PHY321: More on Motion and Forces, begin Work and Energy discussion

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Aims and Overarching Motivation

Monday. We discuss various forces and their pertinent equations of motion

Recommended reading: Taylor 2.1-2.4. Malthe-Sørenssen chapter 6-7 contains many examples. We will cover in particular a falling object in two dimensions with linear air resistance relevant for homework 3.

Wednesday. We discuss other force models with examples such as the gravitational force and a spring force. See Malthe-Sørenssen chapter 7.3-7.5. We start also discussion Energy and work, see Taylor 4.1

Friday. We discuss several examples of energy and work. Taylor 4.1-4.3.

Air Resistance in One Dimension

Last week we considered the motion of a falling object with air resistance. Here we look at both a quadratic in velocity resistance and linear in velocity. But first we give a qualitative argument about the mathematical expression for the air resistance we used last Friday.

Air resistance tends to scale as the square of the velocity. This is in contrast to many problems chosen for textbooks, where it is linear in the velocity. The choice of a linear dependence is motivated by mathematical simplicity (it keeps the differential equation linear) rather than by physics. One can see that the force should be quadratic in velocity by considering the momentum imparted on the air molecules. If an object sweeps through a volume dV of air in time dt, the momentum imparted on the air is

$$dP = \rho_m dV v, \tag{1}$$

where v is the velocity of the object and ρ_m is the mass density of the air. If the molecules bounce back as opposed to stop you would double the size of the term. The opposite value of the momentum is imparted onto the object itself. Geometrically, the differential volume is

$$dV = Avdt, \tag{2}$$

where A is the cross-sectional area and vdt is the distance the object moved in time dt.

Resulting Acceleration

Plugging this into the expression above,

$$\frac{dP}{dt} = -\rho_m A v^2. \tag{3}$$

This is the force felt by the particle, and is opposite to its direction of motion. Now, because air doesn't stop when it hits an object, but flows around the best it can, the actual force is reduced by a dimensionless factor c_W, called the drag coefficient.

$$F_{\text{drag}} = -c_W \rho_m A v^2, \tag{4}$$

and the acceleration is

$$\frac{dv}{dt} = -\frac{c_W \rho_m A}{m} v^2. \tag{5}$$

For a particle with initial velocity v_0, one can separate the dt to one side of the equation, and move everything with vs to the other side. We did this in our discussion of simple motion and will not repeat it here.

On more general terms, for many systems, e.g. an automobile, there are multiple sources of resistance. In addition to wind resistance, where the force is proportional to v^2, there are dissipative effects of the tires on the pavement, and in the axel and drive train. These other forces can have components that scale proportional to v, and components that are independent of v. Those independent of v, e.g. the usual $f = \mu_K N$ frictional force you consider in your first Physics courses, only set in once the object is actually moving. As speeds become higher, the v^2 components begin to dominate relative to the others. For automobiles at freeway speeds, the v^2 terms are largely responsible for the loss of efficiency. To travel a distance L at fixed speed v, the energy/work required to overcome the dissipative forces are fL, which for a force of the form $f = \alpha v^n$ becomes

$$W = \int dx \; f = \alpha v^n L. \tag{6}$$

For $n = 0$ the work is independent of speed, but for the wind resistance, where $n = 2$, slowing down is essential if one wishes to reduce fuel consumption. It is also important to consider that engines are designed to be most efficient at a chosen range of power output. Thus, some cars will get better mileage at higher speeds (They perform better at 50 mph than at 5 mph) despite the considerations mentioned above.

Going Ballistic, Projectile Motion or a Softer Approach, Falling Raindrops

As an example of Newton's Laws we consider projectile motion (or a falling raindrop or a ball we throw up in the air) with a drag force. Even though air resistance is largely proportional to the square of the velocity, we will consider the drag force to be linear to the velocity, $\boldsymbol{F} = -m\gamma\boldsymbol{v}$, for the purposes of this exercise.

Such a dependence can be extracted from experimental data for objects moving at low velocities, see for example Malthe-Sørenssen chapter 5.6.

We will here focus on a two-dimensional problem.

Two-dimensional falling object

The acceleration for a projectile moving upwards, $\boldsymbol{a} = \boldsymbol{F}/m$, becomes

$$\begin{aligned}\frac{dv_x}{dt} &= -\gamma v_x, \\ \frac{dv_y}{dt} &= -\gamma v_y - g,\end{aligned} \tag{7}$$

and γ has dimensions of inverse time.

If you on the other hand have a falling raindrop, how do these equations change? See for example Figure 2.1 in Taylor. Let us stay with a ball which is thrown up in the air at $t = 0$.

Ways of solving these equations

We will go over two different ways to solve this equation. The first by direct integration, and the second as a differential equation. To do this by direct integration, one simply multiplies both sides of the equations above by dt, then divide by the appropriate factors so that the vs are all on one side of the equation and the dt is on the other. For the x motion one finds an easily integrable equation,

$$
\begin{aligned}
\frac{dv_x}{v_x} &= -\gamma dt, \\
\int_{v_{0x}}^{v_x} \frac{dv_x}{v_x} &= -\gamma \int_0^t dt, \\
\ln\left(\frac{v_x}{v_{0x}}\right) &= -\gamma t, \\
v_x(t) &= v_{0x} e^{-\gamma t}.
\end{aligned}
\tag{8}
$$

This is very much the result you would have written down by inspection. For the y-component of the velocity,

$$
\begin{aligned}
\frac{dv_y}{v_y + g/\gamma} &= -\gamma dt \\
\ln\left(\frac{v_y + g/\gamma}{v_{0y} - g/\gamma}\right) &= -\gamma t_f, \\
v_{fy} &= -\frac{g}{\gamma} + \left(v_{0y} + \frac{g}{\gamma}\right) e^{-\gamma t}.
\end{aligned}
\tag{9}
$$

Whereas v_x starts at some value and decays exponentially to zero, v_y decays exponentially to the terminal velocity, $v_t = -g/\gamma$.

Solving as differential equations

Although this direct integration is simpler than the method we invoke below, the method below will come in useful for some slightly more difficult differential equations in the future. The differential equation for v_x is straight-forward to solve. Because it is first order there is one arbitrary constant, A, and by inspection the solution is

$$
v_x = Ae^{-\gamma t}. \tag{10}
$$

The arbitrary constants for equations of motion are usually determined by the initial conditions, or more generally boundary conditions. By inspection $A = v_{0x}$, the initial x component of the velocity.

Differential Equations, contn

The differential equation for v_y is a bit more complicated due to the presence of g. Differential equations where all the terms are linearly proportional to a function, in this case v_y, or to derivatives of the function, e.g., v_y, dv_y/dt, $d^2v_y/dt^2 \cdots$, are called linear differential equations. If there are terms proportional to v^2, as would happen if the drag force were proportional to the square of the velocity, the differential equation is not longer linear. Because this expression has only one derivative in v it is a first-order linear differential equation. If a term were added proportional to d^2v/dt^2 it would be a second-order differential equation. In this case we have a term completely independent of v, the gravitational acceleration g, and the usual strategy is to first rewrite the equation with all the linear terms on one side of the equal sign,

$$\frac{dv_y}{dt} + \gamma v_y = -g. \tag{11}$$

Splitting into two parts

Now, the solution to the equation can be broken into two parts. Because this is a first-order differential equation we know that there will be one arbitrary constant. Physically, the arbitrary constant will be determined by setting the initial velocity, though it could be determined by setting the velocity at any given time. Like most differential equations, solutions are not "solved". Instead, one guesses at a form, then shows the guess is correct. For these types of equations, one first tries to find a single solution, i.e. one with no arbitrary constants. This is called the *particular* solution, $y_p(t)$, though it should really be called "a" particular solution because there are an infinite number of such solutions. One then finds a solution to the *homogenous* equation, which is the equation with zero on the right-hand side,

$$\frac{dv_{y,h}}{dt} + \gamma v_{y,h} = 0. \tag{12}$$

Homogenous solutions will have arbitrary constants.

The particular solution will solve the same equation as the original general equation

$$\frac{dv_{y,p}}{dt} + \gamma v_{y,p} = -g. \tag{13}$$

However, we don't need find one with arbitrary constants. Hence, it is called a **particular** solution.

The sum of the two,

$$v_y = v_{y,p} + v_{y,h}, \tag{14}$$

is a solution of the total equation because of the linear nature of the differential equation. One has now found a *general* solution encompassing all solutions, because it both satisfies the general equation (like the particular solution), and has an arbitrary constant that can be adjusted to fit any initial condition (like the homogeneous solution). If the equations were not linear, that is if there were terms such as v_y^2 or $v_y\dot{v}_y$, this technique would not work.

More details

Returning to the example above, the homogenous solution is the same as that for v_x, because there was no gravitational acceleration in that case,

$$v_{y,h} = Be^{-\gamma t}. \tag{15}$$

In this case a particular solution is one with constant velocity,

$$v_{y,p} = -g/\gamma. \tag{16}$$

Note that this is the terminal velocity of a particle falling from a great height. The general solution is thus,

$$v_y = Be^{-\gamma t} - g/\gamma, \tag{17}$$

and one can find B from the initial velocity,

$$v_{0y} = B - g/\gamma, \quad B = v_{0y} + g/\gamma. \tag{18}$$

Plugging in the expression for B gives the y motion given the initial velocity,

$$v_y = (v_{0y} + g/\gamma)e^{-\gamma t} - g/\gamma. \tag{19}$$

It is easy to see that this solution has $v_y = v_{0y}$ when $t = 0$ and $v_y = -g/\gamma$ when $t \to \infty$.

One can also integrate the two equations to find the coordinates x and y as functions of t,

$$\begin{aligned} x &= \int_0^t dt'\ v_{0x}(t') = \frac{v_{0x}}{\gamma}\left(1 - e^{-\gamma t}\right), \qquad (20)\\ y &= \int_0^t dt'\ v_{0y}(t') = -\frac{gt}{\gamma} + \frac{v_{0y} + g/\gamma}{\gamma}\left(1 - e^{-\gamma t}\right). \end{aligned}$$

If the question was to find the position at a time t, we would be finished. However, the more common goal in a projectile equation problem is to find the range, i.e. the distance x at which y returns to zero. For the case without a drag force this was much simpler. The solution for the y coordinate would have been $y = v_{0y}t - gt^2/2$. One would solve for t to make $y = 0$, which would be $t = 2v_{0y}/g$, then plug that value for t into $x = v_{0x}t$ to find $x = 2v_{0x}v_{0y}/g = v_0 \sin(2\theta_0)/g$. One follows the same steps here, except that the expression for $y(t)$ is more complicated. Searching for the time where $y = 0$, and we get

$$0 = -\frac{gt}{\gamma} + \frac{v_{0y} + g/\gamma}{\gamma}\left(1 - e^{-\gamma t}\right). \tag{21}$$

This cannot be inverted into a simple expression $t = \cdots$. Such expressions are known as "transcendental equations", and are not the rare instance, but are the norm. In the days before computers, one might plot the right-hand side of the above graphically as a function of time, then find the point where it crosses zero.

Now, the most common way to solve for an equation of the above type would be to apply Newton's method numerically. This involves the following algorithm for finding solutions of some equation $F(t) = 0$.

1. First guess a value for the time, $t_{\rm guess}$.
2. Calculate F and its derivative, $F(t_{\rm guess})$ and $F'(t_{\rm guess})$.
3. Unless you guessed perfectly, $F \neq 0$, and assuming that $\Delta F \approx F'\Delta t$, one would choose
4. $\Delta t = -F(t_{\rm guess})/F'(t_{\rm guess})$.
5. Now repeat step 1, but with $t_{\rm guess} \rightarrow t_{\rm guess} + \Delta t$.

If the $F(t)$ were perfectly linear in t, one would find t in one step. Instead, one typically finds a value of t that is closer to the final answer than $t_{\rm guess}$. One breaks the loop once one finds F within some acceptable tolerance of zero. A program to do this will be added shortly.

Motion in a Magnetic Field

Another example of a velocity-dependent force is magnetism,

$$\begin{aligned} \boldsymbol{F} &= q\boldsymbol{v} \times \boldsymbol{B}, \\ F_i &= q\sum_{jk} \epsilon_{ijk} v_j B_k. \end{aligned} \tag{22}$$

For a uniform field in the z direction $\boldsymbol{B} = B\hat{z}$, the force can only have x and y components,

$$\begin{aligned} F_x &= qBv_y \\ F_y &= -qBv_x. \end{aligned} \tag{23}$$

The differential equations are

$$\begin{aligned} \dot{v}_x &= \omega_c v_y, \omega_c = qB/m \\ \dot{v}_y &= -\omega_c v_x. \end{aligned} \tag{24}$$

One can solve the equations by taking time derivatives of either equation, then substituting into the other equation,

$$\begin{aligned} \ddot{v}_x = \omega_c \dot{v}_y = -\omega_c^2 v_x, \\ \ddot{v}_y &= -\omega_c \dot{v}_x = -\omega_c v_y. \end{aligned} \tag{25}$$

The solution to these equations can be seen by inspection,

$$\begin{aligned} v_x &= A\sin(\omega_c t+\phi), \\ v_y &= A\cos(\omega_c t+\phi). \end{aligned} \tag{26}$$

One can integrate the equations to find the positions as a function of time,

$$\begin{aligned} x - x_0 &= \int_{x_0}^{x} dx = \int_0^t dt v(t) \\ &= \frac{-A}{\omega_c}\cos(\omega_c t+\phi), \\ y - y_0 &= \frac{A}{\omega_c}\sin(\omega_c t+\phi). \end{aligned} \tag{27}$$

The trajectory is a circle centered at x_0, y_0 with amplitude A rotating in the clockwise direction.

The equations of motion for the z motion are

$$\dot{v_z} = 0, \tag{28}$$

which leads to

$$z - z_0 = V_z t. \tag{29}$$

Added onto the circle, the motion is helical.

Note that the kinetic energy,

$$T = \frac{1}{2}m(v_x^2+v_y^2+v_z^2) = \frac{1}{2}m(\omega_c^2 A^2 + V_z^2), \tag{30}$$

is constant. This is because the force is perpendicular to the velocity, so that in any differential time element dt the work done on the particle $\boldsymbol{F}\cdot dr = dt\boldsymbol{F}\cdot v = 0$.

One should think about the implications of a velocity dependent force. Suppose one had a constant magnetic field in deep space. If a particle came through with velocity v_0, it would undergo cyclotron motion with radius $R = v_0/\omega_c$. However, if it were still its motion would remain fixed. Now, suppose an observer looked at the particle in one reference frame where the particle was moving, then changed their velocity so that the particle's velocity appeared to be zero. The motion would change from circular to fixed. Is this possible?

The solution to the puzzle above relies on understanding relativity. Imagine that the first observer believes $\boldsymbol{B} \neq 0$ and that the electric field $\boldsymbol{E} = 0$. If the observer then changes reference frames by accelerating to a velocity $\boldsymbol{v}$, in the new frame $\boldsymbol{B}$ and $\boldsymbol{E}$ both change. If the observer moved to the frame where the charge, originally moving with a small velocity v, is now at rest, the new electric field is indeed $\boldsymbol{v}\times\boldsymbol{B}$, which then leads to the same acceleration as one had before. If the velocity is not small compared to the speed of light, additional γ factors come into play, $\gamma = 1/\sqrt{1-(v/c)^2}$. Relativistic motion will not be considered in this course.

Sliding Block tied to a Wall

Another classical case is that of simple harmonic oscillations, here represented by a block sliding on a horizontal frictionless surface. The block is tied to a wall with a spring. If the spring is not compressed or stretched too far, the force on the block at a given position x is

$$F = -kx.$$

The negative sign means that the force acts to restore the object to an equilibrium position. Newton's equation of motion for this idealized system is then

$$m\frac{d^2x}{dt^2} = -kx,$$

or we could rephrase it as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega_0^2 x,$$

with the angular frequency $\omega_0^2 = k/m$.

The above differential equation has the advantage that it can be solved analytically with solutions on the form

$$x(t) = Acos(\omega_0 t + \nu),$$

where A is the amplitude and ν the phase constant. This provides in turn an important test for the numerical solution and the development of a program for more complicated cases which cannot be solved analytically.

With the position $x(t)$ and the velocity $v(t) = dx/dt$ we can reformulate Newton's equation in the following way

$$\frac{dx(t)}{dt} = v(t),$$

and

$$\frac{dv(t)}{dt} = -\omega_0^2 x(t).$$

We are now going to solve these equations using first the standard forward Euler method. Later we will try to improve upon this.

Before proceeding however, it is important to note that in addition to the exact solution, we have at least two further tests which can be used to check our solution.

Since functions like *cos* are periodic with a period 2π, then the solution $x(t)$ has also to be periodic. This means that

$$x(t + T) = x(t),$$

with T the period defined as

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{k/m}}.$$

Observe that T depends only on k/m and not on the amplitude of the solution.

In addition to the periodicity test, the total energy has also to be conserved.

Suppose we choose the initial conditions

$$x(t=0) = 1 \text{ m} \qquad v(t=0) = 0 \text{ m/s},$$

meaning that block is at rest at $t=0$ but with a potential energy

$$E_0 = \frac{1}{2}kx(t=0)^2 = \frac{1}{2}k.$$

The total energy at any time t has however to be conserved, meaning that our solution has to fulfil the condition

$$E_0 = \frac{1}{2}kx(t)^2 + \frac{1}{2}mv(t)^2.$$

We will derive this equation in our discussion on energy conservation.

An algorithm which implements these equations is included below.

- Choose the initial position and speed, with the most common choice $v(t=0)=0$ and some fixed value for the position.
- Choose the method you wish to employ in solving the problem.
- Subdivide the time interval $[t_i, t_f]$ into a grid with step size

$$h = \frac{t_f - t_i}{N},$$

where N is the number of mesh points.

- Calculate now the total energy given by

$$E_0 = \frac{1}{2}kx(t=0)^2 = \frac{1}{2}k.$$

- Choose ODE solver to obtain x_{i+1} and v_{i+1} starting from the previous values x_i and v_i.
- When we have computed $x(v)_{i+1}$ we upgrade $t_{i+1} = t_i + h$.
- This iterative process continues till we reach the maximum time t_f.
- The results are checked against the exact solution. Furthermore, one has to check the stability of the numerical solution against the chosen number of mesh points N.

Work, Energy, Momentum and Conservation laws

Energy conservation is most convenient as a strategy for addressing problems where time does not appear. For example, a particle goes from position x_0 with speed v_0, to position x_f; what is its new speed? However, it can also be applied to problems where time does appear, such as in solving for the trajectory $x(t)$, or equivalently $t(x)$.

Note: See the handwritten notes for more material. This will be added during this week to these slides.

Energy Conservation

Energy is conserved in the case where the potential energy, $V(\boldsymbol{r})$, depends only on position, and not on time. The force is determined by V,

$$\boldsymbol{F}(\boldsymbol{r}) = -\nabla V(\boldsymbol{r}). \tag{31}$$

The net energy, $E = V + K$ where K is the kinetic energy, is then conserved,

$$\begin{aligned}
\frac{d}{dt}(K+V) &= \frac{d}{dt}\left(\frac{m}{2}(v_x^2+v_y^2+v_z^2)+V(\boldsymbol{r})\right) \\
&= m\left(v_x\frac{dv_x}{dt}+v_y\frac{dv_y}{dt}+v_z\frac{dv_z}{dt}\right)+\partial_x V\frac{dx}{dt}+\partial_y V\frac{dy}{dt}+\partial_z V\frac{dz}{dt} \\
&= v_xF_x+v_yF_y+v_zF_z-F_xv_x-F_yv_y-F_zv_z=0.
\end{aligned} \tag{32}$$

The same proof can be written more compactly with vector notation,

$$\begin{aligned}
\frac{d}{dt}\left(\frac{m}{2}v^2+V(\boldsymbol{r})\right) &= m\boldsymbol{v}\cdot\dot{\boldsymbol{v}}+\nabla V(\boldsymbol{r})\cdot\dot{\boldsymbol{r}} \\
&= \boldsymbol{v}\cdot\boldsymbol{F}-\boldsymbol{F}\cdot\boldsymbol{v}=0.
\end{aligned} \tag{33}$$

Inverting the kinetic energy expression

Inverting the expression for kinetic energy,

$$v=\sqrt{2K/m}=\sqrt{2(E-V)/m}, \tag{34}$$

allows one to solve for the one-dimensional trajectory $x(t)$, by finding $t(x)$,

$$t=\int_{x_0}^{x}\frac{dx'}{v(x')}=\int_{x_0}^{x}\frac{dx'}{\sqrt{2(E-V(x'))/m}}. \tag{35}$$

Note this would be much more difficult in higher dimensions, because you would have to determine which points, x,y,z, the particles might reach in the trajectory, whereas in one dimension you can typically tell by simply seeing whether the kinetic energy is positive at every point between the old position and the new position.

EXample of Harmonic Oscillator

Consider a simple harmonic oscillator potential, $V(x)=kx^2/2$, with a particle emitted from $x=0$ with velocity v_0. Solve for the trajectory $t(x)$,

$$\begin{aligned}
t &= \int_0^x \frac{dx'}{\sqrt{2(E-kx^2/2)/m}} \\
&= \sqrt{m/k}\int_0^x \frac{dx'}{\sqrt{x_{\rm max}^2-x'^2}}, \quad x_{\rm max}^2=2E/k.
\end{aligned} \tag{36}$$

Here $E = mv_0^2/2$ and $x_{\max}$ is defined as the maximum displacement before the particle turns around. This integral is done by the substitution $\sin\theta = x/x_{\max}$.

$$\begin{aligned} (k/m)^{1/2} t &= \sin^{-1}(x/x_{\max}), \\ x &= x_{\max} \sin \omega t, \quad \omega = \sqrt{k/m}. \end{aligned} \tag{37}$$