PHY321: Conservative Forces, Momentum and Angular Momentum conservation

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Feb 14, 2021

Aims and Overarching Motivation

Monday. Short repetition from last week. Discussion of momentum and angular momentum. Reading suggestion: Taylor sections 3.1-3.4

Wednesday. More on angular momentum, Taylor sections 3.4 and 3.5. Discussion of potential energy and conservative forces. Reading suggestions: Taylor section 4.2

Friday. Conservative forces and potential energy. Reading suggestion: Taylor sections 4.2-4.4

If you wish to read more about conservative forces or not, Feyman's lectures from 1963 are quite interesting. He states for example that **All fundamental forces in nature appear to be conservative**. This statement was made while developing his argument that *there are no nonconservative forces*. You may enjoy the link to Feynman's lecture.

Work-Energy Theorem and Energy Conservation

Last week (Friday) we observed that energy was conserved for a force which depends only on the position. In particular we considered a force acting on a block attached to a spring with spring constant k. The other end of the spring was attached to the wall.

The force F_x from the spring on the block was defined as

$$F_x = -kx.$$

The work done on the block due to a displacement from a position x_0 to x

$$W = \int_{x_0}^{x} F_x dx' = \frac{1}{2}kx_0^2 - \frac{1}{2}kx^2.$$

Conservation of energy

With the definition of the work-energy theorem in terms of the kinetic energy we obtained

$$W = \frac{1}{2}mv^2(x) - \frac{1}{2}mv_0^2 = \frac{1}{2}kx_0^2 - \frac{1}{2}kx^2,$$

which we rewrote as

$$\frac{1}{2}mv^2(x) + \frac{1}{2}kx^2 = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2.$$

The total energy, which is the sum of potential and kinetic energy, is conserved. We will analyze this interesting result now in more detail when we study energy, momentum and angular momentum conservation.

But before we start with energy conservation, conservative forces and potential energies, we need to revisit our definitions of momentum and angular momentum.

What is a Conservative Force?

A conservative force is a force whose property is that the total work done in moving an object between two points is independent of the taken path. This means that the work on an object under the influence of a conservative force, is independent on the path of the object. It depends only on the spatial degrees of freedom and it is possible to assign a numerical value for the potential at any point. It leads to conservation of energy. The gravitational force is an example of a conservative force.

Two important conditions

First, a conservative force depends only on the spatial degrees of freedom. This is a necessary condition for obtaining a path integral which is independent of path. The important condition for the final work to be independent of the path is that the **curl** of the force is zero, that is

$$\boldsymbol{\nabla} \times \boldsymbol{F} = 0$$

Work-energy theorem to show that energy is conserved with a conservative force

The work-energy theorem states that the work done W by a force $\boldsymbol{F}$ that moves an object from a position $\boldsymbol{r}_0$ to a new position $\boldsymbol{r}_1$

$$W = \int_{\boldsymbol{r}_0}^{\boldsymbol{r}_1} \boldsymbol{F} d\boldsymbol{r} = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2,$$

where v_1^2 is the velocity squared at a time t_1 and v_0^2 the corresponding quantity at a time t_0. The work done is thus the difference in kinetic energies. We can rewrite the above equation as

$$\frac{1}{2}mv_1^2 = \int_{\boldsymbol{r}_0}^{\boldsymbol{r}_1} \boldsymbol{F}d\boldsymbol{r} + \frac{1}{2}mv_0^2,$$

that is the final kinetic energy is equal to the initial kinetic energy plus the work done by the force over a given path from a position $\boldsymbol{r}_0$ at time t_0 to a final position position $\boldsymbol{r}_1$ at a later time t_1.

Conservation of Momentum

Before we move on however, we need to remind ourselves about important aspects like the linear momentum and angular momentum. After these considerations, we move back to more details about conservatives forces.

Assume we have N objects, each with velocity $\boldsymbol{v}_i$ with $i = 1, 2, \ldots, N$ and mass m_i. The momentum of each object is $\boldsymbol{p}_i = m\boldsymbol{v}_i$ and the total linear (or mechanical) momentum is defined as

$$\boldsymbol{P} = \sum_{i=1}^{N} \boldsymbol{p}_i = \sum_{i=1}^{N} m_i \boldsymbol{v}_i,$$

Two objects first

Let us assume we have two objects only that interact with each other and are influenced by an external force.

We define also the total net force acting on object 1 as

$$\boldsymbol{F}_1^{\mathrm{net}} = \boldsymbol{F}_1^{\mathrm{ext}} + \boldsymbol{F}_{12},$$

where $\boldsymbol{F}_1^{\mathrm{ext}}$ is the external force (for example the force due to an electron moving in an electromagnetic field) and $\boldsymbol{F}_{12}$ is the force between object one and two. Similarly for object 2 we have

$$\boldsymbol{F}_2^{\mathrm{net}} = \boldsymbol{F}_2^{\mathrm{ext}} + \boldsymbol{F}_{21}.$$

Newton's Third Law

Newton's third law which we met earlier states that **for every action there is an equal and opposite reaction**. It is more accurately stated as

if two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction.

This means that for two bodies i and j, if the force on i due to j is called $\boldsymbol{F}_{ij}$, then

$$\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}. \tag{1}$$

For the abovementioned two objects we have thus $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$.

Newton's Second Law and Momentum

With the net forces acting on each object we can now related the momentum to the forces via

$$\boldsymbol{F}_1^{\mathrm{net}} = m_1\boldsymbol{a}_i = m_1\frac{d\boldsymbol{v}_i}{dt} = \boldsymbol{F}_1^{\mathrm{ext}} + \boldsymbol{F}_{12},$$

and

$$\boldsymbol{F}_2^{\mathrm{net}} = m_2\boldsymbol{a}_2 = m_1\frac{d\boldsymbol{v}_i}{dt} = \boldsymbol{F}_2^{\mathrm{ext}} + \boldsymbol{F}_{21}.$$

Recalling our definition for the linear momentum we have then

$$\frac{d\boldsymbol{p}_1}{dt} = \boldsymbol{F}_1^{\mathrm{ext}} + \boldsymbol{F}_{12},$$

and

$$\frac{d\boldsymbol{p}_2}{dt} = \boldsymbol{F}_2^{\mathrm{ext}} + \boldsymbol{F}_{21}.$$

The total Momentum

The total momentum $\boldsymbol{P}$ is defined as the sum of the individual momenta, meaning that we can rewrite

$$\boldsymbol{F}_1^{\mathrm{net}} + \boldsymbol{F}_2^{\mathrm{net}} = \frac{d\boldsymbol{p}_1}{dt} + \frac{d\boldsymbol{p}_2}{dt} = \frac{d\boldsymbol{P}}{dt},$$

that is the derivate with respect to time of the total momentum. If we now write the net forces as sums of the external plus internal forces between the objects we have

$$\frac{d\boldsymbol{P}}{dt} = \boldsymbol{F}_1^{\mathrm{ext}} + \boldsymbol{F}_{12} + \boldsymbol{F}_2^{\mathrm{ext}} + \boldsymbol{F}_{21} = \boldsymbol{F}_1^{\mathrm{ext}} + \boldsymbol{F}_2^{\mathrm{ext}}.$$

The derivative of the total momentum is just **the sum of the external forces**. If we assume that the external forces are zero and that only internal (here two-body forces) are at play, we obtain the important result that the derivative of the total momentum is zero. This means again that the total momentum is a constant of the motion and conserved quantity. This is a very important result that we will use in many applications to come.

Newton's Second Law

Let us now general to several objects N and let us also assume that there are no external forces. We will label such a system as **an isolated system**.

Newton's second law, $\boldsymbol{F} = m\boldsymbol{a}$, can be written for a particle i as

$$\boldsymbol{F}_i = \sum_{j\neq i}^{N} \boldsymbol{F}_{ij} = m_i \boldsymbol{a}_i, \tag{2}$$

where $\boldsymbol{F}_i$ (a single subscript) denotes the net force acting on i from the other objects/particles. Because the mass of i is fixed and we assume it does not change with time, one can see that

$$\boldsymbol{F}_i = \frac{d}{dt} m_i \boldsymbol{v}_i = \sum_{j\neq i}^{N} \boldsymbol{F}_{ij}. \tag{3}$$

Summing over all Objects/Particles

Now, one can sum over all the objects/particles and obtain

$$\frac{d}{dt}\sum_i m_i v_i = \sum_{ij,i\neq j}^{N} \boldsymbol{F}_{ij} = 0.$$

How did we arrive at the last step? We rewrote the double sum as

$$\sum_{ij,i\neq j}^{N} \boldsymbol{F}_{ij} = \sum_i^{N}\sum_{j>i}\left(\boldsymbol{F}_{ij} + \boldsymbol{F}_{ji}\right),$$

and using Newton's third law which states that $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$, we obtain that the net sum over all the two-particle forces is zero when we only consider so-called **internal forces**. Stated differently, the last step made use of the fact that for every term ij, there is an equivalent term ji with opposite force. Because the momentum is defined as $m\boldsymbol{v}$, for a system of particles, we have thus

$$\frac{d}{dt}\sum_i m_i \boldsymbol{v}_i = 0, \quad \text{for isolated particles.} \tag{4}$$

Conservation of total Momentum

By "isolated" one means that the only force acting on any particle i are those originating from other particles in the sum, i.e. “no external” forces. Thus, Newton's third law leads to the conservation of total momentum,

$$\boldsymbol{P} = \sum_i m_i \boldsymbol{v}_i,$$

and we have

$$\frac{d}{dt}\boldsymbol{P} = 0.$$

Example: Rocket Science

Consider a rocket of mass M moving with velocity v. After a brief instant, the velocity of the rocket is $v + \Delta v$ and the mass is $M - \Delta M$. Momentum conservation gives

$$\begin{aligned} Mv &= (M - \Delta M)(v + \Delta v) + \Delta M(v - v_e) \\ 0 &= -\Delta M v + M\Delta v + \Delta M(v - v_e), \\ 0 &= M\Delta v - \Delta M v_e. \end{aligned}$$

In the second step we ignored the term $\Delta M \Delta v$ because it is doubly small. The last equation gives

$$\begin{aligned} \Delta v &= \frac{v_e}{M}\Delta M, \qquad (5) \\ \frac{dv}{dt} &= \frac{v_e}{M}\frac{dM}{dt}. \end{aligned}$$

Integrating the Equations

Integrating the expression with lower limits $v_0 = 0$ and M_0, one finds

$$\begin{aligned} v &= v_e \int_{M_0}^{M} \frac{dM'}{M'} \\ v &= -v_e \ln(M/M_0) \\ &= -v_e \ln[(M_0 - \alpha t)/M_0]. \end{aligned}$$

Because the total momentum of an isolated system is constant, one can also quickly see that the center of mass of an isolated system is also constant. The center of mass is the average position of a set of masses weighted by the mass,

$$\bar{x} = \frac{\sum_i m_i x_i}{\sum_i m_i}. \qquad (6)$$

Rate of Change

The rate of change of $\bar{x}$ is

$$\dot{\bar{x}} = \frac{1}{M}\sum_i m_i \dot{x}_i = \frac{1}{M}P_x.$$

Thus if the total momentum is constant the center of mass moves at a constant velocity, and if the total momentum is zero the center of mass is fixed.

Conservation of Angular Momentum

The angular momentum is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m\boldsymbol{r} \times \boldsymbol{v}. \tag{7}$$

It means that the angular momentum is perpendicular to the plane defined by position $\boldsymbol{r}$ and the momentum $\boldsymbol{p}$ via $\boldsymbol{r} \times \boldsymbol{p}$.

Rate of Change of Angular Momentum

The rate of change of the angular momentum is

$$\frac{d\boldsymbol{L}}{dt} = m\boldsymbol{v} \times \boldsymbol{v} + m\boldsymbol{r} \times \dot{\boldsymbol{v}} = \boldsymbol{r} \times \boldsymbol{F}$$

The first term is zero because $\boldsymbol{v}$ is parallel to itself, and the second term defines the so-called torque. If $\boldsymbol{F}$ is parallel to $\boldsymbol{r}$ then the torque is zero and we say that angular momentum is conserved.

If the force is not radial, $\boldsymbol{r} \times \boldsymbol{F} \neq 0$ as above, and angular momentum is no longer conserved,

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times \boldsymbol{F} \equiv \boldsymbol{\tau}, \tag{8}$$

where $\boldsymbol{\tau}$ is the torque.

The Torque, Example 1 (hw 4, exercise 4)

Let us assume we have an initial position $\boldsymbol{r}_0 = x_0\boldsymbol{e}_1 + y_0\boldsymbol{e}_2$ at a time $t_0 = 0$. We add now a force in the positive x-direction

$$\boldsymbol{F} = F_x\boldsymbol{e}_1 = \frac{d\boldsymbol{p}}{dt},$$

where we used the force as defined by the time derivative of the momentum.

We can use this force (and its pertinent acceleration) to find the velocity via the relation

$$\boldsymbol{v}(t) = \boldsymbol{v}_0 + \int_{t_0}^{t} \boldsymbol{a}dt',$$

and with $\boldsymbol{v}_0 = 0$ we have

$$\boldsymbol{v}(t) = \int_{t_0}^{t} \frac{\boldsymbol{F}}{m}dt',$$

where m is the mass of the object.

The Torque, Example 1 (hw 4, exercise 4)

Since the force acts only in the x-direction, we have after integration

$$\boldsymbol{v}(t) = \frac{\boldsymbol{F}}{m}t = \frac{F_x}{m}t\boldsymbol{e}_1 = v_x(t)\boldsymbol{e}_1.$$

The momentum is in turn given by $\boldsymbol{p} = p_x\boldsymbol{e}_1 = mv_x\boldsymbol{e}_1 = F_xt\boldsymbol{e}_1$.

Integrating over time again we find the final position as (note the force depends only on the x-direction)

$$\boldsymbol{r}(t) = (x_0 + \frac{1}{2}\frac{F_x}{m}t^2)\boldsymbol{e}_1 + y_0\boldsymbol{e}_2.$$

There is no change in the position in the y-direction since the force acts only in the x-direction.

The Torque, Example 1 (hw 4, exercise 4)

We can now compute the angular momentum given by

$$\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p} = \left[(x_0 + \frac{1}{2}\frac{F_x}{m}t^2)\boldsymbol{e}_1 + y_0\boldsymbol{e}_2\right] \times \frac{F_xt}{m}\boldsymbol{e}_1.$$

Computing the cross product we find

$$\boldsymbol{l} = -y_0F_xt\boldsymbol{e}_3 = -y_0F_xt\boldsymbol{e}_z.$$

The torque is the time derivative of the angular momentum and we have

$$\boldsymbol{\tau} = -y_0F_x\boldsymbol{e}_3 = -y_0F_x\boldsymbol{e}_z.$$

The torque is non-zero and angular momentum is not conserved.

The Torque, Example 2

One can write the torque about a given axis, which we will denote as $\hat{z}$, in polar coordinates, where

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\cos\phi, \quad z = r\cos\theta, \tag{9}$$

to find the z component of the torque,

$$\begin{aligned} \tau_z &= xF_y - yF_x \\ &= -r\sin\theta\left\{\cos\phi\partial_y - \sin\phi\partial_x\right\}V(x,y,z). \end{aligned} \tag{10}$$

Chain Rule and Partial Derivatives

One can use the chain rule to write the partial derivative w.r.t. ϕ (keeping r and θ fixed),

$$\begin{aligned}\partial_\phi &= \frac{\partial x}{\partial \phi}\partial_x + \frac{\partial_y}{\partial \phi}\partial_y + \frac{\partial z}{\partial \phi}\partial_z \\ &= -r\sin\theta\sin\phi\partial_x + \sin\theta\cos\phi\partial_y.\end{aligned} \tag{11}$$

Combining the two equations,

$$\tau_z = -\partial_\phi V(r,\theta,\phi). \tag{12}$$

Thus, if the potential is independent of the azimuthal angle ϕ, there is no torque about the z axis and L_z is conserved.

System of Isolated Particles

For a system of isolated particles, one can write

$$\begin{aligned}\frac{d}{dt}\sum_i \boldsymbol{L}_i &= \sum_{i\neq j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} \\ &= \frac{1}{2}\sum_{i\neq j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} + \boldsymbol{r}_j \times \boldsymbol{F}_{ji} \\ &= \frac{1}{2}\sum_{i\neq j} (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ij} = 0,\end{aligned} \tag{13}$$

where the last step used Newton's third law, $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$. If the forces between the particles are radial, i.e. $\boldsymbol{F}_{ij} \mid\mid (\boldsymbol{r}_i - \boldsymbol{r}_j)$, then each term in the sum is zero and the net angular momentum is fixed. Otherwise, you could imagine an isolated system that would start spinning spontaneously.

Work, Energy, Momentum and Conservation laws

Energy conservation is most convenient as a strategy for addressing problems where time does not appear. For example, a particle goes from position x_0 with speed v_0, to position x_f; what is its new speed? However, it can also be applied to problems where time does appear, such as in solving for the trajectory $x(t)$, or equivalently $t(x)$.

Energy Conservation

Energy is conserved in the case where the potential energy, $V(\boldsymbol{r})$, depends only on position, and not on time. The force is determined by V,

$$\boldsymbol{F}(\boldsymbol{r}) = -\boldsymbol{\nabla} V(\boldsymbol{r}). \tag{14}$$

Conservative forces

We say a force is conservative if it satisfies the following conditions:

1. The force $\boldsymbol{F}$ acting on an object only depends on the position $\boldsymbol{r}$, that is $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r})$.
2. For any two points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, the work done by the force $\boldsymbol{F}$ on the displacement between these two points is independent of the path taken.
3. Finally, the **curl** of the force is zero $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$.

Forces and Potentials

The energy E of a given system is defined as the sum of kinetic and potential energies,

$$E = K + V(\boldsymbol{r}).$$

We define the potential energy at a point $\boldsymbol{r}$ as the negative work done from a starting point $\boldsymbol{r}_0$ to a final point $\boldsymbol{r}$

$$V(\boldsymbol{r}) = -W(\boldsymbol{r}_0 \to \boldsymbol{r}) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} d\boldsymbol{r}'\boldsymbol{F}(\boldsymbol{r}').$$

If the potential depends on the path taken between these two points there is no unique potential.

Example (relevant for homework 5)

We study a classical electron which moves in the x-direction along a surface. The force from the surface is

$$\boldsymbol{F}(x) = -F_0 \sin(\frac{2\pi x}{b})\boldsymbol{e}_1.$$

The constant b represents the distance between atoms at the surface of the material, F_0 is a constant and x is the position of the electron.

This is indeed a conservative force since it depends only on position and its **curl** is zero, that is $-\boldsymbol{\nabla} \times \boldsymbol{F} = 0$. This means that energy is conserved and the integral over the work done by the force is independent of the path taken. We will come back to this in more detail next week.

Example Continues

Using the work-energy theorem we can find the work W done when moving an electron from a position x_0 to a final position x through the integral

$$W = -\int_{x_0}^{x} \boldsymbol{F}(x')dx' = \int_{x_0}^{x} F_0 \sin(\frac{2\pi x'}{b})dx',$$

which results in

$$W = \frac{F_0 b}{2\pi}\left[\cos\left(\frac{2\pi x}{b}\right) - \cos\left(\frac{2\pi x_0}{b}\right)\right].$$

Since this is related to the change in kinetic energy we have, with v_0 being the initial velocity at a time t_0,

$$v = \pm\sqrt{\frac{2}{m}\frac{F_0 b}{2\pi}\left[\cos\left(\frac{2\pi x}{b}\right) - \cos\left(\frac{2\pi x_0}{b}\right)\right] + v_0^2}.$$

The potential energy from this example

The potential energy, due to energy conservation is

$$V(x) = V(x_0) + \frac{1}{2}mv_0^2 - \frac{1}{2}mv^2,$$

with v given by the velocity from above.

We can now, in order to find a more explicit expression for the potential energy at a given value x, define a zero level value for the potential. The potential is defined, using the work-energy theorem, as

$$V(x) = V(x_0) + \int_{x_0}^{x} (-F(x'))dx',$$

and if you recall the definition of the indefinite integral, we can rewrite this as

$$V(x) = \int (-F(x'))dx' + C,$$

where C is an undefined constant. The force is defined as the gradient of the potential, and in that case the undefined constant vanishes. The constant does not affect the force we derive from the potential.

We have then

$$V(x) = V(x_0) - \int_{x_0}^{x} \boldsymbol{F}(x')dx',$$

which results in

$$V(x) = \frac{F_0 b}{2\pi}\left[\cos\left(\frac{2\pi x}{b}\right) - \cos\left(\frac{2\pi x_0}{b}\right)\right] + V(x_0).$$

We can now define

$$\frac{F_0 b}{2\pi}\cos\left(\frac{2\pi x_0}{b}\right) = V(x_0),$$

which gives

$$V(x) = \frac{F_0 b}{2\pi}\left[\cos\left(\frac{2\pi x}{b}\right)\right].$$

Force and Potential

We have defined work as the energy resulting from a net force acting on an object (or sseveral objects), that is

$$W(\boldsymbol{r} \rightarrow \boldsymbol{r} + d\boldsymbol{r}) = \boldsymbol{F}(\boldsymbol{r})d\boldsymbol{r}.$$

If we write out this for each component we have

$$W(\boldsymbol{r} \rightarrow \boldsymbol{r} + d\boldsymbol{r}) = \boldsymbol{F}(\boldsymbol{r})d\boldsymbol{r} = F_x dx + F_y dy + F_z dz.$$

The work done from an initial position to a final one defines also the difference in potential energies

$$W(\boldsymbol{r} \rightarrow \boldsymbol{r} + d\boldsymbol{r}) = -\left[V(\boldsymbol{r} + d\boldsymbol{r}) - V(\boldsymbol{r})\right].$$

Getting to $\boldsymbol{F}(\boldsymbol{r}) = -\boldsymbol{\nabla} V(\boldsymbol{r})$

We can write out the differences in potential energies as

$$V(\boldsymbol{r} + d\boldsymbol{r}) - V(\boldsymbol{r}) = V(x+dx, y+dy, z+dz) - V(x,y,z) = dV,$$

and using the expression the differential of a multi-variable function $f(x,y,z)$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz,$$

we can write the expression for the work done as

$$W(\boldsymbol{r} \rightarrow \boldsymbol{r} + d\boldsymbol{r}) = -dV = -\left[\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right].$$

Final expression

Comparing the last equation with

$$W(\boldsymbol{r} \rightarrow \boldsymbol{r} + d\boldsymbol{r}) = F_x dx + F_y dy + F_z dz,$$

we have

$$F_x dx + F_y dy + F_z dz = -\left[\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right],$$

leading to

$$F_x = -\frac{\partial V}{\partial x},$$

and

$$F_y = -\frac{\partial V}{\partial y},$$

and

$$F_z = -\frac{\partial V}{\partial z},$$

or just

$$\boldsymbol{F} = -\frac{\partial V}{\partial x}\boldsymbol{e}_1 - \frac{\partial V}{\partial y}\boldsymbol{e}_2 - \frac{\partial V}{\partial z}\boldsymbol{e}_3 = -\boldsymbol{\nabla} V(\boldsymbol{r}).$$

And this connection is the one we wanted to show.