Ch. 1 Representation of numbers in computer systems

1.1. Information classification

binary digit → bit ; byte ; word

- Information
  - instructions
  - Data
    - Numbers
      - Fixed-point
        - Binary
          - Sign-Magnitude
          - One's Complement
          - Two's Complement
        - Decimal
          - Binary Coded Decimal
          - Excess of 3
          - Two-out-of-five
      - Floating-point
        - IEEE 754
        - IBM Floating-point format
    - Non-numerical data
      - ASCII
      - EBCDIC
      - UNICODE: UTF-8, UTF-16, ...

Fixed-point no: integers, fractionals
- high precision
- moderate HW complexity

Floating-point nos = real values
- lower precision ⇒ approximate repres.
- higher HW complexity

$\overset{2^3}{0}\,\overset{2^2}{1}\,\overset{2^1}{0}\,\overset{2^0}{0} = 4$

$\underset{2^0}{0}.\underset{2^{-1}}{1}\,\underset{2^{-2}}{0}\,\underset{2^{-3}}{0} = \frac{1}{2}$

$.0100 = \frac{1}{4}$

Ex: $1e20 = 10^{20}$

$(3.14 + 1e20) - 1e20 \rightarrow 0$ f.p. format

$3.14 + (1e20 - 1e20) \rightarrow 3.14$

! F.p. arithmetic breaks associativity

1.2. Representation of fixed-point no.

$$X = X_{n-1} X_{n-2} \cdots X_1 X_0 . X_{-1} X_{-2} \cdots X_{-m}$$

r = Radix

$\xleftrightarrow{<n>}$ integer part . $\xleftrightarrow{<m>}$ fractional part

value of X $\quad X = \sum_{j=-m}^{n-1} X_j * \underbrace{r^j}_{\text{weight}}$ $\quad 0 \le X_j < r$

$X_{n-1} \equiv msb$

POSITIONAL representation $\quad \boxed{\boxed{r == 2}}$ $\quad X_{-m} = lsb$

Ex: $103_{10} = 1100111._{(2)}$

$2^6 \; 2^5 \; 2^4 \; 2^3 \; 2^2 \; 2^1 \; 2^0$

$68_{10} = 1000100._{(2)}$

integers: $X = X_{n-1} X_{n-2} \cdots X_1 X_0 . = \sum_{i=0}^{n-1} x_i \cdot 2^i$

$\Updownarrow$

fractionals: $X = . X_{n-1} X_{n-2} \cdots X_1 X_0 = \sum_{i=0}^{n-1} x_i \cdot 2^{i-n}$

Ex: $.1100111_{(2)} = 2^{-1} + 2^{-2} + 2^{-5} + 2^{-6} + 2^{-7} =$

$2^{-1} \; 2^{-2} \; 2^{-3} \; 2^{-4} \; 2^{-5} \; 2^{-6} \; 2^{-7}$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} = \frac{64 + 32 + 4 + 2 + 1}{128}$

$= \frac{103}{2^7} = 0.8046875$

1.2.1 Sign-Magnitude

msb $\longrightarrow$ sign $\begin{cases} 0 : \text{positives} \\ 1 : \text{negatives} \end{cases}$

$$X = \underbrace{X_{n-1}}_{\text{sign}} \; \underbrace{X_{n-2} X_{n-3} \cdots X_1 X_0}_{\text{magnitude}}$$

Ex: $+103_{10} = 0\,1100111_{(SM)}$

$-103_{10} = 1\,1100111_{(SM)}$

a) Range of values

- largest integer on n bits: $0\ \underbrace{11 \ldots 111}_{n-1}$

$$0111\ldots11 = 2^0 + 2^1 + \ldots + 2^{n-2}$$

$$= \frac{2^{n-1} - 2^0}{2-1} = 2^{n-1} - 1$$

range of values: $[1 - 2^{n-1} ; 2^{n-1} - 1]$

- largest fractional on n bits

$$0.11\ldots111 = \frac{2^{n-1} - 1}{2^{n-1}} = 1 - 2^{-n+1}$$

range of val.: $[2^{-n+1} - 1 ; 1 - 2^{-n+1}]$

b) Precision: SM no. on n bits

$$2^{n-1} - 1 = 10^p \rightarrow p = \lceil \log_{10} (2^{n-1} - 1) \rceil$$

for large n $p \simeq \lceil \log_{10} (2^{n-1}) \rceil = \lceil (n-1) \log_{10} 2 \rceil$

$$= \lceil (n-1) * 0.3 \rceil$$

Ex: $n = 10$: largest integer $= 511$ 3 digits.

$$p = \lceil 9 \cdot 0.3 \rceil = \lceil 2.7 \rceil = 3$$

c) HW complexity

- SM: moderate complexity
- ! favours multiplication

d) Disadvantages:

(A) Value of 0: $+0$: 0 00 ... 000 = 0; -0: 1 00 ... 000 = 0

right-shift

(B) Addition: $X = 5, Y = 2$ on 4 bits.

$X = +5$: 0101_{SM}
$Y = +2$: 0010_{SM} +
$0111_{SM} = +7$ ✓

$X = +5$: 0101_{SM}
$Y = -2$: 1010_{SM} +
$1111_{SM} = \cancel{-7}$

$X = -5$: 1101_{SM}
$Y = +2$: 0010_{SM} +
$1111_{SM} = \cancel{-7}$

$X = -5$: 1101_{SM}
$Y = -2$: 1010_{SM}
$\cancel{1}\,0111_{SM} = \cancel{+7}$

??

One's Complement:

$$\overline{X} = \begin{cases} 0\ x_{n-2}\ x_{n-3} \dots x_1 x_0, & \text{for } X \geq 0 \\ 1\ \overline{x_{n-2}}\ \overline{x_{n-3}} \dots \overline{x_1}\,\overline{x_0}, & \text{for } X \leq 0 \end{cases}$$

$$\overline{x_i} = 1 - x_i$$

Ex: $+103 = 0\,1\,1\,0\,0\,1\,1\,1_{c1}$

$-103 = 1\,0\,0\,1\,1\,0\,0\,0_{c1}$

a) Range of values: same as for SM

b) HW complexity:

c) Precision:

d) Disadvantages of SM:

(A) $0 <$ $+0$: $0\ 00 \dots 000$.
-1: $1\ 11 \dots 111$

(B) Addition:

$X = +5$: 0101_{c1}
$Y = +2$: 0010_{c1} | +
$0111_{c1} = +7$

$X = +5$: 0101_{c1}
$Y = -2$: 1101_{c1} | +
$1\ 0010_{c1} = +2$
1 | +
$0011_{c1} = +3$

end around carry

$X = -5$: 1010_{c1}
$Y = +2$: 0010_{c1} | +
$1100_{c1} = -3$
1011_{SM}

$X = -5$: 1010_{c1}
$Y = -2$: 1101_{c1} | +
$1\ 0111_{c1} = +7$
1 | +
$1000_{c1} = -7$
1111_{SM}

Q: Disadvantages