = \alpha$ thus $P(\alpha) = \alpha^{p^n} - \alpha = 0$ so $\alpha \in K$.
- K is a field: Let $\alpha, \beta \in K$, we have that $(\alpha^{-1})^{p^n} = (\alpha^{p^n})^{-1} = \alpha^{-1}$, $(\alpha\beta)^{p^n} = \alpha^{p^n}\beta^{p^n} = \alpha\beta$ since \cdot is commutative, $(\alpha + \beta)^{p^n} = \sum_{i=0}^{p^n} \binom{p^n}{i} \alpha^i \beta^{p^n-i}$, since $K \subseteq \overline{\mathbb{F}_p}$ and $\text{Char } \overline{\mathbb{F}_p} = \text{Char } \mathbb{F}_p = p$ then $\forall i \in \llbracket 1, p-1 \rrbracket, p \mid \binom{p^n}{i}$ thus we get $(\alpha + \beta)^{p^n} = \alpha^{p^n} + \beta^{p^n} = \alpha + \beta$.

Thus K is a subfield of $\overline{\mathbb{F}_p}$ containing \mathbb{F}_p and has p^n elements since $\overline{\mathbb{F}_p}$ is algebraically closed and $\deg P = p^n$. \square

Corollary 2.1.7: Let F be a finite field, $F \subseteq \overline{\mathbb{F}_p}$ and $\alpha \in \overline{\mathbb{F}_p}$, then the zeros of $\text{Irr}(\alpha, F, X)$ are distinct, the conjugates of α are all in $F(\alpha)$ and so there are exactly $[F(\alpha) : F]$ embeddings which are all automorphisms.

Proof. we have $\mathbb{F}_p \subseteq F \subseteq F(\alpha)$, $[F : \mathbb{F}_p] = d$ and $[F(\alpha) : F] = n$ then we get $F(\alpha) \cong \mathbb{F}_{p^{nd}}$. The elements of $F(\alpha)$ are zeros of $X^{p^{nd}} - X \in \mathbb{F}_p[X]$ so $\text{Irr}(\alpha, \mathbb{F}_p, X)$ divides $X^{p^{nd}} - X$ whose zeros are distinct so the zeros of $\text{Irr}(\alpha, \mathbb{F}_p, X)$ are distinct. Let σ be an F -embedding of $F(\alpha)$ then $\sigma(\alpha)$ is a conjugate of α so $\# F(\alpha) = \# F(\sigma(\alpha))$ thus $F(\alpha) = F(\sigma(\alpha))$. \square

Example:

- $\mathbb{F}_2 = \{0, 1\}$, the set of cubic polynomials over $\mathbb{F}_2[X]$ are $P_1(X) = X^3 + X + 1$ and $P_2(X) = X^3 + X^2 + 1$.

Proposition 2.1.8: Let $m, n \in \mathbb{N}^*$ and p prime then

$$\mathbb{F}_{p^m} \text{ is a subfield of } \mathbb{F}_{p^n} \Leftrightarrow m \text{ divides } n.$$

Proof. \Rightarrow suppose that \mathbb{F}_{p^m} is a subfield of \mathbb{F}_{p^n} , then \mathbb{F}_{p^n} is an extension of \mathbb{F}_{p^m} and we have that $n = [\mathbb{F}_{p^n} : \mathbb{F}_p] = [\mathbb{F}_{p^n} : \mathbb{F}_{p^m}] \cdot [\mathbb{F}_{p^m} : \mathbb{F}_p] = k \cdot m$ thus m divides n . \Leftarrow suppose that m divides n so $n = k \cdot m$, let $\alpha \in \mathbb{F}_{p^m}$, by definition of \mathbb{F}_{p^m} we have $\alpha^{p^m} = 1$ so $\alpha^{p^n} = \alpha^{p^{km}} = (\alpha^{p^m})^{p^{(k-1)m}} = 1^{p^{(k-1)m}} = 1$ thus $\alpha \in \mathbb{F}_{p^n}$, and since \mathbb{F}_{p^m} is a field then it is a subfield of \mathbb{F}_{p^n} . \square

Corollary 2.1.9: The number of subfields of \mathbb{F}_{p^n} is equal to the number of positive divisors of n .

Proof. It is clear from the previous statement that the only subfields are divisors of n , denote them d_1, \dots, d_k . Then the subfields of \mathbb{F}_{p^n} are $\mathbb{F}_{p^{d_1}}, \mathbb{F}_{p^{d_2}}, \dots, \mathbb{F}_{p^{d_k}}$. \square

Definition 2.1.10 (Frobenius Automorphisms): Let $f : \mathbb{F}_q \rightarrow \mathbb{F}_q$ where $q = p^n$ with $f : \alpha \mapsto \alpha^p$, then f is called the Frobenius automorphism of \mathbb{F}_q .

To justify this nomination, we will prove that it is a \mathbb{F}_p -automorphism

- f is well defined: since for $\alpha \in \mathbb{F}_q$, $\alpha^p \in \mathbb{F}_q$ thus $f(\alpha) \in \mathbb{F}_q$.
- f is a \mathbb{F}_p -embedding: let $\alpha, \beta \in \mathbb{F}_q$, $f(\alpha + \beta) = (\alpha + \beta)^p = \alpha^p + \beta^p = f(\alpha) + f(\beta)$ since $\text{Char } \mathbb{F}_q = p$, $f(\alpha\beta) = (\alpha\beta)^p = \alpha^p\beta^p = f(\alpha)f(\beta)$. f is injective since $\text{Ker } f = \{\alpha \in \mathbb{F}_q \mid f(\alpha) = 0\} = \{\alpha \in \mathbb{F}_q \mid \alpha^p = 0\} = \{0\}$ given its a field. Let $\alpha \in \mathbb{F}_p$, α satisfies $\alpha^p - \alpha = 0$ thus $f(\alpha) = \alpha^p = \alpha$.

Using Proposition 1.1.6, we deduce that f is an automorphism of \mathbb{F}_q .

Proposition 2.1.11: Let α be algebraic over \mathbb{F}_p of degree n ($\mathbb{F}_p(\alpha) = \mathbb{F}_{p^n}$) then the embeddings of $\mathbb{F}_p(\alpha)$ into $\overline{\mathbb{F}_p}$ are f^1, f^2, \dots, f^n where f is the Frobenius automorphism of \mathbb{F}_{p^n} .

Proof. Notice that f is an automorphism, thus $\forall i \in [\![1, n]\!]$, f^i is an automorphism from \mathbb{F}_{p^n} to \mathbb{F}_{p^n} so is an embedding from $\mathbb{F}_p(\alpha)$ to $\overline{\mathbb{F}_p}$. Since α is of degree n then there are exactly n embeddings, we will prove that all of them are of the form f^i .

There n embeddings f^1, \dots, f^n , now we will prove that they are all not equal, that is $\forall 1 \leq i < j \leq n$, $f^i \neq f^j$. Suppose that there exists $i, j \in [\![1, n]\!]$, $i < j$ such that $f^i \equiv f^j$, $\mathbb{F}_p(\alpha)$ is a finite field then there is a generator θ for the multiplicative group $(\mathbb{F}_p(\alpha))^* = \langle \theta \rangle$, $\theta^{p^i} = f^i(\theta) = f^j(\theta) = \theta^{p^j}$, so $\theta^{p^j - p^i} = 1$ but $p^j - p^i < p^n$ and the order of θ is p^n which is a contradiction. Thus we conclude that $\{f^i\}_{i \in [\![1, n]\!]}$ are the only embeddings of $\mathbb{F}_p(\alpha)$ into $\overline{\mathbb{F}_p}$. \square

Note: Some things may be missing here

Chapter 3

Normal Extensions

In what follows K denotes a field, \overline{K} an algebraic closure of K and $P(X)$ a non-constant polynomial.

Definition 3.1 (Splitting Field): Let $\alpha_1, \dots, \alpha_n$ be the zeros of the polynomial $P \in K[X]$ in \overline{K} , then the field $K(\alpha_1, \dots, \alpha_n)$ is called splitting field of P over K . In other words, $K(\alpha_1, \dots, \alpha_n)$ is the smallest subfield of \overline{K} , in which, we can write $P(X) = (X - \alpha_1) \cdots (X - \alpha_n)$.

Example:

- The splitting field of $X^2 - 2$ over \mathbb{Q} in \mathbb{C} is $\mathbb{Q}(\sqrt{2})$.
- The splitting field of $X^3 - 2$ over \mathbb{Q} in \mathbb{C} is $\mathbb{Q}(\sqrt[3]{2}, j\sqrt[3]{2}, j^2\sqrt[3]{2}) = \mathbb{Q}(\sqrt[3]{2}, j)$.
- \mathbb{C} is the splitting field of $X^2 + 1$ in \mathbb{R} because $\mathbb{R}(i, -i) = \mathbb{R}(i) = \mathbb{C}$.
- Any finite field \mathbb{F}_{p^n} is the splitting field some polynomial in $\mathbb{F}_p[X]$, since $\forall \alpha \in \mathbb{F}_{p^n}$, α is a zero of $X^{p^n} - X \in \mathbb{F}_p[X]$ so if we set $\mathcal{Z}(X^{p^n} - X)$ the set of zeros of $X^{p^n} - X$ then $\mathbb{F}_{p^n} = \mathbb{F}_p(\mathcal{Z}(X^{p^n} - X))$.
- The splitting field of $(X^2 - 2)(X^2 - 3)(X^2 - 1) \in \mathbb{Q}[X]$ is the field containing all the roots which is $\mathbb{Q}(\pm\sqrt{3}, \pm\sqrt{2}, \pm 1) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

Definition 3.2 (Splitting Field Of A Family Of Polynomials): Let $(P_i)_{i \in I} \subseteq K[X]$ and $\mathcal{Z} = \cup_{i \in I} \mathcal{Z}(P_i)$ then the field $K(\mathcal{Z})$ is the intersection of all subfields of \overline{K} containing K and \mathcal{Z} which is called the splitting field of the family $(P_i)_{i \in I}$ over K .

Example 3.3:

- \overline{K} is a splitting field of the family of non-constant polynomials in $K[X]$.
- The splitting field of $\{X^2 - 2, X^2 - X - 1, X^3 - 2, X^3 - 1\}$ over \mathbb{Q} is $\mathbb{Q}(\pm\sqrt{2}, (1 \pm \sqrt{5})/2, \sqrt[3]{2}, j\sqrt[3]{2}, j^2\sqrt[3]{2}) = \mathbb{Q}(\sqrt[3]{2}, \sqrt{2}, \sqrt{5}, j)$.

Definition 3.4 (Normal Extension): Let L/K be an algebraic extension, then L/K is said to be a normal extension iff for any $P \in K[X]$ irreducible in K , if P has a root in L , then all its roots are in L .

Proposition 3.5: Let L/K be an algebraic extension, the following statements are equivalent

- L/K is a normal extension.
- $\forall \alpha \in L$, all conjugates of α over K are in L .

Proof.

- $1. \Rightarrow 2.$ is clear given that the conjugates of α are the roots of $\text{Irr}(\alpha, K, X)$, which exists by the fact that L/K is algebraic and has α as a root in L , are contained in L thus all the conjugates of α over K are contained in L .
- $2. \Rightarrow 1.$ Let $P \in K[X]$ be irreducible over K , and suppose P has a zero $\alpha \in L$, since $\text{Irr}(\alpha, K, X)$ divides $P(X)$ in $K[X]$ and P is irreducible then $P(X) = \lambda \text{Irr}(\alpha, K, X)$, any root of P is a root of $\text{Irr}(\alpha, K, X)$ and thus a conjugate of α , so the roots of P are contained in L .

□

Example 3.6:

- \overline{K}/K is normal since \overline{K}/K is algebraic and every polynomial is splitted on \overline{K} .
- \mathbb{C}/\mathbb{Q} is not normal since it is not algebraic, while \mathbb{C}/\mathbb{R} is normal since $\overline{\mathbb{R}} = \mathbb{C}$.
- K/K is normal since $[K : K] = 1$ thus algebraic and $\text{Irr}(\alpha, K, X) = X - \alpha$ thus its only conjugate is α and thus contained in K .
- Let $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not normal since the conjugates of $\sqrt[3]{2}$ which are $j\sqrt[3]{2}$ and $j^2\sqrt[3]{2}$ are not in $\mathbb{Q}(\sqrt[3]{2})$.

Theorem 3.7: Let L/K be an algebraic extension and $L \subseteq \Omega$, then the following statements are equivalent:

1. Any K -embeddings of L into Ω is an automorphism of L .
2. L/K is normal.
3. L is a splitting field over K of a family of $K[X]$.

Proof.

- 1. \Rightarrow 2. Let $\alpha \in L$ and β a conjugate of α over K , by Proposition 1.1.4 there exists a K -embedding τ from $K(\alpha) \subseteq L$ to $K(\beta) \subseteq \Omega$ that sends α to β and since L/K then $L/K(\alpha)$ is algebraic. Let σ be an extension of τ to L , so σ is a K -automorphism of L from assumption, and thus $\sigma(\alpha) = \tau(\alpha) = \beta \in L$.
- 2. \Rightarrow 3. Suppose L/K normal, consider the family of polynomials \mathcal{F} defined by $\mathcal{F} = \{P(X) \in K[X] \mid P \text{ irreducible having a zero in } L\}$ and F be the splitting field of \mathcal{F} over K . Let \mathcal{Z} be the set of zeros of all elements of \mathcal{F} , $F = K(\mathcal{Z})$. Since $\mathcal{Z} \subseteq L$ (because L/K normal), we have $K(\mathcal{Z}) = F \subseteq L$. Now let $\alpha \in L$, α is algebraic and is a zero $\text{Irr}(\alpha, K, X) \in \mathcal{F}$ so $\alpha \in \mathcal{Z} \subseteq F$ thus $L \subseteq F$.
- 3. \Rightarrow 1. Suppose that L is a splitting field over K of a family \mathcal{F} of $K[X]$, $L = K(\mathcal{Z})$ where \mathcal{Z} is the set of zeros of elements of \mathcal{F} in Ω . Let σ be a K -embedding of L into Ω . To show that σ is a K -automorphism of L , it suffices to prove that σ is a K -endomorphism using Proposition 1.1.6. Let $\alpha \in \mathcal{Z}$ then $\sigma(\alpha)$ is a conjugate of α over K thus $\sigma(\alpha) \in \mathcal{Z}$.

□

Example:

- $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is normal. We have that $\text{Irr}(\sqrt{2}, K, X) = X^2 - 2$ then there are two embeddings of $\mathbb{Q}(\sqrt{2}) \rightarrow \overline{\mathbb{Q}}$ which are $\sigma_1 : \sqrt{2} \mapsto \sqrt{2}$ and $\sigma_2 : \sqrt{2} \mapsto -\sqrt{2}$ which are both automorphisms.
- $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not normal, we have that $j\sqrt[3]{2}$ is a conjugate of $\sqrt[3]{2}$ over \mathbb{Q} but is not in $\mathbb{Q}(\sqrt[3]{2})$ thus
- F is a finite field, E a finite extension of F then E/F is normal.

Summary

Chapter 3: Remainders

3: Rings & Homomorphisms

Definition 3.7 (Ring Homomorphism/Kernel/Image): Let R, R' be two rings and $f : R \rightarrow R'$ a map. We say that f is a ring homomorphism if

$$f(a + b) = f(a) + f(b), \quad f(ab) = f(a)f(b).$$

define $\text{Ker } f = f^{-1}(\{0\})$ and $\text{Im } f = f(R)$.

Proposition 3.7: Let $f : R \rightarrow R'$ be a ring homomorphism

- $\text{Ker } f$ is an ideal of R and $\text{Im } f$ is a subring of R' .
- $\text{Ker } f = \{0\} \Leftrightarrow f$ injective and $\text{Im } f = R' \Leftrightarrow f$ surjective.
- If R, R' are fields then $f \equiv 0$ or $f(1) = 1$.

Theorem 3.7 (First Isomorphism Theorem): Let $f : A \rightarrow B$ be a ring homomorphism, then $\text{Im } f \cong A/\text{Ker } f$ and for $I \subseteq \text{Ker } f$ is an ideal then there exists a unique isomorphism $f_* : A/I \rightarrow B$, $f_*(x + I) = f(x)$ with $f = f_* \circ \pi$ and $\pi(x) = x + I$ the canonical surjection.

$$\begin{array}{ccc} & f & \\ A & \xrightarrow{\pi} & A/I & \xrightarrow{f_*} & B \end{array}$$

3: Ideals, UFDs, PIDs & EDs

Definition 3.7 (Elements): Let R be a ring, we have the following:

- **Unit:** $x \in R$ is a unit if $\exists y \in R, xy = 1$, denoted R^* .
- **Zero Divisor:** $x \in R \setminus \{0\}$ is a zero divisor if $\exists y \in R, xy = 0$.

- **Irreducible:** $x \in R$ is irreducible if $x = x_1x_2 \Rightarrow x_1 \in R^* \vee x_2 \in R^*$.

Definition 3.7 (Ideals): Let R be a ring, $I \subseteq R$ ideal, we have:

- **Prime Ideal:** I is prime if $\forall a, b \in R, ab \in I \Rightarrow a \in I \vee b \in I$.
- **Principal Ideal:** I is principal if $\exists x \in R, I = (x) = xR$.
- **Maximal Ideal:** I is maximal if $\forall M$ ideal $I \subseteq M \subseteq R \Rightarrow M = I$ or R .

Proposition 3.7: Let $f : R \rightarrow R'$ be a ring homomorphism and let I' be an ideal of R' then $I = f^{-1}(I')$ is an ideal of R , if I' is prime then I is prime.

Theorem 3.7: Let R be a ring and I, J ideals of R

- I is a prime ideal $\Leftrightarrow R/I$ is an integral domain.
- I is a maximal ideal $\Leftrightarrow R/I$ is a field.
- if $I \subseteq J$ then $(A/I)/(J/I) \cong A/J$.

Definition 3.7 (Domains): Let R be a ring, we have the following:

- **Integral Domain:** R is an integral domain if $xy = 0 \Rightarrow x = 0$ or $y = 0$.
- **Principle Ideal Domain:** R is a PID if for any ideal I in R , I is principle.
- **Euclidean Domain:** R is said to be an ED if $\exists \nu : R/\{0\} \rightarrow \mathbb{N}$ a valuation function, $\forall a, b \in R, \exists q, r \in R, a = bq + r, r = 0$ or $\nu(r) < \nu(b)$.
- **Unique Factoriation Domain:** R is a UFD if any element can be decomposed into a unique product of irreducible elements.

Theorem 3.7:

- If R is a UFD and $x \in R$ is irreducible, then (x) is prime.
- If R is an integral domain and a PID, every prime ideal is maximal.
- ED \Rightarrow PID \Rightarrow UFD.

3: Ring Of Polynomials

Definition 3.7 (Ring Of Polynomials): Let R be a ring, we define

$$R[X] = \left\{ \sum_{i \in I} a_i X^i \mid I \text{ finite}, \{a_i\}_{i \in I} \subseteq R \right\}.$$

to be the ring of polynomials on R , and for any $P \in R[X]$ we set

$$\deg P = \max\{i \in \mathbb{N} \mid X^i \text{ has a non-zero coefficient in } P\}.$$

Definition 3.7 (Polynomials): Let L/K be a field extension and $P \in K[X]$

- **Minimal:** P is the minimal polynomial of $\alpha \in L$ if it is the unique monic polynomial with the smallest degree which vanishes at α denoted $\text{Irr}(\alpha, K, x)$.

Proposition 3.7: Let K be a field, $P(X) \in K[X]$ with $\deg P \in \{2, 3\}$ then we have that P is reducible over $K \Leftrightarrow P$ has a zero in K .

Theorem 3.7: Let R be a ring

- R is an integral domain $\Rightarrow R[X]$ is an integral domain.
- $P = \sum a_i X^i$ is a unit in $R[X] \Leftrightarrow a_0 \in R^*$ and $\forall i \geq 1, a_i$ nilpotent.
- $P \in R[X]$ irreducible $\Rightarrow R[X]/(P) = \left\{ \sum_{i=0}^{\deg P-1} a_i \alpha^i \mid a_i \in R \right\}$.
- Let α a root of $P \in R[X]$ then $\text{Irr}(\alpha, K, X)$ divides P .
- if R is a field, $R[X]$ is a Euclidean domain with the valuation $\nu(P) = \deg P$.

Proposition 3.7 (Eisenstein's Criterions):

- Let $P(x) = a_0 + a_1 x + \dots + a_n x^n \in \mathbb{Z}[X]$, if there is a prime p such that $p \mid a_0, \dots, p \mid a_{n-1}, p^2 \nmid a_0$ and $p \nmid a_n$ then P is irreducible over $\mathbb{Q}[X]$.
- Let $P(X) \in \mathbb{Z}[X]$ and $\varphi : \mathbb{Z}[X] \rightarrow \mathbb{Z}_{n[X]}$ the extension of $k \mapsto k \bmod n$, if $\deg \varphi(P) = \deg P$ and $\varphi(P)$ is irreducible in $\mathbb{Z}_{n[X]}$ then P is irreducible in $\mathbb{Q}[X]$.

3: Field Extensions

Definition 3.7 (Extension/Degree Of Extension): Let L, K be two fields such that $K \subseteq L$, we call L a field extension of K and we denote it L/K , we define the degree of extension of L on K as $\dim_K L$ if it is finite and $+\infty$ if it is infinite, and we denote it $[L : K] = \dim_K L$

Definition 3.7 (Characteristic): we define the characteristic of K to be the smallest n such that $1 + 1 + \dots + 1 = 0$ n times, denoted $\text{Char } K = n$, if n does not exist then we say that $\text{Char } K = 0$.

Definition 3.7 (Elements): Let K be a field and $x \in K$.

- **Algebraic:** x is said to be algebraic if $\exists P \in K[X], \deg P > 0, P(x) = 0$.
- **Transcendental:** x is said to be transcendental if it is not algebraic.
- **Conjugate:** α is said to be the conjugate of β if β is a root of $\text{Irr}(\alpha, K, x)$.

Proposition 3.7:

- If α, β are conjugates then $\text{Irr}(\alpha, K, X) = \text{Irr}(\beta, K, X)$.
- If L/K then $\text{Char } L = \text{Char } K$.
- if K is a field then $\text{Char } K$ is prime.

Definition 3.7 (Fields): Let K be a field

- **Algebraically Closed:** K is said to be algebraically closed if any algebraic extension of K is K .

Definition 3.7 (Extensions):

- **L/K Algebraic Extension:** $\forall x \in L, x$ is algebraic over K .
- **L/K Transcendental Extension:** if it is not an algebraic extension.
- **Algebraic Closure:** \bar{K} is an algebraic closure of K if \bar{K}/K is an algebraic extension and \bar{K} is algebraically closed.

Theorem 3.7: Let L/K be a field extension and $\alpha_1, \dots, \alpha_n \in L$ and set $\alpha = (\alpha_1, \dots, \alpha_n)$ then

- $K[\alpha_1][\alpha_2] \dots [\alpha_n] = K[\alpha_1, \dots, \alpha_n] = \{P(\alpha) \mid P \in K[X_1, \dots, X_n]\}$ is the smallest ring containing K and $\alpha_1, \dots, \alpha_n$.
- $K(\alpha_1)(\alpha_2) \dots (\alpha_n) = K(\alpha_1, \dots, \alpha_n) = \{P(\alpha)/Q(\alpha) \mid P, Q \in K[X_1, \dots, X_n], Q(\alpha) \neq 0\}$ is the smallest field containing K and $\alpha_1, \dots, \alpha_n$.
- any extension of finite degree is algebraic.

Theorem 3.7 (Steinitz):

1. Any field is contained inside of an algebraically closed field.
2. Any two algebraic closures of a field are isomorphic.

Chapter 3: Embeddings

3: Embedding

Definition 3.7 (Embedding): Let $\sigma : K \rightarrow L$ a homomorphism, if $\sigma \not\equiv 0$ then σ is an embedding from K to L .

Definition 3.7 (Extension/Restriction): Suppose E is an extension of K , τ is an embedding of E into L such that $\forall k \in K, \tau(k) = \sigma(k)$, then τ is called an extension σ and σ is called a restriction of τ to K . Moreover if $\sigma = \text{Id}_K$ then τ is called a K -embedding of E into L .

Proposition 3.7: Let τ be a K -embedding of L into E where $K \subseteq L$ and let $\alpha \in L$ be algebraic over K , then $\tau(\alpha)$ is a conjugate of α over K .

Proposition 3.7: Let K be a field, \bar{K} an algebraic closure of K , $\alpha \in \bar{K}$ and let $\beta \in \bar{K}$ be a conjugate of α over K , then there is a K -embedding $\tau : K(\alpha) \rightarrow \bar{K}$ which is a K -isomorphism of $K(\alpha)$ into $K(\beta)$ sending α to β .

Proof Outline:

1. $P(X) = \text{Irr}(\alpha, K, X) = \text{Irr}(\beta, K, X)$ since they are conjugates.
2. $K[\alpha] = K(\alpha), K[\beta] = K(\beta)$ since they are algebraic.
3. $K(\alpha) \cong K[X]/(P(X)) \cong K(\beta)$.
4. Construct using those isomorphisms a map that keeps K invariant.

Corollary 3.7: Let α be algebraic over K of degree n , $\alpha \in \bar{K}$ an algebraic closure of K and let s be the number of distinct conjugates of α over K , then there are exactly s embeddings of $K(\alpha)$ into \bar{K} sending α to its distinct conjugates.

Proposition 3.7: Let L/K be an algebraic extension and σ a K -endomorphism of L , then σ is surjective.

Proof Outline:

1. σ transforms a conjugate to another.
2. the conjugates of an element are finite so σ is a bijection.
3. σ is a permutation thus there is a preimage for any element.

3: Primitive Element Theorem

Definition 3.7 (Derivative/Repeated Factor):

- Let $P \in K[X]$ such that $P(X) = \sum_{i=0}^n k_i X^i$, the formal derivative of $P(X)$ is defined by $P'(X) = \sum_{i=1}^n i k_i X^{i-1}$.
- Let $Q \in K[X]$ of degree ≥ 1 , then Q is said to be a repeated factor of $P \in K[X]$ if $P(X) = Q(X)^m R(X)$ for some $m \geq 2$ and $R \in K[X]$.

Proposition 3.7: Let K be a field with $\text{Char } K = 0$, then we have that $\deg P' = \deg P - 1$, thus P does not divide P' .

Proposition 3.7: Let K be a field with $\text{Char } K = 0$ and $P \in K[X]$, then P has a repeated factor in $K[X] \Leftrightarrow P, P'$ have a common factor.

More precisely, if $Q \in K[X]$ a repeated factor of P then it divides both P and P' . Conversely if Q is irreducible and is a common divisor of P and P' then it is a repeated factor of P .

Corollary 3.7: Let $P \in K[X]$ irreducible and $\text{Char } K = 0$ then $P(X)$ has no repeated zeros in any algebraic closure \overline{K} .

Proof Outline:

1. Consider the Bezout identity $P(X)U(X) + P'(X)V(X) = 1$.
2. Replace with the roots of $P(X)$ and deduce it doesn't nullify P' .
3. Deduce that the factor $(X - \alpha)$ is not present in P' .
4. Deduce that $(X - \alpha)$ is simple factor of P .

Corollary 3.7: Let K be a field of $\text{Char } K = 0$ and α be algebraic over K of degree d and let $\text{Irr}(\alpha, K, X) = (X - \alpha_1) \dots (X - \alpha_d)$ in $\overline{K}[X]$, then there exists d embeddings of $K(\alpha)$ into \overline{K} of the form $\sigma_i : \alpha \mapsto \alpha_i$.

Theorem 3.7 (Primitive Element): Let L be a finite extension of a field K with $\text{Char } K = 0$ then there is $\theta \in L$ such that $L = K(\theta)$.

Proof Outline:

1. Simplify by induction to two elements $L = K(\alpha, \beta)$.
2. Take $P(X) = \text{Irr}(\alpha, K, X)$, $Q(X) = \text{Irr}(\beta, K, X)$.
3. Take $\theta \in K \setminus \{(\alpha - \alpha_i)/(\beta_j - \beta) \mid (i, j) \in [\![1, n]\!] \times [\![2, m]\!]\}$.
4. Consider $\theta = \alpha + k\beta$ and $R(X) = P(\theta - kX)$.
5. Prove that k is the only common factor of R and Q .
6. Deduce that $(X - \beta) \in K(\theta)[X]$ and thus $\beta \in K(\theta)$.

Chapter 3: Extensions Of Finite Fields

3: Finite Fields

Proposition 3.7: Let F be a finite field with $\# F = q$ and E/F a finite extension of degree n then $\# E = q^n$.

Corollary 3.7: Let E be a finite field with $\text{Char } E = p$, then $\# E = p^n$ for some $n \in \mathbb{N}^*$.

Theorem 3.7: Let E be a finite field with $\# E = p^n$, then the elements of E are precisely the zeros of the polynomial $X^{p^n} - X \in \mathbb{F}_p[X]$ in a certain algebraic closure $\overline{\mathbb{F}_p}$.

Theorem 3.7: Let E be a finite field, then the group E^* is cyclic.

Corollary 3.7: Let E be a finite extension of a finite field F , then there exists a primitive element $\theta \in E$, $E = F(\theta)$.

Theorem 3.7: Let $n \in \mathbb{N}^*$ and p a prime number, there exists a unique field of order p^n up to an isomorphism denoted \mathbb{F}_{p^n} .

Corollary 3.7: Let F be a finite field, $F \subseteq \overline{\mathbb{F}_p}$ and $\alpha \in \overline{\mathbb{F}_p}$, then the zeros of $\text{Irr}(\alpha, F, X)$ are distinct, the conjugates of α are all in $F(\alpha)$ and so there are exactly $[F(\alpha) : F]$ embeddings which are all automorphisms.

Proposition 3.7: Let $m, n \in \mathbb{N}^*$ and p prime then

\mathbb{F}_{p^m} is a subfield of $\mathbb{F}_{p^n} \Leftrightarrow m$ divides n .

Corollary 3.7: The number of subfields of \mathbb{F}_{p^n} is equal to the number of positive divisors of n .

Definition 3.7 (Frobenius Automorphisms): Let $f : \mathbb{F}_q \rightarrow \mathbb{F}_q$ where $q = p^n$ with $f : \alpha \mapsto \alpha^p$, then f is called the Frobenius automorphism of \mathbb{F}_q .

Proposition 3.7: Let α be algebraic over \mathbb{F}_p of degree n ($\mathbb{F}_p(\alpha) = \mathbb{F}_{p^n}$) then the embeddings of $\mathbb{F}_p(\alpha)$ into $\overline{\mathbb{F}_p}$ are f^1, f^2, \dots, f^n where f is the Frobenius automorphism of \mathbb{F}_{p^n} .

Note: Some things may be missing here

Chapter 3: Normal Extensions

Definition 3.7 (Splitting Field): Let $\alpha_1, \dots, \alpha_n$ be the zeros of the polynomial $P \in K[X]$ in \overline{K} , then the field $K(\alpha_1, \dots, \alpha_n)$ is called splitting field of P over K . In other words, $K(\alpha_1, \dots, \alpha_n)$ is the smallest subfield of \overline{K} , in which, we can write $P(X) = (X - \alpha_1)\cdots(X - \alpha_n)$.

Definition 3.7 (Splitting Field Of A Family Of Polynomials): Let $(P_i)_{i \in I} \subseteq K[X]$ and $\mathcal{Z} = \cup_{i \in I} \mathcal{Z}(P_i)$ then the field $K(\mathcal{Z})$ is the intersection of all subfields of \overline{K} containing K and \mathcal{Z} which is called the splitting field of the family $(P_i)_{i \in I}$ over K .

Definition 3.7 (Normal Extension): Let L/K be an algebraic extension, then L/K is said to be a normal extension if for any $P \in K[X]$ irreducible in K , if P has a root in L , then all its roots are in L .

Proposition 3.7: Let L/K be an algebraic extension, the following statements are equivalent

1. L/K is a normal extension.
2. $\forall \alpha \in L$, all conjugates of α over K are in L .

Theorem 3.7: Let L/K be an algebraic extension and $L \subseteq \Omega$, then the following statements are equivalent:

1. Any K -embeddings of L into Ω is an automorphism of L .
2. L/K is normal.
3. L is a splitting field over K of a family of $K[X]$.