Chapter VII

PAIRS OF RANDOM VARIABLES - 1st Part

1 Definitions

In an experiment that produces one r.v., events are points or intervals on a line. In an experiment that leads to two r.v. X and Y, each outcome (x, y) is a point in a plane and events are points or areas in the plane.

Definition 1.1. **The joint cumulative distribution function** *(CDF) of r.v. X and Y is:*

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y).$$

Proposition 1.1. *The following properties hold:*

1. $$0 \leq F_{X,Y}(x, y) \leq 1,$$
   for any pair $(x, y) \in \mathbb{R}^2$;
2. $F_X(x) = \lim_{y\to\infty} F_{X,Y}(x, y)$ *and* $F_Y(y) = \lim_{x\to\infty} F_{X,Y}(x, y)$;
3. $\lim_{x\to-\infty} F_{X,Y}(x, y) = \lim_{y\to-\infty} F_{X,Y}(x, y) = 0$;
4. *If $x \leq x_1$ and $y \leq y_1$, then $F_{X,Y}(x, y) \leq F_{X,Y}(x_1, y_1)$ ($F_{X,Y}(x, y)$ is an increasing function);*
5. $\lim_{\substack{x\to\infty \\ y\to\infty}} F_{X,Y}(x, y) = 1$.

Definition 1.2. The joint probability mass function *of discrete r.v. X and Y is:*

$$P_{X,Y}(x,y) = P[X = x, Y = y].$$

We denote by $S_{X,Y}$ the range of the pair (X,Y), meaning the set of possible values of the pair:

$$S_{X,Y} = \{(x,y),\ P(x,y) > 0\}.$$

Proposition 1.2. *For any two discrete r.v. X and Y, and any set $B \subset (xOy)$, the probability of the event $(X,Y) \in B$ is:*

$$P(B) = \sum_{(x,y)\in B} P_{X,Y}(x,y).$$

Proposition 1.3. *If $P_{X,Y}(x,y)$ is the joint PMF for r.v. X and Y, the PMF of the r.v. X is given by*

$$P_X(x) = \sum_{y\in S_Y} P_{X,Y}(x,y),$$

and is called **the marginal PMF for** X.
Obviously, the marginal PMF of Y is:

$$P_Y(y) = \sum_{x\in S_X} P_{X,Y}(x,y).$$

Definition 1.3. The joint probability density function *of the continuous r.v. X and Y is a function $f_{X,Y}(x,y)$ with the property:*

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)dudv.$$

Proposition 1.4. *1. Given the joint CDF $F_{X,Y}(x,y)$ of the continuous r.v. X and Y, the joint PDF of X and Y is the second order partial derivative of joint CDF:*

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}.$$

2. The probability that X takes values in $[a,b]$ and Y takes values in $[c,d]$ is:

$$P[a < X \leq b, c < Y \leq d] = F_{X,Y}(b,d) - F_{X,Y}(b,c) - F_{X,Y}(a,d) + F_{X,Y}(a,c).$$

3. The probability that the continuous r.v. (X,Y) are in $A \subset (xOy)$ is:

$$P(A) = \iint_A f_{X,Y}(x,y)dxdy.$$

Proposition 1.5. *A joint PDF $f_{X,Y}$ has the following properties:*

1. *$f_{X,Y} \geq 0$, for any real pair (x,y);*
2. *$\int_{-\infty}^{\infty} \int_{-\infty}^{f_{X,Y}(u,v)} dudv = 1$.*

Proposition 1.6. *If X and Y are continuous r.v. with joint PMF $f_{X,Y}(x,y)$, then* **the marginal PDF of** X, *respectively Y are given by*

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy,$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx.$$

2 Solved Problems

1. Test two integrated circuits one after the other. On each test, the possible outcomes are a (accepted), and r (rejected).
   Assume that all circuits are acceptable with probability 0.9 and that

the outcomes of successive tests are independent.
Count the number of acceptable circuits X and count the number of successful tests Y before you observe the first reject. If both tests are successful, let $Y = 2$. Find the following:
a) The joint PMF of X and Y.
b) The probability of the event B that X equals Y (the number of acceptable circuits equals the number of tests before observing the first failure);
c) The marginal PMFs.

Solution: a) Let us denote by $S = \{aa, ar, ra, rr\}$ the sample space, and the function $g : S \to \mathbb{R}^2$ that transforms the outcome $s \in S$ into the pair (X, Y).

Then $g(aa) = (2, 2),\ g(ar) = (1, 1),\ g(ra) = (1, 0),\ g(rr) = (0, 0)$.
The corresponding probabilities are computed in the following table:

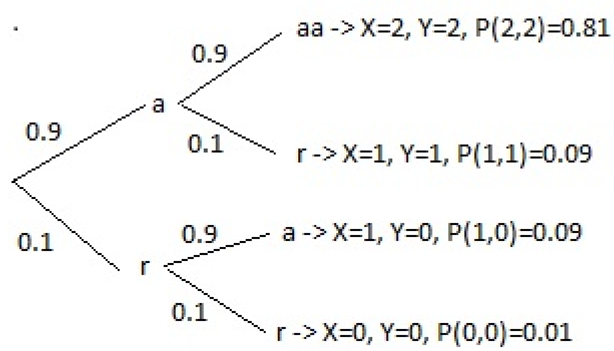

Figure 1

X/Y	$y=0$	$y=1$	$y=2$	$P_X(x)$
$x=0$	0.01	0	0	0.01
$x=1$	0.09	0.09	0	0.18
$x=2$	0	0	0.81	0.81
$P_Y(y)$	0.10	0.09	0.81	1

so the joint PMF of X and Y is:

$$P_{X,Y}(x,y) = \begin{cases} 0.81, & x=2, y=2 \\ 0.09, & x=1, y=1 \\ 0.09, & x=1, y=0 \\ 0.01, & x=0, y=0 \end{cases}$$

b) $B = \{X = Y\}$ so $B \cap S_{X,Y} = \{(0,0),(1,1),(2,2)\}$, therefore

$P(B) = P_{X,Y}(0,0)+P_{X,Y}(1,1)+P_{X,Y}(2,2) = 0.01+0.09+0.81 = 0.91.$

c) The marginal PMF of X can obtained from the last column of the above table:

$$P_X(x) = \begin{cases} 0.01, & x=0 \\ 0.18, & x=1 \\ 0.81, & x=2 \end{cases}$$

The marginal PMF of Y can obtained from the last line of the above table:

$$P_Y(y) = \begin{cases} 0.1, & y=0 \\ 0.09, & y=1 \\ 0.81, & y=2 \end{cases}$$

2. R.v. X and Y have the joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} c, & x \in [0,5], y \in [0,3] \\ 0, & otherwise \end{cases}$$

a) Find $c \in \mathbb{R}$.
b) Compute the probability $P(A) = P(2 \le X < 3, 1 \le Y < 3)$.
c) Compute the probability $P(B) = P(Y > X)$.

Solution: a) Following the properties of the PDF of a r.v., the double integral on $\mathbb{R}^2$ equals 1 :

$$\int_0^5 \int_0^3 c dx dy = 1,$$

so $c = \dfrac{1}{15}$.
b) The probability of the event A is:

$$P(A) = \int_2^3 \int_1^3 \frac{1}{15} du dv = \frac{2}{15}.$$

c)

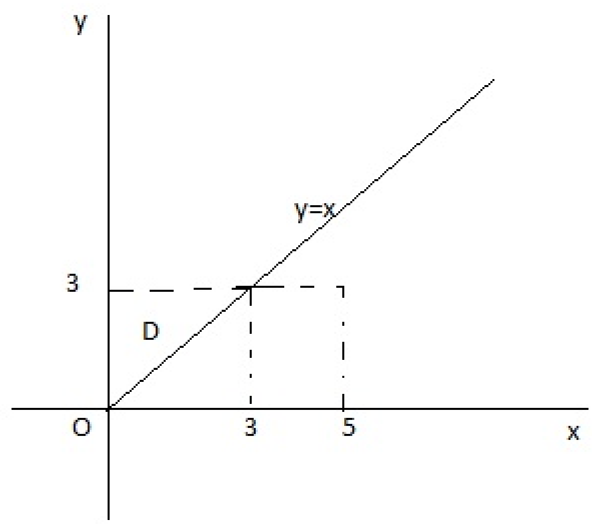

Figure 2

$$P(A) = \iint_A f_{X,Y}(x,y)dxdy = \int_0^3 dx \int_x^3 \frac{1}{15}dy$$
$$= \frac{1}{15}\int_0^3 (y \mid_x^3)dx = \frac{1}{15}\int_0^3 (3-x)dx = \frac{3}{10}.$$

3. R.v. X and Y have the joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} \frac{5y}{4}, & -1 \le x \le 1, -1 \le y \le 1 \\ 0, & otherwise \end{cases}$$

Find the marginal PDFs, $f_X(x)$ and $f_Y(y)$.

Solution: The marginal PDF of r.v. X is:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_{x^2}^{1} \frac{5y}{4}dy = \frac{5}{8}(1-x^4),$$

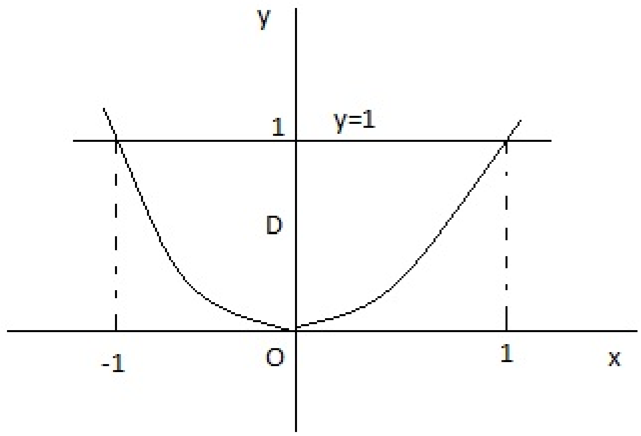

Figure 3

therefore,

$$f_X(x) = \begin{cases} \dfrac{5}{8}(1-x^4), & -1 \le x \le 1, \\ \\ 0, & otherwise \end{cases}$$

Using the same procedure, the marginal PDF of r.v. Y is:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{5y}{4} dx = \frac{5}{2} y\sqrt{y},$$

so

$$f_Y(y) = \begin{cases} \dfrac{5}{2} y\sqrt{y}, & 0 \le y \le 1, \\ \\ 0, & otherwise \end{cases}$$

4. Find the joint CDF $F_{X,Y}(x,y)$ when X and Y have the joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \le y \le x \le 1 \\ 0, & otherwise \end{cases}$$

Solution: As we know, the joint CDF of X and Y is

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)dudv.$$

- If $x, y < 0$, then

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} 0 \; dudv = 0.$$

- If $x, y > 1$, then

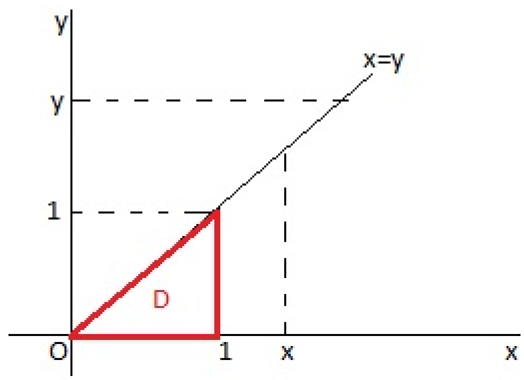

Figure 4

$$F_{X,Y}(x,y) = \int_{0}^{1} dx \int_{0}^{x} 2 \; dy = 1.$$

- If $x > 1, y \in [0,1]$, then

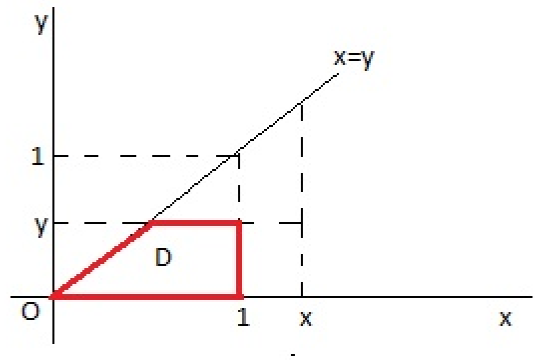

Figure 5

$$F_{X,Y}(x,y) = \int_0^y dv \int_y^1 2\ du = 2(1-y)y.$$

• If $y > 1, x \in [0,1]$, then

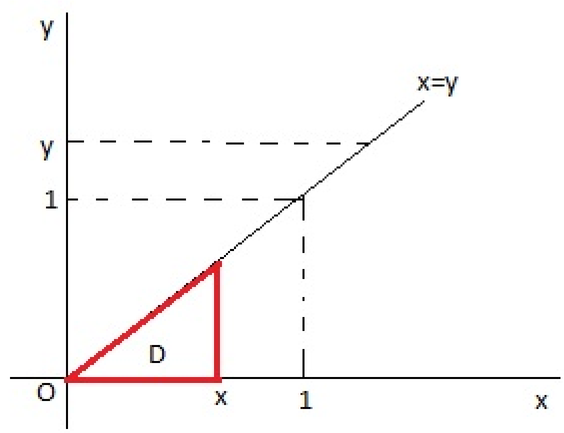

Figure 6

$$F_{X,Y}(x,y) = 2\int_0^x du \int_0^x 2\ dv = 2x^2.$$

• If $y \in [0,1], y < x$ then

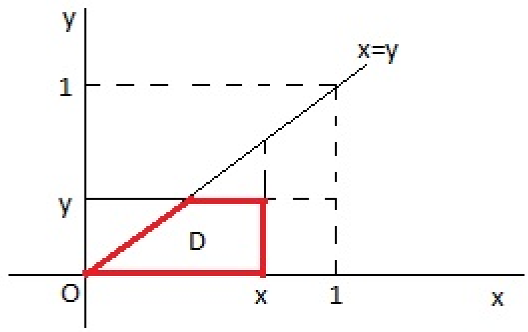

Figure 7

$$F_{X,Y}(x,y) = 2\int_0^y dv \int_y^x du = 2y(x-y),$$

so:

$$F_{X,Y}(x,y) = \begin{cases} 0, \ \ x, y < 0 \\ 2(1-y)y, \ \ x > 1, y \in [0,1] \\ 2x^2, \ \ y > 1, x \in [0,1] \\ 2y(x-y), \ \ y \in [0,1], y < x \\ 1, \ \ x, y > 1. \end{cases}$$