PHY321: Conservative Forces, Momentum and Angular Momentum conservation

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Aims and Overarching Motivation

Monday. Short repetition from last week. Discussion of momentum and angular momentum. Reading suggestion: Taylor sections 3.1-3.4

Wednesday. More on angular momentum, Taylor sections 3.4 and 3.5. Discussion of potential energy and conservative forces. Reading suggestions: Taylor section 4.2

Friday. Conservative forces and potential energy. Reading suggestion: Taylor sections 4.2-4.4

If you wish to read more about conservative forces or not, Feyman's lectures from 1963 are quite interesting. He states for example that **All fundamental forces in nature appear to be conservative**. This statement was made while developing his argument that *there are no nonconservative forces.* You may enjoy the link to Feynman's lecture.

Work-Energy Theorem and Energy Conservation

Last week (Friday) we observed that energy was conserved for a force which depends only on the position. In particular we considered a force acting on a block attached to a spring with spring constant k. The other end of the spring was attached to the wall at the origin $x = 0$.

The force F_x from the spring on the block was defined as

$$F_x = -kx.$$

The work done on the block due to a displacement from a position x_0 to x

$$W = \int_{x_0}^{x} F_x dx' = \frac{1}{2}kx_0^2 - \frac{1}{2}kx^2.$$

Conservation of energy

With the definition of the work-energy theorem in terms of the kinetic energy we obtained

$$W = \frac{1}{2}mv^2(x) - \frac{1}{2}mv_0^2 = \frac{1}{2}k(x_0 - L_0)^2 - \frac{1}{2}k(x - L_0)^2,$$

which we rewrote as

$$\frac{1}{2}mv^2(x) + \frac{1}{2}kx^2 = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2.$$

The total energy, which is the sum of potential and kinetic energy, is conserved. We will analyze this interesting result now in more detail when we study energy, momentum and angular momentum conservation.

But before we start with energy conservation, conservative forces and potential energies, we need to revisit our definitions of momentum and angular momentum.

What is a Conservative Force?

A conservative force is a force whose property is that the total work done in moving an object between two points is independent of the taken path. This means that the work on an object under the influence of a conservative force, is independent on the path of the object. It depends only on the spatial degrees of freedom and it is possible to assign a numerical value for the potential at any point. It leads to conservation of energy. The gravitational force is an example of a conservative force.

Two important conditions

First, a conservative force depends only on the spatial degrees of freedom. This is a necessary condition for obtaining a path integral which is independent of path. The important condition for the final work to be independent of the path is that the **curl** of the force is zero, that is

$$\boldsymbol{\nabla} \times \boldsymbol{F} = 0$$

Work-energy theorem to show that energy is conserved with a conservative force

The work-energy theorem states that the work done W by a force $\boldsymbol{F}$ that moves an object from a position $\boldsymbol{r}_0$ to a new position $\boldsymbol{r}_1$

$$W = \int_{\boldsymbol{r}_0}^{\boldsymbol{r}_1} \boldsymbol{F} d\boldsymbol{r} = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2,$$

where v_1^2 is the velocity squared at a time t_1 and v_0^2 the corresponding quantity at a time t_0. The work done is thus the difference in kinetic energies. We can rewrite the above equation as

$$\frac{1}{2}mv_1^2 = \int_{\boldsymbol{r}_0}^{\boldsymbol{r}_1} \boldsymbol{F}\,d\boldsymbol{r} + \frac{1}{2}mv_0^2,$$

that is the final kinetic energy is equal to the initial kinetic energy plus the work done by the force over a given path from a position $\boldsymbol{r}_0$ at time t_0 to a final position position $\boldsymbol{r}_1$ at a later time t_1.

Conservation of Momentum

Before we move on however, we need to remind ourselves about important aspects like the linear momentum and angular momentum. After these considerations, we move back to more details about conservatives forces.

Newton's third law which we met earlier states that **For every action there is an equal and opposite reaction**, is more accurately stated as

If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction.

This means that for two bodies i and j, if the force on i due to j is called $\boldsymbol{F}_{ij}$, then

$$\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}. \tag{1}$$

Newton's Second Law

Newton's second law, $\boldsymbol{F} = m\boldsymbol{a}$, can be written for a particle i as

$$\boldsymbol{F}_i = \sum_{j \neq i} \boldsymbol{F}_{ij} = m_i \boldsymbol{a}_i, \tag{2}$$

where $\boldsymbol{F}_i$ (a single subscript) denotes the net force acting on i. Because the mass of i is fixed, one can see that

$$\boldsymbol{F}_i = \frac{d}{dt}m_i\boldsymbol{v}_i = \sum_{j \neq i} \boldsymbol{F}_{ij}. \tag{3}$$

Summing over all Particles

Now, one can sum over all the particles and obtain

$$\frac{d}{dt}\sum_i m_i v_i = \sum_{ij, i \neq j} \boldsymbol{F}_{ij} \tag{4}$$
$$= 0.$$

The last step made use of the fact that for every term ij, there is an equivalent term ji with opposite force. Because the momentum is defined as $m\boldsymbol{v}$, for a system of particles,

$$\frac{d}{dt}\sum_i m_i \boldsymbol{v}_i = 0, \quad \text{for isolated particles.} \qquad (5)$$

Conservation of total Momentum

By "isolated" one means that the only force acting on any particle i are those originating from other particles in the sum, i.e. "no external" forces. Thus, Newton's third law leads to the conservation of total momentum,

$$\boldsymbol{P} = \sum_i m_i \boldsymbol{v}_i, \qquad (6)$$

$$\frac{d}{dt}\boldsymbol{P} = 0.$$

Example: Rocket Science

Consider a rocket of mass M moving with velocity v. After a brief instant, the velocity of the rocket is $v + \Delta v$ and the mass is $M - \Delta M$. Momentum conservation gives

$$\begin{aligned}
Mv &= (M - \Delta M)(v + \Delta v) + \Delta M(v - v_e) \\
0 &= -\Delta M v + M\Delta v + \Delta M(v - v_e), \\
0 &= M\Delta v - \Delta M v_e.
\end{aligned}$$

In the second step we ignored the term $\Delta M \Delta v$ because it is doubly small. The last equation gives

$$\begin{aligned}
\Delta v &= \frac{v_e}{M}\Delta M, \qquad (7) \\
\frac{dv}{dt} &= \frac{v_e}{M}\frac{dM}{dt}.
\end{aligned}$$

Integrating the Equations

Integrating the expression with lower limits $v_0 = 0$ and M_0, one finds

$$\begin{aligned}
v &= v_e \int_{M_0}^{M} \frac{dM'}{M'} \\
v &= -v_e \ln(M/M_0) \\
&= -v_e \ln[(M_0 - \alpha t)/M_0].
\end{aligned}$$

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Because the total momentum of an isolated system is constant, one can also quickly see that the center of mass of an isolated system is also constant. The center of mass is the average position of a set of masses weighted by the mass,

$$\bar{x} = \frac{\sum_i m_i x_i}{\sum_i m_i}. \tag{8}$$

Rate of Change

The rate of change of $\bar{x}$ is

$$\dot{\bar{x}} = \frac{1}{M} \sum_i m_i \dot{x}_i = \frac{1}{M} P_x. \tag{9}$$

Thus if the total momentum is constant the center of mass moves at a constant velocity, and if the total momentum is zero the center of mass is fixed.

Conservation of Angular Momentum

Consider a case where the force always points radially,

$$\boldsymbol{F}(\boldsymbol{r}) = F(r)\hat{r}, \tag{10}$$

where $\hat{r}$ is a unit vector pointing outward from the origin. The angular momentum is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m\boldsymbol{r} \times \boldsymbol{v}. \tag{11}$$

Rate of Change of Angular Momentum

The rate of change of the angular momentum is

$$\begin{aligned}
\frac{d\boldsymbol{L}}{dt} &= m\boldsymbol{v} \times \boldsymbol{v} + m\boldsymbol{r} \times \dot{\boldsymbol{v}} \\
&= m\boldsymbol{v} \times \boldsymbol{v} + \boldsymbol{r} \times \boldsymbol{F} = 0.
\end{aligned} \tag{12}$$

Rewriting the Equations

The first term is zero because $\boldsymbol{v}$ is parallel to itself, and the second term is zero because $\boldsymbol{F}$ is parallel to $\boldsymbol{r}$.

As an aside, one can see from the Levi-Civita symbol that the cross product of a vector with itself is zero. Here, we consider a vector

$$\begin{aligned}
\boldsymbol{V} &= \boldsymbol{A} \times \boldsymbol{A}, \\
V_i &= (\boldsymbol{A} \times \boldsymbol{A})_i = \sum_{jk} \epsilon_{ijk} A_j A_k.
\end{aligned} \tag{13}$$

Two main Contributions

For any term i, there are two contributions. For example, for i denoting the x direction, either j denotes the y direction and k denotes the z direction, or vice versa, so

$$V_1 = \epsilon_{123} A_2 A_3 + \epsilon_{132} A_3 A_2. \tag{14}$$

This is zero by the antisymmetry of ϵ under permutations.

If the force is not radial, $\boldsymbol{r} \times \boldsymbol{F} \neq 0$ as above, and angular momentum is no longer conserved,

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times \boldsymbol{F} \equiv \boldsymbol{\tau}, \tag{15}$$

where $\boldsymbol{\tau}$ is the torque.

System of Isolated Particles

For a system of isolated particles, one can write

$$
\begin{aligned}
\frac{d}{dt} \sum_i \boldsymbol{L}_i &= \sum_{i \neq j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} \\
&= \frac{1}{2} \sum_{i \neq j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} + \boldsymbol{r}_j \times \boldsymbol{F}_{ji} \\
&= \frac{1}{2} \sum_{i \neq j} (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ij} = 0,
\end{aligned}
\tag{16}
$$

where the last step used Newton's third law, $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$. If the forces between the particles are radial, i.e. $\boldsymbol{F}_{ij} \parallel (\boldsymbol{r}_i - \boldsymbol{r}_j)$, then each term in the sum is zero and the net angular momentum is fixed. Otherwise, you could imagine an isolated system that would start spinning spontaneously.

The Torque

One can write the torque about a given axis, which we will denote as $\hat{z}$, in polar coordinates, where

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \cos\phi, \quad z = r \cos\theta, \tag{17}$$

to find the z component of the torque,

$$
\begin{aligned}
\tau_z &= x F_y - y F_x \\
&= -r \sin\theta \left\{ \cos\phi \partial_y - \sin\phi \partial_x \right\} V(x, y, z).
\end{aligned}
\tag{18}
$$

Chain Rule and Partial Derivatives

One can use the chain rule to write the partial derivative w.r.t. ϕ (keeping r and θ fixed),

$$
\begin{aligned}
\partial_\phi &= \frac{\partial x}{\partial \phi}\partial_x + \frac{\partial y}{\partial \phi}\partial_y + \frac{\partial z}{\partial \phi}\partial_z & (19) \\
&= -r\sin\theta\sin\phi\,\partial_x + \sin\theta\cos\phi\,\partial_y.
\end{aligned}
$$

Combining the two equations,

$$
\tau_z = -\partial_\phi V(r,\theta,\phi). \qquad (20)
$$

Thus, if the potential is independent of the azimuthal angle ϕ, there is no torque about the z axis and L_z is conserved.

Work, Energy, Momentum and Conservation laws

Energy conservation is most convenient as a strategy for addressing problems where time does not appear. For example, a particle goes from position x_0 with speed v_0, to position x_f; what is its new speed? However, it can also be applied to problems where time does appear, such as in solving for the trajectory $x(t)$, or equivalently $t(x)$.

Energy Conservation

Energy is conserved in the case where the potential energy, $V(\boldsymbol{r})$, depends only on position, and not on time. The force is determined by V,

$$
\boldsymbol{F}(\boldsymbol{r}) = -\nabla V(\boldsymbol{r}). \qquad (21)
$$

Net Energy

The net energy, $E = V + K$ where K is the kinetic energy, is then conserved,

$$
\begin{aligned}
\frac{d}{dt}(K+V) &= \frac{d}{dt}\left(\frac{m}{2}(v_x^2 + v_y^2 + v_z^2) + V(\boldsymbol{r})\right) & (22) \\
&= m\left(v_x\frac{dv_x}{dt} + v_y\frac{dv_y}{dt} + v_z\frac{dv_z}{dt}\right) + \partial_x V\frac{dx}{dt} + \partial_y V\frac{dy}{dt} + \partial_z V\frac{dz}{dt} \\
&= v_x F_x + v_y F_y + v_z F_z - F_x v_x - F_y v_y - F_z v_z = 0.
\end{aligned}
$$

In Vector Notation

The same proof can be written more compactly with vector notation,

$$\frac{d}{dt}\left(\frac{m}{2}v^2 + V(\boldsymbol{r})\right) = m\boldsymbol{v}\cdot\dot{\boldsymbol{v}} + \nabla V(\boldsymbol{r})\cdot\dot{\boldsymbol{r}} \tag{23}$$
$$= \boldsymbol{v}\cdot\boldsymbol{F} - \boldsymbol{F}\cdot\boldsymbol{v} = 0.$$

Inverting the expression for kinetic energy,

$$v = \sqrt{2K/m} = \sqrt{2(E-V)/m}, \tag{24}$$

allows one to solve for the one-dimensional trajectory $x(t)$, by finding $t(x)$,

$$t = \int_{x_0}^{x}\frac{dx'}{v(x')} = \int_{x_0}^{x}\frac{dx'}{\sqrt{2(E-V(x'))/m}}. \tag{25}$$

Note this would be much more difficult in higher dimensions, because you would have to determine which points, x, y, z, the particles might reach in the trajectory, whereas in one dimension you can typically tell by simply seeing whether the kinetic energy is positive at every point between the old position and the new position.

Harmonic Oscillator Potential

Consider a simple harmonic oscillator potential, $V(x) = kx^2/2$, with a particle emitted from $x = 0$ with velocity v_0. Solve for the trajectory $t(x)$,

$$t = \int_{0}^{x}\frac{dx'}{\sqrt{2(E-kx^2/2)/m}} \tag{26}$$
$$= \sqrt{m/k}\int_{0}^{x}\frac{dx'}{\sqrt{x_{\max}^2 - x'^2}}, \quad x_{\max}^2 = 2E/k.$$

Here $E = mv_0^2/2$ and $x_{\max}$ is defined as the maximum displacement before the particle turns around. This integral is done by the substitution $\sin\theta = x/x_{\max}$.

$$(k/m)^{1/2}t = \sin^{-1}(x/x_{\max}), \tag{27}$$
$$x = x_{\max}\sin\omega t, \quad \omega = \sqrt{k/m}.$$

Symmetries and Conservation Laws

When we derived the conservation of energy, we assumed that the potential depended only on position, not on time. If it depended explicitly on time, one can quickly see that the energy would have changed at a rate $\partial_t V(x, y, z, t)$. Note that if there is no explicit dependence on time, i.e. $V(x, y, z)$, the potential energy can depend on time through the variations of x, y, z with time. However, that variation does not lead to energy non-conservation. Further, we just saw that if a potential does not depend on the azimuthal angle about some axis, ϕ, that the angular momentum about that axis is conserved.

Translational Invariance

Now, we relate momentum conservation to translational invariance. Considering a system of particles with positions, $\boldsymbol{r}_i$, if one changed the coordinate system by a translation by a differential distance $\boldsymbol{\epsilon}$, the net potential would change by

$$
\begin{aligned}
\delta V(\boldsymbol{r}_1, \boldsymbol{r}_2 \cdots) &= \sum_i \boldsymbol{\epsilon} \cdot \nabla_i V(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots) \qquad (28) \\
&= -\sum_i \boldsymbol{\epsilon} \cdot \boldsymbol{F}_i \\
&= -\frac{d}{dt} \sum_i \boldsymbol{\epsilon} \cdot \boldsymbol{p}_i.
\end{aligned}
$$

Translational Invariance and Conservation of Momentum

Thus, if the potential is unchanged by a translation of the coordinate system, the total momentum is conserved. If the potential is translationally invariant in a given direction, defined by a unit vector, $\hat{\epsilon}$ in the $\boldsymbol{\epsilon}$ direction, one can see that

$$
\hat{\epsilon} \cdot \nabla_i V(\boldsymbol{r}_i) = 0. \qquad (29)
$$

The component of the total momentum along that axis is conserved. This is rather obvious for a single particle. If $V(\boldsymbol{r})$ does not depend on some coordinate x, then the force in the x direction is $F_x = -\partial_x V = 0$, and momentum along the x direction is constant.

We showed how the total momentum of an isolated system of particle was conserved, even if the particles feel internal forces in all directions. In that case the potential energy could be written

$$
V = \sum_{i,j \leq i} V_{ij}(\boldsymbol{r}_i - \boldsymbol{r}_j). \qquad (30)
$$

In this case, a translation leads to $\boldsymbol{r}_i \rightarrow \boldsymbol{r}_i + \boldsymbol{\epsilon}$, with the translation equally affecting the coordinates of each particle. Because the potential depends only on the relative coordinates, δV is manifestly zero. If one were to go through the exercise of calculating δV for small $\boldsymbol{\epsilon}$, one would find that the term $\nabla_i V(\boldsymbol{r}_i - \boldsymbol{r}_j)$ would be canceled by the term $\nabla_j V(\boldsymbol{r}_i - \boldsymbol{r}_j)$.

The relation between symmetries of the potential and conserved quantities (also called constants of motion) is one of the most profound concepts one should gain from this course. It plays a critical role in all fields of physics. This is especially true in quantum mechanics, where a quantity A is conserved if its operator commutes with the Hamiltonian. For example if the momentum operator $-i\hbar\partial_x$ commutes with the Hamiltonian, momentum is conserved, and clearly this operator commutes if the Hamiltonian (which represents the total

energy, not just the potential) does not depend on x. Also in quantum mechanics the angular momentum operator is $L_z = -i\hbar\partial_\phi$. In fact, if the potential is unchanged by rotations about some axis, angular momentum about that axis is conserved. We return to this concept, from a more formal perspective, later in the course when Lagrangian mechanics is presented.

Bulding a code for the Earth-Sun system

We will now venture into a study of a system which is energy conserving. The aim is to see if we (since it is not possible to solve the general equations analytically) we can develop stable numerical algorithms whose results we can trust!

We solve the equations of motion numerically. We will also compute quantities like the energy numerically.

We start with a simpler case first, the Earth-Sun system in two dimensions only. The gravitational force F_G on the earth from the sun is

$$\boldsymbol{F}_G = -\frac{GM_\odot M_E}{r^3}\boldsymbol{r},$$

where G is the gravitational constant,

$$M_E = 6 \times 10^{24}\text{Kg},$$

the mass of Earth,

$$M_\odot = 2 \times 10^{30}\text{Kg},$$

the mass of the Sun and

$$r = 1.5 \times 10^{11}\text{m},$$

is the distance between Earth and the Sun. The latter defines what we call an astronomical unit **AU**. From Newton's second law we have then for the x direction

$$\frac{d^2x}{dt^2} = -\frac{F_x}{M_E},$$

and

$$\frac{d^2y}{dt^2} = -\frac{F_y}{M_E},$$

for the y direction.

Here we will use that $x = r\cos(\theta)$, $y = r\sin(\theta)$ and

$$r = \sqrt{x^2 + y^2}.$$

We can rewrite

$$F_x = -\frac{GM_\odot M_E}{r^2}\cos(\theta) = -\frac{GM_\odot M_E}{r^3}x,$$

and

$$F_y = -\frac{GM_\odot M_E}{r^2}\sin(\theta) = -\frac{GM_\odot M_E}{r^3}y,$$

for the y direction.

We can rewrite these two equations

$$F_x = -\frac{GM_\odot M_E}{r^2}\cos{(\theta)} = -\frac{GM_\odot M_E}{r^3}x,$$

and

$$F_y = -\frac{GM_\odot M_E}{r^2}\sin{(\theta)} = -\frac{GM_\odot M_E}{r^3}y,$$

as four first-order coupled differential equations

$$\frac{dv_x}{dt} = -\frac{GM_\odot}{r^3}x,$$

$$\frac{dx}{dt} = v_x,$$

$$\frac{dv_y}{dt} = -\frac{GM_\odot}{r^3}y,$$

$$\frac{dy}{dt} = v_y.$$

Building a code for the solar system, final coupled equations

The four coupled differential equations

$$\frac{dv_x}{dt} = -\frac{GM_\odot}{r^3}x,$$

$$\frac{dx}{dt} = v_x,$$

$$\frac{dv_y}{dt} = -\frac{GM_\odot}{r^3}y,$$

$$\frac{dy}{dt} = v_y,$$

can be turned into dimensionless equations or we can introduce astronomical units with 1 AU $= 1.5 \times 10^{11}$.

Using the equations from circular motion (with $r = 1$AU)

$$\frac{M_E v^2}{r} = F = \frac{GM_\odot M_E}{r^2},$$

we have

$$GM_\odot = v^2 r,$$

and using that the velocity of Earth (assuming circular motion) is $v = 2\pi r/\mathrm{yr} = 2\pi \mathrm{AU}/\mathrm{yr}$, we have

$$GM_\odot = v^2 r = 4\pi^2 \frac{(\mathrm{AU})^3}{\mathrm{yr}^2}.$$

Building a code for the solar system, discretized equations

The four coupled differential equations can then be discretized using Euler's method as (with step length h)

$$v_{x,i+1} = v_{x,i} - h\frac{4\pi^2}{r_i^3}x_i,$$

$$x_{i+1} = x_i + hv_{x,i},$$

$$v_{y,i+1} = v_{y,i} - h\frac{4\pi^2}{r_i^3}y_i,$$

$$y_{i+1} = y_i + hv_{y,i},$$

Code Example with Euler's Method

The code here implements Euler's method for the Earth-Sun system using a more compact way of representing the vectors. Alternatively, you could have spelled out all the variables v_x, v_y, x and y as one-dimensional arrays.

```python
# Common imports
import numpy as np
import pandas as pd
from math import *
import matplotlib.pyplot as plt
import os

# Where to save the figures and data files
PROJECT_ROOT_DIR = "Results"
FIGURE_ID = "Results/FigureFiles"
DATA_ID = "DataFiles/"

if not os.path.exists(PROJECT_ROOT_DIR):
    os.mkdir(PROJECT_ROOT_DIR)

if not os.path.exists(FIGURE_ID):
    os.makedirs(FIGURE_ID)

if not os.path.exists(DATA_ID):
    os.makedirs(DATA_ID)

def image_path(fig_id):
    return os.path.join(FIGURE_ID, fig_id)

def data_path(dat_id):
    return os.path.join(DATA_ID, dat_id)

def save_fig(fig_id):
    plt.savefig(image_path(fig_id) + ".png", format='png')


DeltaT = 0.001
#set up arrays
tfinal = 10 # in years
n = ceil(tfinal/DeltaT)
# set up arrays for t, a, v, and x
```

```
t = np.zeros(n)
v = np.zeros((n,2))
r = np.zeros((n,2))
# Initial conditions as compact 2-dimensional arrays
r0 = np.array([1.0,0.0])
v0 = np.array([0.0,2*pi])
r[0] = r0
v[0] = v0
Fourpi2 = 4*pi*pi
# Start integrating using Euler's method
for i in range(n-1):
    # Set up the acceleration
    # Here you could have defined your own function for this
    rabs = sqrt(sum(r[i]*r[i]))
    a =  -Fourpi2*r[i]/(rabs**3)
    # update velocity, time and position using Euler's forward method
    v[i+1] = v[i] + DeltaT*a
    r[i+1] = r[i] + DeltaT*v[i]
    t[i+1] = t[i] + DeltaT
# Plot position as function of time
fig, ax = plt.subplots()
#ax.set_xlim(0, tfinal)
ax.set_ylabel('x[m]')
ax.set_xlabel('y[m]')
ax.plot(r[:,0], r[:,1])
fig.tight_layout()
save_fig("EarthSunEuler")
plt.show()
```

Problems with Euler's Method

We notice here that Euler's method doesn't give a stable orbit. It means that we cannot trust Euler's method. In a deeper way, as we will see in homework 5, Euler's method does not conserve energy. It is an example of an integrator which is not symplectic.

Here we present thus two methods, which with simple changes allow us to avoid these pitfalls. The simplest possible extension is the so-called Euler-Cromer method. The changes we need to make to our code are indeed marginal here. We need simply to replace

```
r[i+1] = r[i] + DeltaT*v[i]
```

in the above code with the velocity at the new time t_{i+1}

```
r[i+1] = r[i] + DeltaT*v[i+1]
```

By this simple caveat we get stable orbits. Below we derive the Euler-Cromer method as well as one of the most utlized algorithms for sovling the above type of problems, the so-called Velocity-Verlet method.

Deriving the Euler-Cromer Method

Let us repeat Euler's method. We have a differential equation

$$y'(t_i) = f(t_i, y_i) \tag{31}$$

and if we truncate at the first derivative, we have from the Taylor expansion

$$y_{i+1} = y(t_i) + (\Delta t)f(t_i, y_i) + O(\Delta t^2), \tag{32}$$

which when complemented with $t_{i+1} = t_i + \Delta t$ forms the algorithm for the well-known Euler method. Note that at every step we make an approximation error of the order of $O(\Delta t^2)$, however the total error is the sum over all steps $N = (b-a)/(\Delta t)$ for $t \in [a, b]$, yielding thus a global error which goes like $NO(\Delta t^2) \approx O(\Delta t)$.

To make Euler's method more precise we can obviously decrease Δt (increase N), but this can lead to loss of numerical precision. Euler's method is not recommended for precision calculation, although it is handy to use in order to get a first view on how a solution may look like.

Euler's method is asymmetric in time, since it uses information about the derivative at the beginning of the time interval. This means that we evaluate the position at y_1 using the velocity at v_0. A simple variation is to determine x_{n+1} using the velocity at v_{n+1}, that is (in a slightly more generalized form)

$$y_{n+1} = y_n + v_{n+1} + O(\Delta t^2) \tag{33}$$

and

$$v_{n+1} = v_n + (\Delta t)a_n + O(\Delta t^2). \tag{34}$$

The acceleration a_n is a function of $a_n(y_n, v_n, t_n)$ and needs to be evaluated as well. This is the Euler-Cromer method.

Exercise: go back to the above code with Euler's method and add the Euler-Cromer method.

Deriving the Velocity-Verlet Method

Let us stay with x (position) and v (velocity) as the quantities we are interested in.

We have the Taylor expansion for the position given by

$$x_{i+1} = x_i + (\Delta t)v_i + \frac{(\Delta t)^2}{2}a_i + O((\Delta t)^3).$$

The corresponding expansion for the velocity is

$$v_{i+1} = v_i + (\Delta t)a_i + \frac{(\Delta t)^2}{2}v_i^{(2)} + O((\Delta t)^3).$$

Via Newton's second law we have normally an analytical expression for the derivative of the velocity, namely

$$a_i = \frac{d^2x}{dt^2}\Big|_i = \frac{dv}{dt}\Big|_i = \frac{F(x_i, v_i, t_i)}{m}.$$

If we add to this the corresponding expansion for the derivative of the velocity

$$v_{i+1}^{(1)} = a_{i+1} = a_i + (\Delta t)v_i^{(2)} + O((\Delta t)^2) = a_i + (\Delta t)v_i^{(2)} + O((\Delta t)^2),$$

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and retain only terms up to the second derivative of the velocity since our error goes as $O(h^3)$, we have

$$(\Delta t)v_i^{(2)} \approx a_{i+1} - a_i.$$

We can then rewrite the Taylor expansion for the velocity as

$$v_{i+1} = v_i + \frac{(\Delta t)}{2}(a_{i+1} + a_i) + O((\Delta t)^3).$$

The velocity Verlet method

Our final equations for the position and the velocity become then

$$x_{i+1} = x_i + (\Delta t)v_i + \frac{(\Delta t)^2}{2}a_i + O((\Delta t)^3),$$

and

$$v_{i+1} = v_i + \frac{(\Delta t)}{2}(a_{i+1} + a_i) + O((\Delta t)^3).$$

Note well that the term a_{i+1} depends on the position at x_{i+1}. This means that you need to calculate the position at the updated time t_{i+1} before the computing the next velocity. Note also that the derivative of the velocity at the time t_i used in the updating of the position can be reused in the calculation of the velocity update as well.

Adding the Velocity-Verlet Method

We can now easily add the Verlet method to our original code as

```
DeltaT = 0.01
#set up arrays
tfinal = 10
n = ceil(tfinal/DeltaT)
# set up arrays for t, a, v, and x
t = np.zeros(n)
v = np.zeros((n,2))
r = np.zeros((n,2))
# Initial conditions as compact 2-dimensional arrays
r0 = np.array([1.0,0.0])
v0 = np.array([0.0,2*pi])
r[0] = r0
v[0] = v0
Fourpi2 = 4*pi*pi
# Start integrating using the Velocity-Verlet  method
for i in range(n-1):
    # Set up forces, air resistance FD, note now that we need the norm of the vecto
    # Here you could have defined your own function for this
    rabs = sqrt(sum(r[i]*r[i]))
    a =  -Fourpi2*r[i]/(rabs**3)
    # update velocity, time and position using the Velocity-Verlet method
    r[i+1] = r[i]  + DeltaT*v[i]+0.5*(DeltaT**2)*a
    rabs = sqrt(sum(r[i+1]*r[i+1]))
    anew = -4*(pi**2)*r[i+1]/(rabs**3)
    v[i+1] = v[i] + 0.5*DeltaT*(a+anew)
```

```
    t[i+1] = t[i] + DeltaT
# Plot position as function of time
fig, ax = plt.subplots()
ax.set_ylabel('x[m]')
ax.set_xlabel('y[m]')
ax.plot(r[:,0], r[:,1])
fig.tight_layout()
save_fig("EarthSunVV")
plt.show()
```

You can easily generalize the calculation of the forces by defining a function which takes in as input the various variables. We leave this as a challenge to you.