Homework 6

Problem 1. *Find an example to verify the claim that '(pairwise) independence does not imply mutual independence'. Pls give a detailed proof.*

Solution. (by S. Bernstein)

Suppose X and Y are two independent tosses of a fair coin, where we designate 1 for heads and 0 for tails. Let the third random variable $Z = (X + Y) \bmod 2$.

Then jointly the triple $\langle X, Y, Z\rangle$ has the following probability distribution:

$$\langle X, Y, Z\rangle = \begin{cases} \langle 0,0,0\rangle & \text{with probability } 1/4 \\ \langle 0,1,1\rangle & \text{with probability } 1/4 \\ \langle 1,0,1\rangle & \text{with probability } 1/4 \\ \langle 1,1,0\rangle & \text{with probability } 1/4 \end{cases}$$

$i, j, k \in \{0, 1\}$.

It is easy to verify that $Pr(X = i) = Pr(Y = j) = Pr(Z = k) = 1/2$ and $Pr(X = i, Y = j) = Pr(X = i, Z = k) = Pr(Y = j, Z = k) = 1/4$. i.e., X, Y, Z are pairwise independent.

However, $Pr(X = i, Y = j, Z = k) \neq Pr(X = i) \cdot Pr(Y = j) \cdot Pr(Z = k)$. For example, the left side equals $1/4$ for $\langle x, y, z\rangle = \langle 0, 0, 0\rangle$ while the right side equals $1/8$.

In fact, any of $\langle X, Y, Z\rangle$ is completely determined by the first two components. That is as far from independence as random variables can get.

□

Problem 2. *Show that, if $E_1, E_2, \ldots, E_n$ are mutually independent, then so are $\overline{E_1}, \overline{E_2}, \ldots, \overline{E_n}$.*

Solution. (sketch) It will be enough to prove that for any $2 \leq k \leq n$, and $\{F_1, F_2, \ldots, F_k\} \subseteq \{E_1, E_2, \ldots, E_n\}$

$$Pr(\bigcap_{i=1}^{k} \overline{F_i}) = \prod_{i=1}^{k} Pr(\overline{F_i})$$

Let $Pr(F_i) = f_i$, then

$$Pr(\bigcap_{i=1}^{k} \overline{F_i}) = 1 - Pr(\bigcup_{i=1}^{k} F_i) = 1 - \sum_{i=1}^{k} f_i + \sum_{1\leq i<j\leq k} f_i f_j - \sum_{1\leq i<j<l\leq k} f_i f_j f_l + \cdots$$

The right hand side of the above equation is $(1 - f_1)(1 - f_2)\cdots(1 - f_k) = \prod_{i=1}^{k} Pr(\overline{F_i})$. □

Problem 3. *Suppose X and Y are two independent discrete random variables, show that*

$$E(X \cdot Y) = E[X] \cdot E[Y]$$

.

Solution.

$$\begin{aligned} E[X \cdot Y] &= \Sigma_i \Sigma_j (i \cdot j) \cdot Pr((X = i) \cap (Y = j)) \\ &= \Sigma_i \Sigma_j (i \cdot j) \cdot Pr(X = i) \cdot Pr(Y = j)) \\ &= (\Sigma_i i \cdot Pr(X = i)) \left(\Sigma_j j \cdot Pr(Y = j)\right) \\ &= E(X) \cdot E(Y) \end{aligned}$$

□

Problem 4. *A monkey types on a 26 -letter keyboard that has lowercase letters only. Each letter is chosen independently and uniformly at random from the alphabet. If the monkey types 1,000,000 letters. what is the expected number of times the sequence "proof" appears?*

Solution. By the linearity of expectation:

$$E[X] = (1/26)^5 \times (1000000 - 4)$$

□

Problem 5. *Let* (Ω, P) *be a finite probability space in which all elementary events have the same probability. Show that if* $|\Omega|$ *is a prime number then no two non-trivial events (distinct from* $\emptyset$ *and* Ω*) can be independent.*

Solution. Suppose $\Omega = \{e_1, e_2, \cdots, e_p\}$ where p is a prime number and $Pr[e_i] = 1/p$ for all $i \in [1, p]$. For two events $A, B \in 2^\Omega$, if $A, B \notin \{\emptyset, \Omega\}$ take $C = A \cap B$. If A, B are independent then $Pr[C] = Pr[A] \times Pr[B]$, which will lead to $\frac{|C|}{n} = \frac{|A|}{n} \times \frac{|B|}{n}$. As C will not be emptyset (otherwise A, B must be dependent), we can finally get $n = \frac{|A| \times |B|}{|C|}$. As n is a prime number we will have a contradiction. □

Problem 6. *We have 27 fair coins and one counterfeit coin (28 coins in all), which looks like a fair coin but is a bit heavier. Show that one needs at least 4 weighings to determine the counterfeit coin. We have no calibrated weights, and in one weighing we can only find out which of two groups of some k coins each is heavier, assuming that if both groups consist of fair coins only the result is an equilibrium.*

Solution. Each weighting has 3 possible outcomes, and hence 3 weightings can only distinguish one among 3^3 possibilities.

□

Problem 7 (Optional.). *A geometric random variable X with parameter p with distribution*

$$Pr[X = n] = (1 - p)^{n-1} p$$

Prove that its variance $Var[X] = \frac{1-p}{p^2}$.

Solution.

[Hint: one can try to use the conditional expectation formula:
$E[X^2] = E[X^2|X^2 = 1] \cdot Pr(X^2 = 1) + E[X^2|X^2 > 1] \cdot Pr(X^2 > 1)$.]
It is clear that $E[X^2|X^2 = 1] = 1$, and $X^2 > 1$ iff $X > 1$. Thus we have

$$E(X^2) = p + E[X^2|X > 1] \cdot Pr(X > 1)$$

$$E(X^2) = p + E[X^2|X > 1] \cdot (1 - p)$$

To the latter item

$$\begin{aligned} E[X^2|X > 1] &= E\left[((X-1)^2 + 2(X-1) + 1)\Big|(X - 1 > 0)\right] \quad \text{(linearity of conditional expectation)} \\ &= E\left[(X-1)^2\Big|(X-1>0)\right] + 2E\left[(X-1)\Big|(X-1>0)\right] + 1 \\ &= E[X^2] + 2E[X] + 1 \\ &= E[X^2] + \tfrac{2}{p} + 1 \end{aligned}$$

Thus we have

$$E[X^2] = p + (1-p)(E[X^2] + \frac{2}{p} + 1)$$

$$i.e., \quad E[X^2 = \frac{2-p}{p^2}]$$

And we have $Var[X] = \frac{1-p}{p^2}$. □