Orthogonal Projection Matrices

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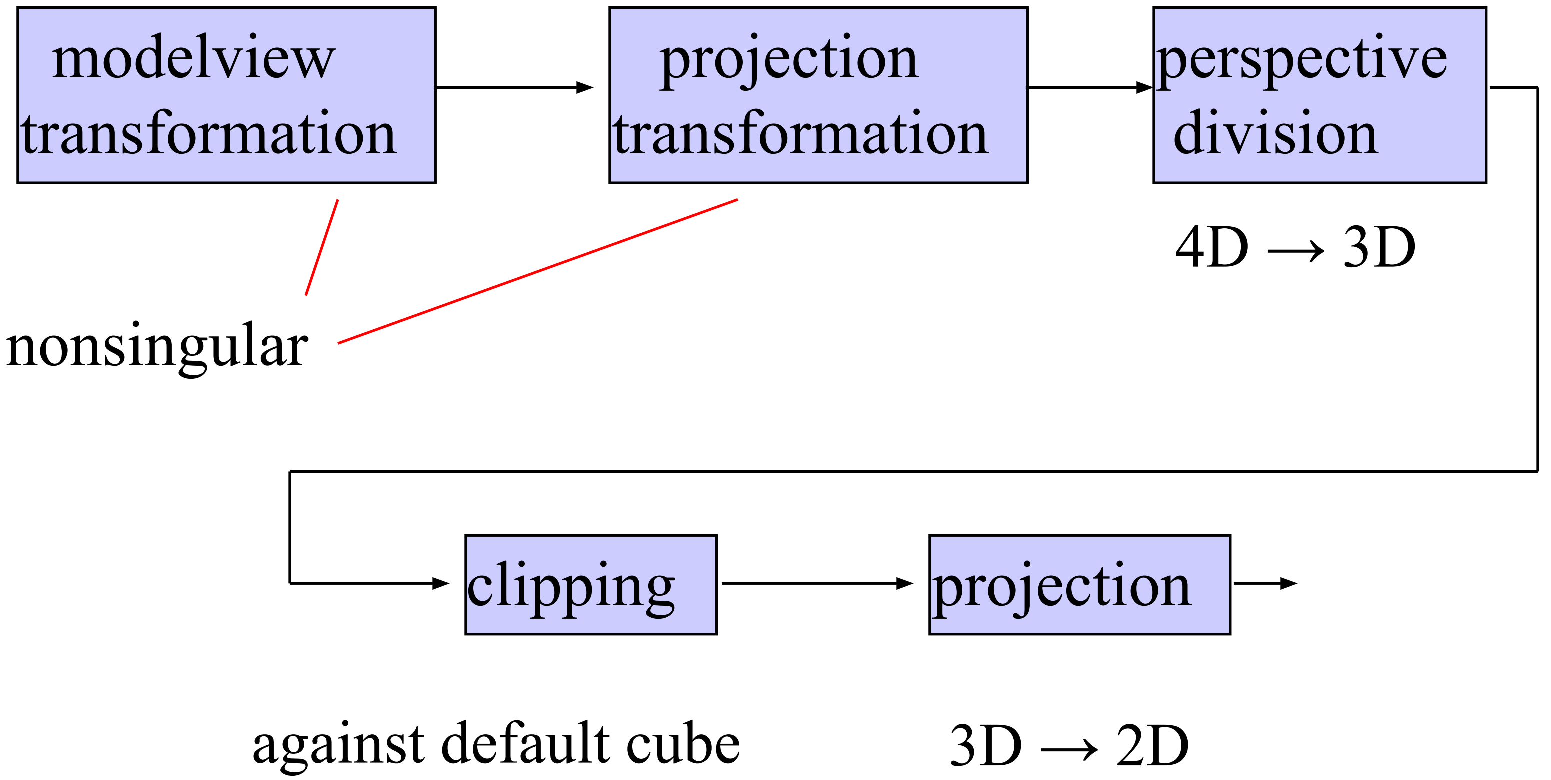
Objectives

- What are the projection matrices used for standard orthogonal projections
- Oblique projections
- Projection normalization

Normalization

- We can convert all projections to orthogonal projections with the default view volume → no different projection matrix for each type of projection
- This strategy allows us to use standard transformations in the pipeline and makes for efficient clipping

Pipeline View



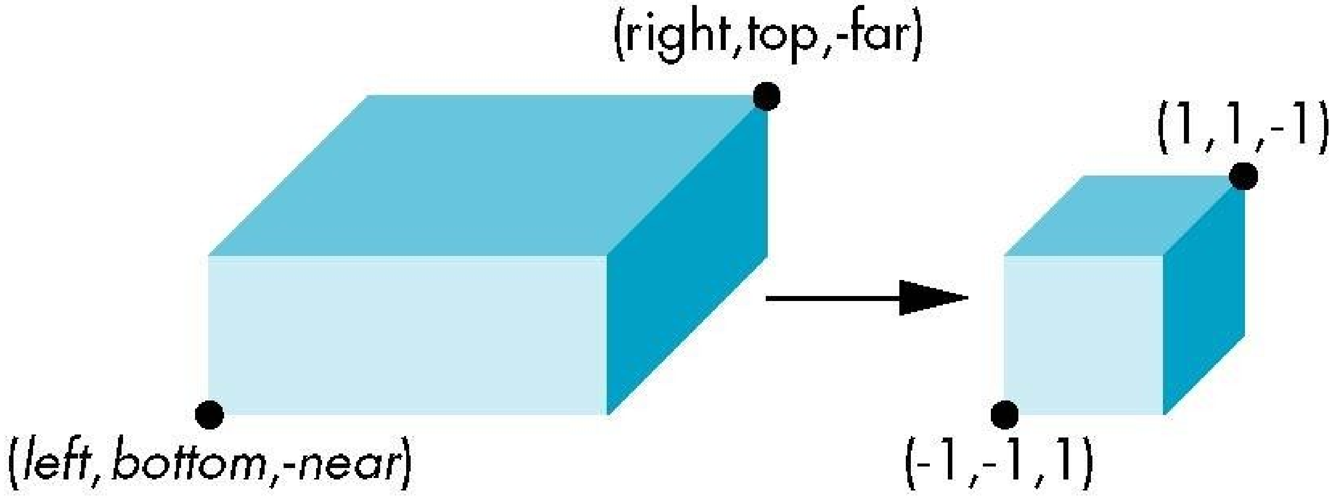
Notes

- We stay in four-dimensional homogeneous coordinates through both the modelview and projection transformations
  - Both these transformations are nonsingular
  - Default to identity matrices (orthogonal view)
- Normalization → clip against simple cube regardless of type of projection
- Delay final projection until end
  - Important for hidden-surface removal to retain depth information as long as possible

Orthogonal Normalization

`ortho(left,right,bottom,top,near,far)`

normalization ⇒ find transformation to convert specified clipping volume to default



Orthogonal Matrix

- Two steps

  Move center to origin

  ```
  T(-(left+right)/2, -(bottom+top)/2,(near+far)/2))
  ```

  Scale to have sides of length 2

  ```
  S(2/(left-right),2/(top-bottom),2/(near-far))
  ```

$$\mathbf{P} = \mathbf{ST} = \begin{bmatrix} \frac{2}{right-left} & 0 & 0 & -\frac{left+right}{right-left} \\ 0 & \frac{2}{top-bottom} & 0 & -\frac{top+bottom}{top-bottom} \\ 0 & 0 & -\frac{2}{far-near} & -\frac{far+near}{far-near} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Final Projection

- Set $z = 0$
- Equivalent to the homogeneous coordinate transformation

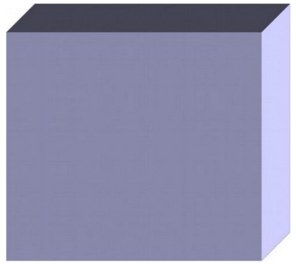
$$\mathbf{M}_{\mathrm{orth}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Hence, general orthogonal projection in 4D is

$$\mathbf{P} = \mathbf{M}_{\mathrm{orth}}\mathbf{ST}$$

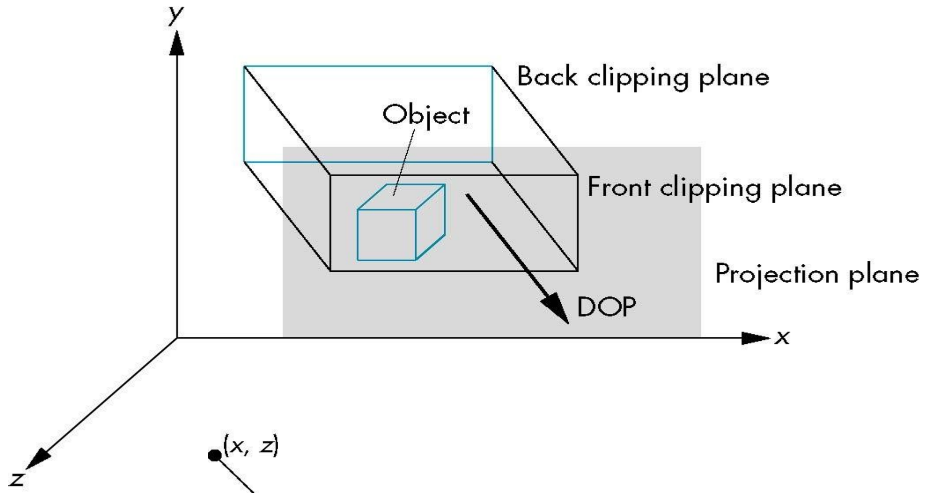
Oblique Projections

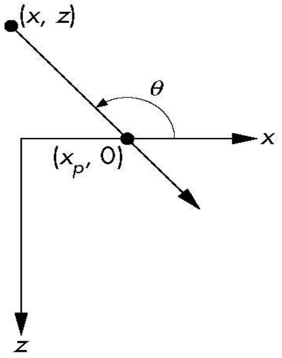
- The OpenGL projection functions cannot produce general parallel projections such as



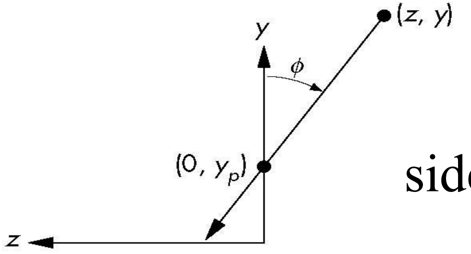
- However if we look at the example of the cube it appears that the cube has been *sheared*
- Oblique Projection = Shear + Orthogonal Projection

General Shear





top view



side view

Shear Matrix

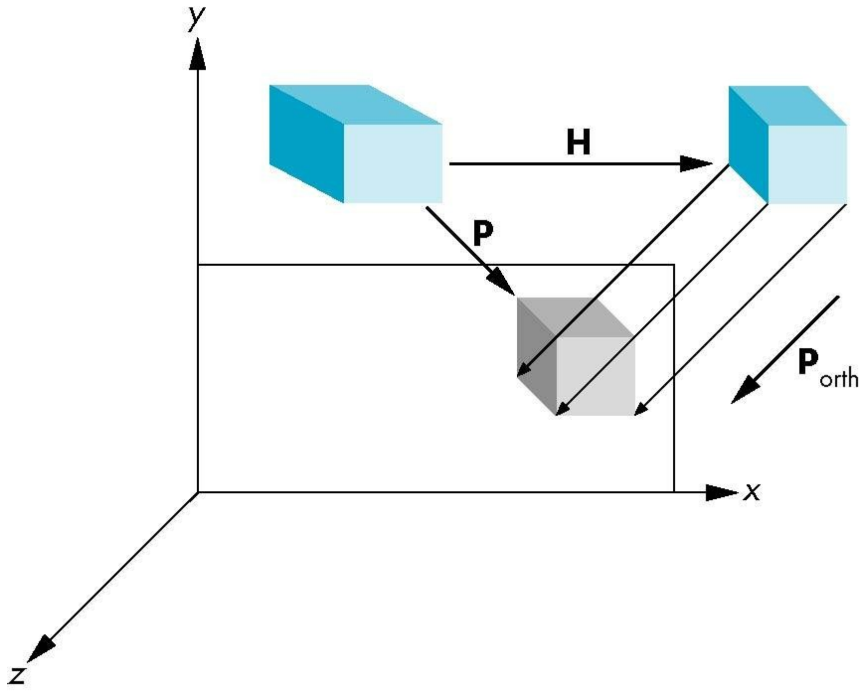
*xy*shear (*z* values unchanged)

$$\mathbf{H}(\theta,\varphi) = \begin{bmatrix} 1 & 0 & -\cot\theta & 0 \\ 0 & 1 & -\cot\varphi & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Projection matrix $\mathbf{P} = \mathbf{M}_{\text{orth}}\,\mathbf{H}(\theta,\varphi)$

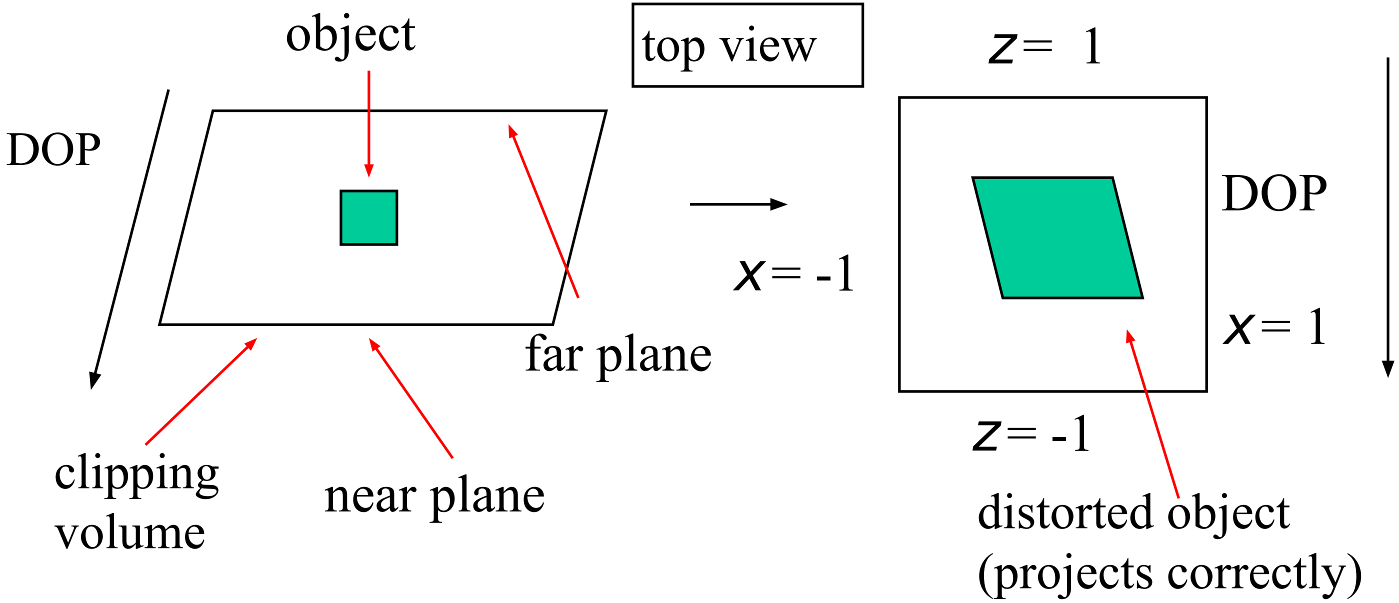
General case: $\mathbf{P} = \mathbf{M}_{\text{orth}}\,\mathbf{STH}(\theta,\varphi)$

Equivalency



Effect on Clipping

The projection matrix $\mathbf{P} = \mathbf{STH}$ transforms the original clipping volume to the default clipping volume



Perspective Projection Matrices

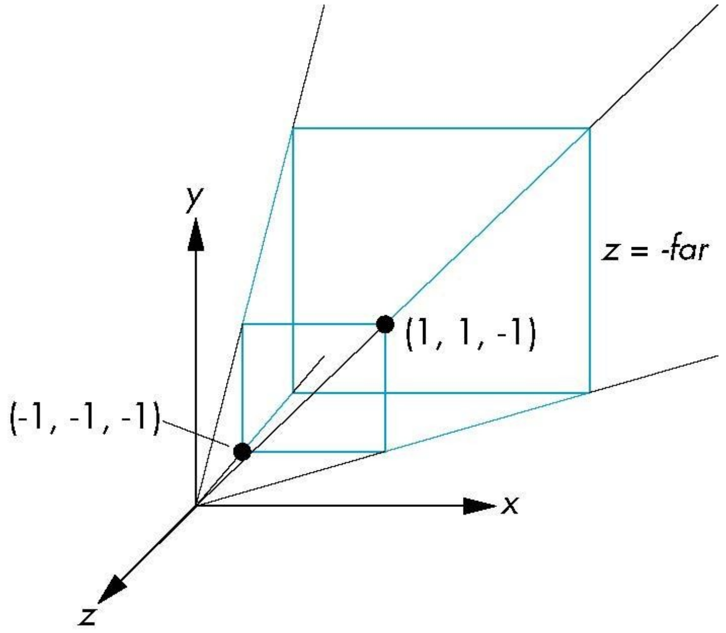
Objectives

- Derive the perspective projection matrices used for standard WebGL projections

Simple Perspective

Consider a simple perspective with the COP at the origin, the near clipping plane at $z = -1$, and a 90 degree field of view determined by the planes

$$x = \pm z, \; y = \pm z$$



Perspective Matrices

Simple projection matrix in homogeneous coordinates

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Note that this matrix is independent of the far clipping plane

Generalization

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

after perspective division, the point $(x, y, z, 1)$ goes to

$x'' = x/z$
$y'' = y/z$
$z'' = -(\alpha+\beta/z)$

which projects orthogonally to the desired point, regardless of α and β

Picking α and β

If we pick

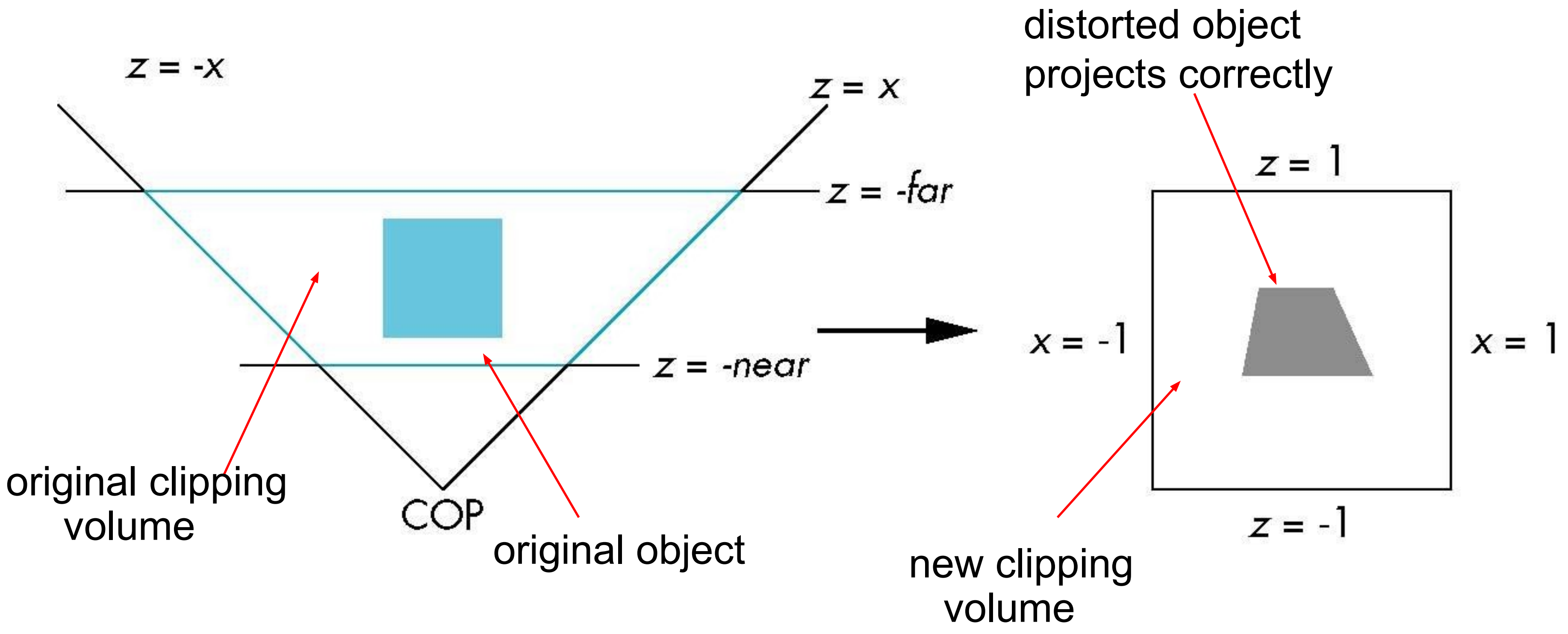
$$\alpha = \frac{\text{near} + \text{far}}{\text{far} - \text{near}}$$

$$\beta = \frac{2\text{near} * \text{far}}{\text{near} - \text{far}}$$

the near plane is mapped to $z = -1$
the far plane is mapped to $z=1$
and the sides are mapped to $x = \pm 1$, $y = \pm 1$

Hence the new clipping volume is the default clipping volume

Normalization Transformation

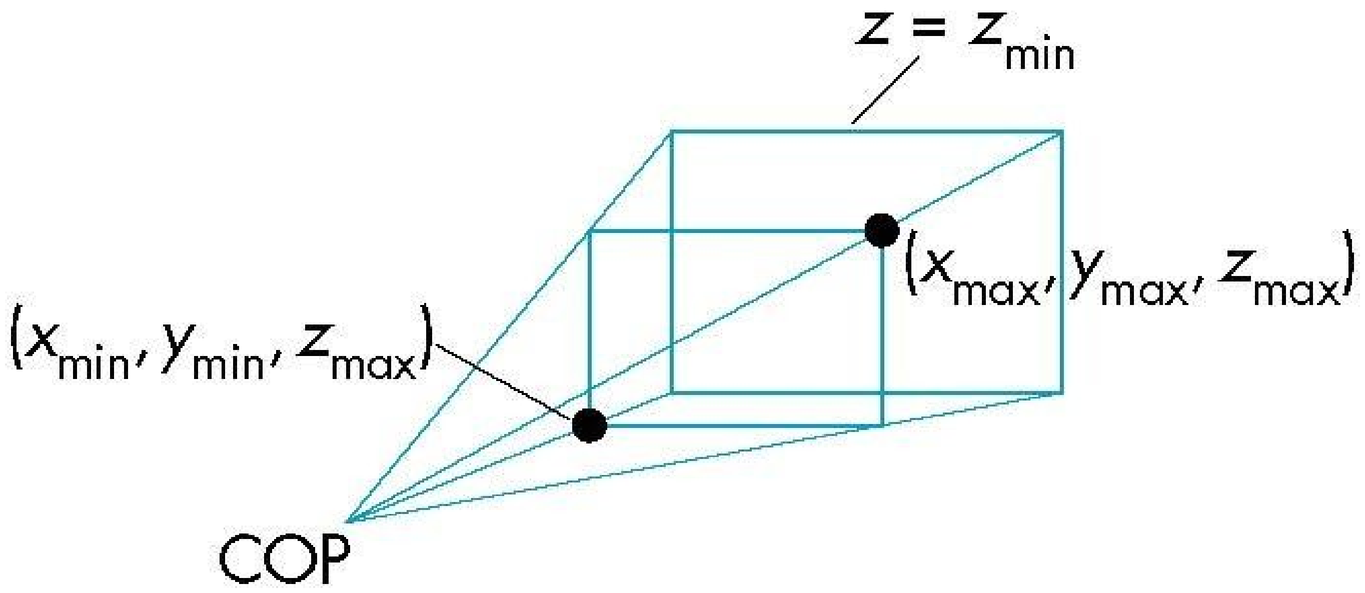


Normalization and Hidden-Surface Removal

- Although our selection of the form of the perspective matrices may appear somewhat arbitrary, it was chosen so that if $z_1 > z_2$ in the original clipping volume then for the transformed points we have $z_1' > z_2'$
- Thus hidden surface removal works if we first apply the normalization transformation
- However, the formula $z'' = -(\alpha+\beta/z)$ implies that the distances are distorted by the normalization which can cause numerical problems especially if the near distance is small

WebGL Perspective

`gl.frustum` allows for an unsymmetric viewing frustum (although **`gl.perspective`** does not)



OpenGL Perspective Matrix

The normalization in `Frustum` requires an initial shear to form a right viewing pyramid, followed by a scaling to get the normalized perspective volume. Finally, the perspective matrix results in needing only a final orthogonal transformation

$$\mathbf{P} = \mathbf{NSH}$$

our previously defined perspective matrix

shear and scale

Why do we do it this way?

- Normalization allows for a single pipeline for both perspective and orthogonal viewing
- We stay in four dimensional homogeneous coordinates as long as possible to retain three-dimensional information needed for hidden-surface removal and shading
- We simplify clipping

Perspective Matrices

frustum

$$\mathbf{P} = \begin{bmatrix} \frac{2 * near}{right - left} & 0 & \frac{right + left}{right - left} & 0 \\ 0 & \frac{2 * near}{top - bottom} & \frac{top + bottom}{top - bottom} & 0 \\ 0 & 0 & -\frac{far + near}{far - near} & -\frac{2 * far * near}{far - near} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

perspective

$$\mathbf{P} = \begin{bmatrix} \frac{near}{right} & 0 & 0 & 0 \\ 0 & \frac{near}{top} & 0 & 0 \\ 0 & 0 & -\frac{far + near}{far - near} & -\frac{2 * far * near}{far - near} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Meshes

Objective

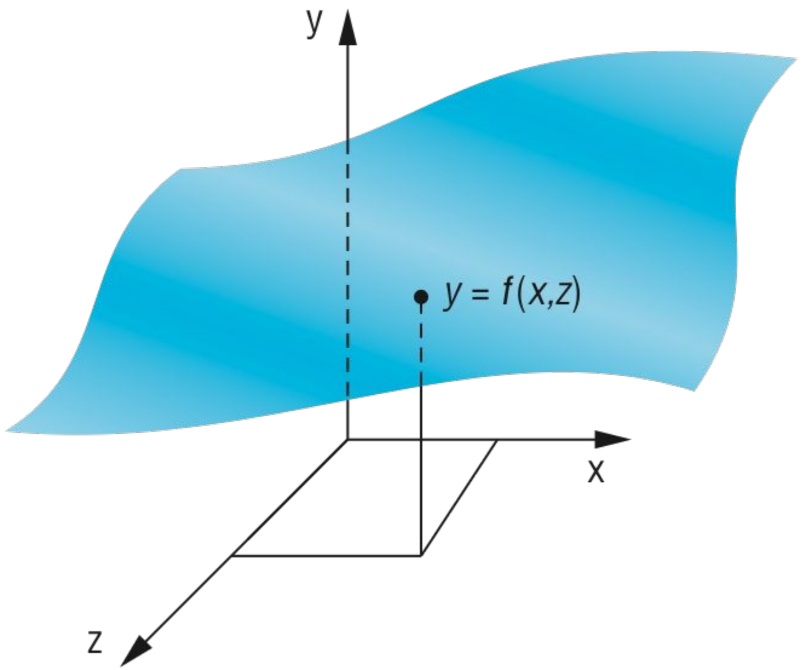
Introduce techniques for displaying polygonal meshes

Meshes

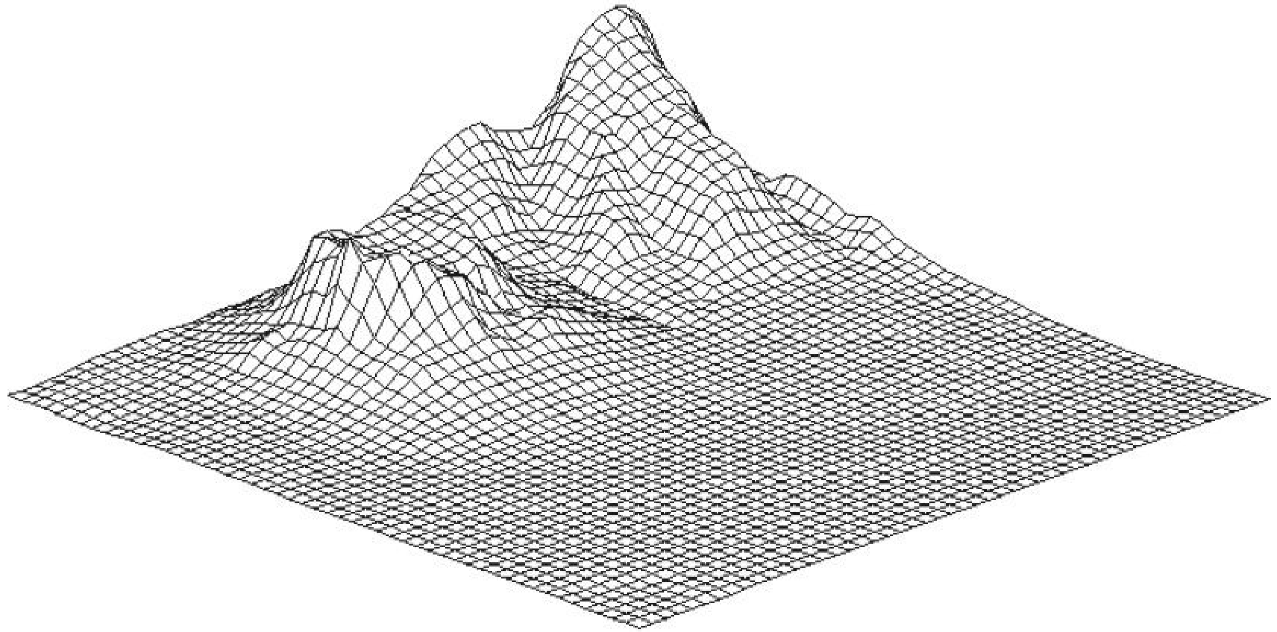
- Polygonal meshes are the standard method for defining and displaying surfaces
  - Approximations to curved surfaces
  - Directly from CAD packages
  - Subdivision
- Most common are quadrilateral and triangular meshes
  - Triangle strips and fans

Height Fields

- For each (x, z) there is a unique y
- Sampling leads to an array of y values
- Display as quadrilateral or triangular mesh using strips



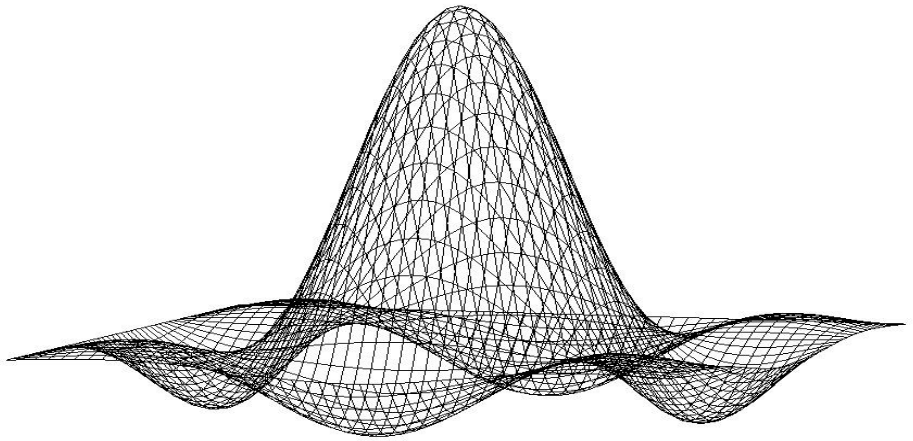
Honolulu Plot Using Line Strips



Plot 3D

- Old 2D method uses fact that data are ordered and we can render front to back
- See each plane of constant z as a flat surface that can block (parts of) planes behind it
- Can proceed iteratively maintaining a visible top and visible bottom
    - Lots of little line intersections
- Lots of code but avoids all 3D

Lines on Back and Hidden Faces

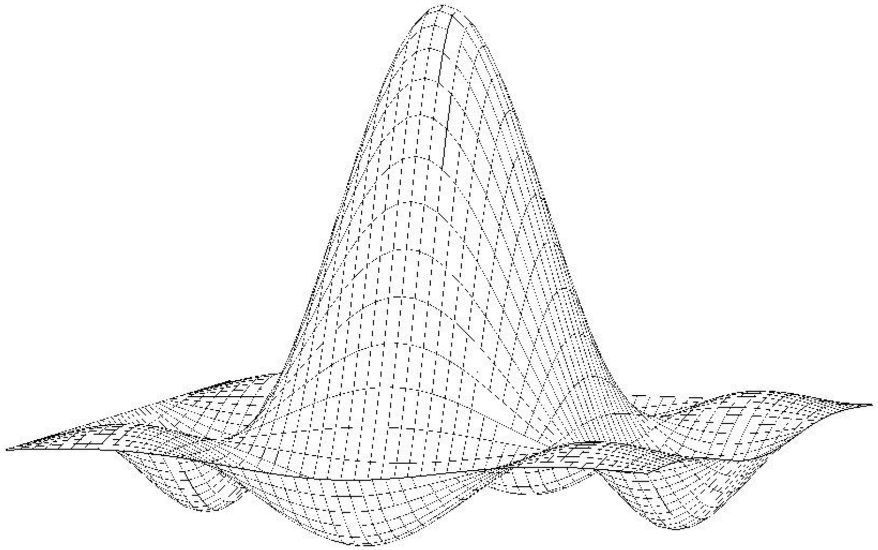


sombrero or Mexican hat function $(\sin \pi r)/(\pi r)$

Using Polygons

- We can use four adjacent data points to form a quadrilateral and thus two triangles which can be shaded
- But what if we want to see the grid?
- We can display each quadrilateral twice
  - First as two filled white triangles
  - Second as a black line loop

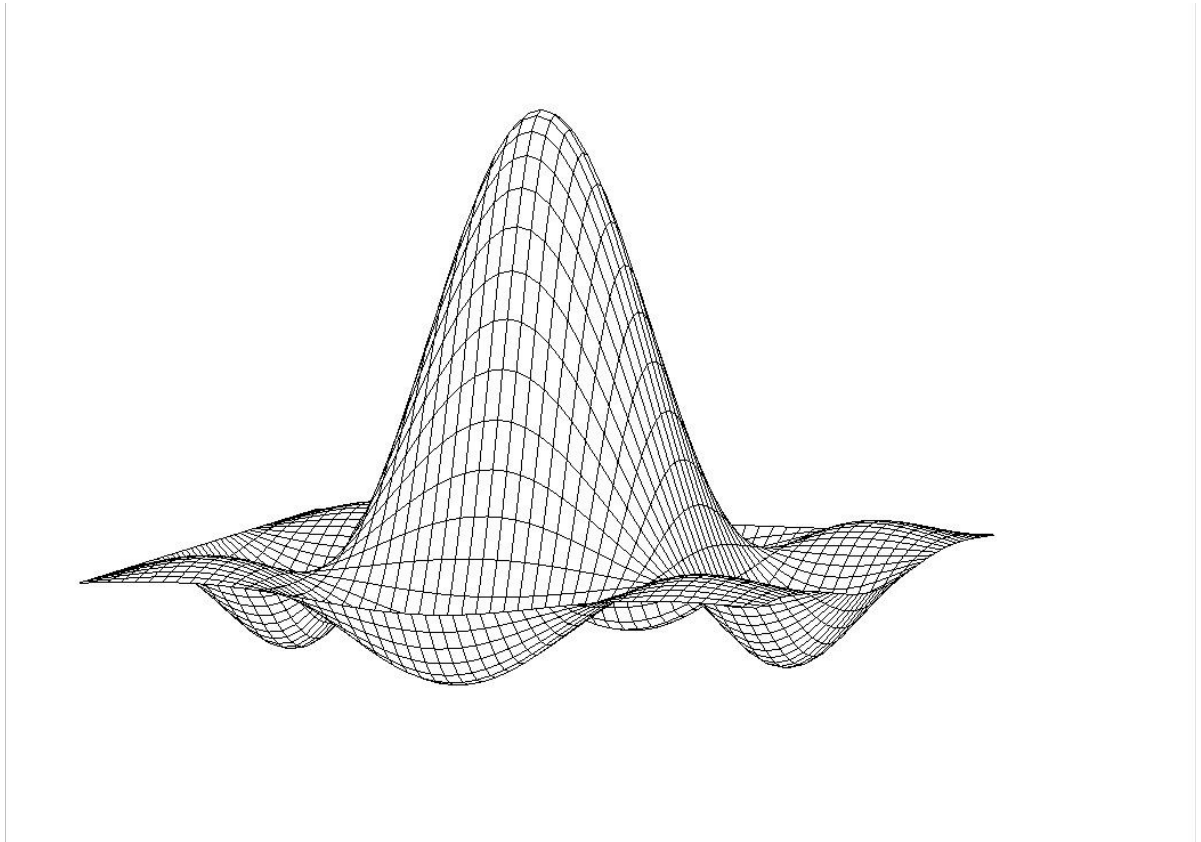
Hat Using Triangles and Lines



Polygon Offset

- Even though we draw the polygon first followed by the lines, small numerical errors can cause some fragments on the line to be displayed behind the corresponding fragment on the triangle
- Polygon offset (**`gl.polygonOffset`**) moves fragments slightly away from camera
- Apply to triangle rendering

Hat with Polygon Offset



Video

http://www.cs.upt.ro/~sorin/webgl/Code/w07/hat.html

Other Mesh Issues

- How do we construct a mesh from disparate data (unstructured points)
- Technologies such as laser scans can produced tens of millions of such points
- Chapter 12: Delaunay triangulation
- Can we use one triangle strip for an entire 2D mesh?
- Mesh simplification

Shadows

Objectives

- Introduce Shadow Algorithms
- Projective Shadows
- Shadow Maps
- Shadow Volumes

Flashlight in the Eye Graphics

- When do we not see shadows in a real scene?
  - When the only light source is a point source at the eye or center of projection
    - Shadows are behind objects and not visible
- Shadows are a global rendering issue
  - Is a surface visible from a source
  - May be obscured by other objects

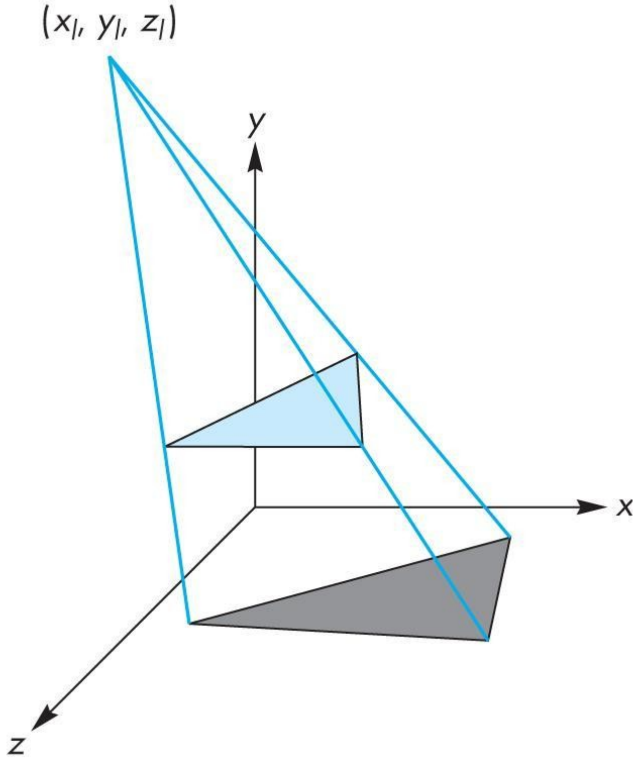
Shadows in Pipeline Renderers

- Note that shadows are generated automatically by a ray tracers
  - feeler rays will detect if no light reaches a point
  - need all objects to be available
- Pipeline renderers work an object at a time so shadows are *not automatic*
  - can use some tricks: projective shadows
  - multi-rendering: shadow maps and shadow volumes

Projective Shadows

- Oldest methods
  - Used in flight simulators to provide visual clues
- Projection of a polygon is a polygon called a **shadow polygon**
- Given a point light source and a polygon, the vertices of the shadow polygon are the projections of the original polygon's vertices from a point source onto a surface

Shadow Polygon



Computing Shadow Vertex

1. Source at (x_l, y_l, z_l)
2. Vertex at (x, y, z)
3. Consider simple case of shadow projected onto ground at $(x_p, 0, z_p)$
4. Translate source to origin with $T(-x_l, -y_l, -z_l)$
5. Perspective projection

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{-y_l} & 0 & 0 \end{bmatrix}$$

6. Translate back

Shadow Process

1. Put two identical triangles and their colors on GPU (black for shadow triangle)
2. Compute two model view matrices as uniforms
3. Send model view matrix for original triangle
4. Render original triangle
5. Send second model view matrix
6. Render shadow triangle
    - Note: shadow triangle undergoes two transformations
    - Note: hidden surface removal takes care of depth issues

Video

http://www.cs.upt.ro/~sorin/webgl/Code/w07/shadow.html

Generalized Shadows

- Approach was OK for shadows on a single flat surface
- Note with geometry shader we can have the shader create the second triangle
- Cannot handle shadows on general objects
- Other methods based on same basic idea
- Methods based on projective textures

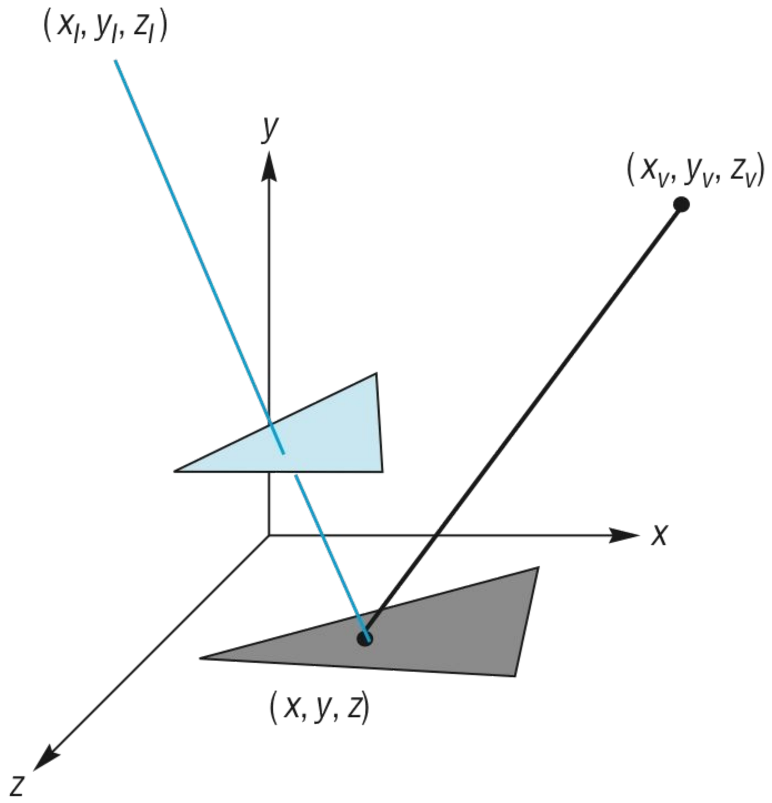
Image Based Lighting

- We can project a texture onto the surface in which case we are treating the texture as a “slide projector”
- This technique is the basis of projective textures and image based lighting
- Supported in desktop OpenGL and GLSL through four dimensional texture coordinates
- …Not yet in WebGL

Shadow Maps

- If we render a scene from a light source, the depth buffer will contain the distances from the source to nearest lit fragment.
- We can store these depths in a texture called a **depth map** or **shadow map**
- Note that although we don’t care about the image in the shadow map, if we render with some light, anything lit is not in shadow.
- Form a shadow map for each source

Shadow Mapping



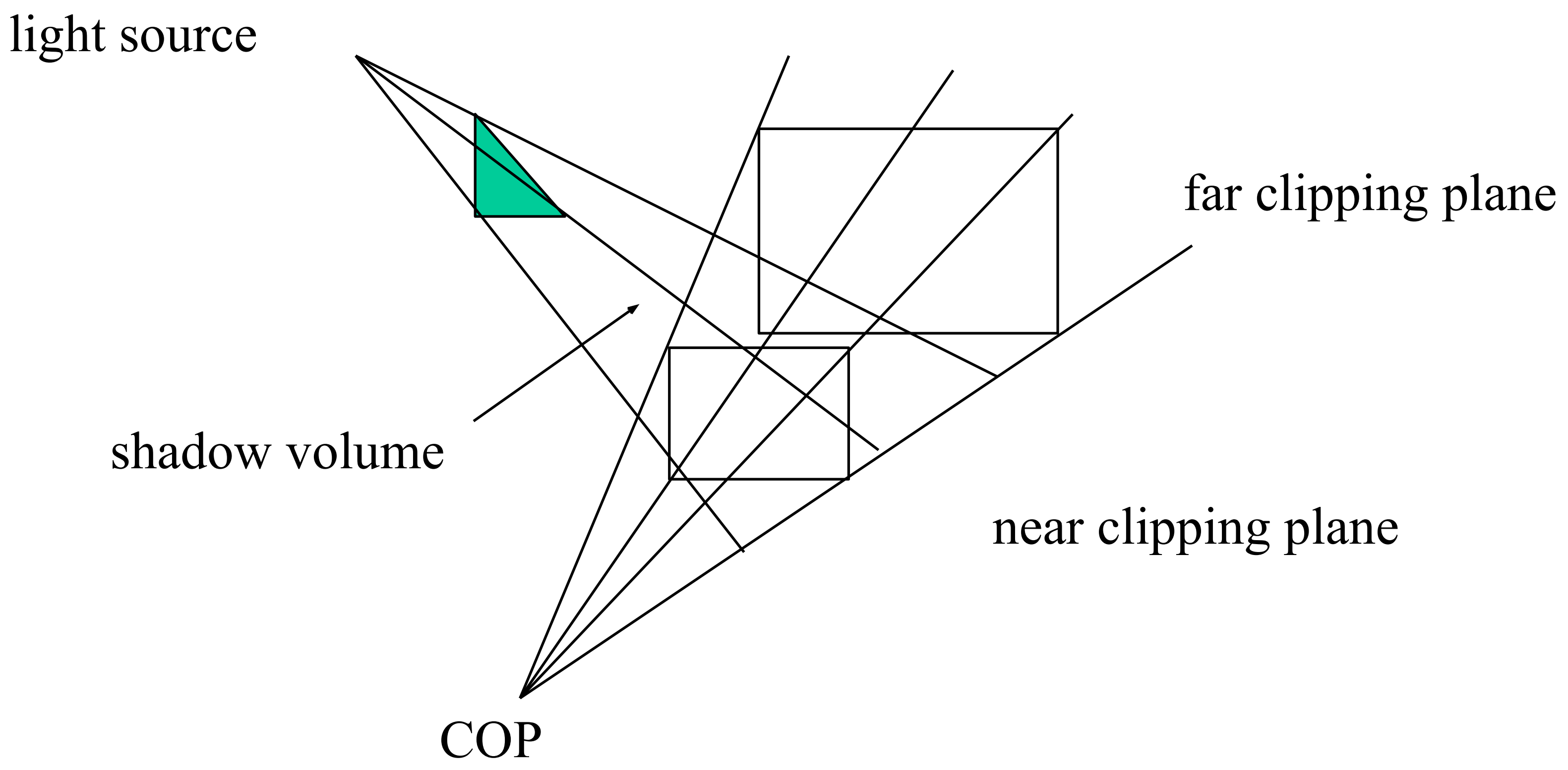
Final Rendering

- During the final rendering we compare the distance from the fragment to the light source with the distance in the shadow map
- If the depth in the shadow map is less than the distance from the fragment to the source the fragment is in shadow (from this source)
- Otherwise we use the rendered color

Implementation

- Requires multiple renderings
- We will look at render-to-texture later
    - gives us a method to save the results of a rendering as a texture
    - almost all work done in the shaders

Shadow Volumes


light source
far clipping plane
shadow volume
near clipping plane
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