Chapter VI

DOUBLE INTEGRALS

1 Iterated Integrals. Definitions and Properties

Consider $D = [a, b] \times [c, d]$ a rectangle in $\mathbb{R}^2$, $g : [a, b] \times [c, d] \to \mathbb{R}$ a function, and partitions

$$a = t_0 < t_1 < ... < t_n = b$$

of $[a, b]$,

$$c = s_0 < s_1 < ... < s_m = d$$

of $[c, d]$,

such that $g(x, y)$ has the constant value K_{ij} if $(x, y) \in D_{i,j} = (t_{i-1}, t_i) \times (s_{j-1}, s_j)$.

The integral of g is defined as a sum of $m \cdot n$ terms:

$$\iint\limits_{D} g(x, y)dxdy = \sum_{i=1,j=1}^{n,m} K_{ij} \cdot area(D_{i,j}),$$

where $D \subset \mathbb{R}^2$.

Remark If $g(x, y) \geq 0$, the integral of g is exactly the area under its graph.

Proposition 1.1. *1. If g is continuous on D, then g is integrable on D.*
2. If $D = D_1 \cup D_2$ and g is integrable on D_1 and D_2, then g is integrable on D, and

$$\iint\limits_{D} g(x, y)dxdy = \iint\limits_{D_1} g(x, y)dxdy + \iint\limits_{D_2} g(x, y)dxdy.$$

3. If g_1, g_2 are two integrable functions on D and if $g_1 \leq g_2$ on D, then

$$\iint\limits_D g_1(x,y)dxdy \leq \iint\limits_D g_2(x,y)dxdy.$$

4. If $g(x,y) = k, \forall (x,y) \in D$, then

$$\iint\limits_D g(x,y)dxdy = k \cdot Area(D).$$

5.

$$\iint\limits_D (\alpha g_1(x,y) + \beta g_2(x,y))dxdy = \alpha \iint\limits_D g_1(x,y)dxdy + \beta \iint\limits_D g_2(x,y)dxdy,$$

for any two scalars $\alpha, \beta \in \mathbb{R}$ and for any two integrable functions g_1 and g_2.

Remark: The iterated integral

$$\int\limits_a^b dx \int\limits_c^d g(x,y)dy$$

or

$$\int\limits_c^d dy \int\limits_a^b g(x,y)dx$$

is evaluated from the inside out. One first holds y fixed and evaluates $\int\limits_a^b g(x,y)dx$ with respect to x; the result is a function of y, which is then integrated from c to d.

Example: Evaluate

$$\int\limits_0^2 dx \int\limits_1^3 x^2 y dy.$$

Solution:

$$\int\limits_0^2 dx \int\limits_1^3 x^2 y dy = \int\limits_0^2 x^2 (\frac{y^2}{2} |_1^3) dx$$

$$= 4 \int\limits_0^2 x^2 dx = \frac{32}{3}.$$

Proposition 1.2. *The double integral equals the iterated integral: for* $D = [a, b] \times [c, d]$, f *integrable on* D, *we have:*

$$\iint\limits_D g(x,y)dxdy = \int\limits_a^b dx \int\limits_c^d g(x,y)dy = \int\limits_c^d dy \int\limits_a^b g(x,y)dx.$$

Example: Evaluate $\iint\limits_D e^{2x+y}dxdy$ on the rectangle $D = [0, 1] \times [0, 3]$.

Solution We express the double integral as an iterated integral, as follows:

$$\iint\limits_D e^{2x+y}dxdy = \int\limits_0^1 dx \int\limits_0^3 e^{2x+ydy}$$

$$= \int\limits_0^1 (e^{2x+y} \mid_0^3)dx = \int\limits_0^1 (e^{2x+3} - e^{2x})dx$$

$$= (\frac{e^{2x+3}}{2}) \mid_0^1 -(\frac{e^{2x}}{2}) \mid_0^1$$

$$= \frac{1}{2}(e^5 - e^3 - e^2 + 1).$$

2 Double Integral Over General Regions

Let $f : D \to \mathbb{R}$ be an integrable function on D, D is **NOT** a rectangle. We assume that $D \subset D^*$ is contained in some rectangle D^*, and $f^* : D^* \to \mathbb{R}$,

$$f^*(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \notin D \end{cases}$$

Definition 2.1. *If* f^* *is integrable on* D^*, *then we say that* f *is integrable on* D, *and we define*

$$\iint\limits_D f(x,y)dxdy = \iint\limits_{D^*} f^*(x,y)dxdy.$$

Definition 2.2. *We shall define two simple type of regions, called* **elementary regions.** *Other regions can be broken into elementary ones.*

$$D = \{(x, y),\ x \in [a, b], y \in [\varphi_1(x), \varphi_2(x)]\},$$

where $\varphi_1(x), \varphi_2(x)$ *two continuous functions on* $[a, b]$. *In this case,* D *is said to be* **of type** 1.

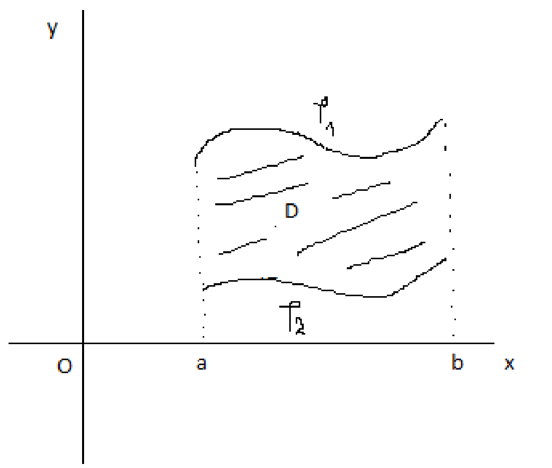

Figure 1

$$D = \{(x, y),\ y \in [c, d], x \in [\psi_1(y), \psi_2(y)]\},$$

where $\psi_1(x), \psi_2(x)$ *two continuous functions on* $[c, d]$. *In this case,* D *is said to be* **of type** 2.

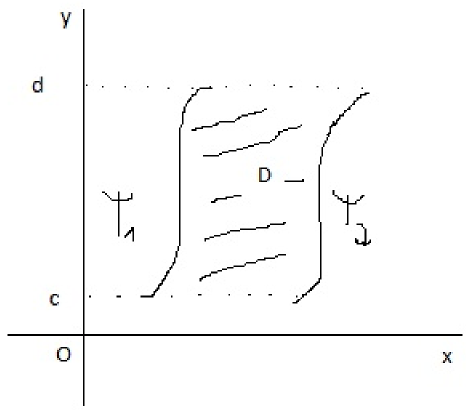

Figure 2

Proposition 2.1. *If f is continuous on the elementary region D, then f is integrable on D and:*

$$\iint\limits_D f(x,y)dxdy = \int\limits_a^b \Big(\int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \Big)dx,$$

if D is of type 1, *or*

$$\iint\limits_D f(x,y)dxdy = \int\limits_c^d \Big(\int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \Big)dy,$$

if D is of type 2.

3 Applications of Double Integrals

1. Center of mass $G(x_G, y_G)$, where

$$x_G = \frac{\iint\limits_D x \cdot \rho(x,y)dxdy}{\iint\limits_D \rho(x,y)dxdy}, \quad y_G = \frac{\iint\limits_D y \cdot \rho(x,y)dxdy}{\iint\limits_D \rho(x,y)dxdy},$$

where $\rho(x,y)$ is the variable density of the plate D.

2. The area of $D \in \mathbb{R}^2$:
$$\mathcal{A}(D) = \iint\limits_D dxdy.$$

3. If f is integrable on D, the ratio
$$\frac{\iint\limits_D f(x,y)dxdy}{\iint\limits_D dxdy}$$
is called **the average value of f on D.**

4 Integrals in Polar Coordinates

If f is an integrable function on the domain $D \in \mathbb{R}^2$, then:

$$\iint\limits_D f(x,y)dxdy = \iint\limits_{D'} f(r\cos\theta, r\sin\theta) \cdot rdrd\theta, r \geq 0, \theta \in [0, 2\pi],$$

where D' is the region corresponding to D in the variables r and θ.