PHY321: Conservative Forces, Examples and start Harmonic Oscillations

Morten Hjorth-Jensen[1,2]

[1]Department of Physics and Astronomy and Facility for Rare Ion Beams (FRIB), Michigan State University, USA
[2]Department of Physics, University of Oslo, Norway

Feb 16, 2021

Aims and Overarching Motivation

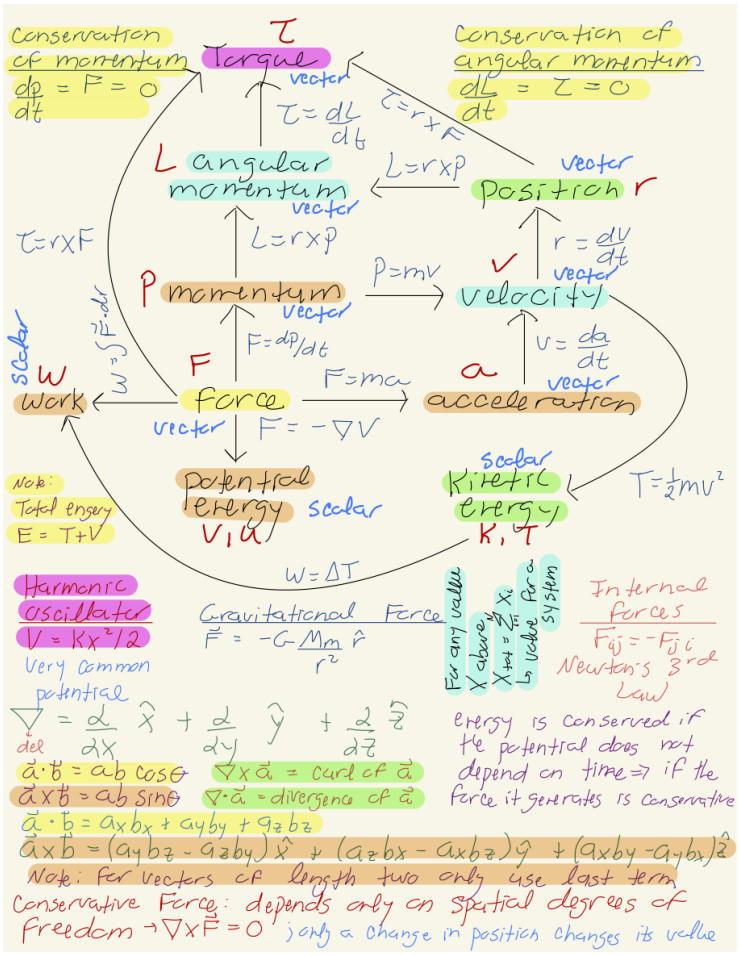
Monday. Short repetition from last week about conservative forces. Discussion of conditions for conservative forces and the Earth-Sun gravitional force example. **Reading suggestion**: Taylor sections 4.3, 4.4 and 4.8.

Wednesday. Potential curves and discussion of the Earth-Sun example, analytical and numerical considerations. **Reading suggestions**: Taylor section 4.6, 4.8 and 4.9.

Friday. Earth-Sun, conservative forces and potential energy. **Reading suggestion**: Taylor sections 4.8 and 4.9.

If we get time, we start with harmonic oscillations and Hooke's law. **Reading suggestion**: Taylor section 5.1.

One Figure to Rule All Forces (thx to Julie)



Repetition from last week: Work, Energy, Momentum and Conservation laws

Energy conservation is most convenient as a strategy for addressing problems where time does not appear. For example, a particle goes from position x_0 with speed v_0, to position x_f; what is its new speed? However, it can also be applied to problems where time does appear, such as in solving for the trajectory $x(t)$, or equivalently $t(x)$.

Energy Conservation

Energy is conserved in the case where the potential energy, $V(\boldsymbol{r})$, depends only on position, and not on time. The force is determined by V,

$$\boldsymbol{F}(\boldsymbol{r}) = -\boldsymbol{\nabla} V(\boldsymbol{r}). \tag{1}$$

Conservative forces

We say a force is conservative if it satisfies the following conditions:

1. The force $\boldsymbol{F}$ acting on an object only depends on the position $\boldsymbol{r}$, that is $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r})$.
2. For any two points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, the work done by the force $\boldsymbol{F}$ on the displacement between these two points is independent of the path taken.
3. Finally, the **curl** of the force is zero $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$.

Forces and Potentials

The energy E of a given system is defined as the sum of kinetic and potential energies,

$$E = K + V(\boldsymbol{r}).$$

We define the potential energy at a point $\boldsymbol{r}$ as the negative work done from a starting point $\boldsymbol{r}_0$ to a final point $\boldsymbol{r}$

$$V(\boldsymbol{r}) = -W(\boldsymbol{r}_0 \to \boldsymbol{r}) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} d\boldsymbol{r}' \boldsymbol{F}(\boldsymbol{r}').$$

If the potential depends on the path taken between these two points there is no unique potential.

Example (relevant for homework 5)

We study a classical electron which moves in the x-direction along a surface. The force from the surface is

$$\boldsymbol{F}(x) = -F_0 \sin\left(\frac{2\pi x}{b}\right) \boldsymbol{e}_1.$$

The constant b represents the distance between atoms at the surface of the material, F_0 is a constant and x is the position of the electron.

This is indeed a conservative force since it depends only on position and its **curl** is zero, that is $-\boldsymbol{\nabla} \times \boldsymbol{F} = 0$. This means that energy is conserved and the integral over the work done by the force is independent of the path taken.

Example Continues

Using the work-energy theorem we can find the work W done when moving an electron from a position x_0 to a final position x through the integral

$$W = \int_{x_0}^{x} \boldsymbol{F}(x')dx' = -\int_{x_0}^{x} F_0 \sin\left(\frac{2\pi x'}{b}\right)dx',$$

which results in

$$W = \frac{F_0 b}{2\pi}\left[\cos\left(\frac{2\pi x}{b}\right) - \cos\left(\frac{2\pi x_0}{b}\right)\right].$$

Since this is related to the change in kinetic energy we have, with v_0 being the initial velocity at a time t_0,

$$v = \pm\sqrt{\frac{2}{m}\frac{F_0 b}{2\pi}\left[\cos\left(\frac{2\pi x}{b}\right) - \cos\left(\frac{2\pi x_0}{b}\right)\right] + v_0^2}.$$

The potential energy from this example

The potential energy, due to energy conservation is

$$V(x) = V(x_0) + \frac{1}{2}mv_0^2 - \frac{1}{2}mv^2,$$

with v given by the velocity from above.

We can now, in order to find a more explicit expression for the potential energy at a given value x, define a zero level value for the potential. The potential is defined, using the work-energy theorem, as

$$V(x) = V(x_0) + \int_{x_0}^{x}(-F(x'))dx',$$

and if you recall the definition of the indefinite integral, we can rewrite this as

$$V(x) = \int(-F(x'))dx' + C,$$

where C is an undefined constant. The force is defined as the gradient of the potential, and in that case the undefined constant vanishes. The constant does not affect the force we derive from the potential.

We have then

$$V(x) = V(x_0) - \int_{x_0}^{x} \boldsymbol{F}(x')dx',$$

which results in

$$V(x) = -\frac{F_0 b}{2\pi}\left[\cos\left(\frac{2\pi x}{b}\right) - \cos\left(\frac{2\pi x_0}{b}\right)\right] + V(x_0).$$

We can now define

$$-\frac{F_0 b}{2\pi}\cos\left(\frac{2\pi x_0}{b}\right) = V(x_0),$$

which gives

$$V(x) = -\frac{F_0 b}{2\pi}\left[\cos\left(\frac{2\pi x}{b}\right)\right].$$

Force and Potential

We have defined work as the energy resulting from a net force acting on an object (or sseveral objects), that is

$$W(\boldsymbol{r}\rightarrow\boldsymbol{r}+d\boldsymbol{r}) = \boldsymbol{F}(\boldsymbol{r})d\boldsymbol{r}.$$

If we write out this for each component we have

$$W(\boldsymbol{r}\rightarrow\boldsymbol{r}+d\boldsymbol{r}) = \boldsymbol{F}(\boldsymbol{r})d\boldsymbol{r} = F_x dx + F_y dy + F_z dz.$$

The work done from an initial position to a final one defines also the difference in potential energies

$$W(\boldsymbol{r}\rightarrow\boldsymbol{r}+d\boldsymbol{r}) = -\left[V(\boldsymbol{r}+d\boldsymbol{r}) - V(\boldsymbol{r})\right].$$

Getting to $\boldsymbol{F}(\boldsymbol{r}) = -\boldsymbol{\nabla}V(\boldsymbol{r})$

We can write out the differences in potential energies as

$$V(\boldsymbol{r}+d\boldsymbol{r}) - V(\boldsymbol{r}) = V(x+dx, y+dy, z+dz) - V(x,y,z) = dV,$$

and using the expression the differential of a multi-variable function $f(x,y,z)$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz,$$

we can write the expression for the work done as

$$W(\boldsymbol{r}\rightarrow\boldsymbol{r}+d\boldsymbol{r}) = -dV = -\left[\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right].$$

Final expression

Comparing the last equation with

$$W(\boldsymbol{r}\rightarrow\boldsymbol{r}+d\boldsymbol{r}) = F_x dx + F_y dy + F_z dz,$$

we have

$$F_x dx + F_y dy + F_z dz = -\left[\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right],$$

leading to

$$F_x = -\frac{\partial V}{\partial x},$$

and

$$F_y = -\frac{\partial V}{\partial y},$$

and

$$F_z = -\frac{\partial V}{\partial z},$$

or just

$$\boldsymbol{F} = -\frac{\partial V}{\partial x}\boldsymbol{e}_1 - \frac{\partial V}{\partial y}\boldsymbol{e}_2 - \frac{\partial V}{\partial z}\boldsymbol{e}_3 = -\boldsymbol{\nabla} V(\boldsymbol{r}).$$

And this connection is the one we wanted to show.

Net Energy

The net energy, $E = V + K$ where K is the kinetic energy, is then conserved,

$$\begin{aligned} \frac{d}{dt}(K+V) &= \frac{d}{dt}\left(\frac{m}{2}(v_x^2+v_y^2+v_z^2) + V(\boldsymbol{r})\right) \\ &= m\left(v_x\frac{dv_x}{dt} + v_y\frac{dv_y}{dt} + v_z\frac{dv_z}{dt}\right) + \partial_x V\frac{dx}{dt} + \partial_y V\frac{dy}{dt} + \partial_z V\frac{dz}{dt} \\ &= v_x F_x + v_y F_y + v_z F_z - F_x v_x - F_y v_y - F_z v_z = 0. \end{aligned} \tag{2}$$

In Vector Notation

The same proof can be written more compactly with vector notation,

$$\begin{aligned} \frac{d}{dt}\left(\frac{m}{2}v^2 + V(\boldsymbol{r})\right) &= m\boldsymbol{v}\cdot\dot{\boldsymbol{v}} + \boldsymbol{\nabla} V(\boldsymbol{r})\cdot\dot{\boldsymbol{r}} \\ &= \boldsymbol{v}\cdot\boldsymbol{F} - \boldsymbol{F}\cdot\boldsymbol{v} = 0. \end{aligned} \tag{3}$$

Inverting the expression for kinetic energy,

$$v = \sqrt{2K/m} = \sqrt{2(E-V)/m}, \tag{4}$$

allows one to solve for the one-dimensional trajectory $x(t)$, by finding $t(x)$,

$$t = \int_{x_0}^{x} \frac{dx'}{v(x')} = \int_{x_0}^{x} \frac{dx'}{\sqrt{2(E-V(x'))/m}}. \tag{5}$$

Note this would be much more difficult in higher dimensions, because you would have to determine which points, x, y, z, the particles might reach in the trajectory, whereas in one dimension you can typically tell by simply seeing whether the kinetic energy is positive at every point between the old position and the new position.

Harmonic Oscillator Potential

Consider a simple harmonic oscillator potential, $V(x) = kx^2/2$, with a particle emitted from $x = 0$ with velocity v_0. Solve for the trajectory $t(x)$,

$$\begin{aligned} t &= \int_0^x \frac{dx'}{\sqrt{2(E - kx^2/2)/m}} \\ &= \sqrt{m/k}\int_0^x \frac{dx'}{\sqrt{x_{\max}^2 - x'^2}}, \quad x_{\max}^2 = 2E/k. \end{aligned} \tag{6}$$

Here $E = mv_0^2/2$ and $x_{\max}$ is defined as the maximum displacement before the particle turns around. This integral is done by the substitution $\sin\theta = x/x_{\max}$.

$$\begin{aligned} (k/m)^{1/2}t &= \sin^{-1}(x/x_{\max}), \\ x &= x_{\max}\sin\omega t, \quad \omega = \sqrt{k/m}. \end{aligned} \tag{7}$$

The Earth-Sun system

We will now venture into a study of a system which is energy conserving. The aim is to see if we (since it is not possible to solve the general equations analytically) we can develop stable numerical algorithms whose results we can trust!

We solve the equations of motion numerically. We will also compute quantities like the energy numerically.

We start with a simpler case first, the Earth-Sun system in two dimensions only. The gravitational force F_G on the earth from the sun is

$$\boldsymbol{F}_G = -\frac{GM_\odot M_E}{r^3}\boldsymbol{r},$$

where G is the gravitational constant,

$$M_E = 6 \times 10^{24}\text{Kg},$$

the mass of Earth,

$$M_\odot = 2 \times 10^{30}\text{Kg},$$

the mass of the Sun and

$$r = 1.5 \times 10^{11}\text{m},$$

is the distance between Earth and the Sun. The latter defines what we call an astronomical unit **AU**.

The Earth-Sun system, Newton's Laws

From Newton's second law we have then for the x direction

$$\frac{d^2x}{dt^2} = -\frac{F_x}{M_E},$$

and

$$\frac{d^2y}{dt^2} = -\frac{F_y}{M_E},$$

for the y direction.

Here we will use that $x = r\cos(\theta)$, $y = r\sin(\theta)$ and

$$r = \sqrt{x^2 + y^2}.$$

We can rewrite

$$F_x = -\frac{GM_\odot M_E}{r^2}\cos(\theta) = -\frac{GM_\odot M_E}{r^3}x,$$

and

$$F_y = -\frac{GM_\odot M_E}{r^2}\sin(\theta) = -\frac{GM_\odot M_E}{r^3}y,$$

for the y direction.

The Earth-Sun system, rewriting the Equations

We can rewrite these two equations

$$F_x = -\frac{GM_\odot M_E}{r^2}\cos(\theta) = -\frac{GM_\odot M_E}{r^3}x,$$

and

$$F_y = -\frac{GM_\odot M_E}{r^2}\sin(\theta) = -\frac{GM_\odot M_E}{r^3}y,$$

as four first-order coupled differential equations

$$\frac{dv_x}{dt} = -\frac{GM_\odot}{r^3}x,$$

$$\frac{dx}{dt} = v_x,$$

$$\frac{dv_y}{dt} = -\frac{GM_\odot}{r^3}y,$$

$$\frac{dy}{dt} = v_y.$$

Building a code for the solar system, final coupled equations

The four coupled differential equations

$$\frac{dv_x}{dt} = -\frac{GM_\odot}{r^3}x,$$

$$\frac{dx}{dt} = v_x,$$

$$\frac{dv_y}{dt} = -\frac{GM_\odot}{r^3}y,$$

$$\frac{dy}{dt} = v_y,$$

can be turned into dimensionless equations or we can introduce astronomical units with 1 AU $= 1.5 \times 10^{11}$.

Using the equations from circular motion (with $r = 1$AU)

$$\frac{M_E v^2}{r} = F = \frac{GM_\odot M_E}{r^2},$$

we have

$$GM_\odot = v^2 r,$$

and using that the velocity of Earth (assuming circular motion) is $v = 2\pi r/\text{yr} = 2\pi\text{AU/yr}$, we have

$$GM_\odot = v^2 r = 4\pi^2 \frac{(\text{AU})^3}{\text{yr}^2}.$$

Building a code for the solar system, discretized equations

The four coupled differential equations can then be discretized using Euler's method as (with step length h)

$$v_{x,i+1} = v_{x,i} - h\frac{4\pi^2}{r_i^3}x_i,$$

$$x_{i+1} = x_i + hv_{x,i},$$

$$v_{y,i+1} = v_{y,i} - h\frac{4\pi^2}{r_i^3}y_i,$$

$$y_{i+1} = y_i + hv_{y,i},$$

Code Example with Euler's Method

The code here implements Euler's method for the Earth-Sun system using a more compact way of representing the vectors. Alternatively, you could have spelled out all the variables v_x, v_y, x and y as one-dimensional arrays.

```
# Common imports
import numpy as np
import pandas as pd
from math import *
import matplotlib.pyplot as plt
import os

# Where to save the figures and data files
PROJECT_ROOT_DIR = "Results"
FIGURE_ID = "Results/FigureFiles"
DATA_ID = "DataFiles/"
```

```
if not os.path.exists(PROJECT_ROOT_DIR):
    os.mkdir(PROJECT_ROOT_DIR)

if not os.path.exists(FIGURE_ID):
    os.makedirs(FIGURE_ID)

if not os.path.exists(DATA_ID):
    os.makedirs(DATA_ID)

def image_path(fig_id):
    return os.path.join(FIGURE_ID, fig_id)

def data_path(dat_id):
    return os.path.join(DATA_ID, dat_id)

def save_fig(fig_id):
    plt.savefig(image_path(fig_id) + ".png", format='png')


DeltaT = 0.001
#set up arrays
tfinal = 10 # in years
n = ceil(tfinal/DeltaT)
# set up arrays for t, a, v, and x
t = np.zeros(n)
v = np.zeros((n,2))
r = np.zeros((n,2))
# Initial conditions as compact 2-dimensional arrays
r0 = np.array([1.0,0.0])
v0 = np.array([0.0,2*pi])
r[0] = r0
v[0] = v0
Fourpi2 = 4*pi*pi
# Start integrating using Euler's method
for i in range(n-1):
    # Set up the acceleration
    # Here you could have defined your own function for this
    rabs = sqrt(sum(r[i]*r[i]))
    a =  -Fourpi2*r[i]/(rabs**3)
    # update velocity, time and position using Euler's forward method
    v[i+1] = v[i] + DeltaT*a
    r[i+1] = r[i] + DeltaT*v[i]
    t[i+1] = t[i] + DeltaT
# Plot position as function of time
fig, ax = plt.subplots()
#ax.set_xlim(0, tfinal)
ax.set_ylabel('x[m]')
ax.set_xlabel('y[m]')
ax.plot(r[:,0], r[:,1])
fig.tight_layout()
save_fig("EarthSunEuler")
plt.show()
```

Problems with Euler's Method

We notice here that Euler's method doesn't give a stable orbit. It means that we cannot trust Euler's method. In a deeper way, as we will see in homework

5, Euler's method does not conserve energy. It is an example of an integrator which is not symplectic.

Here we present thus two methods, which with simple changes allow us to avoid these pitfalls. The simplest possible extension is the so-called Euler-Cromer method. The changes we need to make to our code are indeed marginal here. We need simply to replace

```
r[i+1] = r[i] + DeltaT*v[i]
```

in the above code with the velocity at the new time t_{i+1}

```
r[i+1] = r[i] + DeltaT*v[i+1]
```

By this simple caveat we get stable orbits. Below we derive the Euler-Cromer method as well as one of the most utlized algorithms for sovling the above type of problems, the so-called Velocity-Verlet method.

Deriving the Euler-Cromer Method

Let us repeat Euler's method. We have a differential equation

$$y'(t_i) = f(t_i, y_i) \tag{8}$$

and if we truncate at the first derivative, we have from the Taylor expansion

$$y_{i+1} = y(t_i) + (\Delta t) f(t_i, y_i) + O(\Delta t^2), \tag{9}$$

which when complemented with $t_{i+1} = t_i + \Delta t$ forms the algorithm for the well-known Euler method. Note that at every step we make an approximation error of the order of $O(\Delta t^2)$, however the total error is the sum over all steps $N = (b-a)/(\Delta t)$ for $t \in [a, b]$, yielding thus a global error which goes like $NO(\Delta t^2) \approx O(\Delta t)$.

To make Euler's method more precise we can obviously decrease Δt (increase N), but this can lead to loss of numerical precision. Euler's method is not recommended for precision calculation, although it is handy to use in order to get a first view on how a solution may look like.

Euler's method is asymmetric in time, since it uses information about the derivative at the beginning of the time interval. This means that we evaluate the position at y_1 using the velocity at v_0. A simple variation is to determine x_{n+1} using the velocity at v_{n+1}, that is (in a slightly more generalized form)

$$y_{n+1} = y_n + v_{n+1} + O(\Delta t^2) \tag{10}$$

and

$$v_{n+1} = v_n + (\Delta t) a_n + O(\Delta t^2). \tag{11}$$

The acceleration a_n is a function of $a_n(y_n, v_n, t_n)$ and needs to be evaluated as well. This is the Euler-Cromer method.

Exercise: go back to the above code with Euler's method and add the Euler-Cromer method.

Deriving the Velocity-Verlet Method

Let us stay with x (position) and v (velocity) as the quantities we are interested in.

We have the Taylor expansion for the position given by

$$x_{i+1} = x_i + (\Delta t)v_i + \frac{(\Delta t)^2}{2}a_i + O((\Delta t)^3).$$

The corresponding expansion for the velocity is

$$v_{i+1} = v_i + (\Delta t)a_i + \frac{(\Delta t)^2}{2}v_i^{(2)} + O((\Delta t)^3).$$

Via Newton's second law we have normally an analytical expression for the derivative of the velocity, namely

$$a_i = \frac{d^2x}{dt^2}|_i = \frac{dv}{dt}|_i = \frac{F(x_i, v_i, t_i)}{m}.$$

If we add to this the corresponding expansion for the derivative of the velocity

$$v_{i+1}^{(1)} = a_{i+1} = a_i + (\Delta t)v_i^{(2)} + O((\Delta t)^2) = a_i + (\Delta t)v_i^{(2)} + O((\Delta t)^2),$$

and retain only terms up to the second derivative of the velocity since our error goes as $O(h^3)$, we have

$$(\Delta t)v_i^{(2)} \approx a_{i+1} - a_i.$$

We can then rewrite the Taylor expansion for the velocity as

$$v_{i+1} = v_i + \frac{(\Delta t)}{2}\left(a_{i+1} + a_i\right) + O((\Delta t)^3).$$

The velocity Verlet method

Our final equations for the position and the velocity become then

$$x_{i+1} = x_i + (\Delta t)v_i + \frac{(\Delta t)^2}{2}a_i + O((\Delta t)^3),$$

and

$$v_{i+1} = v_i + \frac{(\Delta t)}{2}\left(a_{i+1} + a_i\right) + O((\Delta t)^3).$$

Note well that the term a_{i+1} depends on the position at x_{i+1}. This means that you need to calculate the position at the updated time t_{i+1} before the computing the next velocity. Note also that the derivative of the velocity at the time t_i used in the updating of the position can be reused in the calculation of the velocity update as well.

Adding the Velocity-Verlet Method

We can now easily add the Verlet method to our original code as

```
DeltaT = 0.01
#set up arrays
tfinal = 10
n = ceil(tfinal/DeltaT)
# set up arrays for t, a, v, and x
t = np.zeros(n)
v = np.zeros((n,2))
r = np.zeros((n,2))
# Initial conditions as compact 2-dimensional arrays
r0 = np.array([1.0,0.0])
v0 = np.array([0.0,2*pi])
r[0] = r0
v[0] = v0
Fourpi2 = 4*pi*pi
# Start integrating using the Velocity-Verlet  method
for i in range(n-1):
    # Set up forces, air resistance FD, note now that we need the norm of the vecto
    # Here you could have defined your own function for this
    rabs = sqrt(sum(r[i]*r[i]))
    a =  -Fourpi2*r[i]/(rabs**3)
    # update velocity, time and position using the Velocity-Verlet method
    r[i+1] = r[i] + DeltaT*v[i]+0.5*(DeltaT**2)*a
    rabs = sqrt(sum(r[i+1]*r[i+1]))
    anew = -4*(pi**2)*r[i+1]/(rabs**3)
    v[i+1] = v[i] + 0.5*DeltaT*(a+anew)
    t[i+1] = t[i] + DeltaT
# Plot position as function of time
fig, ax = plt.subplots()
ax.set_ylabel('x[m]')
ax.set_xlabel('y[m]')
ax.plot(r[:,0], r[:,1])
fig.tight_layout()
save_fig("EarthSunVV")
plt.show()
```

You can easily generalize the calculation of the forces by defining a function which takes in as input the various variables. We leave this as a challenge to you.

Harmonic Oscillator

The harmonic oscillator is omnipresent in physics. Although you may think of this as being related to springs, it, or an equivalent mathematical representation, appears in just about any problem where a mode is sitting near its potential energy minimum. At that point, $\partial_x V(x) = 0$, and the first non-zero term (aside from a constant) in the potential energy is that of a harmonic oscillator. In a solid, sound modes (phonons) are built on a picture of coupled harmonic oscillators, and in relativistic field theory the fundamental interactions are also built on coupled oscillators positioned infinitesimally close to one another in space. The phenomena of a resonance of an oscillator driven at a fixed frequency plays out repeatedly in atomic, nuclear and high-energy physics, when quantum mechanically the evolution of a state oscillates according to e^{-iEt} and exciting

discrete quantum states has very similar mathematics as exciting discrete states of an oscillator.

Harmonic Oscillator, deriving the Equations

The potential energy for a single particle as a function of its position x can be written as a Taylor expansion about some point x_0

$$V(x) = V(x_0) + (x - x_0)\ \partial_x V(x)\big|_{x_0} + \frac{1}{2}(x - x_0)^2\ \partial_x^2 V(x)\big|_{x_0} + \frac{1}{3!}\ \partial_x^3 V(x)\big|_{x_0} + \cdots \tag{12}$$

If the position x_0 is at the minimum of the resonance, the first two non-zero terms of the potential are

$$\begin{aligned} V(x) &\approx V(x_0) + \frac{1}{2}(x - x_0)^2\ \partial_x^2 V(x)\big|_{x_0}\,, \qquad (13)\\ &= V(x_0) + \frac{1}{2}k(x - x_0)^2, \quad k \equiv \partial_x^2 V(x)\big|_{x_0}\,,\\ F &= -\partial_x V(x) = -k(x - x_0). \end{aligned}$$

Put into Newton's 2nd law (assuming $x_0 = 0$),

$$\begin{aligned} m\ddot{x} &= -kx, \qquad (14)\\ x &= A\cos(\omega_0 t - \phi), \quad \omega_0 = \sqrt{k/m}. \qquad (15) \end{aligned}$$

Harmonic Oscillator, Technicalities

Here A and ϕ are arbitrary. Equivalently, one could have written this as $A\cos(\omega_0 t) + B\sin(\omega_0 t)$, or as the real part of $Ae^{i\omega_0 t}$. In this last case A could be an arbitrary complex constant. Thus, there are 2 arbitrary constants (either A and B or A and ϕ, or the real and imaginary part of one complex constant. This is the expectation for a second order differential equation, and also agrees with the physical expectation that if you know a particle's initial velocity and position you should be able to define its future motion, and that those two arbitrary conditions should translate to two arbitrary constants.

A key feature of harmonic motion is that the system repeats itself after a time $T = 1/f$, where f is the frequency, and $\omega = 2\pi f$ is the angular frequency. The period of the motion is independent of the amplitude. However, this independence is only exact when one can neglect higher terms of the potential, $x^3, x^4 \cdots$. Once can neglect these terms for sufficiently small amplitudes, and for larger amplitudes the motion is no longer purely sinusoidal, and even though the motion repeats itself, the time for repeating the motion is no longer independent of the amplitude.

One can also calculate the velocity and the kinetic energy as a function of time,

$$\begin{aligned} \dot{x} &= -\omega_0 A \sin(\omega_0 t - \phi), \\ K &= \frac{1}{2} m \dot{x}^2 = \frac{m\omega_0^2 A^2}{2} \sin^2(\omega_0 t - \phi), \\ &= \frac{k}{2} A^2 \sin^2(\omega_0 t - \phi). \end{aligned} \tag{16}$$

Harmonic Oscillator, Total Energy

The total energy is then

$$E = K + V = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = \frac{1}{2} k A^2. \tag{17}$$

The total energy then goes as the square of the amplitude.

A pendulum is an example of a harmonic oscillator. By expanding the kinetic and potential energies for small angles find the frequency for a pendulum of length L with all the mass m centered at the end by writing the eq.s of motion in the form of a harmonic oscillator.

The potential energy and kinetic energies are (for x being the displacement)

$$\begin{aligned} V &= mgL(1 - \cos\theta) \approx mgL \frac{x^2}{2L^2}, \\ K &= \frac{1}{2} m L^2 \dot{\theta}^2 \approx \frac{m}{2} \dot{x}^2. \end{aligned}$$

For small x Newton's 2nd law becomes

$$m\ddot{x} = -\frac{mg}{L} x,$$

and the spring constant would appear to be $k = mg/L$, which makes the frequency equal to $\omega_0 = \sqrt{g/L}$. Note that the frequency is independent of the mass.

Additional Material: Link between Line Integrals and Conservative forces