Chapter V

CONTINUOUS RANDOM VARIABLES

1 The Cumulative Distribution Function. Probability Density Function

Definition 1.1. The cumulative distribution function *(CDF) of r.v. X is $F_X(x) = P(X \leq x)$.*

Proposition 1.1. *The CDF of r.v. X has the following properties:*
1. $\lim\limits_{x \to -\infty} F_X(x) = 0$;
2. $\lim\limits_{x \to \infty} F_X(x) = 1$;
3. $P(a < X \leq b) = F_X(b) - F_X(a), \ \ \forall a, b \in \mathbb{R}$;
4. $P(X = x) = 0, \ \ \forall x \in \mathbb{R}$ *(the probability of any individual outcome is 0.)*

Definition 1.2. *The r.v. X is* **a continuous r.v.** *if the CDF of $X, F_X(x)$ is a continuous function.*

Definition 1.3. The probability density function *(PDF) of a continuous r.v. X is the function given by $f_X(x) = F'_X(x)$.*

Proposition 1.2. *The PDF of r.v. X has the following properties:*
1. $f_X(x) \geq 0$, *for any* $x \in \mathbb{R}$;
2. $F_X(x) = \int_{-\infty}^{x} f_X(u)du$;
3. $\int_{-\infty}^{\infty} f_X(x)dx = 1$;
4. $P(a < X \leq b) = \int_a^b f_X(x)dx$.

2 Expected Values

Definition 2.1. The expected value of a continuous r.v. *X is the number*

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)dx.$$

Remark: The expected value pf a function of r.v. X, $g(X)$, is

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)dx.$$

Proposition 2.1. *The following properties hold for any r.v.* X :
1. $E(X - E(X)) = 0$;
2. $E(aX + b) = aE(X) + b$;
3. $Var(X) = E(X^2) - (E(X))^2$;
4. $Var(aX + b) = a^2 Var(X)$.

3 Families of Continuous R.V.

Definition 3.1. X **is a uniform** (a, b) **r.v.** *(or* X **is a uniform r.v.** *or* X **is uniformly distributed** *or* X **has a uniform distribution***) if the PDF is*

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x < b \\ 0, & otherwise \end{cases}, \quad b > a.$$

Properties: If X is a uniform (a, b) r.v. then:
a) the CDF of X is

$$F_X(x) = \begin{cases} 0, & x \le a \\ \\ \dfrac{x-a}{b-a}, & a \le x < b \\ \\ 1, & x > b \end{cases}, \quad b > a.$$

b) $E(X) = \dfrac{a+b}{2}$;

c) $Var(X) = \dfrac{(a-b)^2}{12}$.

Definition 3.2. X **is an exponential** (λ) **r.v.** *if the PDF is*

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & otherwise \end{cases}, \lambda > 0.$$

Properties: If X is an exponential (λ) r.v. then:
a) the CDF of X is

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & otherwise; \end{cases}$$

b) $E(X) = \dfrac{1}{\lambda}$;

c) $Var(X) = \dfrac{1}{\lambda^2}$.

Definition 3.3. *X* **is a Gaussian** (μ, σ) **r.v.)** *(or* **normal r.v.)** *if the PDF is*

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\dfrac{(x-\mu)^2}{2\sigma^2}}, \quad \mu \in \mathbb{R}, \sigma > 0.$$

We denote $X \in N(\mu, \sigma^2)$

Remark: The graph of the PDF has a bell shape, where the center of the bell is $x = \mu$ and σ is the width of the bell. The height pf the peak is $\dfrac{1}{\sigma\sqrt{2\pi}}$.

Properties:
1. If $X \in N(\mu, \sigma^2)$ then $E(X) = \mu, Var(X) = \sigma^2$.
2. If $X \in N(\mu, \sigma^2)$ then $aX + b = Y \in N(a\mu + b, a\sigma^2)$.
3. $\mu - \sigma$ and $\mu + \sigma$ are points of inflection for the graph;
4. The subtended area of the curve is equal to 1;
5. The area of the left side of $x = \mu$ is equal to the area of the right side, and equals $\dfrac{1}{2}$;
6. 68% of the subtended area of the curve is between $\mu - \sigma$ and $\mu + \sigma$; 95% of the subtended area is between $\mu - 2\sigma$ and $\mu + 2\sigma$; 99.7% of the subtended area is between $\mu - 3\sigma$ and $\mu + 3\sigma$;

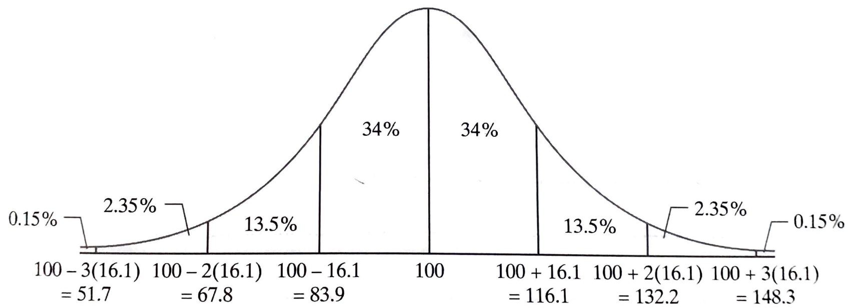

Figure 1: The graph of f_X for $\mu = 100$ and $\sigma = 16.1$.

7. The graph of the curve is asymptotically to Ox axis.

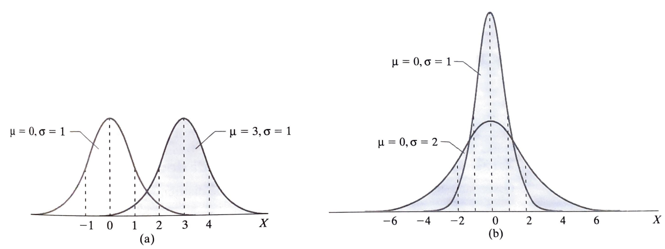

Figure 2: The graph of f_X for different values of μ and σ.

Definition 3.4. The standard normal r.v. *Z is the Gaussian $(0, 1)$ r.v.*

Properties:

1. If $X \in N(0, 1)$ then $E(X) = 0, Var(X) = 1$.
2. The CDF of the standard normal r.v. Z is

$$\Phi(z) = \frac{1}{2\pi} \int_{-\infty}^{z} e^{-\frac{u^2}{2}} du.$$

3. If $X \in N(\mu, \sigma^2)$ then the CDF of X is:

$$F_X(x) = \Phi(\frac{x - \mu}{\sigma}).$$

4. The probability that X is in the interval $(a, b]$ is

$$P(a < X \leq b) = \Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma}).$$

5. For any complex z

$$\Phi(-z) = 1 - \Phi(z).$$

Definition 3.5. X **is a Pareto r.v.** *($X \in (Pareto(\alpha, \beta))$) if the PDF is*

$$f_X(x) = \begin{cases} \frac{\alpha}{\beta} \cdot (\frac{\beta}{x})^{\alpha+1}, & x \geq \beta \\ 0, & x < \beta, \end{cases} , \lambda > 0.$$

for any $\alpha, \beta > 0$.

Remarks: 1. The Pareto distribution, named after the Italian civil engineer, economist, and sociologist Vilfredo Pareto, is a power-law probability distribution that is used in description of social, scientific, geophysical, actuarial, and many other types of observable phenomena.
2. The duration of the processor service for the processes of the UNIX operating system, was proved by experimentation that it has Pareto distribution.
3. The size of the files stored on the WEB servers has Pareto distribution of parameter $\alpha \in [1.1, 1.3]$.
4. The length of time, X, between two successive packets of information in the irregular data bundle in communication systems, is a Pareto parameter variable. Parameter β represents the minimum time interval between two packages, and the parameter α characterize the intensity of the network use.
5. Pareto distribution is used as a model for simulating Internet traffic. This simulation is used to study the parameters of the Internet network under high traffic conditions.

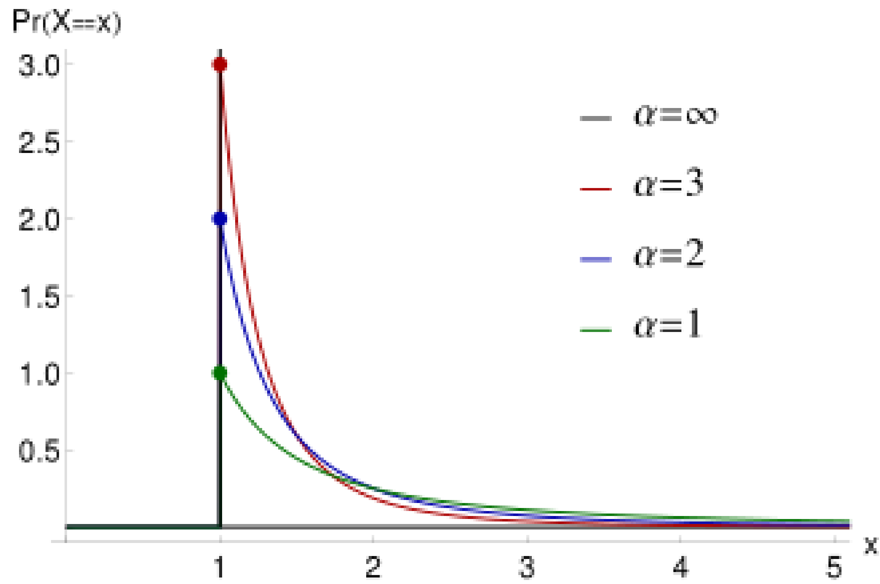

Figure 3: The graph of f_X for different values of α. (by Wikipedia)

Properties:

1. The CDF of X is

$$F_X(x) = \begin{cases} 1 - (\frac{\beta}{x})^\alpha, & x \geq \beta \\ 0, & x < \beta; \end{cases}$$

2. The expected value of X is

$$E(X) = \frac{\alpha\beta}{\alpha - 1},$$

for any $\alpha > 1, \beta > 0$.

3. The variance of X is:

$$Var(X) = \frac{\alpha\beta^2}{(\alpha-1)^2(\alpha-2)},$$

for any $\alpha > 2, \beta > 0$.

4 Conditioning Continuous Random Variables

Definition 4.1. *For a r.v. X with PDF $f_X(x)$ and an event $B \subset S_X$ with probability $P(B) > 0$,* **the conditional PDF of X given B** *is:*

$$f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{P(B)}, & x \in B \\ 0, & otherwise; \end{cases}$$

Proposition 4.1. *Given an event space $\{B_1, B_2, ...B_n\}$ and the conditional PDF $f_{X|B}(x)$, then*

$$f_X(x) = \sum_{i=1}^{n} f_{X|B}(x) \cdot P(B_i).$$

Definition 4.2. *If $x \in B$, the conditional expected value of X is:*

$$E(X \mid B) = \int_{-\infty}^{\infty} x \cdot f_{X|B}(x).$$

5 Probability Models of Derived R.V.

Let X be a continuous r.v. with the PMF $f_X(x)$, g a real function, and $Y = g(X)$ a new r.v.
In order to find the PMF of the new r.v. Y, we follow a two-step procedure:
• find the CDF $F_Y(y) = P(Y \leq y)$;
• compute $f_y(y) = F'_Y(y)$.

Proposition 5.1. *Multiplying a r.v. by a positive constant stretches $(a > 1)$ or shrinks $(a < 1)$ the original PDF: if $Y = aX, a > 0$, then $F_Y(y) = F_X(\frac{y}{a})$, and $f_Y(y) = \frac{1}{a} f_X(\frac{y}{a})$.*

Proposition 5.2. *1. If $Y = aX, a > 0$, then:*
a) if X is uniform (b, c), then Y is uniform (ab, ac);
b) if X is exponential (λ), then Y is exponential $(\frac{\lambda}{a})$;
c) if X is Gaussian (μ, σ) then Y is Gaussian $(a\mu, a\sigma)$.

2. If $Y = X + b$, then:
a) the CDF of Y is $F_Y(y) = F_X(y - b)$;
b) the PDF of Y is $f_Y(y) = f_X(y - b)$.

6 Solved Problems

1. Suppose X is uniformly distributed over $[-1, 3]$ and $Y = X^2$. Find the CDF and the PDF of Y.

   Solution: The PDF of X is given by:

$$f_X(x) = \begin{cases} \frac{1}{4}, & x \in [-1, 3] \\ \\ 0, & otherwise, \end{cases}$$

   and the corresponding CDF is:

$$F_X(x) = \begin{cases} 0, & x < -1 \\ \\ \frac{1}{4}(x+1), & x \in [-1, 3] \\ \\ 1, & x > 3. \end{cases}$$

   The CDF of $Y = X^2$ is:

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

   • If $\sqrt{y} > 3$, then $y > 9$ and $-\sqrt{y} < -3$, so

$$F_X(\sqrt{y}) - F_X(-\sqrt{y}) = 1 - 0 = 1,$$

   if $y > 9$.

   • If $\sqrt{y} < 3$, and $-\sqrt{y} < -1$, then $y < 9$ and $y > 1$, so

$$F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \frac{1}{4}(\sqrt{y} + 1),$$

   if $y \in [1, 9]$.

   • If $\sqrt{y} < 3$, and $-\sqrt{y} > -1$, then $y < 9$ and $y < 1$, so

$$F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \frac{1}{4}(\sqrt{y} + 1) - \frac{1}{4}(-\sqrt{y} + 1) = \frac{1}{2}\sqrt{y},$$

   if $y \in [0, 1]$.

Therefore:

$$F_Y(y) = \begin{cases} 0, \quad y < 0 \\ \\ \dfrac{1}{2}\sqrt{y}, \quad y \in [0,1) \\ \\ \dfrac{1}{4}(\sqrt{y}+1), \quad y \in [1,9) \\ \\ 1, \quad y \geq 9, \end{cases}$$

so the PDF for Y is:

$$f_Y(y) = F'_Y(y) = \begin{cases} \dfrac{1}{4\sqrt{y}}, \quad y \in [0,1) \\ \\ \dfrac{1}{8\sqrt{y}}, \quad y \in [1,9) \\ \\ 0, \quad otherwise. \end{cases}$$

7 Exercises

1. R.v. has PDF

$$f_X(x) = \begin{cases} cxe^{-\frac{x}{2}}, \quad x \geq 0 \\ \\ 0, \quad otherwise. \end{cases}$$

   Find the following:
   a) $c \in \mathbb{R}$;
   b) The probability that X takes values in $[0, 4]$;
   c) The probability that X takes values in $(-2, 2)$;
   d) The CDF for X.
   e) The expected value of $X, E(X)$;
   f) The variance of $X, var(X)$;
   g) The standard deviation of X;
   h) The second moment of Y.

2. If $X \in N(0, 1)$ and $Y \in N(0, 2)$, then find the following:
   a) $P(-1 < X \leq 1)$; b) $P(X > 3.5)$;

c) $P(-1 < Y \le 1)$; d) $P(Y > 3.5)$.

3. The CDF of r.v. X is

$$F_X(x) = \begin{cases} 0, & x < -1 \\ \\ \dfrac{1}{4}(x+1), & x \in [-1, 1) \\ \\ 1, & x \ge 1. \end{cases}$$

Sketch the graph for the CDF and find the following:
a) $P(X \le 1)$; b) $P(X < 1)$;
c) $P(X = 1)$; d) The PDF of X.

4. The PDF of r.v. X is

$$f_X(x) = \begin{cases} \dfrac{1}{10}, & 0 \le x < 10 \\ \\ 0, & otherwise. \end{cases}$$

Find the following:
a) $P(X \le 6)$;
b) The conditional PDF of X given $Y \le 6$;
c) The conditional probability of X given $X > 8$;
d) The conditional expected value of X given $X > 8$.

5. The CDF of r.v. X is

$$F_X(x) = \begin{cases} a, & x \le 1 \\ \\ \dfrac{1}{2}(x+1), & x \in (-1, 1] \\ \\ b, & x > 1. \end{cases}$$

Find the following:
a) The values of the real parameters a and b;
b) The PDF for X;
c) The PDF for $Y = X^2$.

6. The PDF of r.v. X is

$$f_X(x) = \begin{cases} \frac{1}{3}, & -1 < x < 2 \\ 0, & otherwise. \end{cases}$$

Find the following:
a) The CDF for X;
b) The PDF for the r.v. $Y = e^X$;
c) The PDF for the r.v. $Z = \sqrt{X}$.