Homework 1

Problem 1. *Let*$(X, \preceq_1)$, $(Y, \preceq_2)$ *be (partially) ordered sets. We say that they are* isomorphic *if there exists a bijection* $f : X \to Y$ *such that for every* $x, y \in X$, *we have* $x \preceq_1 y$ *if and only if* $f(x) \preceq_2 f(y)$.

1. *Draw Hasse diagrams for all nonisomorphic 3-element posets.*

2. *Prove that any two n-element linearly ordered sets are isomorphic.*

3. *Prove that* $(\mathbb{N}, \leq)$ *and* $(\mathbb{Q}, \leq)$ *are not isomorphic. (where* $\mathbb{N}$ *is the set of natural numbers,* $\mathbb{Q}$ *is the set of rational numbers,* $\leq$ *is the usual 'less or equal to' between numbers).*

Solution.
2. Hint: Always map the minimal/least element in one structure to the other.
3. Suppose there is such an isomorphism function $f : \mathbb{N} \to \mathbb{Q}$. $f(0) = a$, $f(1) = b$. We have $a < b$ for $0 < 1$. Then consider $f^{-1}(\frac{a+b}{2})$. □

Problem 2. *Prove or disprove: If a partially ordered set* $(X, \leq)$ *has a single minimal element, then it is a smallest element as well.*

Solution. Wrong. Consider $(\{a\}, \langle a, a\rangle) \cup (\mathbb{Z}, \leq)$. □

Problem 3. *Let* $(X, \leq)$ *and* $(X', \leq')$ *be partially ordered sets. A mapping* $f : X \to X'$ *is called an embedding of* $(X, \leq)$ *into* $(X', \leq')$ *if the following conditions hold:*

- f *is an injective mapping;*
- $f(x) \leq' f(y)$ *if and only if* $x \leq y$.

Now consider the following problem

a) *Describe an embedding of the set* $\{1, 2\} \times \mathbb{N}$ *with the lexicographic ordering into the ordered set* $(\mathbb{Q}, \leq)$.

b) *Solve the analog of a) with the set* $\mathbb{N} \times \mathbb{N}$ *(ordered lexicographically) instead of* $\{1, 2\} \times \mathbb{N}$.

Solution.

1. $f(i,n) = i.y$ where $i \in \{1,2\}$ and $y = \frac{n}{n+1}$.

2. Similarly.

$\square$

Problem 4. *Prove the following strengthening of the* ***Erdös-Szekeres Lemma****: Let κ, ℓ be natural numbers. Then every sequence of real numbers of length $\kappa\ell + 1$ contains an nondecreasing subsequence of length $\kappa + 1$ or a decreasing subsequence of length $\ell + 1$.*

Solution. Hint: $\alpha(P) \cdot \omega(P) \geq \kappa\ell$. Then similar to the proof of Erdös-Szekeres Lemma: either $\omega(P) > \kappa$, which implies the existence of a nondecreasing subsequence of length $\kappa + 1$, or $\omega(P) < \kappa$, then $\omega(P) > \ell$, which implies the other case. $\square$

Problem 5. *Prove the formula*

1. $\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$

2. $\sum_{k=0}^{n} \binom{m+k-1}{k} = \binom{n+m}{n}$

Solution.

1. Use the equivalence $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$ iteratively.

2. Note that $\binom{m-1}{0} = \binom{m}{0} = 1$. The rest is just like above.

$\square$

Problem 6. *(a) Using* **Problem 5.** *for $r = 2$, calculate the sum $\sum_{i=2}^{n} i(i-1)$ and $\sum_{i=1}^{n} i^2$.*

(b) Use (a) and **Problem 5.** *for $r = 3$, calculate $\sum_{i=1}^{n} i^3$.*

Solution.

1.
$$r = 2 : \qquad \binom{2}{2} + \binom{3}{2} + \cdots + \binom{i}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}$$

Thus $\frac{\sum_{i=2}^{n} i(i-1)}{2!} = \binom{n+1}{3} \therefore \sum_{i=2}^{n} i(i-1) = 2\binom{n+1}{3}$

$$r = 1 : \qquad \binom{1}{1} + \binom{2}{1} + \cdots + \binom{i}{1} + \cdots + \binom{n}{1} = \binom{n+1}{2}$$

Thus $\therefore \sum_{i=1}^{n} i = \binom{n+1}{2}$.

Finally, $\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} (i(i-1) + i) = \sum_{i=1}^{n} i(i-1) + \sum_{i=1}^{n} i = \frac{n(n+1)(2n+1)}{6}$.

2.

$$r = 3 : \qquad \binom{3}{3} + \binom{4}{3} + \cdots + \binom{i}{3} + \cdots + \binom{n}{3} = \binom{n+1}{4}$$

Thus $\frac{\sum_{i=3}^{n} i(i-1)(i-2)}{3!} = \binom{n+1}{4}$. $\therefore \sum_{i=3}^{n} i^3 - 3i^2 + 2i = 6\binom{n+1}{4}$,

...

The final result is $\binom{n+1}{2}^2$.

□