LAB-STA-4. STEADY-STATE (SS) ANALYSIS OF CONTROL SYSTEMS. CALCULATION OF THE STEADY-STATE VALUES OF CONTROL SYSTEMS

A. OBJECTIVES. 1. Gaining knowledge on the properties of control systems (CSs) in steady-state. 2. Finding the SS values of a CS. 3. Calculation of the static coefficient of a CS.

B. THEORETICAL CONSIDERATIONS.

1. Constant steady-state regime. In a system (or CS), the constant SS regime is possible if the following necessary conditions are met:

- the system is stable,
- the system's inputs have constant value in time, that is:

  w_∞=const and v_∞=const, (1)

The subscript ∞ points the constant steady-state (CSS) values of the variables. The SS regime appears after all the transients die out and implies the *cancellation of the integral and derivative effects in the system*.

2. Necessary conditions for the SS regime of a system and specific equations. The constant steady-state values of a system are determined:

- *analytically*: using different MMs of the system written explicitly for the steady-state regime;
- *experimentally:* using measurements on system's inputs/outputs/states.

The analytical condition of reaching the SS has several formulations.

(a) For continuous-time systems:

❑ Given by state-space MMs:

$$\underline{x}' = \underline{x}'_\infty = \underline{0}. \tag{2-a}$$

❑ Given as input-output MMs, cancelling the effects of the derivatives implies:

$$y^{(\nu)} = 0 \quad \text{and} \quad u^{(\mu)} = 0 \quad \text{for } \nu, \mu > 0, \tag{2-b}$$

For a rational transfer function:

$$H(s) = \frac{B(s)}{A(s)} = \frac{b_m s^m + \ldots + b_0}{a_n s^n + \ldots + a_0} \text{ with } m \le n \text{ and } b_0 \ne 0 \text{ and } a_0 \ne 0. \tag{3}$$

Then for t $\to\infty$, using **The Final Value Theorem** (FVT):

$$y_\infty = \lim_{s \to 0} s \cdot H(s) \frac{1}{s} u_\infty = H(0)\, u_\infty \quad \text{and} \quad k = \frac{b_0}{a_0}, \text{ respectively.} \tag{4-a}$$

For any u_∞=ct. $\ne 0$ there exists a value y_∞= ct. $\ne 0$. *k* is also known as the **DC gain**.

The graphical representation:

$$y_\infty = f(u_\infty) \tag{4-b}$$

is called the *static characteristic* (SC) of a system.

The output of the control system given in Fig.B-2 has the transfer function:

$$z(s) = H_{zw}(s)w(s) + H_{zv}(s)v(s)\,; \tag{5}$$

after applying FVT (5) becomes:

$$z_\infty = H_{zw}(0)w_\infty + H_{zv}(0)v_\infty\,, \tag{6}$$

where $H_{zw}(0) = k_1$ and $H_{zv}(0) = k_2$, where k_1 and k_2 are the static coefficients of the control system (the DC gains).

In particular, for P, I and D blocks:

- the proportional-type blocks (P) (and PT1, PDT1,…,) with $a_0 \ne 0$ and $b_0 \ne 0$

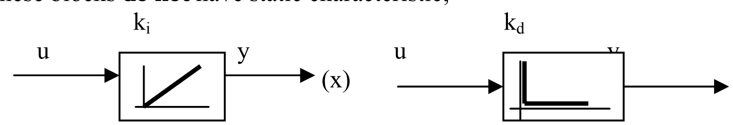
  $$y_\infty = k \cdot u_\infty\,; \tag{7}$$

  these blocks have *static characteristics.*

- the integral-type blocks (I) (or containing a distinct I component), Fig. B.1-a, usually recognized for a_0=0:

  $$u_\infty = 0 \to y_\infty = ct\,, \quad \text{any constant value is possible.} \tag{8}$$

  These blocks **do not** have static characteristic;



(a) (b)

Fig. B-1. The I and D-type blocks.

- The derivative-type blocks (D) (or with distinct D component), Fig. B.1-b, recognized by b_0=0:

  $$u_\infty = cons \to y_\infty = 0\,, \quad u_\infty \text{ takes any possible constant value.} \tag{9}$$

  These blocks also do not have static characteristic.

(b) For discrete-time systems. The following relations are valid in SS:

❑ For state-space representations:

$$\underline{x}_{k+1} = \underline{x}_k = \underline{x}_\infty. \tag{10}$$

❑ For stable discrete-time transfer function given by:

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_m z^m + \ldots + b_0}{a_n z^n + \ldots + a_0}, \tag{11}$$

with $m \le n$ and $\Sigma b_\mu \ne 0$ and $\Sigma a_\nu \ne 0$, then **FVT** gives:

$$y_\infty = \lim_{z \to 1} \frac{z-1}{z} H(z) \frac{z}{z-1} u_\infty \quad \text{and} \quad k = H(1) = \frac{\Sigma b_\mu}{\Sigma a_\nu}, \text{ respectively.} \tag{12}$$

In particular, for:

- P-type blocks: $y_\mu = y_\infty$= const $u_\mu = u_\infty$= const, (13)

$y_\infty = k\, u_\infty$ with $k = H(1)$; (14)

these blocks have *static characteristic.*

- I-type blocks (or with distinct I component):

$$u_\infty = 0 \rightarrow y_\infty = y_k = y_{k+1} = const.\ , \quad \text{any possible value} \tag{15}$$

- D-type blocks (or with distinct D component)

$$u_\infty = u_k = u_{k+1} = const. \rightarrow y_\infty = 0\,. \tag{16}$$

3. Different situations for calculating the SS values of CSs.

3.1. For state-space representation of CS.

Continuous-time systems: ***Discrete-time systems:***

$$\begin{cases} \dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t) \\ \underline{y}(t) = \underline{C}\,\underline{x}(t) + \underline{D}\,\underline{u}(t) \end{cases} \qquad \begin{cases} \underline{x}(t+1) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t) \\ \underline{y}(t) = \underline{C}\,\underline{x}(t) + \underline{D}\,\underline{u}(t) \end{cases}. \tag{17}$$

with r – number of inputs, n – the number of states and q – the dimension of the output. Using (2-a) and (10), the following hold:

$$\begin{cases} 0 = \underline{A}\,\underline{x}_\infty + \underline{B}\,\underline{u}_\infty \\ \underline{y}_\infty = \underline{C}\,\underline{x}_\infty + \underline{D}\,\underline{u}_\infty \end{cases} \qquad \begin{cases} \underline{x}_\infty = \underline{A}\,\underline{x}_\infty + \underline{B}\,\underline{u}_\infty \\ \underline{y}_\infty = \underline{C}\,\underline{x}_\infty + \underline{D}\,\underline{u}_\infty \end{cases}. \tag{18}$$

(18) is a system of (n+q) equations in (n+r+q) steady-state values. If *r* steady-state values are known and the system is compatible, then the rest of $n + q$ steady-state values can be found.

3.2. SS values calculations for connections of systems given by block diagram representations with input-output or state-space MMs. For aforementioned blocks, the calculations are summarized in Table 1.

Table 1.

Block type	Continuous-time	Discrete-time
P	$y_\infty = \dfrac{b_0}{a_0} u_\infty$	$y_\infty = \dfrac{\sum b_i}{\sum a_j} u_\infty$
I	$u_\infty = 0$ $y_\infty = const$	$u_k = 0$ $y_k = const$ for $k > k_0$
D	$u_\infty = 0$ $y_\infty = const$	$u_{k+1} = u_k$ $y_k = 0$ for $k > k_0$

We often:

- Explicit the SS regime functioning conditions for each typical block (I, D, P) and then we write down the SS relations between the variables:

- for I-type blocks: $u_\infty = 0 \rightarrow$ $y_\infty = const$,
- for D-type blocks: $u_\infty = const \rightarrow y_\infty = 0$,
- for P-type blocks: $u_\infty = const \rightarrow y_\infty = k \cdot u_\infty$.

- Again, a system of equations is obtained with dimension depending on the complexity of the system; in principle we have a system of (n+q) equations with (n+q+r) variables.
- If sufficient SS values are known with respect to which the system is compatible (r – SS values, but not any r values) the other SS values can be determined.

4. The SS regime of CSs.

The control system given in Fig. B-2 (with C – controller, ACT – actuator, CP – controlled process, ME – measuring element (sensor) should usually ensure:

- zero steady-state error:

$$e_\infty = w_\infty - y_\infty = 0 \quad \Leftrightarrow \quad y_\infty = 1 \cdot w_\infty; \tag{19}$$

- rejection of constant value disturbance $v_\infty = const$:

$$y_\infty = y_0 + 0 * v_\infty \quad \text{with} \quad y_0 = 1 \cdot w_\infty. \tag{20}$$

- the operation around different operating points and transition between them;
- favorable transient regimes.

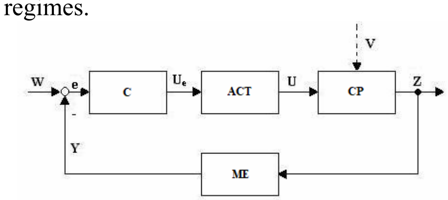


Fig. B-2. Block diagram of a CS.

- ❑ Equations (19) and (20) are valid only for CSs where the controller has an I component (CSs of type-1 and type-2).
- ❑ For CS of type-0 without I component in the controller the steady-state control error e_∞ will be nonzero:

$$y_\infty = \frac{k_0}{1+k_0} \cdot w_\infty + \gamma_n\, v_\infty, \text{ where} \tag{21-a}$$

$$\gamma_n = \left.\frac{y_\infty}{v_\infty}\right|_{w_\infty = 0} \quad \text{is the } \textit{natural static coefficient} \text{ of the CS.} \tag{21-b}$$

and is a measure of the sensitivity of the output with respect to the disturbance.

In this case ($e_\infty \neq 0$) with:

$$e_\infty = w_\infty - y_\infty = \frac{1}{1+k_0} \cdot w_\infty + \gamma_n\, v_\infty\,. \tag{22}$$

For a P-type character of the sensor (or PT_1, etc. ...), in the SS relation is

$y_\infty = k_M z_\infty$, (23)
and next:

$$z_\infty = \frac{1}{k_M} \cdot \frac{k_0}{1+k_0} \cdot w_\infty + \frac{1}{k_M} \gamma_n \cdot v_\infty \,. \qquad (24)$$

$\gamma_{n(z)} = 1/k_M \cdot \gamma_n$ is the natural static coefficient of the CS for the controlled (output) variable *z*. More generally, for a CS one can write

$$\gamma_n = \left. \frac{y_\infty}{v_\infty} \right|_{w\infty=0} = \frac{k_{N(y)}}{1+k_0} \quad [<y>/<v>] \qquad (25\text{-a})$$

$$\gamma_{n(z)} = \left. \frac{z_\infty}{v_\infty} \right|_{w\infty=0} = \frac{k_{N(z)}}{1+k_0} \quad [<z>/<v>] \quad \text{where} \quad k_0 = k_C\, k_{CP}\,,. \qquad (25\text{-b})$$

where $k_{N(y)}$ and $k_{N(z)}$ are some gains depending on the process and the controller.
The static coefficient can be modified by changing the value of k_C*,* but keep in mind that this can affect the closed loop stability*:* for a desired value of the static coefficient γ_{nd}, the necessary k_C can be found given that k_N and k_{CP} are known:

$$k_{Cunk} = \frac{k_N - \gamma_{nd}}{\gamma_{nd}\, k_{CP}}. \qquad (26)$$

Summarizing:

- For CSs with controllers of type I, PI, PID: $e_\infty = 0$ and $\gamma_n = 0$;
- For CSs with controllers of type P, PT_1: $e_\infty \neq 0$ and $\gamma_n \neq 0$.

The graphical representations for the CS are the families of curves:
$y_\infty = f(w_\infty)$ for v_∞= const called reference static map, (27-a)
$y_\infty = f(v_\infty)$ for w_∞=const called load static map. (27-b)
The static coefficient can be also expressed in percents:

$$\gamma_{n(z)}^{\,*} = \frac{\Delta z_\infty / z_n}{\Delta v_\infty / v_n} = \frac{\Delta z_\infty}{\Delta v_\infty} \cdot \frac{v_n}{z_n} = \gamma_{n(z)} \frac{v_n}{z_n} \quad \text{and in [\%]:} \quad \gamma_{n(z)}^{\,\%} = \gamma_{n(z)} \cdot \frac{v_n}{z_n} \cdot 100\% \,. \qquad (28)$$

C. CASE STUDIES TO UNDERSTAND THE KEY CONCEPTS PRESENTED IN THIS LAB.

CS-1. The CS given in Fig. C-1 is considered. It is the simplified scheme of a DC motor position control system. The controller is of PI-type with parameters of the transfer function $H_{PI}(s) = k_C(1+sT_r)/(sT_r)$ being k_C=5 and T_r =1. For the other blocks, the parameters values are:

- for parallel actuators k_{ACT1}=10 (20) k_{ACT2}=15 (20);
- for the DC motor k_1=0.08, T_1=0.05, $1/T_i$= 1/0,1 , k_{em}=0.8;
- for measuring element (sensor) k_{ME}= 0.02.

(a) Calculate $H_{z\text{-}w}(s)$ and $H_{z\text{-}v}(s)$; for the given controller, determine the closed loop system stability and estimate the phase margin of the closed loop system. For specified T_r , find the maximum value of k_C for which the closed-loop system becomes unstable.
(b) (b) Find the SS values in the system {e_∞, $u_{M\infty}$, y_∞, $y_{1\infty}$, $e_{2\infty}$, m_∞, $m_{1\infty}$, $m_{2\infty}$, n_∞, $e_{1\infty}$} for the combination of input values given in Table SC-1 (the subscript ∞ is omitted).

Table SC-1.

w	v	e	u_M	y	y_1	e_2	n	e_1	m	m_1	m_2	z	
0	0												(1)
3	0												(2)
6	0												(3)
6	5												(4)
6	10												(5)

(c) Find the unknown w_∞ that ensures z_∞ = 250 and z_∞ = 350 for v_∞ = 10 and v_∞ = 15 respectively; calculate the other SS values of the CS (hint: use a table similar to table SC-1).
(d) A fault of the actuator ACT1 with k_{ACT1} = 10 (20) , implies $m_{1\infty}$ = 0,0. Accepting that actuator ACT2 saturates at the maximal value of m_{2max} = 1,5 m_{2n} , where $m_{2n} = m_{2\infty}$, ($m_{2\infty}$ taken from 5th row of Table SC-1), analyze if the CS can function in the regimes (4) and (5) given in Table SC-1. Calculate the other SS values in the CS and give an interpretation of the results.
(e) Design a structure that ensures a –5% artificial static coefficient.
(f) Reconsider the case study for a controller of type PDT1 with the transfer function:

$$H_C(s) = \frac{Q(s)}{P(s)} = \frac{b_1 s + b_0}{a_1 s + a_0} \qquad \begin{matrix} b_0 = 5, & b_1 = 12{,}5 \\ a_1 = 0{,}1, & a_0 = 1 \end{matrix}$$

In the first step, express the controller transfer function in parameters k_C , T_f and T_d.

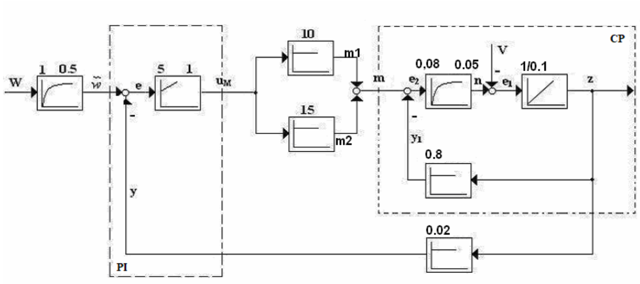


Fig. C-1.

CS-2. The CS given in Fig. C-2 is considered. The block diagram corresponds to a synchronous generator voltage control. The controller is of PI-type with parameters' values indicated in Fig. C-2.

For $w_\infty = 10$ and $v_\infty = 1250$ calculate the SS values of the control system.

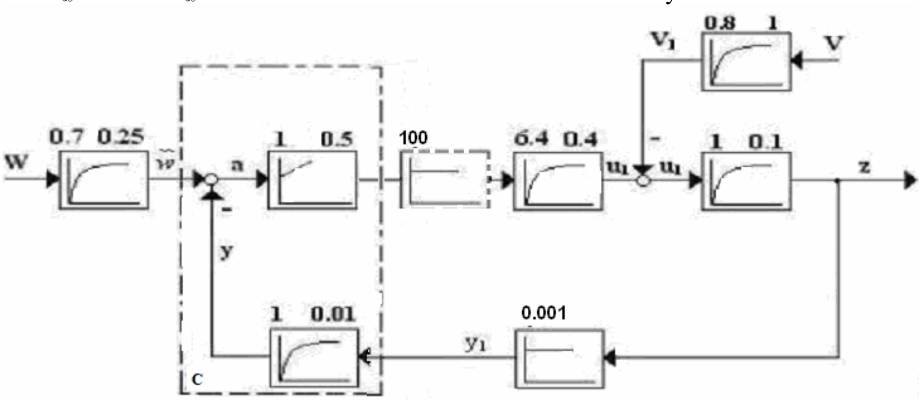


Fig. C-2.

(b) Find w_∞ that ensures $z_\infty = 6000$ for $v_\infty = 1000$; find the SS values of the CS in this case;
(c) Calculate the closed-loop transfer functions $H_{z\text{-}w}(s)$ and $H_{z\text{-}v}(s)$ and analyze the CS stability. Is the system's static coefficient influenced by a 10 time increase of k_C?

CS-3. The discrete-time CS given in Fig. C-3 is considered. The discrete-time controller (CD-N) is described by the following transfer functions (which were obtained by Tustin approximation of continuous-time controllers):

$$H_C(z) = H_C(s)\Big|_{s=\frac{2}{Te}\frac{z-1}{z+1}}$$

F-w: $H_{FW}\left(z^{-1}\right)=\dfrac{0.2+0.2z^{-1}}{1-0.6z^{-1}}$, F-y: $H_{FY}\left(z^{-1}\right)=\dfrac{0.8-0.8z^{-1}}{1-0.6z^{-1}}$,

AR-P: $H_C\left(z^{-1}\right)=\dfrac{2.05-1.95z^{-1}}{1-z^{-1}}$, with T_s=0.5 sec.

(a) Write the algorithm which characterizes the controller CD-N ($u^*_1(k)$, $u^*_2(k)$, $u(k) = u^*_1(k) - u^*_2(k)$ as functions of $w^*(k)$ and $y^*(k)$) and write the program in pseudocode;
(b) The controlled continuous-time process is given by the state-space description:

$$\begin{cases} \dot{x}_1 = -0.2x_1 + 200u_c \\ \dot{x}_2 = 0.6x_1 - x_2 + 2{,}5u_c - 50v \\ y = 0{,}005x_2 \\ z = 5x_2 \end{cases}.$$

Find the SS values for the controlled process for $u_{c\infty} = 10$ and $v_\infty = 50$, namely $\{u_{c\infty}, x_{1\infty}, x_{2\infty}, z_\infty, y_\infty\}$.

Find the transfer function of the process and determine the CD gain of the sensor $k_M = y_\infty/z_\infty$

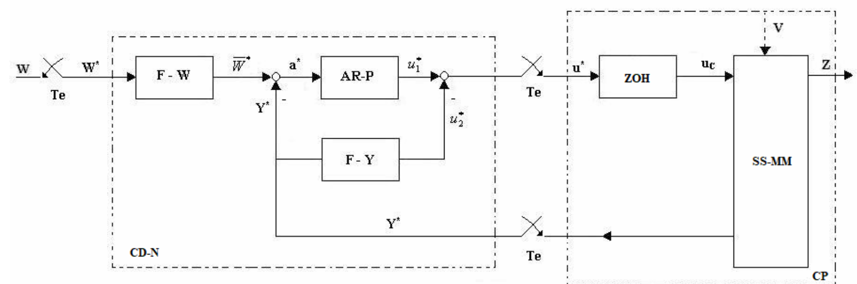


Fig. C-3.

(c) Calculate the SS values of the CS for $u_{c\infty} = 10$ and $v_\infty = 1$:
Calculate the SS values of the CS for $w_\infty = 10$ and $v_\infty = 50$:

$$\{a^*, \overline{w}^*, u_1^*, u_2^*, u^*, u_{c\infty}, x_{1\infty}, x_{2\infty}, z_\infty, y_\infty\}.$$

(d) How can the CD gain of AR-P be increased by a factor of 5 without changing its dynamic behavior? Write down the modified recurrent equation.

CS-4. The CS given in Fig. C-4 is considered.
(a) For a PDT1 controller and w_∞= 7, v_∞= 1.25, find the SS values in the CS.
(b) Calculate the natural static coefficient (γ_n).

(c) Solve again (a) and (b) for a PI controller and calculate k_{bcv} of a feed-forward structure from disturbance *v* ensuring an artificial static coefficient of 5%.

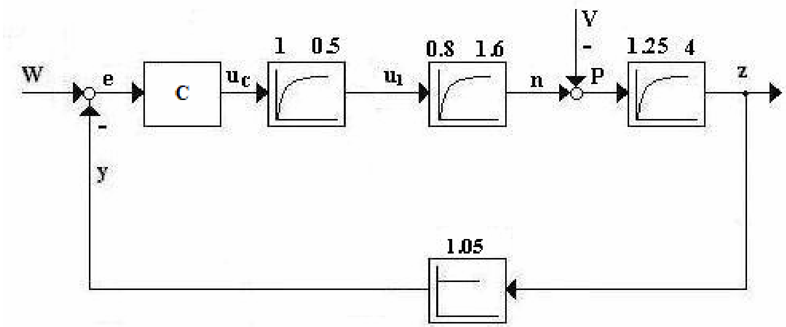


Fig. C-4.

The controllers are:

PDT1: $H_C(s) = \dfrac{2(1+2s)}{1+0.1s}$ (a), PI: $H_C(s) = \dfrac{2(1+2s)}{s}$ (b).

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