Notes on Various Symbolic and Formal Systems

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Table of Contents:

I. Introduction

[Logic *qua* Field of Inquiry]:

[1] Logic is the formal science of truth. (Frege)
[2] Logic is the formal science of logical consequence.

[Formal Logics]:

[1] A *logic* is a *language*, a *semantics* to interpret that language and a *proof system*.
[2] A formal *language* is an *alphabet* and a *grammar*.
[3] An *alphabet* is comprises a set of *logical symbols* and a set of *non-logical symbols*.
[4] A *grammar* is a set of syntactic formation rules.
[5] A *semantics* provides an interpretation of and the truth-conditions for expressions of the language.
[6] A *proof system* is a set of axioms and/or inference rules for making deductions within the language.

[Characteristic Features]:

[1] If a logic L is *classical* then:
  [A] L is truth-functional: Two-Valued.
  [B] The following axiom-schemata hold for every well-formed expression p, q in L:
    [i] *Tertium non datur*: $p \vee \neg p$
    [ii] Non-Contradiction: $\neg(p \wedge \neg p)$
    [iii] Double Negation: $\neg\neg p \leftrightarrow p$
    [iv] Contraposition: $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$
    [v] *Reductio Ad Absurdum*: $((\neg p \rightarrow (q \wedge \neg q)) \rightarrow p)$
    [vi] Monotonicity: $(p \rightarrow q) \rightarrow ((p \wedge r) \rightarrow q)$

[Conventions]:

[1] We shall assume the standard conventions for parenthetical dropping, precedence, quotation and uniform substitution.
[2] 'Logical operator' shall be used interchangeably with 'logical connective'.
[3] 'Scheme' shall be used interchangeably with 'schema'.
[4] 'Proof system' shall be used interchangeably with 'calculus'.
[5] 'Grammar' shall be used interchangeably with 'syntax'.
[6] 'Model Theory' shall be used interchangeably with 'semantics'.
[7] A variety of symbols will be deployed to denote meta-variables.
[8] Arity is the number of arguments that a function or predicate can take.

[Definitions - Axioms]:

[1] A *theorem* is a statement proved from the application of our inference rules and *axiom schemata* alone, that is to say without any additional *premises* (assumptions).

[2] An *axiom* is a wff that is regarded as self-evident without proof.

[3] An *axiom schema* represents infinitely many axioms. An *axiom* is obtained by uniformly substituting any wff into the variables of the schema.

[4] A *theory* is a set of wff.

[Definitions - Proof Systems]:

[1] An axiom system *S* is *sound just in case each* sentence *s* that is provable in system *S* is true.

- [A] An inference rule '$\vdash$' is *sound only if* $P \vdash Q$ *implies* $P \vDash Q$.
- [B] If axiom system *S* has only tautologies as axioms and has *modus ponens* as its only rule of inference then, axiom system *S* is *sound*.

[2] An axiom system *S* is *complete just in case* each sentence *s* that is true is provable in system *S*.

- [A] An inference rule '$\vdash$' is *complete only if* $P \vDash Q$ *implies* $P \vdash Q$.
- [B] By proving that a *complete* system *M* can be proven in *S*, one can show that *S* is also *complete*.

II. Łukasiewicz's Simple Sentential Logic

[Characteristics]:

[1] Zero-order.
[2] Classical.
[3] Complete.
[4] Consistent.
[5] Sentential.

[Logic L_1]:

[1] $L_1 = \{\mathcal{A}, \mathcal{Z}, \mathcal{I}, \Omega\}$

[Language L_1]:

[1] [A] $\mathcal{A}$ is a set of *propositional variables*.
[B] $\mathcal{A} = \{A_0, A_1, \ldots, B_0, B_1, \ldots, \ldots, Z_0, Z_1, \ldots\}$
[2] [A] Ω is the set of primitive *logical connectives* for L_1.
[B] $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$
[C] [i] Ω_0 is the set of logical connectives of *arity* 0.
[ii] $\Omega_0 = \{\bot, \top\}$
[D] [i] Ω_1 is the set of logical connectives of *arity* 1.
[ii] $\Omega_1 = \{\neg\}$
[E] [i] Ω_2 is the set of logical connectives of *arity* 2.
[ii] $\Omega_2 = \{\rightarrow\}$
[3] The set $\mathcal{A} \cup \Omega$ comprises the *alphabet* of L_1.
[4] The *well-formed formulae* (wff) of L_1 are recursively defined as follows:
[A] Any δ, where δ is a sentential variable of L_1, is a formula.
[B] If δ is a formula then, $\neg\delta$ is a formula.
[C] If δ and φ are formulas then, $\delta \rightarrow \varphi$ is a formula.
[D] $\top$ and $\bot$ are formulas.
[E] There are no other wff.
[5] [4] comprises the *grammar* of L_1.
[6] Let *wff*(L_1) denote the set of all wff in L_1.

[L_1 Logical Equivalences]:

[1] The following logical equivalences hold for L_1:
[A] $A \rightarrow \bot \equiv \neg A$
[B] $\top \rightarrow A \equiv A$
[C] $A \rightarrow B \equiv \neg(A \wedge \neg B)$
[D] $A \wedge B \equiv \neg(\neg A \vee \neg B) \equiv \neg(A \rightarrow \neg B)$
[E] $A \vee B \equiv \neg A \rightarrow B$

[F] $A \leftrightarrow B \equiv \neg((A \rightarrow B) \rightarrow \neg(B \rightarrow A)) \equiv (A \rightarrow B) \wedge (B \rightarrow A)$

[L_1 Proof System]:

[1] [A] Z is the set of *inference rules* valid in L_1.

[B] $Z = \{(\delta, \delta \rightarrow \varphi \vdash \varphi)\}$

[2] [A] I is the set of *axiom schemata* for L_1.

[B] $I = AS1 \cup AS2 \cup AS3$

[i] $AS1 = \{A \rightarrow (B \rightarrow A)\}$

[ii] $AS2 = \{(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))\}$

[iii] $AS3 = \{(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)\}$

[L_1 Model Theory]:

[1] A triple $\langle V, \Phi, \Phi^* \rangle$ is an $\boldsymbol{L}^{\mathrm{T}}$ *structure just in case*:

[A] [i] V is a theory.

[ii] $V = A(V) \cup B(V)$ such that:

[a] $A(V) \subseteq A$ and $A(V) \neq \varnothing$; and

[b] $A(V) \subseteq B(V)$; and

[c] $B(V) \subseteq wff(\boldsymbol{L}^{\mathrm{T}})$.

[B] [i] We call Φ a *propositional interpretation* function.

[ii] $\Phi : A(V) \rightarrow \{\top, \bot\}$ such that:

[a] $\Phi(p) = \mathrm{T}$ else $\Phi(p) = \bot$.

[C] [i] We call Φ^* a *sentential interpretation* function.

[ii] $\Phi^* : B(V) \rightarrow \{\top, \bot\}$ such that:

[a] For all $p \in A(V)$, $\Phi^*(p) = \Phi(p)$

[b] $\Phi^*(p) = \top$ *just in case* $\Phi^*(p) \neq \bot$

[c] $\Phi^*(\bot) = \bot$

[d] $\Phi^*(\top) = \top$

[e] $\Phi^*(\neg p) = \top$ *just in case* $\Phi^*(p) = \bot$

[f] $\Phi^*(p \rightarrow q) = \top$ *just in case* $\Phi^*(p) = \bot$ or $\Phi^*(q) = \top$

[g] $\Phi^*(p \,\&\, q) = \top$ *just in case* $\Phi^*(p) = \top = \Phi^*(q)$

[h] $\Phi^*(p \vee q) = \top$ *just in case* $\Phi^*(p) = \top$ or $\Phi^*(q) = \top$

[i] $\Phi^*(p \leftrightarrow q) = \top$ *just in case* $\Phi^*(p) = \Phi^*(q)$

[iii] *If* $\Phi^*(p) = \top$, *then* $\Phi^* \models p$.

[iv] For all $p \in V$, if $\Phi^* \models p$, then Φ^* is a *model* of V.

III. Zero-Order Modal Logic

[Characteristics]:

[1] Zero-order.
[2] Classical.
[3] Complete.
[4] Consistent.
[5] Propositional.
[6] Modal.

[Logic L_2]:

[1] $L_2 = \{\mathcal{A}, Z, I, \Omega\}$

[Language L_2]:

[1] [A] $\mathcal{A}$ is a finite set of *propositional variables*.
[B] $\mathcal{A} = \{A_0^0, A_1^0, \ldots, B_0^0, B_1^0, \ldots, \ldots, Z_0^0, Z_1^0, \ldots\}$
[2] [A] Ω is the set of primitive *logical connectives* for L_1.
[B] $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$
[C] [i] Ω_0 is the set of logical connectives of *arity* 0.
[ii] $\Omega_0 = \{\top, \bot\}$
[D] [i] Ω_1 is the set of logical connectives of *arity* 1.
[ii] $\Omega_1 = \{\neg, \Box\}$
[E] [i] Ω_2 is the set of logical connectives of *arity* 2.
[ii] $\Omega_2 = \{\rightarrow\}$
[3] The set $\mathcal{A} \cup \Omega$ comprises the *alphabet* of L_2.
[4] The *well-formed formulae* (wff) of L_2 are recursively defined as follows:
[A] Any δ, where δ is a sentential variable of L_2, is a formula.
[B] If δ is a formula then, $\neg\delta$ is a formula.
[C] If δ and φ are formulas then, $\delta \rightarrow \varphi$ is a formula.
[D] $\top$ and $\bot$ are formulas.
[E] If δ is a formula then, $\Box\delta$ is a formula.
[F] There are no other wff.
[5] [4] comprises the *grammar* of L_2.
[6] Let *wff*(L_2) denote the set of all wff in L_2.

[L_2 Logical Equivalences]:

[1] The following logical equivalences hold for L_2:
[A] $A \rightarrow \bot \equiv \neg A$
[B] $\top \rightarrow A \equiv A$
[C] $A \rightarrow B \equiv \neg(A \wedge \neg B)$

[D] $A \wedge B \equiv \neg(\neg A \vee \neg B) \equiv \neg(A \rightarrow \neg B)$
[E] $A \vee B \equiv \neg A \rightarrow B$
[F] $A \leftrightarrow B \equiv \neg((A \rightarrow B) \rightarrow \neg(B \rightarrow A)) \equiv (A \rightarrow B) \wedge (B \rightarrow A)$
[G] $\Diamond A \equiv \neg\Box\neg A$

[L_2 Proof System]:

[1] [A] *Z* is the set of *inference rules* valid in L_2.
[B] $Z = \mathbf{MP} \cup \mathbf{NR}$
[i] $\mathbf{MP} = \{(\delta, \delta \rightarrow \varphi \vdash \varphi)\}$
[ii] $\mathbf{NR} = \{\delta \vdash \Box\delta\}$
[2] [A] *I* is the set of *axiom schemata* for L_2.
[B] $I = AS1 \cup AS2 \cup AS3 \cup \mathbf{K}$
[i] $AS1 = \{A \rightarrow (B \rightarrow A)\}$
[ii] $AS2 = \{(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))\}$
[iii] $AS3 = \{(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)\}$
[iv] $\mathbf{K} = \{\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)\}$
[3] Proof system Z is called modal axiom *System K*.
[4] The following axiom schemata are regularly added to *System K*:
[A] $\mathbf{D} = \{(\Box A) \rightarrow (\Diamond A)\}$
[B] $\mathbf{T} = \{(\Box A) \rightarrow A\}$
[C] $\mathbf{B} = \{A \rightarrow (\Box\Diamond A)\}$
[D] $\mathbf{S4} = \{(\Box A) \rightarrow (\Box\Box A)\}$
[E] $\mathbf{S5} = \{(\Diamond A) \rightarrow (\Box\Diamond A)\}$
[5] The following modal axiom systems are obtained by adding the corresponding axiom rules to *System K*:
[A] System $T =_{df}$ System K + **T**
[B] System $S4 =_{df}$ System T **+ S4**
[C] System $S5 =_{df}$ System $S4$ + **B** (alternatively: **T** + **S5**)
[D] System $D =_{df}$ System K + **D**

[L_2 Model Theory]:

[1] A set $\langle W, R, V\rangle$ is a *Kripke Model* for L_2 *just in case*:
[A] [i] $W \neq \varnothing$
[ii] $R \subseteq W \times W$
[iii] $V : A \times W \rightarrow \{\bot, \top\}$.
[B] [i] Each $w \in W$ is called a *possible world*.
[ii] For each $p \in A$: $p \in wff(L_2)$.
[2] Truth of a modal formula *p* at a *possible world w* in a relational structure $M = \langle W, R, V\rangle$ is denoted '$M,w \vDash p$' and is inductively defined as follows:
[A] $M,w \vDash p$ *just in case* $V(p, w) = \top$
[B] $M,w \vDash \top$ and $M,w \nvDash \bot$
[C] $M,w \vDash \neg p$ *just in case* $M,w \nvDash p$

[D] $M,w \vDash p \,\&\, q$ *just in case* $M,w \vDash p \,\&\, M,w \vDash q$

[E] $M,w \vDash \Box p$ *just in case* $(\forall\ v \in W)(wRv \rightarrow M,v \vDash p)$

[F] $M,w \vDash \Diamond p$ *just in case* $(\exists\ v \in W)(wRv \,\&\, M,v \vDash p)$

IV. Simple Supervaluation Theory

[Characteristics]:

[1] Fragment of First-Order Logic.
[2] No quantification.
[3] Complete.
[4] Consistent.

[Logic L_3]:

[1] $L_3 = \{A, Z, I, \Omega\}$

[Language L_3]:

[1] [A] A is the set of non-logical symbols.
[B] $A = A_1 \cup A_2$
[C] A_1 is the set of *individual constants* such that $A_1 = \{a, b, c, ...\}$.
[D] A_2 is a singleton set of a particular *vague unary predicate* such that $A_2 = \{P\}$.
[2] [A] Ω is the set of *logical operators* (*logical connectives*) for L_3.
[B] $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$
[C] [i] Ω_0 is the set of logical connectives of *arity* 0.
[ii] $\Omega_0 = \{\bot, \top\}$
[D] [i] Ω_1 is the set of logical connectives of *arity* 1.
[ii] $\Omega_1 = \{\neg, D\}$
[E] [i] Ω_2 is the set of logical connectives of *arity* 2.
[ii] $\Omega_2 = \{\rightarrow\}$
[3] The set $A \cup \Omega$ comprises the *alphabet* of L_3.
[4] The *well-formed formulae* (wff) of L_3 are recursively defined as follows:
[A] For any individual constant a: $P(a)$ is a formula of L_3.
[B] If ϕ is a wff of L_3 then, so is $\neg\phi$.
[C] If ϕ and φ are wff of L_3 then, $\phi \rightarrow \varphi$ is formula.
[D] $\top$ and $\bot$ are formulas.
[E] If ϕ is a formula of L_3 then, so is $D\phi$.
[F] Nothing else is a formula in L_3.
[5] [4] comprises the *grammar* of L_3.
[6] Let *wff*(L_3) denote the set of all wff in L_3.

[L_3 Logical Equivalences]:

[1] The following logical equivalences hold for L_2:

[A] $A \rightarrow \bot \equiv \neg A$

[B] $\top \rightarrow A \equiv A$

[C] $A \rightarrow B \equiv \neg(A \wedge \neg B)$
[D] $A \wedge B \equiv \neg(\neg A \vee \neg B) \equiv \neg(A \rightarrow \neg B)$
[E] $A \vee B \equiv \neg A \rightarrow B$
[F] $A \leftrightarrow B \equiv \neg((A \rightarrow B) \rightarrow \neg(B \rightarrow A)) \equiv (A \rightarrow B) \wedge (B \rightarrow A)$

[L_3 Proof System]:

[1] [A] Z is the set of *inference rules* valid in L_3.
[B] $Z = \{(\delta, \delta \rightarrow \varphi \vdash \varphi)\}$
[2] [A] I is the set of *axiom schemata* for L_3.
[B] $I = AS1 \cup AS2 \cup AS3$
[i] $AS1 = \{A \rightarrow (B \rightarrow A)\}$
[ii] $AS2 = \{(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))\}$
[iii] $AS3 = \{(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)\}$

[L_3 Model Theory]:

[1] A 4-tuple $\langle D, P, \ulcorner\urcorner_+, \ulcorner\urcorner_-\rangle$ is a *partial model* for L_3 *just in case*:
[A] [i] D is a non-empty domain of objects.
[ii] We write '$|M|$' to denote the domain of *partial model* M.
[B] P is an vague unary predicate.
[C] [i] $\ulcorner\urcorner_+$ is an *extension* function mapping P into a subset of D.
[ii] $\ulcorner\urcorner_-$ is an *anti-extension* function mapping P into a subset of D.
[iii] $\ulcorner P\urcorner_+ \cap \ulcorner P\urcorner_- = \varnothing$.
[2] Partial model M_2 extends partial model M_1 *if*:
[A] $|M_1| = |M_2|$.
[B] $P \in A_2^{M1}$ and $P \in A_2^{M2}$.
[C] $\ulcorner P\urcorner_+^{M1} \subseteq \ulcorner P\urcorner_+^{M2}$.
[D] $\ulcorner P\urcorner_-^{M1} \subseteq \ulcorner P\urcorner_-^{M2}$.
[3] Given an assignment function g, a partial model M then supports a notion of truth in a model ($\models$) and falsity in a model ($\dashv$) with base clauses:
[A] $M, g \models P(x)$ *just in case* $g(x) \in \ulcorner P\urcorner_+$.
[B] $M, g \dashv P(x)$ *just in case* $g(x) \in \ulcorner P\urcorner_-$.
[C] $M, g \models \neg\sigma$ *just in case* $M, g \dashv \sigma$.
[D] $M, g \dashv \neg\sigma$ *just in case* $M, g \models \sigma$.
[E] $M, g \models (\sigma \vee \rho)$ *just in case* $M, g \models \sigma$ or $M, g \models \rho$.
[F] $M, g \models (\sigma \wedge \rho)$ *just in case* $M, g \models \sigma$ and $M, g \models \rho$.
[G] $M, g \models D\varphi$ *just in case* for each partial model R, given an assignment h, *extending* from M: $R, h \models \varphi$.
[4] A partial model M is *complete if* $\ulcorner P\urcorner_+ \cup \ulcorner P\urcorner_- = |M|$.
[5] [A] A *specification space* is an arbitrary collection of partial models.
[B] A *rooted* specification space is a specification space with one model identified as the *root* partial model.

[C] A *complete specification space* S satisfies the following condition: for every partial model M in S there is some complete partial model R in S that extends M.

[6] [A] [i] A wff $\phi \in wff(L_3)$ is *supertrue* in a complete specification space S *if* ϕ is true at each complete extension of a root partial model.

[ii] A wff p is *supertrue just in case* p is evaluated as true at the root partial model.

[iii] A wff p is *supertrue just in case* p is evaluated as true at each complete partial model.

[iv] A wff p is *supertrue just in case* for every specification point M, $M \models p$.

[B] A sentence $\phi \in wff(L_3)$ is *superfalse* in a complete specification space S *if* ϕ is false at each complete extension of a root partial model.

[Validity]:

[1] [A] We shall write '$A \models_L B$' for *local validity* (A is a set of premises and B a conclusion).

[B] '$A \models_L B$' *reads* left-to-right 'A locally entails B' and right-to-left 'B is a local consequence of A'.

[C] [i] $A \models_L B$ *just in case* at every specification point, if A is true so is B.

[ii] $A \models_L B$ *just in case* necessarily, if A is true so is B.

[iii] $A \models_L B$ *just in case* for every specification point M: $M \models A \rightarrow M \models B$.[1]

[2] [A] We shall write '$A \models_G B$' for *global validity* (A is a set of premises and B a conclusion).

[B] '$A \models_G B$' reads left-to-right 'A globally entails B' and right-to-left 'B is a global consequence of A.'

[C] [i] $A \models_G B$ *just in case* the supertruth of A guarantees the supertruth of B.

[ii] $A \models_G B$ *just in case* A*'s* supertruth necessitates B*'s* supertruth.

[iii] $A \models_G B$ *just in case* for every specification point M, $M \models A$ then for every specification point N, $N \models B$.

[D] This is also referred to as *supervalidity*.

[3] $(A \models_L B) \Rightarrow (A \models_G B)$

[Failure of Deduction Theorem]:

[1] The Deduction Theorem: $(A \cup p \models_G q) \Rightarrow (A \models_G p \rightarrow q)$.[2]

[2] The Deduction Theorem fails if '$p \models_G Dp$' succeeds and '$\varnothing \models_G p \rightarrow Dp$' fails.

[3] Dp is true *just in case* p is evaluated as true at each specification point.

[4] Thus, whenever p is supertrue so is Dp. So '$p \models_G Dp$' always succeeds.

[5] Imagine a specification space where p is indeterminate. It follows that Dp is evaluated as false at each specification point.

[6] Thus, there is a specification point where p is evaluated as true, but Dp is evaluated as false.

[7] Hence, there is a specification point where '$p \rightarrow Dp$' is false.

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[1] The unsubscripted turnstile '$M \models A$' *reads* 'M satisfies A' or equivalently 'A is true under M' or still, 'A is evaluated or interpreted as true in M.'

[2] Substitute '$\varnothing$' for 'A' and 'Dp' for 'q'.

[8] Hence, '$p \rightarrow Dp$' is not supertrue.

[9] Hence, '$\varnothing \vDash_{G} p \rightarrow Dp$' fails.