

Notes on Various Symbolic and Formal Systems

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I. Introduction

[Logic *qua* Field of Inquiry]:

- [1] Logic is the formal science of truth. (Frege)
- [2] Logic is the formal science of logical consequence.

[Formal Logics]:

- [1] A *logic* is a *language*, a *semantics* to interpret that language and a *proof system*.
- [2] A formal *language* is an *alphabet* and a *grammar*.
- [3] An *alphabet* is comprises a set of *logical symbols* and a set of *non-logical symbols*.
- [4] A *grammar* is a set of syntactic formation rules.
- [5] A *semantics* provides an interpretation of and the truth-conditions for expressions of the language.
- [6] A *proof system* is a set of axioms and/or inference rules for making deductions within the language.

[Characteristic Features]:

- [1] If a logic L is *classical* then:
 - [A] L is truth-functional: Two-Valued.
 - [B] The following axiom-schemata hold for every well-formed expression p, q in L :
 - [i] *Tertium non datur*: $p \vee \neg p$
 - [ii] Non-Contradiction: $\neg(p \wedge \neg p)$
 - [iii] Double Negation: $\neg\neg p \leftrightarrow p$
 - [iv] Contraposition: $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$
 - [v] *Reductio Ad Absurdum*: $((\neg p \rightarrow (q \wedge \neg q)) \rightarrow p)$
 - [vi] Monotonicity: $(p \rightarrow q) \rightarrow ((p \wedge r) \rightarrow q)$

[Conventions]:

- [1] We shall assume the standard conventions for parenthetical dropping, precedence, quotation and uniform substitution.
- [2] 'Logical operator' shall be used interchangeably with 'logical connective'.
- [3] 'Scheme' shall be used interchangeably with 'schema'.
- [4] 'Proof system' shall be used interchangeably with 'calculus'.
- [5] 'Grammar' shall be used interchangeably with 'syntax'.
- [6] 'Model Theory' shall be used interchangeably with 'semantics'.
- [7] A variety of symbols will be deployed to denote meta-variables.
- [8] Arity is the number of arguments that a function or predicate can take.

[Definitions - Axioms]:

- [1] A *theorem* is a statement proved from the application of our inference rules and *axiom schemata* alone, that is to say without any additional *premises* (assumptions).

- [2] An *axiom* is a wff that is regarded as self-evident without proof.
- [3] An *axiom schema* represents infinitely many axioms. An *axiom* is obtained by uniformly substituting any wff into the variables of the schema.
- [4] A *theory* is a set of wff.

[Definitions - Proof Systems]:

- [1] An axiom system S is *sound just in case* each sentence s that is provable in system S is true.
 - [A] An inference rule ' \vdash ' is *sound only if* $P \vdash Q$ *implies* $P \models Q$.
 - [B] If axiom system S has only tautologies as axioms and has *modus ponens* as its only rule of inference then, axiom system S is *sound*.
- [2] An axiom system S is *complete just in case* each sentence s that is true is provable in system S .
 - [A] An inference rule ' \vdash ' is *complete only if* $P \models Q$ *implies* $P \vdash Q$.
 - [B] By proving that a *complete* system M can be proven in S , one can show that S is also *complete*.

II. Łukasiewicz's Simple Sentential Logic

[Characteristics]:

- [1] Zero-order.
- [2] Classical.
- [3] Complete.
- [4] Consistent.
- [5] Sentential.

[Logic L_1]:

- [1] $L_1 = \{A, Z, I, \Omega\}$

[Language L_1]:

- [1] [A] A is a set of *propositional variables*.
[B] $A = \{A_0, A_1, \dots, B_0, B_1, \dots, Z_0, Z_1, \dots\}$
- [2] [A] Ω is the set of primitive *logical connectives* for L_1 .
[B] $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$
[C] [i] Ω_0 is the set of logical connectives of *arity* 0.
[ii] $\Omega_0 = \{\perp, \top\}$
[D] [i] Ω_1 is the set of logical connectives of *arity* 1.
[ii] $\Omega_1 = \{\neg\}$
[E] [i] Ω_2 is the set of logical connectives of *arity* 2.
[ii] $\Omega_2 = \{\rightarrow\}$
- [3] The set $A \cup \Omega$ comprises the *alphabet* of L_1 .
- [4] The *well-formed formulae* (wff) of L_1 are recursively defined as follows:
[A] Any δ , where δ is a sentential variable of L_1 , is a formula.
[B] If δ is a formula then, $\neg\delta$ is a formula.
[C] If δ and ϕ are formulas then, $\delta \rightarrow \phi$ is a formula.
[D] \top and \perp are formulas.
[E] There are no other wff.
- [5] [4] comprises the *grammar* of L_1 .
- [6] Let $wff(L_1)$ denote the set of all wff in L_1 .

[L_1 Logical Equivalences]:

- [1] The following logical equivalences hold for L_1 :
[A] $A \rightarrow \perp \equiv \neg A$
[B] $\top \rightarrow A \equiv A$
[C] $A \rightarrow B \equiv \neg(A \wedge \neg B)$
[D] $A \wedge B \equiv \neg(\neg A \vee \neg B) \equiv \neg(A \rightarrow \neg B)$
[E] $A \vee B \equiv \neg A \rightarrow B$

$$[F] \quad A \leftrightarrow B \equiv \neg((A \rightarrow B) \rightarrow \neg(B \rightarrow A)) \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

[L₁ Proof System]:

- [1] [A] Z is the set of *inference rules* valid in L_1 .
- [B] $Z = \{(\delta, \delta \rightarrow \varphi \vdash \varphi)\}$
- [2] [A] I is the set of *axiom schemata* for L_1 .
- [B] $I = AS1 \cup AS2 \cup AS3$
 - [i] $AS1 = \{A \rightarrow (B \rightarrow A)\}$
 - [ii] $AS2 = \{(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))\}$
 - [iii] $AS3 = \{(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)\}$

[L₁ Model Theory]:

- [1] A triple $\langle V, \Phi, \Phi^* \rangle$ is an L^T structure just in case:
 - [A] [i] V is a theory.
 - [ii] $V = A(V) \cup B(V)$ such that:
 - [a] $A(V) \subseteq A$ and $A(V) \neq \emptyset$; and
 - [b] $A(V) \subseteq B(V)$; and
 - [c] $B(V) \subseteq \text{dff}(L^T)$.
 - [B] [i] We call Φ a *propositional interpretation* function (for the non-concatenated dff) of L^T .
 - [ii] $\Phi : A(V) \rightarrow \{\top, \perp\}$ such that:
 - [a] $\Phi(p) = \top$ else $\Phi(p) = \perp$.
 - [C] [i] We call Φ^* a *sentential interpretation* function (for the concatenated dff) of L^T – the procedure for constructing that Φ^* is explained below.
 - [ii] $\Phi^* : B(V) \rightarrow \{\top, \perp\}$ such that:
 - [a] For all $p \in A(V)$, $\Phi^*(p) = \Phi(p)$
 - [b] $\Phi^*(p) = \top$ just in case $\Phi^*(p) \neq \perp$
 - [c] $\Phi^*(\perp) = \perp$
 - [d] $\Phi^*(\top) = \top$
 - [e] $\Phi^*(\neg p) = \top$ just in case $\Phi^*(p) = \perp$
 - [f] $\Phi^*(p \rightarrow q) = \top$ just in case $\Phi^*(p) = \perp$ or $\Phi^*(q) = \top$
 - [g] $\Phi^*(p \& q) = \top$ just in case $\Phi^*(p) = \top = \Phi^*(q)$
 - [h] $\Phi^*(p \vee q) = \top$ just in case $\Phi^*(p) = \top$ or $\Phi^*(q) = \top$
 - [i] $\Phi^*(p \leftrightarrow q) = \top$ just in case $\Phi^*(p) = \Phi^*(q)$
 - [iii] If $\Phi^*(p) = \top$, then $\Phi^* \models p$.
 - [iv] For all $p \in V$, if $\Phi^* \models p$, then Φ^* is a *model* of V .

III. Zero-Order Modal Logic

[Characteristics]:

- [1] Zero-order.
- [2] Classical.
- [3] Complete.
- [4] Consistent.
- [5] Propositional.
- [6] Modal.

[Logic L_2]:

- [1] $L_2 = \{A, Z, I, \Omega\}$

[Language L_2]:

- [1] [A] A is a finite set of *propositional variables*.
[B] $A = \{A_0^0, A_1^0, \dots, B_0^0, B_1^0, \dots, Z_0^0, Z_1^0, \dots\}$
- [2] [A] Ω is the set of primitive *logical connectives* for L_1 .
[B] $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$
[C] [i] Ω_0 is the set of logical connectives of *arity* 0.
[ii] $\Omega_0 = \{\top, \perp\}$
[D] [i] Ω_1 is the set of logical connectives of *arity* 1.
[ii] $\Omega_1 = \{\neg, \Box\}$
[E] [i] Ω_2 is the set of logical connectives of *arity* 2.
[ii] $\Omega_2 = \{\rightarrow\}$
- [3] The set $A \cup \Omega$ comprises the *alphabet* of L_2 .
- [4] The *well-formed formulae* (wff) of L_2 are recursively defined as follows:
[A] Any δ , where δ is a sentential variable of L_2 , is a formula.
[B] If δ is a formula then, $\neg\delta$ is a formula.
[C] If δ and ϕ are formulas then, $\delta \rightarrow \phi$ is a formula.
[D] \top and \perp are formulas.
[E] If δ is a formula then, $\Box\delta$ is a formula.
[F] There are no other wff.
- [5] [4] comprises the *grammar* of L_2 .
- [6] Let $wff(L_2)$ denote the set of all wff in L_2 .

[L_2 Logical Equivalences]:

- [1] The following logical equivalences hold for L_2 :
[A] $A \rightarrow \perp \equiv \neg A$

- [B] $\top \rightarrow A \equiv A$
- [C] $A \rightarrow B \equiv \neg(A \wedge \neg B)$
- [D] $A \wedge B \equiv \neg(\neg A \vee \neg B) \equiv \neg(A \rightarrow \neg B)$
- [E] $A \vee B \equiv \neg A \rightarrow B$
- [F] $A \leftrightarrow B \equiv \neg((A \rightarrow B) \rightarrow \neg(B \rightarrow A)) \equiv (A \rightarrow B) \wedge (B \rightarrow A)$
- [G] $\Diamond A \equiv \neg \Box \neg A$

[L₂ Proof System]:

- [1] [A] Z is the set of *inference rules* valid in L_2 .
 [B] $Z = \mathbf{MP} \cup \mathbf{NR}$
 - [i] $\mathbf{MP} = \{(\delta, \delta \rightarrow \varphi \vdash \varphi)\}$
 - [ii] $\mathbf{NR} = \{\delta \vdash \Box \delta\}$
- [2] [A] I is the set of *axiom schemata* for L_2 .
 [B] $I = \mathbf{AS1} \cup \mathbf{AS2} \cup \mathbf{AS3} \cup \mathbf{K}$
 - [i] $\mathbf{AS1} = \{A \rightarrow (B \rightarrow A)\}$
 - [ii] $\mathbf{AS2} = \{(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))\}$
 - [iii] $\mathbf{AS3} = \{(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)\}$
 - [iv] $\mathbf{K} = \{\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)\}$
- [3] Proof system Z is called modal axiom *System K*.
- [4] The following axiom schemata are regularly added to *System K*:
 - [A] $\mathbf{D} = \{(\Box A) \rightarrow (\Diamond A)\}$
 - [B] $\mathbf{T} = \{(\Box A) \rightarrow A\}$
 - [C] $\mathbf{B} = \{A \rightarrow (\Box \Diamond A)\}$
 - [D] $\mathbf{S4} = \{(\Box A) \rightarrow (\Box \Box A)\}$
 - [E] $\mathbf{S5} = \{(\Diamond A) \rightarrow (\Box \Diamond A)\}$
- [5] The following modal axiom systems are obtained by adding the corresponding axiom rules to *System K*:
 - [A] System $T =_{\text{df}} \text{System } K + \mathbf{T}$
 - [B] System $S4 =_{\text{df}} \text{System } T + \mathbf{S4}$
 - [C] System $S5 =_{\text{df}} \text{System } S4 + \mathbf{B}$ (alternatively: $\mathbf{T} + \mathbf{S5}$)
 - [D] System $D =_{\text{df}} \text{System } K + \mathbf{D}$

[L₂ Model Theory]:

- [1] A set $\langle W, R, V \rangle$ is a *Kripke Model* for L_2 just in case:
 - [A] [i] $W \neq \emptyset$
 - [ii] $R \subseteq W \times W$
 - [iii] $V: A \times W \rightarrow \{\perp, \top\}$.
 - [B] [i] Each $w \in W$ is called a *possible world*.
 - [ii] For each $p \in A$: $p \in \text{mff}(L_2)$.
- [2] Truth of a modal formula p at a *possible world* w in a relational structure $M = \langle W, R, V \rangle$ is denoted ' $M, w \models p$ ' and is inductively defined as follows:
 - [A] $M, w \models p$ just in case $V(p, w) = \top$

- [B] $M, w \models \top$ and $M, w \not\models \perp$
- [C] $M, w \models \neg p$ just in case $M, w \not\models p$
- [D] $M, w \models p \ \& \ q$ just in case $M, w \models p \ \& \ M, w \models q$
- [E] $M, w \models \Box p$ just in case $(\forall v \in W)(wRv \rightarrow M, v \models p)$
- [F] $M, w \models \Diamond p$ just in case $(\exists v \in W)(wRv \ \& \ M, v \models p)$

IV. Simple Supervaluation Theory

[Characteristics]:

- [1] Fragment of First-Order Logic.
- [2] No quantification.
- [3] Complete.
- [4] Consistent.

[Logic L_3]:

- [1] $L_3 = \{A, Z, I, \Omega\}$

[Language L_3]:

- [1] [A] A is the set of non-logical symbols.
 [B] $A = A_1 \cup A_2$
 [C] A_1 is the set of *individual constants* such that $A_1 = \{a, b, c, \dots\}$.
 [D] A_2 is a singleton set of a *particular, vague, unary predicate* such that $A_2 = \{P\}$.
- [2] [A] Ω is the set of *logical operators (logical connectives)* for L_3 .
 [B] $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$
 [C] [i] Ω_0 is the set of logical connectives of *arity* 0.
 [ii] $\Omega_0 = \{\perp, \top\}$
 [D] [i] Ω_1 is the set of logical connectives of *arity* 1.
 [ii] $\Omega_1 = \{\neg, D\}$
 [E] [i] Ω_2 is the set of logical connectives of *arity* 2.
 [ii] $\Omega_2 = \{\rightarrow\}$
- [3] The set $A \cup \Omega$ comprises the *alphabet* of L_3 .
- [4] The *well-formed formulae* (wff) of L_3 are recursively defined as follows:
 [A] For any individual constant a : $P(a)$ is a formula of L_3 .
 [B] If ϕ is a wff of L_3 then, so is $\neg\phi$.
 [C] If ϕ and φ are wff of L_3 then, $\phi \rightarrow \varphi$ is formula.
 [D] \top and \perp are formulas.
 [E] If ϕ is a formula of L_3 then, so is $D\phi$.
 [F] Nothing else is a formula in L_3 .
- [5] [4] comprises the *grammar* of L_3 .
- [6] Let $wff(L_3)$ denote the set of all wff in L_3 .

[L_3 Logical Equivalences]:

- [1] The following logical equivalences hold for L_2 :

$$[A] \quad A \rightarrow \perp \equiv \neg A$$

- [B] $\top \rightarrow A \equiv A$
- [C] $A \rightarrow B \equiv \neg(A \wedge \neg B)$
- [D] $A \wedge B \equiv \neg(\neg A \vee \neg B) \equiv \neg(A \rightarrow \neg B)$
- [E] $A \vee B \equiv \neg A \rightarrow B$
- [F] $A \leftrightarrow B \equiv \neg((A \rightarrow B) \rightarrow \neg(B \rightarrow A)) \equiv (A \rightarrow B) \wedge (B \rightarrow A)$

[L₃ Proof System]:

- [1] [A] Z is the set of *inference rules* valid in L_3 .
 - [B] $Z = \{(\delta, \delta \rightarrow \varphi \vdash \varphi)\}$
- [2] [A] I is the set of *axiom schemata* for L_3 .
 - [B] $I = \text{AS1} \cup \text{AS2} \cup \text{AS3}$
 - [i] $\text{AS1} = \{A \rightarrow (B \rightarrow A)\}$
 - [ii] $\text{AS2} = \{(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))\}$
 - [iii] $\text{AS3} = \{(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)\}$

[L₃ Model Theory]:

- [1] A 4-tuple $\langle D, P, \mathbb{P}_+, \mathbb{P}_- \rangle$ is a *partial model* for L_3 just in case:
 - [A] [i] D is a non-empty domain of objects.
 - [ii] We write ' $|M|$ ' to denote the domain of *partial model* M .
 - [B] P is an vague unary predicate.
 - [C] [i] \mathbb{P}_+ is an *extension* function mapping P into a subset of D .
 - [ii] \mathbb{P}_- is an *anti-extension* function mapping P into a subset of D .
 - [iii] $\mathbb{P}_+ \cap \mathbb{P}_- = \emptyset$.
- [2] Partial model M_2 extends partial model M_1 if:
 - [A] $|M_1| = |M_2|$.
 - [B] $P \in \mathcal{A}_2^{M_1}$ and $P \in \mathcal{A}_2^{M_2}$.
 - [C] $\mathbb{P}_+^{M_1} \subseteq \mathbb{P}_+^{M_2}$.
 - [D] $\mathbb{P}_-^{M_1} \subseteq \mathbb{P}_-^{M_2}$.
- [3] Given an assignment function g , a partial model M then supports a notion of truth in a model (\models) and falsity in a model ($\not\models$) with base clauses:
 - [A] $M, g \models P(x)$ just in case $g(x) \in \mathbb{P}_+$.
 - [B] $M, g \not\models P(x)$ just in case $g(x) \in \mathbb{P}_-$.
 - [C] $M, g \models \neg\sigma$ just in case $M, g \not\models \sigma$.
 - [D] $M, g \not\models \neg\sigma$ just in case $M, g \models \sigma$.
 - [E] $M, g \models (\sigma \vee \rho)$ just in case $M, g \models \sigma$ or $M, g \models \rho$.
 - [F] $M, g \models (\sigma \wedge \rho)$ just in case $M, g \models \sigma$ and $M, g \models \rho$.
 - [G] $M, g \models D\varphi$ just in case for each partial model R , given an assignment h , extending from M : $R, h \models \varphi$.
- [4] A partial model M is *complete* if $\mathbb{P}_+ \cup \mathbb{P}_- = |M|$.
- [5] [A] A *specification space* is an arbitrary collection of partial models.
 - [B] A *rooted* specification space is a specification space with one model identified as the *root* partial model.

- [C] A complete specification space S satisfies the following condition: for every partial model M in S there is some complete partial model R in S that extends M .
- [6] [A] [i] A wff $\phi \in \mathcal{WFF}(L_{\Delta})$ is *supertrue* in a complete specification space S if ϕ is true at each complete extension of a root partial model.
- [ii] A wff p is *supertrue just in case* p is evaluated as true at the root partial model.
- [iii] A wff p is *supertrue just in case* p is evaluated as true at each complete partial model.
- [iv] A wff p is *supertrue just in case* for every specification point M , $M \models p$.
- [B] A sentence $\phi \in \mathcal{WFF}(L_{\Delta})$ is *superfalse* in a complete specification space S if ϕ is false at each complete extension of a root partial model.

[Validity]:

- [1] [A] We shall write ' $A \models_L B$ ' for *local validity* (A is a set of premises and B a conclusion).
- [B] ' $A \models_L B$ ' reads left-to-right ' A locally entails B ' and right-to-left ' B is a local consequence of A '.
- [C] [i] $A \models_L B$ *just in case* at every specification point, if A is true so is B .
- [ii] $A \models_L B$ *just in case* necessarily, if A is true so is B .
- [iii] $A \models_L B$ *just in case* for every specification point M : $M \models A \rightarrow M \models B$.¹
- [2] [A] We shall write ' $A \models_G B$ ' for *global validity* (A is a set of premises and B a conclusion).
- [B] ' $A \models_G B$ ' reads left-to-right ' A globally entails B ' and right-to-left ' B is a global consequence of A '.
- [C] [i] $A \models_G B$ *just in case* the supertruth of A guarantees the supertruth of B .
- [ii] $A \models_G B$ *just in case* A 's supertruth necessitates B 's supertruth.
- [iii] $A \models_G B$ *just in case* for every specification point M , $M \models A$ then for every specification point N , $N \models B$.
- [D] This is also referred to as *supervalidity*.
- [3] $(A \models_L B) \Rightarrow (A \models_G B)$

[Failure of Deduction Theorem]:

- [1] The Deduction Theorem: $(A \cup p \models_G q) \Rightarrow (A \models_G p \rightarrow q)$.²
- [2] The Deduction Theorem fails if ' $p \models_G Dp$ ' succeeds and ' $\emptyset \models_G p \rightarrow Dp$ ' fails.
- [3] Dp is true *just in case* p is evaluated as true at each specification point.
- [4] Thus, whenever p is supertrue so is Dp . So ' $p \models_G Dp$ ' always succeeds.
- [5] Imagine a specification space where p is indeterminate. It follows that Dp is evaluated as false at each specification point.
- [6] Thus, there is a specification point where p is evaluated as true, but Dp is evaluated as false.
- [7] Hence, there is a specification point where ' $p \rightarrow Dp$ ' is false.

¹ The unsubscripted turnstile ' $M \models A$ ' reads ' M satisfies A ' or equivalently ' A is true under M ' or still, ' A is evaluated or interpreted as true in M '.

² Substitute ' \emptyset ' for ' A ' and ' Dp ' for ' q '.

- [8] Hence, ' $p \rightarrow Dp$ ' is not supertrue.
- [9] Hence, ' $\emptyset \models_{\mathbf{G}} p \rightarrow Dp$ ' fails.