1 Introduction

Non-relativistic quantum mechanics refers to the mathematical formulation of quantum mechanics applied in the context of Galilean relativity, more specifically quantizing the equations from the Hamiltonian formalism of non-relativistic Classical Mechanics by replacing dynamical variables by operators through the correspondence principle. Therefore, the physical properties predicted by the Schrödinger theory are invariant in a Galilean change of referential, but they do not have the invariance under a Lorentz change of referential required by the principle of relativity. In particular, all phenomena concerning the interaction between light and matter, such as emission, absorption or scattering of photons, is outside the framework of non-relativistic Quantum Mechanics. One of the main difficulties in elaborating relativistic Quantum Mechanics comes from the fact that the law of conservation of the number of particles ceases in general to be true. Due to the equivalence of mass and energy there can be creation or absorption of particles whenever the interactions give rise to energy transfers equal or superior to the rest massses of these particles. This theory, which i am about to present to you, is not exempt of difficulties, but it accounts for a very large body of experimental facts. We will base our deductions upon the six axioms of the Copenhagen Interpretation of quantum mechanics.Therefore, with the principles of quantum mechanics and of relativistic invariance at our disposal we will try to construct a Lorentz covariant wave equation. For this we need special relativity.

1.1 Special Relativity

Special relativity states that the laws of nature are independent of the observer´s frame if it belongs to the class of frames obtained from each other by transformations of the Poincaré group. The latter is generated by space and time translations, spatial rotations and special Lorentz transformations (or boosts), which relate frames moving with constant relative velocity. Furthermore, the speed of light c is an absolute upper bound on the velocity of any signal.

2 Klein Gordon equation

2.1 Derivation

We shall now try to find a relativistic wave equation by modifying the Schrödinger picture to be consistent with special relativity, thus to be consistent with the concepts stated beforehand. The energy of a free massive particle is given in special relativity by the relativistic energy momentum relation.

$$E^2 = \vec{p}^2 c^2 + m^2 c^4 \tag{1}$$

By imposing the correspondence principle, thus replacing the energy and momentum by their respective operators

$$p^\mu \to i\hbar\partial^\mu = \left(i\frac{\hbar}{c}\partial_t,\, -i\hbar\vec{\nabla}\right)^T$$

and by imposing the Schrödinger equation to give the time evolution of the quantum system, we find

$$-\hbar^2\partial_t^2\psi(x) = -\hbar^2 c^2\vec{\nabla}^2\psi(x) + m^2c^4\psi(x) \tag{2}$$

which is the Klein Gordon equation (KGE) for the wavefunction of a free relativistic particle of spin zero. The KGE in a covariant form reads

$$\left[\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right]\psi(x) = 0 \tag{3}$$

The KGE is a differential equation of second order in space and time derivatives and therefore Lorentz invariant.

2.2 Free particle solutions

Any plane wave function

$$\psi(x) = N\exp\left\{-\frac{i}{\hbar}\left(Et - \vec{p}\vec{x}\right)\right\}$$

solves the KGE, where N is a normalization constant and $E = \pm\sqrt{\vec{p}^2c^2 + m^2c^4}$. The negative energy solutions come into play here, because we regarded the square of the relativistic energy momentum relation for the derivation of the Klein Gordon equation. As we will see in the following, the negative energy solutions pose a severe problem if you try to interpret ψ as a wavefuntion. By the existence of negative energy solutions the spectrum is not longer bound from below, that means you can extract arbitrarily large amounts of energy from the system by driving it into ever more negative energy states.

2.3 Propabilistic interpretation

We require a probabilistic interpretation of the wavefunction solving the Klein Gordon equation, in order to interpret ψ as a wave function of a given quantum system. We therefore require said wavefunction to obey a continuity equation. Indeed, one can compute a continuity equation from the Klein Gordon equation similarly as in non-relativistic quantum mechanics and finds:

$$\partial_t \underbrace{\left[\frac{i\hbar}{2mc^2}\left(\psi^*\partial_t\psi - \psi\partial_t\psi^*\right)\right]}_{\rho(x)} + \vec{\nabla}\cdot\underbrace{\left[\frac{\hbar}{2im}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)\right]}_{=:\vec{j}(x)} = 0 \tag{4}$$

From this we can observe, that the density we´ve found is not positive definite, because of the second time derivative appearing here. It is therefore not possible to assign a probabilistic interpretation to the solutions of the Klein Gordon equation. The presence of the second time derivative indicates that ψ has two degrees of freedom rather than the single degree of freedom from the Schrödinger equation. Pauli and Weisskopf have already shown 1934, that the two degrees of freedom of ψ correspond to two different possible charge states. The continuity equation can then be interpreted as charge conservation. It is clear that the charge density measures the difference between the number of positive and the number of negative particles, and in consequence will not be positive definite. Therefore, the insight, that the increase of "degrees of freedom" is

connected with the appearance of a second-order time derivative in the Klein-Gordon equation, corresponds to the simultaneous description of a particle of either positive or negative charge; this means that the value of the charge becomes a degree of freedom of the system.
Note to myself: For plane wave solutions as discussed before we find

$$\rho(x) = 2|N|^2 E$$

which is obviously not positive definite, because negative energy solutions exist. We will now turn towards another relativistic wave equation, which tries to cope with the problems we´ve already encountered with the Klein Gordon equation, namely negative energy solutions and a negative density, by imposing a completely different approach.

3 Dirac equation

3.1 Comment on its derivation

In order to cope with the negative energy solutions Dirac wanted to find a relativistic wave equation which is first order in time and spatial derivatives, thus also guarantueeing a probabilistic interpretation and Lorentz invariance. Dirac therefore imposes a general expression for the Hamiltonian of a free relativistic particle

$$H_D := -i\hbar c\,\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2$$

The solutions must satisfy the relativistic energy dispersion relation and by this constraint, thus by comparing to the KG equation, one can derive algebraic properties of α^k and β. By these constraints one finds, that α^k and β have to be hermitian 4x4 matrices, which in turn requires the wavefunction to be a four component vector. One arrives at a relativistic wave equation, which describes massive particles and anti-particles of spin-1/2 (e.g electron, positron), the Dirac equation:

$$\left[i\hbar\gamma^\mu\partial_\mu - mc\right]\psi(x) = 0 \tag{5}$$

The four by four matrices appearing here are called Dirac matrices, they are combinations of the matrices α^k and β, and they read in the so-called chiral representation

$$\gamma^0 = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix}, \gamma^k = \begin{pmatrix} 0_2 & \sigma^k \\ -\sigma^k & 0_2 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The four-component wavefunction ψ is called a bispinor, by this it is implied, that ψ transforms under a specific representation of the Lorentz group. Its transformation properties manifest themselves for example by having to turn the spinor by 720 in order to arrive at its initial position. We will see in the following, that the four real degrees of freedom manifest themselfs as two degrees of freedom (spin up and down) for the particle and for the anti-particle respectively.

3.2 Probabilistic Interpretation

Again we want to find a continuity equation in order to have a probabilistic interpretation of the bispinors solving the Dirac equation. By defining the Dirac adjoint spinor

$$\bar{\psi} := \psi^\dagger\gamma^0$$

we find a Lorentz invariant continuity equation (by multiplying the hermitian conjugate of the Dirac equation from the right by ψ and subtracting it from the Dirac equation mupltiplied from the left by the conjugate spinor) which reads

$$\frac{1}{c}\partial_t \left[\psi^\dagger\psi\right] + \vec{\nabla}\cdot\left[\bar{\psi}\vec{\gamma}\psi\right] = 0 \tag{6}$$

By aquiring a positive definite density we obtain a probabilistic interpretation of the bispinor solving the Dirac equation.

3.3 The Weyl equations

We will now further discuss the structure of the Dirac equation and its solutions thereafter. The Dirac equation reduces to the so-called Weyl equation for a massles m=0 spin one half fermion. One can reduce the four-component bispinor into two component Weyl spinors:

$$\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$$

This reduction is possible, because the subspaces spanned by these two Weyl spinors transform seperately under spatial rotations or Lorentz boosts. They are coupled in the free Dirac equation by the mass

$$\left(i\hbar\gamma^\nu\partial_\nu - mc\right)\psi(x) = \begin{pmatrix} -mc & i\hbar\left(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}\right)\\ -i\hbar\left(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}\right) & -mc\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

From this we find, that the Weyl spinors decouple for m=0 and that they describe independent degrees of freedom subject to the Weyl equations:

$$i\hbar\left(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}\right)\psi_L(x) = 0 \tag{7}$$

$$i\hbar\left(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}\right)\psi_R(x) = 0 \tag{8}$$

By describing massless spin one half fermions the Weyl equations are important when treating neutrinos and the theory of weak interactions.
The Weyl equations are solved by the two component Weyl spinors ψ_L and ψ_R. With the ansatz $\psi_{L,R} = \chi_{L,R}\exp\{-i(\vec{p}\cdot\vec{r} - Et)/\hbar\}$ where χ is a constant two-component spinor we find with the Weyl equaion that it satisfies:

$$(I_2 E - \vec{\sigma}\cdot\vec{p})\chi = 0$$

3.4 Solutions of the Dirac equation for a free particle

After obtaining a possible probabilistic interpretation for the solutions of the Dirac equation we will try in the following to give them a physical interpretation by further exploring their explicit nature. The plane wave solutions of the Dirac equation are given by two different Dirac spinors

$$\psi_+(x) = u(\vec{p})\exp\left\{-i\frac{Et - \vec{p}\vec{x}}{\hbar}\right\} \quad \text{positive energy solution} \tag{9}$$

$$\psi_-(x) = v(\vec{p}) \exp\left\{ i\frac{Et - \vec{p}\vec{x}}{\hbar} \right\} \quad \text{negative energy solution} \tag{10}$$

Plugging this into the Dirac equation yields

$$(p_\mu \gamma^\mu - mc)\, u(\vec{p}) = 0 \quad \& \quad (p_\mu \gamma^\mu + mc)\, v(\vec{p}) = 0$$

The most general solutions for these bispinors is given by two linearly independent solutions for each bispinor:

$$\begin{aligned}
u^{(1)}(p) &= \left(1, 0, \frac{cp_z}{E+mc^2}, \frac{c(p_x+ip_y)}{E+mc^2}\right)^T \\
u^{(2)}(p) &= \left(0, 1, \frac{c(p_x-ip_y)}{E+mc^2}, \frac{-cp_z}{E+mc^2}\right)^T \\
v^{(1)}(p) &= \left(\frac{-p_z c}{-E+mc^2}, \frac{c(-p_x-ip_y)}{-E+mc^2}, 1, 0\right)^T \\
v^{(2)}(p) &= \left(\frac{c(-p_x+ip_y)}{-E+mc^2}, \frac{p_z c}{-E+mc^2}, 0, 1\right)^T
\end{aligned}$$

Note the funny minus signs, here the interpretation of u representing an electron of positive energy with two possible spin states and v representing an electron of negative energy with two possible spin states becomes apparent. One generally replaces $p \to -p$ for the v bispinors in order to describe a anti particle of positive energy - the positron (the reason for this becomes apparent later on):

$$\tilde{v}^{(1)}(p) = \left(\frac{p_z c}{E+mc^2}, \frac{c(p_x+ip_y)}{E+mc^2}, 1, 0\right)^T$$

$$\tilde{v}^{(2)}(p) = \left(\frac{c(p_x-ip_y)}{E+mc^2}, \frac{-p_z c}{E+mc^2}, 0, 1\right)^T$$

with

$$\psi^{1,2} = \tilde{v}^{1,2}(p) \exp\left\{ -i\frac{Et - \vec{p}\vec{x}}{\hbar} \right\}$$

3.5 Spin

Now we have to justify the statement that the Dirac equation describes spin one half particles. We have to justifty the statement, that the four possible solutions describe positive and negative energy solutions with spin up and spin down respectively. To fully describe all possible states we want a set of commuting operators, to label all different solutions. One such operator is the total angular momentum. The total angular momentum has to commute with the Dirac Hamiltonian in order to be conserved. We can therefore deduce the explicit form of the spin operator by computing the commutator and find:

$$\hat{\vec{S}} = \frac{\hbar}{2}\vec{\Sigma} = \frac{\hbar}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{11}$$

We interpret this operator as an angular momentum intrinsic to the particle, the spin operator of the Dirac particle. Computing its square yields the characteristic property of a particle of spin one half:

$$\hat{\vec{S}} = \frac{\hbar^2}{4}\begin{pmatrix}\vec{\sigma}\cdot\vec{\sigma} & 0 \\ 0 & \vec{\sigma}\cdot\vec{\sigma}\end{pmatrix} = \frac{3\hbar^2}{4}I_4 = s(s+1)\hbar^2 I_4 \quad \text{for} \quad s = \frac{1}{2}$$

Furthermore we find the eigenvalues of the bispinors in z-direction for $\vec{p} = (0,0,p_z)^T$ to be:

$$\begin{aligned}
\hat{S}_z u^{(1)}(p) &= +\frac{\hbar}{2}u^{(1)}(p) \\
\hat{S}_z u^{(2)}(p) &= -\frac{\hbar}{2}u^{(2)}(p) \\
\hat{S}_z \tilde{v}^{(1)}(p) &= +\frac{\hbar}{2}\tilde{v}^{(1)}(p) \\
\hat{S}_z \tilde{v}^{(2)}(p) &= -\frac{\hbar}{2}\tilde{v}^{(2)}(p)
\end{aligned}$$

From this we can conclude, that the Dirac equation describes spin one half particles.

3.6 Helicity

Another operator which commutes with the Dirac Hamiltonian is the helicity operator, which describes the projection of the particles spin onto its direction of flight with eigenvalues ± 1:

$$\hat{h}(p) = \frac{\hbar}{2}\vec{\sigma}\cdot\frac{\vec{p}}{|\vec{p}|} \tag{12}$$

With this operator we can now try to understand the subscript of the Weyl spinors ψ_L and ψ_R for massless particles as defined before. One can construct an explicit form of the Weyl spinors from the Weyl equation and finds their respective helicity to be:

$$\begin{aligned}
\hat{h}\psi_L(\vec{p}) &= -\frac{\hbar}{2}\psi_L(\vec{p}) \quad \rightarrow \quad \text{Left-handed particle} \\
\hat{h}\psi_R(\vec{p}) &= +\frac{\hbar}{2}\psi_R(\vec{p}) \quad \rightarrow \quad \text{Right-handed particle}
\end{aligned}$$

Note that under parity transformation the Weyl spinors transform into each other, because the helicity changes sign. This means, that a theory in which ψ_L has different interactions to ψ_R (such as the standard model in which the weak force only acts on left handed fermions and right handed antifermions) manifestly violates parity.

3.7 The Dirac particle in an electromagnetic field & Charge conjugation

The charge of a particle is another important degree of freedom of the Dirac theory which gives us a further notion of interpreting the solutions of the Dirac equation. We will therefore study the interactions of a Dirac particle with an external (classical) electromagnetic field with a potential $A^\mu = (\frac{\phi}{c}, \vec{A})^T$. The Hamiltonian for a particle of charge q interacting with an electromagnetic field is

$$H_{em} = \frac{|\vec{p} - q\vec{A}|^2}{2m} + q\phi$$

Clearly, the interacting Hamiltonian may be obtained from the free Hamiltonian by subtracting $q\phi$ and replacing $\vec{p} \to \vec{p} - q\vec{A}$. This is equivalent to replacing

$$E \to E - q\phi \quad \text{and} \quad \vec{p} \to \vec{p} - q\vec{A}$$

This is know as the minimal coupling prescription. The Dirac equation then becomes:

$$i\hbar\gamma^{\mu} \underbrace{(\partial_{\mu} + iqA_{\mu})}_{=:D_{\mu}\text{covariant derivative}} \psi(x) - mc\psi(x) = 0 \tag{13}$$

We can therefore describe a particle of mass m and charge $-|e|$ -the electron- and its respective anti-particle of the same mass but opposite charge $+|e|$ -the positron. With this form of the Dirac equation we can now identify negative energy electron solutions with positive energy positron solutions if we find an operator which links the respective solutions - the charge conjugate operator.

We demand of such an operator to transform any realizable physical state, which contains an electron in a potential A_{μ} into another realizable physical state describing a positron in a potential -A_{μ}. The charge conjugate operator needed to go from

$$\begin{aligned} i\hbar\gamma^{\mu} \underbrace{(\partial_{\mu} - ieA_{\mu})}_{=:D_{\mu}\text{covariant derivative}} \psi(x) - mc\psi(x) = 0 \\ \to i\hbar\gamma^{\mu} \underbrace{(\partial_{\mu} + ieA_{\mu})}_{=:D_{\mu}\text{covariant derivative}} \psi_C(x) - mc\psi_C(x) = 0 \end{aligned}$$

is represented by

$$\psi_C = \hat{C}\psi = i\gamma^2\psi^* \tag{14}$$

We shall now further investigate the action of the charge conjugate operator onto a negative energy solution for an electron at rest and spin down :

$$\psi(x) = v^{(2)}(p)\exp\left\{\frac{i(Et - \vec{p}\cdot\vec{x})}{\hbar}\right\} = (0,0,0,1)^T \exp\left\{\frac{i(Et - \vec{p}\cdot\vec{x})}{\hbar}\right\}$$

The corresponding positron solution is then identified by applying the charge conjugate operator:

$$\hat{C}\psi^4(x) = i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\exp\left\{\frac{-i(Et - \vec{p}\cdot\vec{x})}{\hbar}\right\} \tag{15}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\exp\left\{\frac{-i(Et - \vec{p}\cdot\vec{x})}{\hbar}\right\} \tag{16}$$

$$= u^{(1)}(\vec{p})\exp\left\{\frac{-i(Et - \vec{p}\cdot\vec{x})}{\hbar}\right\} = \psi^1(x) \tag{17}$$

We therefore find, that an electron at rest with negative energy and spin down is equivalent to a positron at rest with positive energy and spin up. We therefore find a deeper symmetry within the Dirac equation, namely that the existence of an electron of mass m and charge $-e$ demands the existence of a positron of the same mass but opposite charge.

3.8 Interpretation of negative energy solutions

With these possible interpretations to the solutions of the Dirac equation at our disposal we will now try to put them into a wholesome physical context in the following. The energy spectrum of the solutions of the Dirac equation is made up of two continuous band $(-\infty, -mc^2)$ and (mc^2, ∞) seperated by an interval of $2mc^2$. The first of the bands corresponds to negative energy states : $E = -\sqrt{m^2c^4 + \vec{p}^2c^2}$, and the second to positive energy states $E = \sqrt{m^2c^4 + c^2\vec{p}^2}$. With the existence of negative energy solutions of the Dirac equation one might ask, why matter is stable. Negative energy solutions are a big difficulty, when effects of the electromagnetic field are considered, because a positive-energy electron would be able to shed energy by continuously emitting photons, a process that could continue without limit as the electron descends into lower and lower energy states. Any external perturbation capable of pushing a particle across the energy gap $\Delta E = 2mc^2$ between the positive and negative energy continuum of states can uncover this difficulty. Real electrons clearly do not behave this way. There are two possible explanations for negative energy solutions.

3.8.1 Hole Theory

The major assumption of this hypothesis is, that the vacuuum is the many-body quantum state, in which all the negative-energy electron eigenstates are occupied, such that *Explain this with the diagram from relmodule04, figure 2.2.* With this assumption, the Dirac formalism ceases to be a one-particle theory. This assumption prevents any electron from falling into these negative energy states through the Pauli exclusion principle, and thereby ensures the stability of positive energy states. In turn, an electron of the negative energy continuum, the so called Dirac sea, may be excited to a positive energy state, thus an electron-hole pair is created, that is a positive energy electron, and a hole in the negative energy sea. This hole in the sea of negative energy electrons would respond to an electric field as though it were a positively charged particle - it is seen in nature as a positive energy positron. This theory therefore makes the following predictions besides the existence of the positron:

- The annihilation of an electron-positron pair. A posititve energy electron falls into a hole in the negative energy sea with the emission of radiation. *From energy momentum conservation at least two photons are emitted*

- Conversely, an electron-positron pair may be created from the vacuum by the excitement through radiation.

These predictions have all succesfully been observed. The theory nevertheless has the following problems. Although we started with a single-particle wave equation, the Dirac theory forces us into a many-particle interpretation, for which quantum mechanics with a fixed number of particles is inadequate. The existence of the Dirac sea implies an infinite positive electric charge filling all of space, one must therefore assume the bare vacuum to have an infinite negative charge density which is exactly cancelled by the Dirac sea. The Hole theory completely neglects the interactions between the electrons in the Dirac see, it neglects the inconsistensies by describing an infinitely high mass without any gravitative action and the apparent asymmetry between positrons and electrons seems furthermore to be arbitrary.

3.8.2 Feynman-Stückelberg-interpretation

The interpretation of the Dirac theory is replaced today by the Feynman-Stückelberg-interpretation (QED). Causality forces us to ensure that positive energy states propagate forwards in time. But if we force the negative energy states only to propagate backwards in time then we find a theory that is consistent with the requirements of causality and that has none of the aforementioned problems. A state of negative energy is interpreted as a state of positive energy, an opposite charge, a mirrored space and reversed timedirection. The wavefunction of an electron of negative energy corresponds to the wavefunction of a positron, which moves backwards in time through a mirrored space. *The reversed identification between positron and electron is also possible.* Therefore, between electron and positron there does exist the so-called CPT-symmetry (charge conjugation, parity and time reverse). Therefore, we should interpret the emission of a negative energy particle with momentum p^μ as the absorption of a positive energy antiparticle with momentum $-p^\mu$ or vice versa.
Why do we keep the Dirac equation for describing one-particle problems, even though it essentially describes a multi-particle theory ? Simply because the Dirac equation has made a vast number of true predictions. She gives the correct energy spectrum for Hydrogen and the g-factor of the electron up to very high precision. The predicted positron has been observed back in 1932. Dirac´s idea is furthermore directly applicable to solid state physics, where the valence band in a solid can be regarded as a ”sea” of electrons. Holes in this sea indeed occur, and are extremely important in order to understand the effects of semi-conductors.
Starting from the notion of modifying the Schrödinger equation with respect to special relativity we have found essentially two equations, one of which describes for example electrons with high precision, the Dirac equation, and the other one is used to describe spinless particles like pions. In both cases, we have to reinterpet the negative energy solutions to get stable ground states. This leads us to the notion of anti-particles for negative energy solutions and particles for positive energy solutions. The Hole theory implies a new fundamental symmetry of nature. For every particle exists an anti particle, the existence of electrons demands the existence of positrons, its experimental observations therefore justifies holding on to the Dirac equation.

4 Comments

4.1 Expectation and Eigenvalue

In Schroedinger theory, the eigenvalues are the expectation values of the operator for the corresponding eigenstates. For the spin zero equation this is no longer true. For example, the Hamiltonian in the free particle case has eigenvalues of $\pm E_p$, but expectation values of E_p only. Moreover, this value of E_p is sharp in the sense that the fluctuation vanishes. For the eigenstates of H,

$$< [H- < H >]^2 >= 0$$

One can also show, that $\frac{\partial H}{\partial p_k} = v_k$ is not an observable in relativistic spin zero theory, because it has eigenvalues zero by not forming a complete set (but has expectation value). One sees from this example that in this theory no longer have the direct phyical significance they have in nonrelativistic theory.

4.2 Splitting of KG equation

The Klein-Gordon equation admits of a single particle interpretation in which the particle necessarily possesses a charge degree of freedom. To every solution with a given sign of charge there is a corresponding charge conjugate solution with the opposite sign and both of these solutions are needed to give a complete description of possible solutions of the Klein-Gordon equation representing a particle of a given sign of charge. One can find a representation where the wave equation for a free particle seperates into two uncoupled equations (brakes down for existing em. field)m one describing a particle of positive charge, the second a particle of negative charge. The coupling between the two equations is an effect of the polarization of the vacuum.

4.3 Charge interpretation

We have considered the Klein-Gordon and Dirac equation to be single particle equations in which the particle also possesses the charge degree of freedom. There are limitations to such an interpretation which arise when many particle phenomena intervene, as in the presence of strong fields.

4.4 Zitterbewegung

The "Zitterbewegung" is an oscillatory motion a Dirac wavepacket undergoes whilst moving with a uniform rectilinear motion of velocity (group velocity). It is caused by interference between the posititve and negative energy components of the wave packet. The "Zitterbewegung" term vanishes if the wavepacket is a superposition of only positive or only negative energy waves. The frequency of these oscillations is very high, larger than

$$2m\frac{c^2}{\hbar} \approx 2 \times 10^{21}\mathrm{s}^{-1}$$

4.5 Dirac equation in its original form

The four by four hermitian matrices β and α^i yield the original Dirac equation in natural units:

$$i\hbar\partial_t\psi = \left(-i\hbar c\vec{\alpha} \cdot \vec{\nabla} + \beta mc^2\right)\psi \tag{18}$$

The matrices obey:

$$\gamma^0 = \beta$$
$$\gamma^i = \beta\alpha^i$$
$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

4.6 Electromagnetic coupling with the classical Dirac equation

This can be cast into a more instructive version of the form

$$\begin{aligned} i\partial_t\psi &= \left[\vec{\alpha} \cdot \left(-i\vec{\nabla} - e\vec{A}\right) + \beta m + eA^0\right]\psi \\ &= \left(\vec{\alpha} \cdot \vec{p} + \beta m\right)\psi + \left(-e\vec{\alpha} \cdot \vec{A} + eA^0\right)\psi \\ &= \left(H_0 + H_{int}\right)\psi \end{aligned}$$

We note a strong resemblance of the interaction part H_{int} with the hamiltonian of a classical particle in an external field $-e\vec{v}\cdot\vec{A}+eA^0$, in agreement with the interpretation of $\vec{\alpha}$ as a velocity operator, because from Heisenbergs equation of motion we find $\frac{d\vec{r}}{dt} = c\vec{\alpha}$ (not constant, because $[\vec{\alpha}, H] = 0$)

4.7 Feynman Stückelberg interpretation

In order to get more familiar with this picture, consider a process with a + and a photon in the initial state and final state. In figure 2.1(a) the + starts from the point A and at a later time t1 emits a photon at the point x1. If the energy of the + is still positive, it travels on forwards in time and eventually will absorb the initial state photon at t2 at the point x2. The final state is then again a photon and a (positive energy) +. There is another process however, with the same initial and final state, shown in figure 2.1(b). Again, the + starts from the point A and at a later time t2 emits a photon at the point x1. But this time, the energy of the photon emitted is bigger than the energy of the initial +. Thus, the energy of the + becomes negative and it is forced to travel backwards in time. Then at an earlier time t1 it absorbs the initial state photon at the point x2, thereby rendering its energy positive again. From there, it travels forward in time and the final state is the same as in figure 2.1(a), namely a photon and a (positive energy) +. In today's language, the process in figure 2.1(b) would be described as follows: in the initial state we have an + and a photon. At time t1 and at the point x2 the photon creates a +-− pair. Both propagate forwards in time. The + ends up in the final state, whereas the − is annihilated at (a later) time t2 at the point x1 by the initial state +, thereby producing the final state photon. To someone observing in real time, the negative energy state moving backwards in time looks to all intents and purposes like a negatively charged pion with positive energy moving forwards in time. We have discovered anti-matter!