Homework 3

Problem 1. *Fill in the blanks with either true (✓) or false (×)*

$f(n)$	$g(n)$	$f = O(g)$	$f = \Omega(g)$	$f = \Theta(g)$
$2n^3 + 3n$	$100n^2 + 2n + 100$	×	✓	×
$50n + \log n$	$10n + \log\log n$	✓	✓	✓
$50n \log n$	$10n \log\log n$	×	✓	×
$\log n$	$\log^2 n$	✓	×	×
$n!$	5^n	×	✓	×

Problem 2.

1. *Find two functions $f(x)$ and $g(x)$ such that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$.*
2. *Furthermore, we say a function $h : \mathbb{R} \to \mathbb{R}$ is* monotonically increasing *if it satisfies the property '$x \leq y \implies h(x) \leq h(y)$'.*
   Find two monotonically increasing functions $f(x)$ and $g(x)$ such that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$.

(Please give the detailed proof that your functions satisfy the requirements.)

Solution.

1. $\begin{cases} f(x) = \sin(x); \\ g(x) = \cos(x). \end{cases}$
2. $\begin{cases} f(x) = x^{\sin(x)+x}; \\ g(x) = x^{\cos(x)+x}. \end{cases}$

The detailed proof are omitted. Just stick to the definition of $O(-)$. □

Problem 3. *Prove that*

(a) $\left(1 + \frac{1}{n}\right)^n \leq e$ *for all* $n \geq 1$.

(b) $\left(1 + \frac{1}{n}\right)^{n+1} \geq e$ *for all* $n \geq 1$.

(c) Using (a) *and* (b)*, conclude that* $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

Solution.

(a) $\left(1+\frac{1}{n}\right)^n \leq \left(e^{\frac{1}{n}}\right)^n = e$.

(b) $\left(1+\frac{1}{n}\right)^{n+1} = \left(\frac{n+1}{n}\right)^{n+1} = \left(\frac{1}{\frac{n}{n+1}}\right)^{n+1} = \left(\frac{1}{1-\frac{1}{n+1}}\right)^{n+1} \geq \left(e^{\frac{1}{n+1}}\right)^{n+1} = e$.

(c) $\because \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n}\right)^{n+1}} = 1 \therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}$.

While $e \overset{(b)}{\leq} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) \overset{(a)}{\leq} e \cdot \lim_{n\to\infty}\frac{1}{n} = e$.

□

Problem 4. *Prove* Bernoulli's inequality*: for each natural number n and for every real $x \geq -1$, we have $(1+x)^n \geq 1+nx$.*

Solution. Apply binomial theorem to the left. □

Problem 5. *Prove that for $n = 1, 2, \ldots,$ we have*

$$2\sqrt{n+1} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1.$$

Solution. Proof by induction. □

Problem 6.

a) Show that the product of all primes p with $m < p \leq 2m$ is at most $\binom{2m}{m}$.

b) Using a), prove the estimate $\pi(x) = O(\frac{x}{\ln x})$, where $\pi(x)$ denote the number of primes not exceeding the number x.

Solution.

1. $B = \binom{2m}{m} = \frac{(m+1)\times(m+2)\times\cdots\times(2m)}{1\times2\times\cdots\times m}$. It is easy to find that if p is a prime number and $p \in (m, 2m]$ then $p|B$. Thus $\prod_{m<p\leq 2m} p \mid B$. It follows that the upper bound of the products of prime numbers between m and $2m$ is B.

2. There are several ways to prove the second problem.

First proof: Combing a), w.l.o.g. assume n is even and $n = 2m$. It is obvious that

$$B \leq \sum_{i=0}^{2m} \binom{2m}{i} = 4^m$$

With a) we have $\prod_{m<p\leq 2m} p \leq B \leq 4^m$ (p is prime, as above). It follows that

$$\sum_{m<p\leq 2m} \log p \leq m \log 4 = 2m \qquad (\star)$$

Then count the number of primes between m and $2m$, i.e. the number of $p \in (m, 2m]$,

$$\pi(2m) - \pi(m) = \sum_{m<p\leq 2m} 1 \leq \sum_{m<p\leq 2m} \frac{\log p}{\log m} = \frac{1}{\log m} \left(\sum_{m<p\leq 2m} \log p \right) \overset{(\star)}{\leq} \frac{2m}{\log m}.$$

For any given x, there exists $k \geq 1$ such that $x \in (2^{k-1}, 2^k]$.
Finally with the above analysis

$$\pi(x) \leq \pi(2^k) = \sum_{i=1}^{k} \left(\pi(2^i) - \pi(2^{i-1}) \right) = O\left(\sum_{j=1}^{k} \frac{2^j}{j} \right) = O\left(\frac{2^k}{k} \right) = O\left(\frac{x}{\ln x} \right).$$

Second proof: Proof by contradiction

$\square$