# Homework 5

**Problem 1.** *Prove that, every edge of a r-regular graph lies on an even number of Hamilton cycles, as long as r is an odd number.*

*Solution.* Almost the same for 3-regular grapsh. □

**Problem 2.** *Given a sequence* $(d_1, d_2, \ldots, d_n)$ *of positive integers (where* $n \geq 1$*):*

*(i) There exists a tree with score* $(d_1, d_2, \ldots, d_n)$*.*

*(ii)* $\sum_{i=1}^{n} d_i = 2n - 2$*.*

*Prove that (i) and (ii) are equivalent.*

*Solution.*

1. $(i) \Rightarrow (ii)$ is obvious.

2. To prove $(ii) \Rightarrow (i)$:

   By induction on the number $n$.

   For $n = 1, 2$ the implication holds trivially, so let $n > 2$. Suppose the implication holds for any $n-1$ long positive sequence $(d_1, d_2, \ldots, d_{n-1})$ with $\sum_{i=1}^{n-1} d_i = 2(n-1) - 2$.

   For the induction step, consider an length $n$ positive sequence $\ell = (d_1, d_2, \ldots, d_n)$ with $\sum_{i=1}^{n} d_i = 2n - 2$:

   Since the sum of the $d_i$ is smaller than $2n$, there exists an $i$ with $d_i = 1$. w.l.o.g. we assume $d_1 = 1$. With a similar argument we can also conclude that there must exist some index $j$ such that $d_j \geq 2$. We take $k = \min\{j \mid d_j \geq 2\}$.
   Now the sequence $\ell = (d_1, d_2, \ldots, d_k, \ldots, d_n)$= $(1, d_2, \ldots, d_k - 1 + 1, \ldots, d_n)$, we can derive a new sequence $\ell' = (d_2, \ldots, d_k - 1, \ldots, d_n)$. Obviously $\ell'$ is a $n-1$ length sequence (all positive) with the summation to be $2n - 2 - 1 - 1 = 2(n-1) - 2$. Then according to the induction hypothesis, there exists a tree $\mathcal{T}'$ which corresponds to $\ell'$.
   Then $\mathcal{T} = (V(\mathcal{T}') \cup \{v_1\}, E(\mathcal{T}') \cup \{v_1, v_k\})$ is the tree which witnesses the validity of the sequence $\ell$.

BE CAREFUL: Why is the following 'proof' of the implication (ii)⇒(i) insufficient (or, rather, makes no sense)? We proceed by induction on $n$. The base case $n = 1$ is easy to check, so let us assume that the implication holds for some $n \geq 1$. We want to prove it for $n + 1$.If $D = (d_1, d_2, ..., d_n)$ is a sequence of positive integers with $\sum_{i=1}^{n} d_i = 2n - 2$, then we already know that there exists a tree $T$ on $n$ vertices with $D$ as a score. Add another vertex $v$ to $T$ and connect it to any vertex of $T$ by an edge, obtaining a tree $T'$ on $n+1$ vertices. Let $D'$ be the score of $T'$. We know that the number of vertices increased by 1, and the sum of degrees of vertices increased by 2 (the new vertex has degree 1 and the degree of one old vertex increased by 1). Hence the sequence $D'$ satisfies condition (ii) and it is a score of a tree, namely of $T'$. This finishes the inductive step. □

**Problem 3.** *Let $N_k$ denote the number of spanning trees of $K_n$ in which the vertex $n$ has degree $k$, $k = 1, 2, \ldots, n-1$ (recall that we assume $V(K_n) = \{1, 2, \ldots, n\}$).*

*i) Prove that $(n - 1 - k)N_k = k(n - 1)N_{k+1}$.*

*ii) Using i), derive $N_k = \binom{n-2}{k-1}(n-1)^{n-1-k}$.*

*iii) Prove Cayley's formula from ii).*

*Solution.*

i) Both sides of the equality count the number of pairs spanning trees $(T, T^*)$, where $deg_T(n) = k$, $deg_{T^*}(n) = k + 1$, and $T^*$ arises from $T$ by the following operation: pick an edge $\{i, j\} \in E(T)$ with $i \neq n \neq j$, delete it, and add either the edge $\{i, n\}$ or the edge $\{j, n\}$, depending on which of these edges connects the two components of $T - \{i, j\}$.

- From one $T$ we can get $n - 1 - k$ different $T^*$: the number of different edges in $T$ which are not connected to $n$ at the beginning;
- And one $T^*$ can be obtained from $k(n - 1)$ different $T$: pick any vertex $v \in \{1, 2, \ldots, n - 1\}$. If one deletes all edges incident to $n$ in a spanning tree from $N_{k+1}$, neighbours of $n$ (denoted by $\ell_1, \ell_2, \ldots, \ell_{k+1}$) will lie in exactly $k + 1$ different components. Suppose $v$ lies in the last component, namely $C_{k+1}$. Add an edge between $v$ and some $i$th leaf $\ell_i$ ($i \in \{1, 2, \ldots, k\}$) of $n$ and remove the original edge $(n, \ell_i)$ simultaneously, one will get a different $T$. In all, there are $n - 1$ ways to pick $v$ and $k$ ways to pick $\ell_i$.

ii)

iii) $\sum_k N_k$ happens to be the expansion of $((n-1)+1)^{n-2}$ according to the binomial theorem.

$\square$