

QUANTUM FIELD THEORY

THIMO PREIS[†]

15th October 2019

† thimo.preis@posteo.de

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RELATIVISTIC THEORY AND ITS TENSOR STRUCTURE

We use metric signature $(-, +, +, +)$ such that $\partial_\mu = (\partial_0, \vec{\nabla})$, $\partial^\mu = (\partial_0, -\vec{\nabla})$ and

$$\partial_\mu \partial^\mu = \partial_0 \partial^0 + \partial_i \partial^i = \eta^{00} \partial_0 \partial_0 + \eta^{ii} \partial_i \partial_i = \partial_0^2 - \vec{\nabla}^2. \quad (1.1)$$

We write $\phi(p) = \mathcal{F}[\phi(x)]$ for the Fourier transform of $\phi(x)$, i.e.

$$\phi(p) = \mathcal{F}[\phi(x)](p) := \int d^d x \phi(x) e^{ipx} \quad (1.2)$$

$$\phi(x) = \mathcal{F}^{-1}[\phi(p)](x) := \int \frac{d^d p}{(2\pi)^d} \phi(p) e^{-ipx}. \quad (1.3)$$

The Delta distribution is the inverse Fourier transform of the constant function $\phi(p) \equiv 1$

$$\delta(x - y) = \int \frac{d^d p}{(2\pi)^d} e^{-ip(x-y)}. \quad (1.4)$$

This $\delta(x)$ is the only function we do not use the Fourier notation for, i.e. $\delta(p)$ is not the Fourier transform of $\delta(x)$ (which is $\mathcal{F}[\delta(x)](p) \equiv 1$) but the same function with argument p , i.e. sometimes we write

$$\int d^d x e^{ix(p-q)} = (2\pi)^d \delta(q - p) = (2\pi)^d \delta(p - q), \quad (1.5)$$

which is just a relabelling of $q \leftrightarrow x$ and $p \leftrightarrow y$.

The (infinite) volume of spacetime can then be written in terms of the Delta function as

$$\text{vol } \mathbb{R}^{1,d-1} = \int_{\mathbb{R}^{1,d-1}} d^d x = (2\pi)^d \delta(0). \quad (1.6)$$

To motivate looking into classical field theory on our search for a fundamental theory, we begin with the realization that the Schrödinger equation

$$i\partial_t \psi(x) = - \left[-\frac{1}{2m} \nabla^2 + V(x) \right] \psi(x) \quad (1.7)$$

is linear in ∂_0 but quadratic in ∂_i and therefore not Lorentz forminvariant, i.e. it must fail for high energies. A warning: this motivation of abandoning the Schrödinger equation will end in a plot twist: we will eventually understand that the Schrödinger equation in its most general form

$$i\partial_t |\psi\rangle = H |\psi\rangle \quad (1.8)$$

is really fundamental and can in fact be Lorentz forminvariant if the Hamiltonian transforms like a 0-component because then

$$i\partial_0 |\psi\rangle = P^0 |\psi\rangle. \quad (1.9)$$

Indeed in quantum field theory it is true that $H = P^0$ where P^μ is the momentum operator, the collection of charges associated to spacetime translation invariance, and $|\psi\rangle$ are tensor products of scalars, vectors and spinors under unitary representations. A more physical perspective of why the Schrödinger equation in the form of 1.7 must fail is the fact that its Greens function for $V = 0$ does not vanish for spacelike $(x - y)^2 < 0$, such that it describes superluminal propagation.

The first aim of this section will be to extend Lagrangian point mechanics to fields and to discuss which physical quantities necessarily arise in a translation- and Lorentz invariant Lagrangian theory. We will find the Energy momentum tensor and angular momentum tensor. The second aim of this section will be to find a procedure which allows us to discuss all possible Lorentz invariant classical equations of motion. For this we will make use of the group theory of Lorentztransformations and its representation theory. We will classify all representations of the Lorentzgroup and interpret them physically by introducing the idea of spinors. With the help of these representations we will be able to construct Lorentzscalars by means other than the already known contraction of 4-vectors, which will be left over as the special case of the fundamental representation. Eventually we will be able to discuss physics by constructing Lagrangians out of all possible Lorentzscalars obtained in the spinor language and discussing their implied equations of motion. As a reaffirming byproduct we will completely recover Maxwell's electrodynamics.

1.1 THE PRINCIPLE OF RELATIVITY

Postulate

Laws of nature are the same for all observers at different positions and times, facing different directions or moving at different velocities.

1.2 LORENTZ INVARIANT INTEGRALS

$$d^4 p \rightarrow \det \Lambda d^4 p = d^4 p \quad (1.10)$$

$$\theta(p^0) \rightarrow \theta(\Lambda_\mu^0 p^\mu) = \theta(p^0) \quad (1.11)$$

$$\int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} (\dots)|_{p^0=E_p} \quad (1.12)$$

$$= \int \frac{d^4 p}{(2\pi)^4} (\dots) 2\pi \delta(p^2 - m^2) \theta(p^0) \text{ is invariant under } L_+^\uparrow$$

$$2E_p \delta(\vec{p} - \vec{q}) = \int dp^0 \delta(p^2 - m^2) \theta(p^0) \text{ is invariant under } L_+^\uparrow \quad (1.13)$$

Some useful identities:

$$\delta(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} \quad (1.14)$$

$$\delta(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} e^{-i\vec{p}\cdot\vec{x}} \quad (1.15)$$

$$\theta(x^0) = i \lim_{\epsilon \rightarrow 0} \int \frac{dp^0}{2\pi} \frac{1}{p^0 + i\epsilon} e^{-ip^0 x^0}. \quad (1.16)$$

1.3 LAGRANGIAN FORMALISM

Euler-Lagrange equations

The equations of motion of a field $\phi(x)$ are given by the Euler-Lagrange equations of a lorentz invariant Lagrange density $\mathcal{L} = \mathcal{L}(\phi, \partial^\mu \phi)$ per independent field ϕ :

$$\frac{\partial \mathcal{L}(\phi(x), \partial_\mu \phi(x))}{\partial \phi(x)} = \partial_\mu \left(\frac{\partial \mathcal{L}(\phi(x), \partial_\mu \phi(x))}{\partial (\partial_\mu \phi(x))} \right), \quad (1.17)$$

$$\frac{\delta S}{\delta \phi^i(x)} = \frac{\partial \mathcal{L}}{\partial \phi^i(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^i)} \stackrel{!}{=} 0, \quad (1.18)$$

with $\partial\phi \equiv \partial_\nu \phi(x^\mu)$.

The Lagrangian equations 1.17 are equivalent to the Hamiltonian principle

$$\frac{\delta S[\phi]}{\delta \phi(x)} = 0 \text{ with } S[\phi] = \int d^4 x \mathcal{L}. \quad (1.19)$$

Note that the derivation of the EL eqns 1.17 is as follows:

$$\delta S = \delta \int d^4x \mathcal{L} = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \partial_\mu \phi \right) \quad (1.20)$$

$$\stackrel{PI}{=} \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \delta \phi \quad (1.21)$$

$$\Leftrightarrow \frac{\delta S}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)}. \quad (1.22)$$

Note further that the assumption of vanishing boundary terms for the Least Action Principle does not hold in general for non-trivial space-times. However, in these notes we are only discussing QFTs embedded in a Minkowskian spacetime where this assumption is valid.

1.3.1 Noether's theorem

A *symmetry* is a field transformation by which \mathcal{L} changes at most by a total derivative such that the action stays invariant. This ensures that the equations of motion are also invariant.

Every continuous transformation of a field

$$\phi \rightarrow \phi + \epsilon \delta \phi + \mathcal{O}(\epsilon^2) \quad (1.23)$$

induces a change in the Lagrangian

$$\mathcal{L} \rightarrow \mathcal{L} + \epsilon \delta \mathcal{L} + \mathcal{O}(\epsilon^2) \text{ with } \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi). \quad (1.24)$$

A transformation that leaves the Euler-Lagrange equations of motion 1.17 invariant induces a change of the form $\delta \mathcal{L} = \partial_\mu F^\mu$ and is called a symmetry.

Noether's theorem

Every continuous symmetry in the above sense gives rise to a Noether current $j^\mu(x)$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - F^\mu \text{ with } \delta\mathcal{L} =: \partial_\mu F^\mu \quad (1.25)$$

such that it is conserved on-shell, i.e.

$$\partial_\mu j^\mu = \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} \right) \delta\phi, \quad (1.26)$$

or

$$\partial_\mu j^\mu(x) = 0 \Leftrightarrow \frac{\partial j^0}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0. \quad (1.27)$$

Lemma: Every continuous symmetry whose current j^μ obeys

$$j^i \xrightarrow{|\vec{x}| \rightarrow \infty} 0 \quad (1.28)$$

sufficiently fast enough gives rise to a locally conserved (time conserved in a fixed reference frame), i.e. locally conserved charge Q

$$\frac{dQ}{dt} = 0 \text{ with } Q = \int d^3 j^0. \quad (1.29)$$

The condition 1.28 can be violated in the presence of massive scalars that induce long range forces by spontaneously broken symmetries.

This conserved current can be constructed for a continuous symmetry via

$$\partial^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} X - F^\mu, \quad (1.30)$$

$$j^\mu = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \delta\phi^i - F^\mu \quad (1.31)$$

with the transformation in ϕ and \mathcal{L} :

$$\phi \rightarrow \phi + \epsilon \delta\phi \Rightarrow \delta\phi = X(\phi, \partial_\mu \phi) \quad (1.32)$$

$$\mathcal{L} \rightarrow \mathcal{L} + \epsilon \delta\mathcal{L} \Rightarrow \delta\mathcal{L} = \partial_\mu F^\mu \quad (1.33)$$

$$\Rightarrow \partial\mathcal{L} = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial\phi \right] + \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \delta\phi. \quad (1.34)$$

Symmetry

$\delta\phi$ is a symmetry if the Lagrangian changes by a total derivative:

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} \Rightarrow F^\mu = \text{constant} = 0, \text{ since } \delta\mathcal{L} = \partial_\mu F^\mu. \quad (1.35)$$

Every continuous symmetry whose associated Noether current satisfies $j^i(t, \vec{x}) \rightarrow 0$ sufficiently fast for $|\vec{x}| \rightarrow \infty$ gives rise to a *conserved charge* Q

$$Q = \int_{\mathbb{R}^3} d^3x j^0(t, \vec{x}) \quad (1.36)$$

$$\frac{dQ}{dt} = \int_{\mathbb{R}^3} d^3x \partial_t j^0(t, \vec{x}) = - \int_{\mathbb{R}^3} d^3x \partial_i j^i(t, \vec{x}) = 0 \quad (1.37)$$

Therefore, any charge leaving the volume V must be accounted for by a flow of the current \vec{j} out of the volume:

$$\frac{dQ_V}{dt} \stackrel{\frac{dV}{dt}=0}{=} - \int_V dV \vec{\nabla} \cdot \vec{j} = - \int_{\partial V} \vec{j} \cdot d\vec{s}, \quad (1.38)$$

which reflects *local charge conservation*. This holds in any local field theory.

A generalization to symmetries with multiple parameters is

$$\phi \rightarrow \phi + \epsilon^\nu \delta\phi_\nu \quad (1.39)$$

$$\mathcal{L} \rightarrow \mathcal{L} + \epsilon^\nu \delta\mathcal{L}_\nu \quad (1.40)$$

$$\partial_\mu (j^\mu)^\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi^\nu - F^{\mu\nu} \text{ with } \delta\mathcal{L}_\mu =: \partial_\mu (F^\mu)_\nu. \quad (1.41)$$

Proof of the general formula 1.41:

$$\begin{aligned} \delta\mathcal{L}_\nu &= \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi)_\nu \\ &= \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi)_\nu + \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi_\nu - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi_\nu \\ &\Leftrightarrow \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi_\nu = \delta\mathcal{L}_\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta\phi_\nu - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi_\nu \\ &= \delta\mathcal{L}_\nu - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi_\nu \right) \\ &= \partial_\mu \left((F^\mu)_\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi_\nu \right). \end{aligned}$$

Proof of the conservation 1.36:

$$\begin{aligned} 0 &= \partial_\mu j^\mu \\ \Leftrightarrow 0 &= \int d^3x \partial_\mu j^\mu = \int d^3x (\partial_0 j^0 + \partial_i j^i) = \frac{d}{dt} \int d^3x j^0 - \int d^3x \vec{\nabla} \cdot \vec{j}. \end{aligned}$$

But the integral over the spatial derivative vanishes by Gauss' law if \vec{j} vanishes at infinity

$$\begin{aligned} \int_{\mathbb{R}^3} d^3x \vec{\nabla} \cdot \vec{j} &= \int_{\partial \mathbb{R}^3} d^2s \vec{n} \cdot \vec{j} \stackrel{1.28}{=} 0 \\ \Rightarrow \frac{d}{dt} \int d^3x j^0 &= 0. \end{aligned}$$

1.3.1.1 Energy-momentum tensor

The global spacetime translation transformation gives rise to a conserved current in every component of the transformation. This yields the *canonical energy-momentum tensor*, the conserved Noether current associated with spacetime translations:

Energy-momentum tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}, \quad (1.42)$$

with

$$\partial_\mu T^{\mu\nu} = 0 \quad \text{on - shell.} \quad (1.43)$$

The currents $(j^\mu)_\nu$ induced by a global space-time translation

$$\phi(x) \rightarrow \phi(x - \epsilon) = \phi(x) - \epsilon^\nu \partial_\nu \phi(x) + O(\epsilon^2) \quad (1.44)$$

$$\mathcal{L} \rightarrow \mathcal{L} - \epsilon^\mu \partial_\nu (\delta_\mu^\nu \mathcal{L}) + \epsilon^2 \quad (1.45)$$

are compactly written as the so-called energy momentum tensor

$$T^{\mu\nu} := (j^\mu)^\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}, \quad (1.46)$$

where \mathcal{L} is the on-shell Lagrangian. The implied charges $(Q)^\mu =: P^\mu$ are

$$\frac{d}{dt} P^\mu = 0 \text{ with } P^\mu := \int d^3x T^{0\mu} \quad (1.47)$$

and can be identified with the 4-momentum, i.e. 4-momentum is locally conserved if global space-time translation is a symmetry. $T^{\mu\nu}$ is conserved on-shell in its first index, i.e.

$$\partial_\mu T^{\mu\nu} = 0. \quad (1.48)$$

These 4 equations are invariant under a redefinition of $T^{\mu\nu}$ in the form of

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\sigma K^{\sigma\mu\nu} \quad (1.49)$$

with arbitrary gauge $K^{\sigma\mu\nu} = -K^{\mu\sigma\nu}$.

The energy momentum tensor symmetrized in this way is sometimes called the *Belinfante* tensor.

[1.49](#) is used to symmetrize $T^{\mu\nu}$. The canonical derivation via [1.49](#) usually does not lead to a symmetric energy momentum tensor. A symmetrization is necessary before $T^{\mu\nu}$ can be used as the left hand side of Einstein's field equations since its right side is symmetric. Compare the exploration of the canonical and least action energy momentum tensor in the GR chapter.

This ambiguity in $T^{\mu\nu}$ can lead to conformal theories with non-vanishing T_μ^μ .

In a Hamiltonian formalism the 0-component of the 4-momentum defined by [1.47](#) can be identified with the Hamiltonian.

The canonical derivation via The conserved charges then are

1. Energy

$$H = E = \int_{\mathbb{R}^3} d^3x T^{00} \quad (1.50)$$

associated with time translation invariance.

2. Spatial momentum

$$P^i = \int_{\mathbb{R}^3} d^3x T^{0i} \quad (1.51)$$

associated with *spatial translation invariance*. \Rightarrow Conserved 4-momentum.

3. E.g. for the free scalar field

$$P^\mu = \int_{\mathbb{R}^3} d^3x T^{0\mu}, \quad \frac{dP^\mu}{dt} = 0 \quad (1.52)$$

$$E = \int_{\mathbb{R}^3} d^3x \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \right] \quad (1.53)$$

$$\vec{p} = -\int_{\mathbb{R}^3} d^3x (\dot{\phi}\vec{\nabla}\phi). \quad (1.54)$$

1.3.2 Canonical angular momentum tensor and centre of energy

Infinitesimal Lorentz transformation

$$\phi(x) \rightarrow \phi(\Lambda^{-1}x) = \phi(x) - \omega_\nu^\mu x^\nu \partial_\mu \phi(x) \text{ with } \omega_{\mu\nu} = -\omega_{\nu\mu} \quad (1.55)$$

$$\mathcal{L} \rightarrow \mathcal{L} - \omega_\nu^\mu x^\nu \partial_\mu \mathcal{L} \quad (1.56)$$

leads to the angular momentum tensor

$$\mathcal{J}^{\mu\nu\rho} := (j^\mu)^{\nu\rho} \quad (1.57)$$

with the angular momentum

$$J_i := \epsilon_{ijk} \mathcal{J}^{0jk}. \quad (1.58)$$

1.4 HAMILTONIAN FORMALISM

For a given Lagrangian density, we define the Hamiltonian density \mathcal{H} as the Legendre transform

$$\mathcal{H}(\pi, \phi) := \phi \partial_0 \phi - \mathcal{L} \text{ with } \phi(x) := \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \quad (1.59)$$

with the canonical conjugated momentum density $\pi(x)$. The Hamiltonian H then simply is

$$H := \int d^3x \mathcal{H}. \quad (1.60)$$

These definitions coincide with the definitions of the energy momentum tensor, such that

$$\mathcal{H} = T^00, \quad H = P^0. \quad (1.61)$$

The connection between the interaction Hamiltonian and the Lagrangian is

$$H_{int} = - \int d^3x \mathcal{L}_{int}, \quad (1.62)$$

which can be seen by

$$\begin{aligned} H_{int} &= H - H_0 = \int d^3\mathcal{H} - \int d^3x \mathcal{H}_0 \\ &= \int d^3x (\pi \partial_0 \phi - \mathcal{L}) - \int d^3x (\pi \partial_0 \phi - \mathcal{L}_0) \\ &= - \int d^3x (\mathcal{L} - \mathcal{L}_0) = - \int d^3x \mathcal{L}_{int}. \end{aligned}$$

Hamiltonian equations

The Lagrange equations are equivalent to the Hamiltonian equations

$$\dot{\phi} = + \frac{\partial \mathcal{H}}{\partial \pi} - \partial^\mu \frac{\partial \mathcal{H}}{\partial (\partial^\mu \pi)} \quad (1.63)$$

$$\dot{\pi} = - \frac{\partial \mathcal{H}}{\partial \phi} + \partial^\mu \frac{\partial \mathcal{H}}{\partial (\partial^\mu \phi)}. \quad (1.64)$$

2

GROUP THEORY AND REPRESENTATION OF THE LORENTZ GROUP

2.1 GROUP THEORY

2.1.1 *Introduction*

Symmetries in physics will be related to groups and will constrain the possible terms in the Lagrangian. This means that the theory can be solved exactly in particular cases, this property is called *integrability*. The following continuous symmetries are realized in nature:

1. Rotational symmetry $SO(3)$ (i.e. rotate lab or experiment, invariant)
2. Lorentz symmetry $SO(3, 1)$ (experiment in inertial system same as in laboratory system)
3. Gauge, Flavour symmetry e.g. $SU(3)$ (scattering experiments are invariants)

Continuous symmetries are represented by Lie groups. Discrete symmetries on the other hand are

1. Parity $\vec{x} \rightarrow -\vec{x}$
2. charge conjugation $e^- \rightarrow e^+$
3. time reversal $t \rightarrow -t$

Note that the CPT symmetry always holds for the weak force, compare [3.30](#). CP symmetry holds valid for EM and QCD.

2.1.2 *Lie-Group, Lie-Algebras and their Representations*

A Lie group is a group which is also a differentiable manifold, a group whose elements can be labelled at least locally by a set of continuous variables. Everything (multiplication, inverse, representation etc) should depend smoothly on these parameters. The dimension of a Lie group is given by the number of these parameters.

$O(n), U(n)$

Lie group

A *Lie-group* is a group with the structure of a smooth manifold, i.e. its differentiable (in the sense that it is equipped with a differentiable map) structure is compatible with the group. Lie groups are therefore groups which are also manifolds, such that the group action is a diffeomorphism.

The map goes from parameter space in \mathbb{R}^n to the manifold and thus assigns an element of parameter space to an element of the group, i.e. the manifold. The group action $g_{\alpha_1}g_{\alpha_2} = g_{\alpha_3}, g_{\alpha} \in G$, induces a link between elements of parameter space. The map satisfies the group laws.

2.1.2.1 Lie algebras

The Lie algebra generates (i.e. it is not part of the Lie group) elements of the Lie group infinitesimally close to the identity.

Lie-algebra

A Lie-algebra $\mathfrak{g} = \text{Lie}G$ is a vector space with a bilinear antisymmetric operation

$$\circ : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}, \quad (a, b) \mapsto [a, b] \quad (2.1)$$

with the properties

$$i) \quad A \circ B = -B \circ A \Rightarrow 0 = [,] \quad (2.2)$$

$$ii) \quad A \circ (B \circ C) + B \circ (C \circ A) + C \circ (A \circ B) = 0. \quad (2.3)$$

A Lie-algebra of finite dimension is defined by $[,] = 0$ on the basis of the vector space, i.e. with

$$x_A \circ x_B = f_{AB}^C x_C, \quad (2.4)$$

where f_{AB}^C are called *structure constants* of the Lie algebra. They have the properties:

$$i) \quad f_{AB}^C = -f_{BA}^C \quad (2.5)$$

$$ii) \quad f_{AB}^D f_{CD}^E + f_{CA}^D f_{BD}^E + f_{BC}^D f_{AD}^E = 0. \quad (2.6)$$

In general, a basis T^A of a Lie algebra satisfies

$$[T^A, T^B] = \sum_C i f_C^{AB} T^C. \quad (2.7)$$

2.1.2.2 Example on how to count these parameters

Consider $SO(3)$, it describes rotations around a given axis in \mathbb{R}^3 , hence in the (x, y) , (y, z) , (x, z) plane. These rotations can be generated by the basis elements, i.e. the generators, of the associated Lie group $so(3)$. This basis is the basis for a $3d$ vector space of antisymmetric 3×3 matrices. We make an ansatz for the elements of $SO(3)$:

$$A = \mathcal{I} + \epsilon a + O(\epsilon^2) \quad (2.8)$$

such that we find

$$A^T A = \mathcal{I} = \mathcal{I} + \epsilon(a + a^T) + O(\epsilon^2) \Leftrightarrow a^T = -a$$

such that the antisymmetric matrices form the basis for the vector space, as we stated above

We do not in general get all elements of the Lie group by means of generating them with an exponential map, as we can only map to those elements which are connected to the identity, i.e. the exponential map is not bijective.

However, it should normally suffice, but for the spinor representation of $SU(2)$ we only get a pseudo representation of $SO(3)$ by means of the exponential map, since $SU(2)$ is the double cover of $SO(3)$, cf. below. Now onto counting:

The dimension of a Lie algebra as a vector space is the dimension of the Lie group, which then again is equal to the number of parameters.

Therefore, we can find the dimension of the Lie group, i.e. the number of parameters by e.g. counting the number of basis elements of the associated Lie algebra.

Example: What is the dimension of the vector space of $n \times n$ antisymmetric matrices. We have $a_{\mu\nu} = -a_{\nu\mu}$ plus their vanishing trace, which means that we have to add $(n-1) + (n-2) + \dots + 1$ on both sides of the diagonal and divide by two such that we don't overcount: $\dim SO(n) = \frac{n(n-1)}{2}$.

2.1.3 Representations

Symmetries in QM correspond to unitary operators acting on the Hilbert space of states, compare Wigner's theorem. The symmetry is represented by a (Lie) group, whilst the operator has to *represent* the symmetry group. The operator has to respect the group structure, i.e. closure.

Representations

A representation of dimension $p \in [0, \infty]$ of a group G assigns to each element $g \in G$ a $p \times p$ matrix $D(g)$ acting on a p dimensional vector space called the *representation space*, such that it satisfies the group structure

$$D(g_1 \cdot_G g_2) = D(g_1) \cdot_{matrix\ multpl.} D(g_2) \quad (2.9)$$

$$D(g^{-1}) = [D(g)]^{-1} \quad (2.10)$$

$$D(e) = \mathcal{I}_{p \times p}. \quad (2.11)$$

A function

$$D(g) : V \rightarrow V, \quad (2.12)$$

is called *representation* of the group G with the vector space being the representation space, if for every $g \in G$ we define a linear operator $D(g) : V \rightarrow V$ which realizes the group G on V and which is compatible with the group structure.

Note again that the symmetry is the group and that the particles will be the representations.

Note that the Lie algebra is given by the tangent space defined at one specific point of the manifold, i.e. the Lie group. Thus, Lie algebras are the tangent space to the identity $e \in G$ of the Lie group G . The exponential map is then the key concept relating Lie groups and Lie algebras, since they represent a way of parallel transporting the tangent space to another group element, hence to another point on the manifold.

2.1.3.1 Irreducible representations

Why are irreducible representations so important ?

Schur's lemma

If a matrix B is such that $[B, D(g)] = 0 \forall g \in G$ and with D the irreducible representation of G , then $B = \lambda \mathcal{I}$ with $\lambda \in \mathbb{C}$.

Relation to physics

Any symmetry in physics should commute with the Hamiltonian

$$[H, D(g)] = 0. \quad (2.13)$$

If D is irreducible, then we have

$$H = E\mathcal{J} \quad (2.14)$$

in the representation space.

Thus, states in an irreducible representation of an *exact* symmetry, i.e. the commutator vanishes exactly, form a *multiplet* of the same mass or energy. Then, the dimension of the irreducible representation is equivalent to the *degeneration of the multiplet*.

Example:

Pion has three different states $\Pi^{\pm, 0}$, which makes up a multiplet as it is an irreducible representation of $SU(2) \Rightarrow H = E\mathcal{J} \Rightarrow \Pi^{\pm, 0}$ need to have the same mass. However, experimentally this symmetry was only almost exact, i.e. the commutator didn't vanish completely $[D, H] \ll 1$, then H is not diagonal and the three states don't have the same mass. Note that this almost multiplet has dimension 3.

\mathcal{L} is invariant under a group G implies that the spectrum, i.e. the eigenvalues of the Hamiltonian, will fall in representations of G .

2.1.4 Young-Tableaux

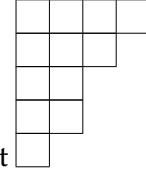
YT is the systematic way to find the sum of irreducible representations if given the tensor product of representations.

1. Pictorial way to characterize irreducible representations (only of $SU(N)$ here)
2. Correspond to a particular symmetrization and antisymmetrization of indices
3. Generate irreps. and it gives directly their dimension

Insert boxes
for Young
tableau

Definition of YT

A YT corresponding to a *tensor* with n indices is an arrangement



of n boxes with rows aligned to the left, such that

1. Each row must contain *no more* boxes than the row above
2. Number of rows must not exceed N for $SU(N)$ irreps ($i = 1, \dots, N$) (since column boxes represent antisymmetrization of indices, i.e. you cannot antisym. $> N$ indices if each index runs over N objects.)

Some examples

1. Fundamental representation $\phi^i \rightarrow U_j^i \phi^j \Rightarrow \square, n = 1$
2. Symmetric representation $\phi^{(ij)} = \phi^{(ji)} \rightarrow U_{i'}^j U_{j'}^i \phi^{(i'j')} \Rightarrow \square\square$
such that we get $n = 2$ boxes in a row since symmetric.
3. Antisymmetric representation $\phi^{[ij]} = -\phi^{[ji]} \Rightarrow \square\square$, i.e. $n = 2$ boxes in a column since antisymmetric
4. Consider $D_{symmetric} \otimes D_{fundamental}$,

$$\phi^{(ij)} \psi^k \rightarrow U_{i'}^i U_{j'}^j U_{k'}^k \phi^{(i'j')} \psi^{k'} \quad (2.15)$$

is not an irreps, since it decomposes into two irreps by symmetrizing all possible indices as follows:

$$\phi^{(ij)} \psi^k = \underbrace{\frac{1}{3} [\phi^{(ij)} \psi^k + \phi^{(jk)} \psi^i + \phi^{(ki)} \psi^j]}_{I. irrep} + \underbrace{\frac{1}{3} 2\phi^{(ij)} \psi^k - \phi^{(jk)} \psi^i - \phi^{(ki)} \psi^j}_{II irrep} \quad (2.16)$$

which then is represented by

$$\underbrace{\square\square}_{\phi^{(ij)}} \otimes \underbrace{\square}_{\psi^k} = \underbrace{\square\square\square}_{=I.=\phi^{(ij)}\psi^k} \oplus \underbrace{\square\square}_{=II.}, \quad (2.17)$$

thus it decomposes into two irreps.

Row indices will be symmetrized and column indices will be antisymmetrized.

What anti-/symmetrization of indices does each YT. represent?

How to set up a YT

Each YT corresponds to a unique anti-/symmetrization procedure. E.g. for the tensor $\psi^{a_1 \dots a_n}$ with $a_j = 1, \dots, N$, assign each index to a box from *left to right and then from top to bottom*, i.e.

$$\phi^{ijk} \leftrightarrow \begin{array}{|c|c|} \hline i & j \\ \hline k \\ \hline \end{array} . \quad (2.18)$$

1. Step Apply the symmetrization operator $P = \sum_{\text{rows}} p$, p is a permutation of the indices in a row, i.e. *symmetrize the indices in a row*. Note that you can symmetrize infinite indices in the row but can only antisymmetrize up to N indices in a column.
2. Step Apply the antisymmetrization operator $Q = \sum_{\text{columns}} \text{sgn}(q)q$, where q is a permutation of the column indices, i.e. *antisymmetrize the indices in a column*.
 1. After 1., all the rows are symmetrized but after 2. this symmetry is "destroyed" and we are left with fully antisymmetrized column indices.
 2. These will produce all the irreducible representations of $SU(N)$
 3. $YT = \hat{Q}\hat{P}(\text{tensor})$.

As an example, consider the three index tensor ϕ^{ijk} and its three different possible YTs:

1. $\begin{array}{|c|c|} \hline i & j \\ \hline k \\ \hline \end{array}$, i.e. something with mixed symmetry, where we assigned the indices from left to right and then from top to bottom. Now we apply the first step and symmetrize the row indices i, j

$$\hat{P}(\phi^{ijk}) = \frac{1}{2} (\underbrace{\phi^{ijk}}_{\begin{array}{|c|c|} \hline i & j \\ \hline k \\ \hline \end{array}} + \underbrace{\phi^{jik}}_{\begin{array}{|c|c|} \hline j & i \\ \hline k \\ \hline \end{array}}). \quad (2.19)$$

For the second step, we antisymmetrize the column indices i, k and j, k

$$\hat{Q}\hat{P}(\phi^{ijk}) = \frac{1}{4} (\underbrace{\phi^{ijk} - \phi^{kji}}_{\begin{array}{|c|c|} \hline i & j \\ \hline k \\ \hline \end{array} \Leftarrow \begin{array}{|c|c|} \hline j & i \\ \hline k \\ \hline \end{array}} + \underbrace{\phi^{jik} - \phi^{kij}}_{\begin{array}{|c|c|} \hline j & i \\ \hline k \\ \hline \end{array} \Leftarrow \begin{array}{|c|c|} \hline i & j \\ \hline k \\ \hline \end{array}}). \quad (2.20)$$

Comparing with 2.16, we see that we exactly get the identification with the $II.$ irreducible representation,i.e.

$$\phi^{ijk} - \phi^{kji} \rightarrow 2\phi^{(ij)}\psi^k, \quad \phi^{jik} \rightarrow \phi^{(kj)}\psi^i, \quad \phi^{kij} \rightarrow \phi^{(ki)}\psi^j. \quad (2.21)$$

2. $\boxed{i \mid j \mid k} \Rightarrow Y = QP(\phi^{ijk}) = \phi^{(ijk)}$ the fully symmetrized YT



3. $\boxed{\begin{matrix} i \\ j \\ k \end{matrix}} \Rightarrow Y = QP(\phi^{[ijk]}) = \phi^{[ijk]}$ the fully antisymmetrized YT



Lecturer sometimes ignores $1/m!$ factors in the anti/symmetrizations.

In $SO(N)$, lower and higher indices can be contracted, we don't consider it in this class.

Then, 1, 2, 3 are the three possible YTs for this $n = 3$ tensor.

Note that symmetrization and antisymmetrizations form invariant subspaces under the $SU(N)$.

2.1.4.1 How to count dimensions ?

As a motivating example, what is the dimension of the mixed symmetric representation 1?

Remember that

$$\boxed{\square} \otimes \boxed{\square} = \underbrace{\boxed{\square \square}}_{\phi^{ij}} \oplus \underbrace{\boxed{\begin{matrix} & \\ & \end{matrix}}}_{\phi_A^{[ij]}} \quad (2.22)$$

has the dimension

$$\dim(N \otimes N) = N^2$$

$$\dim(\boxed{\square \square} \oplus \boxed{\begin{matrix} & \\ & \end{matrix}}) = \dim(\boxed{\square \square}) + \dim(\boxed{\begin{matrix} & \\ & \end{matrix}}) = \frac{1}{2}N(N+1) + \frac{1}{2}N(N-1) = N^2.$$

How does this work in general ?

Dimension is here meant in the sense of dimension of the vector space of the irreducible representations which are labelled by YT

Dimension of YT

Consider $SU(N)$. We begin by labelling the boxes according to the rule *moving in the row increases label by one, moving in the column decreases the label by one*.

N	$N+1$	$N+2$
$N-1$	N	
$N-2$		

The dimension

sion of the representation space is then computed as follows

$$\dim = \frac{\prod \text{all the numbers in the YT}}{\prod_{x \in YT} \text{hook}(x)}, \quad (2.23)$$

where the $\text{hook}(x)$ for an x in a box in a YT is defined as follows
 $\text{hook}(x) = \# \text{ boxes to the right (in the same row)} + \# \text{ boxes below (in the same column)} + 1$ for x itself.

1. Example 1, $SU(6)$

$$\begin{array}{c} \boxed{} \\ \boxed{} \\ \boxed{} \\ \boxed{} \\ \boxed{} \end{array}
 = \boxed{\begin{matrix} 6 & 7 & 8 & 9 \\ 5 & 6 \\ 4 & 5 \\ 3 \\ 2 \end{matrix}}
 \Rightarrow \dim SU(6) = \boxed{\begin{matrix} 6 & 7 & 8 & 9 \\ 5 & 6 \\ 4 & 5 \\ 3 \\ 2 \end{matrix}} / \boxed{\begin{matrix} 8 & 5 & 2 & 1 \\ 5 & 2 \\ 4 & 1 \\ 2 \\ 1 \end{matrix}} = 1701$$

(2.24)

as the hook is calculated on the example of 6 in the top left corner
 $\text{hook}(6) = 3 + 4 + 1 = 8$.

2. In $SU(N)$

$$\begin{aligned}
 \boxed{} \otimes \boxed{} &= \boxed{} \oplus \boxed{} \\
 \dim(\boxed{} \otimes \boxed{}) &= \dim \boxed{} \cdot \dim \boxed{} \\
 \dim \boxed{} &= N \\
 \dim \boxed{} &= \boxed{\begin{matrix} N & | & N+1 \\ & | & \\ & 2 & | & 1 \end{matrix}} / \boxed{2\ 1} = \frac{N(N+1)}{2 \cdot 1} = \frac{N(N+1)}{2} \\
 \dim \boxed{} &= \boxed{\begin{matrix} N & | & N+1 & | & N+2 \\ & | & & | & \\ & 3 & | & 2 & | & 1 \end{matrix}} / \boxed{3\ 2\ 1} = \frac{N(N+1)(N+2)}{3!} \\
 \dim \boxed{} &= \boxed{\begin{matrix} N & | & N+1 & | & 3 & | & 1 \\ & | & & | & & | & \\ & N-1 & | & & 1 & | & \\ & & & & & | & \\ & & & & & 1 & \end{matrix}} / \boxed{1} = \frac{N(N+1)(N-1)}{3 \cdot 1 \cdot 1} = \frac{N(N^2-1)}{3} \\
 \Rightarrow \dim(\boxed{} \otimes \boxed{}) &= \dim(\boxed{} \oplus \boxed{}).
 \end{aligned}$$

Thus, the YT tells us how to identify invariant subspaces, meaning that the dimensions should always add up correctly (check this).

Specific cases with specific questions

1. What happens if a YT of $SU(N)$ has more than N rows ?

$$\begin{array}{c} \boxed{N} \\ \boxed{N-1} \\ \dots \\ \boxed{1} \\ \boxed{0} \end{array}
 \Rightarrow \dim \boxed{0} = 0$$

(2.25)

which comes out when you take the product of the box labels due to the 0 in the $N+1^{th}$ box. This arises due to the impossibility to

antisymmetrize more than N indices for $SU(N)$, this representation can therefore not exist.

Compare $SU(3)$

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \Rightarrow \psi^{[ijkl]} \quad i,j,k,l = 1,2,3 \quad \Rightarrow \quad \psi^{[1231]} = 0 \quad (2.26)$$

does not exist as an irreducible representation of $SU(3)$.

2. What about a YT with exactly $n = N$ boxes, i.e. what if # boxes in a column are N ?

$$\begin{array}{|c|} \hline N \\ \hline N-1 \\ \hline \dots \\ \hline 1 \\ \hline \end{array} \Rightarrow \dim = \frac{\begin{array}{|c|} \hline N \\ \hline N-1 \\ \hline \dots \\ \hline 1 \\ \hline \end{array}}{\begin{array}{|c|} \hline N \\ \hline N-1 \\ \hline \dots \\ \hline 1 \\ \hline \end{array}} = 1 \quad (2.27)$$

thus, this representation is a scalar representation (e.g. $V = \mathbb{R}$), it has no tensor structure. This representation is called the *trivial representation*. This irrep can be removed from the possible representations since it does not really contribute. We will see what this means in the following example.

3. Consider $SU(8)$, such that

$$\begin{array}{|c|} \hline i_1 \\ \hline \dots \\ \hline i_8 \\ \hline \end{array} \Rightarrow \phi^{[i_1 \dots i_8]} = \phi \cdot \epsilon^{i_1 \dots i_8} \quad (2.28)$$

we can choose this scalar representation since we have the maximal columns of N for $SU(N)$ and the maximal antisymmetrization is then given via the Levi-Civita tensor.

Trivial representation reduction scheme

The Levi-Civita tensor is an *invariant tensor* under $SU(N)$, i.e.

$$\epsilon^{i_1 \dots i_8} \rightarrow U_{j_1}^{i_1} U_{j_2}^{i_2} \dots U_{j_8}^{i_8} \epsilon^{j_1 \dots j_8} = \det U \cdot \epsilon^{i_1 \dots i_8} = \epsilon^{i_1 \dots i_8} \quad (2.29)$$

where the determinant comes into play due to the total antisymmetrization of the matrices and is equal to 1 as we are considering $SU(N)$.

Whenever you get a N column structure for $SU(N)$, you can simply remove it from the representation, as it is a trivial representation simply given by the LV-tensor and has dimension 1. Consider $SU(3)$ as an example

$$\begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & e & \\ \hline f & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline b & c \\ \hline e & \\ \hline \end{array}, \quad \dim \begin{array}{|c|} \hline a \\ \hline d \\ \hline f \\ \hline \end{array} = 1 \quad (2.30)$$

since the three indices a, d, f are in the trivial representation and don't carry any info, neglect them and do the simpler case. Compare

$$\dim = \begin{array}{|c|c|c|} \hline 3 & 4 & 5 \\ \hline 2 & 3 & \\ \hline 1 & & \\ \hline \end{array} / \begin{array}{|c|c|c|} \hline 5 & 3 & 1 \\ \hline 3 & 1 & \\ \hline 1 & & \\ \hline \end{array} = 8 \quad \rightarrow \quad \dim = \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 2 & \\ \hline 1 & \\ \hline \end{array} / \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 1 & \\ \hline \end{array} = 8, \quad (2.31)$$

thus the same.

4. What if we have $N - 1$ column boxes for $SU(N)$?

$$\dim(N - 1 \text{ column boxes, i.e. } i = 2, \dots, N)) = \begin{array}{|c|} \hline N \\ \hline \dots \\ \hline 2 \\ \hline \end{array} / \begin{array}{|c|} \hline N - 1 \\ \hline \dots \\ \hline 1 \\ \hline \end{array} \quad (2.32)$$

but we already know a representation of dimension N , the fundamental representation $\dim \square = N$. The anti-fundamental rep is also displayed by this

$$N - 1 \text{ boxes} = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} =: \bar{\square}. \quad (2.33)$$

Consider this more explicitly on for tensors,

$$\phi^{[i_1 \dots i_{N-1}]} = \epsilon^{i_1 \dots i_{N-1} j} \Phi_j \quad (2.34)$$

as $N - 1$ indices are in the trivial (column) representation, this looks like the last N th index j transforms in the anti-fundamental representation (anti-fundamental since it is contracted downwards), is this true ? Transforming yields

$$\phi^{[i_1 \dots i_{N-1}]} \rightarrow U_{j_1}^{i_1} U_{j_2}^{i_2} \dots U_{j_{N-1}}^{i_{N-1}} \phi^{[j_1 \dots j_{N-1}]} = U_{j_1}^{i_1} U_{j_2}^{i_2} \dots U_{j_{N-1}}^{i_{N-1}} \delta_k^j \epsilon^{j_1 \dots j_{N-1}} \Phi_j, \quad (2.35)$$

where we inserted an identity δ_k^j to use the identification of the full anti-symmetrized matrices with the determinant (which is 1) as above. The identity is $U^T U = \mathcal{I}$, such that here the anti-fundamental rep appears, which is therefore hidden in here

$$\begin{aligned} \delta_k^j &= (U^\dagger)_l^j U_k^l \\ \Rightarrow &= \underbrace{U_{j_1}^{i_1} \dots U_{j_{N-1}}^{i_{N-1}} U_k^l \epsilon^{j_1 \dots j_{N-1} l}}_{=\det U \epsilon^{i_1 \dots i_{N-1} l} - \epsilon^{i_1 \dots i_{N-1} l}} (U^\dagger)_l^j \Phi_j \\ &= \epsilon^{i_1 \dots i_{N-1} l} (U^\dagger)_l^j \Phi_j \equiv \epsilon^{i_1 \dots i_{N-1} l} \Phi'_l \\ \Rightarrow \Phi &\in \bar{D} \end{aligned} \quad (2.36)$$

transforms actually only under the anti-fundamental representation. To summarize, the $N - 1$ column representation is represented by $\phi^{i_1 \dots i_{N-1}}$ $N - 1$ index tensor, but this is misleading since only the N th index transforms non-trivially in the anti-fundamental rep., thus just write $\phi^{i_1 \dots i_{N-1}} = \epsilon^{i_1 \dots i_{N-1} j} \Phi_j$, such that the first $N - 1$ transforming trivially is guaranteed by the LC-tensor.
For $SU(N)$, you get all the irreducible representations out of YT., there is no need for any \square .

2.1.4.2 List of irreducible representations

	YT	\square	$\square\square$	$\square\square\square$		
1. $SU(2)$:	$\square\square = 1$	tensor	ϕ^i	$\phi^{(ij)}$	$\phi^{(ijk)}$	eg. $\square'' =$
		dim	2	3	4	

" $\bar{\square}$ in $SU(2)$, i.e. quark and anti-quark.

2. $SU(3)$, can have at most 2 rows. The dictionary is

\square fundamental dim = 3,	$\square\square$ anti – fundamental dim = 3	
--------------------------------	---	--

(2.37)

such that q antifundamental pieces $\square\square\square\dots\square$ and p fundamental pieces $\square\square$ make up the whole of $SU(3)$

$$\begin{array}{c|cccccc} \hline & \square & \square & \square & \square & \square & \square \\ \hline & \square & \square & \square & \square & \square & \square \\ \hline \end{array} \Rightarrow \phi_{(j_1 \dots j_q)}^{(i_1 \dots i_p)} \quad (2.38)$$

where these are the totally traceless tensors such that $\varphi_{kj_2\dots j_q}^{k i_2 \dots i_p} = 0$.

We sometimes refer to a representation by its dimension:

The fundamental representation of $SU(N)$ is $\square = \mathbf{N}$ whereas the anti-fundamental representation of $SU(N)$ $\bar{\square} = \bar{\mathbf{N}}$. For example, consider

$$5 \otimes 5 = \mathbf{15} \oplus \mathbf{10} \quad \text{in } SU(5)$$

$$\square \otimes \square = \square \square \oplus \begin{array}{|c|}\hline \square \\ \hline \end{array}$$

$$\text{since } \dim \square \square = 15, \dim \begin{array}{|c|}\hline \square \\ \hline \end{array} = 10 \text{ in } SU(5).$$

2.2 SYMMETRY BREAKING TO DO

Symmetry breaking only happens in infinite dimensions.

2.3 LORENTZ GROUP $SO(3, 1)$ STRUCTURE AND POINCARÉ GROUP

2.3.1 Lorentz group structure

Proper Lorentz group

The set of proper (Lorentz trafo is called proper if $\det \Lambda = 1$ holds for its representation), orthochronous (Lorentz trafo is called orthochronous if $\Lambda_0^0 \geq 1$ holds true for its representation) Lorentz transformations makes up the six dimensional Lorentz group

$$L_+^\uparrow := \left\{ \Lambda \mid \eta_{\mu\nu} \Lambda_\rho^\mu \Lambda_\sigma^\nu = \eta_{\rho\sigma}, \Lambda_0^0 \geq 1, \det(\Lambda) = 1 \right\}. \quad (2.39)$$

Note that

$$L_+^\uparrow = SO(3, 1). \quad (2.40)$$

It is a subgroup of the Lorentzgroup of which every element is countinously expandable from $\Lambda = \mathcal{I}$.

Equivalent notations of the equation in 2.39 are

$$\Lambda^{\mu\rho} \Lambda_{\mu\sigma} = \delta_\sigma^\rho \quad (2.41)$$

$$\Lambda_\mu^\rho \Lambda_\sigma^\mu = \eta_\sigma^\rho. \quad (2.42)$$

Poincaré group

The Poincaré group \mathcal{P} and the Lorentz group L are made up of L_+^\dagger , parity transformations P , time inversion T and the translations T_4 in \mathbb{M}^4 :

$$L = L_+^\dagger \oplus T \oplus P, \quad (2.43)$$

$$\mathcal{P} = L_+^\dagger \oplus T_4. \quad (2.44)$$

2.3.2 Principle of relativity in the Lagrangian formalism**Principle of relativity take two**

In order to obtain equations of motion whose form is the same for all observers, we demand:

The Lagrangian \mathcal{L} of a relativistic FT is invariant under \mathcal{P}_+^\dagger .
(2.45)

Equations that are form invariant are called *covariant*.

Note the restriction to \mathcal{P}_+^\dagger . Always keep this restriction in mind when reading 'Lorentz invariance'.

This postulate immediately guarantees that the Euler-Lagrange equations are Lorentz covariant and that quantities derived from L in a covariant way are Lorentz tensors. In anticipation of the standard model it should be noted that nature really is not described by a Lagrangian that is invariant under parity or timeinversion transformations and we will use terms violating P and T when building our SM Lagrangian so excluding those here is really necessary.

2.3.3 Generators of boosts and rotations in Minkowski space**2.3.4 The Lorentz algebra $so(1,3)$**

Relativistic fields are classified by their behaviour under Lorentz transformations

$$x^\mu \mapsto x'^\mu = \Lambda_\nu^\mu x^\nu, \quad \Lambda_\nu^\mu \in SO(1,3). \quad (2.46)$$

Generally a field $\phi^a(x)$ transforms then as a representation of the *Lorentz group* $SO(1,3)$. With the field being a map

$$\phi^a : \mathbb{R}^{1,3} \rightarrow V, x \mapsto \phi^a(x), a = 1, \dots, \dim V = n \quad (2.47)$$

we have

$$\phi^a(x) \mapsto \phi'^a(x) = R(\Lambda)_b^a \phi^b(\Lambda^{-1}x') = R(\Lambda)_b^a \phi^b(x) \quad (2.48)$$

with $R(\Lambda)$ being the representation for $\Lambda \in SO(1,3)$.

representation of $so(1,3)$

For a real/complex scalar ($s=0$) field $\phi(x)$ the representation is trivial

$$R(\Lambda) = \mathcal{I} \quad \forall \Lambda \Rightarrow \phi'(x') = \phi(x). \quad (2.49)$$

A vector field ($s=1$) $A^\mu(x)$ transforms in the *vector representation*

$$R(\Lambda)_\nu^\mu = \Lambda_\nu^\mu \quad \forall \Lambda \Rightarrow A'^\mu(x') = \Lambda_\nu^\mu A^\nu(\Lambda^{-1}x'). \quad (2.50)$$

For $V = \mathbb{R}^{1,3} \Rightarrow$
 $a, b \equiv \mu, \nu$

Find a representation
of element of
Lie-group
 $\Lambda \in SO(1,3)$ by
having a
representation for
element of Lie(G):
 $R(\Lambda)e^{R(a)}$, $a \in$
Lie(G).

Basis os $so(1,3)$

We find six antisymmetric 4×4 matrices as elements of the basis of the Lorentz algebra $so(1,3) = \text{Lie}(SO(1,3))$

$$(J^{\rho\sigma})^{\mu\nu} = i [\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\rho\nu}\eta^{\sigma\mu}]. \quad (2.51)$$

The $J^{\rho\sigma}$ are the *generators of the Lie group $SO(1,3)$* , or equivalently form a *basis of the Lie algebra $so(1,3)$* . This is the algebra of infinitesimal Lorentz transformations connected to the identity. We find in this basis

$$\Lambda_\nu^\mu = \left[e^{-i\frac{1}{2}w_{\rho\sigma}J^{\rho\sigma}} \right]_\nu^\mu. \quad (2.52)$$

The defining commutator of the Lie-algebra $so(1,3)$ is

$$[J^{\mu\nu}, J^{\rho\sigma}] = i [\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}]. \quad (2.53)$$

These six basis elements $(J^{\rho\sigma})^{\mu\nu}$ therefore generate the three boosts and three rotations of the Lorentz group $SO(1,3)$.

E.g. Rotation (spatial) by angle α around an axis \vec{n} :

$$\begin{aligned} \omega_{ij} &= \alpha \epsilon_{ijk} n^k & \vec{n} &= (1, 0, 0)^T \\ \Rightarrow w_{\rho\sigma} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix} \Rightarrow \Lambda_\nu^\mu &= \delta_\nu^\mu + \omega_\nu^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \alpha \\ 0 & 0 & -\alpha & 1 \end{pmatrix} \end{aligned}$$

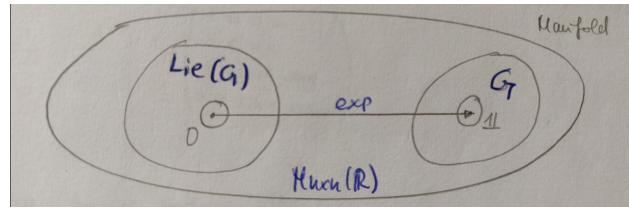


Figure 2.1: Lie groups relation.

For an infinitesimal rotation $\Lambda_\nu^\mu \in SO(1, 3)$. Thus not infinitesimally

$$\Lambda_\nu^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \alpha & -\sin \alpha \\ 0 & 0 & \sin \alpha & \cos \alpha \end{pmatrix} \quad (2.54)$$

2.3.4.1 Connection of representation of Lie group at the example of $so(1, 3)$ as in the chapter before

Consider the relation between matrix groups and Lie algebras: The map "exp" is a diffeomorphism of a small neighbourhood of $O \& \mathcal{J}$ in $M_{n \times n}(\mathbb{R})$ and $GL(n, \mathbb{R})$: $\exp(a) = g, \exp(0) = \mathcal{J}$.

$\text{Lie}(G)$ is the linear subspace of $M_{n \times n}(\mathbb{R})$ generated by $\exp^{-1}(\mathcal{J})$, where \mathcal{J} is a neighbourhood of $\mathcal{J} \in G \subset M_{n \times n}(\mathbb{R})$, compare 2.1. $\Rightarrow \exp(a) = g, a \in \text{Lie}(G), g \in G$ maps a small neighbourhood/patch of $\text{Lie}(G)$ near O to a small patch of G near \mathcal{J} . If $a, b \in \text{Lie}(G)$, then $\exp([a, b]) \in G$.

E.g. $G = SO(3)$,
 $\text{Lie}(G) = \{\text{antisymmetric } 3 \times 3 \text{ matrices}\} \Rightarrow$ If
 $R = \exp(T)$, then
 $RR^T = e^T(e^T)^T = e^Te^{-T} = \mathcal{I}$.

Representation of a Lie-algebra

$$\text{Lie}(G) \xrightarrow{R} M(n), \quad a \mapsto R(a), \quad (2.55)$$

with

$$R(0) = 0, R([a, b]) = R(a)R(b) - R(b)R(a) = [R(a), R(b)]. \quad (2.56)$$

Crucial:

Given some representation R of Lie-algebra $\text{Lie}(G)$, we can always construct an associated representation of G (we call it also R) such that

$$R(A) = \exp(R(a)) \quad \text{if} \quad A = \exp(a). \quad (2.57)$$

Thus by having a representation of an element of $\text{Lie}(G)$

$$w \Rightarrow w_\mu^\nu = -i\frac{1}{2}\Omega_{\rho\sigma}(J^{\rho\sigma})_\mu^\nu \quad (2.58)$$

we find a representation for an element of G

$$\Lambda_\mu^\nu = \left[\exp\left(-i\frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})\right) \right]_\nu^\mu. \quad (2.59)$$

Consider $(J^{\rho\sigma})_\mu^\nu$ as the *canonical basis* of $\text{so}(1,3)$, thus these are the antisymmetric generators of $SO(1,3)$. For $\Lambda \in SO(1,3)$ we find a $\tilde{\omega}$ such that we have a representation

$$\Lambda = e^{\tilde{\omega}} \rightarrow \Lambda_\mu^\nu = \delta_\mu^\nu + \omega_\mu^\nu \quad (2.60)$$

$$\omega_\mu^\nu = -i\frac{1}{2}\Omega_{\rho\sigma}(J^{\rho\sigma})_\mu^\nu. \quad (2.61)$$

2.3.5 The Poincaré group and associated Lie algebra

Poincaré algebra

The Poincaré algebra is given by

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} + \eta^{\mu\sigma}J^{\nu\rho} - \eta^{\nu\sigma}J^{\mu\rho}) \quad (2.62)$$

$$[P^\mu, J^{\rho\sigma}] = -i(\eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho) \quad (2.63)$$

$$[P^\mu, P^\nu] = 0. \quad (2.64)$$

The fundamental representation of the Lorentz group is

$$\Lambda_\nu^\mu = e^{i\frac{1}{2}\omega_{\alpha\beta}(J^{\alpha\beta})_\nu^\mu}, \quad \text{with } (J^{\alpha\beta})_\nu^\mu = i(\delta_\nu^\alpha\delta_\mu^\beta - \delta_\mu^\alpha\delta_\nu^\beta). \quad (2.65)$$

Note that the commutator relations of the Poincaré algebra, together with $P^0 \equiv H$, immediately imply that momentum, angular momentum, and energy are conserved in the sense of QM

$$[\vec{P}, H] = [\vec{J}, H] = [H, H] = 0. \quad (2.66)$$

Note that

$$J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \quad (2.67)$$

and $(J^{\alpha\beta})^{\mu\nu}$ is the antisymmetrized metric, i.e.

$$(J^{\alpha\beta})^{\mu\nu} = i\eta^{\mu[\alpha}\eta^{\beta]\nu}. \quad (2.68)$$

⁵

2.3.6 The Dirac spinor representation

Representation connected to spin

Spin $\frac{1}{2}$ particles are described by fields in the *spinor representation*.

Dirac algebra is a complexification of the real spacetime algebra
 $Cliff(1,3, \mathbb{R}) : Cliff(1,3\mathbb{C}) = Cliff(1,3, \mathbb{R}) \otimes \mathbb{C}$

Dirac/Clifford algebra and spinor representation

Every representation of $Cliff(1,3)$ induces a representation of $so(1,3)$. To find the spinor representation of $so(1,3)$ we start from the Clifford algebra $Cliff(1,3)$ defined as the algebra spanned by $n \times n$ -matrices $(\gamma^\mu)_B^A$, $\mu = 0, 1, 2, 3$ and $A, B = 1, \dots, n$ such that the anti-commutator is

$$\{\gamma^\mu, \gamma^\nu\} := 2\eta^{\mu\nu} \mathcal{I}_{n \times n} \quad (2.69)$$

$$\Rightarrow \gamma^\mu \gamma^\nu = \begin{cases} \eta^{\mu\nu} & \text{if } \mu = \nu \\ -\gamma^\nu \gamma^\mu & \text{if } \mu \neq \nu \end{cases}, \quad (2.70)$$

$$(\gamma^0)^2 = \mathcal{I}, \quad (\gamma^i)^2 = -\mathbf{I}. \quad (2.71)$$

With the following object

$$(S^{\rho\sigma})_B^A := \frac{i}{4}[\gamma^\rho, \gamma^\sigma]_B^A \quad (2.72)$$

we find a basis of $so(1,3)$, i.e.

$$[S^{\rho\sigma}, S^{\tau\kappa}] = -i[\eta^{\rho\kappa} S^{\sigma\tau} + \eta^{\sigma\tau} S^{\rho\kappa} - \eta^{\rho\tau} S^{\sigma\kappa} - \eta^{\sigma\kappa} S^{\rho\tau}]. \quad (2.73)$$

The Dirac algebra, thus the $Cliff(1,3)$ algebra on \mathbb{C} , is then the standard environment the spinors of the Dirac equation live in, rather than the spacetime algebra.

Every representation of $Cliff(1,3)$ induces a representation of $Lie(SO(1,3))$.

By defining $(S^{\rho\sigma})_B^A$ we have constructed a representation of $\text{Lie}(SO(1,3))$ and can from this representation infer a representation of $SO(1,3)$ with $\Lambda \in SO(1,3)$:

$$(\Lambda_{\frac{1}{2}})_B^A = \left(e^{i\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}} \right)_B^A. \quad (2.74)$$

Or more precisely, we have constructed a basis and find a representation of this basis by choosing an explicit representation for the γ^μ matrices. From the basis we can construct the elements of G by $\exp()$ and find therefore a representation of G .

The explicit representation of the Clifford algebra $Cliff(1, d-1)$ is given by the representation for its elements γ^μ by $n \times n$ matrices

$$(\gamma^\mu)_B^A \quad \text{with } \mu \in \{0, 1, \dots, d-1\}, A, B \in \{1, \dots, n\}. \quad (2.75)$$

This will then also give a representation of $\text{Lie}(SO(1, d-1)) \Rightarrow$ The irreducible representations of $Cliff(1, d-1)$ are of dimension $n = 2^{d/2}$ if d is even and $n = 2^{\frac{1}{2}(d-1)}$ if d is odd.

Specialize to $d = 4 \Rightarrow n = 4 \Rightarrow$ Dirac representation (*chiral*)

$$\gamma^0 = \begin{pmatrix} 0 & \mathcal{I}_{2 \times 2} \\ \mathcal{I}_{2 \times 2} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad (2.76)$$

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}, \quad \gamma^5 = \begin{pmatrix} -\mathcal{I}_{2 \times 2} & 0 \\ 0 & \mathcal{I}_{2 \times 2} \end{pmatrix} \quad (2.77)$$

Dirac spinor representation

The complex vector space on which $(\gamma^\mu)_B^A$ acts is called the space of Dirac spinors. A Dirac spinor is a set of fields $\psi^A(x), A \in \{1, 2, 3, 4\}$ transforming as

$$(\psi)^A(x) \mapsto [S(\Lambda)]_B^A(\psi)^B(\Lambda^{-1}x') = [S(\Lambda)]_B^A(\psi)^B(x) \quad (2.78)$$

with $[S(\Lambda)]_B^A = \left[\exp\left(-i\frac{1}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right) \right]_B^A$.

\Rightarrow A Dirac spinor field $(\psi)^A(x)$, with A,B *spinor indices*, behaves like spin $\frac{1}{2}$ -field, because

$$\Lambda(2\pi) = \exp\left(i\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}\right) = -\mathcal{I} : (\psi)^A \mapsto -(\psi)^A. \quad (2.79)$$

The transformation of a Dirac spinor forms a representation of

$$Spin(1,3) \cong \underbrace{SL(2, \mathbb{C})}_{\{M_{2 \times 2}(\mathbb{C}) \text{ and } \det M = 1\}} \quad (2.80)$$

and not of $SO(1,3)$. Furthermore we know, that $Spin(1,3)$ is the *double cover* of $SO(1,3)$.

This is equivalent to the fact that the $j = 1/2$ spinor representation of the algebra of spatial rotations $\text{Lie}(\text{SO}(3))$ does not furnish a representation of the Lie group $\text{SO}(3)$ but only of its double cover $\text{SU}(2)$.

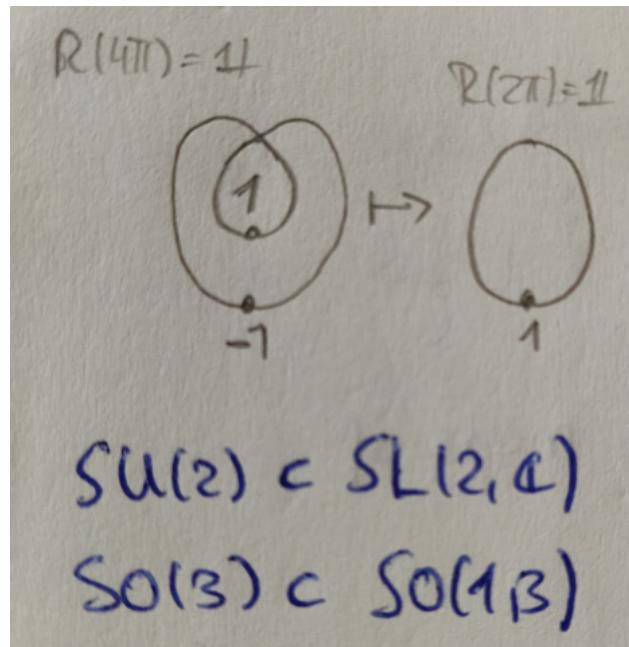


Figure 2.2: Double cover of the group, need to rotate by 4π to get back to the origin.

$\Rightarrow \psi_L, \psi_R$ transform under the irreducible representation of $\text{SO}(1,3)$, $\psi = (\psi_L, \psi_R)^T$ only transforms under the reducible representation of $\text{Spin}(1,3)$.

2.3.7 Lorentz tensor- and spinor-fields

Fields $\psi(x)$ transforming under Lorentz transformations of the coordinates $x^\mu \rightarrow x'^\mu = \Lambda_\nu^\mu x^\nu$ as

$$\psi(x) \rightarrow \psi'(x') = D(\Lambda)\psi(\Lambda^{-1}x') \quad (2.81)$$

Work through
gregors script
on Lorentz
group

with the Spinor representations of L_+^\uparrow are ordinary representations of $SL(2, \mathbb{C})$, which is the double cover of $L_+^\uparrow \equiv SO(3, 1)$.

3

QUANTUM FIELD THEORY, CANONICAL QUANTIZATION

Summary of QFT idea

In quantum field theory of point particles, the force between two particles is mediated by the exchange of virtual (or off-shell) particles. Associated with each force is a charge. Charged particles feel the force by coupling to or interacting with the particles that carry the force. The most familiar example is electrodynamics. The particles that feel the force carry electric charge. The electromagnetic force is mediated by the exchange of spin-1 photons. The photons themselves are uncharged and therefore do not directly couple to each other. The resulting field equations are linear. In QCD, a theory of the strong force built from a Yang-Mills gauge theory (the strong force is responsible for holding together nucleons and thereby the nucleus), the charge is called colour. The fundamental particles that feel the strong force are coloured quarks, and the particles that carry the force are called gluons. The gluons themselves are colour charged, hence, unlike the photon, they can directly interact with each other, and the resulting field equations are nonlinear.

Geometrical Interpretation of gauge QFTs

Presently we have a geometrical interpretation of classical gauge theories such as electrodynamics and Yang-Mills. The vector potential A_μ^a are connection coefficients on a principal fiber bundle where the structure group is the gauge group ($U(1)$ for electromagnetism, $SU(2)$ for Yang-Mills, and $SU(3)$ for classical chromodynamics). The field strengths $F_{\mu\nu}$ (i.e., the electric and magnetic fields in electrodynamics) are the curvatures associated with the connections (the potentials). The charged matter that the fields couple to are associated vector bundles. From this path integral viewpoint, Quantum Electrodynamics and Quantum Chromodynamics amount to integrals over the space of connection on principal fiber bundles. Compare ??.

3.1 RENORMALIZATION

In the process of renormalization, counterterms are generated to cancel the high energy or ultraviolet divergences that are encountered in

the individual terms of the perturbation series. When the renormalization process is successful, the counterterms build a finite effective action that can be thought of as classical field theory that contains all of the quantum effects. The possible counterterms are consistent with the symmetries of the original bare action. In other words, internal symmetries can severely restrict the types of counterterms that can be generated and thereby limit the number of corresponding divergences. Hence, theories with more symmetry are generally more convergent.

3.2 REGULARIZATION

3.3 ON THE TRANSITION TO A QUANTIZED FIELD THEORY

The idea now is to take the equations of classical field theory and develop a quantization scheme analogously to quantizing classical point theory. Recall that in point theory we had the fundamental Poisson-brackets

$$\{q_i, q_k\}_P ==, \quad \{p_i, p_k\}_P = 0, \quad \{q_i, p_k\}_P = \delta_{ik} \quad (3.1)$$

which we modified to quantize the dynamical variable of point theory x and p by demanding

$$[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0 \quad [\hat{x}_i, \hat{p}_j] = i\delta_{ij}. \quad (3.2)$$

You will see how closely we will continue this idea in the context of the dynamical variables of field theory ϕ and the conjugate field π .

Following quantization, we will have to very carefully rethink results of classical field theory (FT), since we are now dealing with completely altered objects. One of the central identities from FT that is no longer true in QFT is the Hamiltonian principle

$$\delta S = 0. \quad (3.3)$$

Even though we will carry the free equations of motion over to QFT, after solving the interacting equations as operator equations we will encounter so called off-shell fields, i.e. fields that do not follow the path of extreme action and therefore do not obey the classical equations of motion. In Feynman's approach of pathintegrals we will eventually be able to replace Hamilton's principle by a much more powerful one which keeps Hamilton's principle as the limit of the most likely (but not exclusive) path.

Another statement that does not necessarily hold for quantized systems is Noether's theorem:

the symmetry of a Lagrangian can be broken by quantization which we will see much later in the context of *anomalous symmetries*. An example will be the axial symmetry of the Dirac field. All of this is most elegantly discussed in Feynman's pathintegral formulation later on.

3.3.1 Principles of quantum theory

The following postulates of QM are taken over to QFT:

1. States are normalized elements of a Hilbert space

$$|\psi_n\rangle \in \mathcal{H} \text{ with } |\langle\psi_n|\psi_n\rangle|^2 = 1 \text{ for all } n, \quad (3.4)$$

2. Observables are eigenvalues of hermitian operators on that Hilbert space

$$O = O^\dagger, \quad (3.5)$$

3. Transition probabilities between states are squares of scalar products on that Hilbert space

$$P(\psi \rightarrow \psi_n) = |\langle\psi|\psi_n\rangle|^2. \quad (3.6)$$

The normalization 3.4 makes it possible to interpret 3.6 as probabilities since only then is

$$\sum_n |\langle\psi|\psi_n\rangle|^2 = 1 \quad (3.7)$$

and the hermiticity of the operators associated with observables makes it possible to interpret its eigenvalues as results of measurements, since only then are the eigenvalues real, i.e.

$$O|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle \text{ with } \lambda \in \mathbb{R} \text{ for all } \psi_\lambda. \quad (3.8)$$

Since these axioms carry over to QFT, QM is not useless at all. In fact all the structures found in QM will carry over to QFT and simply be supplemented by the principle of relativity. It is this combination of axioms that will inevitably lead to QFT. QFT will explain the appearance of the half integer spin, where the Landé factor between the magnetic moment μ and the momentum J was needed initially

$$\mu^i = -g\mu_B J^i \quad (3.9)$$

to match experiment with an input parameter not predicted by QM. In QFT, we will understand it as an implication of Lorentz invariance and even derive the Landé factor from these first principles. The ad hoc definition of momentum operators

$$[J^\alpha, J^\beta] = ie_\gamma^{\alpha\beta} J^\gamma \quad (3.10)$$

can be understood as part of a defining commutator of the Lorentz group.

Furthermore, we will be able to understand the appearance of gauge invariance in the classical field theory of electrodynamics

$$\phi \rightarrow \phi + \frac{\partial \chi}{\partial t} \quad \& \quad \vec{A} \rightarrow \vec{A} + \vec{\nabla} \chi \quad (3.11)$$

as a necessity of implementing Lorentz invariance in the Hilbert space of massless spin 1 particles.

In fact all of QM will be contained in QFT as the limit of zero spatial dimension $d \rightarrow 0$, i.e. QM should be quantum quantum point mechanics as CM should be called classical point mechanics since it is the limit $d \rightarrow 0$ of classical field theory. All of classical field theory will furthermore be contained in QT as the limit $\hbar \rightarrow 0$, which will be most obvious in the pathintegral formalism of QFT. The limit $c \rightarrow \infty$ brings us from relativistic field theory to non-relativistic field theory, and from there $d \rightarrow 0$ finally brings us to classical point mechanics just as $\hbar \rightarrow 0$ brings us there from QM. Everything prior to QFT is therefore just an approximation of QFT in certain limits.

Taking over the principles of QM also entails taking over the discussions around the interpretation of quantum theories, which are still relevant today. For example whter a probabilistic axiom such as 3.6 should exist in a fundamental theory is still discussed.

QM is often called non-deterministic in this context, but this is misleading. QM is probabilistic *and* deterministic in its prediction of these probabilities. The evolution of the states is completely determined by the eom, which we will soon find. It is only non-deterministic in its prediction of a *single* measurement, but does determine exactly the evolution of probability distributions and thereby the result that *infinitely many* measurements will converge to. Of course, this kind of talk about infinitely many measurements is exactly the source of doubt over the usefulness of a probabilistic axiom and the question stans why probabilities should ever come into play if there are deterministic equations working in the background. Maybe more obviously problematic is the introduction of observables via hermitian operators, which makes a fundamental distinction between an "observer" and the system which is described by the observer by mentioning an "observable". Surely if the theory is fundamental it should *describe* the observer and the measurement process and not simply postulate the collapse of a state. The distinction between a measurement apparatus and the measured object should arise from the theory and not be hard-wired into it at the axiomatic level. It seems that mentioning an observer is really the same as mentioning consciousness and doing so in an axiomatic way means giving up on ever explaining how consciousness arises from first principles.

3.3.2 Wigner's theorem

If a system of states $\{\psi, \psi_1, \dots\}$ has a symmetry U such that

$$|\langle \psi | \psi_n \rangle|^2 = |\langle \psi' | \psi'_n \rangle|^2 \text{ for all } n \text{ as } \rightarrow' \quad (3.12)$$

i.e. the postulated probability of transition is invariant, then

$$\langle U\psi | U\psi_n \rangle = \langle \psi | \psi_n \rangle \text{ and } U(\alpha\psi + \beta\psi_n) = \alpha U\psi + \beta U\psi_n \quad (3.13)$$

$$\text{or } \langle U\psi | U\psi_n \rangle = \langle \psi | \psi \rangle^* \text{ and } U(\alpha\psi + \beta\psi_n) = \alpha^* U\psi + \beta^* U\psi_n \quad (3.14)$$

i.e. U is either linear and unitary or antilinear and antiunitary.

3.3.3 Lorentz invariance in quantum theory

Transformation of a state via ∞ -dimensional unitary representation $U(\Lambda)$ (acc. to Wigner)

$$|\alpha\rangle \rightarrow |\alpha'\rangle = U(\Lambda)|\alpha\rangle \text{ with } UU^\dagger = \mathcal{I}. \quad (3.15)$$

Alternatively we can interpret this as a transformation of the field operator where $x \rightarrow \Lambda x$ as

$$\psi(x) \rightarrow \psi'(x') = U^{-1}(\Lambda)\psi(\Lambda^{-1}x')U(\Lambda) \stackrel{2.81}{=} D(\Lambda)\psi(\Lambda^{-1}x'). \quad (3.16)$$

Analogously to $\phi'(x') = \phi(x)$ in classical field theory, we now define a quantized scalar field as

$$\langle \alpha' | \phi(x') | \beta' \rangle = \langle \alpha | \phi(x) | \beta \rangle \Leftrightarrow \phi(x) \text{ is a scalar field} \quad (3.17)$$

or generally analogous to 2.81

$$\langle \alpha' | \psi(x') | \beta' \rangle = \langle \alpha | D(\Lambda)\psi(\Lambda^{-1}x') | \beta \rangle. \quad (3.18)$$

3.3.4 Parity

3.3.5 Time reversal

3.3.6 Charge conjugation

3.3.7 Discrete symmetries (CPT)

Parity and time inversion of a scalar field operator are

$$P^{-1}\phi(t, \vec{x})P = \phi(t, -\vec{x}) \quad (3.19)$$

$$T^{-1}\phi(t, \vec{x})T = \phi(-t, \vec{x}). \quad (3.20)$$

Go through different kinds of discrete and continuous symmetries in QFT

The parity and time inversion of a vector field operator are

$$P^{-1}A^\mu(t, \vec{x})P = P_\nu^\mu A^\nu(t, -\vec{x}) \quad (3.21)$$

$$T^{-1}A^\mu(t, \vec{x})T = T_\nu^\mu A^\nu(t, -\vec{x}) \quad (3.22)$$

with

$$P_\nu^\mu = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad T_\nu^\mu = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1- \end{pmatrix} \quad (3.23)$$

The parity and time inversion of a spinor field operator with $\Lambda^{-1}\gamma^\mu\Lambda = \Lambda_\nu^\mu\gamma^\nu$ is

$$P^{-1}\psi(t, \vec{x})P = P_B^A\psi^B(t, -\vec{x}) \quad (3.24)$$

$$T^{-1}\psi(t, \vec{x})T = T_B^A\psi^B(-t, \vec{x}) \quad \text{with} \quad (3.25)$$

$$P_B^A = (\gamma^0)_B^A \text{ and } T_B^A = -i(\gamma^1\gamma^3)_B^A. \quad (3.26)$$

The infinite dimensional representation of T is anti-unitary and anti-linear.

Charge conjugation is given by

$$C^{-1}\psi(x)C = -i(\bar{\psi}(x)\gamma^0\gamma^2)^T \quad (3.27)$$

$$C^{-1}j^\mu(x)C = -j^\mu(x) \quad (3.28)$$

$$C^{-1}A^\mu(x)C = -A^\mu(x). \quad (3.29)$$

Whether a theory is parity, time inversion or charge conjugation invariant, i.e. whether $P|\Omega\rangle = |\Omega\rangle, T|\Omega\rangle = |\Omega\rangle, C|\Omega\rangle = |\Omega\rangle$ has to be tested case by case, e.g. weak interactions are *not* charge conjugation invariant (but invariant under the combination CPT); the electromagnetic interactions are invariant under P, C, T individually.

3.3.8 CPT -theorem

CPT-theorem

For a Lorentz invariant theory with a stable vacuum, i.e. a vacuum with an energy expectation value that is bounded from below, one has

$$\boxed{CPT|\Omega\rangle = |\Omega\rangle.} \quad (3.30)$$

3.4 THE FREE SCALAR FIELD

3.4.1 Why Quantum Field Theory?

QFT describes a many-body system through fields as fundamental entities. These are abstract objects that penetrate spacetime by describing

the distribution of some physical quantity. Then particles are the excitation of the field, the quanta of oscillation of an abstract field.

QM is then QFT's non-relativistic limit, it's a quantum field theory in zero spatial dimensions (\equiv time dimension), because the combination of QM and SR implies that *particle number is not conserved*, because all particles of the same type are *the same*. Particles are the excitations of one particular specific field that penetrates spacetime, therefore their nature doesn't depend on time and space of their creation.

Universality and renormalization are central concepts of QFT, i.e. different "microscopic" theories can describe the same "macroscopic" observable physics.

3.4.1.1 Natural Units

We use natural units, where

$$\begin{aligned}\hbar = c = 1 &\Rightarrow [\text{energy}] = [\text{mass}] = [\text{length}]^{-1} = [\text{time}]^{-1} \\ \Rightarrow \alpha = \frac{e^2}{\hbar c}.\end{aligned}$$

If X has dimensions $(\text{mass})^d$ we will write $[X] = d$.

3.4.2 Schrödinger equation and the Schrödinger picture

Schrödinger equation

$$i\partial^0 |\psi\rangle = H^0 |\psi\rangle. \quad (3.31)$$

The Schrödinger picture has the advantage that it makes manifest the linear nature of QFT, which is the same as that of QM. Another natural property of this formulation is that the "wavefunction" formalism of old fashioned QM can be obtained as the limit of zero spatial dimensions from the "wavefunctional" formalism, which is obtained from the Schrödinger equation.

3.4.3 Heisenberg equation and Heisenberg picture

Heisenberg equation

$$\frac{dO}{dt} = i[H, O]. \quad (3.32)$$

The Heisenberg picture has the advantage that field operator equations of motion can directly be derived from a manifestly Lorentz invariant variational principle of a quantum action which for free theory coincides with classical field theory. Just like in the Lagrangian picture of

classical field theory, properties of symmetry and locality are manifest in this picture.

3.4.4 Causality

$$[O_i(x), O_j(y)] = 0 \text{ for } (x - y)^2 < 0 \text{ and all operators.} \quad (3.33)$$

Note that this statement holds true for arbitrary x^0, y^0 while the quantizing commutations relations are all equal time relations.

3.4.5 Uncertainty

$$\langle (\Delta O_1)^2 \rangle \langle (\Delta O_2)^2 \rangle \geq \frac{1}{4} |\langle [O_1, O_2] \rangle|^2 \text{ for all states, and } \Delta O_i := O_i - \langle O_i \rangle. \quad (3.34)$$

3.4.6 Classical Scalar Field, Lagrangian Formulation

By starting with a classical action for a finite number of d.o.f. $q_i(t)$

$$S[q_i] = \int_{t_1}^{t_2} dt L(q_i(t), \dot{q}_i(t)), \quad (3.35)$$

we can make the transition to a scalar field $\phi(t, \vec{x}) = \phi(x^\mu)$ by replacing:

First step:

$$q_i \rightarrow \phi(x^\mu), \quad \dot{q}_i \rightarrow \frac{\partial \phi(x^\mu)}{\partial t}, \quad (3.36)$$

where we consider a *real* scalar field $\phi : x^\mu \rightarrow \phi(x^\mu) \in \mathbb{R}$ that describes *spin-zero* particles

$$S = \int d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x)), \quad (3.37)$$

with $[\mathcal{L}] = 4, [S] = 0, [d^4x] = -4$. This system has an infinite number of degrees of freedom (at least for each point \vec{x} in space).

Second Step: Find the Lagrangian:

\mathcal{L} is a Lorentz-scalar and therefore has to depend only on Lorentz-scalars in a relativistic setting.

$$S = \int d^4x \left[\frac{1}{2} \underbrace{\partial_\mu \phi \partial^\mu \phi}_{=(\partial \phi)^2} - V(\phi) + \mathcal{O}(\phi^n (\partial \phi)^m) \right], \quad (3.38)$$

where $m \geq 2 \& n \geq 1$ can be left out doesn't change physics. Note that the Lagrangian is *local*. No terms in \mathcal{L} that couple $\phi(\vec{x})$ to $\phi(\vec{y})$: $\Leftrightarrow \mathcal{L} = \mathcal{L}(\phi(\vec{x}, t), \dot{\phi}(\vec{x}, t), \vec{\nabla}\phi(\vec{x}, t))$. Only consider \mathcal{L} depending on $\nabla\phi$ and not on higher derivatives, because they are not Lorentz-invariant.

$\partial_\mu \partial^\mu \phi$ is a *total derivative* and therefore does not alter e.o.m. due to assumptions of boundary terms which can be left out.

Note that

$$[\phi] = \text{mass}^1 = \frac{d}{2} - 1 \quad (3.39)$$

$$\begin{aligned} \Rightarrow \mathcal{L}(\text{scalar field, real}) &= \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{m^2}{2} \phi^2 \\ &= \frac{1}{2} \phi^2 - \frac{1}{2} (\nabla\phi)^2 - \frac{m^2}{2} \phi^2. \end{aligned} \quad (3.40)$$

By assuming a global minimum for $V(\phi(x))$ at $\phi_0(x)$ we can expand

$$V(\phi(x)) = V_0 + \frac{m^2}{2} \phi^2(x) + \mathcal{O}(\phi^3(x)), \quad (3.41)$$

with V_0 the classical contribution to the ground state or vacuum energy. Higher power of $\phi^{>2}$ in the potential $V(\phi)$ as well as the terms $\mathcal{O}(\phi^n (\partial\phi)^m)$ will give rise to interactions between the particles. Taking these terms into account gives rise to interaction and therefore the field would be described by infinitely anharmonic coupled oscillators:

Free scalar field theory

$$S = \int_{\mathbb{R}^{3,1}} d^4x \left[\frac{1}{2} (\partial\phi)^2 - \frac{m^2 \phi^2}{2} \right], \quad \phi = \phi^*. \quad (3.42)$$

Klein-Gordon equation

I.e. for the free scalar field this yields the

$$(\partial^2 + m^2)\phi(x) = 0, \quad (3.43)$$

3.4.7 Canonical Quantization in the Schrödinger Picture

Begin with the replacement $q_i \rightarrow \phi(\vec{x}, t)$ and the conjugate momentum

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \rightarrow \Pi(\vec{x}, t) := \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\vec{x}, t)} \quad (3.44)$$

with Π the *conjugate momentum density*.

The Hamiltonian is

$$H = \int d^3x \mathcal{H} = \int d^3x [\Pi(\vec{x}, t) \dot{\phi}(\vec{x}, t) - \mathcal{L}], \quad (3.45)$$

The fields vanish at spatial boundaries. They also vanish inside action evaluated at time boundaries, because action has fixed timepoints.

which for the free scalar field is

$$H = \int_{\mathbb{R}^3} d^3x \left[\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\vec{\nabla}\phi)^2 + \frac{m^2}{2} \phi^2 \right]. \quad (3.46)$$

To quantize in the Schrödinger picture we go over to

$$\phi^{(s)}(\vec{x}) = (\phi^{(s)}(\vec{x}))^\dagger, \quad \Pi^{(s)}(\vec{x}) = (\Pi^{(s)}(\vec{x}))^\dagger \quad (3.47)$$

self-adjoint and time independent (all time dependence lies in the states) scalar field operators with the *canonical commutation relations*

$$[\phi(\vec{x}), \Pi(\vec{y})] = i\hbar \delta_D^{(3)}(\vec{x} - \vec{y}) = i\delta_D^{(3)}(\vec{x} - \vec{y}), \quad (3.48)$$

$$[\phi(\vec{x}), \phi(\vec{y})] = 0, \quad [\Pi(\vec{x}), \Pi(\vec{y})] = 0. \quad (3.49)$$

Note that this is analogous to the canonical commutation relations of QM, where we quantized x and p . This is no longer demanded in QFT. There is no second quantization as historically misunderstood. The only quantized objects from CFT are the fields. Momentum and space are those of CFT, where they are mere labels of the fields.

Also note that although this method of quantization is not manifestly Lorentz invariant (since $\pi(x)$ transforms like the 0-component of a 4-vector), it will nevertheless lead to a Lorentz invariant formulation of observables, as will become apparent in the following. For a manifestly Lorentz invariant formulation compare pathintegral quantization.

3.4.8 Mode Expansion

For the example of the free scalar field theory:

The classical Lagrangian describes an infinity of coupled harmonic oscillators. Because the coupling is described by $\vec{\nabla}$ with eigenfunctions $e^{i\vec{p}\vec{x}}$, interaction will be diagonal through a Fourier transform

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \tilde{\phi}(\vec{p}) e^{i\vec{p}\vec{x}} \quad (3.50)$$

$$\Pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \tilde{\Pi}(\vec{p}) e^{i\vec{p}\vec{x}}, \quad (3.51)$$

with the reality of the fields ensuring $\tilde{\phi}^\dagger(\vec{p}) = \tilde{\phi}(-\vec{p})$, $\tilde{\Pi}^\dagger(\vec{p}) = \tilde{\Pi}(-\vec{p})$. One uses the identity of the Dirac distribution

$$\int d^3x e^{i(\vec{p}+\vec{q})\vec{x}} = (2\pi)^3 \delta_D^{(3)}(\vec{p} + \vec{q}). \quad (3.52)$$

This implies the form of the quantized Hamiltonian

$$H = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left[\underbrace{\tilde{\Pi}(\vec{p}) \tilde{\Pi}^\dagger(\vec{p})}_{=|\tilde{\Pi}(\vec{p})|^2} + \omega_p^2 \underbrace{\tilde{\phi}(\vec{p}) \tilde{\phi}^\dagger}_{=|\tilde{\phi}(\vec{p})|^2} \right], \quad (3.53)$$

with $\omega_p^2 = \vec{p}^2 + m^2$. This is a collection of *decoupled harmonic oscillators* with frequency ω_p with momentum p .

The collective Hilbert space of all these oscillators is thus constructed using *creation* and *annihilation* operators constructed from these modes. We define the operators:

$$\hat{a}(\vec{p}) = \frac{1}{2} \left[\sqrt{2\omega_p} \tilde{\phi}(\vec{p}) + o\sqrt{\frac{2}{\omega_p}} \tilde{\Pi}(\vec{p}) \right]; \quad (3.54)$$

$$\tilde{\phi}(\vec{p}) = \frac{1}{\sqrt{2\omega_p}} [\hat{a}(\vec{p}) + \hat{a}^\dagger(-\vec{p})] \quad (3.55)$$

$$\hat{a}^\dagger(\vec{p}) = \frac{1}{2} \left[\sqrt{2\omega_p} \tilde{\phi}(-\vec{p}) - i\sqrt{\frac{2}{\omega_p}} \tilde{\Pi}(-\vec{p}) \right], \quad (3.56)$$

$$\tilde{\Pi}(\vec{p}) = -i\sqrt{\frac{\omega_p}{2}} [\hat{a}(\vec{p}) - \hat{a}^\dagger(-\vec{p})], \quad (3.57)$$

$$\text{with } [\tilde{\phi}(\vec{p}), \tilde{\Pi}(\vec{q})] = (2\pi)^3 i \delta_D^{(3)}(\vec{p} + \vec{q}). \quad (3.58)$$

Quantized space and momentum operator

$$\phi(\vec{x}) = \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} [\hat{a}(\vec{p}) e^{i\vec{x}\vec{p}} + \hat{a}^\dagger(\vec{p}) e^{-i\vec{x}\vec{p}}] \quad (3.59)$$

$$\Pi(\vec{x}) = \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} [\hat{a}(\vec{p}) e^{i\vec{p}\vec{x}} - \hat{a}^\dagger(\vec{p}) e^{-i\vec{p}\vec{x}}]. \quad (3.60)$$

The ladder operators obey the commutation relation

Commutation relations

*Leave hats off of operators as of now,
 $\hat{a} \equiv a$.*

$$[a(\vec{p}), a^\dagger(\vec{q})] = (2\pi)^3 \delta_D^{(3)}(\vec{p} - \vec{q}) \quad (3.61)$$

$$[a^\dagger(\vec{p}), a^\dagger(\vec{q})] = 0 = [a(\vec{p}), a(\vec{q})], \quad \forall \vec{p}, \vec{q}. \quad (3.62)$$

Therefore, the Hamiltonian in its final form is given by

$$H = \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} \omega_p a^\dagger(\vec{p}) a(\vec{p}) + \Delta_H, \quad \omega_p = E_p = \sqrt{\vec{p}^2 + m^2}, \quad (3.63)$$

where $a^\dagger(\vec{p}) a(\vec{p})$ may be interpreted as the number operator N_p giving the number of particles in a state with momentum p .

With the *divergent ground state/zero-point energy* $\frac{\hbar\omega_p}{2}$ in

$$\Delta_H = \frac{1}{2} \int d^3 p \omega_p \delta_D^{(3)}(\vec{0}). \quad (3.64)$$

In a theory without gravity, absolute energy has no meaning, only energy differences do. This Hilbert space possesses a state of lowest energy, the vacuum $|0\rangle$. The vacuum corresponds to the absence of any

excitations of the field ϕ .

The Hamiltonian obeys the commutation relations

$$[H, a(\vec{p})] = -\omega_p a(\vec{p}) \quad (3.65)$$

$$[H, a^\dagger(\vec{p})] = \omega_p a^\dagger(\vec{p}). \quad (3.66)$$

Or rather by combining H and the spatial momentum operator

$$p^i = \int \frac{d^3 p}{(2\pi)^3} p^i a^\dagger(\vec{p}) a(\vec{p}) + \underbrace{\Delta_{p_i}}_{=\frac{1}{2} \int d^3 p p^i \delta_D^{(0)}(0) \equiv 0} \quad (3.67)$$

to the *4-momentum operator*

$$P^\mu = \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} p^\mu a^\dagger(\vec{p}) a(\vec{p}) + \Delta_{p^\mu}, \quad \Delta p^\mu = \begin{cases} \Delta H & \mu = 0 \\ 0 & \mu = 1, 2, 3. \end{cases} \quad (3.68)$$

with $p^\mu = (p^0, \vec{p}) = (\omega_p, \vec{p})$.

Momentum operator of the real scalar field

The momentum operator is hermitian $P^\mu = (P^\mu)^\dagger$.

There exists a common set of eigenstates of H and \vec{P} , i.e. \vec{P} is locally conserved, the system is translation invariant and

$$[H, \vec{P}] = 0. \quad (3.69)$$

The momentum operator obeys

$$\partial^\mu \phi = i[P^\mu, \phi] \quad (3.70)$$

which generalizes the Heisenberg equation

$$\phi(x+a) = e^{ia_\mu P^\mu} \phi(x) e^{-ia_\mu P^\mu} \quad (3.71)$$

which then again identifies the 4-momentum as the generator of space-time translations.

Note that as for any result obtained by using Noether's theorem, we could have obtained the energy-momentum tensor in CFT already. Only after plugging in the field operators we obtain results such as the momentum operator 3.68 exclusive to QFT.

One can look at $[H, \vec{P}] = 0$ as an immediate implication of the axiom of Poincaré invariance where it is one of the defining commutators of the Poincaré algebra and *prove* the quantization rules (i.e. the canonical commutation relations). Alternatively one can look at lorentz invariant Lagrangians and use the quantization rules axiomatically to *prove* $[H, \vec{P}] = 0$.

Construction of the Hilbert space

We get the commutation relations

$$[P^\mu, a^\dagger(\vec{p})] = p^\mu a^\dagger(\vec{p}), \quad (3.72)$$

$$[P^\mu, a(\vec{p})] = -p^\mu a(\vec{p}). \quad (3.73)$$

What did we do ?

We have started with the assertion that spacetime (here $\mathbb{R}^{1,3}$) is filled with the real scalar field $\phi(\vec{x})$, which we have taken to be a free field with Lagrangian $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2$. This field is interpreted as a field operator, i.e. in the Schrödinger picture at every space point \vec{x} the object $\phi(\vec{x})$ represents a self-adjoint operator that acts on a Hilbert space. This Hilbert space possesses a state of lowest energy, the vacuum $|0\rangle$.

3.4.9 The Fock space

QM asks the question "which particle is on which state". In Qm, the particles are identical, such that exchanging two particles ($\vec{r}_i \leftrightarrow \vec{r}_j$) does not lead to different many-body quantum states.

spin-statistics-theorem

The spin-statistics theorem states, that a many-body wave function is either symmetric (bosons) or antisymmetric (fermions) under particle exchange.

In QFT one asks "how many particles are there on each state". In this approach, the many-body state is represented in the occupation number basis and the basis state is labelled by the set of occupation numbers.

The occupation number states $|[n_\alpha]\rangle$ are known as *Fock states*

$$|[n_\alpha]\rangle \equiv |n_1, n_2, \dots, n_\alpha\rangle \quad (3.74)$$

meaning that there are n_α particles in the single-particle state $|\alpha\rangle$ (or as ψ_α). The occupation numbers sum up to total number of particles

$$N = \sum_\alpha n_\alpha. \quad (3.75)$$

The *number operator* N counts the number of particles in a given state in the Fock space

$$N = \int \frac{d^3p}{(2\pi)^3} a^\dagger(\vec{p}) a(\vec{p}) \quad \Rightarrow \quad N |p_1, \dots, p_n\rangle = n |p_1, \dots, p_n\rangle \quad (3.76)$$

and

$$[N, H] = 0 \quad \leftrightarrow \quad \text{particle number is conserved.} \quad (3.77)$$

This is only a property of *free theories*, but will no longer be true when we consider interactions.

For fermions, the occupation number n_α can only be 0 or 1, due to the Pauli exclusion principle, while for bosons its a non-negative integer

$$n_\alpha = \begin{cases} 0, 1 & \text{fermions} \\ 0, 1, 2, 3, \dots & \text{bosons} \end{cases}. \quad (3.78)$$

All the Fock states form a complete set of basis of the many-body Hilbert space, or the *Fock space*. Any generic quantum many-body state can be expressed as a linear combination of Fock states.

The Fock state with all occupation numbers equal to zero is called *vacuum state*, $|0\rangle \equiv |..., 0_\alpha, ... \rangle$. $|0\rangle \leftrightarrow$ state containing zero particles, in a non-interacting (free) field theory.

$$\psi_\alpha \underbrace{\otimes \cdots \otimes \psi_\alpha}_{n\text{-times}} \quad |n_\alpha\rangle \leftrightarrow$$

Let $\{|k^\mu\rangle\}_{k,\mu}$ be a set of eigenstates of P^μ , it the follows that

$$P^\mu a^\dagger(\vec{q}) |k^\mu\rangle = (k^\mu + q^\mu) a^\dagger(\vec{q}) |k^\mu\rangle \quad (3.79)$$

$$P^\mu a(\vec{q}) |k^\mu\rangle = (k^\mu - q^\mu) a(\vec{q}) |k^\mu\rangle \quad (3.80)$$

due to the commutation relations. Thus, $a(\vec{q}), a^\dagger(\vec{q})$ are ladder operators which add/subtract 4-momentum q^μ from $|k^\mu\rangle$.

Because the Hamiltonian is non-negative $\langle \psi | H | \psi \rangle \geq 0$, the vacuum is annihilated as

$$a(\vec{q}) |0\rangle = 0 \quad \forall \vec{q}, \quad P^\mu |0\rangle = \Delta_{p^\mu} |0\rangle = \begin{cases} \Delta_H & \mu = 0 \\ 0 & \mu = 1, 2, 3 \end{cases} \quad (3.81)$$

with $P^\mu a^\dagger(\vec{W}) |0\rangle = p^\mu a^\dagger(\vec{p}) |0\rangle$.

We thus define

$$P^\mu := \tilde{P}^\mu := P^\mu - \Delta_{p^\mu} = \int \frac{d^3 p}{(2\pi)^3} p^\mu a^\dagger(\vec{p}) a(\vec{p}), \quad (3.82)$$

such that $\tilde{P}^\mu |0\rangle = 0$. Therefore, an N -particle state with energy $E = E_{p_1} + \dots + E_{p_N}$ and momentum $\vec{p} = \vec{p}_1 + \dots + \vec{p}_N$ is given by

$$a^\dagger(\vec{p}_1) \dots a^\dagger(\vec{p}_N) |0\rangle = |p_1, \dots, p_N\rangle = |p_N, \dots, p_1\rangle, \quad (3.83)$$

since all $a^\dagger(\vec{p}_i)$ commute \Rightarrow state is symmetric under particle exchange. This definition of a state is not normalized as of yet, see further down below. If one pumps E_p, \vec{p} into some region of spacetime such that the relativistic dispersion relation $E_p = \sqrt{\vec{p}^2 + m^2}$ holds, a particle $a^\dagger(\vec{p}) |0\rangle$ is created as an excitation of $\phi(\vec{x})$. Since $\phi(x)$ is a scalar field, $a^\dagger(\vec{p}) |0\rangle$ is a scalar particle.

Field excitation interpretation as a particle

The field $\phi(\vec{x})$ is the property of spacetime that in the presence of energy and momentum (E_p, \vec{p}) a particle of energy (E_p, \vec{p}) can be created.

Scalar particles obey Bose statistics, therefore the N -particle bosonic wavefunction is symmetric under permutation. $|p_1, p_2\rangle = |p_2, p_1\rangle$ state is symmetric under particle exchange because $[a^\dagger(\vec{p}_1), a^\dagger(\vec{p}_2)] = 0 \Rightarrow$ particles are bosons.

Note that the construction of the Fock space is based on the existence of a mode decomposition of field operators into ladder operators, which only exists for free theories.

3.4.10 Spin 0 and Bose statistics of the scalar field

The 1-particle state at rest has no angular momentum, i.e. the eigenvalue of the angular momentum operator of $(E_{\vec{p}}, \vec{\hat{0}})$ vanishes:

$$J_i a^\dagger(\vec{0}) |0\rangle = 0. \quad (3.84)$$

We say the *scalar field has spin 0*.

N -particle states 3.86 are invariant under permutations, i.e.

$$|p, q\rangle = |q, p\rangle. \quad (3.85)$$

From this we will later obtain Bose statistics in the context of finite temperature QFT.

It may seem like we artificially put 3.85 into our theory by quantizing the scalar field via commutators rather than anticommutators, but in fact the bosonic way is the only way to quantize the scalar field consistently. Imposing anticommutator relations for the scalar field would lead to a Hamiltonian that is unbound from below and therefore has no stable vacuum. This can easily be seen from 3.68 where a global minus sign appears for $\{a(\vec{p}), a^\dagger(\vec{q})\} = (2\pi)^3 \delta(\vec{p} - \vec{q})$.

$$\phi(\vec{x}) |0\rangle = |\vec{x}\rangle.$$

3.4.11 Technicalities

3.4.11.1 Normalization

Normalization of momentum eigenstates

The N -particle momentum eigenstates are normalized via

$$|\vec{p}_1, \dots, \vec{p}_N\rangle = \sqrt{2E_{p_1} \dots 2E_{p_N}} a^\dagger(\vec{p}_1) \dots a^\dagger(\vec{p}_N) |0\rangle \quad (3.86)$$

$$\Rightarrow \langle \vec{q} | \vec{p} \rangle = (2\pi)^3 2E_p \delta_D^{(3)}(\vec{p} - \vec{q}). \quad (3.87)$$

These states are relativistically normalized, thus Lorentz invariant. We can interpret this as the N -particle state with energy $E_{\vec{p}_1} + \dots + E_{\vec{p}_N}$ and momentum $\vec{p}_1 + \dots + \vec{p}_N$:

$$P^\mu |p_1, \dots, p_N\rangle = (p_1^\mu + \dots + p_N^\mu) |p_1, \dots, p_N\rangle \text{ with } p_i^0 = \sqrt{|\vec{p}_i|^2 + m^2}. \quad (3.88)$$

The space spanned by $|p_1\rangle, |p_1, p_2\rangle, \dots, |p_1, \dots, p_N\rangle$ for $N \rightarrow \infty$ is called the Fock space

$$\mathcal{F} := \bigoplus_{n=1}^{\infty} \mathcal{H}_n \quad (3.89)$$

and the identity is in fact the projection operator on the one-particle Hilbert space \mathcal{H}_1 .

A complete set of eigenstates of \mathcal{F} is

$$\mathcal{I} = \sum_{n=1}^{\infty} \left(\prod_{i=1}^n \int \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{p_i}} \right) |p_1, \dots, p_n\rangle \langle p_1, \dots, p_n|. \quad (3.90)$$

Similar to QM, the momentum and position eigenstates are not normalizable. Neither the operator $\phi(x)$, nor $a(\vec{p})$ are good operators acting on the Fock space, because they produce non-normalizable states. They are operator valued distributions \Rightarrow WE can construct well defined operators by having them form a wavepacket

$$\begin{aligned} \langle 0 | a(\vec{p}) a^\dagger(\vec{p}) | 0 \rangle &= \\ \langle \vec{p} | \vec{p} \rangle &= \\ (2\pi)^3 \delta_D(0) 2E_p. \end{aligned}$$

$$|f\rangle = \int \frac{d^3 p}{(2\pi)^3} e^{-i\vec{x}\cdot\vec{p}} f(\vec{p}) |\vec{p}\rangle, \quad \text{e.g. } f(\vec{p}) = \exp\left\{-\frac{\vec{p}^2}{2m^2}\right\} \text{ Gaussian.} \quad (3.91)$$

3.4.11.2 Identity

The identity operator on the 1-particle Hilbert-space (hence the projection operator onto the 1-particle Hilbert space) is

$$\mathcal{I}_{1\text{-particle}} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} |\vec{p}\rangle \langle \vec{p}|. \quad (3.92)$$

It is Lorentz-invariant, because the measure is Lorentz invariant

$$\int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} = \int \frac{d^4 p}{(2\pi)^4} \delta_D(p^2 - m^2) \theta(p^0). \quad (3.93)$$

3.4.11.3 Position-space representation

For the free theory only

$$|\vec{x}\rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-i\vec{p}\vec{x}} |\vec{p}\rangle \quad (3.94)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [e^{-i\vec{p}\vec{x}} a^\dagger(\vec{p}) + e^{i\vec{p}\vec{x}} a(\vec{p})] |0\rangle \quad (3.95)$$

$$|\vec{x}\rangle = \phi(\vec{x}) |0\rangle. \quad (3.96)$$

Therefore, the field operator acting on the vacuum *creates* a 1-particle position eigenstate.

3.4.12 On the vacuum energy

There are two different infinities in QFT:

1. Infra-red (IR) divergence:

The divergence of $\delta_D^{(3)}(0)$ is rooted in the fact that the volume of \mathbb{R}^3 is infinite. This divergent factor thus results from the long-distance, i.e. small energy, behaviour of the theory. IR divergences signal that we are either *making a mistake or ask an unphysical question*.

Only appears in a theory with massless particles.

One can regularize the IR-divergence by instead considering the theory in a given, but finite volume. Thus, one can deal with it by performing:

- a) Impose an infra-red cutoff and take the limit as the cutoff approaches zero.
- b) Then refine the question

The vacuum density is free of IR-divergence

$$\epsilon_0 = \frac{E_0}{V_{\mathbb{R}^3}} = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \omega_p \rightarrow \infty, \quad (3.97)$$

diverges as well, but due to an UV-divergence.

We can renormalize this divergence by absorbing ϵ_0 into V_0 in the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial\phi)^2 - \frac{1}{2} m^2 \phi^2 - V_0.$$

Free scalar vacuum energy renormalization by cutoff regularization

Vacuum momentum eigenvalues are $P^\mu |0\rangle = \Delta^\mu |0\rangle$ and the vacuum energy density is

$$\langle 0 | \mathcal{H} | 0 \rangle = \frac{\Delta^0}{\text{vol}(\mathbb{R}^3)} = \frac{1}{2} \frac{4\pi}{(2\pi)^3} \int_0^\infty d|\vec{p}| |\vec{p}|^2 \sqrt{|\vec{p}|^2 + m^2} \rightarrow \infty \text{ like } |\vec{p}|^4. \quad (3.98)$$

We can add constants to Lagrangian and Hamiltonian with no physical effect

$$\tilde{\mathcal{L}} = \mathcal{L} - V_0, \quad \tilde{H} = H + \int d^3x V_0. \quad (3.99)$$

The cutoff regularization (breaks Lorentz symmetry and therefore leads to incorrect result for \mathcal{H}) now entails

$$V_0 = V_0(\Lambda) = \rho - \langle \mathcal{H} \rangle(\Lambda) \quad (3.100)$$

with

$$\langle \mathcal{H} \rangle(\Lambda) = \frac{1}{(2\pi)^2} \int_0^\Lambda d|\vec{p}| |\vec{p}|^2 \sqrt{|\vec{p}|^2 + m^2} \propto \Lambda^4 \quad (3.101)$$

and experimentally determine the renormalized vacuum energy density ρ that is finite as $\Lambda \rightarrow \infty$

$$\frac{\tilde{H}}{\text{vol}\mathbb{R}^3} |0\rangle = \rho |0\rangle. \quad (3.102)$$

Note that because cutoff regularization of the 3-momentum $|\vec{p}|$ breaks Lorentz symmetry, the result $\langle \mathcal{H} \rangle \propto \Lambda^4$ is incorrect. A regularization scheme that respects Lorentz symmetry is dimensional regularization and leads to $\langle \mathcal{H} \rangle \propto m^4$. Nevertheless, adding an infinite constant to the Hamiltonian (and Lagrangian) such that the difference of two infinities becomes a finite observable is the basic idea of renormalization. We will come back to a more detailed discussion of vacuum energy regularization and renormalization and its connection to the cosmological constant of GR

do this, refer
there

2. Ultra-violet (UV) divergence:

This is a high frequency, i.e. short distance or high energy, infinity. It arises because the theory breaks down at high energies (equivalently at short distances). Because we're only interested in energy differences, P^μ was redefined by subtracting the divergence.

In a good QFT the UV divergences can be removed by the powerful machinery of regularization and renormalization.

a) Regularization:

Is a method of modifying observables which have singularities in order to make them finite by introduction of a suitable parameter called cutoff. The correct physical result is obtained in the limit in which the cutoff goes away, but the virtue of the cutoff is that for its finite value, the result is finite.

b) Renormalization:

Is a collection of techniques in QFT that are used to treat infinities arising in calculated quantities by altering values of quantities to compensate for effects of their self-interactions. The divergence is thus absorbed into a term such that the measured quantity is finite. Regularizing a theory by e.g a cutoff and absorbing the divergence thus via renormalization into quantity comes at a price: We lose the prediction of one observable *per type* of UV divergence as a result of the inherent arbitrariness of the renormalization step. While the divergence can be removed by renormalizing the original Lagrangian, the actual value of the physical observable associated with the divergence must be taken as an input parameter from experiment or from other considerations.

You give up on predicting N parameters in order to hide N divergences.

3.4.13 The complex scalar field

The formalism of the free, real scalar field can be extended to the theory of a *complex scalar field* by describing it as a linear combination of independent real scalar fields ϕ_1, ϕ_2 :

$$\phi(x) = \frac{1}{\sqrt{2}} [\phi_1(x) + i\phi_2(x)] \quad (3.103)$$

$$\Rightarrow \mathcal{L} = \partial_\mu \phi^\dagger(x) \partial^\mu \phi(x) - m^2 \phi^\dagger(x) \phi(x). \quad (3.104)$$

Thus, treat (Π, ϕ) & $(\Pi^\dagger, \phi^\dagger)$ as *independent* in H

$$\Pi(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\vec{x}, t)} = \dot{\phi}^\dagger(\vec{x}, t) \quad (3.105)$$

$$\Pi^\dagger(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^\dagger(\vec{x}, t)} = \dot{\phi}(\vec{x}, t). \quad (3.106)$$

The fields are promoted to time-independent self-adjoint operators in Schrödinger picture (quantization)

Commutation relations for the scalar field

$[\phi(\vec{x}), \Pi(\vec{y})] = i\delta_D^{(3)}(\vec{x} - \vec{y}) = [\phi^\dagger(\vec{x}), \Pi^\dagger(\vec{y})], \quad (3.107)$

and all other commutators vanishing.

Since the classical field ϕ is not real, the corresponding quantum field ϕ is not hermitian, thus different operators $a(\vec{p}), b^\dagger(\vec{p})$ appear in the positive and negative frequency parts. It thus has the mode expansion:

$$\phi(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [a(\vec{p})e^{i\vec{p}\vec{x}} + b^\dagger(\vec{p})e^{-i\vec{p}\vec{x}}], \quad (3.108)$$

where $a = 1/\sqrt{2}(a_1 + ia_2), b^\dagger = 1/\sqrt{2}(a_1^\dagger + ia_2^\dagger)$.

Commutation relations of ladder operators of complex scalar field

$$[a(\vec{p}, a^\dagger(\vec{q}))] = (2\pi)^3 \delta_D^{(3)}(\vec{p} - \vec{q}) = [b(\vec{p}), b^\dagger(\vec{q})], \quad (3.109)$$

with all other commutators vanishing.

Quantizing a complex scalar field gives rise to two creation operators $b^\dagger(\vec{p}), a^\dagger(\vec{p})$. These have the interpretation of creating two types of particle, both of mass m and both spin zero. They are interpreted as particles and anti-particles, for a real scalar field, the particle is its own anti-particle.

⇒ This Lagrangian is *invariant* under the global continuous $U(1)$ symmetry $\phi \rightarrow e^{i\alpha} \phi$ with the conserved current

$$j^\mu = -i [\phi^\dagger \partial^\mu \phi - \partial^\mu \phi^\dagger \phi] \quad (3.110)$$

and the conserved charge

$$Q = \int d^3x j^0 = - \int \frac{d^3 p}{(2\pi)^3} [a^\dagger(\vec{p})a(\vec{p}) - b^\dagger(\vec{p})b(\vec{p})] \quad (3.111)$$

with

$$Q(a^\dagger(\vec{p})|0\rangle) = -a^\dagger(\vec{p})|0\rangle : \text{particle of charge } -1 \quad (3.112)$$

$$Q(b^\dagger(\vec{p})|0\rangle) = +b^\dagger(\vec{p})|0\rangle : \text{particle of charge } +1. \quad (3.113)$$

Here, $a^\dagger(\vec{p})|0\rangle$ and $b^\dagger(\vec{p})|0\rangle$ both are momentum eigenstates of $P^\mu = \int \frac{d^3 p}{(2\pi)^3} p^\mu [a^\dagger(\vec{p})a(\vec{p}) + b^\dagger(\vec{p})b(\vec{p})]$ and have energy $E_p^2 = \vec{p}^2 + m^2$.

⇒ $Q = N_c - N_b$ counts the number of anti-particles minus the number of particles. We have $[H, Q] = 0$. This is nothing special, because in free theory $[H, N_b] = 0 = [H, N_c]$. But $[H, Q] = 0$ holds still in interacting theory, while $[H, N_c] = [H, N_b] = 0$ does not.

3.4.14 Quantization in the Heisenberg picture

It is not trivial to assume Lorentz invariance of time independent Schrödinger operators. Therefore, switch to Heisenberg picture with the time-dependence now being carried by the operators:

$$A^{(H)}(t) = e^{iH^{(s)}(t-t_0)} A^{(s)} e^{-iH^{(s)}(t-t_0)} \quad (3.114)$$

$$\frac{d}{dt} A^{(H)}(t) = i[H, A^{(H)}(t)] \quad (3.115)$$

$$H^{(H)}(t) = H^{(s)} \quad \forall t, \quad A^{(H)}(t_0) = A^{(s)}. \quad (3.116)$$

The Heisenberg operators obey *equal-time* canonical commutation relations:

$$[\phi(t, \vec{x}), \Pi(t, \vec{y})] = i\delta_D^{(3)}(\vec{x} - \vec{y}), \quad (3.117)$$

$$[\phi(t, \vec{x}), \phi(t, \vec{y})] = 0 = [\Pi(t, \vec{x}), \Pi(t, \vec{y})], \quad \phi(x) = e^{iH_s t} \phi_s(\vec{x}) e^{-iH_s t}. \quad (3.118)$$

For a *real* scalar field:

$$\frac{\partial}{\partial t} \phi(t, \vec{x}) = i[H, \phi(t, \vec{x})] = \Pi(t, \vec{x}) \quad (3.119)$$

$$\frac{\partial}{\partial t} \Pi(t, \vec{x}) = i[H, \Pi(t, \vec{x})] = \nabla^2 \phi(t, \vec{x}) - m^2 \phi(t, \vec{x}), \quad (3.120)$$

the *field operator* ϕ then satisfies the *operator Klein-Gordon equation* at the quantum level

$$[\partial_\mu \partial^\mu + m^2] \hat{\phi}(x) = 0. \quad (3.121)$$

We find an expression in the mode expansion by putting

$$\partial^\mu \phi(x) = i[P^\mu, \phi(x)] \Rightarrow \phi(x^\mu + a^\mu) = e^{ia^\mu P_\mu} \phi(x) e^{-ip^\rho P_\rho} \quad (3.122)$$

with the 4-momentum P_μ as the *generator of the set of translations, hence the translation operator*.

Mode expansion of the real quantized field in the Heisenberg picture

We thus find

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [a(\vec{p}) e^{-ip \cdot x} + a^\dagger(\vec{p}) e^{ip \cdot x}], \quad (3.123)$$

thus now the particle and antiparticle are moving also in the time direction in the Heisenberg picture, i.e. $e^{-ip \cdot x} \rightarrow e^{-ip \cdot x}$. With the coefficient of $e^{-ip \cdot x}$ in the mode expansion corresponding to the annihilator and the coefficient of $e^{ip \cdot x}$ to the creator.

The inverted expression reads

$$a(\vec{q}) = \frac{1}{\sqrt{2E_q}} \int d^3 x e^{iq \cdot x} \overset{\leftrightarrow}{\partial}_0 \phi(x), \quad u(x) \overset{\leftrightarrow}{\partial}_0 v(x) := u(x) \partial_0 v(x) - (\partial_0 u(x)) v(x). \quad (3.124)$$

3.4.15 Causality and Propagators

3.4.15.1 Causality

For causality to hold we need to measurements at spacelike distance not to affect each other. This is guaranteed if any two local observables $O_1(x)$ and $O_2(y)$ at spacelike separation commute, i.e.

$$[O_1(x), O_2(y)] \stackrel{!}{=} 0, \quad \text{for } (x-y)^2 < 0. \quad (3.125)$$

local \equiv local operators only depend on a local neighbourhood of the spacetime point x .

This ensure that a measurement at x cannot affect a measurement at y when x and y are *not causally connected*.

This theory is *indeed causal* with commutators vanishing outside the lightcone

$$\Delta(x-y)|_{(x-y)^2 < 0} = [\phi(x), \phi(y)]|_{(x-y)^2 < 0} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} [e^{-ip(x-y)} - e^{+ip(x-y)}]|_{(x-y)^2 < 0} = 0 \quad (3.126)$$

The states of QFT are non-local objects.

because one can always make a Lorentz transformation such that $(x^0 - y^0) = 0$ for spacelike separation and change the minus sign of the integration variable. This property will continue to hold in interacting theories, it is usually given as an axiom of local QFTs.

The fact that $[\phi(x), \phi(y)]$ is a \mathbb{C} -number function, rather than an operator, is a property of *free fields only*.

3.4.15.2 Propagators, for a real scalar field

Propagator

The probability amplitude for a particle emitted at y to propagate to x is given by the propagator

$$D(x-y) := \langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (x-y)}. \quad (3.127)$$

Thus, the propagator is the *two-point correlation function* and has this form in a free scalar theory.

For spacelike separations $(x-y)^2 < 0$ one can show that the propagator decays exponentially quickly outside the lightcone, but it is nonetheless non-vanishing. Yet we've seen, that spacelike measurements commute and the theory is causal. Consistency?

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x) \Rightarrow [\phi(x), \phi(y)] = 0 \quad \text{for } (x-y)^2 < 0. \quad (3.128)$$

\Rightarrow At spacelike distances, both processes $x \rightarrow y, y \rightarrow x$ can occur and cancel each other in the sense of a destructive quantum mechanical interference out in $\Delta(x-y)$.

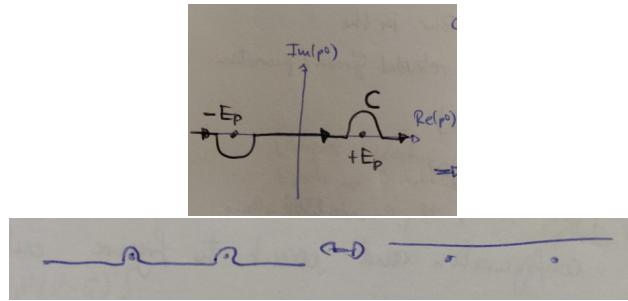


Figure 3.1: Integration contour for Feynman propagator of the real scalar field.

For a *complex scalar field*

$[\phi(x), \phi^\dagger(y)] = 0$ for $(x - y)^2 < 0$ implies to a particle travelling from y to x and an anti-particle travelling from x to y that cancels the amplitude of the respective other.

Thus, the field formalism saves causality in the QM sense even though the QM probability for a propagation $x \rightarrow y$ itself is non-zero if $(x - y)^2 < 0$. This is why a single particle approach must fail.

The Feynman-propagator for a real scalar field

$$D_F(x-y) := \langle 0 | T\phi(x)\phi(y) | 0 \rangle = \theta(x^0 - y^0)D(x-y) + \theta(y^0 - x^0)D(y-x), \quad (3.129)$$

where the time/normal ordering operator t puts latest times to the left

$$T\phi(x)\phi(y) = \begin{cases} \phi(x)\phi(y) & x^0 \geq y^0 \\ \phi(y)\phi(y) & y^0 \geq x^0 \end{cases}. \quad (3.130)$$

The Feynman-propagator can be evaluated by means of a complex integration, for a free theory

$$D_F(x-y) = \oint_C \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)} \quad (3.131)$$

$$= \lim_{\epsilon \rightarrow 0} \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}, \quad (3.132)$$

with p^0 integration along the real axis and with the limit $\epsilon \rightarrow 0$ after performing the integral. The $i\epsilon$ term represents time ordering. Important is only the position of the poles with respect to the contour, thus the relative position.

3.4.15.3 Propagators as Green's functions

The Feynman propagator $D_F(x - y)$ is a Green's function for the Klein-Gordon equation:

$$(\partial_x^2 + m^2)D_F(x - y) = -i\delta_D^{(4)}(x - y). \quad (3.133)$$

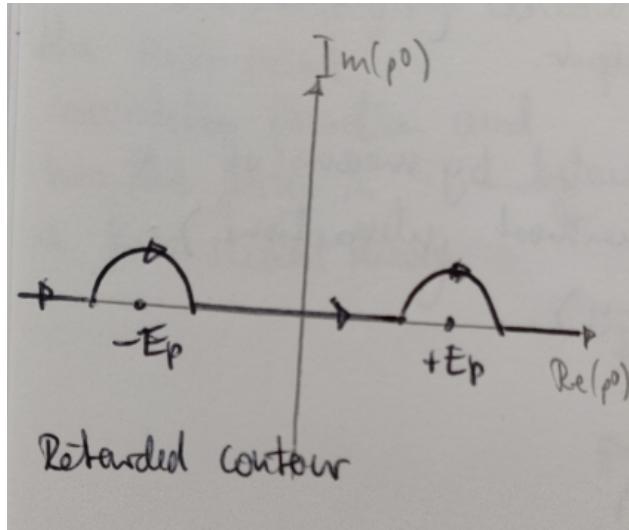


Figure 3.2: Retarded contour.

1. By avoiding both poles along a contour in the upper half-plane, the solution is the retarded Green's function

$$D_R(x - y) = \theta(x^0 - y^0)[D(x - y) - D(y - x)] \equiv \theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle. \quad (3.134)$$

D_R is useful in classical field theory if we know the initial value of some field configuration and want to figure out what it evolves into in the presence of the source. It propagates information backward in time.

2. Avoiding both poles $p^0 = \pm\sqrt{E_{\vec{p}}^2}$ in the lower half plane yields the advanced Green's function

$$D_A(x - y) = \theta(y^0 - x^0)[D(x - y) - D(y - x)]. \quad (3.135)$$

D_A is useful if we know the end point of a field configuration and want to figure out where it came from. It propagates forward in time.

3. $D_F(x - y)$ propagates positive frequency modes e^{-ipx} forward in time and negative frequency modes e^{ipx} backward in time.

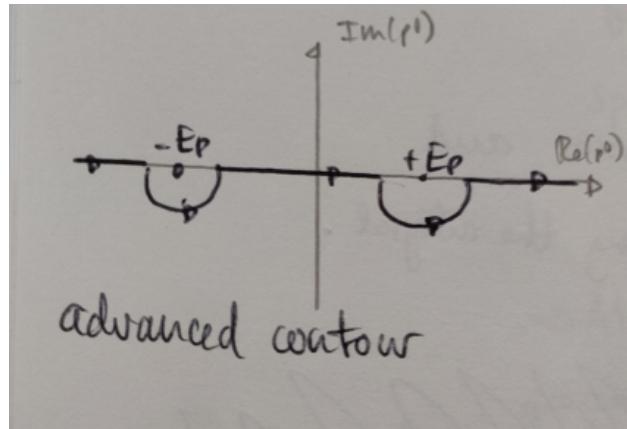


Figure 3.3: Advanced contour.

3.4.15.4 How to perform the contour integral

Define a function $g(z)$ with poles z_0 and apply Residue theorem:

$$\begin{aligned}
 g(z) &= \frac{1}{E_p + z} e^{-iz(x^0 - y^0)}, \quad z_0 = E_p, z = p^0. \\
 \Rightarrow \theta(x^0 - y^0) &= g(z_0) = \theta(x^0 - y^0) \left[\frac{1}{2\pi i} \oint_{C_1} \frac{g(z) dz}{z - z_0} \right] \\
 &= \theta(x^0 - y^0) \left[\frac{1}{2\pi i} \oint_{C_1} \left(\frac{1}{E_p + p^0} e^{-ip^0(x^0 - y^0)} \right) \frac{dp^0}{p^0 - E_p} \right] \\
 &= \theta(x^0 - y^0) \left[\frac{1}{2\pi i} \oint_{C_1} dp^0 \frac{e^{-ip^0(x^0 - y^0)}}{(p^0 + E_p)(p^0 - E_p)} \right] \\
 &= -\theta(x^0 - y^0) \frac{1}{2\pi i} \oint_C dp^0 \frac{e^{-ip^0(x^0 - y^0)}}{(p^0 + E_p)(p^0 - E_p)},
 \end{aligned}$$

where the minus comes about since the integral is performed clockwise ($\Rightarrow \times(-1)$) and because it picks up the pole at $+E_p \Rightarrow (-1) \times (+1) = -1$.

$$\begin{aligned}
 \theta(y^0 - x^0) \frac{1}{2E_p} e^{iE_p(x^0 - y^0)} &= \theta(y^0 - x^0) g(z_0) \\
 &= \theta(y^0 - x^0) \left[\frac{1}{2\pi i} \oint_{C_2} \left(\frac{1}{p^0 + E_p} e^{-ip^0(x^0 - y^0)} \right) \frac{dp^0}{p^0 - E_p} \right] \\
 &= -\theta(y^0 - x^0) \frac{1}{2\pi i} \oint_{C_2} dp^0 \frac{e^{-ip^0(x^0 - y^0)}}{(p^0 + E_p)(p^0 - E_p)},
 \end{aligned}$$

where the minus sign comes about since the integral is performed counter-clockwise ($\Rightarrow \times + 1$) and because it picks up pole at $-E_p \Rightarrow +1 \times (-1) =$

-1. Thus, the Feynman propagator is given by the addition of both contours

$$\begin{aligned}
D_F(x - y) &= \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}(\vec{x} - \vec{y})} \left[-\theta(x^0 - y^0) \frac{1}{2\pi i} \oint_{C_1} dp^0 \frac{e^{-ip^0(x^0 - y^0)}}{(p^0 + E_p)(p^0 - E_p)} \right. \\
&\quad \left. - \theta(y^0 - x^0) \frac{1}{2\pi i} \oint_{C_2} dp^0 \frac{e^{-ip^0(x^0 - y^0)}}{(p^0 - E_p)(p^0 + E_p)} \right] \\
&\stackrel{R \rightarrow \infty}{=} - \oint \frac{d^4 p}{(2\pi)^4} \frac{1}{i} e^{-ip \cdot (x-y)} \underbrace{[\theta(x^0 - y^0) + \theta(y^0 - x^0)]}_{=1} \frac{1}{(p^0 + E_p)(p^0 - E_p)} \\
&= \oint \frac{d^4 p}{(2\pi)^4} \underbrace{\frac{ie^{-ip \cdot (x-y)}}{(p^0 + E_p)(p^0 - E_p)}}_{=p^2 - m^2}.
\end{aligned}$$

3.5 INTERACTING SCALAR THEORY

3.5.1 Introduction

Our consideration of free scalar field theories showed, that the theory is *exactly solvable*, we can determine the spectrum (Hilbert space is the Fock space of multi-particle states created from the vacuum $|0\rangle$) and the fields have particle excitations which do not interact.

Interactions are described in QFT by potentials $V(\phi)$ beyond quadratic order

$$V(\phi) = \underbrace{\frac{1}{2}m_0^2\phi^2}_{V_0(\phi)} + \sum_{n \geq 3} \frac{\lambda_n}{n!}\phi^n. \quad (3.136)$$

The coefficients λ_n are called *coupling constants*. Here we restrict ourselves to

$$V(\phi) = \frac{1}{2}m_0^2\phi^2 + \underbrace{\frac{1}{3!}g\phi^3 + \frac{1}{4!}\lambda\phi^4}_{V_{\text{int}}}, \quad [\lambda_n] = 4 - n \neq 0!, \quad (3.137)$$

with the decomposition

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \quad \mathcal{L}_{\text{int}} = -V_{\text{int}}, \quad \mathcal{L}_{\text{int}} = -\mathcal{H}_{\text{int}}, \quad H = H_0 + H_{\text{int}}. \quad (3.138)$$

Introducing interaction terms leads to changes in the theory:

1. The Hilbert space is different from the free theory
 - a) $|0\rangle \leftrightarrow$ vacuum of H_0 : $H_0|0\rangle = E_0|0\rangle$.
 - b) $|\Omega\rangle \leftrightarrow$ vacuum of H : $H|\Omega\rangle = E_\Omega|\Omega\rangle$ with $|\Omega\rangle \neq |0\rangle$ in general.
 - c) Bound states (e.g. hydrogen) may exist in the spectrum.
2. The states interact.

The coupling constants are characterized as follows:

1. $[\lambda_3 = g] = 1$: The dimensionless parameter is λ_3/E , E being the energy scale of the process of interest. This means that $\lambda_3 \frac{\phi^3}{3!}$ is a small perturbation at high energies $E \gg \lambda_3$, but a large perturbation at low energies $E \ll \lambda_3$. Terms that we add to the Lagrangian with this behaviour are called *relevant* because they are most relevant at low energies.

Only doing weakly coupled field theories here, because they can be considered as small perturbation of free field theory.

2. $[\lambda_4 = \lambda] = 0$: This term is small if $\lambda_4 \ll 1$. Such perturbations are called *marginal*. If $\lambda_4 \ll 1$ then perturbation theory is applicable.
3. $[\lambda_n] < 0$ for $n \geq 5$: The dimensionless parameter is $(\lambda_n E^{n-4})$, which is small at low-energies and large at high energies. Such perturbations are called *irrelevant*.
4. Of the infinite number of interaction terms that we could write down, only $g\lambda$ are needed (for real scalar field, else some more), because the irrelevant couplings become small at low-energies, our field of interest.

Exact solution of non-free QFT is often not possible.

3.5.2 Källén-Lehmann spectral representation

Here we take a look at the spectrum of an interacting real scalar field theory in a manner valid for all types of interactions and without relying on perturbation theory. This representation gives a *general expression* for the time ordered two-point function of an interacting quantum field theory as a sum of free propagators.

In interacting theory $[H, \vec{P}] = 0$ still holds due to Lorentz invariance. Their mutual eigenstates $|\lambda_p\rangle$ with

$$H|\lambda_p\rangle = E_p(\lambda)|\lambda_p\rangle, \quad \vec{P}|\lambda_p\rangle = \vec{p}|\lambda_p\rangle \quad (3.139)$$

correspond via a Lorentz boost to the state at rest, called $|\lambda_0\rangle$.

We can have the following types of $|\lambda_{\vec{p}}\rangle$:

1. 1-particle states with $E_p^2 = \vec{p}^2 + m^2$, they have $p^\nu p_\nu = m^2$, $m \neq m_0$ (even in vacuum $m \neq m_0$ because of self-interactions).
2. Bound states with no analogue in the free theory.
3. 2-and N -particle states formed out of 1-particle and the bound states. In this case, we take \vec{p} to the centre-of-mass momentum of the multi-particle state.

All these are created from the vacuum $|\Omega\rangle$. The crucial difference to the free theory is, that $\phi(x)$ cannot simply be written as a superposition of its Fourier amplitudes $a(\vec{p})$ and $a^\dagger(\vec{p})$, because $\phi(x)$ does not obey the free e.o.m, rather

$$(\partial^2 + m^2)\phi = j. \quad (3.140)$$

Thus, acting with ϕ on $|\Omega\rangle$ does not simply create a 1-particle state as in the free theory!

Completeness relation of this Hilbert space:

$$\mathcal{I} = |\Omega\rangle\langle\Omega| + \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} |\lambda_p\rangle\langle\lambda_p|, \quad (3.141)$$

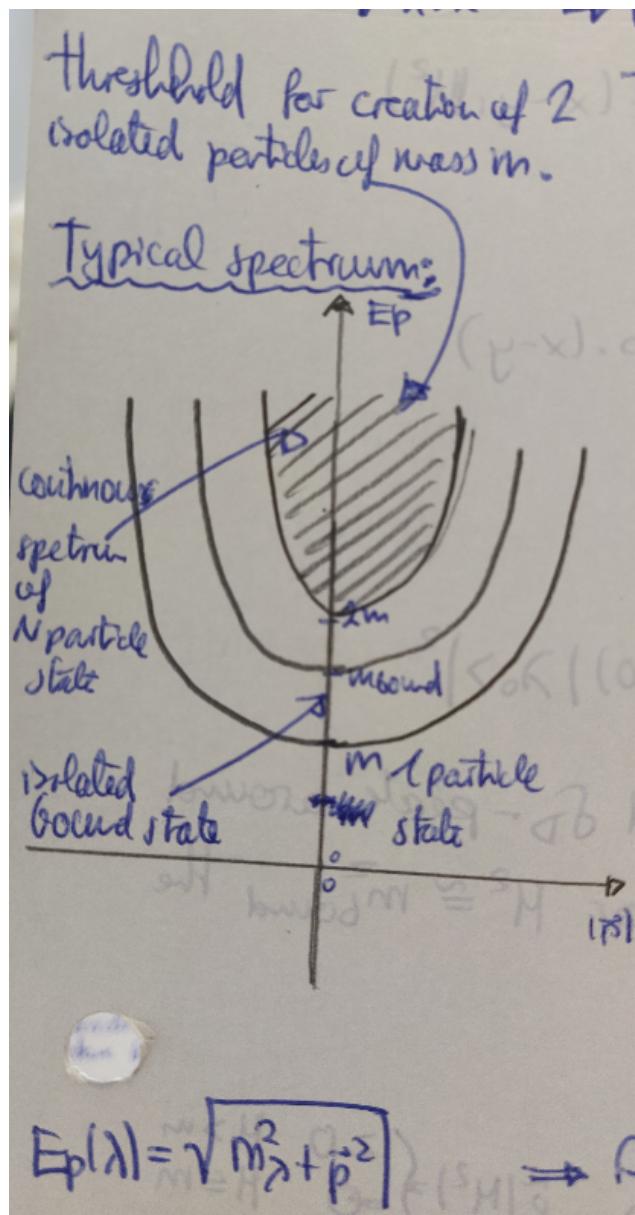


Figure 3.4: Typical spectrum in the interacting theory.

where $E_p(\lambda) = \sqrt{m_\lambda^2 + \vec{p}^2}$ and \sum includes a sum over 1-particle states, over all types of bound states as well as over all multiparticle states. $\int d^3p \dots$ refer to the centre-of-mass momentum of a state of species λ .

Feynman-propagator in interacting scalar theory

We find the time-ordered interacting Feynman-propagator to be

$$\langle \Omega | T\phi(x)\phi(y) | \Omega \rangle = \sum_{\lambda} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_{\lambda}^2 + i\epsilon} e^{-i\cdot(x-y)} |\langle \Omega | \phi(0) | \lambda_0 \rangle|^2, \quad (3.142)$$

or rather in the Kallén-Lehmann spectral representation

$$\langle \Omega | T\phi(x)\phi(y) | \Omega \rangle = \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) D_F(x-y, M^2), \quad (3.143)$$

where we defined

$$D_F(x-y, M^2) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - M^2 + i\epsilon} e^{-ip\cdot(x-y)} \quad (3.144)$$

and the *spectral function* (density)

$$\rho(M^2) = \sum_{\lambda} 2\pi\delta_D(M^2 - m_{\lambda}^2) |\langle \Omega | \phi(0) | \lambda_0 \rangle|^2. \quad (3.145)$$

The 1-particle state lead to an isolated δ_D -peak around $M^2 = m^2$. Therefore, below $M^2 \approx (2m)^2$ or $M^2 \approx m_{\text{bound}}^2$ the spetral function takes the form

$$\rho(M^2) = 2\pi\delta_D(M^2 - m^2) Z \quad (3.146)$$

$$= 2\pi\delta_D(M^2 - m^2) Z + \tilde{\rho}(M^2), \quad \tilde{\rho}(M^2) \begin{cases} \geq 0 & M > m \\ = 0 & M \leq m. \end{cases} \quad (3.147)$$

with the *wavefunction renormalization* Z which takes as a rescale factor the effects of interactions or quantum fields into account

$$Z = |\langle \Omega | \phi(0) | 1_0 \rangle|^2, \quad (3.148)$$

where $|1_0\rangle$ is the 1-particle state at rest.

We find the following statements

1. The 1-particle state is the first analytic pole of the fully Feynman propagator at $m^2 \Rightarrow$ The mass-square m^2 of the particle is the location of the lowest-lying pole of the Fourier transformed propagator.

In general
 $\int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) = 1$
holds.

Calculation of the propagator yields the mass of the particle !

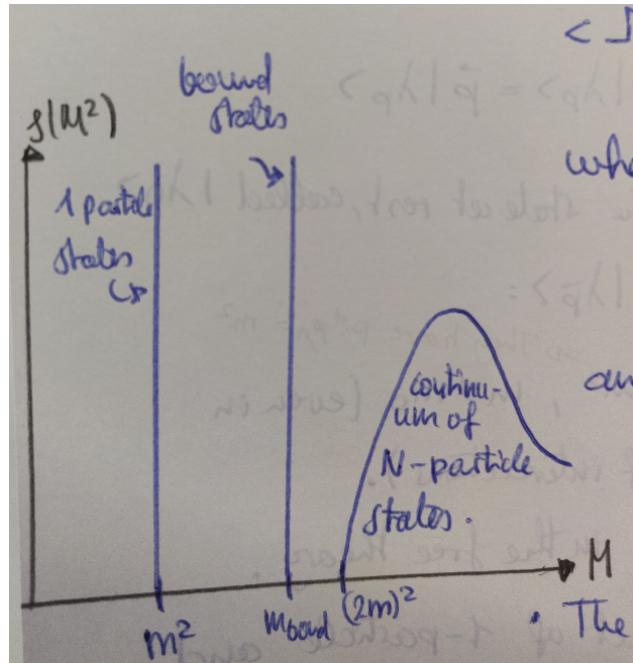


Figure 3.5: Particle spectrum with respect to the mass.

2. Bound states appear at higher isolated poles
3. N-particle states give rise to a branch cut beginning at $p^2 = em^4$.

The field strength renormalization in a free theory, i.e. $Z = 1$, because $\phi(0)$ just creates the free particle from vacuum. In an interacting theory

$$1 > \sqrt{Z} = |\langle \Omega | \phi(0) | 1_0 \rangle|, \quad (3.149)$$

because ϕ creates not only 1-particle states and thus the overlap with the 1-particle states is smaller. Thus,

Statements 3.1 $Z = 1$ if and only if the theory is free.

In the derivation of the interacting Feynman propagator we made use of the transformation behaviour of a scalar field under a Lorentz transformation

$$x \mapsto x' = \Lambda x \Rightarrow U^{-1}(\Lambda) \phi(x') U(\Lambda) = \phi(x), \quad (3.150)$$

because classically $\phi(x) \mapsto \phi'(x') = \phi(x)$ has its analogue in

$$\Leftrightarrow \langle \alpha' | \phi(x') | \beta' \rangle = \langle \alpha | \phi(x) | \beta \rangle = \langle \alpha | U^{-1} \phi(x') U | \beta \rangle. \quad (3.151)$$

3.5.3 S-matrix and asymptotic in/out-states

Consider scattering of incoming states $|\alpha, in\rangle$ to outgoing states $|\beta, out\rangle$ with the aim of computing the QM transition amplitude, i.e. the probability amplitude for scattering of $|\alpha, in\rangle$ to $|\beta, out\rangle$.

Can be derived in the interaction picture as well, but here with in & out states.

In and out states

Associated to the in and out states are the in and out field operators ϕ_{in}, ϕ_{out} , which satisfy

$$(\partial^2 + \frac{m^2}{\neq m_0^2} \phi_{in,out}(x) = 0!!.) \quad (3.152)$$

They have the same equal time commutation relations

$$[\phi_{in,out}(x), \Pi_{in,out}(y)]_{x^0=y^0} = i\delta_D^{(3)}(\vec{x} - \vec{y}), \quad (3.153)$$

with others vanishing.

Associated to the in and out fields are two sets of creation and annihilation operators, $a_{in}^\dagger(\vec{p}), a_{in}(\vec{p}), a_{out}^\dagger(\vec{p}), a_{out}(\vec{p})$, acting in the same Hilbert space, on two complete sets (Fock spaces, initial \mathcal{F}_{in} and final space \mathcal{F}_{out}). These operators satisfy the commutation relations

$$[a_{in,out}(\vec{p}), a_{in,out}^\dagger(\vec{p}')] = i\delta_D(\vec{p} - \vec{p}'), \quad (3.154)$$

with others vanishing.

The action of the creation operators on the respective vacua and states with a finite number of particles in the in and out states is given by

$$|in, q_1, \dots, q_n\rangle = a_{in}^\dagger(\vec{q}_1) \dots a_{in}^\dagger(\vec{q}_n) |\Omega, in\rangle \quad (3.155)$$

$$|p_1, \dots, p_n, out\rangle = a^\dagger(\vec{p}_1) \dots a^\dagger(\vec{p}_n) |\Omega, out\rangle \quad (3.156)$$

$$\mathcal{H}_i = \text{span}\{|in, q_1, \dots, q_n\rangle\} \quad (3.157)$$

$$\mathcal{H}_f = \text{span}\{|out, p_1, \dots, p_n\rangle\}, \quad (3.158)$$

where issues of normalization have been ignored!

Relation between free vacuum $|0\rangle$ and interacting vacuum $|\Omega\rangle$

$$|0, in\rangle = |0, out\rangle = |\Omega\rangle. \quad (3.159)$$

In the *asymptotic past*, $t \rightarrow -\infty$, the in-states $|i, in\rangle$ are described as distinct wave packets corresponding to well-separated single particle states. Being far apart for $t \rightarrow -\infty$, they travel freely as individual states.

As these states approach each other, they start to interact and scatter into the final states. For $t \rightarrow \infty$ these final states are again asymptotically free and well-separated 1-particle states.

We can expand

$$\phi_{in}(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [a_{in}(\vec{p}) e^{-ip \cdot x} + a_{in}^\dagger(\vec{p}) e^{ip \cdot x}]. \quad (3.160)$$

We can identify

$$\lim_{t \rightarrow -\infty} \langle \alpha | \phi | \beta \rangle = \lim_{t \rightarrow -\infty} \sqrt{Z} \langle \alpha | \phi_{in} | \beta \rangle \Leftrightarrow \begin{aligned} \langle \alpha | \phi | \beta \rangle &\xrightarrow{t \rightarrow -\infty} \sqrt{Z} \langle \alpha | \phi_{in} | \beta \rangle \\ \langle \alpha | \phi | \beta \rangle &\xrightarrow{t \rightarrow +\infty} \sqrt{Z} \langle \alpha | \phi_{out} | \beta \rangle \end{aligned} . \quad (3.161)$$

3.5.3.1 The S-matrix and the scattering process

The S-matrix

The S-matrix (scattering) maps the out-states onto the in-states (because the Fock spaces are isomorphic):

$$|\alpha, in\rangle = S |\alpha, out\rangle, \quad (3.162)$$

with the properties

1. S is unitary $S^\dagger = S^{-1}$,
2. $\phi_{in}(x) = S \phi_{out}(x) S^{-1}$,
3. $|vac, in\rangle = |vac, out\rangle = |\Omega\rangle$ and $S |\Omega\rangle = |\Omega\rangle$.

Thus,

$$|\langle f, in | S | i, in \rangle|^2 \quad (3.163)$$

is the probability for scattering from initial states to the final states !.

Formally, the scattering process is described by the following: We have one Hilbert space for the whole scattering process from $t = -\infty$ to $t = +\infty$. There exist two Fock spaces $\mathcal{F}_{in}, \mathcal{F}_{out}$ in this Hilbert space, they are not disjoint because states can simply not participate in the scattering process. Long before the collisions, i.e. in the asymptotic past, we have well separated, free and independent wavepackets, the $|\alpha, in\rangle$ states with $\mathcal{F}_{in} = \text{span}\{|\alpha, in\rangle\}$. Long after the collisions, we again have well separated, free and independent wavepackets, the $|\beta, out\rangle$ states with $\mathcal{F}_{out} = \text{span}\{|\beta, out\rangle\}$. Then there exists an isomorphism between $|\alpha, in\rangle$ and $|\beta, out\rangle$, the S-matrix, with

$$|\alpha, in\rangle = S |\alpha, out\rangle, \quad S : \mathcal{F}_{out} \rightarrow \mathcal{F}_{in}. \quad (3.164)$$

$$\mathcal{H} = \mathcal{F}_{out} \cup \mathcal{F}_{in}, \quad \mathcal{F}_{in} \cap \mathcal{F}_{out} \neq \emptyset.$$

Then again, $\mathcal{F}_{in} = \text{span}\{\alpha, in\} = \phi_{in}(x) |\Omega, in\rangle\}$ and $\mathcal{F}_{out} = \text{span}\{\beta, out\} = \phi_{out}(x) |\Omega, out\rangle\}$ with $|\Omega, in\rangle = |\Omega, out\rangle = |\Omega\rangle$.

We thus find

$$S_{\beta\alpha} := \langle \beta, out | \alpha, in \rangle, \quad |\alpha, in\rangle = \sum_{\beta} S_{\beta\alpha} |\beta, out\rangle \quad (3.165)$$

$$\Rightarrow \hat{S} |\alpha, out\rangle = \sum_{\beta} S_{\beta\alpha} |\beta, out\rangle = |\alpha, in\rangle \quad (3.166)$$

$$\Rightarrow S_{\beta\alpha} = \langle \beta, out | \alpha, in \rangle = \langle \beta, out | \hat{S} |\alpha, out\rangle \quad (3.167)$$

$$\langle \beta, out | S^{\dagger} = \langle \beta, in | \quad SS^{\dagger} = \sum_{\beta} |\beta, in\rangle \langle \beta, in| = \mathcal{I}. \quad (3.168)$$

The probability amplitude for scattering from initial to final state.

3.5.4 The LSZ reduction formular

As it is always the case for in and out states we regard the fully interacting theory, $|\Omega\rangle$.

The LSZ reduction formular

The aim is to compute a S-matrix element $\langle p_1, \dots, p_n, out | q_1, \dots, q_r, in \rangle$ for a real scalar field. Note that p_1, \dots, p_n and q_1, \dots, q_r are *on-shell* since they correspond to the physical 4-momentum of the out-and incoming 1-particle states. We find

$$\begin{aligned} \langle p_1, \dots, p_n, out | q_1, \dots, q_r, in \rangle &= \langle p_1, \dots, p_n, in | S | q_1, \dots, q_r, in \rangle \\ &= (\sum \text{disconnected terms}) + (iZ^{-\frac{1}{2}})^{n+r} \int d^4y_1 \dots d^4y_n \\ &\quad \int d^4x_1 \dots d^4x_r \exp \left\{ i \left[\sum_{k=1}^n p_k \cdot y_k - \sum_{l=1}^r q_l \cdot x_l \right] \right\} \\ &\quad \times (\partial_{y_1}^2 + m^2) \dots (\partial_{y_n}^2 + m^2) \\ &\quad \langle \Omega | T\phi(y_1) \dots \phi(y_n) \phi(x_1) \dots \phi(x_n) | \Omega \rangle .. \end{aligned}$$



Figure 3.6: Disconnected diagram
This LSZ-formula reduces the computation of the S-matrix to the computation of time-ordered correlation functions $\langle \Omega | T\phi(y_1) \dots \phi(y_n) \phi(x_1) \dots \phi(x_n) | \Omega \rangle$ of the full interacting theory.

The first term describes a process where one of the in-and outgoing states are identical and do not participate in scattering. Such an amplitude corresponds to a disconnected diagram, compare fig. 3.6, and its computation reduces to computing an S-matrix element involving only $(r - 1)$ in-and $(n - 1)$ out-states.

For the connected term we find

$$\begin{aligned} &\prod_{k=1}^n \int_{\mathbb{R}^{3,1}} d^4y_k e^{ip_k \cdot y_k} \prod_{l=1}^r \int_{\mathbb{R}^{3,1}} d^4x_l e^{-iq_l \cdot x_k} \times \langle \Omega | T \prod_k \phi(y_k) \prod_l \phi(x_l) | \Omega \rangle \\ &= \left(\prod_{k=1}^n \frac{i\sqrt{Z}}{p_k^2 - m^2} \right) \left(\prod_{l=1}^r \frac{i\sqrt{Z}}{q_l^2 - m^2} \right) \times \langle p_1, \dots, p_n | S | q_1, \dots, q_r \rangle |_{\text{connected}}. \end{aligned}$$

Statements 3.2 *S-matrix \leftrightarrow residues of on-shell correlation functions.*

$\langle \Omega | T\tilde{\phi}(p_1) \dots \tilde{\phi}(p_n) \tilde{\phi}(q_1) \dots \tilde{\phi}(q_r) | \Omega \rangle$ will in general be a sum of terms with different poles in the momenta. Only the term with the pole structure given precisely by $\prod_{k=1}^n \frac{1}{p_k^2 - m^2} \prod_{l=1}^r \frac{1}{q_l^2 - m^2}$ contributes to the connected S-matrix element.

3.5.5 Correlators in the interaction picture

The computation of the full correlator shall now be reduced to a calculation in terms of free-field creation/annihilation operators and the free-field vacuum. This is achieved in the interaction picture:

$$H = H_0 + H_{int}, \\ [H_0, H] \neq 0.$$

Operator fields in the interaction picture

The time dependence of operators is governed by H_0 , while the time dependence of states is governed by H_{int} :

$$\phi_I(t, \vec{x}) = e^{iH_0(t-t_0)} \phi(\vec{x}, t_0) e^{-iH_0(t-t_0)} \quad (3.169)$$

$$\Pi_I(t, \vec{x}) = e^{iH_0(t-t_0)} \Pi(\vec{x}, t_0) e^{-iH_0(t-t_0)} \quad (3.170)$$

$$|\psi(t)\rangle_I = e^{iH_0 t} |\psi(t)\rangle_S \quad (3.171)$$

$$H_I \equiv (H_{int})_I = e^{iH_0 t} (H_{int})_S e^{-iH_0 t}. \quad (3.172)$$

Then $\phi_I(t, \vec{x})$ satisfies the free Klein-Gordon equation

$$(\partial^2 + m_0^2) \phi_I(t, \vec{x}) = 0, \quad (3.173)$$

thus a *free mode expansion* is possible.

The free mode expansion reads

$$\phi_I(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [a_I(\vec{p}) e^{-ip \cdot x} + a_I^\dagger(\vec{p}) e^{ip \cdot x}] \quad (3.174)$$

with

$$[\phi_I(t, \vec{x}), \Pi_I(t, \vec{y})] = i\delta_D^{(3)}(\vec{x} - \vec{y}), \quad [a_I(\vec{p}), a_I^\dagger(\vec{q})] = (2\pi)^3 \delta_D^{(3)}(\vec{p} - \vec{q}). \quad (3.175)$$

Therefore the results of the free theory carry over:

$$H_- |0\rangle = 0, \quad a_I(\vec{p}) |0\rangle = 0. \quad (3.176)$$

The transition to Heisenberg picture can now be done with

$$\phi(t, \vec{x}) = U^\dagger(t, t_0) \phi_I(t, \vec{x}) U(t, t_0) \quad (3.177)$$

with the time-evolution operator

$$U(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)} = \hat{T} e^{-i \int_{t_0}^t H_I(t') dt'}. \quad (3.178)$$

How to compute the correlators

The logic is now to replace the Heisenberg picture operators $\phi(x)$ on the correlator $\langle \Omega | T\phi(y_1) \dots \phi(y_n)\phi(x_1) \dots \phi(x_r) | \Omega \rangle$ by the interaction picture operators $\phi_I(x)$ because they obey a *free mode expansion*.

We find *Dyson's formula* for the time-evolution operator

$$U(t, t_0) = \mathcal{I} + \sum_{n=1}^{\infty} \left(\frac{1}{i} \right)^n \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n \underbrace{H_I(t_1)H_I(t_2) \dots H_I(t_n)}_{\text{these are time-ordered}} \quad (3.179)$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_n} dt_n T H_I(t_1) H_I(t_2) \dots H_I(t_n) \quad (3.180)$$

$$= T e^{-i \int_{t_0}^t dt' H_I(t')} \quad (3.181)$$

With the properties

$$U^\dagger(t_1, t_2) = U^{-1}(t_1, t_2) = U(t_2, t_1) \quad (3.182)$$

$$U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3) \quad \text{for } t_1 \geq t_2 \geq t_3. \quad (3.183)$$

There is an equivalent representation of the time-evolution operator

$$U(t) = \lim_{n \rightarrow \infty} \{ [1 - iH_{I,int}(\tau_{n-1})(\tau_n - \tau_{n-1})] \quad (3.184)$$

$$[1 - iH_{I,int}(\tau_{n-2})(\tau_{n-1} - \tau_{n-2})] \dots [1 - iH_{I,int}(\tau_0)(\tau_1 - \tau_0)] \}. \quad (3.185)$$

Furthermore we find a relation between the free vacuum $|0\rangle$ and the interacting vacuum $|\Omega\rangle$. The time-evolution of the free vacuum is

$$e^{-iHT} |0\rangle = e^{-iHT} \sum_{|n\rangle} |n\rangle \langle n|0\rangle = e^{-iE_\Omega T} |\Omega\rangle \langle \Omega|0\rangle + \sum_{|n\rangle \neq |\Omega\rangle} e^{-iE_n T} |n\rangle \langle n|0\rangle. \quad (3.186)$$

The idea is, that the second term must vanish for $T \rightarrow \infty$, because excited states will go down eventually.

If $H_0 |0\rangle = 0$, then $H |\Omega\rangle = E_\Omega |\Omega\rangle$ with $E_\Omega \neq 0$ and $E_n > E_\Omega \forall |n\rangle \neq$

$|\Omega\rangle$. So if we formally take the limit $T \rightarrow \infty(1 - i\epsilon)$, then $e^{-iE_n T}$ is stronger suppressed and only the vacuum $|\Omega\rangle$ survives:

$$|\Omega\rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \left[e^{-iE_\Omega(t_0 - (-T))\langle\Omega|0\rangle} \right]^{-1} U(t_0, -T) |0\rangle \quad (3.187)$$

$$\Rightarrow \langle\Omega|\hat{T}\phi(x)\phi(y)|\Omega\rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0|\hat{T}\left(\phi_I(x)\phi_I(y)e^{-i\int_{-T}^T dt H_I(t)}\right)|0\rangle}{\langle 0|\hat{T}e^{-i\int_{-T}^T dt H_I(t)}|0\rangle} \quad (3.188)$$

with the same reasoning for higher n-point correlators.

Thus, gauge E_0 energy to be zero: $H_0|0\rangle = E_0|0\rangle = 0$ such that $H|\Omega\rangle = E_\Omega|\Omega\rangle \neq 0$.

Logic on how to compute correlators

Replace Heisenberg with interaction picture operators, from this we get $e^{-i\hbar\int H_I}$. Then go over from $|\Omega\rangle \rightarrow |0\rangle$ with given relation. Then compute $\langle 0|\phi \dots \phi|0\rangle$ explicitly via Wick normal ordering \Rightarrow express results in terms of Feynman diagrams.

3.5.6 Wick's Theorem

From Dyson's formula, we want to compute quantities like $\langle f|T[H_I(x_1) \dots H_I(x_n)]|i\rangle$, where $|i\rangle$ and $|f\rangle$ are eigenstates of the free theory. Since the H_I 's contain certain creation and annihilation operators, calculation would be way easier if all annihilation operators were ordered to the right.

Normal ordering

An operator O is *normal-ordered* if all creation/annihilation operators appear on the left/right. For such O we write : O :

$$\Rightarrow \langle 0|:O:|0\rangle = 0 \quad (3.189)$$

for every non-trivial operator $O \neq c\mathcal{I}, c \in \mathbb{C}$.

Contraction

$$\overline{\phi_I(x)\phi_I(y)} := D_F^{(0)}(x-y) = D_F^{(0)}(y-x) \quad (3.190)$$

with the free Feynman propagator

$$D_F^{(0)}(x-y) = \oint_C \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m_0^2} e^{ip \cdot (x-y)}. \quad (3.191)$$

Wick's theorem

We can formulate Wick's theorem now

$$\begin{aligned}
 & \hat{T}\phi_I(x_1) \dots \phi_I(x_n) \\
 & = : \phi_I(x_1) \dots \phi_I(x_n) : + \text{:all contractions of distinct pairs:} \\
 & = : \phi_I(x_1) \dots \phi_I(x_n) : + \sum_{i < j} \overline{\phi_I(x_i)} \overline{\phi_I(x_j)} \\
 & \quad : \phi_I(x_1) \dots \phi_I(x_{i-1}) \phi_I(x_{i+1}) \dots \phi_I(x_{j-1}) \phi_I(x_{j+1}) \dots \phi_I(x_n) : \\
 & \quad + \sum_{i < j, k < l} \overline{\phi_I(x_i)} \overline{\phi_I(x_j)} \overline{\phi_I(x_k)} \overline{\phi_I(x_l)} \\
 & \quad : \phi_I(x_1) \dots \phi_I(x_n) : + \dots \\
 & \quad \text{with } : \overline{\phi_1} \overline{\phi_2} \overline{\phi_3} \overline{\phi_4} : \Leftrightarrow D_F^{(0)}(x_1 - x_3) : \phi_2 \phi_4 :
 \end{aligned}$$

Alternatively:

$$\int_{\mathbb{R}^{3,1}} \frac{D_F^{(0)}(x - y)}{\frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m_0^2 + i\epsilon}} e^{-ip(x-y)}. \quad \text{Wick's theorem has two important consequences}$$

1) $\langle 0 | \hat{T}\phi_I(x_1) \dots \phi_I(x_{2N+1}) | 0 \rangle = 0 \quad \forall N \in \mathbb{N} \quad (3.192)$

2) $\langle 0 | \hat{T}\phi_I(x_1) \dots \phi_I(x_{2N}) | 0 \rangle = D_F^{(0)}(x_1 - x_2) D_F^{(0)}(x_3 - x_4) \dots D_F^{(0)}(x_{2N-1} - x_{2N}) \sum_{\text{all other contractions}}. \quad (3.193)$

Therefore, Wick's theorem allows us to turn any expression of the form

$$\langle 0 | \hat{T}\{\phi_I(x_1) \dots \phi_I(x_n)\} | 0 \rangle \quad (3.194)$$

into a sum of products of Feynman propagators.

How to compute correlators

Correlation function in the full interacting theory $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$:

$$\langle \Omega | \hat{T} \prod_{i=1}^n \phi(x_i) | \Omega \rangle = \frac{\langle \Omega | \hat{T} \prod_{i=1}^n \phi(x_i) e^{-i \int_{\mathbb{R}^{3,1}} d^4 x \mathcal{L}_{int}} | \Omega \rangle}{\langle \Omega | \hat{T} e^{-i \int_{\mathbb{R}^{3,1}} d^4 x \mathcal{L}_{int}} | \Omega \rangle}. \quad (3.195)$$

3.5.7 Feynman diagrams

There exists a graphical representation of the systematics of contractions in terms of Feynman diagrams. First introduce some definitions.

1. The symmetry factors $S = |G|$ represent the order (Kardinalität) of the symmetry group of the diagrams.

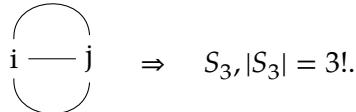
Symmetry factors

The symmetry factors are given by the number of ways one can exchange components of the diagram without changing the diagram, where by components we mean either the two ends of a line starting and ending on the same point, entire lines between points or internal points. $\Rightarrow G \times \mathbb{Z}_2$ for every possible change + the ones described in the following.

It holds in general, that a Feynman diagram with symmetry group G always carries a combinatorial factor $\frac{1}{|G|}$. This is because if some symmetry remains, the factor of $\frac{1}{n!}$ in the interaction term $\frac{-\lambda_n}{n!} \phi^n$ is only partially cancelled by counting the various contractions that yield the same diagram.

How to calculate symmetry factors

In addition, if k vertices are identical, the symmetry group includes the group of permutations of these k vertices of order $|S_k| = k!$. One also has to take the symmetry group S_k for k lines between two connected vertices into account, e.g.



The symmetry comes from the internal points, because the external points are distinct in their connections to the other points.

2. Number of pairings of $(2n)$ fields

$$\# = \frac{C_2^{2n} C_2^{2n-2} \dots C_2^2}{n!}, \quad C_m^n = \binom{n}{m}. \quad (3.196)$$

Check the total number of fields you have found by this formula, to be correct. Or, equivalently for pairings of n fields

$$\# = \frac{(2n)!}{n! 2^n}. \quad (3.197)$$

3. The x, y associated to $\phi(x)\phi(y)$ in the correlator, which are not integrated over, are called *external points*.
4. All other points over which is integrated over are called *internal points*.

Vertices

For ϕ^n interaction, at an internal point n lines meet. Such points are called *vertices*.

5.

3.5.7.1 Position space Feynman-rules

In the following, rules are summarized for the computation of

$$\langle 0 | \hat{T} \left\{ \prod_{i=1}^m \phi_I(x_i) \exp \left[-i \frac{\lambda}{n!} \int d^4 z \phi_I^n(z) \right] \right\} | 0 \rangle \quad (3.198)$$

at order λ^k , where we are assuming that all points x_i are distinct:

1. Draw one external point for all x_i and k internal points for all $z_j \cdot \begin{pmatrix} i = 1, \dots, m \\ j = 1, \dots, n \end{pmatrix}$.
2. Connect the points by lines such that
 - a) To each external point x_i 1 line is attached.
 - b) To each internal point z_j n lines are attached.
3. To each line between points y_i and y_j (both external and internal) we associate a free propagator

$$D_F^{(0)}(y_i - y_j) = D_F^{(0)}(y_j - y_i). \quad (3.199)$$

4. To each vertex z_j associate a factor

$$-i\lambda \int d^4 z_j. \quad (3.200)$$

5. To each external point associate a factor of 1.
6. Multiply all factors, Feynman propagators etc. and divide by the symmetry factor of the diagram.
7. Then sum up all distinct such Feynman diagrams.

3.5.7.2 Momentum space Feynman-rules

The momentum-space Feynman rules for the computation of the n -point correlator at order λ^k

$$\langle 0 | \hat{T} \left\{ \prod_{i=1}^m \phi_I(x_i) \exp \left[-i \frac{\lambda}{n!} \int d^4 z \phi_I^n(z) \right] \right\} | 0 \rangle \quad (3.201)$$

are the following. This is an equivalent set of rules for the computation of the correlator where the integral over the vertex positions has been performed explicitly!

1. Draw one external point for all x_i and k internal points $z_j, j = 1, \dots, k$.
2. Connect the points by lines such that
 - a) To each external point x_i , 1 line is attached.
 - b) To each internal point z_j , n lines are attached.
3. to each line between points y_i and y_j (both external and internal) we associate a free propagator

$$D_F^{(0)}(y_i - y_j) \quad (3.202)$$

with one (arbitrary) choice of direction and to each such $D_F^{(0)}(y_i - y_j)$ we associate directed momentum p from y_i to y_j and a factor

$$\frac{i}{p^2 - m_0^2 + i\epsilon}. \quad (3.203)$$

Note that

$$\begin{aligned} D_F^{(0)}(x - y) &= \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m_0^2 + i\epsilon} e^{-ip(x-y)} \\ &= \text{x} \rightarrow \text{y} \\ &= \text{y} \rightarrow \text{x} \\ &= D_F^{(0)}(y - x). \end{aligned}$$

4. At each vertex, 4-momentum is conserved! For each vertex we thus multiply a factor of

$$(-i\lambda) (2\pi)^4 \delta_D^{(4)} \left(\sum_{\text{ingoing}} p_i - \sum_{\text{outgoing}} p_k \right). \quad (3.204)$$

5. For each external point we multiply a factor of $e^{-ip \cdot x}$ for momentum pointing out of the external point ($e^{-ip \cdot x} = \text{x} \rightarrow \text{---}$), or for momentum pointing into the point we multiply a factor $e^{+ip \cdot x}$ ($e^{ip \cdot x} = \text{---} \rightarrow \text{x}$).
6. Integrate over each momentum $\int \frac{d^4 p}{(2\pi)^4}$ and divide by the symmetry factor.
7. Then sum up all distinct such Feynman diagrams.

Summary of the calculation of Feynman diagrams

For example, the two-point correlator factorizes into connected and disconnected diagrams:

$$\begin{aligned} & \lim_{T \rightarrow \infty(1-i\epsilon)} \langle 0 | \hat{T} \phi(x) \phi(y) e^{-i \int_{-T}^T dt H_I(t)} | 0 \rangle \\ &= \sum (\text{connected diagrams}) \cdot e^{\sum (\text{disconnected diagrams})}. \quad (3.205) \end{aligned}$$

where the sum of connected diagrams is equal to

$$\sum (\text{connected diagrams}) = \begin{array}{c} x \longrightarrow z \\ \nearrow \\ \text{x} \longrightarrow y + y \longrightarrow \dots \end{array} \quad (3.206)$$

the sum of at least partially connected diagrams with n external points.

And where $\sum(\text{disconnected terms})$ is equal to the sum of all entirely disconnected diagrams without external points.

Such that

$$\langle \Omega | \hat{T} \phi(x) \phi(y) | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | \hat{T} \phi(x) \phi(y) e^{-i \int_{-T}^T dt H_I(t)} | 0 \rangle}{\langle 0 | \hat{T} e^{-i \int_{-T}^T dt H_I(t)} | 0 \rangle} \quad (3.207)$$

$$= \sum (\text{connected diagrams}) \quad (3.208)$$

$$= \dots \quad (3.209)$$

Or more generally

$$\langle \Omega | \hat{T} \prod_{i=1}^n \phi(x_i) | \Omega \rangle = \sum_{\substack{\text{over all partially connected} \\ \text{diagrams with } n \text{ external points}}} . \quad (3.210)$$

Insert correct
Feynman dia-
grams here
from page 16.

3.5.8 Disconnected diagrams

A typical diagram contains *disconnected pieces*, i.e. subdiagrams which are not connected to any of the external points.

A disconnected piece contains only internal points.

Partially connected diagram

By contrast, the part of the diagram which is connected to at least one external point is called *partially connected diagram*.

If we sum up all Feynman diagrams that contribute to

$$\langle 0 | \hat{T} \prod_{i=1}^k \phi_i(x_i) e^{-i \frac{\lambda}{N!} \int d^4x \phi_I^N(x)} | 0 \rangle, \quad (3.211)$$

the result factorizes into the sum of all partially connected diagrams multiplied by the sum of all disconnected diagrams.

Let $\{V_i\}_i = \{\dots\}$ denote the set of all individual disconnected pieces. Then the k -point correlator becomes

insert Feynman diagrams

$$\sum (\text{at least partially connected pieces}) \cdot \prod_i \sum_{n_i=0}^{\infty} (V_i)^{n_i} \frac{1}{n_i!} \quad (3.212)$$

$$= \sum (\text{at least partially connected pieces}) \cdot e^{\sum_i V_i}. \quad (3.213)$$

With the full correlator

$$\langle \Omega | \hat{T} \prod_i \phi_i | \Omega \rangle = \frac{\langle 0 | \hat{T} \prod_i \phi_i e^{-i \int dt H_I(t)} | 0 \rangle}{\langle 0 | \hat{T} e^{-i \int dt H_I(t)} | 0 \rangle} \quad (3.214)$$

we find, that the denominator contains no external points:

$$\langle 0 | \hat{T} e^{-i \int dt H_I(t)} | 0 \rangle = e^{\sum_i V_i} \quad (3.215)$$

which is the *partition function*.

Therefore, the denominator cancels exactly all disconnected diagrams and divergent factors of the nominator and we arrive at

$$\langle \Omega | \hat{T} \prod_{i=1}^m \phi_i | \Omega \rangle = \sum (\text{all partially connected diagrams with } n \text{ external points}). \quad (3.216)$$

3.5.9 One-Particle-Irreducible Diagrams 1PI

1PI diagrams

A *one-particle-irreducible Feynman diagram* (1PI) is a diagram, out of which one cannot produce two separate non-trivial diagrams (diagrams containing more than just one line) by cutting a single line.

We introduce the notion

Draw circle around 1PI

$$(1PI) := \sum (\text{all non-trivial 1PI diagrams}) \quad (3.217)$$

where it is understood that we do not attach external points to both ends from straight lines the left or right of $(1PI) \Rightarrow -iM^2(p^2)$ is defined to be the value of $(1PI)$, where p^2 denotes the in-and outgoing momentum, this quantity may be divergent.

The Fourier transform $D_F(p^2)$ of the Feynman propagator

$$\langle \Omega | \hat{T} \phi(x) \phi(y) | \Omega \rangle = D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} D_F(p^2) \quad (3.218)$$

yields the *resummed propagator via Dyson resummation*

$$D_F(p^2) = \frac{i}{p^2 - [m_0^2 + M^2(p^2)] + i\epsilon} \equiv \text{---} \circlearrowleft \text{---} \quad (3.219)$$

this denotes all Feynman diagrams in p -space without vacuum bubbles.

By extracting the first analytic pole of the resummed propagator at m^2 with the residue being Z , we can compute the physical 1-particle mass m^2 and Z *perturbatively* to given order in λ .

$\Rightarrow m$ (mass of 1-particle momentum eigenstate) $\neq m_0$, because of the self-interactions of the field, which are resummed as above to shift the pole of the full propagator from m_0 to $-m$. We cannot switch off the interactions of the asymptotic particles with themselves:

These are precisely the 1PI-contributions to $D_F(p^2)$ and thus the in-and out-states *do have the fully resummed mass* $m^2 \neq m_0^2$.

3.5.9.1 A note on self energy

A particle's *self energy* represents the contribution to the particle's energy, or *effective mass*, due to interactions between the particle and the system it is part of. The self-energy is the energy that a particle has as a result of changes that itself causes in its environment:

Statements 3.3 $m = m_0 + \text{self-energy} \Rightarrow \text{self-energy} \equiv M^2(p^2)!$

The self-energy is equal to the on-the-mass shell value of the proper mass operator

$$\text{---} \rightarrow \text{---} \circlearrowleft \text{---} \xrightarrow{\text{amputating}} \text{---} \circlearrowleft = (1PI) = M^2(p^2). \quad (3.220)$$

In general, M^2 is complex. In such a case, it is the real part of this self-energy that is defined as the particle's self-energy. The inverse of the imaginary part is a measure for the *lifetime* of the particle under investigation.

The photon and the gluon don't get a mass through renormalization, because gauge symmetry protects them from getting a mass (Ward identity). W, Z -bosons get their masses through the Higgs-mechanism.

3.5.10 Scattering amplitudes

By the LSZ-formula we could reduce the computation of S-matrix element to the computation of the connected S-matrix elements, which are then again computed via n -point correlation functions. Due to physical interpretation, we find that only those Feynman diagrams are relevant with exactly $(n+r)$ poles at m^2 for the computation of the connected S-matrix element. Therefore, we find the final result for the computation of scattering amplitudes to be

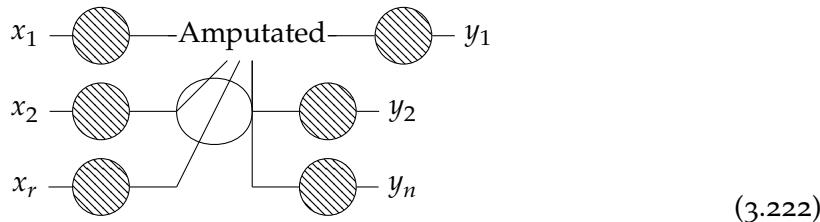
Where all q_l and p_k are on-shell.

$$\langle p_1, \dots, p_n | S | q_1, \dots, q_r \rangle_{\text{connected}} = (\sqrt{Z})^{(n+r)} \quad (3.221)$$

$$\left[\prod_{k=1}^n \int d^4 y_k e^{ip_k \cdot y_k} \prod_{l=1}^r \int d^4 x_l e^{-iq_l \cdot x_l} \langle \Omega | \hat{T} \phi(y_1) \dots \phi(x_1) \dots | \Omega \rangle \right]_{\text{Amputated}}.$$

With amputation having the following meaning:

A fully connected correlation function has the following structure :



By amputated correlator we mean the Feynman diagram after cutting

all external legs carry , or . Since each external leg carries a factor of $\frac{iZ}{p^2 - m^2 + i\epsilon}$ near m^2 if p^2 is on-shell, all $(n+r)$ external legs yield together the right singularity structure.

Insert graphic or draw Feynman diagram p.19

Wavefunction renormalization in perturbation theory

In perturbation theory

$$Z = 1 + \mathcal{O}(\lambda), \quad (3.223)$$

hence to leading order in λ , Z plays no role as only $\mathcal{O}(\lambda)$ diagrams can be fully connected.

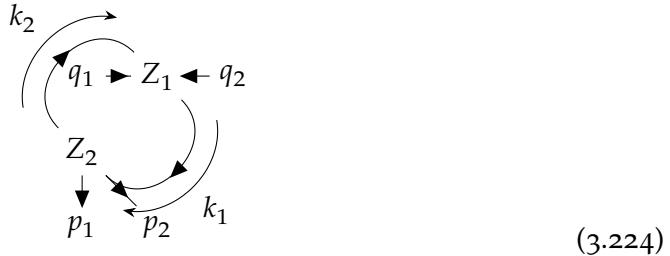
Thus, amputating means discarding all propagators from external lines:

E.g. for 2 – 2 scattering:

$$\begin{aligned} \langle p_1, p_2 | S | q_1, q_2 \rangle_{\text{connected}} &= \dots \\ &= \left[(-i\lambda)(2\pi)^4 \delta_D^{(4)}(q_1 + q_2 - p_1 - p_2) \prod_{j=1}^2 \frac{i}{q_j^2 - m_0^2 + i\epsilon} \frac{i}{p_j^2 - m_0^2 + i\epsilon} \right]_{\text{Amputated}} \\ &= (-i\lambda)(2\pi)^4 \delta_D^{(4)}(q_1 + q_2 - p_1 - p_2). \end{aligned}$$

Feynman rules for the computation of $\langle p_1, \dots, p_n | S | q_1, \dots, q_r \rangle_{\text{connected}}$

1. Draw the relevant fully connected Feynman diagrams with $(n + r)$ external points to given order in λ .
2. Assign ingoing momenta q_i and outgoing momenta p_k and label momenta of internal lines with k_j (virtual particles):



3. Each vertex carries

$$(-i\lambda)(2\pi)^4 \delta_D^{(4)}(\sum \text{ingoing momenta} - \sum \text{outgoing momenta}). \quad (3.225)$$

4. Each internal line carries

$$\frac{i}{k_j^2 - m_0^2 + i\epsilon} \quad (3.226)$$

5. Integrate over all internal momenta, i.e.

$$\prod_j \int \frac{d^4 k_j}{(2\pi)^4} \quad (3.227)$$

and divide by the symmetry factor.

6. Sum up all diagrams and multiply by $(\sqrt{Z})^{n+r}$ to given order in λ .

3.5.10.1 Deeper Interpretation of Feynman rules:

1. A line — corresponds to the worldline of a particle.
2. $\rightarrow e^{-ip \cdot x}$ is the wavefunction for the momentum-eigenstate.
3. A vertex Z_1 —

— Z_1 —

is a localized interaction at a spacetime point Z_1 .

4. Summing up diagrams and integration over $\int d^4z$ amounts to coherently summing up the QM probability amplitudes for all possible processes - called *channel*- with the same macroscopic result.
5. Intermediate particles, e.g. those running in the loop as k_1 and k_2 , are called *virtual* because they are generally off-shell:
i.e. $k_j^2 \neq m_0^2$ in general $\Rightarrow E_j^2 \neq \vec{k}_j^2 + m_0^2$ for virtual particles. This is allowed for sufficiently short times, i.e. allowed by the energy-time uncertainty relation in QM perturbation theory.

3.5.11 Cross-sections

The S-matrix

The S-matrix can be decomposed into contributions, where no scattering takes place, and contributions where a transition indeed takes place

$$S = \mathcal{I} + iT, \quad (3.228)$$

where T is the *transition matrix*.

S-matrix elements are therefore in general of the form

$$\langle f | S | i \rangle = \underbrace{\delta_{fi}}_{\text{no scattering}} + \underbrace{i(2\pi)^4 \delta_D^{(4)}(p_f - p_i) \cdot M_{fi}}_{\substack{\text{Momentum conservation} \\ \text{scattering amplitude}}} .$$

= $\langle f | iT | i \rangle$

(3.229)

The *transition rate* is the normalized QM probability for a scattering of a given initial state $\{|i\rangle\}$ into a range of final states $\{|f\rangle\}$

$$\omega_{fi} = \frac{\mathcal{P}_{|i\rangle \rightarrow |f\rangle}}{Vol_{\mathbb{R}^{3,1}}} = \sum_{|f\rangle \in \{|f\rangle\}} \frac{(2\pi)^4 \delta_D^{(4)}(p_f - p_i) (2\pi)^4 \delta_D^{(4)}(0) |M|}{Vol_{\mathbb{R}^{3,1}}} \quad (3.230)$$

$$= \frac{1}{N!} \prod_{n=1}^N \int \frac{d^3 k_n}{(2\pi)^3} \frac{1}{2E_n} (2\pi)^4 \delta_D^{(4)} \left(\sum_i p_i - \sum_n k_n \right) |M_{fi}|^2, \quad (3.231)$$

where the latter equality holds for scattering into N identical particles.

Cross-section

The *cross-section* σ is the effective area of the beam B that participates in the scattering.

Consider a $2 \rightarrow N$ scattering process, such that all out-going states $|k\rangle_j$ are momentum eigenstates and the initial states are momentum eigenstates $|p\rangle_A$ and $|p\rangle_B$.

The *differential cross section* then is

$$\mathrm{d}\sigma = \frac{(2\pi)^4}{4E_A E_B |\vec{v}_A - \vec{v}_B|} \underbrace{\frac{1}{N!} \prod_{n=1}^N \int \frac{\mathrm{d}^3 k_n}{(2\pi)^3} \frac{1}{2E_n} \delta_D \left(p_A + p_B - \sum_{i=1}^N k_i \right) |M_{fi}|^2}_{\mathrm{d}\Pi_N} \underbrace{}_{\text{Lorentz invariant phase space}} \quad (3.232)$$

with $|\vec{v}_A| = \frac{|\vec{p}_A|}{E_A}$.

In the lab frame $4E_A E_B |\vec{v}_B^{(L)}| = 4E_A E_B |\vec{v}_A^{(L)} - \vec{v}_B^{(L)}|$ with m_A the rest mass of particle A at rest.

Thus $4E_A E_B |\vec{v}_A^{(L)} - \vec{v}_B^{(L)}| \sigma$ is the correct general expression for the transition rate in any frame. \Rightarrow For $2 - 2$ scattering we find with the Mandelstam variable $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_3} = \frac{1}{2!} \frac{1}{64\pi^2} \frac{1}{s} |M|^2. \quad (3.233)$$

Thus, the differential cross-section for hard scattering off point-like target (i.e. a target with no substructure of length $\ell \geq \frac{1}{\sqrt{s}}$) falls off as $\frac{1}{s}$.

The *centre-of-mass frame* says

$$\sum_{i=1}^N p_i = (\sqrt{s}, 0, 0, 0)^T \quad (3.234)$$

for p_i incoming particles and

$$\sum_{j=1}^M q_j = (\sqrt{s}, 0, 0, 0)^T \quad (3.235)$$

for q_j outgoing particles.

4

QUANTIZING SPIN $\frac{1}{2}$ FIELDS

4.0.1 The Dirac action

We now have a new field, the Dirac spinor, and want to construct a covariant action for it.

Define the *conjugate spinor* $\psi^\dagger := (\psi^*)^T = ((\psi^1)^*, (\psi^2)^*, (\psi^3)^*, (\psi^4)^*)$. In the Dirac representation it is satisfied, that

$$(\gamma^0)^\dagger = \gamma^0, (\gamma^i)^\dagger = -\gamma^i \Rightarrow (\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0 \quad (4.1)$$

and therefore we find

$$\gamma^0 S[\Lambda]^\dagger \gamma^0 = S[\Lambda]^{-1}. \quad (4.2)$$

To find the Lorentz-scalar for this theory, we define the *Dirac conjugate spinor*

$$\bar{\psi} = \psi^\dagger \gamma^0 \Rightarrow \bar{\psi}(x) \psi(x) \quad (4.3)$$

is a Lorentz scalar! Furthermore it holds, that $\bar{\psi} \gamma^\mu \psi$ is a Lorentz vector with

$$\bar{\psi}(x) \gamma^\mu \psi(x) \rightarrow \bar{\psi}(\Lambda^{-1}x') S^{-1}(\Lambda) \gamma^\mu S(\Lambda) \psi(\Lambda^{-1}x') \quad (4.4)$$

and thus

$$S^{-1}(\Lambda) \gamma^\mu S(\Lambda) = \Lambda_\nu^\mu \gamma^\nu. \quad (4.5)$$

Only a first order \mathcal{L} , we had second order \mathcal{L} for scalar fields.

Dirac field equations

With those bilinears of the Dirac field $\bar{\psi} \gamma^\mu \psi, \bar{\psi} \psi$, each of which transforms covariantly under the Lorentz group, we find the *Dirac action*

$$S = \int d^4x \bar{\psi}(x) [i\gamma^\mu \partial_\mu - m] \psi(x) \quad (4.6)$$

with i needed for S to be real and $|m|$ the mass of the Dirac spinor particle.

We find the *Dirac equation*

$$(i\gamma^\mu \partial_\mu - m) \psi(x) = 0 \quad (4.7)$$

$$(-i\partial_\mu \bar{\psi} \gamma^\mu - m \bar{\psi})(x) = 0. \quad (4.8)$$

If ψ solves the Dirac equation, then ψ solves the Klein-Gordon-equation ("Dirac eq. = $\sqrt{KG eq.}$ ").

4.0.2 Chirality and Weyl spinors

Weyl spinors

The Dirac spinor representation of $Cliff(1,3)$ is not irreducible as a representation of $Spin(1,3)$, as the subspaces

$$\psi_- = (\psi_1, \psi_2, 0, 0)^T \quad \psi_+ = (0, 0, \psi_3, \psi_4)^T \quad (4.9)$$

transform *seperately* under Lorentz transformations. The Weyl spinors u_- u_+ form irreducible representations of $Spin(1,3)$: $u_- = (\psi_1, \psi_2)^T$, $u_+ = (\psi_3, \psi_4)^T$.

A Dirac field can be projected onto its left-handed chirality and right-handed chirality components by

$$\mathbb{P}_\pm := \frac{1}{2}(\mathcal{I} \pm \gamma^5), \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \Rightarrow \psi_\pm = \mathbb{P}_\pm \psi. \quad (4.10)$$

In the context of these subspaces the Dirac action reads

$$S = \int d^4x \left[u_+^\dagger i\sigma^\mu \partial_\mu u_+ + u_-^\dagger i\bar{\sigma}^\mu \partial_\mu u_- - m(u_+^\dagger u_- + u_-^\dagger u_+) \right]. \quad (4.11)$$

Weyl equations

If $m = 0$, u_+ and u_- decouple and describe independent degrees of freedom subject to the *Weyl equations*:

$$i\sigma^\mu \partial_\mu u_+(x) = 0, \quad i\bar{\sigma}^\mu \partial_\mu u_-(x) = 0. \quad (4.12)$$

With the definition of the *helicity* (=Chirality for $m = 0$)

$$\hat{h} = \frac{1}{2}\hat{\vec{p}} \cdot \vec{\sigma} = \frac{1}{2} \begin{pmatrix} \hat{\vec{p}} \cdot \vec{\sigma} & 0 \\ 0 & \hat{\vec{p}} \cdot \vec{\sigma} \end{pmatrix}. \quad (4.13)$$

We find for $u_\pm(x) = u_\pm(p)e^{-ip \cdot x}$

$$hu_\pm(p) = \pm \frac{1}{2}u_\pm(p) \left\{ \begin{array}{ll} \Rightarrow u_+ & \text{right-handed spinor} \\ \Rightarrow u_- & \text{left-handed spinor} \end{array} \right\}. \quad (4.14)$$

4.0.3 Classical plane wave solutions

There generally exist two solutions to the Dirac equation

$$\psi(x) = u(\vec{p})e^{-ip \cdot x}, \quad u(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \xi \end{pmatrix} \quad \text{positive energy solution} \quad (4.15)$$

$$\psi(x) = v(\vec{p})e^{ip \cdot x}, \quad v(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ -\sqrt{(p \cdot \bar{\sigma})} \xi \end{pmatrix} \quad \text{negative energy solution} \quad (4.16)$$

with ξ being a 2-component Weyl spinor.

Completeness relation for four-spinors

$$\sum_s u_s(\vec{p}) \bar{u}_s(\vec{p}) = \gamma \cdot p + m \quad (4.17)$$

$$\sum_s v_s(\vec{p}) \bar{v}_s(\vec{p}) = \gamma \cdot p - m \quad (4.18)$$

with other relations being

$$\begin{aligned} \bar{u}_s(\vec{p}) u_{s'}(\vec{p}) &= 2m\delta_{ss'}, & u_s^\dagger(\vec{p}) u_{s'}(\vec{p}) &= 2p^0\delta_{ss'}, \\ \bar{v}_s(\vec{p}) v_{s'}(\vec{p}) &= -2m\delta_{ss'}, & v_s^\dagger(\vec{p}) v_{s'}(\vec{p}) &= 2p^0\delta_{ss'}, \\ \bar{u}_s(\vec{p}) v_{s'}(\vec{p}) &= 0. \end{aligned}$$

With a basis of the 2-spinors given by $\xi_s : \begin{cases} \xi_{1/2} = (1, 0)^T \\ \xi_{-1/2} = (0, 1)^T \end{cases} \Rightarrow u_s$ and v_s with $s = \pm \frac{1}{2}$ describe spinors with spin $\pm \frac{1}{2}$ in direction x_3 .

4.0.4 Quantization of the Dirac field

Quantization procedure for spin $\frac{1}{2}$ fields

Quantizing by imposing commutator relations is not possible, because the b-mode excitations would be negative norm states or equally because of unboundedness of the energy spectrum from below, i.e. instability of the vacuum.

The correct procedure for quantization of spin- $\frac{1}{2}$ fields is to impose the canonical anti-commutation relations:

$$\{\psi^A(\vec{x}), \psi_B^\dagger(\vec{x}')\} = \delta_B^A \delta_D^{(3)}(\vec{x} - \vec{x}') \quad (4.19)$$

$$\{\psi^A(\vec{x}), \psi^B(\vec{x}')\} = 0 = \{\psi_A^\dagger(\vec{x}), \psi_B^\dagger(\vec{x})\}. \quad (4.20)$$

This induces the mode relations

$$\{a_r(\vec{p}), a_s^\dagger(\vec{q})\} = (2\pi)^3 \delta_{rs} \delta_D^{(3)}(\vec{p} - \vec{q}) \quad (4.21)$$

$$\{b_r(\vec{p}), b_s^\dagger(\vec{q})\} = (2\pi)^3 \delta_{rs} \delta_D^{(3)}(\vec{p} - \vec{q}) \quad (4.22)$$

$$\{a_r(\vec{p}), b_s^\dagger(\vec{q})\} = 0. \quad (4.23)$$

It follows that the Hamiltonian is given by

$$H = \int \frac{d^3 p}{(2\pi)^3} E_p \sum_s [a_s^\dagger(\vec{p}) a_s(\vec{p}) + v_s^\dagger(\vec{p}) b_s(\vec{p}) - (2\pi)^3 \delta^{(3)}(\vec{0})] \quad (4.24)$$

with

$$H, a_s(\vec{p}) = -E_p a_s(\vec{p}), \quad [H, a_s^\dagger(\vec{p})] = E_p a_s^\dagger(\vec{p}) \quad (4.25)$$

$$[H, b_s(\vec{p})] = -E_p b_s(\vec{p}), \quad [H, b_s^\dagger(\vec{p})] = E_p b_s^\dagger(\vec{p}). \quad (4.26)$$

$a_s^\dagger(\vec{p})$ creates a *fermion* of momentum \vec{p} and spin s , and $b_s^\dagger(\vec{q})$ creates an *anti fermion* of momentum \vec{q} and spin r .

The general field $\psi(x)$ is now seen to be a weighted summation over all possible spins and momenta for creating fermions and anti fermions

$$\psi(x) = \sum_s \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [a_s(\vec{p}) u_s(\vec{p}) e^{-ip \cdot x} + b_s^\dagger(\vec{p}) v_s(\vec{p}) e^{ip \cdot x}] . \quad (4.27)$$

Its conjugate field $\bar{\psi}$ is a weighted summation over all possible spins and momenta for annihilating fermions and anti fermions $\bar{\psi} = \psi^\dagger \gamma^0$

$$\psi^\dagger(x) = \sum_s \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [b_s(\vec{p}) v_s^\dagger(\vec{p}) e^{-ip \cdot x} + a_s^\dagger(\vec{p}) u_s^\dagger(\vec{p}) e^{ip \cdot x}] . \quad (4.28)$$

We furthermore know

$$\Pi = \frac{\partial \mathcal{L}_D}{\partial(\partial_0 \psi)} = i\psi^\dagger \quad \text{canonically conjugated momentum} \quad (4.29)$$

$$\mathcal{H}_D = \bar{\psi} [-i\vec{\gamma} \cdot \vec{\nabla} + m] \psi \quad \text{Hamiltonian density.} \quad (4.30)$$

Note further, that the divergent vacuum energy has *opposite sign* compared to a scalar theory.

From the vacuum

$$a_s(\vec{p}) |0\rangle = 0 = b_s(\vec{p}) |0\rangle \quad \forall \vec{p} \quad (4.31)$$

a-modes \equiv "particle sector".

1. The state $|\vec{p}, s\rangle := \sqrt{2E_p} a_s^\dagger(\vec{p}) |0\rangle$ is a 1-particle state with momentum \vec{p} , energy $E_p = \sqrt{\vec{p}^2 + m^2}$ and spin $s = \frac{1}{2}, -\frac{1}{2}$ in x_3 -direction.

$$\langle \vec{p}, s | \vec{q}, r \rangle = 2E_p (2\pi)^3 \delta_D^{(3)}(\vec{p} - \vec{q}) \delta^{rs} \quad (4.32)$$

$$|\vec{p}_1, s_1; \dots; \vec{p}_N, s_N\rangle = \prod_{i=1}^N \sqrt{2E_{p_i}} a_{s_1}^\dagger(\vec{p}_1) \dots a_{s_N}^\dagger(\vec{p}_N) |0\rangle . \quad (4.33)$$

2. By the anti-commutation relations we state the theorems:

Statements 4.1 *The wavefunction of N-particle states of spin $\frac{1}{2}$ particles is anti-symmetric under particle exchange.*

\Rightarrow spin- $\frac{1}{2}$ particle obey Fermi-Dirac statistics, i.e. they are fermions.

\Rightarrow Pauli-exclusion principle:

No two fermionic states of exactly the same quantum numbers are possible.

Spin-Statistics Theorem

This can be cast more generally into the *Spin-Statistics-Theorem*:

- Particles of half-integer spin are *fermions* and
- particles of integer spin are *bosons*.

3.

4. The Dirac Lagrangian satisfies a global $U(1)$ symmetry with $j^\mu = \bar{\psi} \gamma^\mu \psi$

$$Q = \int \frac{d^3 p}{(2\pi)^3} \sum_s [a_s^\dagger(\vec{p}) a_s(\vec{p}) - b_s^\dagger(\vec{p}) b_s(\vec{p})] \quad (4.34)$$

$$\Rightarrow \left\{ \begin{array}{ll} Q a_s^\dagger(\vec{p}) |0\rangle &= +a_s^\dagger(\vec{p}) |0\rangle \leftrightarrow \text{fermion of spins, momentum } \vec{p} \\ Q b_s^\dagger(\vec{p}) |0\rangle &= -b_s^\dagger(\vec{p}) |0\rangle \leftrightarrow \text{anti fermion of spins, momentum } \vec{p}. \end{array} \right\} \quad (4.35)$$

4.0.5 *Propagator*

In the Heisenberg picture we find the propagator to be

$$S_B^A(x - y) := \{\psi^A(x), \bar{\psi}_B(y)\} \quad (4.36)$$

$$\Rightarrow S(x - y) = (i\gamma^\mu \partial_{x^\mu} + m) [D^{(0)}(x - y) - D^{(0)}(y - x)] \quad (4.37)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} [(\gamma_\nu p^\nu + m_0) e^{-ip(x-y)} + (\gamma_\nu p^\nu - m_0) e^{-ip(y-x)}]. \quad (4.38)$$

With this we have $S(x - y) = 0$ for $(x - y)^2 < 0$ outside the lightcone; this in turn guarantees $[O_1(x), O_2(y)] = 0$ for $(x - y)^2 < 0$, because all observables are bilinear in fermions, these still commute outside the lightcone. This establishes causality of the Dirac theory.

The propagator satisfies $(i\partial_x - m)S(x - y) = 0$ (because $p^2 = m^2$). This follows from $(\partial_x^2 + m^2)D^{(0)}(x - y) = 0$.

The time-ordering symbol in the fermionic theory is defined by an *additional minus sign*

$$\hat{T}[\psi(x), \bar{\psi}(y)] = \left\{ \begin{array}{ll} \psi(x) \bar{\psi}(y) & \text{if } x^0 \geq y^0 \\ -\bar{\psi}(y) \psi(x) & \text{if } y^0 > x^0. \end{array} \right\} \quad (4.39)$$

Feynman propagator in fermionic theory

The Feynman propagator is

$$S_F(x-y) = \langle 0 | \hat{T}[\psi(x), \bar{\psi}(y)] | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\gamma \cdot p + m_0)}{p^2 - m_0^2 + i\epsilon} e^{-ip \cdot (x-y)}. \quad (4.40)$$

The Feynman propagator is the Green's function of the Dirac operator

$$(i\cancel{D} - m) S_F(x-y) = i\delta_D^{(4)}(x-y). \quad (4.41)$$

For computations use

$$\not{p} - m \frac{\not{p} + m}{p^2 - m^2} = \mathcal{I}, \quad (4.42)$$

$$\frac{\not{p} + m}{p^2 - m^2} = \frac{\mathcal{I}}{\not{p} - m}, \quad (4.43)$$

these are matrices, beware.

4.0.6 Wick's theorem and Feynman diagrams**Fermionic time ordering**

The time ordering of several fields picks up a minus sign whenever 2 fermionic fields are exchanged, i.e. multiply by (-1) for every adjacent permutation of two fields.

Fermionic normal ordering

We define *normal-ordered* products as expressions with all annihilation/creation operators to the right/left, but here each exchange of two operators induces a minus sign, beware !

$$\hat{T}[\psi(x), \bar{\psi}(y)] = : \psi(x) \bar{\psi}(y) : + \overline{\psi(x)} \overline{\bar{\psi}(y)} \quad (4.44)$$

$$\overline{\psi(x)} \overline{\bar{\psi}(y)} = \langle 0 | \hat{T}[\psi(x) \bar{\psi}(y)] | 0 \rangle = S_F(x-y) \quad (4.45)$$

$$\overline{\psi(x)} \overline{\psi(y)} = 0 = \overline{\bar{\psi}(x)} \overline{\bar{\psi}(y)}. \quad (4.46)$$

Wick's theorem for fermions

$$\hat{T}[\bar{\psi}_1 \bar{\psi}_2 \psi_3 \dots] = : \bar{\psi}_1 \bar{\psi}_2 \psi_3 \dots : + \text{:all contractions with signs: .} \quad (4.47)$$

In particular

$$\langle 0 | \hat{T} \left[\prod_i \psi(x_i) \prod_j \bar{\psi}(\bar{x}_j) \right] | 0 \rangle \neq 0 \quad (4.48)$$

only for equal numbers of ψ and $\bar{\psi}$ fields. Physically this just reflects *charge conservation*.

Feynman-diagrams for fermions

To compute a $2n$ -point function, we draw corresponding Feynman diagrams, but now

1. Label the points x_i associated with $\psi(x_i)$ and \bar{x}_j associated with $\bar{\psi}(\bar{x}_j)$ separately.
2. Only connect x_i and \bar{x}_j .
3. Associate each directed line from \bar{x}_j to x_i with a propagator

$$S_F(x_i - \bar{x}_j) = \begin{array}{c} x_i \\ \longleftarrow \\ \bar{x}_j \end{array} \quad (4.49)$$

Always draw the arrow from \bar{x}_j to x_i in order to account for the correct sign in $S_F(x_i - \bar{x}_j)$.

Statements 4.2 *The relative sign between the diagrams is equal to the numbers of crossing lines.*

4.0.7 LSZ and Feynman rules

In the presence of interactions we define asymptotic in-and out-fields satisfying the free Dirac equation with mass $m \neq m_0$, where m_0 is the mass of the Dirac action.

We then express the creation and annihilation modes by the in-and out-fields:

$$a_{in,s}(\vec{q}) = \frac{1}{\sqrt{2E_q}} \int d^3x \bar{u}_s(\vec{q}) e^{iq \cdot x} \gamma^0 \psi_{in}(x) \quad (4.50)$$

$$a_{in,s}^\dagger(\vec{q}) = \frac{1}{\sqrt{2E_q}} \int d^3x \bar{\psi}_{in}(x) \gamma^0 e^{-iq \cdot x} u_s(\vec{q}) \quad (4.51)$$

$$b_{in,s}(\vec{q}) = \frac{1}{\sqrt{2E_q}} \int d^3x \bar{\psi}_{in}(x) \gamma^0 e^{iq \cdot x} v_s(\vec{q}) \quad (4.52)$$

$$b_{in,s}^\dagger(\vec{q}) = \frac{1}{\sqrt{2E_q}} \int d^3x \bar{v}_s(\vec{q}) e^{-iq \cdot x} \gamma^0 \psi_{in}(x). \quad (4.53)$$

LSZ formula for fermions

Consider incoming fermions $|q, s, +\rangle$ and anti fermions $|q', s', -\rangle$ and outgoing fermions $\langle p, r, +|$ and anti fermions $\langle q', r', -|$: Note that $n = \# \text{ fermions}$ and $n' = \# \text{anti-fermions}$ as well as $m = \text{fully renormalized physical mass}$:

$$\begin{aligned} & \langle \dots (p, r, +) \dots (p', r', -) \dots | S | \dots (q, s, +) \dots (q', s', -) \dots \rangle_{\text{connected}} \\ &= (-iZ)^{-n \frac{1}{2}} (iZ)^{-n' \frac{1}{2}} \int d^4x \dots \int d^4x' \dots \int d^4y \dots \int d^4y' \\ & \exp(\{-i(q \cdot x + q' \cdot x' - p \cdot y - p' \cdot y' + \dots)\}) \\ & \bar{u}_r(\vec{p})(i\gamma \cdot \partial_y - m) \dots \bar{v}_{s'}(\vec{q}') (i\gamma \cdot \partial_{x'} - m) \\ & \langle \Omega | \hat{T} [\dots \bar{\psi}(y') \dots \psi(y) \dots \bar{\psi}(x) \dots \psi(x') \dots] | \Omega \rangle \\ & (-i\gamma \cdot \overleftarrow{\partial}_x - m) u_s(\vec{q}) \dots (-i\gamma \cdot \overleftarrow{\partial}_y - m) v_{r'}(\vec{p}'). \end{aligned} \quad (4.54)$$

Feynman diagrams and S-matrix for fermions

Thus to compute the S-matrix we compute the Fourier transform of the amputated fully connected associated Feynman diagram, where for each external particle we include

1. $u_s(\vec{q})$ for an incoming particle of spin s ,
2. $\bar{v}_{s'}(\vec{q}')$ for an incoming anti-particle of spin $-s'$,
3. $\bar{u}_r(\vec{p})$ for an outgoing particle of spin r ,
4. $v_{r'}(\vec{p}')$ for an outgoing anti-particle of spin $-r'$.

4.1 QUANTIZING SPIN 1-FIELDS

4.1.1 Maxwell's equations (classical)

The action for Maxwell's equation is

$$S = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu \right] \quad (4.55)$$

with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, $j^\mu = (\rho, \vec{j})^T$, $A^\mu = (\phi, \vec{A})^T$.

The *inhomogeneous Maxwell equations* follow with

$$\partial_\mu F^{\mu\nu} = j^\nu. \quad (4.56)$$

Two of the Maxwell equations are given by the Bianchi identity

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \Rightarrow \partial_\mu j^\mu = 0 \quad (4.57)$$

follows (S is gauge invariant if and only if this charge conservation holds).

$F^{\mu\nu}$ and thus \vec{E} and \vec{B} are invariant under a *local gauge transformation*

$$A^\mu(x) \rightarrow A^\mu(x) + \partial^\mu \alpha(x) \quad (4.58)$$

for any function $\alpha(x)$ with $\lim_{x \rightarrow \infty} \alpha(x) = 0$. Configurations related by gauge transformations are physically equivalent. Gauge symmetries merely denote a redundancy in the description of the system.

Lorenz gauge

Pick A^μ such that

$$\partial_\mu A^\mu = 0, \quad (4.59)$$

which still leaves us with the freedom of a residual gauge transformation

$$A^\mu \rightarrow A^\mu + \partial^\mu \phi \quad \text{with} \quad \partial^2 \phi = 0. \quad (4.60)$$

\Rightarrow In Lorenz gauge the equation of motions is

$$\partial^2 A^\mu = j^\mu. \quad (4.61)$$

Lorenz gauge is implemented in the action by a Lagrange multiplier

$$S = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\alpha A^\alpha)^2 - A_\beta j^\beta \right] \quad (4.62)$$

$$\stackrel{e.o.m.}{\Rightarrow} \left\{ \begin{array}{l} \partial^2 A^\mu - (1 - \lambda)\partial^\mu(\partial \cdot A) = j^\mu \\ \partial \cdot A = 0 \end{array} \right\} \quad (4.63)$$

This equation (Lorenz gauge) will be implemented at a specific point during quantization, not before.

4.1.2 Canonical Quantization of the free field

Complications in quantization procedure

Quantizing a free ($j^\mu = 0$) non-gauge fixed Lagrangian doesn't work, because A^0 is not a dynamical field $\Rightarrow \Pi_\mu = F_{0\mu} \Rightarrow \Pi_0 = 0$. Instead quantization starts from the gauge fixed Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial \cdot A)^2 \quad (4.64)$$

$$\Rightarrow \Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} = F_{\mu 0} - \lambda \eta_{\mu 0}(\partial \cdot A). \quad (4.65)$$

With $\lambda = 1$, this choice is called *Feynman gauge*, the equation of motion of λ , namely $\partial \cdot A = 0$, has now to be imposed by hand as a constraint:

$$\Rightarrow \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu \quad \text{with} \quad \partial \cdot A = 0. \quad (4.66)$$

The equations of motion for A^μ are

$$\partial^2 A^\mu = 0 \quad \text{together with } \partial \cdot A = 0. \quad (4.67)$$

And the canonical momentum density is

$$\Pi_\mu = -\dot{A}_\mu. \quad (4.68)$$

Canonical equal-time commutators for photons

Quantize by promoting A_μ, Π_μ to Heisenberg operator field with *canonical equal-time commutators*

$$[A^\mu(\vec{x}, t), \dot{A}^\nu(\vec{y}, t)] = -i\eta^{\mu\nu}\delta_D^{(3)}(\vec{x} - \vec{y}), \quad (4.69)$$

$$[A^\mu(\vec{x}, t), A^\nu(\vec{y}, t)] = 0 = [\dot{A}^\mu(\vec{x}, t), \dot{A}^\nu(\vec{y}, t)], \quad (4.70)$$

which imply the mode expansion for the photon field

$$A^\mu(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int_{\lambda=0}^3 \epsilon^\mu(\vec{p}, \lambda) [a_\lambda(\vec{p}) e^{-ip \cdot x} + a_\lambda^\dagger(\vec{p}) e^{ip \cdot x}], \quad (4.71)$$

which solves $\partial^2 A^\mu = 0$ for $p^2 = 0 \Rightarrow E_p = |\vec{p}|$ for photons and it implies $\Rightarrow p^\mu$ lightlike!

The vectors $\epsilon^\mu(\vec{p}, \lambda)$ are *four linearly independent real polarization vectors*. We pick their normalization to be

$$\epsilon(\vec{p}, \lambda) \cdot \epsilon(\vec{p}, \lambda') = \eta_{\lambda, \lambda'} = \delta_{\lambda, \lambda'}. \quad (4.72)$$

We pick the four ϵ^μ to be the following, let n^μ denote the times axis with $n^2 = 1$:

The basis of polarization vectors depends on the concrete momentum p with $p^2 = 0$.

1. $\epsilon^\mu(\vec{p}, 0) = n^\mu$: timelike/scalar polarization vector,
2. $\epsilon^\mu(\vec{p}, i)$, $i = 2, 1$: transverse polarization vectors with $\epsilon(\vec{p}, i) \cdot n = 0 = \epsilon(\vec{p}, i) \cdot p$ & $\epsilon(\vec{p}, i) \cdot \epsilon(\vec{p}, j) = -\delta_{ij}$,
3. $\epsilon^\mu(\vec{p}, 3)$ longitudinal polarization vector with $\epsilon(\vec{p}, 3) = \frac{p-n(p \cdot n)}{p+n}$.

Photon ladder operator commutation relations

In this theory we find

$$[a_\lambda(\vec{p}), a_{\lambda'}^\dagger(\vec{p}')] = -\eta_{\lambda,\lambda'}(2\pi)^3 \delta_D^{(3)}(\vec{p} - \vec{p}') \quad (4.73)$$

$$[a_\lambda(\vec{p}), a_{\lambda'}(\vec{p}')] = 0 = [a_\lambda^\dagger(\vec{p}), a_{\lambda'}^\dagger(\vec{p}')]. \quad (4.74)$$

Fock space for photons

As usual, from these the Hamiltonian and construction of the Fock space follows straightforwardly:

$$\begin{aligned} H &= \frac{1}{2} \int d^3x \left[-\dot{A}^\mu \dot{A}_\mu + \partial_i A_\mu \partial^i A^\mu \right] \\ &= \int \frac{d^3p}{(2\pi)^3} |\vec{p}| \left[\sum a_i^\dagger(\vec{p}) a_i(\vec{p}) - a_0^\dagger(\vec{p}) a_0(\vec{p}) \right] \end{aligned} \quad (4.75)$$

where the latter equality follows *after dropping the vacuum energy*. The Fock space follows then with

$$[H, a_\lambda^\dagger(\vec{p})] = +|\vec{p}| a_\lambda^\dagger(\vec{p}) \quad (4.76)$$

$$[H, a_\lambda(\vec{p})] = -|\vec{p}| a_\lambda(\vec{p}) \quad (4.77)$$

$$\Rightarrow a_\lambda(\vec{p}) |0\rangle = 0. \quad (4.78)$$

The *1-particle* state is here defined via

$$|\vec{p}, \lambda\rangle := \sqrt{2E_p} a_\lambda^\dagger(\vec{p}) |0\rangle \quad (4.79)$$

as the states of momentum \vec{p} and polarization λ . The corresponding photons are called *photons*.

Ghosts and complications with unitarity

Due to the minus sign in the commutator we find time-like polarization ($\lambda = 0$) states to have a negative norm

$$\langle \vec{p}, 0 | \vec{q}, 0 \rangle \propto \langle 0 | [a(\vec{p}, 0), a^\dagger(\vec{q}, 0)] | 0 \rangle = -(2\pi)^3 \delta_D^{(3)}(\vec{p} - \vec{q}).$$

Such negative norm states are called *ghosts* and spoil unitarity.

⇒ The problems are solved if we implement now the Lorenz gauge. It will remove timelike, negative norm states and furthermore *cut the physical polarizations down to two*.

4.1.3 Gupta-Bleuler quantization

The *Gupta-Bleuler* formalism directly implements the Lorenz gauge condition on the physical states of our Hilbert space. We define a physical Hilbert space by the *Gupta-Bleuler condition*

$$\phi \in \mathcal{H}_{physical} \leftrightarrow \partial^\mu A_\mu^+ |\phi\rangle = 0 \quad (4.80)$$

with $A_\mu(x) = A_\mu^+(x) + A_\mu^-(x)$.

For a 1-photon state of general polarization ζ^μ with

$$\zeta^\mu = \sum_{\lambda, \lambda'} \alpha_\lambda \eta_{\lambda, \lambda'} \epsilon^\mu(\vec{p}, \lambda) \quad (4.81)$$

$$|\vec{p}, \zeta\rangle := \sqrt{2|\vec{p}|} \sum_\lambda \alpha_\lambda a_\lambda^\dagger(\vec{p}) |0\rangle. \quad (4.82)$$

Physical states and polarizations

This condition equivalently reads

$$|\vec{p}, \zeta\rangle \in \mathcal{H}_{physical} \leftrightarrow p^\mu \zeta_\mu = 0. \quad (4.83)$$

One can furthermore decompose the photon state like

$$|\vec{p}, \zeta\rangle = |\vec{p}, \zeta_T\rangle + |\vec{p}, \zeta_s\rangle \quad (4.84)$$

where

1. $|\vec{p}, \zeta_T\rangle$ describes 2 transverse degrees of freedom of positive norm, $\|\vec{p}, \zeta_T\rangle\| > 0$.
2. $|\vec{p}, \zeta_s\rangle$ describes 1 combined timelike and longitudinal degree of freedom of zero no norm, $\|\vec{p}, \zeta_s\rangle\| = 0$.
3. \Rightarrow One finds this theorem to be true:

Statements 4.3 *The state $|\vec{p}, \zeta_s\rangle$ decouples from all physical processes*

Such a zero-norm state that decouples from all physical processes is called *spurious*:

$$\langle \vec{p}, \zeta_s | O | \vec{p}, \zeta_s \rangle = 0 \quad \forall \text{observables in a free theory}. \quad (4.85)$$

Generally, as long as the interactions respect gauge invariance, a spurious state $|\vec{p}, \zeta_s\rangle$ decouples from the S-matrix as an external (in or out) state \Rightarrow Only the 2 transverse polarizations are physically relevant as external state.

Statements 4.4 *Massless vector fields have two physical degrees of freedom.*

4.1.4 Coupling vector fields to matter

On connection of mass and gauge invariance

Lorentz invariance requires invariance of the action under gauge transformations, because any Lorentz vector field describing 2 polarization states transforms under a Lorentz transformation

$$A^\mu(x) \rightarrow \Lambda_\nu^\mu A^\nu(\Lambda^{-1}x') + \partial^\mu \epsilon(x, \Lambda) \quad (4.86)$$

Furthermore, $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the *unique* Lorentz invariant and gauge invariant Lagrangian for a massless free vector field.

Statements 4.5 *The mass term for a massive vector boson explicitly breaks gauge invariance. Massless vector field theories must be gauge theories.*

Note that massless vector fields have two physical degrees of freedom.

Coupling the massless vector theory to matter

We would like to couple a massless vector field $A^\mu(x)$ to a matter sector, e.g. a Dirac fermion or a scalar field (denoted with ϕ here). We can construct such a coupled action

$$S = S_A^0[A] + S_{int}[A, \phi] + S_{matter}^{rest}[\phi]. \quad (4.87)$$

→ If $S_{int}[A, \phi]$ is chosen such, that the successful decoupling of negative norm states and zero norm states in the free vector theory is not spoiled by the interaction. This constraint is formulated by the *Ward identity* for QED

$$k_\mu M^\mu = 0 \quad (4.88)$$

Such that external photons of polarization $\zeta_\mu(k) = k_\mu$ decouple from the interactions. This uncoupling is equivalent to the statement, that the vector theory must couple to a conserved current

$$-\frac{\delta S_{int}[A, \phi]}{\delta A^\mu} = -j^\mu, \quad (4.89)$$

with j^μ for $U(1)$ global symmetry.

If $S_A^0[A]$ is *invariant* under the gauge symmetry

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \alpha(x) \quad (4.90)$$

then the coupling to a conserved $U(1)$ current is equivalent to the statement that the full action S is invariant under the combined gauge transformation, and therefore *equivalent* to the Ward identity

$$\phi(x) \rightarrow \phi(x) - e\alpha(x)\delta\phi(x) \quad \& \quad A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \alpha(x). \quad (4.91)$$

The process of promoting the global continuous $U(1)$ symmetry to a combined gauge symmetry is called *gauging*.

4.1.4.1 Coupling to Dirac fermions

We now want to apply this gauge process to the free Dirac fermion action

$$S_{\text{matter}}^{\text{rest}} = \int d^4x \bar{\psi}(i\gamma \cdot \partial - m_0)\psi. \quad (4.92)$$

Gauging the free Dirac theory \Rightarrow Quantum Electrodynamics

The Dirac Lagrangian has a global $U(1)$ symmetry with an associated conserved current. We find the gauge invariant QED Lagrangian if we gauge the Dirac theory by promoting the global $U(1)$ symmetry to a local one:

$$\psi(x) \rightarrow e^{-ie\alpha(x)}\psi(x), \quad j^\mu = e\bar{\psi}\gamma^\mu\psi. \quad (4.93)$$

The interaction term is then *invariant under the combined gauge transformation*! Thus the decoupling of unphysical photon states is satisfied:

$$\psi(x) \rightarrow e^{-ie\alpha(x)}\psi(x), \quad A_\mu \rightarrow A_\mu + \partial_\mu\alpha(x), \quad (4.94)$$

which can be written in terms of the *covariant derivative*

$$D_\mu := \partial_\mu + ieA_\mu \quad \Rightarrow \quad D_\mu\psi(x) \rightarrow e^{-i\alpha(x)}D_\mu\psi(x). \quad (4.95)$$

Thus, the Lagrangian is *manifestly gauge-invariant* off-shell

$$\mathcal{L}_{\text{QED}} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{=-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m_0)\psi. \quad (4.96)$$

In other words, the equations of motion of QED read

$$\partial_\nu F^{\nu\mu} = e\bar{\psi}\gamma^\mu\psi \quad (4.97)$$

$$\overset{\partial \cdot A = 0}{\Leftrightarrow} \partial^2 A^\mu = e\bar{\psi}\gamma^\mu\psi. \quad (4.98)$$

In QED we set $e = -|e|$ equal to the *elementary charge of one electron*, such that

$$\left\{ \begin{array}{ll} Qa_s^\dagger(\vec{p})|0\rangle = -|e|a_s^\dagger(\vec{p})|0\rangle & \text{for an electron} \\ Qb_s^\dagger(\vec{p})|0\rangle = +|e|b_s^\dagger(\vec{p})|0\rangle & \text{for a positron} \end{array} \right\}, \quad (4.99)$$

where $Q = e \int d^3x \bar{\psi}\gamma^0\psi = e \int d^3x j^0$.

4.1.4.2 A subtle note on gauge symmetry

There is a big difference between the interpretation of a global symmetry and a gauge symmetry. The former takes you from one physical

state to another with the same properties and results in a conserved current through Noether's theorem. The latter is redundancy in our description of the system. Yet here it seems like gauge symmetry leads to a conservation law, namely the *conservation of electric charge*. This is because among the infinite number of gauge symmetries parametrized by a function $\alpha(x)$, there is also a *single* global symmetry: that with $\alpha(x) = \text{constant}$. This is a *true* symmetry of the system, meaning that it takes us to another physical state.

Note that coupling to complex scalars with $U(1)$ symmetry is possible by the *minimal coupling prescription* $\partial_\mu \rightarrow D_\mu$. By this, the gauge invariance of the combined matter and gauge sector is ensured.

4.1.5 Feynman rules for QED

We will now study interactions of QED, whose Lagrangian is given by

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \underbrace{\frac{\lambda}{2}(\partial \cdot A)^2}_{=0 \text{ in Feynman gauge}} + \bar{\psi}(i\gamma \cdot \partial - m_0)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi. \quad (4.100)$$

This theory describes *the coupling of the Maxwell $U(1)$ gauge potential to electro-magnetically charged spin $\frac{1}{2}$ particles of free mass m_0* .

Feynman propagator for the photon/gauge field

The Feynman propagator for the gauge field in Feynman gauge is

$$\begin{aligned} \langle 0 | \hat{T}\{A^\mu(x)A^\nu(y)\} | 0 \rangle &= -\eta^{\mu\nu}D_F^{(0)}(x-y)|_{m_0^2=0} \\ &= \int \frac{d^4 p}{(2\pi)^4} \frac{-i\eta^{\mu\nu}}{p^2 + i\epsilon} e^{-ip \cdot (x-y)}. \end{aligned} \quad (4.101)$$

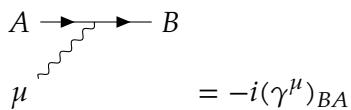
The Feynman rules for QED are the following:

To compute the *scattering amplitude* iM_{fi} of a given process

$$\langle f | S | i \rangle = \delta_{fi} + (2\pi)^4 \delta_D^{(4)}(p_f - p_i) M_{fi} \quad (4.102)$$

we draw all fully connected, amputated Feynman diagrams to the given order in the coupling constant e and read off M_{fi} as follows:

1. Each interaction vertex has the form



$$= -i(\gamma^\mu)_{BA} \quad (4.103)$$

try to get
feynman dia-
grams cor-
rectly posi-
tioned as a
wedge

(with the arrows denoting fermion number flow) and carries a factor $-ie(\gamma^{mu})_{BA}$,

2. Each internal photon lines carries

$$\mu \sim \sim \sim \sim \nu = -\frac{i\eta_{\mu\nu}}{p^2 + i\epsilon} \quad (4.104)$$

3. Each internal fermion line with the arrow denoting fermion (as opposed to anti-fermion) number flow carries a factor

$$A \rightarrow \rightarrow B = \left(\frac{i(\gamma \cdot p + m_0)}{p^2 - m_0^2 + i\epsilon} \right)_{BA}, \quad (4.105)$$

4. Momentum conservation is imposed at each vertex, multiply per vertex $(2\pi)^4 \delta_D^{(4)}(p_f - p_i)$,
5. We integrate over each (undetermined) internal momentum p with

$$\int \frac{d^4 p}{(2\pi)^4} \text{ measure,} \quad (4.106)$$

6. Each ingoing photon of polarization λ carries a factor $\epsilon_\mu(\vec{p}, \lambda) Z_A^{\frac{1}{2}}$, where $Z_A = 1 + O(e)$. Each outgoing photon of polarization λ carries a factor $\epsilon_\mu^*(\vec{p}, \lambda) Z_{A'}^{\frac{1}{2}}$,
7. Each ingoing fermion of spin s carries a factor $u_s(\vec{p}) Z_e^{\frac{1}{2}}$, each ingoing anti-fermion of spin s carries a factor $\bar{v}_{-s}(\vec{p}) Z_e^{\frac{1}{2}}$, Each outgoing fermion of spin s carries $\bar{u}_s(\vec{p}) Z_e^{\frac{1}{2}}$, each outgoing anti-fermion of spin s carries a factor $v_{-s}(\vec{p}) Z_e^{\frac{1}{2}}$,
8. The overall sign of a given diagram is easiest determined directly in the interaction picture. IF 2 diagrams are related by the exchange of n fermion lines, then the relative sign is $(-1)^n$. If we are only interested in $|M_{fi}|^2$ this is often enough to determine the cross section.

Interactions in QED are mediated by the exchange of gauge bosons.

4.2 QUANTUM ELECTRODYNAMICS

4.2.1 Radiative Corrections in QED

Up to this point we've only calculated interactions at leading order, i.e. tree level Feynman diagrams. We will now turn towards loop corrections of QED, thus higher orders in the perturbation theory, which

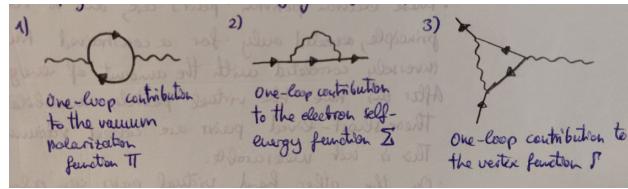


Figure 4.1: Diverging integrals in QED appearing higher than tree level.

typically give infinity. Some tools of renormalization and regularization will be introduced to deal with these infinities.

Higher order terms display diagrams with closed loops, which imply the presence of diverging integrals, such as A theory is renormalizable, thus meaningful after renormalization, if the number of diverging diagrams is finite. The reason for this is that to get observables renormalized one needs a finite number of constants to maintain the predictive value of the theory untouched.

4.2.1.1 Corrections to the fermion propagator

Taking QED interactions into account the Feynman propagator $S_F(x - y) = \overset{y}{\longrightarrow} \overset{x}{\longrightarrow}$ is corrected like

$$\langle \Omega | \hat{T}\{\psi(x)\bar{\psi}(y)\} | \Omega \rangle = \overset{y}{\longrightarrow} \overset{x}{\longrightarrow} + \overset{y}{\longrightarrow} \text{loop} \overset{x}{\longrightarrow} \\ + \overset{y}{\longrightarrow} \text{loop} \overset{x}{\longrightarrow} \quad (4.107)$$

$$= \overset{y}{\longrightarrow} \text{shaded circle} \overset{x}{\longrightarrow}. \quad (4.108)$$

Electron self-energy

Let

$$A - (1PI) - B \equiv -i \sum (\not{p})_{AB} \quad (4.109)$$

denote the amputated 1PI diagram. $\sum(\not{p})$ is called self-energy of the electron. \Rightarrow

$$\langle \Omega | \hat{T}\{\psi(x)\bar{\psi}(y)\} | \Omega \rangle = \overset{y}{\longrightarrow} \overset{x}{\longrightarrow} \rightarrow (\sum) \rightarrow \\ + \rightarrow (\sum) \rightarrow (\sum) \rightarrow + \dots \quad (4.110)$$

Where the full Feynman propagator for interacting QED theory takes the form (via Dyson resummation)

$$y - \text{---} \circlearrowleft \text{---} x = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} \frac{i}{\gamma \cdot p - m_0 - \sum(p) + i\epsilon}. \quad (4.111)$$

4.2.1.2 Corrections to the photon propagator

Photon self-energy

The Fourier transform of the photon propagator, denoted by



, can be derived via Dyson resummation from the

1PI diagram

$$\mu \text{---} \sim \sim \sim \sim \nu \equiv i\Pi^{\mu\nu}(q^2) = \quad (4.112)$$

which is the *self-energy of the photon or vacuum polarization*.

Vacuum polarization describes the process in which a background electromagnetic field produces virtual electron-positron pairs that change the distribution of charges and currents that generated the original electromagnetic field. It is also referred to as the *self-energy of the gauge boson(photon)*.

These electron-positron pairs are, due to Heisenberg's uncertainty principle, created only for a constrained time, having duration inversely correlated with the amount of energy of the fluctuation. After this time the virtual particles annihilate each other. These short-lived pairs are called *vacuum bubbles*. This is not measurable.

On the other hand virtual pairs can also occur as a photon progress, as seen in 4.1; the effect on other processes is measurable.

⇒ Such charged pairs act as an electric dipole. In the presence of an electric field, e.g. the electromagnetic field around an electron, these particle-antiparticle pairs reposition themselves, thus partially counteracting the field.

The field therefore will be weaker than would be expected if the vacuum were completely empty. This reorientation of the short-lived particle-antiparticle pairs is referred to as *vacuum polarization*.

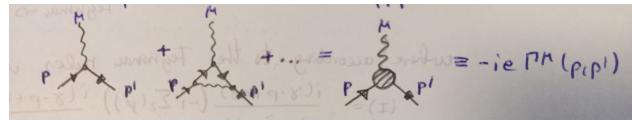


Figure 4.2: Definition of the vertex function, summing up all loop corrections.

4.2.1.3 Corrections to the interaction vertex

Vertex function

We define an effective vertex by summing up all loop-corrections. This is the *vertex function* Γ , which describes the coupling between a photon and an electron beyond leading order, it is the one particle irreducible correlation function which involves $\psi, \bar{\psi}$ and A_μ :

We will compute these corrections to 1-loop order. The diagrams will exhibit:

1. Ultraviolet (UV) divergences from integrating the momenta of particles in the loop up to infinity.
2. Infra-red (IR) divergences if the diagram contains massless particles (i.e. photons) running in the loop

The general status of these divergences is as follows

1. UV divergences require regularization of the integral and can be absorbed in a clever definition of the parameters via renormalization.
2. IR divergences in loop-diagrams cancel if physical observables are computed.

4.2.2 Self-energy of the electron at 1-loop

At 1-loop order the electron propagator takes the form

$$\langle \Omega | \hat{T} \{ \psi(x) \bar{\psi}(y) \} | \Omega \rangle = \overbrace{y \rightarrow y} + \underbrace{\overbrace{y \rightarrow x} \atop \text{Feynman}}_{\Rightarrow \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} \cdot (I)} \quad (4.113)$$

where according to the Feynman rules we have

$$(I) = \frac{i(\gamma \cdot p + m_0)}{p^2 - m_0^2 + i\epsilon} \left(-i \sum_2 (\not{p}) \right) \frac{i(\gamma \cdot p + m_0)}{p^2 - m_0^2 + i\epsilon}. \quad (4.114)$$

⇒ The amputated 1-loop contribution corresponds to omitting the two outer fermion propagators and is thus given by:

$$-i \sum_2 (\not{p}) = (-ie)^2 \int \frac{d^4 k}{(2\pi)^4} \gamma^\mu \frac{i(\not{k} + m_0)}{k^2 - m_0^2 + i\epsilon} \gamma^\nu \frac{(-i\eta_{\mu\nu})}{(p-k)^2 + i\epsilon}. \quad (4.115)$$

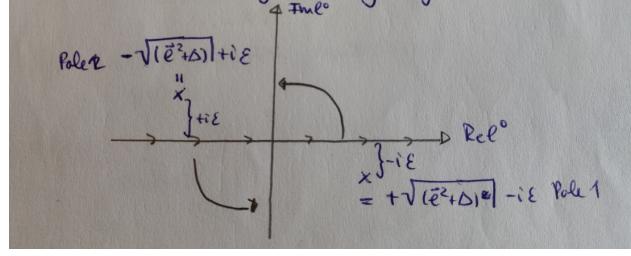


Figure 4.3: How to perform integral in the renormalization procedure by a Wick rotation.

This integral has a IR-divergence near $k = 0$ if $p \rightarrow 0$. We will introduce a fictitious small photon mass μ to regulate the IR divergence:

$$\Rightarrow i \sum_2 (\not{p}) = (-ie)^2 \int \frac{d^4 k}{(2\pi)^4} \gamma^\mu \frac{i(k + m_0)}{k^2 - m_0^2 + i\epsilon} \gamma_\mu \frac{-i}{(p - k)^2 - \mu^2 + i\epsilon}. \quad (4.116)$$

The evaluation of such typical momentum integrals proceeds in 3 steps.

4.2.2.1 Step 1: Feynman parameters

The integrand contains a fraction of the form $\frac{1}{AB}$ with $A = (p - k)^2 - \mu^2 + i\epsilon$ and $B = k^2 - m_0^2 + i\epsilon$. Any $A, B \in \mathbb{C} \setminus \{0\}$ satisfies

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2} \quad (4.117)$$

with x : Feynman parameter.

\Rightarrow in our case applying the identity with some algebra yields

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[l^2 - \Delta + i\epsilon]^2}, \quad l = k - xp, \quad \Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2. \quad (4.118)$$

With some more Dirac algebra manipulations we find

$$-i \sum_2 (\not{p}) = -e^2 \int_0^1 dx \int \frac{d^4 l}{(2\pi)^4} \frac{-2\not{l} + 4m_0}{[l^2 - \Delta + i\epsilon]^2}. \quad (4.119)$$

4.2.2.2 Step 2: Wick rotation

The Wick rotation relates typical integrals appearing in loops like $\int \frac{d^4 l}{(2\pi)^4} \frac{1}{[l^2 - \Delta + i\epsilon]^n}$ to Euclidean integrals.

The 0 integral is in fact a complex contour integral along the real axis. The value of this integral is unchanged if we deform the contour without hitting any pole. Therefore we can rotate the contour by 90° counter-clockwise to lie along the imaginary axis: Introducing the Euclidean

4-momentum $l_E = (l_E^0, \vec{l}_E)$ as

$$l^0 = il_E^0, \vec{l} = \vec{l}_E \Rightarrow l^2 = -l_E^2 \equiv -i \sum_j (l_E^j)^2 \quad (4.120)$$

we can write the integral as

$$\int \frac{d^4l}{(2\pi)^4} \frac{1}{[l^2 - \Delta + i\epsilon]^m} = (-1)^m i \int \frac{d^4l}{(2\pi)^4} \frac{1}{[l_E^2 + \Delta - i\epsilon]^m}. \quad (4.121)$$

Since we won't need the $i\epsilon$ any longer we can omit it at this stage. We can now perform the integral as a spherical integral in \mathbb{R}^4 .

5

ADVANCED QUANTUM FIELD THEORY

5.1 PATH INTEGRAL QUANTIZATION

5.1.1 Path integral in Quantum Mechanics

The path integral provides a formulation of quantum theory completely equivalent to the canonical quantization.

A quantum mechanical transition amplitude $\langle q_f, t_f | q_i, t_i \rangle = \langle q_f | e^{i\hat{H}(t_f - t_i)} | q_i \rangle$ can, by partitioning of the *transition time* $\delta t = \frac{t_f - t_i}{N+1}$ and insertion of complete sets of states $\mathcal{I} = \int_{\mathbb{R}} dq_k |q_k\rangle \langle q_k|$ between each partition, be expressed as

$$\langle q_f, t_f | q_i, t_i \rangle = \lim_{N \rightarrow \infty} \int \frac{dp_0}{2\pi} \prod_{k=1}^N \frac{dp_k dq_k}{2\pi} \exp \left[i \sum_{k=0}^N \left(p_k \frac{q_{k+1} - q_k}{\delta t} - H \right) \delta t \right] \quad (5.1)$$

which is per definition equivalent to

$$\langle q_f, t_f | q_i, t_i \rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q(t) \mathcal{D}p(t) \exp \left[i \int_{t_i}^{t_f} dt (p\dot{q} - H(p, q)) \right] \quad (5.2)$$

with

$$S[p, q] \equiv \int_{t_i}^{t_f} dt L(p, q) \quad \text{and} \quad L(p, q) \equiv p\dot{q} - H(p, q) \quad (5.3)$$

which holds true for every Weyl-ordered Hamiltonian, e.g. $H = f(p) + V(q)$.

Thus, we have boundary conditions for $q(t)$, whereas $p(t)$ is free at the endpoints.

Feynman path integral

Analytic continuation by rotating t onto the lower half-plane via $\delta t \rightarrow \delta t(1 - ie)$ followed by performing the momentum path integral as a Gaussian yields the Feynman-Kac formula

$$\langle q_f, t_f | q_i, t_i \rangle = \lim_{N \rightarrow \infty} C^{N+1} \prod_{k=1}^N \int dq_k e^{i \int_{t_i}^{t_f} dt L(q, \dot{q})} = \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q(t) e^{i \int_{t_i}^{t_f} dt L(q, \dot{q})} \quad (5.4)$$

where the factor $C^{N+1} = \left(\frac{-im}{2\pi\delta t}\right)^{\frac{N+1}{2}}$ is absorbed into $\mathcal{D}q(t)$, using $C := \sqrt{\frac{-im}{2\pi\delta t}} \cdot s$

This is interpreted as:

The transition amplitude $\langle q_f, t_f | q_i, t_i \rangle$ counts all possible continuous path from q_i to q_f weighted by $\exp\left[\frac{i}{\hbar} S\right]$. In the *classical limit* $S[q, \dot{q}] \gg \hbar$ and due to the strongly oscillating phase of the integrand, the right-hand side is dominated by paths for which the action becomes *stationary*

$$\frac{\delta S}{\delta q} = 0 \Leftrightarrow \text{principle of least action.} \quad (5.5)$$

This is the *classical path*, quantum physics is obtained by summing up all additionally possible paths with the classical path via the path integral.

The limit $N \rightarrow \infty$ and the ensuing interpretation of the path integral as an integral over all continuous paths from q_i to q_f can be made mathematically rigorous - at least for $H = \frac{p^2}{2m} + V(q)$ - by passing to the Euclidean theory.

N-point correlators

For n-point correlation functions we find in the interacting theory, that the time ordering appears automatically, which means that under the path-integral the operators are merely \mathbb{C} -numbers, such that ordering is not necessary anymore:

$$\langle \Omega | \hat{T} \prod_i \hat{q}_H(t_i) | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\int \mathcal{D}q(t) \mathcal{D}p(t) e^{-i \int_{-T}^T dt L(p, q)} \prod_i q(t_i)}{\int \mathcal{D}q(t) \mathcal{D}p(t) e^{i \int_{-T}^T dt L(p, q)}}. \quad (5.6)$$

5.1.2 The path integral for scalar fields

Path integral of a real scalar field

The path integral for real scalar fields $\phi(x)$ (as opposed to particles) is given by

$$\langle \phi_f(\vec{x}), t_f | \phi_i(\vec{x}), t_i \rangle = \int_{\phi(\vec{x}, t_i) = \phi_i(\vec{x})}^{\phi(\vec{x}, t_f) = \phi_f(\vec{x})} \mathcal{D}\phi(x) e^{i \int_{t_i}^{t_f} dt \int_{\mathbb{R}^3} d^3x \mathcal{L}[\phi(x)]}. \quad (5.7)$$

Master formula for cumulants

The master formula for an n-point quantum correlation function reads

$$\begin{aligned} G(x_1, \dots, x_n) &= \langle \Omega | \hat{T} \prod_{j=1}^n \hat{\phi}_H(x_j) | \Omega \rangle \\ &= \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi(x) \prod_{j=1}^n \phi(x_j) e^{-i \int_{-T}^T d^4x \mathcal{L}(\phi)(x)}}{\int \mathcal{D}\phi(x) e^{i \int_{-T}^T d^4x \mathcal{L}[\phi(x)]}}. \end{aligned} \quad (5.8)$$

In a perturbatively renormalizable theory, the action can be renormalized order by order such that the perturbatively expanded correlation functions are finite in the continuum limit, order by order. One can evaluate the path-integral in an exact, non-perturbative manner by e.g. *Lattice Field Theory* (c.f. page 181).

5.1.3 Generating Functional for Correlation Functions**Generating functional**

The generating functional $Z[J]$ of Green's functions $G(x_1, \dots, x_n)$ for some source $J(x)$ reads

$$Z[J] = \int \mathcal{D}\phi e^{iS[\phi] + iJ \cdot \phi}, \quad (5.9)$$

where the functional inner product is defined as

$$J \cdot \phi = \int_{\mathbb{R}^{3,1}} d^4x J(x) \phi(x). \quad (5.10)$$

$Z[J]$ maps the function $\phi(x)$ to a number in \mathbb{C} . It is called generating functional because

$$\frac{Z[J]}{Z[0]} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(\prod_{j=1}^n \int_{\mathbb{R}^{3,1}} d^4x_j J(x_j) \right) G(x_1, \dots, x_n). \quad (5.11)$$

Calculating cumulants from the generating functional

This can be solved for $G(x_1, \dots, x_n)$ using the tools of functional calculus:

$$G(x_1, \dots, x_n) = \frac{1}{Z[0]} \left[\prod_{j=1}^n \frac{\delta}{i\delta J(x_j)} \right] Z[J]|_{J=0}, \quad (5.12)$$

with the functional derivative

$$\frac{\delta \phi(y)}{\delta \phi(x)} = \delta_D(x - y). \quad (5.13)$$

$Z[0]$ contains no external points and represents the partition function $Z[0] = e^{\sum_i V_i}$, where the sum runs over all vacuum bubbles V_i of the theory. Consequently, $\frac{Z[J]}{Z[0]}$ contains no vacuum bubbles.

5.1.4 Perturbative Expansion in Interacting Theory

$Z_0[0]$ is in general a divergent quantity, but it can be rigorously defined by a regularization procedure.

All physical quantities are computed in the regularized theory and the expression $Z_0[0]$ cancels in all such expressions because the correlation functions derive from $\frac{Z[J]}{Z[0]}$.

To compute a $2n$ -point function one expands the exponential precisely to n -th order, while the $2n + 1$ -functions vanish.

Consider an action with an interaction

$$S[\phi] = S_0[\phi] + \int_{\mathbb{R}^{3,1}} d^4x \mathcal{L}_{int}[\phi] \quad (5.14)$$

$$\Rightarrow Z[J] = \exp \left[i \int_{\mathbb{R}^{3,1}} d^4x \mathcal{L}_{int} \left[-\frac{i\delta}{\delta J} \right] \right] Z_0[J]. \quad (5.15)$$

Wick theorem in path integral language

We can recover the Feynman rules and Wick's theorem equivalently as in the canonical quantization with the definition for function $F[\phi]$

$$\langle F[\phi] \rangle_0 := e^{\frac{1}{2} \frac{\delta}{\delta \phi} \cdot D_F \cdot \frac{\delta}{\delta \phi}} F[\phi]|_{\phi=0} \quad (5.16)$$

with the scalar Feynman propagator D_F , such that the following hold true for *scalar* theories:

$$\Rightarrow G(x_1, \dots, x_n) : \frac{\langle \phi(x_1) \dots \phi(x_n) \exp [i \int d^4x \mathcal{L}_{int}] \rangle_0}{\langle \exp [i \int d^4x \mathcal{L}_{int}] \rangle_0}. \quad (5.17)$$

We can identify :

$$\langle \phi(x_1) \phi(x_2) \rangle_0 = \langle 0 | \hat{T} \hat{\phi}(x_1) \hat{\phi}(x_2) | 0 \rangle = \phi(x_1) \phi(x_2), \quad (5.18)$$

which gives us the results of Wick's theorem

$$\begin{aligned} \langle \phi_1 \dots \phi_{2n} \rangle_0 &= \phi_1 \overbrace{\phi_2 \phi_3} \overbrace{\phi_4 \dots \phi_{2n-1}} \phi_{2n} \\ &\quad + \text{allothercontractions.} \end{aligned} \quad (5.19)$$

$$\langle \phi_1 \dots \phi_{2n+1} \rangle_0 = 0. \quad (5.20)$$

With Wick's theorem at hand we recover the exact same Feynman rules as in operator language from the Gell-Mann-Low formula in the interaction picture for position and momentum space.

Note, that $\phi(x_i)$ or $\frac{\delta}{i\delta F(x_i)}$ corresponds to an external point in the Feynman diagram, that a line between x_i and x_j denotes as always a factor $D_F(x_i - x_j)$:

$$x_i \longrightarrow x_j \quad (5.21)$$

and that interactions only occur if there is a dot on crossing lines:



5.1.4.1 On the counting of loops

A fully connected Feynman diagram in momentum space with E external and I internal lines, V vertices, and L loops (number of unfixed momentum integrals) satisfies *Euler's formula*

$$L = I - V + 1 \leftrightarrow \underbrace{x_i \longrightarrow \bullet \longrightarrow x_j}_{1=2-2+1} \quad (5.23)$$

⇒ For $\mathcal{L}_{int}(\phi) = \frac{\lambda^n}{n!} \phi^n(x)$ we have

$$E + 2I = nV, \quad (5.24)$$

because every vertex connects to n lines, while every external line connects to one and every internal line to two vertices. Together with Euler's formula we have

$$(n - 2)V = 2L + (E - 2). \quad (5.25)$$

Hence for fixed E , an expansion in L corresponds to an expansion in V .

5.1.5 The Schwinger-Dyson Equation

Symmetries in path integral language

An advantage of the path-integral method is that symmetries are more transparent. It becomes clear that classical symmetries carry over to the quantum theory - but only provided the path integral measure $\mathcal{D}\phi = \mathcal{D}\phi'$ is invariant under a given transformation corresponding to the symmetry. In that case, the Schwinger-Dyson equation

$$\left(\frac{\delta S}{\delta \phi} \Big|_{\phi=\frac{\delta}{i\partial J}} + J \right) Z[J] = 0 \quad (5.26)$$

$$= \int \mathcal{D}\phi \left[\frac{\delta S}{\delta \phi(x)} + J(x) \right] e^{i(S[\phi]+J\cdot\phi)} \quad (5.27)$$

states that the classical equation of motion (in presence of a source J)

$$0 = \frac{\delta S}{\delta \phi} + J \quad (5.28)$$

holds as an operator equation in the quantum theory, i.e. it holds inside the path integral. They provide the e.p.m. for Green's functions/ for given n-point correlators. Note that these equations of motion are therefore preserved under the transformation $\phi \rightarrow \phi'$.

Note that this is only true as long as there are no contact terms $\propto \delta_D(x - x_j)$.

⇒ The idea is that the interactions of a theory are also represented in its Green's functions. The full Green's functions containing the interaction should also contain the free Green's functions in the *limit of the free theory*.

The full 1-point function can be understood in terms of full higher n-point functions and bare interactions as encoded in $S[\phi]$.

Ward-Takahashi identity

For a continuous global classical symmetry $\phi \rightarrow \phi' = \phi + \delta\phi$ with conserved Noether current $j^\mu(x)$ given by $\frac{\delta S}{\delta\phi(x)}\delta\phi(x) = -\partial_\mu j^\mu(x) = 0$ on-shell we can find the *Ward-Takahashi identity*

$$\partial_\mu \langle \Omega | \hat{T}j^\mu \prod_{i=1}^n \phi(x_i) | \Omega \rangle = \quad (5.29)$$

$$- i \sum_{i=1}^n \langle \Omega | \hat{T}\phi(x_1) \dots \phi(x_{i-1}) \delta\phi(x) \delta_D^{(4)}(x - x_i) \phi(x_{i+1}) \dots \phi(x_n) | \Omega \rangle \quad (5.30)$$

which is derived from the Schwinger-Dyson equation, i.e. the statement of current conservation up to contact terms inside correlation functions.

Like the Schwinger-Dyson equation, the Ward-Takahashi identity only holds for classical symmetries of $S[\phi]$ that leave the measure invariant. If $\mathcal{D}\phi$ is affected, the symmetry is *anomalous* and current conservation (up to contact terms) does not hold at quantum level.

5.1.6 Connected diagrams

$G(x_1, \dots, x_n)$ receives contributions from partially connected Feynman diagrams. These are the diagrams that factor into subdiagrams each of which is connected only to some of the n external points x_1, \dots, x_n . As established by the LSZ formalism, what enters the computation of scattering amplitudes are only the fully connected Green's functions $G^{(c)}(x_1, \dots, x_n)$ corresponding to fully connected Feynman graphs, i.e. to those Feynman diagrams which do not factor into subdiagrams.

⇒

Generating functional of connected diagrams/cumulants

The generating functional of $G^{(c)}(x_1, \dots, x_n)$ is called *effective action* and denoted by $iW[J]$. It is given by

$$\frac{Z[J]}{Z[0]} = e^{iW[J]} \quad (5.31)$$

Thus,

$$\tau(x_1, \dots, x_n) = \frac{\delta}{i\delta J(x_1)} \dots \frac{\delta}{i\delta J(x_n)} iW[J] \quad (5.32)$$

$$\Rightarrow G^{(c)}(x_1, \dots, x_n) = \tau(x_1, \dots, x_n)|_{J=0}. \quad (5.33)$$

Summarizing,

1. $G^{(c)}$ = cumulants.
2. $\tau(x_1, \dots, x_n)$ denotes a fully connected n-point Green's function in the presence of a source J . The fully connected are a subset of all Green's functions.

1PI effective action

An even stronger reduction are the fully connected, amputated 1PI Green's functions $\tilde{\Gamma}(x_1, \dots, x_n)$ in the presence of the field φ ($\Gamma(x_1, \dots, x_n) = \tilde{\Gamma}(x_1, \dots, x_n)|_{\varphi=0}$) generated by the 1PI effective action $\Gamma[\varphi]$.

- 3.

5.1.7 The 1PI effective action

Effective action, general idea

In QFT, the effective action is a modified expression for the action, which takes into account quantum-mechanical corrections in the following sense:

In classical mechanics, the equation of motion can be derived from the principle of least action. This is not the case in QM, where the amplitudes of all possible motions are added up in the path integral. However, if the action is replaced by the effective action, the equation of motion for the vacuum expectation value of the field

$$\varphi = \langle \phi \rangle \quad (5.34)$$

can be derived from the requirement that the effective action be stationary.

An important subclass of fully connected Feynman diagrams are the 1-particle-irreducible (1PI) diagrams, which cannot be cut into two non-trivial diagrams by cutting a single (internal) line.

1PI effective action

The generating function for the 1PI connected diagrams is called the *1PI effective action* $\Gamma[\varphi]$, which is defined as the Legendre transform of $W[J]$:

$$\Gamma[\varphi] := W[J_\varphi] - \varphi \cdot J_\varphi \equiv W[J_\varphi] - \int d^4x' \varphi(x') J_\varphi(x') \quad (5.35)$$

where $\varphi(x)$ is the 1-point function of $\phi(x)$ in the presence of a source J

$$\varphi(x) \equiv \frac{\delta W[J]}{\delta J(x)} = \langle \Omega | \hat{\phi}(x) | \Omega \rangle_J = \text{vacuum expectation value} \quad (5.36)$$

and we assumed there to be a bijection between J and φ .

$\Gamma[\varphi]$ is now also the generating functional for the 1PI connected amputated Green's functions $\Gamma_n(x_1, \dots, x_n)$,

$$i\Gamma[\varphi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \varphi(x_1) \dots \varphi(x_n) \Gamma_n(x_1, \dots, x_n). \quad (5.37)$$

Now, $\Gamma_n(x_i)$, $n \geq 3$, is the sum of the tree-level vertex plus of the 1- and higher-loop 1PI amputated corrections, while $\Gamma_2(x_1, x_2)$ consists of the 1- and higher loop 1PI amputated diagrams minus the tree-level propagator.

Connected diagrams in the full quantum theory

To compute a connected n-point function in the full quantum theory, we use the tree-level Feynman rules but replace each k -vertex (with $k \geq 3$) of the classical action $S[\phi]$ with the 1PI amputated k -vertex Γ_k as encoded in $\Gamma[\varphi]$ and replace the free propagator with G_2 , the *fully resummed propagator*.

⇒ Replacing $S[\phi]$ by $\Gamma[\varphi]$ and computing at tree-level gives the full quantum theory. Gives classical equation of motion as provided by $S[\phi]$ + additional quantum mechanical corrections.

$\Gamma[\varphi]$ and $S[\phi]$ are the same functionals at tree-level, i.e.

$$\Gamma[\varphi] = S[\varphi] + \hbar K[\varphi] \quad (5.38)$$

for some $K[\varphi]$ starting at one-loop. The equation

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = -J(x), \text{ i.e. } \frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = 0, \text{ for } J(x) = 0 \quad (5.39)$$

is the *quantum effective equation of motion* replacing $\frac{\delta S}{\delta\phi} = 0$ in the full quantum theory.

The quantum mechanical corrections start at 1-loop; before, the full theory is provided by $\frac{\delta S}{\delta\phi} = 0$.

5.1.8 Background Field Method

Background field method in QFT:

A general field can be decomposed into the vacuum expectation value of the quantum operator $\hat{\phi}(x)$, thus $\varphi(x)$, and the *quantum fluctuation* $f(x)$ around the *background* $\varphi(x)$:

$$\phi(x) = \varphi(x) + f(x). \quad (5.40)$$

The 1PI-effective action is then obtained by integrating out the vacuum fluctuations

$$\begin{aligned} \Gamma[\varphi] &= S[\varphi] \\ &- i\hbar \ln \left[\int \tilde{\mathcal{D}}f \exp \left\{ \frac{i}{\hbar} \left(\frac{1}{2} f \cdot \frac{\delta^2 S}{\delta \varphi^2} \cdot f - \hbar \frac{\delta K}{\delta \varphi} \cdot f + \mathcal{O}(f^3) \right) \right\} \right] \end{aligned} \quad (5.41)$$

$$\equiv S[\varphi] + \hbar K[\varphi]. \quad (5.42)$$

This can be solved perturbatively

$$\Gamma[\varphi] = S[\varphi] + \hbar K^{(1-loop)}[\varphi] + \text{higher-loop corrections} \quad (5.43)$$

with

$$K^{(1-loop)}[\varphi] = -i \ln \left[\tilde{\mathcal{D}}f \exp \left\{ -\frac{1}{2} f \cdot \left(-\frac{i}{\hbar} \frac{\delta^2 S}{\delta \varphi^2} \right) \cdot f \right\} \right]. \quad (5.44)$$

This way one can already solve the whole theory up to x -loops perturbatively, if quantum fluctuations and action are known.

5.1.9 Grassmann algebra calculus

Fermionic anticommutation relations $\{\hat{\psi}^A(t, \vec{x}), \hat{\psi}_B^\dagger(t, \vec{y})\} = \delta_B^A \delta_D^{(3)}(\vec{x} - \vec{y})$, $\{\hat{\psi}^A(t, \vec{x}), \hat{\psi}_B(t, \vec{y})\} = 0 = \{\hat{\psi}_A^\dagger(t, \vec{x}), \hat{\psi}_B^\dagger(t, \vec{y})\}$ can be implemented into the path integral formalism by using anticommuting, nilpotent, Grassmann-valued fields $\psi(x)$ out of a Grassmann algebra \mathbb{A} . This is the case, because these anticommutation relations imply the need for *anti-commuting numbers*, thus

$$\psi_i^A(t) \psi_j^B(t) = -\psi_j^B(t) \psi_i^A(t), \quad (5.45)$$

these ψ_i take values in a so-called Grassmann-algebra \mathbb{A} defined as follows:

1. Let $\theta_i, i = 1, \dots, n$ be a basis of an n -dimensional complex vector space V , i.e. we have the notion of scalar multiplication $a\theta_i = \theta_i a$, $a \in \mathbb{C}$ and vector addition $a\theta_i + b\theta_j \in V$.
2. Define then a bilinear anti-commutative multiplication

$$\theta_i \theta_j = -\theta_j \theta_i \quad (5.46)$$

as a map from $V \times V \mapsto \Lambda^2 V$, where $\Lambda^2 V$ is the antisymmetric tensor product of V .

3. Iteratively we can build higher rank anti-symmetric product spaces $\Lambda^k V$ up to $\Lambda^n V$.
4. Then the Grassman (or exterior) algebra is defined as the space

$$\mathbb{A} = \bigoplus_{k=0}^n \Lambda^k V = \mathbb{C} \oplus V \oplus \Lambda^2 V \oplus \dots \oplus \Lambda^n V. \quad (5.47)$$

A typical element of \mathbb{A} is of the form

$$a + a_i \theta_i + \frac{1}{2} a_{ij} \theta_i \theta_j + \dots + \frac{1}{n!} a_{i_1 \dots i_n} \theta_{i_1} \dots \theta_{i_n}. \quad (5.48)$$

5. The coefficients are completely antisymmetric $a_{ij} = -a_{ji}$. The elements of the abstract Grassmann algebra \mathbb{A} , θ_i namely, are called Grassmann numbers and they are *nilpotent*:

$$\theta_i^2 = 0. \quad (5.49)$$

\Rightarrow An important example of a Grassmann algebra is the space Ω of differential forms defined on an n -dimensional manifold endowed with the structure wedge (or exterior product) (compare Differential geometry section 1.2 under GR).

6. A Grassmann algebra is a graded algebra:

- a) An element of $\Lambda^{2k} V$ has grade (or degree) $s = 0$ (even).
- b) An element of $\Lambda^{2k+1} V$ has grade (or degree) $s = 1$ (odd).

\Rightarrow Note that a general element of \mathbb{A} can always be written as the sum of an even and an odd element.

7. Two elements $A, B \in \mathbb{A}$ of definite s_A and s_B satisfy

$$AB = (-1)^{s_A s_B} BA. \quad (5.50)$$

Thus, even elements are commuting and therefore sometimes called bosonic, whereas odd elements are dubbed fermionic due to their anti-commuting nature.

8. To set up a notion of calculus on the space of functions

$$f : \mathbb{A} \rightarrow \mathbb{C}, \underline{\theta} \mapsto f(\underline{\theta}) = a + a_i \theta_i + \dots + \frac{1}{n!} a_{i_1 \dots i_n} \theta_{i_1} \dots \theta_{i_n} \quad (5.51)$$

one defines differentiation w.r.t. θ_i as follows:

a)

$$\frac{\partial}{\partial \theta_i} \theta_j = \delta_{ij}, \quad \frac{\partial}{\partial \theta_j} a = 0 \quad \forall a \in \mathbb{C}, \quad (5.52)$$

b)

$$\frac{\partial}{\partial \theta_i} [a_1 f_1(\underline{\theta}) + a_2 f_2(\underline{\theta})] = a_1 \frac{\partial}{\partial \theta_i} f_1(\underline{\theta}) + a_2 \frac{\partial}{\partial \theta_i} f_2(\underline{\theta}) \text{ Linearity} \quad (5.53)$$

- c) If $f_1(\underline{\theta})$ has definite grade s , then the graded Leibniz rule holds

$$\frac{\partial}{\partial \theta_i} [f_1(\underline{\theta}) f_2(\underline{\theta})] = \left(\frac{\partial}{\partial \theta_i} f_1(\underline{\theta}) \right) f_2(\underline{\theta}) + (-1)^s f_1(\underline{\theta}) \frac{\partial}{\partial \theta_i} f_2(\underline{\theta}). \quad (5.54)$$

9. We define an integral $I[f(\underline{\theta})]$ as a functional of $f(\underline{\theta})$. Consider first a Grassmann algebra with $n = 1 : \mathbb{A} = \mathbb{C} \oplus V$:

$$I[f(\theta)] = \int d\theta f(\theta) = \int d\theta [a + b\theta]. \quad (5.55)$$

Linearity and translation hold for this integral

- a) Translation invariance means invariance under a shift of the integration variable $\theta \rightarrow \theta + C$, this implies

$$\int d\theta C = 0 \quad \forall C \in \mathbb{C} \quad \text{Normalize } \int d\theta \theta = 1 \quad (5.56)$$

$$\Rightarrow \int d\theta [a + b\theta] = b = \frac{\partial}{\partial \theta} [a + b\theta] \quad (5.57)$$

$$\Rightarrow ! \int d\theta_i \theta_j = \delta_{ij} = \frac{\partial}{\partial \theta_i} \theta_j.. \quad (5.58)$$

- b) The general Grassmann measure is defined as

$$d^n \theta := d\theta_n d\theta_{n-1} \dots d\theta_1, \text{ with order } d\theta_i d\theta_j = -d\theta_j d\theta_i \quad (5.59)$$

$$\Rightarrow \int d^n \theta \theta_{i_1} \dots \theta_{i_n} = \epsilon_{i_1 \dots i_n} \quad (5.60)$$

$$\Rightarrow \int d^n \theta f(\underline{\theta}) = \frac{1}{n!} a_{i_1 \dots i_n} \epsilon_{i_1 \dots i_n} = a_{123 \dots n} \quad (5.61)$$

Only those terms contribute for which the Grassmann integral is "saturated" such that each $d\theta_i$ is paired with precisely are θ_i .

10. For a linear change of Grassmann integration variables we find for A being an $(n \times n)$ -matrix the Jacobian to be

$$\underline{\theta}' = A\underline{\theta} \quad \Rightarrow \quad d^n \theta = \det A \, d^n \theta'. \quad (5.62)$$

11. Complexification: Consider θ_i and θ_i^* from now on as independent degrees of freedom and define the complex conjugate via

$$(\theta_i \theta_j)^* = \theta_j^* \theta_i^* \quad (5.63)$$

$$\Rightarrow d^n \theta d^n \theta^* := d\theta_1 d\theta_1^* \dots d\theta_n d\theta_n^*. \quad (5.64)$$

12. Let us compare bosonic and fermionic Gaussian integrals:

- a) Real case:

$$\int d^n x e^{-\frac{1}{2} x_i A_{ij} x_j} = \sqrt{\frac{(2\pi)^2}{\det A}} \quad (\text{bosonic integration}) \quad (5.65)$$

$$\int d^n \theta e^{\frac{1}{2} \theta_i A_{ij} \theta_j} = \sqrt{\det A} = Pf(A) \quad (\text{fermionic integration}). \quad (5.66)$$

b) Complex case:

$$\int d^n z d^n z^* e^{-z_i^* B_{ij} z_j} = \frac{(2\pi)^n}{\det B} \quad (\text{bosonic}) \quad (5.67)$$

$$\int d^n \theta^* d^n \theta e^{-\theta_i^* B_{ij} \theta_j} = \det B \quad (\text{fermionic}). \quad (5.68)$$

5.1.10 The fermionic path integral

The fermionic path integral

The fermionic path integral is derived in the same fashion as the bosonic one w.r.t. the anticommutation relation. Thus, partitioning of the transition time $\delta t = \frac{t_f - t_i}{N}$ and insertion of complete sets of states $\mathcal{I} = \int d\psi^* d\psi |\psi\rangle e^{-\psi^* \bar{\psi}} \langle \psi|$, with ψ being a complex Grassmann number, leads to the path integral for Dirac fermionic fields $\psi(x)$, $\bar{\psi}(x) = \psi^\dagger(x) \gamma^0$

$$\langle \psi_f(\vec{x}_f, t_f) | \psi_i(\vec{x}_i, t_i) \rangle = \int_{\psi(x, t_i) = \psi_i(x)}^{\psi(x, t_f) = \psi_f(x)} \mathcal{D}\bar{\psi}(t, \vec{x}) \mathcal{D}\psi(t, \vec{x}) e^{i \int_{t_i}^{t_f} d^4x \mathcal{L}(\psi, \bar{\psi})}, \quad (5.69)$$

where $\mathcal{L}(\psi, \bar{\psi}) = \bar{\psi}(x)[i\partial - m_0]\psi(x) + \mathcal{L}_{int.}$

The four $n \times n$ -gamma-matrices (one for every spacetime dimension) span the Clifford algebra $Cl^n(\mathbb{C})$ defined by the anticommutator

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathcal{J}_{4 \times 4}, \quad \text{with } n = 2^{d/2} = 4. \quad (5.70)$$

To project initial and final states to the interacting vacuum $|\Omega\rangle$, the trick $m_0 \rightarrow m_0 - i\epsilon$ can be used.

Generating functional for fermionic correlators

The generating functional for fermionic correlators is defined as

$$Z[\eta, \bar{\eta}] = \langle \Omega | \hat{T} \exp \left[i \int_{\mathbb{R}^{3,1}} d^4x (\mathcal{L}(\psi, \bar{\psi}) + \bar{\psi}(x)\eta(x) + \bar{\eta}(x)\psi(x)) \right] | \Omega \rangle \quad (5.71)$$

$$= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp [iS[\psi, \bar{\psi}] + i\bar{\eta} \cdot \psi + i\bar{\psi} \cdot \eta] \quad (5.72)$$

with the external sources $\bar{\eta}(x), \eta(x)$ as Grassmann-valued classical fields.

5.1.11 Executive Summary of QFT

Executive Summary of QFT

Start with a classical action $S[\phi]$ in which the field $\phi(x)$ arises as the continuum limit $N \rightarrow \infty$ of a system of N harmonic oscillators.

In the classical limit $\hbar \rightarrow 0$, $\phi(x)$ is a definite function given by the classical equation of motion $\frac{\delta S[\phi]}{\delta \phi(x)} = 0$.

In Euclidean Signature.

Interpretation of the path integral

For \hbar finite, quantum fluctuations arise. These are encoded in $Z[J]$, where the path integral takes into account all possible functions $\phi(x)$ could assume (thus classical+quantum fluctuations).

What we can compute in the quantum theory are correlation functions. In particular, the quantum expectation value of the field $\phi(x)$ in the presence of a source J is

$$\varphi_J(x) := \langle \phi(x) \rangle_J = \langle \Omega | \hat{\phi}(x) | \Omega \rangle_J = \frac{1}{Z[0]} \int \mathcal{D}\phi \phi(x) e^{-\frac{1}{\hbar} [S[\phi] + \phi \cdot J]}. \quad (5.73)$$

With our definition of $W[J]$, we can compute this as

$$\varphi_J(x) = -\frac{\delta W[J]}{\delta J(x)}. \quad (5.74)$$

In terms of the Légendre transform $\Gamma[\varphi]$ of $W[J]$, we have

$$\frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = J(x). \quad (5.75)$$

⇒ By integrating out quantum fluctuations $f(x) = \phi(x) - \varphi(x)$, $\Gamma[\varphi]$ gives a quantum effective action

$$e^{-\frac{1}{\hbar} \Gamma[\varphi]} = \frac{1}{Z[0]} \int \mathcal{D}f \exp \left[-\frac{1}{\hbar} \left(S[\varphi + f] + \frac{\delta \Gamma}{\delta \varphi} \cdot f \right) \right]. \quad (5.76)$$

Replacing $S[\phi]$ by $\Gamma[\varphi]$ introduces 1PI amputated vertices and fully resummed propagators. Thus, computing at tree-level with $\Gamma[\varphi]$ already gives the full quantum theory !

5.2 RENORMALIZATION OF QFT

5.2.1 Superficial divergence

Renormalizability

A general QFT is called renormalizable if only a finite number of resummed amputated $1PI$ diagrams is UV divergent.

Suppose the renormalizable QFT contains m different fields and suppose that the UV divergent $1PI$ diagrams give rise to n divergent constants order by order in perturbation theory. Then $(n - m)$ of these constants can be absorbed in the definition of $(n - m)$ unphysical parameters, the so-called *bare couplings*.

This procedure requires specifying the outcome of $(n - m)$ physical observables as external input, i.e. experiment \Rightarrow bare couplings. The remaining m constants can be absorbed in the definition of the kinetic terms of the m fields without reducing the predictability of the theory further. Thus in a renormalizable theory only a finite number $(n - m)$ of physical observables must be specified order by order in perturbation theory, and predictive power is retained for all remaining observables, which can be computed and are finite as we remove the cutoff.

However, the price to pay for the appearance of the UV divergences in the first place is that the $(n - m)$ observables cannot be computed by the theory even in principle! For instance, QED cannot make any predictions whatsoever for the absolute value of the electron mass or the charge at $q^2 = 0$. However, once we take $e(q^2 = 0)$ from experiment, QED does predict the logarithmic running of the effective charge as a function of q^2 . As a result the renormalized QFT necessarily contains free parameters that must be fitted to experiment. Another example for such an observable for which no prediction can be made in QFT with divergent partition function is the vacuum energy (cosmological constant). A non-UV finite, but renormalizable QFT is an *effective theory*: The UV divergences hint at a breakdown of the theory at high energies where it does not describe the microscopic degrees of freedom correctly. Renormalization hides our ignorance about the true physics at high energies in the $(n - m)$ observables and we can fit the theory to experiment as one typically does with a phenomenological model.

Superficial degree of divergence

For a scalar theory in d dimensions with (bare) Lagrangian $\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2 - \frac{m_0^2}{2}\phi^2 - \frac{\lambda_0}{n!}\phi^n$, the naive UV structure of a diagram \mathcal{D} with L loops $\propto \int_{\mathbb{R}^d} d^d k$ and I propagators $\propto (k^2 - m^2)^{-1}$ is

$$\mathcal{D} \xrightarrow{k \rightarrow \infty} \underbrace{\int_{\mathbb{R}^d} d^d k_1 \dots \int_{\mathbb{R}^d} d^d k_L}_{\times L} \underbrace{\frac{1}{k_1^2 \dots k_I^2}}_{\times I}. \quad (5.77)$$

The *superficial degree of divergence* D of \mathcal{D} is defined as the difference in powers of momentum between numerator and denominator, i.e.

$$D = dL - 2I. \quad (5.78)$$

1. Regularizing the divergence with a momentum cutoff Λ , i.e. $\int_{-\infty}^{\infty} dk \rightarrow \lim_{\Lambda \rightarrow \infty} \int_{-\Lambda}^{\Lambda} dk$, diagrams fall into three categories of UV behaviour:
 - a) $D > 0 \Rightarrow \mathcal{D} \propto \Gamma^D$ superficially divergent
 - b) $D < 0 \Rightarrow \mathcal{D} \propto \Gamma^{-|D|}$ superficially finite
 - c) $D = 0 \Rightarrow \mathcal{D} \propto \ln(\Lambda)$ superficially log-divergent

According to this reasoning, as $\Lambda \rightarrow \infty$ only diagrams with $D \geq 0$ are divergent. Therefore D is called superficial degree of divergence: The term superficial indicates that D does not always reflect the actual divergence or finiteness properties of a diagram, due to these 3 possible reasons listed above.

2. The actual UV-behaviour may differ from the superficial one for three reasons:
 - a) For $D \geq 0$, a diagram may still be actually finite if a sufficient amount of symmetry constrains the form of the amplitude or leads to cancellations among infinite terms.
 - b) For $D < 0$, a diagram may still be divergent if it contains a divergent subdiagram.
 - c) Tree-level diagrams have $D = 0$, but are finite.

For $\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2 - \frac{m_0^2}{2}\phi^2 - \frac{\lambda_0}{n!}\phi^n$ as above, D depends on the mass dimension of the coupling $[\lambda_0] = d - \frac{d-2}{2} \cdot n$ (n from $\phi^{(n)}$ and correction λ_0) as

$$D = d - [\lambda_0] \cdot V - \left(\frac{d-2}{2} \right), \quad (5.79)$$

where one can use

$$[\phi] = \frac{d-2}{2} \quad [M^{(E)}] = [\lambda^{(E)}] = d - E \left(\frac{d-2}{2} \right). \quad (5.80)$$

5.2.2 Renormalizability and BPHZ Theorem

Renormalizability

The UV properties of a theory are decisively determined by (the sign of) the prefactor of V .

Thus, following this reasoning:

$\lambda_0^{(4)}$ is scale independent, $\lambda^{(5)}$ is $\propto \frac{1}{E}$ thus IR dominant and $\lambda^{(3)}$ is $\propto E$, thus UV dominant.

Renormalizability

1. A theory is called *renormalizable* if the number of superficially divergent amplitudes is finite, but superficial divergences appear at every order in perturbation theory.

Statements 5.1 $\stackrel{D}{\Leftrightarrow}$ *Renormalizable if its coupling has vanishing mass dimension, e.g. ϕ^4 .*

2. A theory is called *super renormalizable* if the number of superficially divergent Feynman diagrams is finite.

Statements 5.2 $\stackrel{D}{\Leftrightarrow}$ *super-renormalizable if its coupling has positive mass dimension, e.g. ϕ^3 .*

3. A theory is called *non-renormalizable* if the number of superficially divergent amplitudes is infinite

Statements 5.3 $\stackrel{D}{\Leftrightarrow}$ *non-renormalizable if its coupling has negative mass dimension, e.g. ϕ^5 .*

Altogther:

1. \Leftrightarrow If $D \propto +V$, there exists an infinite number of E , superficially divergent amplitudes since for every ??, diagrams with high enough V diverge. The theory is thus *non-renormalizable* $\Leftrightarrow [\lambda_0] < 0$.
2. If $D \neq V$ (and $d \geq 2$), only a finite number of diagrams is divergent but divergences appear at every loop-order. Such theories are called *renormalizable* and arise for $[\lambda] = 0$.
3. If $D \propto -V$, for high-enough loop order, all diagrams become superficially finite, making the theory *super-renormalizable* $\Leftrightarrow [\lambda_0] > 0$.

E.g. for $n = d = 4$, we have $[\lambda_0] = 0$ and $D = 4 - E$ independent of L or V . Hence, ϕ^4 -theory in $d = 4$ is renormalizable with only three superficially divergent diagrams (*at every loop order*).

BPHZ Theorem

Ignore the issue of divergent subgraphs for a moment. Then if a theory is (power-counting) renormalisable, at each order in perturbation theory only a finite number of divergent diagrams, parametrized by a finite number of divergent constants, appear. One can absorb these divergences order by order in the counterterms of the renormalized Lagrangian such that all physical amplitudes are finite.

The counterterms create new Feynman diagrams relevant at the next order. These will cancel the divergences of the divergent subdiagrams (if present) at the next order.

All of this leads to a perturbative adjustment of a finite number of counterterms in the renormalised Lagrangian, and thus predictivity is maintained.

By the BPHZ theorem, (power counting) renormalizability is sufficient for a theory to maintain predictability. The non-trivial aspect of this theorem concerns the complete cancellation of divergent subdiagrams by counterterms of the previous loop order (example ϕ^4 theory to 2nd-loop renormalized).

5.2.3 Regularization

Regularization

Regularization is the practice of isolating divergences. The three common methods in QFT are:

1. 1) *Cutoff regularization*: regularizes divergent momentum integrals via $\int_{-\infty}^{\infty} dk \rightarrow \lim_{\Lambda \rightarrow \infty} \int_{-\infty}^{\infty} dk$. However, this is for QED inconsistent with the Ward identities and gauge invariance because transformations of the sort $A^\mu \rightarrow A^\mu + \partial^\mu \alpha(x)$ cannot be carried out at the cutoff, making it a useless method in QED.
2. *Dimensional regularization*: evaluates divergent integrals in $d = 4 - \epsilon$ dimensions. The result is expanded in power of ϵ , which isolates the divergence as a pole as $\epsilon \rightarrow 0$.
3. *Pauli-Villars regularization* takes a divergent diagram and subtracts it from the same diagram, but with a fictitious massive particle in the loop, e.g. a photon of mass Λ . This removes the divergence, because for $k \rightarrow \infty$ the mass in the loop becomes irrelevant and both diagrams asymptote to the same value. But for $\Lambda \rightarrow \infty$, the auxiliary diagram vanishes and we recover the actual process.

The *dimensional regularization* can be sued in the context of the following steps

1. Transform momentum dependent integrand via *Feynman parametrization*

$$\int_0^1 dx \frac{1}{[xA + (1-x)B]^2} = \frac{1}{AB}. \quad (5.81)$$

2. Following 1., we encounter loop-integrals of the typical form $\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - \Delta + i\epsilon)^n}$, which can be easily solved by a *Wick rotation* to Euclidean space (then $i\epsilon$ can be dropped since the poles are not met anymore)

$$l^0 = il_E^0, \quad \vec{l} = \vec{l}_E. \quad (5.82)$$

3. Now impose dimensional regularization by going over to $d = 4 - \epsilon$ dimensions and carry out the loop-integral in d -dimensional spherical coordinates.
4. Substitute $x = \frac{\Delta}{|l_e|^2 + \Delta}$ and use *Euler-β-function*

$$\mathcal{B}(\alpha, \beta) := \int_0^1 dx x^{\alpha-1} (1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}. \quad (5.83)$$

5. Final result with limit $\epsilon \rightarrow 0$ and use of

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma + O(\epsilon) \quad (5.84)$$

$$x^{\frac{\epsilon}{2}} = 1 + \frac{\epsilon}{2} \ln(x) + O(\epsilon^2). \quad (5.85)$$

5.2.4 *Renormalization of QED*

The QED Lagrangian with symmetry group $U(1)$ describes the coupling of spin- $\frac{1}{2}$ bispinor fields $\psi(x)$ (electron, positron) to a covariant spin-1 gauge field $A_\mu(x)$ (photon) generated by the transformation behaviour of the spinors themselves. \mathcal{L}_{QED} can be expressed in terms of the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the covariant derivative as $D_\mu = \partial_\mu + ieA_\mu$ as

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(iD - m)\psi \quad (5.86)$$

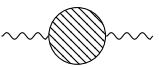
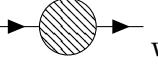
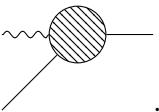
$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(iD - m)\psi - A_\mu j^\mu, \quad (5.87)$$

where m is the fermion mass, e is the coupling constant equal to the (electric) charge of the bispinor field and $j^\mu = e\bar{\psi}\gamma^\mu\psi$ is the conserved fermion current associated with the $U(1)$ symmetry ($\psi(x) \rightarrow e^{-i\epsilon\alpha}\psi(x), \alpha \in \mathbb{R}$).

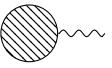
A QED diagram with $E_e(E_\gamma)$ external fermions (photons) has superficial degree of divergence

$$D = 4 - E_\gamma - \frac{3}{2}E_e. \quad (5.88)$$

Since $[e] = 0$, QED is renormalizable with seven superficially (four actually) divergent diagrams. These are the following

1. The vacuum energy  with $D = 4$.
2. The photon propagator  with $D = 2$; the electron propagator  with $D = 1$ and the $D = 0$ vertex .

(5.89)

3. The 1-photon amplitude 

4. The 3-photon amplitude with $D = 1$ 5. The 4-photon amplitude with $D = 0$ 

Only the diagram from 1. and 2. are actually divergent. This is due to the symmetries at work in QED preventing some of QED's superficial divergencies:

1. Discrete charge conjugation $j^\mu \rightarrow -j^\mu$ and $A^\mu \rightarrow -A^\mu$,
2. Chiral symmetry (arises for $m = 0$)
3. Ward identity $k_\mu \mathcal{D}^\mu = 0$ for a diagram $\mathcal{D} = \xi^\mu \mathcal{D}_\mu$ involving an external photon of momentum k^μ ($k^2 = 0$) and polarization ξ^μ ,
4. Gauge symmetry $A^\mu(x) \rightarrow A^\mu(x) + \partial^\mu \alpha(x)$.

The other diagrams vanish as follows:

1. Amplitudes 3. and 4. vanish to all orders due to *charge conservation*. A diagram with only an odd number of external photons vanishes, since each external photon couples via its current j^μ .

2. Diagram 5. is non-zero, but actually finite as a consequence of gauge symmetry, can be shown by exploiting the Ward identities.

Note that this diagram is responsible for the non-linearity of QED due to the scattering of photons with each other induced by loop effects.

→ As far as the UV divergences are concerned it suffices to consider the diagram 2., because the contribution of 1. is easily absorbed into the vacuum energy term.

3. By dimensional analysis it is found that all diagrams of 2. are logarithmically divergent.

Note on Chiral Symmetry

If $m_e \equiv 0$, then the theory possesses chiral or axial symmetry, because the Lagrangian can be written as

$$\mathcal{L} = \bar{\psi}_L i\gamma \cdot \partial \psi_L + \bar{\psi}_R i\gamma \cdot \partial \psi_R - ej^\mu A_\mu \quad (5.92)$$

and is invariant under independent rotations of the left-and right-chiral spinors

$$\psi_L \rightarrow e^{i\alpha} \psi_L, \quad \psi_R \rightarrow e^{i\beta} \psi_R. \quad (5.93)$$

This symmetry forbids a mass term, which must be of the form $m_e \bar{\psi}_R \psi_L$.

5.2.5 The renormalization scale**The renormalization scale**

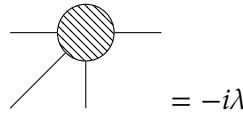
Renormalization automatically introduces a mass scale μ - *the renormalization scale* - into the quantum theory via the renormalization conditions (even when the classical theory was scale-free).

The renormalization scale can be an independent scale since the renormalization conditions are arbitrary.

Most often we renormalize $\phi_r = Z^{-\frac{1}{2}} \phi_0$ with the wavefunction renormalization factor Z as the residue of the propagator at the physical mass $p^2 = m^2$. We impose in this context the renormalization condition

$$M^2(p^2)|_{p^2=m^2} = 0 = \frac{d}{dp^2} M^2(p^2)|_{p^2=m^2} \quad (5.94)$$

and



$$= -i\lambda \quad (5.95)$$

at scale $s = 4m^2, t = u = 0$, which implies $\mu = m$, in order to specify the counterterms $\delta_Z, \delta_m, \delta_\lambda$ appearing due to the renormalization. These will then cancel the divergences of higher order diagrams (BPHZ).

5.2.6 The Callan-Symanzyk (CS) equation

$$G_n(x_1, \dots, x_n) = \\ G_n(x, \lambda, m, \mu)$$

The Callan-Symanzyk equation

To quantify the dependence of coupling constants on the renormalization scale μ , we can study the Callan-Symanzik (or renormalization group) equation (here for massive ϕ^4 -theory)

$$\left[\mu \frac{\partial}{\partial \mu} + \beta_\lambda \frac{\partial}{\partial \lambda} + \beta_{m^2} \frac{\partial}{\partial m^2} + n \cdot \gamma_\phi \right] G_n(x_1, \dots, x_n) = 0 \quad (5.96)$$

where

$$\beta_\lambda := \mu \frac{d\lambda}{d\mu}|_{\lambda_0, m_0 \text{ fixed}}, \quad \beta_{m^2} := \mu \frac{dm^2}{d\mu}|_{\lambda_0, m_0 \text{ fixed}} \quad (5.97)$$

$$\gamma_\phi := \frac{\mu}{2} \frac{d \ln(Z)}{d\mu}|_{\lambda_0, \mu \text{ fixed}}. \quad (5.98)$$

The β function

β_λ for example describes how the physical coupling λ changes as we change the energy scale μ at which we can perform an experiment.

The CS equation allows us to (perturbatively) compute $\beta_\lambda, \beta_{m^2}, \gamma_\phi$ explicitly by first computing $G_n(x_1, \dots, x_n)$ and then plugging it into the CS equation:

E.g.

$$\beta_\lambda = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3) < s \quad (5.99)$$

for massless ϕ^4 theory.

5.2.7 The running coupling

Renormalization group flow

The change in $\lambda(\mu)$ as we change μ is called *renormalization group flow* or running coupling.

$\lambda(\mu)$ gives the strength of the interaction at energy scale μ .

The renormalization conditions define the meaning of the physical coupling λ at the renormalization scale μ .

E.g. in massless ϕ^4 -theory we defined the dimensionless coupling λ by declaring that

$$iM(p_1 p_2 \rightarrow p_3 p_4) = -i\lambda \quad \text{at} \quad s = t = u = -\mu^2. \quad (5.100)$$

Effective coupling

The so-defined coupling is therefore really a function of the renormalization scale $\mu, \lambda(\mu)$, and this $\lambda(\mu)$ is the *effective coupling* relevant for processes with typical momenta of order μ :

Statements 5.4 $\lambda(\mu)$ gives the strength of the interaction at energy scale μ .

Depending on the sign of β there are three qualitatively different renormalization group (RG) behaviours:

1. If $\beta(\lambda) > 0, \lambda(\mu)$ increases as μ increases. If we start with a perturbative value λ_0 at μ_0 and follow the RG flow for increasing μ , then at some scale, $\lambda(\mu)$ ceases to be perturbative and perturbation theory is no more reliable. If by a non-perturbative analysis beyond that point one finds $\beta(\lambda) > 0 \forall \lambda$, then $\lambda(\mu)$ increases indefinitely. This can result in a divergent coupling $\lambda \rightarrow \infty$, either asymptotically as $\mu \rightarrow \infty$, or even for finite values of $\mu \rightarrow \mu_L$. The latter instance is referred to as a *Landau pole*.

One appears e.g. in QED at $\mu_L = \mu_0 \exp\left(\frac{3\pi}{2\alpha_0}\right)$ since

$$\alpha(\mu) = \frac{\alpha_0}{1 - \frac{2\alpha_0}{3\pi} \ln\left(\frac{\mu}{\mu_0}\right)} \xrightarrow{\mu \rightarrow \mu_L} \infty. \quad (5.101)$$

On the other hand, $\beta(\lambda) > 0$ means the theory is perturbatively well-defined in the infrared, where λ becomes small. If $\lambda \rightarrow 0$ as $\mu \rightarrow 0$, the theory even becomes free in the infrared.

Such a non-interacting fixed point is called *Gaussian fixed point*.

2. If $\beta|\lambda| < 0, \lambda(\mu)$ decreases as μ increases. The theory is perturbative in the UV, but may cease to be perturbative in the IR. If $\lambda \rightarrow 0$ as $\mu \rightarrow \infty$, the theory becomes free in the UV. This is called *asymptotic freedom*.

In $d = 4$, the only known example for asymptotic freedom is *Yang-Mills-theory*.

3. If $\beta \equiv 0 \forall \mu, \lambda$ is independent of μ . Such a theory is *conformal*, i.e. *scale-independent*. Since the counterterms do not induce any scale dependence, there cannot be any UV divergences altogether and the theory is UV finite.

5.2.8 Wilsonian interpretation

The original understanding of renormalization was:

1. The cutoff Λ is merely a way to regulate divergent integrals without any physical meaning.

2. Renormalization is a trick to remove the cutoff-dependence in physical amplitudes. This procedure allows us to take $\Lambda \rightarrow \infty$ without encountering any divergences.
3. This comes at the cost of losing predictability for a number of physical masses and coupling.
4. In a renormalizable theory, only a finite number of such physical couplings must be taken as input parameters from experiment to end up with a well-defined (otherwise predictive) theory.

The Wilsonian approach gives a different interpretation:

We should think of QFT as an *effective description* accurate only for energies below an intrinsic cutoff Λ_0 . At energies beyond Λ_0 the field theory picture does not correctly model the microscopic degrees of freedom. For example, QFT neglects gravity but all matter gravitates and the effects of gravity become non-negligible (compared to other forces) near the Planck scale

$$\Lambda_0 \propto M_{pl} = \frac{1}{\sqrt{8\pi G_N}} \approx 10^{18} \text{GeV}. \quad (5.102)$$

The only known theory that is UV finite and asymptotes to a weakly coupled QFT in the infrared is string theory, which abandons the concept of pointlike particles, replacing them with excitations of a one-dimensional string of length l_s . The string length is the intrinsic cutoff of the low-energy effective QFT. At distances near l_s , the theory deviates from a regular field theory in that it becomes non-local, thus avoiding UV divergences and arbitrary input parameters.

Integrating out the degrees of freedom between the regulator Λ and an even small cutoff $\Lambda_0 < \Lambda$ (thus avoiding large log corrections for $\Lambda_0 \ll \Lambda$) yields the Wilsonian effective action S_W^{eff} (which only accounts for the remaining degrees of freedom below Λ_0).

When computing correlators at scales below Λ_0 via S_W^{eff} , only momenta $|k| \leq \Lambda_0$ appear in the loops since all effects of the modes with $\Lambda_0 < |k| < \Lambda$ are already encoded in S_W^{eff} . This *Wilsonian effective action* is not to be confused with the quantum effective action $\Gamma[\varphi]$, which gives the full quantum theory (= includes all quantum effects) already at tree-level (no loops!).

S_W^{eff} includes only those quantum effects due to the integrated-out modes k between $\Lambda_0 < |k| < \Lambda$ and loops must still be performed.

1. The successive application of integrating out the degrees of freedom with $\Lambda_0 < |k| < \Lambda$ gives rise to the renormalization (semi-)group (semi because we can only lower the cutoff; there does not exist an inverse operation).
2. The running couplings in the Wilsonian picture are interpreted as the dependency of the couplings in S_W^{eff} on the cutoff. This

- identifies Λ_0 as the renormalization scale μ .
⇒ The effective couplings $\lambda(\Gamma_0)$ are defined by specifying observables computed from the effective action cutoff Λ_0 .
This replaces our previous renormalization condition fixing $\lambda(\mu)$.
3. ⇒ Even with renormalization our theory remains an effective theory to the extent that all couplings are really input parameters and cannot be computed from first principles without knowing the underlying theory.

5.3 NON-ABELIAN GAUGE THEORY

5.3.1 *Geometric perspective on abelian gauge theory*

In a local QFT

6

NON-EQUILIBRIUM QFT

6.1 KINETIC EQUATIONS FOR INITIAL DYNAMICS

Kinetic equations are able to describe settings where the dynamics of enormous amounts of particles are relevant. Rather than focusing on 'a few', say $1 - 10^4$, asymptotic particle products, i.e. the computation of S-matrix elements, a kinetic description is able to keep track of out-of-equilibrium details of what is happening before 'final states' form. This is achieved by distribution functions $f(X, \vec{p})$, which give the occupation of phase space (\vec{X}, \vec{p}) of arbitrarily many, $N(X^0) = \int d^3X \int d^3p f(X, \vec{p})$, particles at time X^0 . This is necessary, for example, if one is interested in the dynamics *during* a collision event as opposed to just the few asymptotic products which are measured in the detectors on the outside of colliders *at the end* of a collision. High energy collisions initially cause a region of plasma with massive fields and occupations up to the energy scale of the primary collision. Such a far-from-equilibrium system has to transport huge amounts of occupations in a distribution function in order to bring it into the form of the equilibrium Bose or Fermi distribution. Under certain circumstances this motion of occupations can be described by kinetic theory in terms of secondary collisions between the constituents of the plasma. The asymptotic end products leave the collision as the plasma expands and equilibrates via secondary collisions. In fact, in a beautiful analogy to cosmology, all matter in the universe can be understood as the asymptotic product of a collision, the big bang, followed by expansion, i.e. inflation, of the resulting plasma.

6.1.1 Spectrum of a theory

How many species of particles contribute to a kinetic description is decided by the so called *spectral function* $\rho(X, p)$ of the theory. Each particle species corresponds to a peak in the energy dependence of a spectral function and the lifetime of the particle is encoded in the width of this peak. Fundamentally, spectral functions are part of the two-point functions or 'propagators' of a quantum field theory. The exact computation of the spectral functions of even the simplest theories in equilibrium with no occupations, i.e. vacuum, is currently an open problem. In fact, the proof that the vacuum spectral function of Yang-Mills theory has no continuum of states near vanishing energy, i.e. is 'gapped', is one of the millennium problems. Ultimately any bound state can be understood in terms of spectral functions and the periodic

tables of nuclides and elements could be seen as a phenomenological approach to the spectral functions of the standard model. One of many assumptions of kinetic theory is that energy peaks of spectral functions can be exaggerated to delta peaks. These peaks can correspond to fundamental particles or to emergent degrees of freedom such as bound states. This ‘quasiparticle picture’ reduces the dynamics of the system to the ‘hopping’ of occupations due to collisions: a change in momentum of a particle corresponds to hopping along the momentum (\vec{p}) direction of one energy (p^0) peak, while a change in particle species corresponds to hopping between, say, fermion and photon peaks. Each particle species is assigned a distribution function and a kinetic equation that keeps track of these occupations. For example, a scattering $ee \rightarrow \gamma$ removes two fermions from the fermion peaks and puts a photon on the photon peak. The details of this ‘moving of occupations’, i.e. how exactly the distribution functions change due to a scattering, are encoded in collision terms, which couple the kinetic equations to each other in a non-linear fashion. For situations with spectral functions that can not be approximated by delta peaks, the concept of particles is no longer useful to describe the dynamics and a kinetic description breaks down. In such settings, it is simply not possible to capture ‘many body dynamics’ with the concept of particle motion. In this case, conventional kinetic theory is not sufficient to describe the out of equilibrium situation during a primary collision. The next best description, usually called ‘transport theory’, extends the concept of the classical phase space to off-shell energies p^0 by allowing momentum transfer between a continuum of states. These off-shell occupations are kept track of by so called *statistical propagators* $F(X, p)$ or *off-shell distribution functions* $f(X, p)$. In a derivation from nonequilibrium quantum field theory, one can chose to stop at transport equations and to not rely on a particle concept.

Historically, a particle picture had been the center of intuition in the development of quantum field theory. Exploring the validity of kinetic theory in the context of quantum field theory also means exploring the validity of these historical concepts. A particle focused interpretation of quantum field theory has been driven both by experiment and theory. Experimentally, the particle idea is tailored around colliders, which are constructed to test quantum field theory in terms of particle degrees of freedom. On the theoretical side, perturbation theory around free theory facilitates a particle picture because the spectral functions of free quantum field theories are always delta peaks. In fact, the first formulation of a quantum field theory was in terms of a collection of quantized harmonic, i.e. free, oscillators. Generations of development of experiment and theory have brought to light that our desire to extend our intuition about bowling balls to build fundamental truths had blinded us. Our understanding of quantum field

theory has since expanded drastically to out of equilibrium situations and non-perturbative methods.

6.2 REDUCTION TO VACUUM QFT

The initial conditions for all correlation functions are contained in the definition of the average

$$\langle O(x) \rangle := \text{Tr}\{\rho_D(t_0)O(x)\} \quad (6.1)$$

via the density matrix $\rho_D(t_0)$. The defining properties

$$\rho_D^\dagger = \rho_D, \quad \text{tr } \rho_D = 1, \quad \langle \rho_D \rangle \leq 1 \quad (6.2)$$

allow for its interpretation as a probability distribution functional of correlation functions. The choice of the initial condition decides over the complexity of the system. An immense reduction of complexity is commonly assumed in thermal theory, $\rho_D \propto e^{-\beta H}$, and vacuum theory, $\rho_D \propto |\Omega\rangle\langle\Omega|$, which is just thermal theory at zero temperature $\beta = 1/T \rightarrow \infty$. Such choices of ρ_D as a functional of the Hamiltonian imply a complete loss of dependence on the initial time t_0 and a reduction of complexity in all correlation functions via

$$\text{tr}\{e^{-\beta H}O(x)\} = \text{const.} \quad (6.3)$$

$$\text{tr}\{e^{-\beta H}O(x_1)O(x_2)\} = g(x_1 - g_x 2) \quad (6.4)$$

$$\dots \quad (6.5)$$

by virtue of the commutator

$$[e^{-\beta H}, P^\mu] = 0 \quad (6.6)$$

for the translation generator P^μ and the Hamiltonian $H = P^0$. Nonequilibrium theory does not possess this invariance and no such reduction of complexity occurs because in general

$$[\rho_D(t_0), P^\mu] \neq 0. \quad (6.7)$$

Therefore, nonequilibrium theory is the top of a hierarchy of reduction of complexity

$$\rho_D(t_0) \xrightarrow{\text{thermalization}} \frac{e^{-\beta H}}{\text{tr } e^{-\beta H}} \xrightarrow{\beta \rightarrow \infty} |\Omega\rangle\langle\Omega| \quad (6.8)$$

where all the open questions about *thermalization*, such as **the emergence of an arrow of time and dissipation from time reversal evolution**, are hidden in the first step. Most textbook calculations of scattering probabilities are formulated in terms of S-matrix elements which are objects of vacuum theory. Such scattering probabilities also appear in

the collision terms of kinetic theory, however accompanied by dynamical equations for phase-space distribution functions $f(X, \vec{p})$, i.e.

$$\partial_{X^0} f + \vec{v} \cdot \vec{\nabla}_X f - \vec{\nabla}_X V \cdot \vec{\nabla}_p f = C[f]. \quad (6.9)$$

Such Boltzmann equations describe the drifting in phasespace $\partial_{X^0} f + \vec{v} \cdot \vec{\nabla}_X f$ in the presence of an "external force" or "Vlasov term" $- \vec{\nabla}_X V \cdot \vec{\nabla}_p f$ and collisions $C[f]$. Kinetic equations automatically describe thermalization and become trivial in equilibrium since their left and right hand sides vanish identically for thermal Bose- or Fermi-distributions $f = f_{eq}$.

In general, the density matrix includes nonvanishing initial conditions for all correlation functions $\rho_D \propto e^{if[\Phi]}$ via arbitrary $f[\Phi]$. An initial condition is called *Gaussian* if $f[\Phi]$ is quadratic such that only one and two point functions, i.e. fields and propagators have non-trivial initial conditions. In some physical situations, e.g. immediately after cosmological inflation, Gaussian initial conditions can be justified immediately by the central limit theorem, which states that a superposition of independent random variables is generically Gaussian. In thermal equilibrium, $\rho_D \propto e^{-\beta H}$, Gaussian initial conditions imply a quadratic Hamiltonian, i.e. free theory, such that interesting physics only arise in a combination with out-of-equilibrium scenarios. Given the success of a plethora of effective descriptions from classical mechanics, kinetic theory, hydrodynamics to quantum mechanics and Gibbs' thermodynamics and many more, we are faced with the reality that **there has to exist some mechanism that provides a reduction of sensitivity to the details of initial conditions**. Clearly, to evolve a system in time, we don't usually have to specify an infinity of correlation functions with increasingly many variables. In fact, this would make a physical description impossible in practice. Natural science, as we know it, could not exist. Instead of infinitely many correlation functions, classical mechanics only requires you to specify the geometry of the problem and the initial position and velocity of the shapes; kinetic theory only requires you to specify an initial distribution function of phase space, which we will show is part of the propagators. Quantum mechanics only requires you to specify an initial wave function, which is enough for some one or few particle problems. In a stretch of the term, even vacuum quantum field theory, which requires only the specification of asymptotic states, is 'effective' in the sense that it relies on a reduction of complexity and its validity and success is not a priori clear. **Any effective description obtains part of its power from the practicability of obtaining its initial conditions experimentally. To derive an effective description from the first principles of nonequilibrium quantum field theory means to somehow incorporate a loss of sensitivity on initial conditions. How this loss of sensitivity occurs is a central question in nonequilibrium quantum field theory.** In a derivation of kinetic theory we will achieve this loss

of sensitivity mainly by employing the 'late time limit'. Of course, this is not a mechanism, but rather an uncontrolled forceful way to obtain such insensitivity. It is however no more forceful than say assuming thermal equilibrium, as vacuum theory does, and much less forceful than simply taking for granted the validity of a kinetic description.

6.3 NON-EQUILIBRIUM QFT FORMALISM - TODO

6.4 CLASSICAL STATISTICAL THEORY FROM NONEQ QFT - TODO

We will briefly address two other theories which are contained in quantum nonequilibrium theory, classical statistical theory and hydrodynamics, to demonstrate the power and usefulness of nonequilibrium QFT.

It is instructive to understand how classical descriptions emerge in a nonequilibrium setting. Interestingly, this is possible in a controlled and systematic way by 'classical statistical reweighting' sometimes referred to as 'truncated Wigner approximation' in the context of scalar theories. One first integrates out the fermionic degrees of freedom which are never classical via

$$\int \mathcal{D}\mathcal{A} \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{iS[\mathcal{A}, \bar{\psi}, \psi]} =: \int \mathcal{D}\mathcal{A} e^{iS^{eff}[\mathcal{A}]} \quad (6.10)$$

to obtain an effective action of QED

$$S^{eff}[\mathcal{A}] = S[\mathcal{A}] - i \text{tr}_C \ln(\Delta_0^{-1}[\mathcal{A}]) \equiv \int_{x,C} \left[-\frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} - i \text{tr}(\ln(\Delta_0^{-1}[\mathcal{A}](x, x))) \right]. \quad (6.11)$$

Because fermions have no self interactions, the pathintegral in 6.10 is Gaussian in the fermionic fields and the effective action can be obtained exactly. Next, one introduces the separate field degrees of freedom

$$\mathcal{A}^\mu(x) = \theta_C(x^0) \mathcal{A}_+^\mu(x) + \theta_C(-x^0) \mathcal{A}_-^\mu(x) \quad (6.12)$$

and changes variables to a "classical field" \bar{A} and a "quantum field" B via

$$\mathcal{A}^\mu(x) = \bar{A}^\mu(x) + \frac{1}{2} B^\mu(x) \text{sgn}_C(x^0). \quad (6.13)$$

In this way, one can systematically expand the effective action in powers of B , by expanding the logarithm in the effective action

$$\text{tr}_C \ln\left(\Delta_C^{-1}[\bar{A} + \frac{1}{2}B \text{sgn}]\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(ie)^n}{n 2^n} \text{tr}_C\{(\Delta_0[\bar{A}]B \text{sgn})^n\}, \quad (6.14)$$

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thesis for himo_v 420
page 9 for
statistical
propagator
definition etc.

where we have dropped the zeroth order in B since it vanishes under the Keldysh trace. The quantum and classical fields get their names from the fact that at linear order in B , the *classical statistical* equations of motion emerge from

$$\frac{\delta S^{eff}}{\delta B}|_{\bar{A}=A_{cl}} = 0, \quad (6.15)$$

where the effective action is the classical statistical action

$$S^{cl}[\bar{A}, B] = \int_{t_0}^{\infty} dx^0 \int d^3x B_\nu (\partial_\mu \bar{\mathcal{F}}^{\mu\nu} - \bar{j}^\nu) \text{ with } \bar{j}^\mu = \frac{e}{2} \text{tr}\{\gamma^\mu \Delta_0[\bar{A}](x, x)\} \quad (6.16)$$

and $\bar{\mathcal{F}}^{\mu\nu} = \partial^\mu \bar{A}^\nu - \partial^\nu \bar{A}^\mu$. Similarly, the classical statistical expectation values $\langle \dots \rangle_{cl}$ of the Martin-Siggia-Rose formalism emerge at the linear order in B via

$$\langle \dots \rangle = \langle \dots \rangle_{cl} + \mathcal{O}(B). \quad (6.17)$$

The name "rewheighting" can now be understood as the change of the weight of expectation values

$$\langle \dots \rangle = \langle \dots e^{i\Delta S} \rangle_{cl} \quad \text{with } \Delta S = S^{eff} - S^{cl}. \quad (6.18)$$

With this, one can expand $e^{i\Delta S}$ in powers of B to systematically provide quantum corrections to the classical statistical theory. This is similar to the standard rewheighting of perturbation theory w.r.t. the free action S_0

$$\langle \dots \rangle =: \langle \dots e^{iS^{int}} \rangle_0 \quad \text{with } S^{int} = S - S_0, \quad (6.19)$$

which corresponds to an expansion of the exponential $e^{iS^{int}}$ in powers of the coupling. By comparing classical and exact self energies one finds that the classical statistical theory captures the exact quantum evolution whenever

$$F^2 \gg \rho^2. \quad (6.20)$$

For details on a derivation of classical statistical theory at the example of scalar ϕ^4 theory, where no complications due to fermions arise, see Berges noneq. notes cold atoms to cosmology.

Importantly, for large classical fields

$$\langle \bar{A} \rangle \propto \frac{1}{e} \quad \langle B \rangle \propto e \quad (6.21)$$

one can identify an expansion in B with a coupling expansion and physically justify the truncation at linear order by the smallness of e . This recovers, from first principles, the fact that large occupations behave classically. At higher orders in B no mapping to a classical theory exists. Nevertheless an expansion in B systematically gives quantum corrections to the classical statistical theory via $\mathcal{O}(B) = \mathcal{O}(\hbar)$.

6.5 HYDRODYNAMICS FROM NONEQ QFT

Besides kinetic theory, hydrodynamics has proven itself as a powerful effective nonequilibrium theory. At the centre of this description is the energy momentum tensor. Its exact microscopic form can be obtained in a 2PI description as

$$T^{\mu\nu}[\mathcal{A}, D, \Delta](x) = 2 \frac{\delta \Gamma[\mathcal{A}, D, \Delta]}{\delta g_{\mu\nu}}|_{g=\eta}. \quad (6.22)$$

Systems with translation invariant dynamics adhere to the conservation of energy and momentum

$$\partial_\mu T^{\mu\nu} = 0. \quad (6.23)$$

Typically, systems for which the entirety of the microscopic dynamics (i.e. the eom of the statistical propagators for the different particle species involved) can be reduced to the mere effect of the conservation of $T_{\mu\nu}$ are called hydrodynamic. Derivations of hydrodynamics commonly choose the degrees of freedom of kinetic theory or even take kinetic theory as an intermediate step. Indeed, kinetic theory can be used to prepare initial conditions as hydrodynamics takes over the description towards thermalization. Even though kinetic theory also describes thermalization, hydrodynamics carries less redundant information, as details of the kinetic description become irrelevant to the conservation of the energy momentum tensor. This can be understood in the language of moments of the statistical propagator

$$M_k^{\mu_1 \dots \mu_k} := \int_p p^{\mu_1} \dots p^{\mu_k} F(X, p). \quad (6.24)$$

The first moment is the number density and the second moment is commonly identified with an energy momentum tensor. Applying a Kadanoff-Baym ansatz with free spectral functions to these moments reduces them to specific kinetic moments and one can go from a kinetic to a hydrodynamic description with this approach. In this way, kinetic equations can be phrased as an infinite tower of equations for the moments M_1, M_2, M_3 etc. and it becomes clear that hydrodynamics truncate this tower after M_2 . A hydrodynamic description requires that only a few moments M_n contribute to the dynamics which can only be valid in the presence of a separation of scales.

To apply hydrodynamics to the energy momentum tensor of a microscopic theory, a common approach is to expand $T_{\mu\nu}$ around the form of an ideal fluid with velocity field u^μ , energy e and pressure p , i.e.

$$T^{\mu\nu} = eu^\mu u^\nu + p(e)(\eta^{\mu\nu} + u^\mu u^\nu) + \Pi^{\mu\nu}. \quad (6.25)$$

Such an expansion comes with additional assumptions, most notably the existence of an equation of state $p(e)$. The non-ideal part Π can

be approached via a 'hydrodynamic gradient expansion'. Even though conceptually unnecessary, the range of validity of hydrodynamics is then commonly identified with the range of validity of this expansion, e.g. in the case of heavy ion collisions the smallness of

$$\frac{\eta/s}{\tau T} \stackrel{!}{\ll} 1 \quad (6.26)$$

with shear viscosity η , entropy density s , proper time τ and effective temperature T . An interesting connection of hydrodynamics to AdS/CFT correspondence arises here from the conjectured lower bound of $\eta/s \geq \frac{1}{4\pi}$.

6.6 DISTRIBUTION FUNCTIONS

In thermal equilibrium, $\rho_D \propto e^{-\beta H}$, the Fourier transforms with respect to $(x - y)$ of statistical propagators and spectral functions are related by the famous Kubo-Martin-Schwinger (KMS) relations or fluctuation-dissipation theorems

$$F_{eq}^{\mu\nu}(p) = -i[\frac{1}{2} + f_B(p^0)]\rho_{eq}^{\mu\nu}(p) \quad (6.27)$$

$$F_{\Psi,eq}(p) = -i[\frac{1}{2} - f_F(p^0)]\rho_{\Psi,eq}(p) \quad (6.28)$$

with Bose-and Fermi-distributions

$$f_B(p^0) = \frac{1}{e^{\beta p^0} - 1}, \quad f_F(p^0) = \frac{1}{e^{\beta p^0} + 1}. \quad (6.29)$$

This reflects the fact that in thermal equilibrium all distributions are homogeneous and isotropic. It also allows us to highlight another key property of nonequilibrium theory: it is not possible to completely describe a nonequilibrium system in terms of just one kind of propagator per field species. Instead multiple independent two-point functions exist per field species, which are only related to each other in equilibrium or if one assumes the validity of an asymptotic in-out state formulation which factorizes the upper and lower branches of the Keldysh contour.

Another limit where distribution functions arise is the limit of zero coupling. In this case of free theory, distribution functions arise as occupation numbers

$$n(\vec{p}) \langle a^\dagger(\vec{p})a(\vec{p}) \rangle, \quad (6.30)$$

with the ladder operator a by which the free vacuum $|0\rangle$ is defined as

$$a(\vec{p})|0\rangle = 0 \quad \forall \vec{p}. \quad (6.31)$$

In fact, in the absence of conserved currents, free theories are the only theories that have well defined occupation numbers as opposed to distribution functions. Of course systems without interactions can not

thermalize and the intuition obtained from such systems has to be taken with great care when applied to interacting theories. Nevertheless, in the presence of nearly free spectral functions, free perturbation theory is a useful tool to obtain intuition for interacting systems.

6.7 GEODESIC EQUATION FROM QUANTUM FIELD THEORY

Classical trajectories are characteristic curves of collisionless transport equations. Let us explore this statement at the example of the Vlasov equation

$$\left[p_\mu \frac{\partial}{\partial X_\mu} - e p_\mu \mathcal{F}^{\mu\nu}(X) \frac{\partial}{\partial p^\nu} \right] f_\Psi(X, p) = 0. \quad (6.32)$$

If we interpret X and p as trajectories in a phase space $X(\tau)$ and $p(\tau)$, then the curves along which f_Ψ is constant in proper time τ , the characteristic curves of 6.32, are the classical trajectories

Look into:
the geodesics
are just the
character-
istics of the
field equa-
tion

Pauli principle:
 $f_\Psi \leq 1$.

$$\frac{d}{d\tau} f_\Psi(X(\tau), p(\tau)) = 0 \Leftrightarrow \frac{dp_\mu}{d\tau} = e \mathcal{F}_{\mu\nu}(X) \frac{dX^\nu}{d\tau} \text{ with } p_\mu = \frac{dX_\mu}{d\tau}. \quad (6.33)$$

This can be seen by applying the chain rule to $\frac{df_\Psi}{d\tau}$ and comparing coefficients with 6.32. Of course, this Lorentz equation 6.33 is more commonly obtained without kinetic theory as a geodesic equation by variational principle of a classical action in relativistic point-mechanics. However its interpretation in terms of kinetic theory is quite intuitive: **the characteristic curves are curves of constant particle number, such that they naturally trace the motion of a particle.** This intimate relation between trajectories and kinetic theory means that a derivation of kinetic theory from first principles automatically provides a derivation of geodesic equations and the Newtonian force concept from field theory. Even in the presence of collisions, trajectories can be obtained numerically by this method of characteristics. This connection between a field and trajectory description had been sought for, e.g. by Einstein and Rosen who wrote in the context of general relativity in 1935: "One of the imperfections of the original relativistic theory of gravitation was that as a field theory it was not complete; it introduced the independent postulate that the law of motion of a particle is given by the equation of the geodesic. A complete field theory knows only fields and not the concepts of particle and motion. For these must not exist independently of the field but are to be treated as part of it. On the basis of the description of a particle without singularity one has the possibility of a logically more satisfactory treatment of the combined problem: The problem of the field and that of motion coincide."