PHY321: Harmonic Oscillations

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Aims and Overarching Motivation

Monday. Summary on forces and conservation laws, with additional examples. Begin harmonic oscillations.

Reading suggestion: Taylor chapers 3 and 4 for the summary on forces and conservation laws. Taylor section 5.1 for start harmonic oscillations.

Wednesday. Harmonic oscillations, basic equations and formalism **Reading suggestions**: Taylor sections 5.1-5.2

Friday. Harmonic oscillations, damped motion. **Reading suggestion**: Taylor sections 5.3-5.4.

Harmonic Oscillator

The harmonic oscillator is omnipresent in physics. Although you may think of this as being related to springs, it, or an equivalent mathematical representation, appears in just about any problem where a mode is sitting near its potential energy minimum. At that point, $\partial_x V(x) = 0$, and the first non-zero term (aside from a constant) in the potential energy is that of a harmonic oscillator. In a solid, sound modes (phonons) are built on a picture of coupled harmonic oscillators, and in relativistic field theory the fundamental interactions are also built on coupled oscillators positioned infinitesimally close to one another in space. The phenomena of a resonance of an oscillator driven at a fixed frequency plays out repeatedly in atomic, nuclear and high-energy physics, when quantum mechanically the evolution of a state oscillates according to e^{-iEt} and exciting discrete quantum states has very similar mathematics as exciting discrete states of an oscillator.

Harmonic Oscillator, deriving the Equations

The potential energy for a single particle as a function of its position x can be written as a Taylor expansion about some point x_0

$$V(x) = V(x_0) + (x - x_0)\ \partial_x V(x)\big|_{x_0} + \frac{1}{2}(x - x_0)^2\ \partial_x^2 V(x)\big|_{x_0} + \frac{1}{3!}\ \partial_x^3 V(x)\big|_{x_0} + \cdots \tag{1}$$

If the position x_0 is at the minimum of the resonance, the first two non-zero terms of the potential are

$$\begin{aligned} V(x) &\approx V(x_0) + \frac{1}{2}(x - x_0)^2\ \partial_x^2 V(x)\big|_{x_0}, \qquad (2) \\ &= V(x_0) + \frac{1}{2}k(x - x_0)^2, \quad k \equiv \partial_x^2 V(x)\big|_{x_0}, \\ F &= -\partial_x V(x) = -k(x - x_0). \end{aligned}$$

Put into Newton's 2nd law (assuming $x_0 = 0$),

$$m\ddot{x} = -kx, \tag{3}$$

$$x = A\cos(\omega_0 t - \phi), \quad \omega_0 = \sqrt{k/m}. \tag{4}$$

Harmonic Oscillator, Technicalities

Here A and ϕ are arbitrary. Equivalently, one could have written this as $A\cos(\omega_0 t) + B\sin(\omega_0 t)$, or as the real part of $Ae^{i\omega_0 t}$. In this last case A could be an arbitrary complex constant. Thus, there are 2 arbitrary constants (either A and B or A and ϕ, or the real and imaginary part of one complex constant. This is the expectation for a second order differential equation, and also agrees with the physical expectation that if you know a particle's initial velocity and position you should be able to define its future motion, and that those two arbitrary conditions should translate to two arbitrary constants.

A key feature of harmonic motion is that the system repeats itself after a time $T = 1/f$, where f is the frequency, and $\omega = 2\pi f$ is the angular frequency. The period of the motion is independent of the amplitude. However, this independence is only exact when one can neglect higher terms of the potential, $x^3, x^4 \cdots$. Once can neglect these terms for sufficiently small amplitudes, and for larger amplitudes the motion is no longer purely sinusoidal, and even though the motion repeats itself, the time for repeating the motion is no longer independent of the amplitude.

One can also calculate the velocity and the kinetic energy as a function of time,

$$
\begin{aligned}
\dot{x} &= -\omega_0 A \sin(\omega_0 t - \phi), \qquad (5)\\
K &= \frac{1}{2} m \dot{x}^2 = \frac{m\omega_0^2 A^2}{2} \sin^2(\omega_0 t - \phi),\\
&= \frac{k}{2} A^2 \sin^2(\omega_0 t - \phi).
\end{aligned}
$$

Harmonic Oscillator, Total Energy

The total energy is then

$$
E = K + V = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 = \frac{1}{2} kA^2. \qquad (6)
$$

The total energy then goes as the square of the amplitude.

A pendulum is an example of a harmonic oscillator. By expanding the kinetic and potential energies for small angles find the frequency for a pendulum of length L with all the mass m centered at the end by writing the eq.s of motion in the form of a harmonic oscillator.

The potential energy and kinetic energies are (for x being the displacement)

$$
\begin{aligned}
V &= mgL(1 - \cos\theta) \approx mgL \frac{x^2}{2L^2},\\
K &= \frac{1}{2} mL^2 \dot{\theta}^2 \approx \frac{m}{2} \dot{x}^2.
\end{aligned}
$$

For small x Newton's 2nd law becomes

$$
m\ddot{x} = -\frac{mg}{L} x,
$$

and the spring constant would appear to be $k = mg/L$, which makes the frequency equal to $\omega_0 = \sqrt{g/L}$. Note that the frequency is independent of the mass.