Homework 2

Problem 1. *How many functions* $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ *are there that are* monotone*; that is, for* $i < j$ *we have* $f(i) \leq f(j)$*?*

Solution.

Set $k_i = f(i+1) - f(i)$, $i = 0, 1, \ldots, n$, where we add $f(0) = 1$ and $f(n+1) = n$. Then the desired number is the number of nonnegative integer solutions to the equation $k_0 + k_1 + \cdots k_n = n - 1$.

Thus the final solution will be $\binom{2n-1}{n}$.

□

Problem 2. *How many ways are there to seat n married couples at a round table with* $2n$ *chairs in such a way that the couples never sit next to each other?*

Solution.(hint)

$A_i = \{$the ways of seating in which the ith couple is adjacent.$\}$

- $|A_i| = (2n-1)! \cdot 2^1/(2n-1)$;
- $|A_i \cap A_j| = (2n-2)! \cdot 2^2/(2n-2)$ for $i \neq j$;
- $\cdots$;
- $|A_{i1} \cap \cdots \cap A_{ik}| = (2n-k)! \cdot 2^k/(2n-k)$ for different A_{ij}s.

The final result should be $(2n)!/(2n) - |A_1 \cup A_2 \cup \cdots \cup A_k|$. Then by PIE....

□

Problem 3. *1. Determine the coefficient of* x^4 *in* $(2+3x)^5\sqrt{1-x}$

Solution.

1. $$= \sum_{k=0}^{5}\binom{5}{k}2^k(3x)^{5-k}\sum_{j=0}^{\infty}\binom{1/2}{j}(-x)^j$$

   The coefficient of x^4 is $\sum_{k=1}^{5}\binom{5}{k}2^k(3)^{5-k}\binom{1/2}{k-1}(-1)^{k-1}$

□

Problem 4. *Find generating functions for the following sequences (express them in a closed form, without infinite series!):*

1. $0, 0, 0, 0, -6, 6, -6, 6, -6, \cdots.$
2. $1, 0, 1, 0, 1, 0, \cdots .$
3. $1, 2, 1, 4, 1, 8 \cdots$

Solution.

Sequence	Generating Function
$(1, 1, 1, 1, \ldots)$	$\frac{1}{1-x}$
$(1, -1, 1, -1, \ldots)$	$\frac{1}{1+x}$
$(-6, 6, -6, 6, \ldots)$	$\frac{-6}{1+x}$
$(0, 0, 0, 0, -6, 6, -6, 6, \ldots)$	$\frac{-6x^4}{1+x}$
$(1, 0, 1, 0, \ldots)$	$\frac{\frac{1}{1-x}+\frac{1}{1+x}}{2} = \frac{1}{1-x^2}$
$(0, 1, 0, 1, \ldots)$	$\frac{\frac{1}{1-x}-\frac{1}{1+x}}{2} = \frac{x}{1-x^2}$
$(1, 2, 4, 8, \ldots)$	$\frac{1}{1-2x}$
$(2, 4, 8, \ldots)$	$\frac{\frac{1}{1-2x}-1}{x} = \frac{2}{1-2x}$
$(1, 0, 2, 0, 4, 0, 8, \ldots)$	$\frac{1}{1-2x^2}$
$(1, 1, 2, 1, 4, 1, 8, \ldots)$	$\frac{1}{1-2x^2} + \frac{x}{1-x^2}$
$(1, 2, 1, 4, 1, 8, \ldots)$	$\frac{\frac{1}{1-2x^2}+\frac{x}{1-x^2}-1}{x} = -\frac{2x^3+2x^2-2x-1}{(1-2x^2)(1-x^2)}$

□

Problem 5. *Let a_n be the number of ordered triples $\langle i, j, k\rangle$ of integer numbers such that $i \geq 0, j \geq 1, k \geq 1$, and $i + 3j + 3k = n$. Find the generating function of the sequence $(a_0, a_1, a_2, \ldots)$ and calculate a formula for a_n.*

Solution.

$(1 + x + x^2 + x^3 + \cdots)(x^3 + x^6 + x^9 + \cdots)(x3 + x^6 + x^9 + \cdots)$
$= \frac{1}{1-x}\frac{x^3}{1-x^3}\frac{x^3}{1-x^3}$
$= \frac{x^6(1+x+x^2)}{(1-x^3)^3} = x^6(1 + x + x^2)(1 - x^3)^{-3}$.
Then use the generalized binomial theorem. □

Problem 6. *Express the n^{th} term of the sequences given by the following recurrence relations*

1. $a_0 = 2, a_1 = 3, a_{n+2} = 3a_n - 2a_{n+1}$ $(n = 0, 1, 2, \ldots)$.
2. $a_0 = 1, a_{n+1} = 2a_n + 3$ $(n = 0, 1, 2, \ldots)$.

Solution.

1. Characteristic function is $x^2 + 2x - 3 = (x + 3)(x - 1) = 0$.

   Let $f_n = a(-3)^n + b \cdot 1^n$. Then $\left\{\begin{array}{rcl} 2 &=& a + b \\ 3 &=& -3a + b \end{array}\right. \Rightarrow a = -1/4, b = 9/4$.

   $\therefore$ the $n-$th term is f_n.

2. Characteristic function for the homogeneous part is $x = 2$. Take $a_n = p2^n+\lambda$

   $a_0 = 1, a_1 = 5$. Now $\left\{\begin{array}{rcl} 1 &=& p + \lambda \\ 5 &=& 2p + \lambda \end{array}\right. \Rightarrow p = 4, \lambda = -3$.

□

Problem 7. *Solve the recurrence relation $a_{n+2} = \sqrt{a_{n+1}a_n}$ with initial conditions $a_0 = 2, a_1 = 8$ and find $\lim_{n\to\infty} a_n$.*

Solution. Consider the sequence $b_n = \log_2 a_n$. Then

$$2\log_2 a_{n+2} = \log_2 a_{n+1} + \log_2 a_n$$

i.e. $2b_{n+2} = b_{n+1} + b_n$. $b_0 = 1, b_1 = 3$. One can find $b_n = (-\frac{4}{3})(-\frac{1}{2})^n + \frac{7}{3}$. $\therefore a_n = 2^{(-\frac{4}{3})(-\frac{1}{2})^n+\frac{7}{3}}$. $\lim_{n\to\infty} a_n = 2^{\frac{7}{3}}$. □

Problem 8. *Show that for any $n \geq 1$, the number $\frac{1}{2}[(1+\sqrt{2})^n + (1-\sqrt{2})^n]$ is an integer.*

Solution. Consider $\lambda_1 = (1+\sqrt{2})^n$, $\lambda_2 = (1-\sqrt{2})^n$. They are solutions to the characteristic function $(x-1-\sqrt{2})\cdot(x-1+\sqrt{2}) = x^2-2x-1$.

Thus the original sequence satisfies the recurrence $a_{n+2} = 2a_{n+1} + a_n$, with $a_0 = a_1 = 1$. □