Practice Set 1: Probability Theory

Instruction: Practice all questions using the principles of probability, including combinatorics, conditional probability, and Bayes' Theorem

1. Twenty-five people, consisting of 15 women and 10 men are lined up in a random order. Find the probability that the ninth woman to appear is in position 17. That is, find the probability there are 8 women in positions 1 thru 16 and a woman in position 17.
2. An urn contains *M* white and *N* black balls. If a random sample of size *r* is chosen, what is the probability that it contains exactly *k* white balls?
3. A system is composed of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector $(x_1, x_2, x_3, x_4, x_5)$, where x_i is equal to 1 if component i is working and is equal to 0 if component i is failed.
   a) How many outcomes are in the sample space of this experiment?
   b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components $1, 3$, and 5 are all working. Let W be the event that the system will work. Specify all the outcomes in W.
   c) Let A be the event that components 4 and 5 are both failed. How many outcomes are contained in the event A?
   d) Write out all the outcomes in the event AW.
4. Let *E*, *F*, and *G* be three events. Find expressions for the events so that, of *E*, *F*, and *G*,
   a) Only *E* occurs;
   b) Both *E* and *G*, but not *F*, occur;
   c) At least one of the events occurs;
   d) At least two of the events occurs;
   e) All three events occurs;
   f) None of the events occurs;
   g) At most one of the events occurs;
   h) At most two of the events occur;
   i) Exactly two of the events occur;
   j) At most three of the event occur.
5. Prove that
$$P(E|F) = P(E|FG)P(G|F) + P(E|FG^c)P(G^c|F)$$
6. Prove that
$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E^cFG) - P(EF^cG) - P(EFG^c) - 2P(EFG)$$

7. A certain town with a population of $100,000$ has 3 newspapers: I, II, and III. The proportions of townspeople who read these papers are as follows:

   I: 10%; II: 30%; III: 5%

   I and II: 8%; I and III: 2%; II and III: 4%

   I and II and III: 1%

   a) Find the number of people who read only one newspaper.
   b) How many people read at least two newspapers?
   c) If I and III are morning papers and II is an evening paper, how many people read at least one morning paper plus an evening paper?
   d) How many people do not read any newspaper?
   e) How many people read only one morning paper and one evening paper?
8. The union of events A and B make up the whole sample space S. If $P(A) = 0.6$ and $P(B) = 0.8$, find the probability of
   a) The event of all possibilities that A shares with B.
   b) The event of those possibilities that are exclusively A's.
   c) The events whose possibilities are either those A's or those of B's that it does not share with A.
9. There are 5 hotels in a certain town. If 3 people check into hotels in a day, what is the probability that they each check into a different hotel? What assumptions are you making?
10. Rahul is 80% sure he forgot his textbook either at the library or in the Canteen. He is 40% sure that the book is at the library, and 40% sure that it is in the Canteen. Given that Rahul already went to the Canteen and noticed his textbook is not there, what is the probability that it is at the library?
11. A 3 person basketball team consists of a guard, a forward, and a center.
    a) If a person is chosen at random from each of three different such teams, what is the probability of selecting a complete team?
    b) What is the probability that all 3 players selected play the same position?
12. A woman has n keys, of which one will open her door.
    a) If she tries the keys at random, discarding those that do not work, what is the probability that she will open the door on her kth try?
    b) What if she does not discard previously tried keys?
13. Phan wants to take either a biology course or a Chemistry course. His adviser estimates that the probability of scoring an $\boldsymbol{A}$ in Biology is $4/5$, while the probability of scoring an $\boldsymbol{A}$ in chemistry is $1/7$. If Phan decides randomly, by a coin toss, which course to take, what is his probability of scoring $\boldsymbol{A}$ in Chemistry?

14. An insurance company classifies insured policyholders into *accident prone* or *non-accident prone*. Their current risk model works with the following probabilities.
    - The probability that an *accident-prone* insured has an accident within a year is 0.4
    - The probability that a *non-accident-prone* insured has an accident within a year is 0.2

    If 30% of the population is *accident prone*,

    a) What is the probability that a policy holder will have an accident within a year?
    b) Suppose now that the policy holder has had accident within one year. What is the probability that he or she is accident prone?
15. A bag contains a white and b black balls. Balls are chosen from the bag according to the following method:
    Step 1: A ball is chosen at random and is discarded.
    Step 2: A second ball is then chosen. If its color is different from that of the preceding ball, it is replaced in the bag and the process is repeated from the beginning. If its color is the same, it is discarded and we start from step 2
    In other words, balls are sampled and discarded until a change in color occurs, at which point the last ball is returned to the urn and the process starts anew. Let $P_{a,b}$ denote the probability that the last ball in the bag is white. Prove that
    $$P_{a,b} = \frac{1}{2}$$
16. Suppose the test for a Disease is 99% accurate in both directions and 0.3% of the population is the disease positive. If someone tests positive, what is the probability they actually are disease positive?
17. Suppose 36% of families own a dog, 30% of families own a cat, and 22% of the families that have a dog also have a cat. A family is chosen at random and found to have a cat. What is the probability they also own a dog?
18. A worker has asked her supervisor for a letter of recommendation for a new job. She estimates that there is an 80% chance that she will get the job if she receives a strong recommendation, a 40% chance if she receives a moderately good recommendation, and a 10% chance if she receives a weak recommendation. She further estimates that the probabilities that the recommendation will be strong, moderate, and weak are $0.7, 0.2$, and 0.1 respectively.
    a) How certain is she that she will receive the new job offer?
    b) Given that she does receive the offer, how likely should she feel that she received a strong recommendation? A moderate recommendation? A weak recommendation?
    c) Given that she does not receive the job offer, how likely should she feel that she received a strong recommendation? A moderate recommendation? A weak recommendation?

19. Prostate cancer is the most common type of cancer found in males. As an indicator of whether a male has prostate cancer, doctors often perform a test that measures the level of the prostate-specific antigen (PSA) that is produced only by the prostate gland. Although PSA levels are indicative of cancer, the test is notoriously unreliable. Indeed, the probability that a noncancerous man will have an elevated PSA level is approximately $.135$, increasing to approximately 0.268 if the man does have cancer. If, on the basis of other factors, a physician is 70% certain that a male has prostate cancer, what is the conditional probability that he has the cancer given that
    a) The test indicated an elevated PSA level?
    b) The test did not indicate an elevated PSA level?

    Repeat the preceding calculation, this time assuming that the physician initially believes that there is a 30% chance that the man has prostate cancer.

20. Ms. Aquina has just had a biopsy on a possibly cancerous tumor. Not wanting to spoil a weekend family event, she does not want to hear any bad news in the next few days. But if she tells the doctor to call only if the news is good, then if the doctor does not call, Ms. Aquina can conclude that the news is bad. So, being a student of probability, Ms. Aquina instructs the doctor to flip a coin. If it comes up heads, the doctor is to call if the news is good and not call if the news is bad. If the coin comes up tails, the doctor is not to call. In this way, even if the doctor doesn't call, the news is not necessarily bad. Let α be the probability that the tumor is cancerous; let β be the conditional probability that the tumor is cancerous given that the doctor does not call.
    a) Which should be larger α or β?
    b) Find β in terms of α, and prove your answer in part (a).