LAB-STA-3. STABILITY OF DYNAMICAL SYSTEMS

A. OBJECTIVES. 1. Gaining understanding of the concept of stability of a linear system. 2. Gaining experience in application of algebraic stability criteria for continuous-time and discrete-time systems.

B. THEORETICAL CONSIDERATIONS.
The necessary condition for a technical system to be exploitable is to be stable. The concept of stability emphasizes the property of a system to maintain a state of equilibrium or to evolve from one state of equilibrium to another. The stability of a system can be checked against:
- a change in the inputs of that system,
- a change of the parameters of the system, or
- a change in the structure of the system.

C. THE CONCEPT OF STABILITY.
Let the state-space linear system be described by:

$$\begin{aligned} \mathbf{x}'(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,u(t), \\ y(t) &= \mathbf{c}^T\,\mathbf{x}(t), \end{aligned} \qquad (1\text{-a})$$

with:

$$\mathbf{x}'(t) = \begin{cases} \dot{\mathbf{x}}(t), \ t \in T \subset \mathbf{R} \ \text{for continuous-time systems,} \\ \mathbf{x}_{k+1}, \ k \in \mathbf{Z}\,(\mathbf{N}) \ \text{for discrete-time systems,} \end{cases}$$

and the corresponding transfer function (t.f.) given by:

$$H(\lambda) = \mathbf{c}^T(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}, \qquad (1\text{-b})$$

with

$$\lambda = \begin{cases} s, \ \text{for continuous-time systems,} \\ z, \ \text{for discrete-time systems.} \end{cases}$$

1. Input-output stability (external stability) of a system
This is also known as **B**ounded-**I**nput **B**ounded-**O**utput (**BIBO**) stability.

Definition 1. A dynamical system is called BIBO stable or input-output stable if for any initial time moment $t_0 \in T \subset \Re$ and initial state $\mathbf{x}_0 = \mathbf{0}$, for a bounded variation of the input $u(t)$:

$$|u(t)| < L_u, \qquad (1.2)$$

the system's output response $y(t)$ is also bounded:

$$|y(t)| < L_y. \qquad (1.3)$$

Otherwise, the system is called **unstable**.

This stability can intuitively be tested using the step-response method. Several waveforms corresponding to stable (1, 2, 3) and unstable systems (4, 5) are given in Fig. 1.

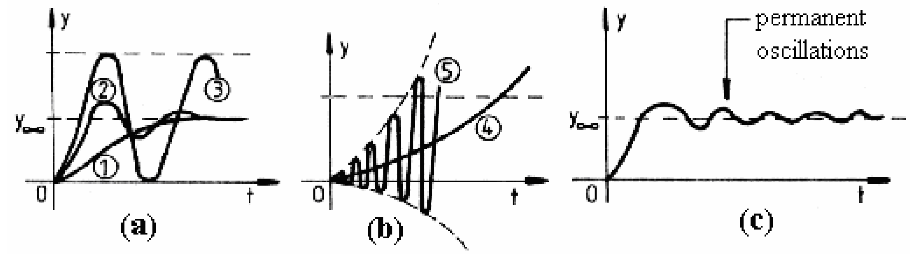

Fig. 1. System responses corresponding to stable (1, 2, 3) and unstable systems (4, 5).

For practical reasons, although simple, this technique for stability validation can sometimes be dangerous if some variables exit the safety boundaries.

2. Internal stability of a system (state stability)
Definition 2. The equilibrium state (or point) $\mathbf{x}_0 = \mathbf{0} \in \Re^n$ of a is **stable state** if the system gets out of this state or a vicinity of the state due to an external input, that is,

$$|\Delta x_i(0_+)| < L_{x0}, \ i = 1...n, \qquad (2.1)$$

with $L_{x0} > 0$, the system will evolve back as follows after removing the cause:
- in the initial stable state $\mathbf{x}_0$ or
- in an acceptable vicinity of this state,

and the resulting state trajectories, $\Delta\mathbf{x}(t)$, $t > 0$, fulfill the condition

$$|\Delta x_i(t)| < L_x, \ i = 1...n, \qquad (2.2)$$

where $L_x > 0$, $L_x = f(L_{x0})$ such that $L_x > L_{x0}$.
Otherwise the system is called unstable.

This concept is illustrated in Fig. 2 in terms of a phase trajectory interpretation.

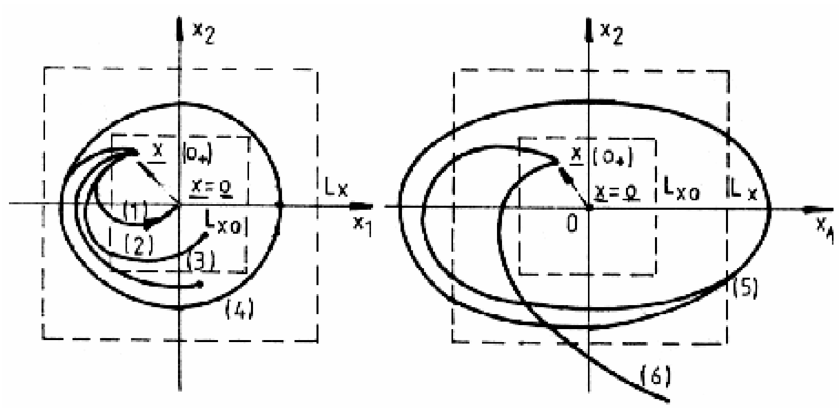

Fig. 2. Phase trajectories.

The state trajectories (4) and (5) represent limit-cycles, but (5) exceeds the boundary imposed by L_x. So (4) is called stable trajectory, while (5) is called unstable.

3. The fundamental stability theorem of continuous-time linear time invariant systems (C-LTIS)

Let the continuous-time SISO dynamical system be described by the state-space (SS) form or by the input-output (IO) form:

SS-MM:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{b}\,u, \qquad y = \mathbf{c}^T\,\mathbf{x}, \tag{3.1-a}$$

or IO-MM:

$$\sum_{\nu=0}^{n} a_\nu y^{(\nu)}(t) = \sum_{\mu=0}^{m} b_\mu u^{(\mu)}(t), \; m < n. \tag{3.1-b}$$

Both models can also be reflected in the t.f. form $H(s)$:

$$H(s) = \begin{cases} \mathbf{c}^T(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b} = \mathbf{c}^T \dfrac{\mathrm{adj}(s\mathbf{I}-\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})}\mathbf{b}, \\ \dfrac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a_0}. \end{cases} \tag{3.2}$$

The characteristic equation $\Delta(s) = 0$ can be expressed as:

$$\Delta(s) = \begin{cases} \det(s\mathbf{I}-\mathbf{A}) = 0, \\ \sum_{\nu=0}^{n} a_\nu s^\nu = 0. \end{cases} \tag{3.3}$$

Theorem 1: The system (3.1) is stable if and only if all roots of the characteristic equation have negative real parts, i.e.

$$\mathrm{Re}(s_\nu) < 0, \; \nu = 1 \ldots n. \tag{3.4}$$

This theorem is applicable to any system, even if the system is a result of an interconnection of systems. However several aspects have to be taken care of. Let a generic control system be given by the feedback interconnection of the process and the controller according to Fig. 3.

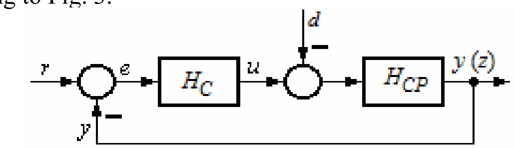

Fig. 3. Conventional control system structure (control loop).

The closed-loop t.f. from the reference input r to the controlled output y is given by:

$$H_{zr}(s) = \frac{H_C(s)H_{CP}(s)}{1 + H_C(s)H_{CP}(s)} = \frac{H_0(s)}{1 + H_0(s)}, \tag{3.5}$$

with $H_C(s)H_{CP}(s) = H_0(s)$. $H_0(s)$ is also called the open-loop t.f. The stability is thus investigated for the characteristic equation $\Delta(s) = 0$ which is equivalent to $1+H_0(s)=0$. While calculating the open-loop t.f. as the product of the controller transfer function and the controlled process transfer function, simplifications may be carried out between common factors at the numerator and the denominator. **The simplification will give false results concerning the stability of the closed-loop system and is therefore prohibited**.

Example 1:

- case (I):

$$H_C(s) = \frac{1+4s}{1+s}, H_{CP}(s) = \frac{1}{(1+2s)(1+4s)};$$

- case (II):

$$H_C(s) = \frac{1-4s}{1+s}, H_{CP}(s) = \frac{1}{(1+2s)(1-4s)}.$$

Solution:

- Case (I):

The characteristic equation is

$$\Delta(s) = 1 + H_0(s) = 1 + \frac{1+4s}{1+s} \cdot \frac{1}{(1+2s)(1+4s)} = 0.$$

(1) Without simplifying (1+4*s*) (**correct**) the result is

$$\Delta(s) = (1+4s)(2+3s+2s^2) = 0,$$

with the roots

$$s_1 = -\frac{1}{4},\ s_{2,3} = \frac{-3 \pm \sqrt{7}}{4} \rightarrow \text{stable system.}$$

(2) Simplifying the common factor (1+4*s*) (**incorrect**), the characteristic equation becomes

$$\Delta^{'}(s) = (1+2s)(1+s) + 1 = 2s^2 + 3s + 2 = 0,$$

with the roots

$$s^{'}_{1,2} = \frac{-3 \pm \sqrt{7}}{4} \rightarrow \text{stable system.}$$

- Case (II):

The characteristic equation is

$$\Delta(s) = 1 + H_0(s) = 1 + \frac{1-4s}{1+s} \cdot \frac{1}{(1+2s)(1-4s)} = 0.$$

(1) Without simplifying common factors (1–4*s*) (**correct**) the result is

$$\Delta(s) = (1-4s)(2+3s+2s^2) = 0,$$

with the roots

$$s_1 = \frac{1}{4},\ s_{2,3} = \frac{-3 \pm \sqrt{7}}{4} \rightarrow \text{unstable system.}$$

(2) Simplifying (1–4*s*), the characteristic equation becomes

$$\Delta^{'}(s) = (1+2s)(1+s) + 1 = 2s^2 + 3s + 2 = 0,$$

with the roots:

$$s^{'}_{1,2} = \frac{-3 \pm \sqrt{7}}{4} \rightarrow \text{stable system, which } \textbf{is false} \text{ (wrong conclusion).}$$

Remark: If time delay (or dead time) is included the t.f., the characteristic equation becomes a transcendental equation and it does not have an order. Then the correct formulation of the fundamental stability theorem is that **all roots of the characteristic equation must have negative real parts**.

4. The Hurwitz stability criterion

Theorem 2: For the roots of an algebraic equation of the form

$$\Delta(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0 \qquad (4.1)$$

to have negative real parts, it is necessary (but not sufficient) that all the coefficients of the equation to be strictly positive.
Therefore, if at least one coefficient is not strictly positive, then the system characterized by $\Delta(s)$ **is unstable**.

The **Hurwitz stability criterion** requires the construction of the Hurwitz matrix of coefficients:

$$\begin{array}{ccccc} \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_3 & \mathbf{H}_{n-1} & \end{array}$$

$$\mathbf{H} = \left[\begin{array}{c|c|c|c|c} a_{n-1} & a_{n-3} & a_{n-5} & \ldots 0 & 0 \\ \cline{1-1} a_n & a_{n-2} & a_{n-4} & \ldots 0 & 0 \\ \cline{1-2} 0 & a_{n-1} & a_{n-3} & \ldots 0 & 0 \\ \cline{1-3} 0 & a_n & a_{n-2} & \ldots 0 & 0 \\ \ldots & & & & \\ 0 & 0 & 0 & \ldots a_0 & 0 \\ 0 & 0 & 0 & \ldots a_1 & 0 \\ \cline{1-4} 0 & 0 & 0 & \ldots a_2 & a_0 \end{array} \right]. \qquad (4.2)$$

Using **H**, the following calculations will be carried out:

- the Hurwitz determinant det(**H**),
- the leading principal minors of **H**: det($\mathbf{H}_1$), det($\mathbf{H}_2$), ...

The sufficient conditions such that the system characterized by $\Delta(s)$ **is stable** is that **the Hurwitz determinant and all of its leading principal minors are strictly positive**.

Remarks:

1) The Hurwitz criterion is not applicable to time delay systems. Instead, if the dead time is small with respect to the large time constants of the system, then a Padé approximation of the delay can be employed in the form of a rational t.f., and the Hurwiz criterion is next applied in a straightforward manner.

2) Even if nowadays efficient numerical tools are available for finding the roots of the characteristic polynomial, the Hurwitz criterion is still useful when analyzing the stability for parameters variations as illustrated by the next example:

Example 2: Let a control system be given by the feedback connection of the controller and the process with the t.f.s

$$H_C(s) = \frac{k_R(1+8s)}{1+20s},\ H_{CP}(s) = \frac{1-4s}{(1+2s)(1+7s)},$$

with the controller gain $k_C > 0$. Find the domain of k_C for which the closed-loop system is stable.

Solution: The characteristic polynomial is

$$\Delta(s) = 1 + H_0(s),$$

where

$$H_0(s) = H_C(s)H_{CP}(s) = \frac{k_C(1+8s)(1-4s)}{(1+20s)(1+2s)(1+7s)}.$$

The characteristic equation is

$$\Delta(s) = 280s^3 + (194 - 32k_C)s^2 + (29 + 4k_C)s + 1 + k_C.$$

The necessary conditions specified in Theorem 2 are imposed:

$$194 - 32k_C > 0,$$
$$29 + 4k_C > 0,$$
$$1 + k_C > 0.$$

Since $k_C > 0$, it follows from this system of inequations that $k_C \in (0, 6.0625)$.

The Hurwitz matrix (n=3) is next built, and the Hurwitz determinant is

$$\det(\mathbf{H}) = \det(\mathbf{H}_3) = \begin{vmatrix} 194 - 32k_C & 1 + k_C & 0 \\ 280 & 29 + 4k_C & 0 \\ 0 & 194 - 32k_C & 1 + k_C \end{vmatrix}.$$

The stability conditions are imposed:

$$\det(\mathbf{H}_1) = 194 - 32k_C > 0 \Rightarrow k_C < 194/32 \Leftrightarrow k_C < 6.0625; \qquad (1)$$

$$\det(\mathbf{H}_2) = \begin{vmatrix} 194 - 32k_C & 1 + k_C \\ 280 & 29 + 4k_C \end{vmatrix} > 0 \qquad (2)$$

$$\Leftrightarrow (194 - 32k_C)(29 + 4k_C) - 280(1 + k_C) > 0 \Rightarrow k_C \in (-8.3668, 4.9919);$$

$$\det(\mathbf{H}_3) = \begin{vmatrix} 194 - 32k_C & 1 + k_C & 0 \\ 280 & 29 + 4k_C & 0 \\ 0 & 194 - 32k_C & 1 + k_C \end{vmatrix} > 0 \Leftrightarrow (1 + k_C)\det(\mathbf{H}_2) > 0 \qquad (3)$$

$$\Leftrightarrow k_C \in (-1, 0) \cap (-8.3668, 4.9919) \Rightarrow k_C \in (-1, 4.9919).$$

Since $k_C > 0$, the intersection of this condition and the conditions (1), (2) and (3) results in $k_C \in (0, 4.9919)$. The intersection with the necessary condition leads to the stability domain $k_C \in (0, 4.9919)$.

5. Stability of discrete-time systems

Theorem 2 has a corresponding version for discrete-time systems. For a system with a given characteristic polynomial

$$\Delta(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0, \qquad (5.1)$$

the necessary and sufficient stability condition is that **all roots are placed inside the unit disk of the z plane**, i.e.

$$|z_\nu| < 1,\ \nu = 1 \ldots n. \qquad (5.2)$$

Two practical criteria can be used to determine the stability of a discrete-time system, 1) and 2):

1) A **conformal mapping** from the interior of the unit disk in the z plane to the left-hand side of the complex s plane. The mappings

$$z = \frac{r+1}{r-1} \text{ or } z = \frac{1+w}{1-w} \qquad (5.3)$$

can be used in this regard. The resulting t.f. is pseudo-continuous, $H(w)$ or $H(r)$. Considering these t.f.s, the Hurwitz stability criterion can be applied in a straightforward manner.

2) **The Jury stability criterion** for discrete-time systems which is basically a coefficient-based calculation scheme described as follows.

6. The Jury stability criterion

The system characteristic equation

$$\Delta(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0 \text{ with } a_n > 0 \qquad (6.1)$$

is used to build **the array for Jury's stability** test (also called **the Routh array**) given in Table 1.

As shown in Table 1, the elements of the even-numbered rows are the elements of the preceding row in reverse order. The elements of the odd-numbered rows are computed in terms of

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix},\ c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix},\ d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}, \ldots, \tag{6.2}$$

$$q_0 = \begin{vmatrix} p_0 & p_3 \\ p_3 & p_0 \end{vmatrix},\ q_1 = \begin{vmatrix} p_0 & p_2 \\ p_3 & p_1 \end{vmatrix},\ q_2 = \begin{vmatrix} p_0 & p_1 \\ p_3 & p_2 \end{vmatrix}.$$

Table 1. Array for Jury's stability test.

Row	z^0	z^1	z^2	$\ldots z^{n-k} \ldots$	z^{n-2}	z^{n-1}	z^n
1 2	a_0 a_n	a_1 a_{n-1}	a_2 a_{n-2}	$\ldots a_{n-k} \ldots$ $\ldots a_k \ldots$	a_{n-2} a_2	a_{n-1} a_1	a_n a_0
3 4	b_0 b_{n-1}	b_1 b_{n-2}	b_2 b_{n-3}	$\ldots b_{n-k} \ldots$ $\ldots b_k \ldots$	b_{n-2} b_1	b_{n-1} b_0	– –
5 6	c_0 c_{n-2}	c_1 c_{n-3}	c_2 c_{n-4}	$\ldots c_{n-k} \ldots$ $\ldots c_k \ldots$	c_{n-2} c_0	– –	– –
…	…	…	…	…	–	–	–
$2n-5$ $2n-5$	p_0 p_3	p_1 p_2	p_2 p_1	p_3 p_0	– –	– –	– –
$2n-3$	q_0	q_1	q_2	–	–	–	–

Using the array for Jury's stability test given in Table 1, **the Jury stability criterion is expressed as follows**: The linear system with the characteristic polynomial (6.1) is **stable** (i.e., all roots are placed inside the unit disk) if and only if the following n+1 **conditions** are fulfilled (with $a_n > 0$):

$\Delta(1) > 0$, (1)

$\Delta(-1) > 0$ if n is even, (2)
$\quad < 0$ if n is odd,

$|a_0| < a_n$, (3)

$|b_0| > |b_{n-1}|$, (4)

$|c_0| > |c_{n-2}|$, (5)

$|d_0| > |d_{n-3}|$, (6)

…

$|q_0| > |q_2|$. (n+1)

Remarks: 1. The coefficients b_k are not related to the coefficients in the nominator of the system t.f.

2. For a second-order system, the array contains only one row.

3. As in the case of the Hurwitz criterion, the Jury criterion has also the shortcoming of not giving information on the stability degree of the system. Since the number of inequalities is rather high, it is difficult to conduct a stability analysis that depends on one or more system parameters.

Steps to apply the Jury criterion:

- the system whose stability is analyzed is separated, its characteristic polynomial $\Delta(z)$ and the system order n is identified;
- $\Delta(1)$ and $\Delta(-1)$ are computed and the conditions (1), (2) and (3) are tested;
- if one of the conditions (1), (2) and (3) is not satisfied, the criterion is stopped and the system is unstable;
- otherwise, the Routh array is built and the rest of (n–2) conditions are tested one by one; if one of the conditions is not satisfied, the criterion is stopped and the system is unstable; otherwise, the system is stable.

Example 3: Conduct the stability analysis of the discrete-time linear system with the t.f.

$$H(z) = \frac{11z^2 - 3z + 0.5}{z^3 + 3z^2 + 4z + 0.5}.$$

Solution: The characteristic polynomial of the system is

$$\Delta(z) = z^3 + 3z^2 + 4z + 0.5,$$

with $n = 3$ and $a_3 = 1 > 0$. The first three stability conditions are tested:

$\Delta(1) = 8.5 > 0$, (1)

$\Delta(-1) = -1.5 < 0$ ($n = 3$ is odd), (2)

$|a_0| = 0.5 < a_3 = 1$. (3)

Since all these conditions are satisfied, the Jury array is built. It is given in Table 2, and its elements are computed as follows:

$$b_0 = \begin{vmatrix} a_0 & a_3 \\ a_3 & a_0 \end{vmatrix} = a_0^2 - a_3^2 = -0.75,$$

$$b_1 = \begin{vmatrix} a_0 & a_2 \\ a_3 & a_1 \end{vmatrix} = a_0 a_1 - a_2 a_3 = -1,$$

$$b_2 = \begin{vmatrix} a_0 & a_1 \\ a_3 & a_2 \end{vmatrix} = a_0 a_2 - a_1 a_3 = -2.5.$$

Table 2. Jury array for the example 3.

Row	z^0	z^1	z^2	z^3
1	0.5 (a_0)	4 (a_1)	3 (a_2)	1 (a_3)
2	1 (a_3)	3 (a_2)	4 (a_1)	0.5 (a_0)
3	–0.75 (b_0)	–1 (b₁)	–2.5 (b_2)	–
4	–2.5 (b_2)	–1 (b_1)	–0.75 (b_0)	–

The last, namely fourth stability condition (n+1 = 4) is next tested:

$|b_0| = 0.75 < |b_2| = 2.5.$ (4)

Since this condition is not satisfied, the conclusion is that the system is unstable.

Example 5:

a) The email server model treated in Lab 1 given by the discrete transfer function $H(z)$=0.47/(z–0.43) is stable since the single root z=0.43 is inside the unit disk.

b) The discrete-time model of the bank account also presented in Lab 1 has an associated t.f. $H(z)$=1/(z–1.1). The root z=1.1 is located outside the unit disk indicating that the system unstable.

Homework:

1) Investigate the stability of the following continuous-time systems: DC motor, electrically heated room temperature controlled process (including actuators and instrumentation measuring dynamics) and HIV-virus dynamic mathematical model given in Lab 1, and mass-spring-damper system and electrical system given in Lab 2.

2) Find the range of k for which the continuous-time system with the characteristic polynomial

$$\Delta(s) = s^3 + 3k\, s^2 + (k+2)s + 4$$

is stable.

3) Let the control loop be characterized by the t.f. of the controlled process

$$H_{CP}(s) = \frac{2}{s^3 + 4s^2 + 5s + 2}.$$

Design a continuous-time controller that stabilizes the closed-loop system in two cases, a) and b):

a) Proportional (P) controller, with the t.f. $H_C(s) = k$, and the range of the controller gain k should be determined.

b) Proportional-Integral (PI) controller, with the t.f. $H_C(s) = k_P + \frac{k_I}{s}$, k_P – the proportional gain, k_I – the integral gain, and the domain in the <k_P, k_I> plane should be determined.

4) Investigate the stability of the following discrete-time systems: predator-prey model, student dynamics model and a supply chain model given in Lab 1.

5) Check the stability of the system with the open-loop t.f.

$$H_0(z) = \frac{0.2z + 0.5}{z^2 - 1.2z + 0.2}.$$

6) Determine the value of k for which the system with the open-loop t.f.

$$H_0(z) = \frac{k(0.2z + 0.5)}{z^2 - 1.2z + 0.2}$$

is stable.

7) The characteristic equation of a discrete-time system is given by

$$\Delta(z) = z^3 - 2z^2 + 1.4z - 0.1.$$

Investigate the stability of this system.

References

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