Chapter III

EXPERIMENTS, MODELS AND PROBABILITIES

1 Definitions and Important Results

We denote by S the universal set and by $\emptyset$ the null set.
A collection of sets $A_1, A_2, ..., A_n$ is **mutually exclusive** iff

$$A_i \cap A_j = \emptyset, (\forall)\, i \neq j,\ i, j = \overline{1, n}.$$

A collection of sets $A_1, A_2, ..., A_n$ is **collectively exhaustive** iff

$$A_1 \cup A_2 \cup ... \cup A_n = S.$$

An experiment consists of the following: procedure, observation, and model.

Example Procedure: flip a coin and let it land on a table.
Observation: Observe which side (head or tail) faces to you after the coin is landed.
Model: Heads and tails are equally likely; the result of each flip is unrelated to the results of the previous flip.

Remark: Two experiments with the same procedure but different observations are different experiments.

An outcome of an experiment is any possible observation of that experiment.

The sample space of an experiment is the finest-grain, mutually exclusive, collectively exhaustive set of all possible outcomes.

Example For the previous example, the sample space consists of $S = \{h, t\}$, where h is the outcome "observe head", and t is the outcome "observe tail".

Example Flip a coin three times and observe the sequence of heads and tails. The sample space consists of:

$$S = \{hhh, hht, hth, thh, ttt\}.$$

Example Flip a coin three times and observe the number of heads. The sample space consists of:

$$S = \{0, 1, 2, 3\}.$$

An event is a set of outcomes of an experiment.

An event space K is a collectively exhaustive, mutually exclusive set of events, that is the following axioms holds:

1. $S \in K$;
2. $A \in K \Rightarrow \bar{A} \in K$;
3. $A_1, A_2, ..., A_n \in K \Rightarrow \bigcup\limits_{i=1}^{n} A_i \in K$.

Remark: The members of the sample space are outcomes (is a set of elements). The members of an event space are events (is a set of sets).

A probability measure P is a function that maps events in a sample space to real numbers $P : K \to \mathbb{R}$ such that:

1. $P(A) \geq 0, (\forall)\, A \in K$;
2. $P(S) = 1$;
3. $(\forall)\, A_1, A_2, ..., A_n, ..., A_i \cap A_j = \emptyset, i \neq j \Rightarrow P\left(\bigcup\limits_{i=1}^{\infty} A_i\right) = \sum\limits_{i=1}^{\infty} P(A_i)$.

Properties:

1. $P(A) \in [0, 1]$;
2. $A_i \cap A_j = \emptyset, i \neq j \Rightarrow P\left(\bigcup\limits_{i=1}^{n} A_i\right) = \sum\limits_{i=1}^{n} P(A_i)$;
3. $A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$;

Otherwise: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
and

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C);$$

4. $P(\emptyset) = 0$;
5. $P(\bar{A}) = 1 - P(A)$;
6. $A \subseteq B \Rightarrow P(A) \leq P(B)$;
7. $P(B \backslash A) = P(B) - P(A \cap B)$;

If $A \subset B$ then $P(B \backslash A) = P(B) - P(A)$.

Proposition 1.1. *For any event A, and event space $\{B_1, B_2, ..., B_n\}$*

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i).$$

2 Conditional Probability

The conditional probability of the event A given by the occurrence of the event B is

$$P(A \mid B) \overset{not}{=} P_B(A) = \frac{P(A \cap B)}{P(A)}.$$

Properties

1. $P_B(A) \geq 0$;
2. $P_A(A) = 1$;
3. If $A = \left(\bigcup\limits_{i=1}^{\infty} A_i\right), A_i \cap A_j = \emptyset, i \neq j$, then

$$P_B(A) = P_B(A_1) + P_B(A_2) + ... = \sum_{i=1}^{\infty} P_B(A_i).$$

Law of Total Probability

Let $A \in K$, and $\{B_1, B_2, ..., B_n\}$ an event space, then

$$P(A) = P(B_1) \cdot P_{B_1}(A) + P(B_2) \cdot P_{B_2}(A) + ... + P(B_n) \cdot P_{B_n}(A).$$

Bayes' Theorem

$$P_B(A) = \frac{P_A(B) \cdot P(A)}{P(B)}.$$

Multiplication Rule

$$P(A \cap B) = P(B) \cdot P_B(A)$$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = P(A_1) \cdot P_{A_1}(A_2) \cdot P_{A_1 \cap A_2}(A_3) \cdot ... \cdot P_{A_1 \cap A_2 \cap ... \cap A_{n-1}}(A_n).$$

The events A and B are **independent** iff $P(A \cap B) = P(A) \cdot P(B)$. The events A_1, A_2, A_3 are **independent** iff two by two are independent, and

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3).$$

Remark Independent and disjoint are not synonyms. In most situations, independent events are not disjoint.

3 Counting Methods

Fundamental Principle of Counting If subexperiment A has n_1 possible outcomes, and subexperiment B has n_2 possible outcomes, then there are $n_1 \cdot n_2$ possible outcomes when you perform both subexperiments.
Generally: If an experiment E has k subexperiments $E_1, E_2, ..., E_k$, where E_i has n_i outcomes, then E has $n_1 \cdot n_2 \cdot ... \cdot n_k$ outcomes.

Example Shuffle a deck and observe each card starting from the top. The outcome of the experiment is an ordered sequence of the 52 cards of the deck. How many possible outcomes are there?

The procedure consists in 52 subexperiments. The observation is the identity of the card. The first subexperiment has 52 possible outcomes corresponding to the 52 cards that could be drown. After the first card is drown, the second subexperiment has 51 possible outcomes (51 remaining cards). Therefore, the total outcomes is $52 \cdot 51 \cdot ... \cdot 1 = 52!$.

An ordered sequence of k distinguishable objects is called **a $k-$permutation** (or n permutation of size k.) Denote by $(n)_k$ the number of possible $k-$permutation of n distinguishable objects.

Choosing objects from a collection is called **sampling**, and the chosen objects are known as a **sample.**

If we choose an object for a $k-$permutation and we remove this object from the collection, then we call this **sampling without replacement.** When an object can be chosen repeatedly, we have **sampling with replacement.**

The number of $k-$combinations of n objects is denoted

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}, \quad k \leq n,$$

("n choose k".)

Example We draw seven cards. What is the probability of getting a hand without any queens?

We have $\binom{52}{7}$ possible hands and $\binom{48}{7}$ hands that have no qeens.

Therefore, the probability is $P = \frac{\binom{48}{7}}{\binom{52}{7}} = 0.5504$.

Given m distinguishable objects, there are m^n ways to choose with replacement an ordered sample of n objects.

Example 1. How many binary sequences of length 10 there exists? 2^{10}
2. How many four letter words there exists? 26^4

For n repetitions of a subexperiment with sample space $S = \{s_0, s_1, ..., s_{m-1}\}$, there are m^n possible observation sequences.

The number of observation sequences for n subexperiments with $S = \{0, 1\}$ sample space, with 0 appearing n_0 times and 1 appearing $n_1 = n - n_0$ times is $\binom{n}{n_1}$.

Example How many observation sequences are there in which 0 appears two times and 1 appears three times? $\binom{5}{2} = 10$.

Generally For n trials of a subexperiment with sample space $S = \{s_0, s_1, ..., s_{m-1}\}$, the number of observation sequences in which s_0 appears n_0 times, s_1 appears n_1 times, and so on $(n_0 + n_1 + ... + n_{m-1} = m)$, is

$$\binom{n}{n_0, n_1, ..., n_{m-1}} = \frac{n!}{n_0! n_1! ... n_{m-1}!}.$$

(**the multinomial coefficient**).

Example Consider a binary code with four bits (0 or 1) in each code word.
1. How many different code words are there? $2^4 = 32$
2. How many different code words have exactly two zeros? $\binom{4}{2} = 6$.
3. How many code words begin with a zero? $2^3 = 8$
4. In a constant ratio binary code, each code has eight bits. In every word, three of the eight bits are 1 and the other are 0. How many different code words are? $\frac{8!}{3!5!}$

Consider a simple subexperiment in which there are two outcomes. The results of all trials are mutually independent and a success occurs with probability p, while a failure occurs with probability $1-p$.
We denote by S_{n_0,n_1} the event "n_0 failures and n_1 successes in $n = n_0 + n_1$ trials," then the probability of S_{n_0,n_1} is:

$$P(S_{n_0,n_1}) = \binom{n}{n_1}(1-p)^{n_0} \cdot p^{n_1}.$$

Generally A subexperiment has sample space $S = \{s_0, s_1, ..., s_{m-1}\}$, with $P(s_i) = p_i,\ i = \overline{0, m-1}$. For $n = n_0 + n_1 + ... + n_{m-1}$ independent trials, the probability of n_i occurs of $s_i,\ i = \overline{0, m-1}$ is:

$$P(S_{n_0,n_1,...,n_{m-1}}) = \binom{n}{n_0, n_1, ..., n_{m-1}} p_0^{n_0} \cdot ... \cdot p_{m-1}^{n_{m-1}}.$$

Example To communicate one bit of information reliably, cellular phones transmit the same binary symbol 5 times (the information 0 is transmitted as 00000 and 1 as 11111). The receiver detects the correct information if 3 ore more binary symbols are receive correctly. What is the information error probability $P(E)$ if the binary error probability is $q = 0.1$?

The error event E occurs when the number of successes is strictly less than 3:

$$P(E) = P(S_{0,5}) + P(S_{1,4}) + P(S_{2,3}) = \binom{5}{0} q^5 + \binom{5}{1} q^4 p \binom{5}{2} q^3 p^2,$$

where $p = 1 - q = 0.09$, therefore

$$P(E) = 10(0.1)^2(0.9)^3 + 5(0.1)(0.9)^4 + (0.9)^5.$$

4 Solved Problems

1. A company has three machines M_1, M_2, M_3 for making resistors. It has been observed that 20% of resistors produced by M_1 are technically

inadequate. Machine M_2 produces 10% unacceptable resistors, and M_3 produces 40%.

Each hour, machine M_1 produces 3000 resistors, M_2 produces 4000 resistors, and M_3 3000 resistors. All the resistors are mixed together at random in one bin and packed for shipment.

a) What is the probability that the company ships an acceptable resistor?
b) What is the probability that an acceptable resistor comes from machine M_3?

Solution: a) Consider the events A : "a resistor is acceptable", and B_i : "the resistor is produced by M_i", $i = \overline{1,3}$.

Then:

$$P_{B_1}A = 0.8, P_{B_2}A = 0.9, P_{B_3}A = 0.6$$

Using the Law of total probabilities, we obtain:

$$P(A) = P(B_1) \cdot P_{B_1}(A) + P(B_2) \cdot P_{B_2}(A) + P(B_3) \cdot P_{B_3}(A),$$

where $P(B_1) = \frac{3000}{10000} = 0.3, P(B_2) = \frac{4000}{10000} = 0.4$ and $P(B_3) = 0.3$.

Therefore, we obtain:

$$P(A) = 0.8 \cdot 0.3 + 0.9 \cdot 0.4 + 0.6 \cdot 0.3 = 0.78.$$

This means that for the whole factory, 78% of resistors are acceptable.

b) Using Bayes' Theorem:

$$P_{B_3}(A) = \frac{P_A(B_3) \cdot P(B_3)}{P(A)} = \frac{0.6 \cdot 0.3}{0.78} = 0.23.$$

2. Suppose traffic engineers have coordinated the timing of two traffic lights to encourage a run of green lights.
In particular, the timing was designed so that with probability 0.8 a driver will find the second light to have the same color as the first.
a) Assuming the first light is equally likely to be red or to be green, what is the probability that the second light is green?
b) What is the probability that you wait for at least one light?
c) What is the conditional probability of a green first light given a red second light?

Solution: a) Consider the events:
G_1 : "the first light is green"
G_2 : "the second light is green"
R_1 : "the first light is red"
R_2 : "the second light is red."

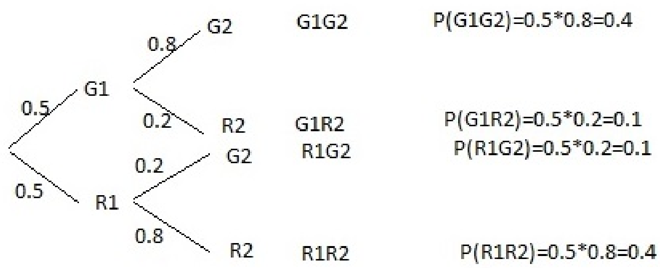

Figure 1

Following the above Figure, the probability of G_2 is:

$$P(G_2) = P(G_1 \cap G_2) + P(R_1 \cap G_2) = 0.4 + 0.1 = 0.5.$$

b) The event that we asked for is $W = (G_1 \cap R_2) \cup (R_1 \cap G2) \cup (R_1 \cap R_2)$, so:

$$P(W) = P(G_1 \cap R_2) + P(R_1 \cap G_2) + P(R_1 \cap R_2) = 0.1 + 0.1 + 0.4 = 0.6.$$

c)

$$P_{R_2}(G_1) = \frac{P(G_1 \cap R_2)}{P(R_2)} = \frac{P(G_1 \cap R_2)}{P(G_1 \cap R_2) + P(R_1 \cap R_2)} = \frac{0.1}{0.1 + 0.4} = \frac{1}{5}.$$

3. Suppose you have two coins, one biased, one fair, but you don't know which coin is which.

   Coin 1 is biased. It comes up heads with probability $\frac{3}{4}$, while coin 2 will flip heads with probability $\frac{1}{2}$.

   Suppose you pick a coin at random and flip it. Denote by C_i the event that coin i is picked, and by H, respectively T, the possible outcomes of the flip.

a) Given that the outcome of the flip is a head, what is the probability that you pick the biased coin?

b) Given that the outcome is a tail, what is the probability that you picked the biased coin?

Solution:

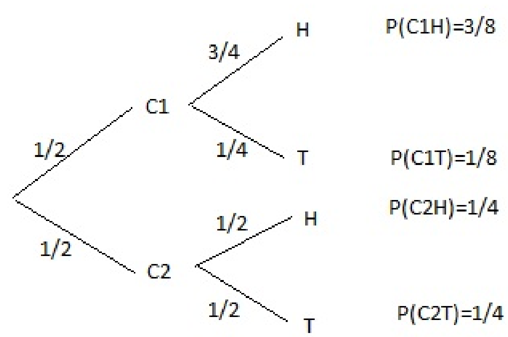

Figure 1

a)

$$P_H(C_1) = \frac{P(C_1 \cap H)}{P(H)} = \frac{P(C_1 \cap H)}{P(C_1 \cap H) + P(C_2 \cap H)} = \frac{\frac{3}{8}}{\frac{3}{8} + \frac{1}{4}} = \frac{3}{5}.$$

b)

$$P_T(C_1) = \frac{P(C_1 \cap T)}{P(T)} = \frac{P(C_1 \cap T)}{P(C_1 \cap T) + P(C_2 \cap T)} = \frac{\frac{1}{8}}{\frac{1}{8} + \frac{1}{4}} = \frac{1}{3}.$$

4. The probabilities that three archers A, B, C hit the mark, independently of one another, are respectively $\frac{1}{6}, \frac{1}{4}$, and $\frac{1}{3}$. Every one shoots an arrow.
a) Find the probability that only one hits the mark.
b) If only one hits the mark, what is the probability he is archer A?

Solution: a)

$$P(A \cup B \cup C) = P(A \cap \bar{B} \cap \bar{C}) + P(\bar{A} \cap B \cap \bar{C}) + P(\bar{A} \cap B \cap \bar{C})$$

$$= \frac{1}{6}\cdot\frac{3}{4}\cdot\frac{2}{3}+\frac{5}{6}\cdot\frac{1}{4}\cdot\frac{2}{3}+\frac{1}{6}\cdot\frac{3}{4}\cdot\frac{2}{3}=\frac{31}{72}.$$

b)

$$P_X(A)=\frac{P(X\cap A)}{P(X)}=\frac{\frac{1}{6}}{\frac{31}{72}}=\frac{12}{31},$$

where $X = A\cup B\cup C$.

5. Deer ticks can carry both Lyme disease and HGE. In a study of ticks, it was found that 16% carried Lyme, 10% had HGE and 10% of the ticks that had either Lyme or HGE, carried both disease.
a) What is the probability that a tick carries both Lyme and HGE?
b) What is the conditional probability that a tick has HGE given that it has Lyme disease?

Solution: a) Let us denote by L the event "the tick has Lyme disease" and by H the event "the tick has HGE". Then, $P(L) = 0.16, P(H) = 0.1$, and $P_{L\cup H}(L\cap H) = 0.1$.
a)

$$0.1 = P_{L\cup H}(L\cap H)=\frac{P(L\cap H)}{P(L\cup H)}=\frac{P(L\cap H)}{P(L)+P(H)-P(L\cap H)}=\frac{P(L\cap H)}{0.26-P(L\cap H)}.$$

Therefore, $P(L\cap H) = 0.023$.

b) $P_L(H)=\dfrac{P(L\cap H)}{P(L)}=\dfrac{0.023}{0.16}=0.14.$

6. Assume women and men exist in equal number, and assuming that 5% of the men are color blind and that 0.25% of the women are color blind.
a) Evaluate the probability that a person drawn at random is color blind.
b) Evaluate the probability that, having drawn a color-blind person, this is a man.

Solution: a) Let us denote by M the event that the person drawn is a men, and by W the event that the person drawn is a women. Then, $\{M, W\}$ is a space event.
Let B be the event that the drawn person is color blind. Using the Law of Total Probabilities, we obtain:

$$P(B)=P(M)\cdot P_M(B)+P(W)\cdot P_W(B)=\frac{1}{2}\cdot\frac{0.25}{100}+\frac{1}{2}\cdot\frac{5}{100}=0.02625.$$

b) Using Bayes' Theorem:

$$P_B(M) = \frac{P_M(B) \cdot P(M)}{P(B)} = \frac{\frac{5}{100} \cdot \frac{1}{2}}{0.02625} = 0.952.$$

5 Exercises

1. How many different ID cards can be made if there are 6 digits on a card and no digit can be used more than one?

2. Suppose you are going to order an ice cream with two different flavored scoops. You are going to take a picture of your ice cream cone for post it on FB. The ice cream shop has five flavors to choose from: chocolate, vanilla, orange, strawberry, and mint.

   How many different different ice cream cone photos are possible?
   R: 20

3. a) In how many ways can a student select five questions to answer from an exam containing 9 questions?
   b) In how many ways can a student select five questions to answer from an exam containing nine questions if the student is required to answer the first and the last question?

4. An electronic car door lock has five buttons on it and each button has a different letter A, B, C, D, E. Suppose the combination to unlock the door is four letters long.
   a) How many different combinations are possible if a letter may be repeated?
   b) How many different combinations are possible if a letter may not be repeated?

5. a) In how many ways can a committee of three people be chosen if there are eight men and four women available for selection and we require that two men and one woman be on the committee?
   b) But if we require at least two women on the committee?

6. The year book editor must select two photos out of 42 juniors and two out 45 seniors for a page in the yearbook. How many photo combinations are possible?

7. There are 20 candidates in the Timişoara Contest. How many ways could the jury choose the winner, first runner-up and second runner-up?

8. A student has to learn 20 subjects for an exam, but he learns only 15. If the exam consists of three questions, what is the probability:
a) that the student answer to two from three questions?
b) that the student answer at least two questions?

9. Let M be the set of all functions $f : \{1, 2, 3\} \to \{5, 6, ..., 10\}$. Find the probability that if we choose one function from M it is:
a) injective
b) bijective
c)strictly increasing
d) strictly monotone

10. Let A and B two events with $P(A) = 0.8, P(B) = 0.5$ and $P(A \cap B) = 0.4$. Find $P(\bar{A} \cap \bar{B})$.

11. Shuffle a deck of cards, flip a coin and roll a dice. Find the probability to pick a 10, a H and an even number.

12. Roll three dices. What is the probability that the sum of the points is 6?