Homework 7

Problem 1. 1. *Prove that, for every integer n, there exists a coloring of the edges of the complete graph K_n by two colors so that the total number of monochromatic copies of K_4 is at most $\binom{n}{4}2^{-5}$.*

2. *Give a randomized algorithm for finding a coloring with at most $\binom{n}{4}2^{-5}$ monochromatic (i.e. single-color) copies of K_4 that runs in expected time polynomial in n.*

Solution.

1. Coloring every edge in K_4 by red or blue with probability $1/2$. The expected value of the total number of monochromatic copies of K_4 is then $2 \times \binom{n}{4} \times \left(\frac{1}{2}\right)^6$. Then there must exist some coloring scheme where the total number of monochromatic copies of K_4 is less or equal to $\binom{n}{4}2^{-5}$ (otherwise the expectation would be strictly larger than $\binom{n}{4}2^{-5}$.

2. Color each edge independently and uniformly. Let $p = Pr\left(X \le \binom{n}{4}2^{-5}\right)$ where X is the number of chromatic K_4.

$$\begin{aligned}
\binom{n}{4}2^{-5} &= \mathbf{E}(X) \\
&= \textstyle\sum_{i \le \binom{n}{4}2^{-5}} i \cdot Pr(X = i) + \sum_{i > \binom{n}{4}2^{-5}} i \cdot Pr(X = i) \\
&\ge p + (1-p)\left(\binom{n}{4}2^{-5} + 1\right)
\end{aligned}$$

which implies $p \ge \frac{32}{\binom{n}{4}}$. The expected number of sampling before finding a suitable coloring is $1/p = \frac{\binom{n}{4}}{32}$. For each sampling, the time needs to count the number of chromatic K_4 is bounded by $\binom{n}{4}$ which is also polynomial. Thus the expected running time of this algorithm is polynomial.

□

Problem 2. *Use the Lovasz local lemma to show that if*

$$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \le 1$$

then it is possible to color the edges of K_n with two colors so that it has no monochromatic (i.e. single color) K_k subgraph.

Solution. E_i: the $i-$th K_k is monochromatic. $Pr(E_i) = 2^{1-\binom{k}{2}}$. Consider the dependency graph, for any different E_i and E_j, they are adjacent if the corresponding K_k share at least one edge. Thus the degree of the dependency graph is bounded by $\binom{k}{2}\binom{n}{k-2}$.

According to the Lovasz local lemma, it is possible that none of the E_i happens under the given inequality. □

Problem 3. *We can generalize the problem of finding a large cut to finding a large k-cut. A k-cut is a partition of the vertices into k disjoint sets, and the value of a cut is the weight of all edges crossing from one of the k sets to another. In class we considered 3-cuts when all edges had the same weight 1, showing via the probabilistic method that any graph G with m edges has a cut with value at least* $m/2$*. Generalize this argument to show that any graph G with m edges has a k-cut with value at least* $(k-1)m/k$*.*

Solution.[sketch] The probability of an edge crossing two of the k sets is $(k-1)/k$ (the same as the probability of the following ball-and-bin probem: the probability of putting two balls into two different bins, where we have k bins in all). Then by the linearity of expectation, we have the result. □