SHANGHAI JIAOTONG UNIVERSITY

ADVANCED ICT PROJECT 1

Paper Reading

Real-time Approximation of Clothoids With Bounded Error
for Path Planning Applications

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1 INTRODUCTION

Common path planning methods usually generate accessible paths, but there are few concerns about the feasibility or optimization of the path, so some form of algorithm is usually required to smooth this path. Various path-smoothing algorithms are proposed in the literature : cubic splines, intrinsic splines, Bezier's curves, quintic Bezier splines, and clothoids. In path planning applications, the main advantage of the clothoids over other smoothing methods is the linear change in its curvature, which is very important for the transportation of people or heavy and sensitive loads because it can prevent the car from suddenly changing in the centripetal acceleration and forces experienced by a vehicle increasing driving comfort.

As shown in Figure 1, without smoothing, there are only inflection points

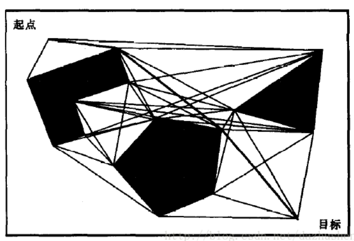

Figure 1: Original

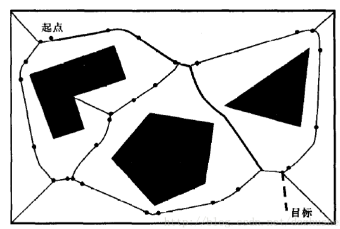

Figure 2: Smoothing

and lines connecting the inflection points on the graph. However, it's hard to turn around in place for a real-world car and the real-world car can't be thought of as a particle. Although the minimum distance between the start point and the end point can be represented by these lines, we have to smooth it in order to apply to life.

2 APPROXIMATION TO A CLOTHOID

The expression for coordinates of a general clothoid are

$$x(s) = x_0 + \int_0^s cos(\theta_0 + \kappa_0\epsilon + \frac{1}{2}c\epsilon^2)d\epsilon \tag{1}$$

$$y(s) = y_0 + \int_0^s sin(\theta_0 + \kappa_0\epsilon + \frac{1}{2}c\epsilon^2)d\epsilon \tag{2}$$

$(x_0,\ y_0)$ is the initial point, θ_0 is the initial tangent angle, κ_0 is the initial curvature, c is the parameter called sharpness, and s $\geq$ 0 is the parameter that denotes arc length. The properties of a clothoid are as follows:

- Curvature $\kappa(s) = \kappa_0 + cs$
- Tangent angle

$$\theta(s) = \theta_0 + \kappa_0 s + \frac{1}{2}cs^2$$

On the circumstance of s = 6,

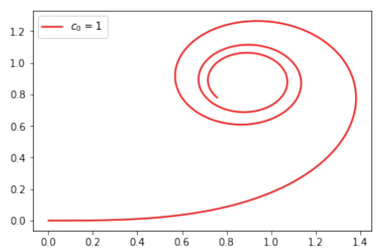

Figure 3: $c_0 = 1$, $\theta_0 = 0$, $\kappa_0 = 0$

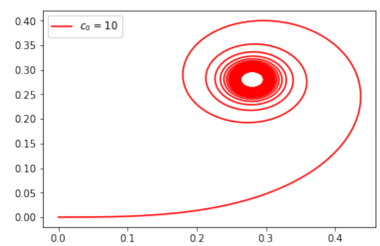

Figure 4: $c_0 = 10$, $\theta_0 = 0$, $\kappa_0 = 0$

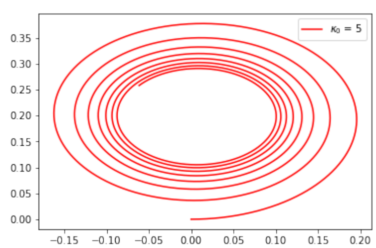

Figure 5: $c_0 = 1$, $\theta_0 = 0$, $\kappa_0 = 5$

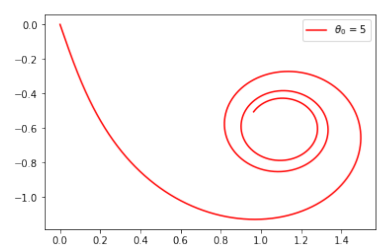

Figure 6: $c_0 = 1$, $\theta_0 = 5$, $\kappa_0 = 0$

Equation for clothoid coordinates are transcendental function. This paper presents a method for real-time computation of clothoid coordinate. Instead of storing coordinates of all needed clothoids, we set all initial conditions of the basic clothoid to zero and its sharpness to some constant greater than zero, a transformation method between a basic clothoid and any general clothoid is shown as following :

$$\begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + R(\frac{-\kappa_0^2}{2c} + \theta_0)\sqrt{\frac{c_L}{|c|}} \begin{bmatrix} x_L(\sqrt{\frac{c_L}{|c|}}(s + \frac{\kappa_0}{c})) - x_L(\sqrt{\frac{c_L}{|c|}}(\frac{\kappa_0}{c})) \\ sgn(c)[y_L(\sqrt{\frac{c_L}{|c|}}(s + \frac{\kappa_0}{c})) - y_L(\sqrt{\frac{c_L}{|c|}}(\frac{\kappa_0}{c}))] \end{bmatrix} \tag{3}$$

(x_L, y_L) are coordinates of the basic clothoid, R(θ) is a rotation matrix defined as :

$$R(\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \tag{4}$$

x_0, y_0, κ_0 and θ_0 are defined in the specific clothoid. c_L is the sharpness of the basic clothoid. With these defined parameters, we can obtain the position (x, y) of the sharpness c in the length s. Thus, we don't need to reserve all of clothoids, but these parameters.

3 INTERPOLATION

3.1 STRAIGHT LINE INTERPOLATION

We obtain an exact transformation method, but the obtained point coordinates are approximate. As shown in Figure 7, a series of points can roughly restore a clothoid. To compute clothoid coordinates that fall between two successive points of the lookup table, an interpolation is used.
On the circumstance $c_0 = 1$, $\theta_0 = 0$, $\kappa_0 = 0$,

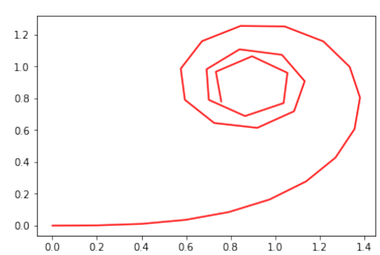

Figure 7: Straight line interpolation

Linear interpolation results in higher interpolation error as length of the clothoid s grows, as is visible in Figure 8. We define the interpolation error as the Euclidean distance between the exact point and the interpolated point.

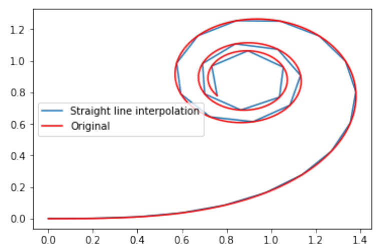

Figure 8: Comparison of straight line interpolation and the original clothoid

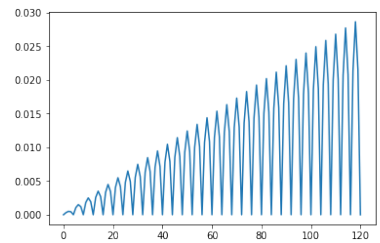

Figure 9: Straight line interpolation error

The x axis of the Figure 9 is the number of the interval. For this situation $\Delta s_L = 0.2$cm, we take 4 samples in 0.2 cm, which means 0.05 cm per interval. The reason of the growth of the straight line interpolation error is the increasing curvature, which results in larger distance of the clothoid curve from the interpolation line. $\kappa(s) = \kappa_0 + cs$, the initial κ_0 is 0, the curvature of the straight line is always 0, so the difference between these two curvatures depends on the length of the clothoid.

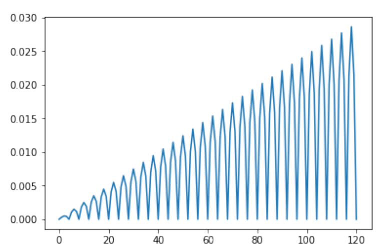

Figure 10: $c_0 = 1$

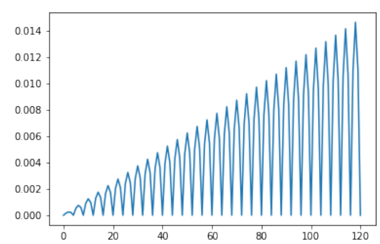

Figure 11: $c_0 = 0.5$

On the same circumstance except the initial sharpness, the interpolation error at the length 6cm of clothoid is 0.0146 ($c_0 = 0.5$) and 0.0286 ($c_0 = 1$) . The ratio of these two interpolation errors is approximately proportional to the the ratio of two sharpness. Thus we verify the correlation between the interpolation error and the curvature.

Because of the high interpolation error, the straight-line interpolation is a poor choice when dealing with clothoids. However, it can be used in cases with a high computation speed. In this paper, a interpolation with circle arcs is presented, as a clothoid can be viewed as an infinite succession of circular arc segments with linearly growing curvature, which has been proved in the previous part.

3.2 CIRCULAR INTERPOLATION

All of this part present the algorithm 1 **GETBASICCLOTHOIDCOORDS** in this paper.

To answer a query that falls between two successive points of the lookup table the circular interpolation is used. To perform circular interpolation between jth and (j+1)th point of the basic clothoid, where j ≥ 0, we calculate the radius of curvature at the middle of the segement :

$$r_j = \frac{1}{\kappa_{mid}} \tag{5}$$

On the circumstance of $\kappa_0 = 0$:

$$r_j = (c_L(j + 0.5)\Delta s_L)^{-1} \tag{6}$$

According to the formula of tangent angle :

$$\theta(s) = \theta_0 + \kappa_0 s + \frac{1}{2}cs^2 \tag{7}$$

With $\theta_0 = 0$ and $\kappa_0 = 0$,

$$\theta_L^j = \frac{1}{2}c_L(j\Delta s_L)^2 \tag{8}$$

Thus, the center of the curvature in the jth point is,

$$x_c^j = x_L^j + r_j cos(\theta_L^j + \frac{\pi}{2}), y_c^j = y_L^j + r_j sin(\theta_L^j + \frac{\pi}{2}) \tag{9}$$

from where interpolated coordinates $(x_{Li}(s), y_{Li}(s))$ at length s are

$$x_{Li}(s) = x_L^j + 2r_j cos(\theta_L^j + \frac{\Delta s}{2r_j}) sin(\frac{\Delta s}{2r_j}) \tag{10}$$

$$y_{Li}(s) = y_L^j + 2r_j sin(\theta_L^j + \frac{\Delta s}{2r_j}) sin(\frac{\Delta s}{2r_j}) \tag{11}$$

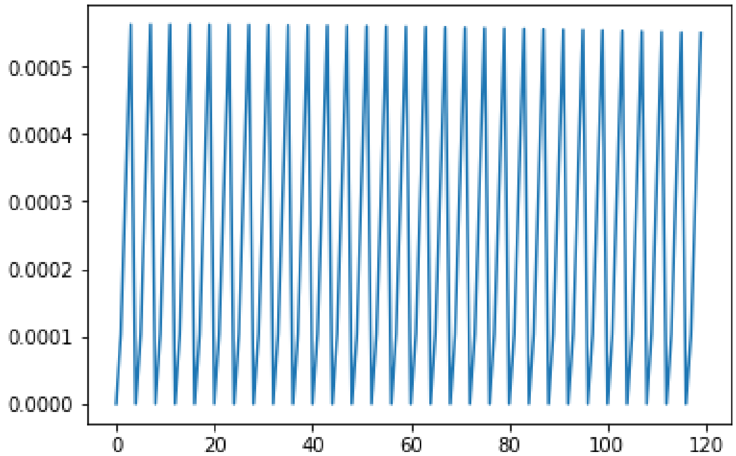

Figure 12: Circular interpolation

where Δs = s - jΔs_L is the distance along the basic clothod between jth point in the lookup table and an interpolated point.

As shown in Figure 12, the circular interpolation is substantially lower than linear interpolation error, and for this case $c_L = 1, \Delta s_L = 0.2, s_L = 6$. This decreases efficiency and complicates estimation of the maximum error at a sampling interval is not monotonically decreasing with s, which is presented by Figure 13. To find out how the value of parameters are related with the interpolation error, we use control variate method.

I sample four cases to visualize the relation between Δs_L and $e_{max}/\Delta s_L^3$ on the circumstance $c_L = 1$. According to the Figure 14, the ratio $e_{im}/\Delta s_L^3$ is around the constant 0.08, which means that halving Δs_L can reduce the error of interpolation by $\frac{1}{8}$. With a fixed length of clothoid, if the lookup table has n points, which is inversely proportional to Δs_L, the approximation error is of order O(Δs_L^3), which identify with O($1/n^3$).[1]

In order to determine the relation between the basic clothoid sharpness c_L and e_{im}, I sample four cases on the circumstance $\Delta s_L = 0.2$, as shown in Figure 15, it is visible that the ratio e_{im}/c_L is around the constant 0.000562, which means that the maximum interpolation error is proportional with c_L.Because the approximation error is proportional to Δs_L^3, the Figure 15 is on the circumstance of $\Delta s_L = 0.2$, the real ratio e_{im}/c_L is 0.0702. According-

[1]This is the first idea about derivation, but not the calculation.

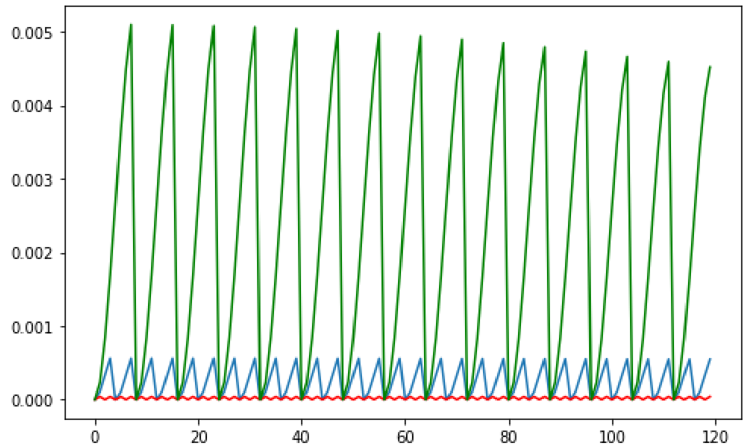

Figure 13: Interpolation error e_i for three different values of lookup table sampling interval Δs_L

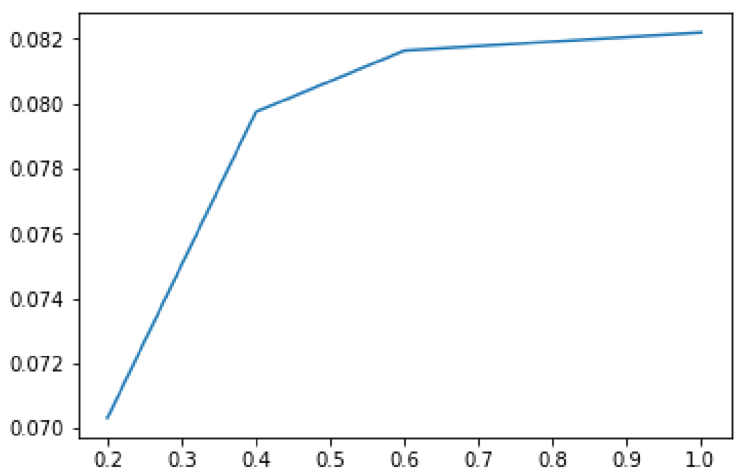

Figure 14: Δs_L and $e_{max}/\Delta s_L^3$

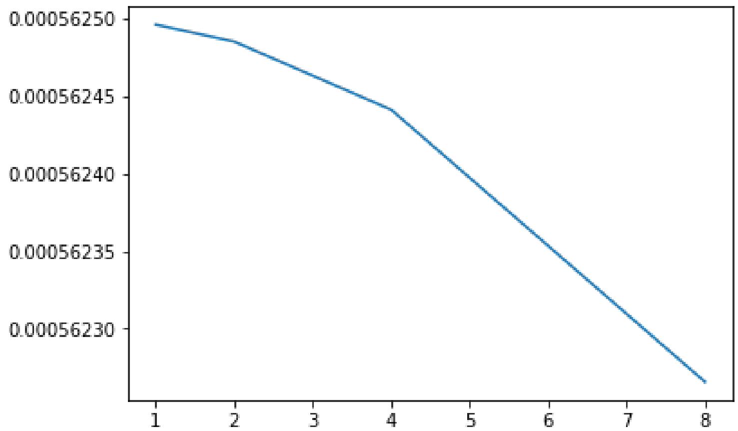

Figure 15: c_L and e_{max}/c_L on the circumstance $\Delta s_L = 0.2$

ing to these previous analysis, we can obtain,

$$e_{im} \leq 0.0702 c_L \Delta s_L^3$$

There is a contradiction, the ratio of $e_{im}/\Delta s_L^3$ is around 0.08, but the ratio of e_{im}/c_L is around 0.07.

4 DETERMINATION OF A REQUIRED SET OF CLOTHOID

When wording with clothoids, a set of clothoids are required, thus we have to find an allowed range of clothoid parameters which is suitable for particular application. In the following section, we discuss clothoids in the first quadrant, which means all parameters of a clothoid are nonegative.

- Bounding the clothoid Orientation Change and Length
  According to the formula of the Tangent angle,

  $$\theta(s) = \theta_0 + \kappa_0 s + \frac{1}{2} c s^2 \tag{12}$$

  The orientation change $\Delta\theta$ that occurs along a single clothoid can be bounded. For example, in the path planning an orientation change

greater than $\Delta\theta_{max} = \frac{\pi}{2}$ is rarely required.
On the basis of the formula for finding roots of a quadratic equation and $C^2 = \frac{1}{c}$,

$$s \leq -C^2\kappa_0 + C\sqrt{C^2\kappa_0^2 + 2\Delta\theta_{max}} \tag{13}$$

Using (13), as shown in Figure 12 and Figure 13, because of the $\kappa_0 = 0$ and $C = 1$, we obtain the maximum length of the clothoid with $\Delta\theta_{max} = \pi/2$ and $\Delta\theta_{max} = \pi$ is,

$$s_{\Delta\theta_{max}=\pi/2} = \sqrt{\pi} = 1.772, s_{\Delta\theta_{max}=\pi} = \sqrt{2 * \pi} = 2.506 \tag{14}$$

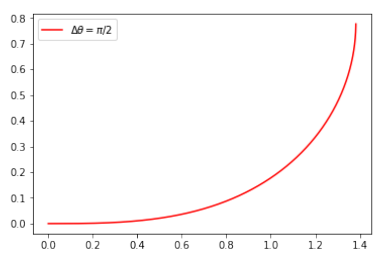

Figure 16: $\Delta\theta_{max} = \pi/2$

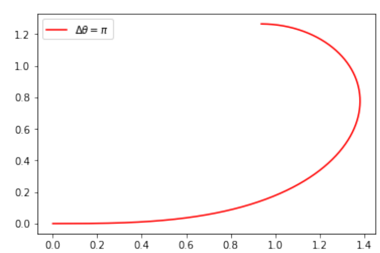

Figure 17: $\Delta\theta_{max} = \pi$

The maximum length of the clothoid can be additionally limited by some fixed upper-bound s_{max}, which can be dtermined. Using (13), the clothoid length is now upper bounded by

$$s \leq min(-C^2\kappa_0 + C\sqrt{C^2\kappa_0^2 + 2\Delta\theta_{max}}, s_{max}) \tag{15}$$

- Determination of the Minimum Scaling
  To be on the safe side, we have to consider extreme cases and exactly predict what happens then in order not to exceed maximum allowed approximation error. As scaling C decreases, this bounding circle becomes smaller. The difference between a small C and a big C has been presented in Figure 3 and Figure 4. In the limit case C = 0, a clothoid reduces to a point, which means that to scale smaller than some value C_{min}, clothoid can be approximated by a point without exceeding specified maximum approximation error. As shown in Figure 14, the scale

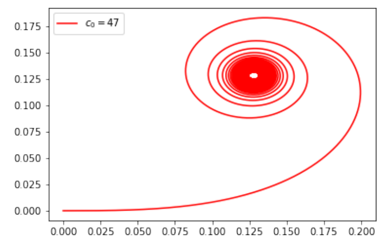

Figure 18: With a smaller C

of clothoid is small and the tendency of the clothoid is a point.
We define approximation error e as the Euclidean distance between an exact clothoid point (x_e, y_e), and its approximating point(x_a, y_a)

$$e = \sqrt{((x_e - x_a)^2 + (y_e - y_a)^2)} \tag{16}$$

A corresponding approximation error e_{pt} is the Euclidean distance of the most distant clothoid point to the origin, which can be expressed as a function of C proved in the part of straight line interpolation,

$$e_{pt}(C) = \sqrt{(x_d(C)^2 + y_d(C)^2)} \tag{17}$$

On the circumstance of $x_0 = 0, \kappa_0 = 0, \theta_0 = 0$ so that,

$$x_d(s) = \int_0^{s_d} cos(\frac{\epsilon^2}{2C^2})d\epsilon \tag{18}$$

$$y_d(s) = \int_0^{s_d} sin(\frac{\epsilon^2}{2C^2})d\epsilon \tag{19}$$

$$s_d = argmax_s(x_d(C)^2 + y_d(C)^2) \tag{20}$$

We can find the minimum scaling C_{min} by computing the scaling at which the approximation error e_{pt} is equal to maximum allowed error e_{max}. By solving nonlinear equation $e_{pt}(C_{min}) = e_{max}$. As shown in Figure 19, s_d is proportional to C.

$$e_{pt}(C) = \sqrt{(x_d(C)^2 + y_d(C)^2)} \tag{21}$$

so

$$e_{pt}(C_{min}) = C_{min} e_{pt}(1) \tag{22}$$

$$C_{min} = e_{max}/e_{pt}(1) \tag{23}$$

Without exceeding maximum allowed error e_{max}, we have the range of

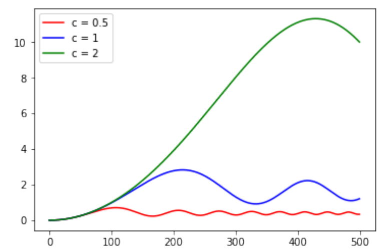

Figure 19: s_d in different choice of c

the low scaling parameter C_{min}.

- Determination of the Maximum Scaling

  In another extreme case, there is a high approximation error with a large scaling size. Thus, scaling should be upper-bounded. On the circumstance of $\kappa_0 = 0$ and $C = \infty$, a clothoid is reduced to a straight line as shown in 20.

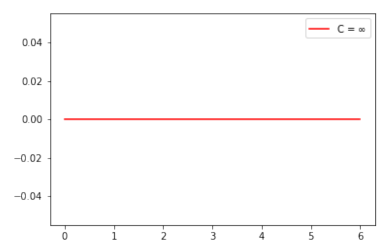

Figure 20: s_d in different choice of c

Although y is always zero, the position of x equal to the length s. According to the equation (16),

$$e_{ln}(C, s) = \sqrt{(s - \int_0^s cos(\frac{\epsilon^2}{2C^2})d\epsilon)^2 + (\int_0^s sin(\frac{\epsilon^2}{2C^2})d\epsilon)^2} \tag{24}$$

The error grows with clothoid length, because clothoid departs more from approximating line as length grows. We can find the worst case at maximum clothoid length s = s_{max}. Thus, we have to solve the following nonlinear equation for C_{max}

$$e_{ln}(C_{max}, s_{max}) = e_{max} \tag{25}$$

If there is a query occurs with C > C_{max}, without violating the maximum allowed error e_{max}, the formula can be simplified on the circumstance of $C = \infty$ and $\kappa_0 = 0$:

$$\begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} cos(\theta_0) \\ sin(\theta_0) \end{bmatrix} s \tag{26}$$

- Determination of Maximum Initial Curvature

  In some application, we need the case when clothoid initial curvature is nonzero. The clothoid is reduced to a circle with radius $1/\kappa_0$. K = $\kappa_0 C$, which serve as a measure of similarity of clothoid to a circle. As shown in 21 and 22, by increasing K, a clothoid become more similar to a circle.

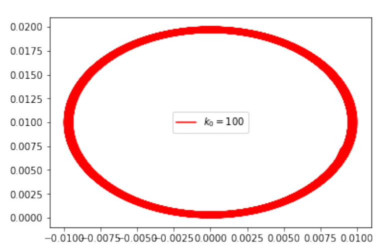

Figure 21: $c_0 = 1$, $\kappa_0 = 100$

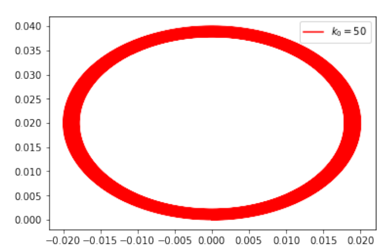

Figure 22: $c_0 = 1$, $\kappa_0 = 50$

Assume $\kappa_0 \neq 0, C = \infty, x_0 = 0, y_0 = 0$ and $\theta_0 = 0$,

$$x(s) = x_0 + \int_0^s cos(\theta_0 + \kappa_0\epsilon + \frac{1}{2}c\epsilon^2)d\epsilon = \int_0^s cos(\kappa_0\epsilon)d\epsilon$$

$$x(s) = \frac{1}{\kappa_0} sin(\kappa_0 s)$$

A clothoid reduces to a circular arc of radius $1/|\kappa_0|$,

$$y(s) = \frac{1}{\kappa_0}(1 - cos(\kappa_0 s))$$

So, the error of approximating clothoid by a circular arc at length s is,

$$\begin{aligned} e_{arc}^2 = (\frac{1}{\kappa_0} sin(\kappa_0 s) - \int_0^s cos(\kappa_0 \epsilon + \frac{1}{2} c\epsilon^2) d\epsilon)^2 + \\ (\frac{1}{\kappa_0}(1 - cos(\kappa_0 s)) - \int_0^s sin(\kappa_0 \epsilon + \frac{1}{2} c\epsilon^2) d\epsilon)^2 \end{aligned} \tag{27}$$

This error becomes higher as length of the clothoid grow. The worst case occurs at maximum clothoid length $s = s_{max}$ or at a maximum orientation change $\Delta\theta = \Delta\theta_{max}$, thus

$$s \leq min(\frac{\Delta\theta_{max}}{|\kappa_0|}, s_{max}) \tag{28}$$

- Determination of the lookup table parameters

  In order to obtain required length of the basic clothoid s_L, maximum possible values of the lookup table arguments must be found. We have defined x_L and y_L in equation (3), which contain κ_0, s and C.

$$\sqrt{\frac{|c|}{c_L}}(s + \frac{\kappa_0}{c}) = \frac{C_L}{C}(s + \kappa_0 C) = C_L(\frac{s}{C} + K)$$

The absolute of the part of $\sqrt{\frac{|c|}{c_L}}(\frac{\kappa_0}{c})$ is always less than or equal to absolute value of the previous formula. We have obtain the upper-bound of the length s.

$$\frac{s}{C}_{max} = min(\sqrt{2\Delta\theta_{max}}, \frac{s_{max}}{C_{min}})$$

$$s_L = C_L(\frac{s}{C} + K)$$

$$s_{Lmax} = C_L(min(\sqrt{2\Delta\theta_{max}}, \frac{s_{max}}{C_{min}}) + K_{max})$$

which means,

$$s_L \leq C_L(min(\sqrt{2\Delta\theta_{max}}, \frac{s_{max}}{C_{min}}) + K_{max})$$

[2]

Though there is a contradiction of the ratio $e_{im}/(\Delta s_L^3 c_L)$, we obey the result of this paper.

$$e_{im} \leq 0.084 c_L \Delta s_L^3$$

e_{im} is proportional with c_L, so e $= \frac{C}{C_L} e_{im}$, where e_{im} is the maximum interpolation error. We bound the maximum allowed error as $e \leq e_{max}$ and consider the worst case $C = C_{max}$, thus

$$\Delta s_{Lmax} \approx (0.084)^{\frac{1}{3}} C_L \sqrt{\frac{e_{max}}{C_{max}}}$$

Besides, the real Δs_{Lmax} depends on the typical case. So in this paper,

$$\Delta s_{Lmax} \approx (0.084)^{\frac{1}{3}} C_L \sqrt{min(\frac{e_{max}}{C_{max}}, \frac{e_{typ}}{C_{typ}})}$$

[2]This is a place that contradicts the thesis.