

Advanced Graph Theory

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Disclaimer

This document contains the lectures given by Dr. MEHDAOUI.

To separate the contents of the course to actual additions or out of context information, a black band will be added by its side like the globing this comment.

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Chapter 1

Fundamental Concepts

1.1. Types Of Graphs

Definition 1.1.1 (Undirected Graph): An undirected graph G is an ordered pair $G = (V, E)$ where V is a finite set called the vertex set (vertices) and $E \subset \{\{u, v\} \mid u, v \in V\}$ called the set of edges.

Definition 1.1.2 (Directed Graph): A directed graph G is an ordered pair $G = (V, A)$ where V is the set of vertices and $A \subset V \times V$ called the set of arcs or directed edges.

Definition 1.1.3 (Unweighted Graph): An unweighted graph G is a graph is a graph where all edges are considered equal.

Definition 1.1.4 (Weighted Graph): A weighted graph $G = (V, E)$ is a graph where there is a function $w : E \rightarrow \mathbb{R}$ where for each edge $e \in E$, the weight of e is $w(e)$.

1.2. Concepts On Vertices & Edges

Definition 1.2.5 (Adjacent/Incident/Isolated): Let $G = (V, E)$ a graph:

1. Two vertices $u, v \in V$ are adjacent or neighbors if $\{u, v\} \in E$.
2. A vertex v is called incident to an edge e if v is an endpoint of e .

3. A vertex with no incident edges is called isolated.
4. The order of G is $\# V$.
5. The size of G is $\# E$.
6. The open neighborhood of $v \in V$ is $\mathcal{N}(v) = \{u \in V \mid \{u, v\} \in E\}$.
7. The closed neighborhood of $v \in V$ is $[v] = \mathcal{N}(v) \cup \{v\}$.
8. A degree of a vertex v denoted $d(v)$ or $\deg(v)$ is the number of edges incident to v , that is $\deg(v) = \# N(v)$.
9. If G is directed then we define the following:
 1. The out neighborhood of $v \in V$ is $\mathcal{N}^+(v) = \{u \in V \mid (v, u) \in A\}$.
 2. The in neighborhood of $v \in V$ is $\mathcal{N}^-(v) = \{u \in V \mid (u, v) \in A\}$.
 3. The in/out degree are $\deg^-(v) = \# \mathcal{N}^-(v)$, $\deg^+(v) = \# \mathcal{N}^+(v)$.
10. The number of edges incident to v with maximum degree of G , $\Delta(G) = \max_{v \in V} d(v)$ and minimum degree is $\delta(G) = \min_{v \in V} d(v)$.

Lemma 1.2.6 (Handshake): For an undirected graph $G = (V, E)$ then

$$\sum_{v \in V} \deg(v) = 2 \# E$$

Proof. Every edge $\{u, v\} \in E$ contributes exactly 1 to $\deg(u)$ and 1 to $\deg(v)$ thus adding 2 to $\sum_{v \in V} \deg(v)$, so all edges counted twice in the sum of all edges. \square

Theorem 1.2.7: The number of vertices with odd degree is even.

Proof. Let $V_o = \{v \in V \mid d(v) \text{ odd}\}$ and $V_e = \{v \in V \mid d(v) \text{ even}\}$, by the handshake lemma we have

$$2 \# E = \sum_{v \in V} \deg(v) = \sum_{v \in V_o} \deg(v) + \sum_{v \in V_e} \deg(v)$$

given that $2 \# E$ is even and $\sum_{v \in V_e} \deg(v)$ is even then necessarily $\sum_{v \in V_o} \deg(v)$ is even, and since the sum of odd numbers is even if and only if the number of odd numbers is even then $\# V_o$ is even so there is an even number of odd degree vertices. \square

1.3. Simple & Multigraphs

Definition 1.3.8 (Simple/Multi Graph): A graph G is said to be simple if it has no loops and no multiple edges. A multigraph is a graph that may have loops and have multiple edges.

Theorem 1.3.9: In any simple graph G with order $n \geq 2$, there exists at least two vertices with same degree.

Proof. The possible degrees in a simple graph is $\{0, \dots, n-1\}$, however a graph cannot have a vertex with degree 0 and $n-1$ at the same time, so either it has vertices of degree $\{1, \dots, n-1\}$ or $\{0, \dots, n-2\}$, and since there are n vertices and $n-1$ choices for degrees, then by the pigeonhole principle, there are at least two vertices with the same degrees. \square

Proposition 1.3.10: There are $\binom{n}{2}$ maximum edges in a simple graph.

Definition 1.3.11 (Walk/Trail): Let $G = (V, E)$ be a graph

1. A walk of length k is a sequence $v_1, e_2, v_2, \dots, e_k, v_k$ with $e_i = \{v_{i-1}, v_i\} \in E$.
2. A trail is a walk with no repeated edges.
3. A path is a walk with no repeated vertices.
4. A walk is closed if $v_0 = v_k$.

Theorem 1.3.12: If there is a walk from vertex u to v then there is also a path from u to v .

Proof. You can do it, I suppose \square

Definition 1.3.13 (Circuit/Cycle):

1. A circuit is a closed trail.
2. A cycle is a closed path.

3. A cycle of length k is denoted C_k , unless otherwise specified.

4. The length of the cycle is the number of edges in it.

5. A graph G is connected if for every pair of vertices $u, v \in V$ there exists a path from u to v , otherwise it is disconnected.

6. A subgraph H of G is a graph such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$.

7. A connected component of G is a maximum connected subgraph of G .

8. A spanning subgraph H of G is a subgraph such that $V(H) = V(G)$.

Theorem 1.3.14: In a connected graph, any two longest paths share at least one vertex.