$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$$

LIE THEORY

Classical Groups, Highest-Weight Theory and

Homogeneous Space

Thread 478

$$\begin{tikzcd}
\tilde{G} \arrow[r] \arrow[d, "\psi"'] & Spin(n) \arrow[d, "\rho"] \\
G \arrow[r, "\phi"'] & SO(n)
\end{tikzcd}$$

Due to the fact that the author's undergraduate institution did not offer courses on Lie theory, this note has been created to supplement the lacking foundation. It primarily references the first three chapters of GTM 255 Symmetry, Representations, and Invariants, which essentially covers the content of Lie theory at the undergraduate level in China, including classical groups, the highest weight theory and homogeneous space.

August 17, 2025

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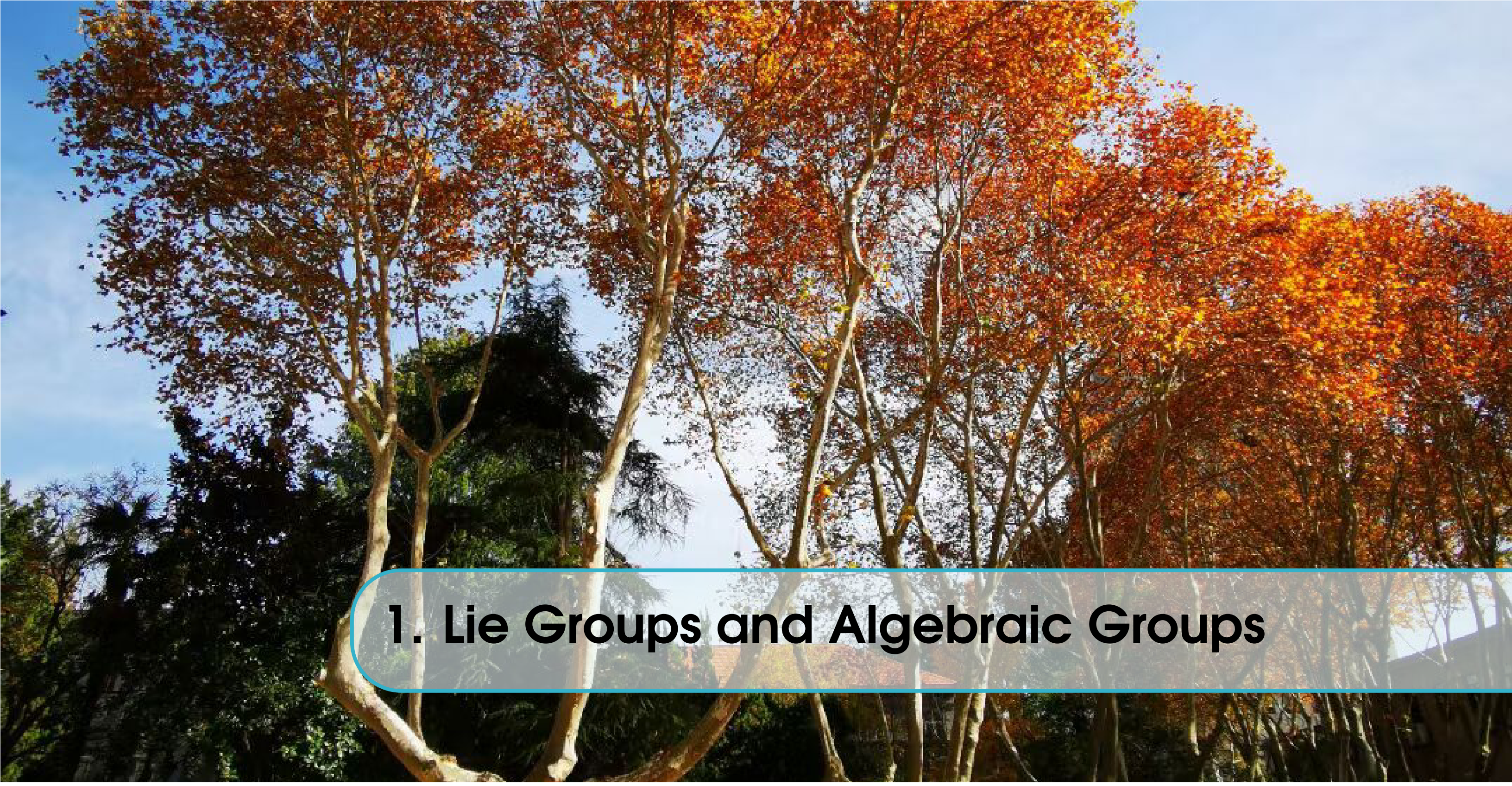


1. Lie Groups and Algebraic Groups

1.1 the Classical Groups

The classical groups are the groups of invertible linear transformations of finite dimensional vector space over the real, complex, and quaternion division algebras, together with the subgroups that preserve a volume forms, a bilinear form, or a sesquilinear form.

1.1.1 General and Special Linear Groups

Let $\mathbb{F}$ represent the field of the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$ and V be a finite dimensional vector space over $\mathbb{F}$. The set of invertible linear transformationsis denoted by $\mathbf{GL}(V)$. This set has a group structure under the composition of transformations, with identity element the identity transformation $I(x) = x$ for all $x \in V$.

The group $\mathbf{GL}(V)$ is a classical group.

Let V and W be finite dimensional vector space over $\mathbb{F}$. Let $\{v_1, v_2, \cdots, v_n\}$ and $\{w_1, w_2, \cdots, w_m\}$ be bases for V and W and $T : V \to W$ be a linear map, then

$$Tv_j = \sum_{i=1}^{m} a_{ij} w_i \quad \text{for} \quad j = 1, 2, \cdots, n \tag{1.1}$$

with $a_{ij} \in \mathbb{F}$. The numbers a_{ij} is called the matrix coefficients or elements of T with respect to the two bases, and the $m \times n$ array

$$A = [a_{ij}]_{m\times n} \tag{1.2}$$

is the matrix of T with respect to the two bases. When use the given bases to identify the elements of V and W, with the column vectors in $\mathbb{F}^n$ and $\mathbb{F}^m$, then action of T becomes multiplication with the matrix A.

Let $S : W \to U$ be another linear transformation, with U is a l dimensions vector space with the base $\{u_1, u_2, \cdots, u_l\}$ and let B is the matrix of S with respect to the bases $\{w_1, w_2, \cdots, w_m\}$ and $\{u_1, u_2, \cdots, u_l\}$. Then the matrix of $S \circ T$ with respect to the bases $\{v_1, v_2, \cdots, v_n\}$ and $\{u_1, u_2, \cdots, u_l\}$ is given BA, there the multiplication is the usual product of matrices.

We denote the space consists of all $n \times n$ matrices on $\mathbb{F}$ by $M_n(\mathbb{F})$, and the $n \times n$ identity matrix by I. Let V be a vector space over $\mathbb{F}$ with the base $\{v_1, v_2, \cdots, v_n\}$. If $T : V \to V$ is a linear map and we use $\mu(T)$ to represent the matrix of T with respect to this base. If $T, S \in \mathbf{GL}(V)$, we have $\mu(ST) = \mu(S)\mu(T)$. Furthermore, if $T \in \mathbf{GL}(V)$, then $\mu(T \circ T^{-1}) = \mu(T^{-1} \circ T) = \mu(\mathrm{Id}) = I$.

If there's a matrix $B \in M_n(\mathbb{F})$ s.t. $AB = BA = I$, then the matrix $A \in M_n(\mathbb{F})$ is called invertible matrix and we notice that the linear map $T \in \mathbf{GL}(V)$ if and only if the matrix $\mu(T)$ is invertible if and only if the determinant is non-zero.

We shall denote by $\mathbf{GL}(n, \mathbb{F})$ the group of $n \times n$ invertible matrices with coefficients in $\mathbb{F}$. Under the product of matrices, $\mathbf{GL}(n, \mathbb{F})$ is a group with the identity matrix as identity element. We note that is V is a n dimensions vector space over $\mathbb{F}$ with basis $\{v_1, v_2, \cdots, v_n\}$, then the map $\mu : \mathbf{GL}(V) \to \mathbf{GL}(n, \mathbb{F})$ corresponding to this basis is a group isomorphism . The group $\mathbf{GL}(n, \mathbb{F})$ is called the general linear group of rank n.

If $\{w_1, w_2, \cdots, w_n\}$ is another basis of V, then there's a matrix $g \in \mathbf{GL}(n, \mathbb{F})$ s.t.

$$w_j = \sum_{1 \le i \le n} g_{ij} v_i \quad \text{and} \quad v_j = \sum_{1 \le i \le n} h_{ij} w_i \quad j = 1, 2, \cdots, n, \tag{1.3}$$

with $[h_{ij}]$ the inverse matrix $[g_{ij}]$. Suppose T is a linear transformationof V into V, $A = [a_{ij}]$ is the matrix of T under the basis $\{v_1, v_2, \cdots, v_n\}$, $B = [b_{ij}]$ is another matrix under another basis $\{w_1, w_2, \cdots, w_n\}$, then

$$B = g^{-1} A g \tag{1.4}$$

is similar to the matrix A.

Special Linear Group

The special linear group $\mathbf{SL}(n, \mathbb{F})$ is the set of all the elements A satisfies $\det(A) = 1$ in $M_n(\mathbb{F})$. It's a subgroup of $\mathbf{GL}(n, \mathbb{F})$

If V is a n dimensions vector space over $\mathbb{F}$ with the basis $\{v_1, v_2, \cdots, v_n\}$ and if $\mu : \mathbf{GL}(V) \to \mathbf{GL}(n, \mathbb{F})$ is the map defined previously, then the group

$$\mu^{-1}(\mathbf{SL}(n, \mathbb{F})) = \{T \in \mathbf{GL}(V) : \det \mu(T) = 1\} \tag{1.5}$$

s independent of the choice of basis, by the change of basis formula. We denote it by $\mathbf{SL}(V)$.

1.1.2 Isometry Groups of Bilinear Forms

Let V be an n-dimensional vector space over $\mathbb{F}$. A bilinear $B : V \times V \to \mathbb{F}$ is called a bilinear form. We denote by $\mathbf{O}(V, B)$ or $\mathbf{O}(B)$ the set of all $g \in \mathbf{GL}(V)$ s.t. for all $v, w \in V$, there is $B(gv, gw) = B(v, w)$. We note $\mathbf{O}(V, B)$ is a subgroup of $\mathbf{GL}(V)$, it's called the isomorohic group of the form B.

Let $\{v_1, v_2, \cdots, v_n\}$ is a basis of V and $\Gamma \in M_n(\mathbb{F})$ is the matrix with the element $\Gamma_{ij} = B(v_i, v_j)$. Then if denote A^{T} the transposed matrix, then $g \in \mathbf{O}(B)$ if and only if

$$\Gamma = A^{\mathrm{T}} \Gamma A. \tag{1.6}$$

Orthogonal Groups

Let $\mathbf{O}(n,\mathbb{F})$ denote the set of all the $g \in \mathbf{GL}(n,\mathbb{F})$ s.t. $gg^{\mathrm{T}} = I$. It's obvious that $\mathbf{O}(n,\mathbb{F})$ is a subgroup of $\mathbf{GL}(n,\mathbb{F})$ and is called the orthogonal group of $n \times n$ matrices over $\mathbb{F}$. If $\mathbb{F} = \mathbb{R}$, we introduce the indefinite orthogonal group $\mathbf{O}(p,q)$ with $p+q=n, p,q \in \mathbb{N}$.

Let

$$I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \tag{1.7}$$

and define

$$\mathbf{O}(p,q) = \{g \in M_n(\mathbb{R}) : g^{\mathrm{T}} I_{p,q} g = I_{p,q}\}. \tag{1.8}$$

We note that $\mathbf{O}(n,0) = \mathbf{O}(0,n) = \mathbf{O}(n,\mathbb{R})$. If

$$s = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}_{n \times n} \tag{1.9}$$

then $s = s^{-1} = s^{\mathrm{T}}$ and $sI_{p,q}s^{-1} = sI_{p,q}s = I_{q,p}$. Thus there's an isomorphism

$$\varphi : \mathbf{O}(p,q) \to \mathbf{O}(q,p) \tag{1.10}$$

given by $\varphi(g) = sgs^{-1}$.

Definition 1.1.1 Let V be a vector space over $\mathbb{R}$ and M is a symmetric bilinear form on V. The form M is said to be positive definite if $M(v,v) > 0$ for all non-zero $v \in V$.

Lemma 1.1.1 Let V be a n dimensions vector space over $\mathbb{F}$ and B be a symmetric nondegenerate bilinear form over $\mathbb{F}$.

- 1.If $\mathbb{F} = \mathbb{C}$ then there's a basis $\{v_1, v_2, \cdots, v_n\}$ s.t. $B(v_i, v_j) = \delta_{ij}$
- 2.If $\mathbb{F} = \mathbb{R}$, then there's integers $p, q \geq 0$ satisfy $p+q=n$ and a basis $\{v_1, v_2, \cdots, v_n\}$ of V s.t. $B(v_i, v_j) = \varepsilon_i \delta_{ij}$, where $\varepsilon_i = 1$ for $i \leq p$ holds and $\varepsilon_i = -1$ for $i > p$. Furthermore, if we have another basis then the corresponding integers (p,q) is the same.

R The basis of V in (2) is called a pseudo-orthogonal basis relative to B and the number $p-q$ is called the signature of the form. A form is positive definite if and only if it's signature is n. In this case, a pseudo-orthogonal basis is an orthogonal basis in usual sense.

Proposition 1.1.2 Let B be a nondegenerate symmetric bilinear form on a n dimensional vector space V over $\mathbb{F}$

- 1.Let $\mathbb{F} = \mathbb{C}$. If $\{v_1, v_2, \cdots, v_n\}$ is an orthonormal basis for V with respect to B, then $\mu : \mathbf{O}(V,B) \to \mathbf{O}(n,\mathbb{F})$ defines a group isomorphism.
- Let $\mathbb{F} = \mathbb{R}$. If B has signature $(p, n-p)$ and $\{v_1, v_2, \cdots, v_n\}$ is a pseudo-orthogonal basis of V, then $\mu : \mathbf{O}(V,B) \to \mathbf{O}(p, n-p)$ is a group isomorphism.

Here $\mu(g)$ is the matrix of g with respect to the given basis for $g \in \mathbf{GL}(V)$.

The special orthogonal group $\mathbb{F}$ is the subgroup

$$\mathbf{SO}(n,\mathbb{F}) = \mathbf{O}(n,\mathbb{F}) \cap \mathbf{SL}(n,\mathbb{F}) \tag{1.11}$$

of $\mathbf{O}(n,\mathbb{F})$. The indefinite special orthogonal groups are groups

$$\mathbf{SO}(p,q) = \mathbf{O}(p,q) \cap \mathbf{SL}(p+q,R) \tag{1.12}$$

Symplectic Group

We set $J = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$ with I is $n \times n$ identity matrix. The symplectic group of rank n over $\mathbb{F}$ is defined to be

$$\mathbf{Sp}(n,\mathbb{F}) = \{g \in M_{2n}(\mathbb{F}) : g^{\mathrm{T}} J g = J\}. \tag{1.13}$$

$\mathbf{Sp}(n,\mathbb{F})$ is a subgroup of $\mathbf{GL}(2n,\mathbb{F})$.

A bilinear B is said to be skew-symmetric if $B(v,w) = -B(w,v)$. If B is skew-symmetric and nondegenerate, then $m = \dim V$ must be even.

Lemma 1.1.3 Let V be a $2n$-dimensional vector space over $\mathbb{F}$, and B is a nondegenerate skew-symmetric bilinear form. Then there's a basis $\{v_1, v_2, \cdots, v_{2n}\}$ s.t. the $|B(v_i, v_j)|$ equals J.

Proposition 1.1.4 Let V be a $2n$-dimensional vector space, and let B be a nondegenerate skew-symmetric bilinear form on V. Fix a B-symplectic basis of V and let $\mu(g)$ be the matrix of g with respect to this basis for $g \in \mathbf{GL}(V)$. Then $\mu : \mathbf{O}(V,B) \to \mathbf{Sp}(n,\mathbb{F})$ is a group isomorphism.

1.1.3 Unitary Groups

We denote $A^* = \overline{A^{\mathrm{T}}}$.

The unitary group of rank n is the group

$$\mathbf{U}(n) = \{g \in M_n(\mathbb{C}) : g^* g = I\}. \tag{1.14}$$

The special unitary group is $\mathbf{SU}(n) = \mathbf{U}(n) \cap \mathbf{SL}(n,\mathbb{C})$. We define the indefinite unitary group of signature (p,q) to be

$$\mathbf{U}(p,q) = \{g \in M_n(\mathbb{C}) : g^* I_{p,q} g = I_{p,q}\}. \tag{1.15}$$

The special indefinite unitary group of signature (p,q) is $\mathbf{SU}(p,q) = \mathbf{U}(p,q) \cap \mathbf{SL}(n,\mathbb{C})$.

We will now obtain a coordinate-free description of these groups.

Definition 1.1.2 — Hermitian Form. Let V be a n dimensional vector space on $\mathbb{C}$. A $\mathbb{R}$ bilinear map $B : V \times V \to \mathbb{C}$ is said to be a Hermitian form if it satisfies:

- 1.$B(av,w) = aB(v,w)$ for all $a \in \mathbb{C}$ and $v,w \in V$.
- 2.$B(w,v) = \overline{B(v,w)}$ for all $v,w \in V$.

We define the $\mathbf{U}(V,B)$ or $\mathbf{U}(B)$ is the group consists of all the elements $g \in \mathbf{GL}(V)$ satisfy $B(gv,gw) = B(v,w)$ for all $v,w \in V$. We call $\mathbf{U}(B)$ is the unitary group of B.

Lemma 1.1.5 Let V is an n-dimensionalvector space over $\mathbb{C}$ and B be a nondegenerate Hermitian form on V. Then there's an integer p satisfies $n \geq p \geq 0$ and a basis $\{v_1, v_2, \cdots, v_n\}$ of V, s.t. $B(v_i, v_j) = \varepsilon_i \delta_{ij}$, with $\varepsilon_i = 1$ for $i \leq p$ and $\varepsilon_i = -1$ for $i > p$. The number depends only on B and not on the choice of basis.

If V is an n-dimensionalvector space over $\mathbb{C}$ and B is a nondegenerate Hermitian form on V, then the basis above is called a pseudo-orthogonal basis. The pair $(p, n-p)$ will be called the signature of B

Proposition 1.1.6 Let V be a finite vector space over $\mathbb{C}$ and B is a nondegenerate Hermitian form with the signature (p, q). Let $\mu(g)$ be the matrix of g with respect to the pseudo-orthogonal basis relative to B in V, for $g \in \mathbf{GL}(V)$. Then $\mu : \mathbf{U}(V, B) \to \mathbf{U}(p, q)$ is a group isomorphism.

1.1.4 Quaternionic Groups

The Quaternion Algebra $\mathbb{H}$

The quaternion algebra, denoted by $\mathbb{H}$, is a 4-dimensional division algebra over the real numbers $\mathbb{R}$. Its standard basis is $\{1, i, j, k\}$, satisfying the multiplication rules:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1.16}$$

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j \tag{1.17}$$

An element $q \in \mathbb{H}$ can be written as $q = a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$.

Alternatively, any quaternion q can be uniquely expressed as $q = z_1 + z_2 j$, where $z_1, z_2 \in \mathbb{C}$. This representation is particularly useful for connecting with complex linear spaces.

The quaternion conjugation is defined as $q^* = a - bi - cj - dk$. For $q = z_1 + z_2 j$, the conjugate is $q^* = \overline{z_1} - z_2 j$.

Complex Matrix Representation of Quaternions

To embed quaternion groups into the more familiar complex matrix groups, we construct a 2×2 complex matrix representation. The basis elements $\{1, i, j, k\}$ are mapped as follows:

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{1.18}$$

$$j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{1.19}$$

By linear extension, a quaternion $q = x + yj$ (where $x, y \in \mathbb{C}$) is represented by the matrix:

$$q \mapsto \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \tag{1.20}$$

This set of 2×2 complex matrices forms a 4-dimensional division algebra over $\mathbb{R}$ that is isomorphic to $\mathbb{H}$.

The Quaternionic Linear Group $\mathbf{GL}(n, \mathbb{H})$

We consider $\mathbb{H}^n$ as a right n-dimensional $\mathbb{H}$-module. The group of invertible $n \times n$ matrices over the quaternions forms the **quaternionic general linear group**, denoted $\mathbf{GL}(n, \mathbb{H})$.

To relate this to complex matrix groups, we identify the n-dimensional quaternion vector space $\mathbb{H}^n$ with the $2n$-dimensional complex vector space $\mathbb{C}^{2n}$. A vector $v \in \mathbb{H}^n$ can be written as $v = \mathbf{x} + \mathbf{y}j$, where $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. We establish the correspondence:

$$v = \mathbf{x} + \mathbf{y}j \quad \longleftrightarrow \quad \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathbb{C}^{2n} \tag{1.21}$$

The action of an $n \times n$ quaternion matrix $C = A + Bj$ (where $A, B \in M_n(\mathbb{C})$) on v is:

$$Cv = (A + Bj)(\mathbf{x} + \mathbf{y}j) = (A\mathbf{x} - B\bar{\mathbf{y}}) + (A\mathbf{y} + B\bar{\mathbf{x}})j \tag{1.22}$$

In the $\mathbb{C}^{2n}$ model, this linear transformation corresponds to the $2n \times 2n$ complex matrix:

$$\begin{pmatrix} A & -B \\ B & \bar{A} \end{pmatrix} \in M_{2n}(\mathbb{C}) \tag{1.23}$$

This matrix satisfies a concise condition. Define the standard symplectic matrix $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. A matrix $T \in M_{2n}(\mathbb{C})$ is the complex representation of a quaternionic linear map if and only if it satisfies:

$$TJ = J\bar{T} \tag{1.24}$$

Thus, we can identify $\mathbf{GL}(n, \mathbb{H})$ with a subgroup of $\mathbf{GL}(2n, \mathbb{C})$:

$$\mathbf{GL}(n, \mathbb{H}) \cong \{g \in \mathbf{GL}(2n, \mathbb{C}) \mid gJ = J\bar{g}\} \tag{1.25}$$

This subgroup is an example of a **real form**, a concept of fundamental importance in Lie theory.

Other Quaternionic Groups

- **Quaternionic Unitary Group $\mathbf{Sp}(p, q)$:** This is the group that preserves the standard quaternionic Hermitian form $B(w, z) = w^* I_{p,q} z$, where $I_{p,q}$ is a diagonal matrix with p ones and q negative ones. Its definition is:

$$\mathbf{Sp}(p, q) = \{g \in \mathbf{GL}(p + q, \mathbb{H}) : g^* I_{p,q} g = I_{p,q}\} \tag{1.26}$$

  When $q = 0$, we get the compact group $\mathbf{Sp}(n) = \mathbf{U}(n, \mathbb{H})$.
- **The Group $\mathbf{SO}^*(2n)$:** This group is defined as a real form of the complex special orthogonal group $\mathbf{SO}(2n, \mathbb{C})$. It consists of matrices satisfying a specific conjugation condition:

$$SO^*(2n) = \{g \in \mathbf{SO}(2n, \mathbb{C}) : \theta(\bar{g}) = g\}, \quad \text{where } \theta(g) = -JgJ \tag{1.27}$$

  It can be viewed as the isomorohic group of a certain quaternionic skew-Hermitian form.

1.2 the Classical Lie Algebras

Let V is a vector space on $\mathbb{F}$, and $\operatorname{End} V$ denote the algebra of $\mathbb{F}$-linear maps of V into V. If $X, Y \in \operatorname{End} V$, then we set $[X,Y] = XY - YX$. This defines a new product on $\operatorname{End} F$ that satisfies the following two properties:

- (1)$[X,Y] = -[Y,X]$ for all X, Y (skew symmetry).
- (2)$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$ (Jacobi identity).

Definition 1.2.1 Let $\mathfrak{g}$ be a vector space on $\mathbb{F}$, and there's a bilinear map $X, Y \mapsto [X,Y]$ of $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$ is said to be a Lie algebra if conditions (1) and (2) hold.

In particular, $\operatorname{End} V$ is a Lie algebraunder the binary operation $[X,Y] = XY - YX$. Condition (2) says that for fixed X, the linear transformation$Y \mapsto [X,Y]$ is a derivation of the nonassociative algebra $(\mathfrak{g}, [\cdot,\cdot])$.

If $\mathfrak{g}$ is a Lie algebraand $\mathfrak{h}$ is a subspace s.t. $X, Y \in \mathfrak{h}$ implies that $[X,Y]$ belongs to $\mathfrak{h}$, then $\mathfrak{h}$ is aLie algebraunder the restriction of $[\cdot,\cdot]$. We say $\mathfrak{h}$ is a Lie subalgebra.

Suppose $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebraon $\mathbb{F}$. A Lie algebrahomomorphism of $\mathfrak{g}$ into $\mathfrak{h}$ is a map $T : \mathfrak{g} \to \mathfrak{h}$, s.t. $T[X,Y] = [TX, TY]$ for all $X, Y \in \mathfrak{g}$. If a Lie algebrahomomorphism is bijective, it's an isomorphism.

1.2.1 General and Special Linear Lie Algebras

If V is a vector space of $\mathbb{F}$, we write as $\mathfrak{gl}(V)$ for $\operatorname{End} V$ when it is viewed as a Lie algebra under $[X,Y] = XY - YX$. We write as $\mathfrak{gl}(n,\mathbb{F})$ for the Lie algebraof $M_n(\mathbb{F})$ under the matrix commutator bracket.

If $\dim V = n$ and fix a basis of V, then there's an Lie algebraisomorphism of linear transformationand it's matrix $\mathfrak{gl}(V) \simeq \mathfrak{gl}(n,\mathbb{F})$. These Lie algebraare called general Lie algebra.

If $A = [a_{ij}] \in M_n(\mathbb{F})$ and its trace is $\operatorname{tr} A = \sum_i a_{ii}$. Note that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, and if A is a matrix of a linear transformation$T \in \operatorname{End} V$ with respect to some basis, then $\operatorname{tr} A$ is independent of the choice of basis. We write $\operatorname{tr} T = \operatorname{tr} A$ and define:

$$\mathfrak{sl}(V) = \{T \in \operatorname{End} V : \operatorname{tr} T = 0\}. \tag{1.28}$$

We can know $\mathfrak{sl}(V)$ is a Lie subalgebra of $\mathfrak{gl}(V)$. Fix a basis for V, we can identify this Lie algebrawith

$$\mathfrak{sl}(n,\mathbb{F}) = \{A \in \mathfrak{gl}(n,\mathbb{F}) : \operatorname{tr} A = 0\}. \tag{1.29}$$

These Lie algebras will be called the special linear Lie algebras.

1.2.2 Lie Algebras Associated with Bilinear Forms

Let V be a vector space over $\mathbb{F}$ and let $B : V \times V \to \mathbb{F}$ be a bilinear map, we define

$$\mathfrak{so}(V,B) = \{X \in \operatorname{End} V : B(Xv,w) = -B(v,Xw)\}. \tag{1.30}$$

Thus $\mathfrak{so}(V,B)$ consists of the skew-symmetric linear transformationwith respect to the form B, and it's obvious a linear subspace of $\mathfrak{gl}(V)$.

If $X, Y \in \mathfrak{so}(V,B)$, then

$$B(XYv,w) = -B(Yv,Xw) = B(v,YXw). \tag{1.31}$$

It follows that $B([X,Y]v,w) = -B(v,[X,Y]w)$, and hence $\mathfrak{so}(V,B)$ is a Lie subalgebra of $\mathfrak{gl}(V)$.

Suppose V is finite dimensional and fix a basis $\{v_1, v_2, \cdots, v_n\}$ for V, and let Γ is the $n\times n$ matrix with the elements $\Gamma_{ij} = B(v_i, v_j)$. We know $T \in \mathfrak{so}(V,B)$ if and only if its matrix A with respect to this basis satisfies

$$A^{\mathrm{T}}\Gamma + \Gamma A = 0. \tag{1.32}$$

When B is nondegenerate, Γ is invertible and it can be written as the form $A^{\mathrm{T}} = -\Gamma A\Gamma^{-1}$. In particular, it implies that for all $T \in \mathfrak{so}(V,B)$, there's $\operatorname{tr} T = 0$.

Orthogonal Lie Algebras

Take $V = \mathbb{F}^n$ and the bilinear form B with the matrix $\Gamma = I_n$ relative to the standard basis of $\mathbb{F}^n$. Define

$$\mathfrak{so}(n,\mathbb{F}) = \{X \in M_n(\mathbb{F}) : X^{\mathrm{T}} = -X\}. \tag{1.33}$$

Since B is nondegenerate, $\mathfrak{so}(n,\mathbb{F})$ is a Lie subalgebra of $\mathfrak{sl}(n,\mathbb{F})$

When $\mathbb{F} = \mathbb{R}$, we take integers $p, q \geq 0$ s.t.$p+q = n$ and let B is the bilinear form whose matrix is $I_{p,q}$ relative tothe standard basis. Define

$$\mathfrak{so}(p,q) = \{X \in M_n(\mathbb{R}) : X^{\mathrm{T}} I_{p,q} + I_{p,q}X = 0\}. \tag{1.34}$$

Since B is nondegenerate, $\mathfrak{so}(p,q)$ is a Lie subalgebra of $\mathfrak{sl}(n,\mathbb{R})$.

Let B is a nondegenerate symmetric bilinear form on a n-dimensionalvector space V over $\mathbb{F}$. Let $\{v_1, v_2, \cdots, v_n\}$ be a basis for V which is orthogonal (when $\mathbb{F} = \mathbb{C}$) or pseudo-orthogonal (when $\mathbb{F} = \mathbb{R}$) relative to$B$. Let $\mu(T)$ be the matrix of $T \in \operatorname{End} V$relative tothis basis. When $\mathbb{F} = \mathbb{C}$, then μ defines a Lie algebraisomorphism of $\mathfrak{so}(V,B)$ into $\mathfrak{so}(n,\mathbb{C})$. When $\mathbb{F} = \mathbb{R}$ and has a signature (p,q), then μ defines a Lie algebraisomorphism of $\mathfrak{so}(V,B)$ into $\mathfrak{so}(p,q)$.

Symplectic Lie Algebras

Let J be the $2n \times 2n$ skew-symmetric matrix as Section 1.1.2. We define

$$\mathfrak{sp}(n,\mathbb{F}) = \{X \in M_{2n}(\mathbb{F}) : X^{\mathrm{T}}J + JX = 0\}. \tag{1.35}$$

This subspace of $\mathfrak{gl}(n,\mathbb{F})$ is a Lie subalgebra that we call it as the symplectic Lie algebraof rank n.

Let B be a nondegenerate skew-symmetric bilinear form on the $2n$-dimensional vector space V over $\mathbb{F}$, and $\{v_1, v_2, \cdots, v_{2n}\}$ be a B-symplectic basis of V. The map μ which assigns to an endomorphism of V defines an isomorphism of the Lie algebra of the isomorohic group of B into $\mathfrak{sp}(n,\mathbb{F})$.

1.2.3 Unitary Lie Algebras

Let p, q be integers satisfy $p+q = n$ and $I_{p,q}$ be the $n\times n$ matrix from Section 1.1.2. We define

$$\mathfrak{u}(p,q) = \{X \in M_n(\mathbb{C}) : X^* I_{p,q} + I_{p,q}X = 0\} \tag{1.36}$$

We can check it is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$ and we define $\mathfrak{su}(p,q) = \mathfrak{u}(p,q) \cap \mathfrak{sl}(n,\mathbb{C})$. Let V is a vector space over $\mathbb{C}$ and B is a nondegenerate Hermitian form, we define

$$\mathfrak{u}(V,B) = \{T \in \operatorname{End}_{\mathbb{C}} T : B(Tv,w) = -B(v,Tw) \text{ for all } v,w \in V\}. \tag{1.37}$$

R [On the Condition for Skew-Adjointness] It is crucial to be precise about the properties of the linear operator T and the form B. Since B is a sesquilinear form, it is not linear in its second argument, but rather conjugate-linear. For the defining condition $B(Tv,w) = -B(v,Tw)$ to be well-posed and consistent, the operator T must be **$\mathbb{C}$-linear**.

The condition for an operator to be skew-adjoint with respect to B is that its adjoint $T^\dagger$ satisfies $T^\dagger = -T$. The defining property of the adjoint is $B(Tv,w) = B(v,T^\dagger w)$. For a skew-Hermitian (skew-adjoint) operator, this becomes $B(Tv,w) = -B(v,Tw)$. This is precisely the condition used to define the Lie algebra of the unitary group $\mathbf{U}(V,B)$, whose elements are skew-Hermitian operators. The key point is that T is a $\mathbb{C}$-linear map, a fact essential for the rigor of the definition.

We set $\mathfrak{su}(V,B) = \mathfrak{u}(V,B) \cap \mathfrak{sl}(V)$. If the signature of B is (p,q) and if $\{v_1, v_2, \cdots, v_n\}$ is a pseudo-orthogonal basis of V relative to$B$, then the assignment $T \mapsto \mu(T)$ defines a Lie algebraisomorphism of $\mathfrak{u}(V,B)$ into $\mathfrak{u}(p,q)$ and of $\mathfrak{su}(V,B)$ into $\mathfrak{su}(p,q)$.

1.2.4 Quaternionic Lie Algebras

Quaternionic General and Special Linear Lie ALgebras

Consider the $n \times n$ matrices over the quaternions with the usual matrix commutator. We call this Lie algebraas $\mathfrak{gl}(n,\mathbb{H})$, and seem it as the Lie algebraover $\mathbb{R}$.

We can identity $\mathbb{H}^n$ and $\mathbb{C}^{2n}$ using one of the isomorphic copies of $\mathbb{C}$ in $\mathbb{H}$. Define

$$\mathfrak{sl}(n,\mathbb{H}) = \{X \in \mathfrak{gl}(n,,\mathbb{H}) : \operatorname{tr} X = 0\}. \tag{1.38}$$

Then $\mathfrak{sl}(n,\mathbb{H})$ is the real Lie algebrawhich is usually denoted by $\mathfrak{su}^*(2n)$.

Quaternionic Unitary Lie Algebras

For the nonnegative integers p,q satisfy $n = p+q$, we define

$$\mathfrak{sp}(p,q) = \{X \in \mathfrak{gl}(n,\mathbb{H}) : X^* I_{p,q} + I_{p,q} X = 0\}. \tag{1.39}$$

$\mathfrak{sp}(p,q)$ is a real Lie subalgebra of $\mathfrak{g}(n,\mathbb{H})$, $\mathfrak{sp}(p,q)$ is defined by the matrix $X \in M_n(\mathbb{H})$ satisfy

$$B(Xx,y) = -B(x,X^*y) \text{ for all } x,y \in \mathbb{H}^n. \tag{1.40}$$

the Lie Algebra $\mathfrak{so}^*(2n)$

Let the automorphism θ of $M_{2n}(\mathbb{C})$ be as defined in Section 1.1.4. Define

$$\mathfrak{so}^*(2n) = \{X \in \mathfrak{so}(2n,\mathbb{C}) : \theta(\overline{X}) = X\}. \tag{1.41}$$

This real vector space of $\mathfrak{so}(2n,\mathbb{C})$ is a real Lie subalgebra of $\mathfrak{so}(2n,\mathbb{C})$. If identify $\mathbb{C}^{2n}$ and $\mathbb{H}^n$, and the quaternionic skew-Hermitian $C(x,y)$ is defined as above, then $\mathfrak{so}^*(2n)$ consists of the matrix $X \in M_n(\mathbb{H})$ which satisfy

$$C(Xx,y) = -C(x,X^*y) \text{ for all } x,y \in \mathbb{H}^n. \tag{1.42}$$

1.2.5 Lie Algebra Associated with Classical Groups

This passage from a Lie group to a Lie algebra, which is fundamental to Lie theory, arises by differentiating the defining equations for the group.

In short, eachclassical group is a subgroup of $\mathbf{GL}(V)$ and it's defined by a pair of algebraic equations $\mathcal{R}$. The corresponding Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ is determined by taking differentiable curves $\sigma : (-\varepsilon, \varepsilon) \to \mathbf{GL}(v)$ such that $\sigma(0) = I$ and $\sigma(t)$ satisfies the equations defining G. Then $\sigma'(0) \in \mathfrak{g}$ and all elements of $\mathfrak{g}$ can be obtained in this way. This why $\mathfrak{g}$ is called the infinitesimal form of G.

■ **Example 1.1** If G is the subgroup $\mathbf{O}(V, B)$ of $\mathbf{GL}(V)$ defined by the bilinear form B, then the curve σ must satisfies $B(\sigma(t)v, \sigma(t)w) = B(v, w)$ for all $v, w \in V$. If we differentiate them we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}\,t} B(\sigma(t)v, \sigma(t)w)\bigg|_{t=0} = B(\sigma'(0)v, \sigma(0)w) + B(\sigma(0)v, \sigma'(0)w) \tag{1.43}$$

for all $v, w \in V$. Since $\sigma(0) = I$, we see that $\sigma'(0) \in \mathfrak{so}(V, B)$. ■

1.3 Closed Subgroups of $\mathbf{GL}(n, \mathbb{R})$

In this section, we endow the classical groups with topological and differential structures, allowing us to study them as Lie groups. The core idea is to transition from the group (a nonlinear object) to its tangent space at the identity—the Lie algebra (a linear object)—via the matrix exponential map.

1.3.1 Topological Groups

Definition 1.3.1 — Topological Group. A group G is called a **Topological Group** if it is endowed with a Hausdorff topology such that its group operations:

- Multiplication map: $G \times G \to G$, $(g, h) \mapsto gh$
- Inversion map: $G \to G$, $g \mapsto g^{-1}$

are continuous.

■ **Example 1.2** $\mathbf{GL}(n, \mathbb{R})$ is a topological group. Its topology comes from its status as an open subset of $M_n(\mathbb{R})$, namely $\{X \in M_n(\mathbb{R}) \mid \det(X) \neq 0\}$. Matrix multiplication is clearly continuous, and the inversion map is also continuous by Cramer's rule. ■

Definition 1.3.2 — Topological Subgroup. If H is a subgroup of a topological group G, it is a topological subgroup under the relative topology. If H is also a closed set in G, it is called a **closed subgroup**.

All classical groups, such as $\mathbf{SL}(n, \mathbb{R})$, $\mathbf{O}(p, q)$, and $\mathbf{Sp}(n, \mathbb{R})$, are defined by polynomial or continuous equations in their matrix entries. Consequently, they are all closed subgroups of $\mathbf{GL}(n, \mathbb{F})$.

1.3.2 The Exponential Map

We can define a norm on $M_n(\mathbb{R})$ by $||X|| = \sqrt{\mathrm{tr}(XX^{\mathrm{T}})}$, making it a complete normed vector space.

Definition 1.3.3 — Matrix Exponential. For any matrix $X \in M_n(\mathbb{R})$, the **exponential map** is defined by the convergent power series:

$$\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!} = I + X + \frac{1}{2}X^2 + \dots \tag{1.44}$$

Similarly, for a matrix with $||X|| < 1$, the **logarithm map** is defined by:

$$\log(I + X) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{X^k}{k} = X - \frac{1}{2}X^2 + \frac{1}{3}X^3 - \dots \tag{1.45}$$

Key properties of the exponential map include:

- If X and Y commute (i.e., $XY = YX$), then $\exp(X + Y) = \exp(X)\exp(Y)$.
- For any X, $\exp(X)$ is always invertible, with $(\exp(X))^{-1} = \exp(-X)$. Thus, exp maps $M_n(\mathbb{R})$ to $\mathbf{GL}(n, \mathbb{R})$.
- For any X, the map $t \mapsto \exp(tX)$ is a continuous group homomorphism from $(\mathbb{R}, +)$ to $\mathbf{GL}(n, \mathbb{R})$, called the **one-parameter subgroup** generated by X.
- $\det(\exp(X)) = e^{\mathrm{tr}(X)}$.
- The exponential map is a diffeomorphism between a neighborhood of the origin in $M_n(\mathbb{R})$ and a neighborhood of the identity in $\mathbf{GL}(n, \mathbb{R})$.

Theorem 1.3.1 Any continuous homomorphism $\varphi : \mathbb{R} \to \mathbf{GL}(n, \mathbb{R})$ can be uniquely written in the form $\varphi(t) = \exp(tX)$ for some $X \in M_n(\mathbb{R})$.

1.3.3 Lie Algebra of a Closed Subgroup

We can now provide the central definition of a Lie algebra.

Definition 1.3.4 — Lie Algebra of a Closed Subgroup. Let G be a closed subgroup of $\mathbf{GL}(n, \mathbb{R})$. The **Lie Algebra** of G, denoted $\mathfrak{g}$ or $\mathfrak{g}$, is defined as:

$$\mathfrak{g} = \{X \in M_n(\mathbb{R}) \mid \exp(tX) \in G \text{ for all } t \in \mathbb{R}\} \tag{1.46}$$

Intuitively, the Lie algebra consists of all matrices that generate one-parameter subgroups lying within G.

R [Equivalence of Analytic and Algebraic Definitions] It is a cornerstone of Lie theory that for linear algebraic groups, this "analytic" definition of the Lie algebra is equivalent to the "algebraic" definition that can be formulated in the language of algebraic geometry (e.g., as the space of derivations on the ring of regular functions). This powerful theorem connects the geometric and topological structure of the group with its purely algebraic properties, providing two different yet complementary perspectives on the same fundamental object. This equivalence will be essential for bridging results from analysis and algebra.

The most important feature of a Lie algebra is its closure under the Lie bracket.

Theorem 1.3.2 If G is a closed subgroup of $\mathbf{GL}(n,\mathbb{R})$, then $\mathfrak{g} = \mathfrak{g}$ is a Lie subalgebra of $M_n(\mathbb{R})$. This means:
1. $\mathfrak{g}$ is a real vector subspace.
2. If $X, Y \in \mathfrak{g}$, then their Lie bracket $[X,Y] = XY - YX$ is also in $\mathfrak{g}$.

The proof of this theorem relies on the famous **Lie-Trotter Product Formulas**, which are a consequence of the Baker-Campbell-Hausdorff formula:

Proposition 1.3.3 — Lie-Trotter Formulas. For any $X, Y \in M_n(\mathbb{R})$:

$$\exp(X+Y) = \lim_{k\to\infty}\left(\exp\left(\frac{X}{k}\right)\exp\left(\frac{Y}{k}\right)\right)^k \tag{1.47}$$

$$\exp([X,Y]) = \lim_{k\to\infty}\left(\exp\left(\frac{X}{k}\right)\exp\left(\frac{Y}{k}\right)\exp\left(-\frac{X}{k}\right)\exp\left(-\frac{Y}{k}\right)\right)^{k^2} \tag{1.48}$$

Since G is a closed subgroup, if $X, Y \in \mathfrak{g}$, then each term on the right-hand side is in G, and their limit must also be in G. The properties of the exponential map then imply that $X+Y$ and $[X,Y]$ are in $\mathfrak{g}$.

1.3.4 Lie Algebras of the Classical Groups

Using the definition above, we can compute the Lie algebras of the classical groups by differentiating the groups' defining equations.

■ **Example 1.3 — Lie algebra of $\mathbf{SL}(n,\mathbb{R})$.** A matrix X belongs to $\mathfrak{sl}(n,\mathbb{R})$ if and only if $\exp(tX) \in \mathbf{SL}(n,\mathbb{R})$ for all $t \in \mathbb{R}$. This means $\det(\exp(tX)) = 1$.

$$\det(\exp(tX)) = e^{\operatorname{tr}(tX)} = e^{t\cdot\operatorname{tr}(X)} = 1 \quad \forall t \in \mathbb{R} \tag{1.49}$$

This holds only if $\operatorname{tr}(X) = 0$. Therefore:

$$\mathfrak{sl}(n,\mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid \operatorname{tr}(X) = 0\} \tag{1.50}$$

■

■ **Example 1.4 — Lie algebra of $\mathbf{U}(p,q)$.** A matrix X belongs to $\mathfrak{u}(p,q)$ if and only if $\exp(tX) \in \mathbf{U}(p,q)$ for all $t \in \mathbb{R}$. The defining equation is $g^* I_{p,q} g = I_{p,q}$.

$$(\exp(tX))^* I_{p,q} \exp(tX) = I_{p,q} \tag{1.51}$$

Noting that $(\exp(tX))^* = \exp(tX^*)$, we differentiate both sides with respect to t and evaluate at $t = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\exp(tX^*) I_{p,q} \exp(tX)\right)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}\,t} I_{p,q}\Big|_{t=0} \tag{1.52}$$

$$(X^*\exp(tX^*))|_{t=0} \cdot I_{p,q} \cdot \exp(tX)|_{t=0} + \exp(tX^*)|_{t=0} \cdot I_{p,q} \cdot (X\exp(tX))|_{t=0} = 0 \tag{1.53}$$

$$X^* I_{p,q} I + I_{p,q} X = 0 \tag{1.54}$$

This yields the defining condition for the Lie algebra:

$$\mathfrak{u}(p, q) = \{X \in M_{p+q}(\mathbb{C}) \mid X^* I_{p,q} + I_{p,q} X = 0\} \tag{1.55}$$

■

Similar methods can be used to derive the Lie algebras for all classical groups, and the results agree with the algebraic definitions given in §1.2.

1.3.5 Exponential Coordinates and Lie Groups

Theorem 1.3.4 Let G be a closed subgroup of $\mathbf{GL}(n, \mathbb{R})$. There exists a neighborhood V of the origin in $\mathfrak{g}$ and a neighborhood Ω of the identity in G such that the exponential map $\exp : V \to \Omega$ is a diffeomorphism.

This theorem implies that, locally, we can parameterize the Lie group (a manifold) using coordinates from its Lie algebra (a vector space). This is the foundation of the definition of a Lie group.

Definition 1.3.5 — Lie Group. A topological group G is a **Lie Group** if it also has the structure of a smooth manifold, such that its multiplication and inversion maps are smooth (C^∞).

Theorem 1.3.5 Any closed subgroup of $\mathbf{GL}(n, \mathbb{R})$ has a unique real-analytic Lie group structure.

1.3.6 Differentials of Homomorphisms

Homomorphisms between Lie groups correspond to homomorphisms between their Lie algebras.

Proposition 1.3.6 Let $G \subseteq \mathbf{GL}(n, \mathbb{R})$ and $H \subseteq \mathbf{GL}(m, \mathbb{R})$ be closed subgroups, and let $\varphi : G \to H$ be a continuous group homomorphism. Then there exists a unique Lie algebra homomorphism $d\varphi : \mathfrak{g} \to \mathrm{Lie}(H)$, called the **differential** of φ, such that:

$$\varphi(\exp(X)) = \exp(d\varphi(X)) \quad \text{for all } X \in \mathfrak{g} \tag{1.56}$$

This differential $d\varphi$ is precisely the tangent map of the manifold map φ at the identity element.

1.3.7 Lie Algebras and Vector Fields

There is an equivalent geometric interpretation of a Lie algebra: it is isomorphic to the space of left-invariant vector fields on the Lie group.

For any $A \in \mathfrak{g}$, we can define a vector field X_A on G:

$$(X_A f)(g) = \frac{\mathrm{d}}{\mathrm{d}\, t} f(g \exp(tA))\bigg|_{t=0} \qquad \text{for } f \in C^\infty(G), g \in G \tag{1.57}$$

This vector field X_A is **left-invariant**, meaning it is unchanged by the action of left multiplication on the group.

Proposition 1.3.7 The map $A \mapsto X_A$ is a Lie algebra isomorphism from $\mathfrak{g}$ to the Lie algebra of left-invariant vector fields on G. That is, it preserves the Lie bracket:

$$[X_A, X_B] = X_{[A,B]} \tag{1.58}$$

This geometric viewpoint is crucial for studying the global properties of Lie groups and for more abstract theories.

R [Equivalence of Analytic and Algebraic Definitions] It is a cornerstone of Lie theory that for linear algebraic groups, this "analytic" definition of the Lie algebra (as the set of generators of one-parameter subgroups) is equivalent to the "algebraic" definition that can be formulated in the language of algebraic geometry (e.g., as the space of derivations on the ring of regular functions). This powerful theorem connects the geometric and topological structure of the group with its purely algebraic properties, providing two different yet complementary perspectives on the same fundamental object. This equivalence will be essential for bridging results from analysis and algebra.

1.4 Linear Algebraic Groups

We take the filed $\mathbb{F}$ as $\mathbb{C}$, and the equation which defines G is the polynomial with complex coefficients, that's they don't involve complex conjugation.

1.4.1 Definitions and Examples

Definition 1.4.1 — Linear Algebraic Group. A subgroup G of $\mathbf{GL}(n, \mathbb{C})$ is a linear algebraic group, if there exists a set of polynomial functions $\mathcal{A}$ on $M_n(\mathbb{C})$ such that

$$G = \{g \in \mathbf{GL}(n, \mathbb{C}) : f(g) = 0 \text{ for all } f \in \mathcal{A}\}. \tag{1.59}$$

Here a function f is a polynomial function if

$$f(y) = p(x_{11}(y), x_{12}(y), \cdots x_{nn}(y)) \text{ for all } y \in M_n(\mathbb{C}), \tag{1.60}$$

where $p \in \mathbb{C}[x_{11}, x_{12}, \cdots, x_{nn}]$ is a polynomial and x_{ij} are the matrix entry functions on $M_n(\mathbb{C})$.

Given a complex vector space V with $\dim V = n$, and let $\mu : \mathbf{GL}(V) \to \mathbf{GL}(n, \mathbb{C})$ is an isomorphism. If $\mu(\mathbb{C})$ is a algebraic group, we call the subgroup $G \subseteq \mathbf{GL}(V)$ is a linear algebraic group.

■ **Example 1.5** The basic example of linear algebraic groupis $\mathbf{GL}(n, \mathbb{C})$, when we take $\mathcal{A}$ be the set consists of zero polynomial. In the case $n = 1$, we have $\mathbf{GL}(1, \mathbb{C}) = \mathbb{C} - \{0\} = \mathbb{C}^\times$. ■

■ **Example 1.6** The special linear group $\mathbf{SL}(n, \mathbb{C})$ is algebraic and defined by a polynomial equation $\det(g) - 1 = 0$. ■

■ **Example 1.7** Let $D_n \subseteq \mathbf{GL}(n, \mathbb{C})$ be the subgroup of diagonal matrices, it's algebraic because the defining equations are $x_{ij}(g) = 0$ for all $i \neq j$. ■

■ **Example 1.8** Let $N_n^+ \subseteq \mathbf{GL}(n, \mathbb{C})$ be the subgroup of upper-triangular matrices with diagonal entries 1. The defining equations of it is $x_{ii}(g) = 1$ for all i and $x_{ij}(g) = 0$ for all $i > j$. When $n = 2$, N_2^+ is isomorphic to additive group of the field $\mathbb{C}$, via the map $z \mapsto \left[\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right]$. ■

■ **Example 1.9** Let B_n be the subgroup of upper-triangular matrices, the defining equations are $x_{ij}(g) = 0$ for all $i > j$, so B_n is a linear algebraic group. ■

■ **Example 1.10** Let $\Gamma \in \mathbf{GL}(n, \mathbb{C})$ and we have $B_\Gamma(x, y) = x^{\mathrm{T}}\Gamma y$, for $x, y \in \mathbb{C}^n$. Then B_Γ is an nondegenerate bilinear form on $\mathbb{C}^n$. Let $G_\Gamma = \{g \in \mathbf{GL}(n, \mathbb{C}) : g^{\mathrm{T}}\Gamma g = \Gamma\}$ is the subgroup preserves this form. Since G_Γ is defined by quadratic equations in the matrix entries, it's a linear algebraic group. This shows that the orthogonal groups and the symplectic group are linear algebraic group. ■

For the orthogonal or symplectic groups, there's another description of G_Γ. Define

$$\sigma_\Gamma(g) = \Gamma^{-1}(g^{\mathrm{T}})^{-1}\Gamma \text{ for } g \in \mathbf{GL}(n, \mathbb{C}). \tag{1.61}$$

Then $\sigma_\Gamma(gh) = \sigma_\Gamma(g)\sigma_\Gamma(h)$ for $g, h \in \mathbf{GL}(n, \mathbb{C})$, $\sigma_\Gamma(I) = I$ and

$$\sigma_\Gamma(\sigma_\Gamma(g)) = \Gamma^{-1}(\Gamma^{\mathrm{T}}g(\Gamma^{\mathrm{T}})^{-1})\Gamma = g, \tag{1.62}$$

since $\Gamma^{-1}\Gamma^{\mathrm{T}} = \pm I$. Such a map σ_S is called an involutive automorphism of $\mathbf{GL}(n, \mathbb{C})$. We have $g \in G_\Gamma$ if and only if $\sigma_\Gamma(g) = g$, and the group G_Γ is the set of fixed points of σ_Γ.

1.4.2 Regular Functions

Definition 1.4.2 For the group $\mathbf{GL}(n, \mathbb{C})$, the algebra of regular functions is defined as

$$\mathcal{O}[\mathbf{GL}(n, \mathbb{C})] = C\left[x_{11}, x_{12}, \cdots, x_{nn}, \det(x)^{-1}\right]. \tag{1.63}$$

It is the commutative algebra over $\mathbb{C}$ generated by the matrix entry $\{x_{ij}\}$ and the function $\det(x)^{-1}$, with the relation $\det(x) \cdot \det(x)^{-1} = 1$.

For any complex vector space V of dimension n, let $\varphi : \mathbf{GL}(V) \to \mathbf{GL}(n, \mathbb{C})$ be the group isomorphism defined by a basis of V. The algebra $\mathcal{O}[\mathbf{GL}(V)]$ is defined as all functions $f \circ \varphi$, where f is the regular function on $\mathbf{GL}(n, \mathbb{C})$. This definition is clearly independent of the choice of basis for V.

The regular functions on $\mathbf{GL}(V)$ that are linear combination of the matrix entry functions x_{ij}, relative tosome basis of V, can be described in the following basis-free way: Given $B \in \operatorname{End} V$, we define a function f_B :

$$f_B(Y) = \operatorname{tr}_V(YB), \text{ for } Y \in \operatorname{End} B. \tag{1.64}$$

■ **Example 1.11** When $V = \mathbb{C}$, and $B = e_{ij}$, then $f_{e_{ij}}(Y) = x_{ji}(Y)$. Since the map $B \mapsto f_B$ is linear, every function f_B on $\mathbf{GL}(n, \mathbb{C})$ is the linear combination of the matrix entry functions, so it's regular. Furthermore, algebra $\mathcal{O}[\mathbf{GL}(n, \mathbb{C})]$ is generated by $\{f_B : B \in M_n(\mathbb{C})\}$ and $(\det)^{-1}$. Thus for any finite dimensional vector space V, the algebra $\mathcal{O}[\mathbf{GL}(V)]$ is generated by $(\det)^{-1}$ and function f_B, where $B \in \operatorname{End} V$. ■

An element $g \in \mathbf{GL}(V)$ acts on $\operatorname{End} V$ by left and right multiplication, and we have

$$f_B(gY) = f_{Bg}(Y), f_B(Yg) = f_{gB}(T) \text{ for } B, Y \in \operatorname{End} V. \tag{1.65}$$

Thus, we see the function f_B can transfer properties of the linear action of g on $\operatorname{End} V$ to the properties of the action of g on functions on $\mathbf{GL}(V)$.

Definition 1.4.3 Let $G \subseteq \mathbf{GL}(V)$ be an algebraic subgroup. A complex-valued function f is regular if it's the restriction on G of a regular function on $\mathbf{GL}(V)$.

The set $\mathcal{O}[G]$ is a commutative algebra on $\mathbb{C}$ under pointwise multiplication. It has a finite set of generators: the restriction of $(\det)^{-1}$ and the function f_B, where B varies over any linear basis for $\operatorname{End} V$.

Set

$$\mathcal{J}_G = \{f \in \mathcal{O}[\mathbf{GL}(V)] : f(G) = 0\} \tag{1.66}$$

is an ideal in $\mathcal{O}[\mathbf{GL}(V)]$ that we can describe it by the algebra $\mathcal{P}(\operatorname{End} V)$ of polynomials on $\operatorname{End} V$:

$$\mathfrak{J}_G = \bigcup_{p \geq 0} \left\{(\det)^{-p} f : f \in \mathcal{P}(\operatorname{End} V), f(G) = 0\right\}. \tag{1.67}$$

The map $f \mapsto f|_G$ gives an algebra isomorphism

$$\mathcal{O}[G] \simeq \mathcal{O}[\mathbf{GL}(V)]/\mathcal{J}_G. \tag{1.68}$$

Let G and H be linear algebraic groupand let $\varphi : G \to H$ be a map. For $f \in \mathcal{O}[H]$, define the function $\varphi^*(f)$ on G via $\varphi^*(f)(g) = f(\varphi(g))$. We say φ is a regular map if $\varphi(\mathcal{O}[H]) \subseteq \mathbf{O}(G)$

Definition 1.4.4 An algebraic group homomorphism $\varphi : G \to H$ is a group homomorphism which is regular. We say G and H are isomorohic as an algebraic group if there's an algebraic group homomorphism $\varphi : G \to H$ which has a regular inverse.

Given linear algebraic group$G \subseteq \mathbf{GL}(m, \mathbb{C})$ and $H \subseteq \mathbf{GL}(n, \mathbb{C})$ we make the group-theoretic direct product $K = G \times H$into an algebraic group by the natural block-diagonal embedding $K \to \mathbf{GL}(m+n, \mathbb{C})$ as the block-diagonal matrices

$$k = \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix} \text{ with } g \in G \text{ and } h \in H. \tag{1.69}$$

Since the polynomials in the matrix entries of g and h are polynomials in the matrix entries of k, we see K is an algebraic subgroup of $\mathbf{GL}(m+n, \mathbb{C})$. The algebra homomorphism that maps $f' \otimes f'' \in \mathcal{O}[G] \otimes \mathcal{O}[H]$ to the function $(g, h) \mapsto f'(g)f''(h)$ on $G \times H$ gives an isomorphism

$$\mathcal{O}[G \times H] \cong \mathcal{O}[G] \otimes \mathcal{O}[H]. \tag{1.70}$$

In particular, $G \times G$ is an algebraic group with the algebra of regular functions $\mathcal{O}[G \times G] \simeq \mathcal{O}[G] \otimes \mathcal{O}[G]$.

Proposition 1.4.1 The maps $\mu : G \times G$ and $\eta : G \to G$ given by the multiplication and the inversion are regular. If $f \in \mathcal{O}[G]$, then there's an integer p aand $f_i', f_p'' \in \mathcal{O}[G]$ for $i = 1, 2, \cdots, p$ s.t.

$$f(gh) = \sum_{1 \leq i \leq p} f_i'(g) f_i''(h) \text{ for } g, h \in G. \tag{1.71}$$

Furthermore, for fixed $g \in G$ the maps $x \mapsto L_g(x) = gx$ and $x \mapsto R_g(x) = xg$ on G are regular.

■ **Example 1.12** Let D_n be the subgroup of diagonal matrices in $\mathbf{GL}(n,\mathbb{C})$. The map

$$(x_1, x_2, \cdots, x_n) \mapsto \mathrm{diag}[x_1, x_2, \cdots, x_n] \tag{1.72}$$

from $(\mathbb{C}^\times)^n$ into D_n is an isomorohic. Since $\mathcal{O}[\mathbb{C}^\times] = \mathbb{C}[x, x^{-1}]$ consists of the Laurent polynomials in one variable, it follows that

$$\mathcal{O}[D_n] \simeq \mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}, \cdots, x_n, x_n^{-1}] \tag{1.73}$$

is the algebra of the Laurent polynomials in n variables. We call an algebraic group H an algebraic torus of rank n which is isomorphic to D_n. ■

■ **Example 1.13** Let $N_n^+ \subseteq \mathbf{GL}(n,\mathbb{C})$ be the subgroup of upper-triangular matrices with unit diagonal. It's easy tuples show that the functions x_{ij} for $i > j$ and $x_{ii} - 1$ generate $\mathcal{J}_{N_n^+}$ and

$$\mathcal{O}[N_n^+] \simeq \mathbb{C}[x_{12}, x_{13}, \cdots x_{n-1,n}] \tag{1.74}$$

is the algebra of polynomials in the $\frac{n(n-1)}{2}$ variables $\{x_{ij} : i < j\}$. ■

1.4.3 Lie Algebra of an Algebraic Group

In Section 1.3, we defined the Lie algebra for a closed subgroup of $\mathbf{GL}(n,\mathbb{R})$ using an analytic approach based on the exponential map. We now introduce a purely algebraic definition, which is more powerful as it does not rely on topology and is suitable for linear algebraic groups over any field of characteristic zero. We will show that for groups that are both closed subgroups of $\mathbf{GL}(n,\mathbb{C})$ and linear algebraic groups, these two definitions yield the same object.

The algebraic approach identifies the Lie algebra with the tangent space at the identity element, which is characterized by its action as derivations on the algebra of regular functions.

Definition 1.4.5 — Derivation. Let $\mathcal{A}$ be a commutative algebra over a field $\mathbb{F}$. A **derivation** of $\mathcal{A}$ is an $\mathbb{F}$-linear map $D : \mathcal{A} \to \mathcal{A}$ that satisfies the Leibniz rule:

$$D(ab) = D(a)b + aD(b) \quad \text{for all } a, b \in \mathcal{A} \tag{1.75}$$

The set of all derivations of $\mathcal{A}$, denoted $\mathrm{Der}(\mathcal{A})$, forms a Lie algebra under the commutator bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$.

Let G be a linear algebraic group. The key idea is to define its Lie algebra as the space of left-invariant derivations on its algebra of regular functions, $\mathcal{O}[G]$.

First, let's consider the case $G = \mathbf{GL}(n,\mathbb{C})$, where $\mathcal{O}[G] = \mathbb{C}[x_{11}, \ldots, x_{nn}, \det^{-1}]$. For any matrix $A \in M_n(\mathbb{C})$, we define a **left-invariant vector field** X_A acting on $\mathcal{O}[G]$. For a function $f \in \mathcal{O}[G]$ and a point $u \in G$, its action is defined as the directional derivative at u in the direction of uA:

$$(X_A f)(u) = \frac{d}{dt} f(u(I + tA))\bigg|_{t=0} \tag{1.76}$$

where the derivative is defined formally for rational functions. This map $A \mapsto X_A$ is a complex linear Lie algebra isomorphism from $M_n(\mathbb{C})$ (with the matrix commutator) to the Lie algebra of left-invariant regular vector fields on $\mathbf{GL}(n,\mathbb{C})$.

A crucial property is how these vector fields act on the coordinate functions $f_B(u) = \mathrm{tr}(uB)$:

$$(X_A f_B)(u) = \frac{d}{dt}\mathrm{tr}(u(I+tA)B)\Big|_{t=0} = \mathrm{tr}(uAB) = f_{AB}(u) \tag{1.77}$$

This formula provides a purely algebraic way to understand the action of X_A.

Now, let $G \subseteq \mathbf{GL}(n,\mathbb{C})$ be any linear algebraic group. Let $\mathcal{I}_G \subset \mathcal{O}[\mathbf{GL}(n,\mathbb{C})]$ be the ideal of regular functions that vanish on G.

Definition 1.4.6 — Lie Algebra of an Algebraic Group. The **Lie algebra** of an algebraic group $G \subseteq \mathbf{GL}(n,\mathbb{C})$, denoted $\mathfrak{g}$, is the set of all matrices $A \in M_n(\mathbb{C})$ such that the corresponding left-invariant vector field X_A maps the ideal $\mathcal{I}_G$ into itself. That is:

$$\mathfrak{g} = \{A \in M_n(\mathbb{C}) \mid X_A(\mathcal{I}_G) \subseteq \mathcal{I}_G\} \tag{1.78}$$

This condition is equivalent to saying that for any regular function f that vanishes on G, the new function $X_A f$ also vanishes on G.

Proposition 1.4.2 The set $\mathfrak{g}$ defined above is a Lie subalgebra of $M_n(\mathbb{C})$. The map $A \mapsto X_A$ gives an injective Lie algebra homomorphism from $\mathfrak{g}$ into $\mathrm{Der}(\mathcal{O}[G])$.

■ **Example 1.14 — Lie Algebra of the Diagonal Group D_n.** Let $G = D_n$, the group of invertible diagonal matrices. The ideal $\mathcal{I}_{D_n}$ is generated by the coordinate functions $\{x_{ij} \mid i \neq j\}$. A matrix $A = [a_{ij}]$ is in the Lie algebra $\mathfrak{d}_n$ if and only if the left-invariant vector field X_A preserves this ideal, meaning $X_A(x_{ij})$ vanishes on D_n for all $i \neq j$.

Let's compute the action. The vector field X_A acts on the coordinate function x_{ij} as $X_A(x_{ij}) = \sum_{p=1}^{n} x_{ip}a_{pj}$[cite: 289]. When we evaluate this on an element $u \in D_n$, the matrix for x_{ip} is diagonal, so $x_{ip}(u) = 0$ if $i \neq p$, and $x_{ii}(u)$ is the i-th diagonal entry. The sum therefore reduces to a single term:

$$(X_A x_{ij})(u) = x_{ii}(u)a_{ij}. \tag{1.79}$$

For this expression to be zero for all $u \in D_n$ and all $i \neq j$, we must have $a_{ij} = 0$, since u is invertible and thus its diagonal entries $x_{ii}(u)$ are non-zero[cite: 291]. This shows that A must be a diagonal matrix. Therefore, the Lie algebra $\mathfrak{d}_n$ consists of all diagonal matrices. ■

■ **Example 1.15 — Lie Algebra of the Unipotent Group N_n^+.** Let $G = N_n^+$, the group of upper-triangular matrices with ones on the diagonal. The ideal $\mathcal{I}_{N_n^+}$ is generated by $\{x_{ij} \mid i > j\}$ and $\{x_{ii} - 1 \mid 1 \leq i \leq n\}$. A matrix $A = [a_{ij}]$ is in the Lie algebra $\mathfrak{n}_n^+$ if X_A preserves this ideal.

1. **Condition for off-diagonal entries ($i > j$):** We need $X_A(x_{ij})|_{N_n^+} = 0$. The action is $X_A(x_{ij}) = \sum_{p=1}^{n} x_{ip}a_{pj}$. On N_n^+, we have $x_{ip} = 0$ if $i > p$. The sum becomes $\sum_{p=i}^{n} x_{ip}a_{pj}$. For $p = i$, the term is $x_{ii}a_{ij} = 1 \cdot a_{ij}$. A careful analysis of the subsequent terms shows that for the entire sum to be zero, we must have $a_{ij} = 0$ for all $i > j$[cite: 301].
2. **Condition for diagonal entries:** We need $X_A(x_{ii} - 1)|_{N_n^+} = 0$, which is equivalent to $X_A(x_{ii})|_{N_n^+} = 0$. The action is $X_A(x_{ii}) = \sum_{p=1}^{n} x_{ip}a_{pi}$. On N_n^+, this sum becomes $\sum_{p=i}^{n} x_{ip}a_{pi}$. The first term is $x_{ii}a_{ii} = 1 \cdot a_{ii}$[cite: 302]. The subsequent terms $x_{ip}a_{pi}$ (for $p > i$) create a system of equations that forces $a_{ii} = 0$ for all i[cite: 303].

Combining these conditions, we find that A must be a strictly upper-triangular matrix. Thus, $\mathfrak{n}_n^+$ consists of all strictly upper-triangular matrices. ■

1.5 Rational Representations

1.5.1 Definitions and Examples

Definition 1.5.1 — Representation and Regular. Let G be a linear algebraic group, a representation of G is a pair (ρ, V), where V is a complex vector space, not necessarily finite dimensional, and $\rho : G \to \mathbf{GL}(V)$ is a group homomorphism. We say that the representation is regular if $\dim V < \infty$ and the functions on G

$$g \mapsto \langle v^*, \rho(g)v \rangle \tag{1.80}$$

we call the matrix coefficients of ρ, are regular for all $v \in V$ and $v^* \in V^*$.

If we fix a basis for V and write out the matrix for $\rho(g)$ in this basis, $d = \dim V$:

$$\rho(g) = [\rho_{ij}(g)]_{d\times d}\,, \tag{1.81}$$

then all the functions $\rho_{ij}(g)$ on G are regular. Furthermore, ρ is a regular homomorphism of G into $\mathbf{GL}(V)$, since the regular functions on $\mathbf{GL}(V)$, are generated by the matrix entry functions and $\det^{-1}$, and we have $(\det \rho(g))^{-1} = \det \rho(g^{-1})$, its a regular function on G. Regular representation is usually called rational representation, since every entry $\rho_{ij}(g)$ is a rational function of the matrix entries of g.

We can also describe the definition of regularity in a convenient way: On $\operatorname{End} V$ we have the symmetric bilinear form $A, B \mapsto \operatorname{tr}_V(AB)$. This form is nondegenerate, so if $\lambda \in \operatorname{End} V^*$, then there's $A_\lambda \in \operatorname{End} V$ s.t. $\lambda(X) = \operatorname{tr}_V(A_\lambda X)$. For $B \in \operatorname{End} V$ define the function f_B^ρ on G by

$$f_B^\rho(g) = \operatorname{tr}_V(\rho(g)B). \tag{1.82}$$

Then (ρ, V) is regular if and only if for all $B \in \operatorname{End} V$, f_B^ρ is a regular function on G. We set

$$E^\rho = \left\{ f_B^\rho : B \in \operatorname{End} V \right\}. \tag{1.83}$$

This is a linear subspace of $\mathbf{O}(\mathcal{G})$, which is spanned by the functions in the matrix for ρ.

Definition 1.5.2 — Locally Regular. We say (ρ, V) is locally regular if every finite dimensional subspace $E \subseteq V$ is contained by a finite dimensional G-invariant subspace F s.t. the restriction of ρ to F is a regular representation.

If (ρ, V) is a regular representation and $W \subseteq V$ is a linear subspace, then we say that W is G-invariant if $\rho(g)w \in W$ for all $g \in G$ and $w \in W$. In this case, we obtain a representation σ of G on W by restriction of $\rho(g)$. We also set $\tau(g)(v + W) = \rho(g)v + W$, obtain a representation of τ of G on the quotient space V/W. If we take a basis of W and complete it to a basis of V, then relative tothis basis, the matrix of $\rho(g)$ has the block form

$$\rho(g \begin{bmatrix} \sigma(g) & * \\ 0 & \tau(g) \end{bmatrix}. \tag{1.84}$$

This matrix form shows that the representations (σ, W) and $(\tau, V/W)$ are regular.

Definition 1.5.3 — Intertwining Map. If (ρ, W) and (τ, W) are representations of G and $T \in \mathrm{Hom}(V, W)$, we say that T is a G intertwining map if

$$\tau(g)T = T\rho(g) \text{ for all } g \in G. \tag{1.85}$$

We denote by $\mathrm{Hom}_G(V, W)$ the vector space of all G intertwining maps. The representation ρ and τ are equivalent if there is an invertible G intertwining map. In this case, we write $\rho \simeq \tau$.

We say that a representation (ρ, V) with $V \neq \{0\}$ is reducible if there's a G-invariant subspace $W \subseteq V$ s.t. $W \neq \{0\}$ and $W \neq V$. This means that there's a basis of V s.t. $\rho(g)$ has a block form with all blocks of size at least 1×1. A representation that isn't reducible is called irreducible.

Consider the representations L and R of G on the infinite dimensional vector space $\mathcal{O}[G]$ given by left ad right translations:

$$L(x)f(y) = f(x^{-1}y), R(x)f(y) = f(yx) \text{ for } f \in \mathcal{O}[G] \tag{1.86}$$

Proposition 1.5.1 The representations $(L, \mathcal{O}[G])$ and $(R, \mathcal{O}[G])$ are locally regular.

We note that $L(x)R(y)f = R(y)L(x)f$ for $f \in \mathcal{O}[G]$. Thus we can define a locally regular representation τ of the product group $G \times G$ on $\mathcal{O}[G]$ by $\tau(x, y) = L(x)R(y)$. We can recover the left and right representations by the restriction of τ on the subgroups $G \times \{1\}$ and $\{1\} \times G$, each of which is isomorphic to G.

We may also embed G into G as $\Delta(G) = \{(x, x) : x \in G\}$. The restriction of τ to G gives the conjugation representation of G on $\mathcal{O}[G]$, which we denote by Int. It acts by

$$\mathrm{Int}(x)f(y) = f(x^{-1}yx) \text{ for } f \in \mathcal{O}[G] \text{ for } x \in G. \tag{1.87}$$

By the above, $(\mathrm{Int}, \mathcal{O}[G])$ is also a locally regular representation.

1.5.2 Differential of a Rational Representation

Definition 1.5.4 — the Differential of the Representation. Let G be a linear algebraic groupwith Lie algebra$\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{C})$. Let (π, V) be a rational representation of G. Viewing G and $\mathbf{GL}(n, V)$ as Lie groups, we can obtain a homomorphism of real Lie algebra:

$$\mathrm{d}\,\pi : \mathfrak{g} \to \mathfrak{gl}(V). \tag{1.88}$$

We call $\mathrm{d}\,\pi$ is the differential of the representation π.

Since in this case, $\mathfrak{g}$ is a Lie algebraon $\mathbb{C}$, we have $\pi(\exp(tA)) = \exp(\mathrm{d}\,\pi(tA))$ for $A \in \mathfrak{g}$ and $t \in \mathbb{C}$. The entries of the matrix $\pi(g)$ relative toany basis are regular functions on G, so we have $t \mapsto \pi(\exp(tA))$ is an analytic matrix-valued function of the complex variable t. Thus

$$\mathrm{d}\,\pi(A) = \frac{\mathrm{d}}{\mathrm{d}\,t}\pi(\exp(tA))\bigg|_{t=0}, \tag{1.89}$$

and the map $A \mapsto \mathrm{d}\,\pi(A)$ is complex linear. Thus $\mathrm{d}\,\pi$ is a homomorphism of Lie algebrawhen G is a linear algebraic group.

We can also define $\mathrm{d}\,\pi$ in a purely algebraic fashion: Viewing the elements of $\mathfrak{g}$ as left-invariant vector field on G by the map $A \mapsto X_A$, and differentiate the entries in the matrices for π using X_A. We know every linear transformation $B \in \operatorname{End} V$, defines a linear function f_C in

$$f_C(B) = \mathrm{tr}_V(BC) \text{ for } B \in \operatorname{End} V. \tag{1.90}$$

The representative function $f_C^\pi = f_c \circ \pi$ on G is a regular function.

Theorem 1.5.2 The differential of a rational representation (π, V) is a unique linear map $\mathfrak{g} \to \operatorname{End} V$ s.t.

$$X_A(f_C \circ \pi)(I) = f_{\mathrm{d}\,\pi(A)C}(I) \text{ for all } A \in \mathfrak{g} \text{ and } C \in \operatorname{End} V. \tag{1.91}$$

Furthermore, for $A \in \mathrm{Lie}(G)$, has

$$X_A(f \circ \pi) = (X_{\mathrm{d}\,\pi(A)} f) \circ \pi \text{ for all } f \in \mathcal{O}[\mathbf{GL}(V)]. \tag{1.92}$$

Let G and H be linear algebraic groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively, and let $\pi : G \to H$ be a regular homomorphism. If $H \subseteq \mathbf{GL}(V)$, then we can view G as the regular representation with the differential $\mathrm{d}\,\pi : \mathfrak{g} \to \operatorname{End} V$ of G.

Proposition 1.5.3 The range of $\mathrm{d}\,\pi$ is contained by $\mathfrak{h}$, hush $\mathrm{d}\,\pi$ is a homomorphism of Lie algebraof $\mathfrak{g}$ into $\mathfrak{h}$. Furthermore, if $K \subseteq \mathbf{GL}(W)$ is a linear algebraic groupand $\rho : H \to K$ is a regular homomorphism, then $\mathrm{d}(\rho \circ \pi) = \mathrm{d}\,\rho \circ \mathrm{d}\,\pi$. In particular, isomorphic linear algebraic groups have isomorphic Lie algebras.

Corollary 1.5.4 Suppose $G \subseteq H$ is the algebraic subgroup of $\mathbf{GL}(n, \mathbb{C})$, If (π, V) is a regular representation of H, then the differential of $\pi|_G$ is $\mathrm{d}\,\pi|_{\mathfrak{g}}$.

■ **Example 1.16** Let $G \subseteq \mathbf{GL}(V)$ be a linear algebraic group. By the definition of $\mathcal{O}[G]$, the representation $\pi(g)g$ on V is regular. We call (π, V) is the defining representation of G. It follows directly from the definition that $\mathrm{d}\,\pi(A) = A$ for $A \in \mathfrak{g}$. ■

■ **Example 1.17** Let (π, V) be a regular representation. Define its dual representation (π^*, V^*) by $\pi^*(g)v^* = v^* \circ \pi(g^{-1})$. Then π^* is obviously regular, since

$$\langle v^*, \pi(g)v \rangle = \left\langle \pi^*(g^{-1})v^*, v \right\rangle \text{ for } v \in V \text{ and } v^* \in V^*. \tag{1.93}$$

If $\dim V = d$ and V is identified with $d \times 1$ column vectors by a choice of basis, hence V^* is identified with $1 \times d$ row vectors. Viewing $\pi(g)$ as a $d \times d$ matrix using basis, we have

$$\langle v^*, \pi(g)v \rangle = v^* \pi(g) v \text{ (matrix multiplication)} \tag{1.94}$$

Thus $\pi^*(g)$ acts by right multiplication on row vectors by the matrix $\pi(g^{-1})$. The space of representative functions for π^* consists of the function $g \mapsto f(g^{-1})$, which f is a representative function of π. If $W \subseteq V$ is a G-invariant subspace, then

$$W^\perp = \{v^* \in V^* : \langle v^*, w \rangle = 0 \text{ for all } w \in W\} \tag{1.95}$$

is a G-invariant subspace of V^*. In particular, if (π, V) is irreducible then so is (π^*, V^*). The natural vector space isomorphism $(V^*)^* \simeq V$ gives an equivalence $(\pi^*)^* \simeq \pi$. Let $A \in \mathfrak{g}, v \in v$ and $v^* \in V^*$. Then

$$\langle \mathrm{d}\,\pi^*(A)v^*, v\rangle = \frac{\mathrm{d}}{\mathrm{d}\,t} \langle \pi^*(\exp tA\rangle\, v^*, v\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}\,t} \langle v^*, \pi(\exp(-tA))v\rangle\Big|_{t=0} = -\langle v^*, \mathrm{d}\,\pi(A)v\rangle\,. \tag{1.96}$$

We conclude that

$$\mathrm{d}\,\pi^*(A) = -\,\mathrm{d}\,\pi(A)^{\mathrm{T}} \text{ for } A \in \mathfrak{g}. \tag{1.97}$$

■

■ **Example 1.18** Let (π_1, V_1) and (π_2, V_2) be regular representations of G. Define the direct sum representation $\pi_1 \oplus \pi_2$ on $V_1 \oplus V_2$ by

$$(\pi_1 \oplus \pi_2)(g)(v_1 \oplus v_2) = \pi_1(g)v_2 \oplus \pi_2(g)v_2 \text{ for } g \in G \text{ and } v_i \in V_i. \tag{1.98}$$

Then $\pi_1 \oplus \pi_2$ is obviously a representation and it's regular, since

$$\langle v_1^* \oplus v_2^*, (\pi_1 \oplus \pi_2)(g)(v_1 \oplus v_2)\rangle = \langle v_1^*, \pi_1(g)v_1\rangle + \langle v_2^*, \pi_2(g)v_2\rangle \tag{1.99}$$

for $v_i \in V_2$ and $v_i^* \in V_i^*$. This shows that the space of representative functions for $\pi_1 \oplus \pi_2$ is $E^{\pi_1 \oplus \pi_2} = E^{\pi_1} + E^{\pi_2}$. If $\pi = \pi_1 \oplus \pi_2$, the in matrix form we have

$$\pi(g) = \begin{bmatrix} \pi_1(g) & 0 \\ 0 & \pi_2(g) \end{bmatrix}. \tag{1.100}$$

Differentiating the matrices entries, we have

$$\mathrm{d}\,\pi(A) = \begin{bmatrix} \mathrm{d}\,\pi_1(A) & 0 \\ 0 & \mathrm{d}\,\pi_2(A) \end{bmatrix} \text{ for } A \in \mathfrak{g}. \tag{1.101}$$

Thus $\mathrm{d}\,\pi(A) = \mathrm{d}\,\pi_1(A) \oplus \mathrm{d}\,\pi_2(A)$. ■

■ **Example 1.19** Let (π_1, V_1) and (π_2, V_2) be regular representations of G. Define the tensor product representation $\pi_1 \otimes \pi_2$ on $V_1 \otimes V_2$ by

$$(\pi_1 \otimes \pi_2)(g)(v_1 \otimes v_2) = \pi_1(g)v_1 \otimes \pi_2(g)v_2 \tag{1.102}$$

for $g \in G$ and $v_i \in V_i$. It's clear that $\pi_1 \otimes \pi_2$ is a representation and it's regular, since

$$\langle v_1^* \otimes v_2^*, (\pi_1 \otimes \pi_2)(g)(v_1 \otimes v_2)\rangle = \langle v_1^*, \pi_1(g)v\rangle\, \langle v_2^*, \pi_2(g)v_2\rangle \tag{1.103}$$

for $v_i \in V_i$ and $v_i^* \in V_i^*$. We have

$$E^{\pi_1 \otimes \pi_2} = \mathrm{Span}(E^{\pi_1} \cdot E^{\pi_2}). \tag{1.104}$$

Set $\pi = \pi_1 \otimes \pi_2$, then

$$\mathrm{d}\,\pi(A) = \mathrm{d}\,\pi_1(A) \otimes I + I \otimes \mathrm{d}\,\pi_2(A). \tag{1.105}$$

■

■ **Example 1.20** Let (π, V) be a regarded representation of G, and let $\rho = \pi \otimes \pi^*$ on $V \otimes V^*$, then we have

$$\mathrm{d}\,\rho(A) = \mathrm{d}\,\pi(A) \otimes I - I \otimes \mathrm{d}\,\pi(A)^{\mathrm{T}}. \tag{1.106}$$

However, there's a canonical isomorphism $T : V \otimes V^* \simeq \mathrm{End}\, V$, with

$$T(v \otimes v^*)(u) = \langle v^*, u \rangle\, v. \tag{1.107}$$

Set $\sigma(g) = T\rho(g)T^{-1}$. If $Y \in \mathrm{End}\, V$ then $T(Y \otimes I) = YT$ and $T(I \otimes Y^{\mathrm{T}}) = TY$. Thus $\sigma(g)(Y) = \pi(g)Y\pi(g)^{-1}$ and

$$\mathrm{d}\,\sigma(A)(Y) = \mathrm{d}\,\pi(A)Y - Y\,\mathrm{d}\,\pi(A) \text{ for } A \in \mathfrak{g}. \tag{1.108}$$

■

R [Connection to the Adjoint Representation] This example is of fundamental importance. It provides a concrete realization of the **adjoint representation**. Specifically, this is the adjoint representation of the general linear group $\mathbf{GL}(V)$ on its Lie algebra $\mathfrak{gl}(V) = \mathrm{End}(V)$, which is then restricted to the subgroup $G \subset \mathbf{GL}(V)$. Understanding this example is key to grasping the more abstract definition of the adjoint representation of a Lie group G acting on its own Lie algebra $\mathfrak{g}$.

■ **Example 1.21** Let (π, V) be a regular representation of G, and $\rho = \pi^* \otimes \pi^*$ on $V^* \otimes V^*$. Then we have

$$\mathrm{d}\,\rho(A) = -\,\mathrm{d}\,\pi(A)^{\mathrm{T}} \otimes I - I \otimes \mathrm{d}\,\pi(A)^{\mathrm{T}}. \tag{1.109}$$

However, there's a canonical isomorphism between $V^* \otimes V^*$ and the space of bilinear forms on V, where $g \in G$ acts on a bilinear form B by

$$g \cdot B(x, y) = B(\pi(g^{-1})x, \pi(g^{-1})y). \tag{1.110}$$

If V is identified with column vectors by a choice of a basis and $B(x, y) = y^{\mathrm{T}}\Gamma x$, then $g \cdot \Gamma = \pi(g^{-1})^{\mathrm{T}}\Gamma\pi(g^{-1})$. The action of $A \in \mathfrak{g}$ on B is

$$A \cdot B(x, y) = -B(\mathrm{d}\,\pi(A)x, y) - B(x, \mathrm{d}\,\pi(A)y). \tag{1.111}$$

We say that a bilinear form B is invariant under G if $g \cdot B = B$ for all $g \in G$. Likewise, we say that B is invariant under $\mathfrak{g}$ if $A \cdot B = 0$ for all $A \in \mathfrak{g}$. This invariance property can be expressed as

$$B(\mathrm{d}\,\pi(A)x, y) + B(x, \mathrm{d}\,\pi(A)Y) = 0 \text{ for all } x, y \in V \text{ and } A \in \mathfrak{g}. \tag{1.112}$$

■

1.5.3 The Adjoint Representation

Let $G \subseteq \mathbf{GL}(n, \mathbb{C})$ be an algebraic group with Lie algebra$\mathfrak{g}$. The representation of $\mathbf{GL}(n, \mathbb{C})$ on $M_n(\mathbb{C})$ by similarity $A \mapsto gAg^{-1}$ is regular.
Lemma 1.5.5 Let $A \in \mathfrak{g}$ and $g \in G$. Then $gAg^{-1} \in \mathfrak{g}$.

Definition 1.5.5 — Adjoint Representation. We define $\mathrm{Ad}(g)A = gAg^{-1}$ for $g \in G$ and $A \in \mathfrak{g}$. Then we have $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$. The representation $(\mathrm{Ad}, \mathfrak{g})$ is called the adjoint representation of G.

For , $B \in \mathfrak{g}$, we have

$$\mathrm{Ad}(g)[A, B] = [\mathrm{Ad}(g)A, \mathrm{Ad}(g)B]. \tag{1.113}$$

Thus $\mathrm{Ad}(g)$ is a Lie algebraautomorphism and $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$.

If $H \subseteq \mathbf{GL}(n, \mathbb{C})$ is another algebraic group with Lie algebra$\mathfrak{h}$, we denote the adjoint representations of G and H by Ad_G and Ad_H, respectively. Suppose that $G \subseteq H$, since $\mathfrak{g} \subseteq \mathfrak{h}$ by property, we have

$$\mathrm{Ad}_H(g)A = \mathrm{Ad}_G(g)A \text{ for } g \in G \text{ and } A \in \mathfrak{g}. \tag{1.114}$$

Theorem 1.5.6 The differential of the adjoint representation of G is the representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}\,\mathfrak{g}$ given by

$$\mathrm{ad}(A)(B) = [A, B] \text{ for } A, B \in \mathfrak{g}. \tag{1.115}$$

Furthermore, $\mathrm{ad}(A)$ is a derivation of $\mathfrak{g}$, and hence $\mathrm{ad}(\mathfrak{g}) \subseteq \mathrm{Der}(\mathfrak{g})$.

R If $G \subseteq \mathbf{GL}(n, \mathbb{R})$ is any closed subgroup, then $gAg^{-1} \in \mathrm{Lie}(G)$ for all $g \in G$ and $A \in \mathrm{Lie}(G)$. Thus we can define the adjoint representation Ad of G on the real vector space $\mathrm{Lie}(G)$ as for algebraic groups. Clearly $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is a homomorphism from G to the group of automorphisms of $\mathrm{Lie}(G)$.

1.6 Jordan Decomposition

1.6.1 Rational Representations of $\mathbb{C}$

We have given the additive group $\mathbb{C}$ the structure of a linear algebraic groupby embedding it into $\mathbf{SL}(2, \mathbb{C})$ with the homomorphism

$$z \mapsto \varphi(z) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} = I + ze_{12}. \tag{1.116}$$

The regular functions on $\mathbb{C}$ are polynomials in z, and the Lie algebras of $\mathbb{C}$ is spanned by the matrix e_{12} which satisfies $(e_{12})^2 = 0$. Thus $\varphi(z) = \exp(ze_{12})$.

A matrix $A \in M_n(\mathbb{C})$ is said to be nilpotent if $A^k = 0$ holds for some positive integer k. The trace of a nilpotent matrices is 0, since 0 is the unique eigenvalue. If $u - I$ is a nilpotent matrices, then $u \in M_n(\mathbb{C})$ is called unipotent. Note a unipotent matrix is nonsingular and has determinant 1, since 1 is the unique eigenvalue.

Let $A \in M_n(\mathbb{C})$ be nilpotent. Then $A^n = 0$ and for $t \in \mathbb{C}$ we have

$$\exp tA = I + Y, \text{ where } Y = tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^{n-1}}{(n-1)!}A^{n-1} \tag{1.117}$$

is also nilpotent. Hence the matrices $\exp tA$ is a unipotent matrix, and $t \mapsto \exp(tA)$ is a regular homomorphism of the additive group $\mathbb{C}$ into $\mathbf{GL}(n, \mathbb{C})$.

Conversely, if $u = I + Y \in \mathbf{GL}(n, \mathbb{C})$ is unipotent, then $Y^n = 0$ define

$$\log u = \sum_{1 \leq k \leq n-1} (-1)^{k+1} \frac{Y^k}{k}. \tag{1.118}$$

Thus the exponential function is a bijection polynomial map of the nilpotent elements of $M_n(\mathbb{C})$ into the unipotent elements of $\mathbf{GL}(n, \mathbb{C})$, the inverse is $u \mapsto \log u$.

Lemma 1.6.1 — Taylor's Formula. Suppose $A \in M_n(\mathbb{C})$ is nilpotent and f is a regular function on $\mathbf{GL}(n, \mathbb{C})$. Then there's a integer k s.t. $(X_A)^k f = 0$ and

$$f(\exp A) = \sum_{0 \leq m \leq k-1} \frac{X_A^m f}{m!}(I). \tag{1.119}$$

Theorem 1.6.2 Let $G \subseteq \mathbf{GL}(n, \mathbb{C})$ is the linear algebraic grouphas the Lie algebra$\mathfrak{g}$. If $A \in M_n(\mathbb{C})$ is nilpotent then $A \in \mathfrak{g}$ if and only if $\exp A \in G$. Furthermore, if $A \in \mathfrak{g}$ is a nilpotent matrix and (ρ, V) is a regular representation of G, then $\mathrm{d}\,\rho(A)$ is a nilpotent transformation on V, and

$$\rho(\exp A) = \exp \mathrm{d}\,\rho(A). \tag{1.120}$$

Corollary 1.6.3 If (π, V) is a regular representation of the addition group $\mathbb{C}$, then there's a unique nilpotent $A \in \operatorname{End} V$ s.t. for all $z \in \mathbb{C}$, we have $\pi(z) = \exp(zA)$.

1.6.2 Rational Representations of $\mathbb{C}^\times$

The regular representation of $\mathbb{C}^\times = \mathbf{GL}(1, \mathbb{C})$ have the following form:

Lemma 1.6.4 Let $(\varphi, \mathbb{C}^\times)$ be a regular representation. For $p \in \mathbb{Z}$ define $E_p = \{v \in \mathbb{C}^n : \varphi(z)v = z^p v$ for all $z \in \mathbb{C}$
Then

$$\mathbb{C}^n = \bigoplus_{p \in \mathbb{Z}} E_p \tag{1.121}$$

and hence $\varphi(z)$ is a semisimple transformation. Conversely, given a direct sum decomposition of $\mathbb{C}^n$, define $\varphi(z)v = z^p v$ for $z \in \mathbb{C}^\times$ and $v \in E_p$. Then φ is a regular representation of $\mathbb{C}^\times$ on $\mathbb{C}^n$ that is determined by the set odd integers $\{\dim E_p : p \in \mathbb{Z}\}$.

1.6.3 Jordan-Chevalley Decomposition

A matrix $A \in M_n(\mathbb{C})$ has a unique additive Jordan decomposition $A = S + N$ with S semisimple, N nilpotent and $SN = NS$. Likewise, $g \in \mathbf{GL}(n, \mathbb{C})$ has a unique multiplication Jordan decomposition $g = su$ with s semisimple, u unipotent and $su = us$.

Theorem 1.6.5 Let $G \subseteq \mathbf{GL}(n,\mathbb{C})$ be a algebraic group has the Lie algebra$\mathfrak{g}$. If $A \in \mathfrak{g}$ and $A = S + N$ is its additive Jordan decomposition, then $S, N \in \mathfrak{g}$. Furthermore, if $g \in G$ and $g = su$ is its multiplication decomposition, then $s, u \in G$.

Theorem 1.6.6 Let G be a algebraic group has the Lie algebra$\mathfrak{g}$. Suppose (ρ, V) is a regular representation of G.

- 1.If $A \in \mathfrak{g}$ and $A = S + N$ is its additive Jordan decomposition, then $\mathrm{d}\,\rho(S)$ is semisimple, $\mathrm{d}\,\rho(N)$ is nilpotent and $\mathrm{d}\,\rho(A) = \mathrm{d}\,\rho(S) + \mathrm{d}\,\rho(N)$ is the additive Jordan decomposition of $\mathrm{d}\,\rho(A)$ in $\operatorname{End} V$.
- If $g \in G$ and $g = su$ is its multiplication Jordan multiplication in G, them $\rho(s)$ is semisimple, $\rho(u)$ is unipotent and $\rho(g) = \rho(s)\rho(u)$ is the multiplication Jordan decomposition of $\rho(g)$ in $\mathbf{GL}(V)$.

We see every element g in G has a semisimple component g_s and a unipotent component g_u s.t. $g = g_s g_u$. Furthermore, this factorization is independent of the choice of defining representation $G \subseteq \mathbf{GL}(V)$. Likewise, every element $Y \in \mathfrak{g}$ has a unique semisimple component Y_s and a unique nilpotent component Y_n s.t. $Y = Y_s + Y_n$.

We denote the set of all the semisimple elements in G by G_s and the set of all the unipotent elements in G by G_u. Likewise, we denote the set of all the semisimple elements in $\mathfrak{g}$ by $\mathfrak{g}_s$ and the set of all the nilpotent elements in $\mathfrak{g}$ by $\mathfrak{g}_n$. Suppose $G \subseteq \mathbf{GL}(n,\mathbb{C})$ is a algebraic subgroup. Since $T \in M_n(\mathbb{C})$ is nilpotent if and only if $T^n = 0$, we have

$$\mathfrak{g}_n = \mathfrak{g} \cap \{T \in M_n(\mathbb{C}) : T^n = 0\}\,, \tag{1.122}$$

$$G_u = G \cap \{g \in \mathbf{GL}(n, \mathbb{C} : (I - g)^n = 0)\}\,. \tag{1.123}$$

Thus $\mathfrak{g}_n$ is an algebraic subset of $M_n(\mathbb{C})$ and G_u is an algebraic subset of $\mathbf{GL}(n,\mathbb{C})$.

Corollary 1.6.7 Suppose G and H are algebraic groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Let $\rho : G \to H$ be a regular homomorphism s.t. $\mathrm{d}\,\rho : \mathfrak{g} \to \mathfrak{h}$ is surjective. Then $\rho(G_u) = H_u$.

1.7 Real Forms of Complex Algebraic Groups

1.7.1 Real Forms and Complex Conjugations

We begin with a definition that refers to subgroups of $\mathbf{GL}(n,\mathbb{C})$, and we shall get a more general notion of a real form later.

Definition 1.7.1 — Defined on $\mathbb{R}$. Let $G \subseteq \mathbf{GL}(n,\mathbb{C})$ be an algebraic subgroup. Then the ideal $\mathcal{J}_G$ is generated by $\mathcal{J}_{\mathbb{R},G} = \{f \in \mathcal{J}_G : f(\mathbf{GL}(n,\mathbb{R}))\} \subseteq \mathbb{R}$. If G is defined on $\mathbb{R}$, then we suppose $G_{\mathbb{R}} = G \cap \mathbf{GL}(n,\mathbb{R})$ and call $G_{\mathbb{R}}$ the group of $\mathbb{R}$-rational points of G.

■ **Example 1.22** The group $\mathbf{GL}(n,\mathbb{C})$ is defined on $\mathbb{R}$ since $\mathcal{I}_G = 0$ and $G_{\mathbb{R}} = \mathbf{GL}(n,\mathbb{R})$. ■

■ **Example 1.23** The group $G = B_n$ is defined on $\mathbb{R}$ consists of $n \times n$ invertible upper-triangular matrices, since $\mathcal{J}_G$ is generated by the matrix entries functions $\{x_{ij} : 1 \leq i < j \leq n\}$, which are real valued on $\mathbf{GL}(n, \mathbb{R})$. In this case $G_{\mathbb{R}}$ is the group consists of $n \times n$ invertible upper-triangular matrices. ■

For $g \in \mathbf{GL}(n, \mathbb{C})$, we define $\sigma(g) = \overline{g}$. Then σ is an involutive automorphism of $\mathbf{GL}(n, \mathbb{C})$ as a real Lie group and $\mathrm{d}\,\sigma(A) = \overline{A}$. If $f \in \mathcal{O}[\mathbf{GL}(n, \mathbb{C})]$, then we define

$$\overline{f}(g) = \overline{f(\sigma(g))}. \tag{1.124}$$

if and only if $\overline{f} = f$, $f(\mathbf{GL}(n, \mathbb{R})) \subseteq \mathbb{R}$.

Lemma 1.7.1 Let $G \subseteq \mathbf{GL}(n, \mathbb{C})$ be an algebraic subgroup. Then G is defined on $\mathbb{R}$ if and only if $\mathcal{J}_G$ is invariant under $f \mapsto \overline{f}$.

Suppose that $G \subseteq \mathbf{GL}(n, \mathbb{C})$ is the group defined on $\mathbb{R}$. Let $\mathfrak{g} \subseteq M_n(\mathbb{C})$ be the Lie algebraof G. Since $\mathcal{J}_{\mathbb{R},G}$ generates $\mathcal{J}_G$ and σ^2 is identity map, which means that $\sigma(G) = G$. Thus, σ defines a Lie group automorphism of G and for all $A \in \mathfrak{g}$ we have $\mathrm{d}\,\sigma(A) = \overline{A} \in \mathfrak{g}$. By definition, $G_{\mathbb{R}} = \{g \in G : \sigma(g) = g\}$. Hence $G_{\mathbb{R}}$ is a subgroup of G and

$$\mathrm{Lie}(G_{\mathbb{R}}) = \left\{A \in \mathfrak{g} : \overline{A} = A\right\}. \tag{1.125}$$

If $A \in \mathfrak{g}$ then $A = A_1 + \mathrm{i}\, A_2$, where $A_1 = \frac{A+\overline{A}}{2}$ and $A_2 = \frac{A-\overline{A}}{2\,\mathrm{i}}$ are in $\mathrm{Lie}(G_{\mathbb{R}})$. Thus

$$\mathfrak{g} = \mathrm{Lie}(G_{\mathbb{R}}) \oplus \mathrm{i}\,\mathrm{Lie}(G_{\mathbb{R}}) \tag{1.126}$$

as a real vector space, so $\dim_{\mathbb{R}} \mathrm{Lie}(G_{\mathbb{R}}) = \dim_{\mathbb{C}} \mathfrak{g}$. Therefore the dimension of the Lie group $G_{\mathbb{R}}$ is the same as the dimension of G as a linear algebraic groupon $\mathbb{C}$.

R If the linear algebraic group$G$ is defined on $\mathbb{R}$, then there's a set $\mathcal{A}$ of polynomials with real coefficients s.t. G consists of the common zeros of these polynomials in $\mathbf{GL}(n, \mathbb{C})$. The converse assertion is more subtle, since it's possible that $\mathcal{A}$ may doesn't generates the ideal $\mathcal{J}_G$.

Definition 1.7.2 Let G be a linear algebraic group, τ is an automorphism of G as a real Lie group s.t. τ^2 is the identity. For $f \in \mathcal{O}[G]$ define f^τ by

$$f^\tau(g) = \overline{f(\tau(g))}. \tag{1.127}$$

Then τ is complex conjugation on G if $f^\tau \in \mathcal{O}[G]$ for all $f \in \mathcal{O}[G]$.

When $G \subseteq \mathbf{GL}(n, \mathbb{C})$ is defined on $\mathbb{R}$, the map $\sigma(g) = \overline{g}$ is a complex conjugation.

Theorem 1.7.2 Let G be a linear algebraic group, τ be a complex conjugation on G, then there's a linear algebraic group$H \in \mathbf{GL}(n,)\mathbb{C}$ defined on $\mathbb{R}$ and an automorphism $\rho : G \to H$ s.t. $\rho(\tau(g)) = \sigma(\rho(g))$, where σ is the conjugation of $\mathbf{GL}(n, \mathbb{C})$ given by the complex conjugation of the matrix entries.

Definition 1.7.3 Let G be a linear algebraic group. A subgroup K of G is called a real form of G is there's a complex conjugation on G s.t.

$$K = \{g \in G : \tau(g) = g\}. \tag{1.128}$$

Let K be a real form of G, then K is a closed subgroup of G and the dimension of K as a real Lie group is the same with the dimension of G as a complex linear algebraic group, and

$$\mathfrak{g} = \mathrm{Lie}(K) \oplus \mathrm{i}\,\mathrm{Lie}(K) \tag{1.129}$$

as a real vector space.

Let G be a linear algebraic groupand G° be the connected component of the identity of G as a Lie group. Let K be a real form of G and $\mathfrak{k} = \mathrm{Lie}(K)$.

Proposition 1.7.3 Suppose (ρ, V) is a regular representation of G, then the subspace $W \subseteq V$ is invariant under $\mathrm{d}\,\rho(\mathfrak{k})$ if and only if it's invariant under G°. In particular, V is invariant under $\mathfrak{k}$ if and only if it is irreducible under G°.

1.7.2 Real Forms of the Classical Groups

Type AI

Let $G = \mathbf{GL}(n,\mathbb{C})$ (resp. $\mathbf{SL}(n,\mathbb{C})$) and define $\tau(g) = \overline{g}$ for $g \in G$. Then $f^\tau = f$ for $f \in \mathbb{C}[G]$, and so τ is a complex conjugation on G. The associated real form is $\mathbf{GL}(n,\mathbb{R})$ (resp. $\mathbf{SL}(n,\mathbb{R})$).

Type AII

Let $G = \mathbf{GL}(2n,\mathbb{C})$ (resp. $\mathbf{SL}(2n,\mathbb{C})$) and let J be the $2n \times 2n$ skew-symmetric matrix from Section 1.1.2. We define $\tau(g) = J\overline{g}J^{\mathrm{T}}$ for $g \in G$. Since $J^2 = -I$, we have τ^2 is the identity. Also if f is the regular function on G, then $f^\tau(g) = \overline{f}(JgJ^{\mathrm{T}})$, so f^τ is also the regular function on G. Thus τ is a complex conjugation on G. The equation $\tau(g) = g$ can be written as $Jg = \overline{g}J$. Thus, the real form relative to$G$ is the group $\mathbf{GL}(n,\mathbb{H})$ (resp. $\mathbf{SL}(n,\mathbb{H})$), where we view $\mathbb{H}^n$ as a $2n$-dimensional vector space over $\mathbb{C}$.

Type AIII

Let $G = \mathbf{GL}(n,\mathbb{C})$ (resp. $\mathbf{SL}(n,\mathbb{C})$) and $p, q \in \mathbb{N}$ be s.t. $p + q = n$. Let $I_{p,q} = \mathrm{diag}\,[I_p, -I_q]$ as in Section 1.1.2 and define $\tau(g) = I_{p,q}(g^*)^{-1}I_{p,q}$ for $g \in G$. Since $I_{p,q}^2 = I_n$, we have τ^2 is the identity. Likewise, if f is a regular function on G then $f^\tau = \overline{f}(I_{p,q}(g^{\mathrm{T}})^{-1}I_{p,q})$ is also a regular function on G. Thus τ is a complex conjugation on G. The equation $\tau(g) = g$ can be written as $g^*I_{p,q}g = I_{p,q}$, so the unitary group $\mathbf{U}(p,q)$ (resp. $\mathbf{SU}(p,q)$) is the real form of G defined by τ. The unitary group $\mathbf{U}(n,0) = \mathbf{U}(n)$ (resp. $\mathbf{SU}(n)$) is a compact real form of G.

Type BDI

Let G be $\mathbf{O}(n,\mathbb{C}) = \left\{g \in \mathbf{GL}(n,\mathbb{C}) : gg^{\mathrm{T}} = I\right\}$ (resp. $\mathbf{SO}(n,\mathbb{C})$), and $p, q \in \mathbb{N}$ satisfy that $p + q = n$. Let matrix $I_{p,q}$ be as in Type AIII. Define $\tau(g) = I_{p,q}\overline{g}I_{p,q}$ for $g \in G$. Since $(g^{\mathrm{T}})^{-1} = g$ for $g \in G$, τ is the restriction of the complex conjugation in Example 3. We have the corresponding real form is isomorphic to the group $\mathbf{O}(p,q)$ (resp. $\mathbf{SO}(p,q)$). When $p = n$, we obtain the compact real form $\mathbf{O}(n)$ (resp. $\mathbf{SO}(n)$).

Type DIII

Let G be $\mathbf{SO}(2n,\mathbb{C})$ and J be $2n \times 2n$ skew-symmetric matrix as in Type AII. Define $\tau(g) = J\overline{g}J^{\mathrm{T}}$ for $g \in G$. We have τ is the complex conjugation of G. The corresponding real form is the group $\mathbf{SO}^*(2n)$.

Type CI

Let G be $\mathbf{Sp}(n,\mathbb{C}) \subseteq \mathbf{SL}(2n,\mathbb{C})$. The equation defining G is $g^{\mathrm{T}}Jg = J$, where J is the skew-symmetric matrix in Type AII. Since J is real, we can define $\tau(g) = \overline{g}$ for $g \in G$ and obtain a complex conjugation on G. The corresponding real form is $\mathbf{Sp}(n,\mathbb{R})$.

Type CII

Let $p, q \in \mathbb{N}$ be s.t. $p+q = n$ and let $K_{p,q} = \operatorname{diag}[I_{p,q}, I_{p,q}] \in M_{2n}(\mathbb{R})$ as in Section 1.1.4. Let Ω be the nondegenerate skew form on $\mathbb{C}^{2n}$ with

$$K_{p,q}J = \begin{bmatrix} 0 & I_{p,q} \\ -I_{p,q} & 0 \end{bmatrix}, \tag{1.130}$$

with J as in Type CI.Let $G = \mathbf{Sp}(\mathbb{C}^{2n}, \Omega)$ and define $\tau(g) = K_{p,q}(g^*)^{-1}K_{p,q}$ for $g \in G$. The corresponding real form is the group $\mathbf{Sp}(p,q)$. When $p = n$, we denote $\mathbf{Sp}(n) = \mathbf{Sp}(n,0)$. Since $K_{n,0} = I_{2n}$, we have $\mathbf{Sp}(n) = \mathbf{SU}(2n) \cap \mathbf{Sp}(n,\mathbb{C})$. Thus $\mathbf{Sp}(n)$ is a compact real form of $\mathbf{Sp}(n,\mathbb{C})$.

1.7.3 Summary

We have shown that the classical group can be viewed as:

- the complex linear algebraic group$\mathbf{SL}(n,\mathbb{C}), \mathbf{SO}(n,\mathbb{C})$, and $\mathbf{Sp}(n,\mathbb{C})$ together with their real forms, or alternatively as
- the special linear groups over the fields $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, together with the special isomorohic groups of nondegenerate forms (symmetric or skew symmetric, Hermitian or skew Hermitian) over these fields.

Thus we have the following families of classical groups:

Special linear groups: $\mathbf{SL}(n,\mathbb{R}), \mathbf{SL}(n,\mathbb{C})$, and $\mathbf{SL}(n,\mathbb{H})$. Only $\mathbf{SL}(n,\mathbb{C})$ is an algebraic group over $\mathbb{C}$, whereas the other two are forms of $\mathbf{SL}(n,\mathbb{C})$ (resp. $\mathbf{SL}(2,\mathbb{C})$).

Automorphism groups of forms: On a real vector space, a Hermitian (resp. skew-Hermitian) form is the same as a symmetric (resp. skew-symmetric) form. On a complex vector space skew-Hermitian forms become Hermitian after multiplication by i, and vice versa. On a quaternionic vector space there are no nonzero bilinear forms at all (by the noncommutativity of quaternionic multiplication), so the form must be either Hermitian or skew-Hermitian. Taking these restrictions into account, we see that the possibilities for unimodular isomorohic groups are those in following table:

Table 1.1: Classical Groups and Their Important Real Forms

Type	Group G	Lie Algebra $\mathfrak{g}$	Compact Form K	Other Important Real Forms
A_{n-1}	$\mathbf{SL}(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{C})$	$\mathbf{SU}(n)$	$\mathbf{SL}(n,\mathbb{R})$, $\mathbf{SU}(p,q)$, $\mathbf{SL}(n/2,\mathbb{H})$
B_n	$\mathbf{SO}(2n+1,\mathbb{C})$	$\mathfrak{so}(2n+1,\mathbb{C})$	$\mathbf{SO}(2n+1)$	$\mathbf{SO}(p,q)$ $(p+q=2n+1)$
C_n	$\mathbf{Sp}(2n,\mathbb{C})$	$\mathfrak{sp}(2n,\mathbb{C})$	$\mathbf{Sp}(n)$	$\mathbf{Sp}(2n,\mathbb{R})$, $\mathbf{Sp}(p,q)$
D_n	$\mathbf{SO}(2n,\mathbb{C})$	$\mathfrak{so}(2n,\mathbb{C})$	$\mathbf{SO}(2n)$	$\mathbf{SO}(p,q)$ $(p+q=2n)$, $\mathbf{SO}^*(2n)$

The group $\mathbf{SU}(p,q)$ isn't the linear algebraic groupdefined over $\mathbb{C}$, since its defining equation involve complex conjugation. Likewise, the groups for the field $\mathbb{H}$ aren't algebraic groups over $\mathbb{C}$.

2. Structure of Classical Groups

2.1 Semisimple Elements

2.1.1 Torus Groups

Definition 2.1.1 — (Algebraic) Torus. An (algebraic) torus is an algebraic group T isomorphic to $(\mathbb{C}^\times)^l$. The integer l is called the rank of T. The rank of T is uniquely determined by the algebraic group structure of T.

Definition 2.1.2 — Rational Character. A rational character of a linear algebraic group$K$ is a regular homomorphism $\chi : K \to \mathbb{C}^\times$.

The set $\mathfrak{X}(K)$ of rational character of K has a natural structure of an abelian group with $(\chi_1\chi_2)(k) = \chi_1(k)\chi_2(k)$ dorm $k \in K$. The identity of $\mathfrak{X}(K)$ is the trivial character $\chi_0(k) = 1$ for all $k \in K$.

Lemma 2.1.1 Let T be a algebraic torus with the rank l. The group $\mathfrak{X}(T)$ is isomorphic to $\mathbb{Z}^l$. Furthermore, $\chi(T)$ is a basis of $\mathcal{O}[T]$ as a vector space over $\mathbb{C}$

Proposition 2.1.2 Let T be a algebraic torus and suppose (ρ, V) is a regular representation of T, then there's a finite subset $\Psi \subseteq \mathfrak{X}(T)$ s.t.

$$V = \bigoplus_{x \in \Psi} V(x) \tag{2.1}$$

where $V(x) = \{v \in V : \rho(t)v = \chi(t)v \text{ for all } t \in T\}$ is the χ weight space of T on V. If $g \in \operatorname{End} V$ commute with $\rho(t)$ for all $t \in T$, then $gV(\chi) \subseteq V(\chi)$.

Lemma 2.1.3 Let T be an algebraic toral, then there's an element $t \in T$ with the following properties: $f \in \mathcal{O}[T]$ and $f(t^n) = 0$, then $f = 0$.

2.1.2 Maximal Torus in a Classical Group

Definition 2.1.3 — Maximal Torus. If G is a linear algebraic group, then a torus $H \subseteq G$ is maximal if it's not contained by any larger torus in G.

When G is one of the classical groups $\mathbf{GL}(n), \mathbb{C}, \mathbf{SL}(n,\mathbb{C}), \mathbf{Sp}(\mathbb{C}^n,\Omega), \mathbf{SO}(\mathbb{C}^n,B)$ where Ω is a nondegenerate skew-symmetric bilinear form, B is a nondegenerate symmetric bilinear form. We would like the subgroup H of the diagonal matrices in G is a maximal torus, for this, we choose:

We denote by s_l the $l \times l$ matrix

$$s_l = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{2.2}$$

with 1 on the skew diagonal and 0 elsewhere. Let $n = 2l$ be even, set

$$J_+ = \begin{bmatrix} 0 & s_l \\ s_l & 0 \end{bmatrix}, J_- = \begin{bmatrix} 0 & s_l \\ -s_l & 0 \end{bmatrix}, \tag{2.3}$$

and define bilinear form:

$$B(x,y) = x^{\mathrm{T}} J_+ y, \Omega(x,y) = x^{\mathrm{T}} J_- y, \text{ for } x, y \in \mathbb{C}^n. \tag{2.4}$$

The form B is nondegenerate and symmetric, the form Ω is nondegenerate and skew-symmetric. We have that the Lie algebraof $\mathbf{SO}(\mathbb{C}^{2l}, B)$ consists of all the matrices

$$A = \begin{bmatrix} a & b \\ c & -s_l a^{\mathrm{T}} s_l \end{bmatrix}, a, b, c \in M_l(\mathbb{C}), b^{\mathrm{T}} = -s_l b s_l, c^{\mathrm{T}} = -s_l c s_l. \tag{2.5}$$

Likewise, the Lie algebra$\mathfrak{sp}(\mathbb{C}^{2l},\Omega)$ of $\mathbf{Sp}(\mathbb{C}^{2l},\Omega)$ consists of all the matrices

$$A = \begin{bmatrix} a & b \\ c & -s_l a^{\mathrm{T}} s_l \end{bmatrix}, a, b, c \in M_l(\mathbb{C}), b^{\mathrm{T}} = s_l b s_l, c = s_l c s_l. \tag{2.6}$$

Finallywe consider the orthogonal group on $\mathbb{C}^n$ when $n = 2l+1$ is odd. We take the symmetric bilinear form

$$B(x,y) = \sum_{i+j=n_1} x_i y_i \text{ for } x, y \in \mathbb{C}^n. \tag{2.7}$$

We can write it as $B(x,y) = x^{\mathrm{T}} S y$, where the $n \times n$ symmetric matrix $S = s_{2l+1}$ has block form

$$S = \begin{bmatrix} 0 & 0 & s_l \\ 0 & 1 & 0 \\ s_l & 0 & 0 \end{bmatrix}. \tag{2.8}$$

We have the Lie algebraof $\mathbf{SO}(\mathbb{C}^{2l+1}, B)$ consists of all the matrices

$$A = \begin{bmatrix} a & w & b \\ u^t & 0 & -w^t s_l \\ c & -s_l u & -s_l a^t s_l \end{bmatrix}, \left\{ \begin{array}{l} a, b, c \in M_l(\mathbb{C}), \\ b^t = -s_l b s_l, c^t = -s_l c s_l, \\ \text{ and } u, w \in \mathbb{C}^l. \end{array} \right. \tag{2.9}$$

Suppose G is $\mathbf{GL}(n,\mathbb{C}), \mathbf{SL}(n,\mathbb{C}), \mathbf{Sp}(\mathbb{C}^n,\Omega)$ or $\mathbf{SO}(\mathbb{C}^n,B)$ with Ω and B chosen as above. Let H is the subgroup of G of the diagonal matrices, denote $\mathfrak{g}=\mathrm{Lie}(G)$ and $\mathfrak{h}=\mathrm{Lie}(H)$, we know $\mathfrak{h}$ consists of all the diagonal matrices belong to $\mathfrak{g}$. We have the following description of H and $\mathfrak{h}$:

- 1.When $G=\mathbf{SL}(l+1,\mathbb{C})$ (type $\mathbf{A}_\ell$), then

$$\{\mathrm{diag}[x_1,\dots,x_{l+1}] \mid \prod_{i=1}^{l+1} x_i = 1\}, \tag{2.10}$$

$$\mathfrak{h}=\left\{\mathrm{diag}\left[a_1,a_2,\cdots,a_{l+1}\right] : a_i\mathbb{C}, \sum_i a_i = 0\right\}. \tag{2.11}$$

- When $G-\mathbf{Sp}(\mathbb{C}^{2l},\Omega)$ (type $\mathbf{C}_\ell$) or $G=\mathbf{SO}(\mathbb{C}^{2l},B)$ (type $\mathbf{D}_\ell$), we have

$$H=\left\{\mathrm{diag}\left[x_1,x_2,\cdots,x_l,x_l^{-1},x_2^{-1},\cdots,x_l^{-1}\right] : x_i\in\mathbb{C}^\times\right\}, \tag{2.12}$$

$$\mathfrak{h}=\left\{\mathrm{diag}\left[a_1,a_2,\cdots,a_l,-a_1,-a_2,\cdots,-a_l\right] : a_i\in\mathbb{C}\right\}. \tag{2.13}$$

- When $G=\mathbf{SO}(\mathbb{C}^{2l+1},B)$ (type $\mathbf{B}_\ell$), we have

$$H=\left\{\mathrm{diag}\left[x_1,x_2,\cdots,x_l,1,x_1^{-1},x_2^{-1},\cdots,x_l^{-1}\right] : x_i\in\mathbb{C}^\times\right\}, \tag{2.14}$$

$$\mathfrak{h}=\left\{\mathrm{diag}\left[a_1,a_2,\cdots,a_l,0,-a_1,-a_2,\cdots-a_l\right] : a_i\in\mathbb{C}\right\} \tag{2.15}$$

In all cases, H is isomorphic to l copies of $\mathbb{C}^\times$ as an algebraic group, so it's a torus with rank l. The Lie algebra$\mathfrak{h}$ is isomorphic to the vector space $\mathbb{C}^\times$ with all Lie brackets zero. Define coordinate functions $x_1,x_2,\cdots,x_l$ on H as above, then $\mathcal{O}[H]=\mathbb{C}\left[x_1,x_2,\cdots,x_l,x_1^{-1},x_2^{-1},\cdots,x_l^{-1}\right]$.

Theorem 2.1.4 Let G be $\mathbf{GL}(n,\mathbb{C}), \mathbf{SL}(n,\mathbb{C}), \mathbf{Sp}(\mathbb{C}^n,\Omega)$ or $\mathbf{SO}(\mathbb{C}^n,B)$ in the form given above, where H is the diagonal subgroup of G. Suppose $g\in G$ and $gh=hg$ for all $h\in H$, then $g\in H$.

The choice of the maximal toral H depends on choosing a particular matrix form of G. We shall prove that if T is anyone maximal toral in G, then there's an element $\gamma\in G$ s.t. $T=\gamma H\gamma^{-1}$.

Theorem 2.1.5 If $g\in G$ is semisimple then there's $\gamma\in G$ s.t. $\gamma g\gamma^{-1}\in H$.

Corollary 2.1.6 If T is any torus in G, then there's $\gamma\in G$ s.t. $\gamma T\gamma^{-1}\subseteq H$. In particular, if T is a maximal torus in G, then $\gamma T\gamma^{-1}=H$.

We see the integer $l=\dim H$ doesn't depend on the choice of a particular maximal torus in G, we call l the rank of G.

2.2 Unipotent Elements

2.2.1 Low-Rank Examples

we shall show that the classical groups $\mathbf{SL}(n,\mathbb{C}),\mathbf{SO}(n,\mathbb{C})$ and $\mathbf{Sp}(n,\mathbb{C})$ are generated by their unipotent elements, and we begin with the case $G=\mathbf{SL}(2,\mathbb{C})$. Let $N^+=\{u(z):z\in\mathbb{C}\}$ and $N^-=\{v(z):z\in\mathbb{C}\}$, where

$$u(z)=\begin{bmatrix}1 & z\\ 0 & 1\end{bmatrix}, \text{ and } v(z)=\begin{bmatrix}1 & 0\\ z & 1\end{bmatrix}. \tag{2.16}$$

The groups N^+ and N^- are isomorphic to the additive group of the field $\mathbb{C}$.

Lemma 2.2.1 The group $\mathbf{SL}(2,\mathbb{C})$ is generated by $N^+\cup N^-$.

The orthogonal group and symmetric groups of low rank are closely relative to$\mathbf{GL}(1,\mathbb{C})$ and $\mathbf{SL}(2,\mathbb{C})$. Define a skew-symmetric bilinear form Ω on $\mathbb{C}^2$ by

$$\Omega(v,w)=\det[v,w], \tag{2.17}$$

where $[v,w]\in M_2(\mathbb{C})$ has column v,w. We have $\det[e_1,e_1\det[e_2,e_2]=0$ and $\det[e_1,e_2]=1$, this shows Ω is nondegenerate. Since the determinant function is multiplicative, the form Ω satisfies

$$\Omega(gv,gw)=(\det g)\Omega(v,w) \text{ for } g\in\mathbf{GL}(2,\mathbb{C}). \tag{2.18}$$

Hence g preserves Ω if and only if $\det g=1$. This prove that $\mathbf{Sp}(\mathbb{C}^2,\Omega)=\mathbf{SL}(2,\mathbb{C})$.

Now we consider the group $\mathbf{SO}(\mathbb{C}^2,B)$, where B is the bilinear form given by matrix s_2 in (2.2). We have

$$g^{\mathrm{T}}s_2g=\begin{bmatrix}2ac & ad+bc\\ ad+bc & 2bd\end{bmatrix} \text{ for } g=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in\mathbf{SL}(2,\mathbb{C}). \tag{2.19}$$

Since $ad-bc=1$, it follows $ad+bc=2ad-1$. Thus, $g^{\mathrm{T}}s_2g=s_2$ holds if and only if $ad=1$ and $b=c=0$. So $\mathbf{SO}(\mathbb{C}^2,B)$ consists of the following matrices:

$$\begin{bmatrix}a & 0\\ 0 & a^{-1}\end{bmatrix} \text{ for } a\in\mathbb{C}^\times. \tag{2.20}$$

This furnishes an isomorphism $\mathbf{SO}(\mathbb{C}^2,B)\simeq\mathbf{GL}(1,\mathbb{C})$.

R [Detailed Derivation] The derivation of the condition on the elements of this group can be made more explicit. Let $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathbf{SL}(2,\mathbb{C})$ and let the matrix defining the bilinear form be $s_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. The condition $g^{\mathrm{T}}s_2g=s_2$ expands as follows:

$$\begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}c & a\\ d & b\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}2ac & ad+bc\\ ad+bc & 2bd\end{pmatrix} \tag{2.21}$$

Setting this equal to $s_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ yields the system of equations:

1. $2ac=0$

2. $2bd = 0$
3. $ad + bc = 1$

We also have the condition $ad - bc = 1$ from $g \in \mathbf{SL}(2, \mathbb{C})$. Adding and subtracting equation (3) and the $\mathbf{SL}(2, \mathbb{C})$ condition gives:

- $(ad + bc) + (ad - bc) = 1 + 1 \implies 2ad = 2 \implies ad = 1$.
- $(ad + bc) - (ad - bc) = 1 - 1 \implies 2bc = 0 \implies bc = 0$.

Since $ad = 1$, neither a nor d can be zero. From $2ac = 0$ and $a \neq 0$, we must have $c = 0$. From $2bd = 0$ and $d \neq 0$, we must have $b = 0$. This rigorously shows that g must be a diagonal matrix with $ad = 1$ and $b = c = 0$.

Now we consider the group $G = \mathbf{SO}(\mathbb{C}^3, B)$, where B is the bilinear form on $\mathbb{C}^3$ with the matrix s_3 as in (2.2). We know the diagonal subgroup of $G = \mathbf{SO}(\mathbb{C}^3, B)$ is a maximal torus

$$H = \left\{ \operatorname{diag}\left[x, 1, x^{-1}\right] : x \in \mathbb{C}^\times \right\}. \tag{2.22}$$

Set $\tilde{G} = \mathbf{SL}(2, \mathbb{C})$ and let

$$\tilde{H} = \left\{ \operatorname{diag}\left[x, x^{-1}\right] : x \in \mathbb{C}^\times \right\} \tag{2.23}$$

be the subgroup of diagonal matrices in $\tilde{G}$.

We now define a homomorphism $\rho : \tilde{G} \to G$ that maps $\tilde{H}$ into H. Set

$$V = \{X \in M_2(\mathbb{C}) : \operatorname{tr} X = 0\} \tag{2.24}$$

and let $\tilde{G}$ act on V by $\rho(g) = gXg^{-1}$. The symmetric bilinear form

$$\omega(X, Y) = \frac{1}{2} \operatorname{tr}(XY) \tag{2.25}$$

is obviously invariant under $\rho(\tilde{G})$, since $\operatorname{tr}(XY) = \operatorname{tr}(YX)$ for all $X, Y \in M_n(\mathbb{C})$. The basis

$$v_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, v_1 = \begin{bmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}, v_{-1} = \begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix} \tag{2.26}$$

for ω is isotropic. We identify V with $\mathbb{C}^3$ via the map $v_1 \mapsto e_1, v_0 \mapsto e_2$, and $v_{-1} \mapsto e_3$. Then ω becomes B. We know that any element in the subgroups N^+ and $N^-$is carried by the homomorphism ρ to a unipotent matrix. Thus we have $\det(\rho(g)) = 1$ for all $g \in \tilde{G}$. Thus, $\rho(\tilde{G}) \subseteq G$. If $h = \operatorname{diag}\left[x, x^{-1}\right] \in \tilde{H}$, then $\rho(h)$ has the matrix $\operatorname{diag}\left[x^2, 1, x^{-2}\right]$, relative tothe ordered basis $\{v_1, v_0, v_{-1}\}$ for V. Thus $\rho(\tilde{H}) = H$.

Finally, we consider $G = \mathbf{SO}(\mathbb{C}^4, B)$, where B is the symmetric bilinear form with the matrix s_4 on $\mathbb{C}^4$. We know that the diagonal matrix subgroup

$$H = \left\{ \operatorname{diag}\left[x_1, x_2, x_1^{-1}, x_2^{-1}\right] : x_1, x_2 \in \mathbb{C}^\times \right\} \tag{2.27}$$

of $G = \mathbf{SO}(\mathbb{C}^4, B)$ is a maximal torus. Set $\tilde{G} = \mathbf{SL}(2, \mathbb{C}) \times \mathbf{SL}(2, \mathbb{C})$ and let $\tilde{H}$ be the product of the diagonal subgroups of the factors of $\tilde{G}$. We now define a homomorphism $\pi : \tilde{G} \to G$ which maps $\tilde{H}$ onto H. Set $V = M_2(\mathbb{C})$ and let $\tilde{G}$ act on V by $\pi(a, b)X = aXb^{-1}$. From the quadratic form $Q(X) = 2 \det X$ on V we obtain the symmetric bilinear form $\beta(X, Y) = \det(X + Y) - \det X - \det Y$. Set

$$v_1 = e_{11}, v_2 = e_{12}, v_3 = -e_{21}, v_4 = e_{22} \tag{2.28}$$

Clearly $\beta(\pi(g)X, \pi(g)Y) = \beta(X, Y)$ for $g \in \tilde{G}$. The vectors v_j are β-isotropic and $\beta(v_1, v_4) = \beta(v_2, v_3) = 1$. If we identify V with $\mathbb{C}^4$ via the basis $\{v_1, v_2, v_3, v_4\}$, then β becomes B.

Let $g \in \tilde{G}$ be the form (I, b) or (b, I), where b is either in the subgroup N^+ or N^-. We know $\pi(g)$ is a unipotent matrix, thus we have $\det(\pi(g)) = 1$ for all $g \in \tilde{G}$. So $\pi(\tilde{G}) \subseteq \mathbf{SO}(\mathbb{C}^4, B)$. Given $h = (\operatorname{diag}\left[x_1, x_2^{-1}\right], \operatorname{diag}\left[x_2, x_2^{-1}\right]) \in \tilde{H}$, we have

$$\pi(h) = \operatorname{diag}\left[x_1 x_2^{-1}, x_1 x_2, x_1^{-1} x_2^{-1}, x_1^{-1} x_2\right]. \tag{2.29}$$

Since the map $(x_1, x_2) \mapsto (x_1 x_2^{-1}, x_1 x_2)$ is a surjection, we have shown that $\pi(\tilde{H}) = H$.

Low-Dimensional Isomorphisms

As observed from the Dynkin diagrams, certain low-rank classical Lie algebras are isomorphic. This is not a coincidence but a reflection of deeper structural similarities.

- **Type** $A_1 = B_1 = C_1$: The rank-one simple Lie algebras are all isomorphic, a fact established in the preceding examples:

$$\mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sp}(2, \mathbb{C}) \tag{2.30}$$

- **Type** $B_2 = C_2$: The Dynkin diagrams for B_2 and C_2 are identical (a double bond with an arrow can be reversed without changing the root system's geometry). This corresponds to the Lie algebra isomorphism:

$$\mathfrak{so}(5, \mathbb{C}) \cong \mathfrak{sp}(4, \mathbb{C}) \tag{2.31}$$

- **Type** $A_3 = D_3$: The isomorphism between the Dynkin diagrams of A_3 and D_3 reveals one of the most remarkable low-dimensional coincidences:

$$\mathfrak{sl}(4, \mathbb{C}) \cong \mathfrak{so}(6, \mathbb{C}) \tag{2.32}$$

  This isomorphism can be constructed explicitly using representation theory. Let $V = \mathbb{C}^4$ be the standard representation of $\mathfrak{sl}(4, \mathbb{C})$. Consider the induced action on the 6-dimensional exterior square space $\Lambda^2 V$. There is a natural non-degenerate symmetric bilinear form on $\Lambda^2 V$ given by the wedge product:

$$B(v_1 \wedge v_2, w_1 \wedge w_2) = \det(v_1, v_2, w_1, w_2) \tag{2.33}$$

  The action of any element $X \in \mathfrak{sl}(4, \mathbb{C})$ on $\Lambda^2 V$ preserves this form B. This gives a Lie algebra homomorphism $\rho : \mathfrak{sl}(4, \mathbb{C}) \to \mathfrak{so}(\Lambda^2 V, B) \cong \mathfrak{so}(6, \mathbb{C})$. Since both algebras are simple and have the same dimension (15), this homomorphism is an isomorphism.

2.2.2 Unipotent Generation of Classical Groups

The differential of a regular representation of an algebraic group G gives a representation of $\operatorname{Lie}(G)$. On the nilpotent elements in $\operatorname{Lie}(G)$ the exponential map is algebraic and maps them to unipotent elements in G. This gives an algebraic link from Lie algebra representations to group representations, provided the unipotent elements generate G. We now prove that this is the case for the following families of classical groups.

Theorem 2.2.2 Suppose G is $\mathbf{SL}(l+1,\mathbb{C}), \mathbf{SO}(2l+1,\mathbb{C})$, or $\mathbf{Sp}(l,\mathbb{C})$ with $l>1$, or that G is $\mathbf{SO}(2l,\mathbb{C})$ with $l \geq 2$. Then G is generated by its unipotent elements.

2.2.3 Connected Groups

Definition 2.2.1 — Connected. A linear algebraic group$G$ is said to be connected in the sense of algebraic group if the ring $\mathcal{O}[G]$ has no zero divisors.

■ **Example 2.1** The rings $\mathbb{C}[t]$ and $[t,t^{-1}]$ obviously has no zero divisors; hence the additive group $\mathbb{C}$ and the multiplicative group $\mathbb{C}^{\times}$ is connected. Likewise, the torus D_n of the diagonal matrices and the group N_n^+ of the upper-triangular matrices are connected. ■

■ **Example 2.2** If G and H are connected groups, then the group $G \times H$ is connected, since $\mathcal{O}[G\times H] \simeq \mathcal{O}[G] \otimes \mathcal{O}[H]$. ■

■ **Example 2.3** If G is a connected linear algebraic groupand there's a surjective regular homomorphism $\rho : G \to H$, then H is connected, since $\rho^* : \mathcal{O}[H] \to \mathcal{O}[H]$ is injective. ■

Theorem 2.2.3 Let G be a linear algebraic groupgenerated by unipotent elements, then G is connected as a algebraic group and a Lie group.

Theorem 2.2.4 The groups $\mathbf{GL}(n,\mathbb{C}), \mathbf{SL}(n,\mathbb{C}), \mathbf{SO}(n,\mathbb{C})$ and $\mathbf{Sp}(n,\mathbb{C})$ are connected as linear algebraic groups and Lie groups for all $n \geq 1$.

Theorem 2.2.5 Suppose G is a linear algebraic grouphas the Lie algebra$\mathfrak{g}$, and let (π, V) is a regular representation of G, $W \subseteq V$ is a subspace.
- 1.If $\pi(g)W \subseteq W$ for all $g \in G$ then $\mathrm{d}\,\pi(A)W \subseteq W$ for all $A \in \mathfrak{g}$.
- 2.Suppose G is generated by unipotent elements. If $\mathrm{d}\,\pi(X)W \subseteq W$ for all $xX \in \mathfrak{g}$, then we have $\pi(g)W \subseteq W$ for all $g \in G$. Thus V is irreducible under the action of G if and only if it's irreducible under the action of $\mathfrak{g}$.

2.3 Regular Representations of $\mathbf{SL}(2,\mathbb{C})$

2.3.1 Irreducible Representations of $\mathfrak{sl}(2,\mathbb{C})$

Recall that the representation of a complex Lie algebra$\mathfrak{g}$ on a complex vector space V is a linear map $\pi : \mathfrak{g} \to \operatorname{End} V$ s.t.

$$\pi([A,B]) = \pi(A)\pi(B) - \pi(B)\pi(A) \text{ for all } A,B \in \mathfrak{g}. \tag{2.34}$$

When $v \in V$, we call V a $\mathfrak{g}$-module, and write $\pi(A)v$ as Av. Thus even if $\mathfrak{g}$ is a Lie subalgebra of $M_n(\mathbb{C})$, an expression such as $A^k v$ means $\pi(A)^k v$ for a nonnegative integer k.

Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. The matrices

$$x = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], y = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right], h = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \tag{2.35}$$

is a basis of $\mathfrak{g}$, and satisfy the commutation relations

$$[h,x] = 2x, [h,y] = -2y, [x,y] = h. \tag{2.36}$$

Any triple $\{x,y,h\}$ of nonzero elements satisfies above is called a TDS triple.

Lemma 2.3.1 Let V be a $\mathfrak{g}$-module and $v_0 \in V$ be s.t.$xv_0 = 0$ and $hv_0 = \lambda v_0$ for some $\lambda \in \mathbb{C}$. Set $v_j = y^j v_o$ for $j \in \mathbb{N}$ and $v_j = 0$ for $j < 0$. Then $yv_j = v_{j+1}, hv_j = (\lambda - 2j)v_j$, and

$$xv_j = j(\lambda - j + 1)v_{j-1} \text{ for } j \in \mathbb{N}. \tag{2.37}$$

Let V be a finite dimensional $\mathfrak{g}$- module. We decompose V into generalized eigenspaces for the action of h:

$$V = \bigoplus_{\lambda \in \mathbb{C}} V(\lambda), \text{ where } V(\lambda) = \bigcup_{k \geq 1} \operatorname{Ker}(h - \lambda)^k. \tag{2.38}$$

If $v \in V(\lambda)$, then we have $(h - \lambda)^k v = 0$ for some $k \geq 1$. As linear transformations on V,

$$x(h - \lambda) = (h - \lambda - 2)x \text{ and } y(h - \lambda) = (h - \lambda + 2)x. \tag{2.39}$$

Hence $(h - \lambda - 2)^k xv = x(h - \lambda)^k v = 0$ and $(h - \lambda + 2)^k yv = y(h - \lambda)^k v = 0$. Thus

$$xV(\lambda) \subseteq V(\lambda + 2) \text{ and } yV(\lambda) \subseteq V(\lambda - 2) \text{for all } \lambda \in \mathbb{C}. \tag{2.40}$$

If $V(\lambda) \neq 0$, then λ is called a weight of V with the weight space $V(\lambda)$.

Lemma 2.3.2 Suppose V is a finite dimensional $\mathfrak{g}$-module and $0 \neq v_0 \in V$ satisfies $hv_0 = \lambda v_0$ and $xv_0 = 0$. Let k be the smallest nonnegative integer s.t. $y^k v_0 \neq 0$ and $y^{k+1} v_0 = 0$. Then $\lambda = k$ and the space $W = \{v_0, yv_0, y^2 v_0, \cdots, y^k v_0\}$ is a $(k+1)$ dimensional $\mathfrak{g}$-module.

We can describe the action of $\mathfrak{g}$ on the subspace W in matrix form as follows: For $k \in \mathbb{N}$ define the $(k+1) \times (k+1)$ matrices

$$X_k = \begin{bmatrix} 0 & k & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2(k-1) & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3(k-2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & k \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, Y_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{2.41}$$

and $H = \operatorname{diag}[k, k-2, \cdots, 2-k, -k]$. We have

$$[X_k, Y_k] = H_k, [H_k, X_k] = 2X_k, \text{ and } [H_k, Y_k] = -2Y_k. \tag{2.42}$$

Proposition 2.3.3 Let $k \geq 0$ be an integer, the representation $(\rho_k, F^{(k)})$ of $\mathfrak{g}$ on $\mathbb{C}^{k+1}$ define ny

$$\rho_k(x) = X_k, \rho_k(h) = H_k, \text{ and } \rho_k(y) = Y_k \tag{2.43}$$

is irreducible. Furthermore, if $(\simeq, W)$ is an irreducible representation of $\mathfrak{g}$ with $\dim W = k + 1 > 0$,then (σ, W) is equivalent to $(\rho_k, F^{(k)})$. In particular, W is equivalent to W^* as a $\mathfrak{g}$-module.

Corollary 2.3.4 The weights of the $\mathfrak{g}$-module is integers.

2.3.2 Irreducible Regular Representations of $\mathbf{SL}(2,\mathbb{C})$

Let $d(a) = \operatorname{diag}\left[a, a^{-1}\right]$ for $a \in \mathbb{C}^\times$.

Proposition 2.3.5 For every integer $k \geq 0$, there's a uniquely $(k+1)$ dimensional irreducible regular representation of $\mathbf{SL}(2,\mathbb{C})$, the differential of it is the representation ρ_k, which satisfies:

- 1.The semisimple operator $\pi(d(a))$ has eigenvalues $a^k, a^{k-2}, \cdots, a^{-k+2}, a^{-k}$.
- 2.$\pi(d(a))$ acts on the one dimensional space V^{N^+} of N^+-fixed vectors by the scalar a^k.
- 3.$\pi(d(a))$ acts on the one dimensional space V^{N^-} of N^--fixed vectors by the scalar a^{-k}.

2.3.3 Complete Reducibility of $\mathbf{SL}(2,\mathbb{C})$

Theorem 2.3.6 Let V be a finite dimensional $\mathfrak{g}$-module with $\dim V > 0$, then there's integers $k_1, k_2, \cdots, k_r$ not necessarily distinct s.t. V is equivalent to $F^{(k_1)} \oplus F^{(k_2)} \oplus \cdots \oplus F^{(k_r)}$.

Corollary 2.3.7 Let (ρ, V) be a finite dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$. There's a regular representation (π, W) of $\mathbf{SL}(2,\mathbb{C})$ s.t. $(\mathrm{d}\,\pi, W)$ is equivalent to (ρ, V). Furthermore, every regular representation of $\mathbf{SL}(2,\mathbb{C})$ is a direct sum of irreducible subrepresentations.

2.4 The Adjoint Representation

2.4.1 Roots with Respect to a Maximal Torus

In this section, G denote a connected classical group with the rank l, thus G is $\mathbf{GL}(l,\mathbb{C})$, $\mathbf{SL}(l+1,\mathbb{C})$, $\mathbf{Sp}(\mathbb{C}^{2l},\Omega)$, $\mathbf{SO}(\mathbb{C}^{2l},B)$ or $\mathbf{SO}(\mathbb{C}^{2l+1},B)$, where we take as Ω and B the bilinear form we like. We set $\mathfrak{g} = \operatorname{Lie}(G)$. The subgroup H of the diagonal matrices in G is a maximal torus with the rank l, we denote its Lie algebra$\mathfrak{h}$. We shall consider the regular representation π of H on the vector space $\mathfrak{g}$, given by $\pi(h)X = hXh^{-1}$, where $h \in H$ and $X \in \mathfrak{g}$.

Let $x_1, x_2, \cdots, x_l$ be the coordinate functions on H. Using these coordinates we obtain an isomorphism between the group $X(H)$ of rational representations of H and the additive group $\mathbb{Z}^l$. Under this isomorphism, $\lambda = [\lambda_1, \lambda_2, \cdots, \lambda_l] \in \mathbb{Z}^l$ corresponds to the character $h \mapsto h^\lambda$ where

$$h^\lambda = \prod_{1\leq k\leq l} x_k(h)^{\lambda_k} \tag{2.44}$$

For $\lambda, \mu \in \mathbb{Z}^l$ and $h \in H$, we have $h^\lambda h^\mu = \lambda^{\lambda+\mu}$

Fix the following bases for $\mathfrak{h}^*$:

- (a)Let $G = \mathbf{GL}(l,\mathbb{C})$. Define $\langle \varepsilon_i, A\rangle = a_i$ for $A = \operatorname{diag}[a_1, a_2, \cdots, a_l] \in \mathfrak{h}$. Then $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_l\}$ is a basis for $\mathfrak{h}^*$.
- (b)Let $G = \mathbf{SL}(l+1,\mathbb{C})$. Then $\mathfrak{h}$ consists of all the matrices with the trace 0. We will continue to denote the restrictions to $\mathfrak{h}$ of the linear functional in (a) by ε_i. The elements of $\mathfrak{h}^*$ can then be written uniquely as $\sum_{1\leq i\leq l+1} \lambda_i \varepsilon_i$ with $\lambda_i \in \mathbb{C}$ and $\sum_{1\leq i\leq l+1} \lambda_i = 0$. A basis of $\mathfrak{h}^*$ is given by the functional:

$$\varepsilon_i - \frac{1}{l+1}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{l+1}) \text{ for } i = 1, 2, \cdots, l. \tag{2.45}$$

- (c)Let G be $\mathbf{Sp}(\mathbb{C}^{2l},\Omega)$ or $\mathbf{SO}(\mathbb{C}^{2l},B)$. For $i=1,2,\cdots,l$, define $\langle\varepsilon_i,A\rangle=a_i$, where $A=[a_1,a_2,\cdots,a_l,-a_1,-a_2,\cdots,-a_l]\in\mathfrak{h}$, then $\{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_l\}$ is a basis for $\mathfrak{h}^*$.
- (d) Let G be $\mathbf{SO}(\mathbb{C}^{2l+1},B)$. For $A=\operatorname{diag}[a_1,a_2,\cdots,a_l,0,-a_1,-a_2,\cdots,-a_l]\in\mathfrak{h}$ and $i=1,2,\cdots,l$ define $\langle\varepsilon_i,A\rangle=a_i$. THen $\{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_l\}$ is a basis of $\mathfrak{h}^*$.

We define $P(G)=\{\mathrm{d}\,\theta:\theta\in\mathfrak{X}(H)\}\subseteq\mathfrak{h}^*$. With the above functionals ε_i, we have

$$P(G)=\bigoplus_{1\le k\le l}\mathbb{Z}\varepsilon_k. \tag{2.46}$$

Indeed, given $\lambda=\lambda_1\varepsilon_1+\lambda_2\varepsilon_2+\cdots+\lambda_l\varepsilon_l$ with $\lambda_i\in\mathbb{Z}$, let e^λ denote the rational character of H determined by $[\lambda_1,\lambda_2,\cdots,\lambda_l]\in\mathbb{Z}^l$ as above. Every element in $\mathfrak{X}(H)$ is og this form, and we claim that $\mathrm{d}\,\mathrm{e}^\lambda(A)=\langle\lambda,A\rangle$ for $A\in\mathfrak{h}$.

The map $\lambda\mapsto\mathrm{e}^\lambda$ is an isomorphism between the additive group $P(G)$ and the character group $\mathfrak{X}(H)$. $P(G)$ is a lattice (free abelian subgroup of rank l) in $\mathfrak{h}^*$, which is called the weight lattice of G.

We now study the adjoint action of H and $\mathfrak{h}$ on $\mathfrak{g}$. From $\alpha\in P(G)$ let

$$\mathfrak{g}_\alpha=\left\{X\in\mathfrak{g}:hXh^{-1}=h^\alpha X\text{ for all }h\in H\right\}=\{X\in\mathfrak{g}:[A,X]=\langle\alpha,A\rangle\,X\text{ for all }A\in\mathfrak{h}\}. \tag{2.47}$$

For $\alpha=0$, we have $\mathfrak{g}_0=\mathfrak{h}$. If $\alpha\neq0$ and $\mathfrak{g}_\alpha\neq0$, then α is called a root of H on $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is called a root space. If α is a root, then a nonzero element in $\mathfrak{g}_\alpha$ is called a root vector of $\mathfrak{g}_\alpha$. We call the set Φ of roots the root system of $\mathfrak{g}$. Its definition requires fixing a choice of maximal torus, so we write $\Phi=\Phi(\mathfrak{g},\mathfrak{h})$ when we want to make it explicit.

We have the root space decomposition

$$\mathfrak{g}=\mathfrak{h}\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha. \tag{2.48}$$

Theorem 2.4.1 Let $G\subseteq\mathbf{GL}(n,\mathbb{C})$ is a classical connected group and $H\subseteq G$ is a maximal torus with the Lie algebra$\mathfrak{h}$. Let $\Phi\subseteq\mathfrak{h}^*$ is the root system of $\mathfrak{g}$.

- 1.dim $\mathfrak{g}_\alpha=1$ for all $\alpha\in\Phi$
- 2.If $\alpha\in\Phi$ and $cx\in\Phi$ for some $\alpha\in\Phi$, then $c=\pm1$.
- The symmetric bilinear form $(X,Y)=\operatorname{tr}_{\mathbb{C}^n}(XY)$ is invariant on $\mathfrak{g}$:

$$([X,Y],X)=-(Y,[X,Z])\text{ for }X,Y,Z\in\mathfrak{g} \tag{2.49}$$

- 4. Let $\alpha,\beta\in\Phi$ and $\alpha\neq-\beta$. Then $(\mathfrak{h},\mathfrak{g}_\alpha)=0$ and $(\mathfrak{g}_\alpha,\mathfrak{g}_\beta)=0$.
- 5.The form (X,Y) is nondegenerate on $\mathfrak{g}$.

2.4.2 Commutation Relations of Root Spaces

We observe that

$$[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]\subseteq\mathfrak{g}_{\alpha+\beta}\text{ for }\alpha,\beta\in\mathfrak{h}^*. \tag{2.50}$$

Indeed, let $A\in\mathfrak{h}$, then

$$[A,[X,Y]]=[[A,X],Y]+[X,[A,Y]]=\langle\alpha+\beta,A\rangle[X,Y] \tag{2.51}$$

for $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$. Thus $[X, Y] \in \mathfrak{g}_{\alpha+\beta}$. In particular, if $\alpha + \beta$ isn't a root, then $\mathfrak{g}_{\alpha+\beta} = 0$, so in this case X and Y commute. We also see

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_0 = \mathfrak{h}. \tag{2.52}$$

When $\alpha, \beta, \alpha+\beta$ are all root, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$ holds, thus the inclusion is an equality.

Lemma 2.4.2 For each $\alpha \in \Phi$ there's $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ s.t. the element $h_\alpha = [e_\alpha, f_\alpha]$ satisfies $\langle \alpha, h_\alpha \rangle = 2$. Thus

$$[h_\alpha, e_\alpha] = 2e_\alpha, [h_\alpha, f_\alpha] = -2f_\alpha, \tag{2.53}$$

so that $\{e_\alpha, f_\alpha, h_\alpha\}$ is a TDS triple.

If $\{e_\alpha, f_\alpha, h_\alpha\}$ is a TDS triple satisfies the condition relative to α, then we can take $e_{-\alpha} = f_\alpha$ and $f_{-\alpha} = e_\alpha$.

Definition 2.4.1 — Coroot. The element $h_\alpha := [e_\alpha, f_\alpha]$ is called the **coroot** associated with the root α. It is an element of the Cartan subalgebra $\mathfrak{h}$. There is a natural duality between roots and coroots. While the roots α are linear functionals on $\mathfrak{h}$ (i.e., $\alpha \in \mathfrak{h}^*$), the coroots h_α are elements of $\mathfrak{h}$ itself. This duality is central to the structure of the root system, with the defining relation being $\langle \alpha, h_\alpha \rangle = 2$.

Type A:
Let $\alpha = \varepsilon_i - \varepsilon_j$ with $1 \leq i < j \leq l+1$. Set $e_\alpha = e_{ij}$ and $f_\alpha = e_{ji}$. Then $h_\alpha = e_{ii} - e_{jj}$.

Type B:
(a) For $\alpha = \varepsilon_i - \varepsilon_j$ with $1 \leq i < j \leq l$ set $e_\alpha = e_{i,j} - e_{-j,-i}$ and $f_\alpha = e_{j,i} - e_{-i,-j}$. Then $h_\alpha = e_{i,i} - e_{j,j} + e_{-j,-j} - e_{-i,-i}$.
(b) For $\alpha = \varepsilon_i + \varepsilon_j$ with $1 \leq i < j \leq l$ set $e_\alpha = e_{i,-j} - e_{j,-i}$ and $f_\alpha = e_{-j,i} - e_{-i,j}$. Then $h_\alpha = e_{i,i} + e_{j,j} - e_{-j,-j} - e_{-i,-i}$.
(c) For $\alpha = \varepsilon_i$ with $1 \leq i \leq l$ set $e_\alpha = e_{i,0} - e_{0,-i}$ and $f_\alpha = 2e_{0,i} - 2e_{-i,0}$. Then $h_\alpha = 2e_{i,i} - 2e_{-i,-i}$.

Type C:
(a) For $\alpha = \varepsilon_i - \varepsilon_j$ with $1 \leq i < j \leq l$ set $e_\alpha = e_{i,j} - e_{-j,-i}$ and $f_\alpha = e_{j,i} - e_{-i,-j}$. Then $h_\alpha = e_{i,i} - e_{j,j} + e_{-j,-j} - e_{-i,-i}$.
(b) For $\alpha = \varepsilon_i + \varepsilon_j$ with $1 \leq i < j \leq l$ set $e_\alpha = e_{i,-j} + e_{j,-i}$ and $f_\alpha = e_{-j,i} - e_{-i,j}$. Then $h_\alpha = e_{i,i} + e_{j,j} - e_{-j,-j} - e_{-i,-i}$.
(c) For $\alpha = 2\varepsilon_i$ with $1 \leq i \leq l$ set $e_\alpha = e_{i,-i}$ and $f_\alpha = e_{-i,i}$. Then $h_\alpha = e_{i,i} - e_{-i,-i}$.

Type D:
(a) For $\alpha = \varepsilon_i - \varepsilon_j$ with $1 \leq i < j \leq l$ set $e_\alpha = e_{i,j} - e_{-j,-i}$ and $f_\alpha = e_{j,i} - e_{-i,-j}$. Then $h_\alpha = e_{i,i} - e_{j,j} + e_{-j,-j} - e_{-i,-i}$.
(b) For $\alpha = \varepsilon_i + \varepsilon_j$ with $1 \leq i < j \leq l$ set $e_\alpha = e_{i,-j} - e_{j,-i}$ and $f_\alpha = e_{-j,i} - e_{-i,j}$. Then $h_\alpha = e_{i,i} + e_{j,j} - e_{-j,-j} - e_{-i,-i}$.

We call h_α the coroot to α. Since the dimension of the space $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is 1, h_α can be determined by the properties $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ and $\langle \alpha, h_\alpha \rangle = 2$. For $X, Y \in \mathfrak{g}$, let the bilinear form (X, Y) be as the Theorem 2.4.1. This form is nondegenerate on $\mathfrak{h} \times \mathfrak{h}$, thus we can equivalent $\mathfrak{h}$ to $\mathfrak{h}^*$. So we have h_α is proportional to α. Furthermore, $(h_\alpha, h_\alpha) =$

$\langle \alpha, h_\alpha \rangle (e_\alpha, f_\alpha) \neq 0$. Thus we have when $\mathfrak{h}$ is identified to $\mathfrak{h}^*$

$$\alpha = \frac{2}{(h_\alpha, h_\alpha)} h_\alpha. \tag{2.54}$$

We also use the notation $\check{\alpha}$ for the coroot h_α.

For $\alpha \in \Phi$, we use $\mathfrak{s}(\alpha)$ denote the algebra generated by $\{e_\alpha, f_\alpha, h_\alpha\}$, it is isotropic to $\mathfrak{sl}(2, \mathbb{C})$ under the map $e \mapsto e_\alpha, f \mapsto f_\alpha, h \mapsto h_\alpha$. The algebra $\mathfrak{g}$ becomes a module of $\mathfrak{s}(\alpha)$ by the restriction of the adjoint representation of $\mathfrak{g}$ on $\mathfrak{s}(\alpha)$.

Let $\alpha, \beta \in \Phi$ with $\alpha \neq \pm\beta$. We observe that $\beta + k\alpha \neq 0$. Hence for every $k \in \mathbb{Z}$,

$$\dim \mathfrak{g}_{\beta+k\alpha} = \begin{cases} 1 \text{ if } \beta + k\alpha \in \Phi, \\ 0 \text{ otherwise.} \end{cases} \tag{2.55}$$

Let

$$R(\alpha, \beta) = \{\beta + k\alpha : k \in \mathbb{Z}\} \cap \Phi, \tag{2.56}$$

which we call the α root string through β. The number of elements of a root string is called the length of the string. Define

$$V_{\alpha,\beta} = \sum_{\gamma \in R(\alpha,\beta)} \mathfrak{g}_\gamma. \tag{2.57}$$

Lemma 2.4.3 For every $\alpha, \beta \in \Phi$ with $\alpha \neq \pm\beta$, the space $V_{\alpha,\beta}$ is invariant and irreducible under $\mathrm{ad}(\mathfrak{s}(\alpha))$.

Corollary 2.4.4 Let $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$. Let p be the largest integer $j \geq 0$ s.t. $\beta + j\alpha \in \Phi$ and q be the largest integer $k \geq 0$ s.t. $\beta - k\alpha \in \Phi$. Then

$$\langle \beta, h_\alpha \rangle = q - p \in \mathbb{Z}, \tag{2.58}$$

and for all integer r satisfies $-q \leq r \leq q$ we have $\beta + r\alpha \in \Phi$. In particular, $\beta - \langle \beta, h_\alpha \rangle \alpha \in \Phi$.

R From the case-by-case calculations for types A ? D made above we see that

$$\langle \beta, h_\alpha \rangle \in \{0, \pm 1, \pm 2\} \text{ for all } \alpha, \beta \in \Phi. \tag{2.59}$$

2.4.3 Structure of Classical Root Systems

Let Φ be the root system of a classical Lie algebra $\mathfrak{g}$ of type A_ℓ, B_ℓ, C_ℓ, or D_ℓ ($\ell \geq 2$). The set of roots Φ spans the dual space of the Cartan subalgebra, $\mathfrak{h}^*$.

Definition 2.4.2 — Simple Roots. A subset $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_l\} \subseteq \Phi$ is a set of **simple roots** if every root $\gamma \in \Phi$ can be written uniquely as a linear combination $\gamma = \sum n_i \alpha_i$, where the coefficients n_i are integers all of the same sign.

A choice of simple roots Δ partitions the full root system Φ into two disjoint sets: the **positive roots** Φ^+ (where all coefficients $n_i \geq 0$) and the **negative roots** Φ^- (where all $n_i \leq 0$).

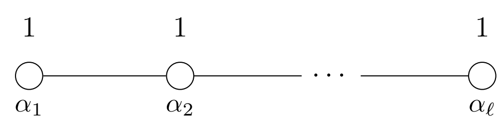
Definition 2.4.3 — Cartan Matrix. The integers $C_{ij} = \langle \alpha_j, h_{\alpha_i} \rangle$ form the entries of the $l \times l$ **Cartan matrix** C. Note that the diagonal entries are always $C_{ii} = \langle \alpha_i, h_{\alpha_i} \rangle = 2$. For $i \neq j$, the entries are non-positive integers.

The Cartan matrix is efficiently encoded by a **Dynkin diagram**, a graph where nodes represent simple roots. The number of lines between nodes i and j is $C_{ij}C_{ji}$. If roots have different lengths, an arrow points from the longer root to the shorter one. We also label each node with the coefficient of the corresponding simple root in the expansion of the highest root.

Here are the standard choices for simple roots, the resulting positive roots, the highest root $\tilde{\alpha}$, and the corrected Dynkin diagrams for each classical type.

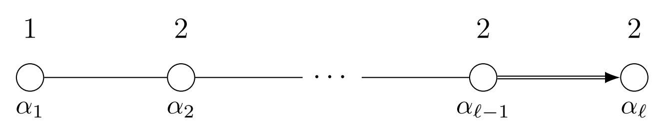
Type $\mathbf{A}_\ell$ ($\mathfrak{sl}(\ell+1, \mathbb{C})$), $\ell \geq 1$:

- **Simple Roots** Δ: $\{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq \ell\}$.
- **Positive Roots** Φ^+: $\{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq \ell + 1\}$.
- **Highest Root** $\tilde{\alpha}$: $\varepsilon_1 - \varepsilon_{\ell+1} = \alpha_1 + \cdots + \alpha_\ell$.



Type $\mathbf{B}_\ell$ ($\mathfrak{so}(2\ell+1, \mathbb{C})$), $\ell \geq 2$:

- **Simple Roots** Δ: $\{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i < \ell\} \cup \{\varepsilon_\ell\}$.
- **Positive Roots** Φ^+: $\{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq \ell\} \cup \{\varepsilon_i \mid 1 \leq i \leq \ell\}$.
- **Highest Root** $\tilde{\alpha}$: $\varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_\ell$.



Type $\mathbf{C}_\ell$ ($\mathfrak{sp}(2\ell, \mathbb{C})$), $\ell \geq 3$:

- **Simple Roots** Δ: $\{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i < \ell\} \cup \{2\varepsilon_\ell\}$.
- **Positive Roots** Φ^+: $\{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq \ell\} \cup \{2\varepsilon_i \mid 1 \leq i \leq \ell\}$.
- **Highest Root** $\tilde{\alpha}$: $2\varepsilon_1 = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$.

Type $\mathbf{D}_\ell$ ($\mathfrak{so}(2\ell, \mathbb{C})$), $\ell \geq 4$:

- **Simple Roots** Δ: $\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \le i < \ell\} \cup \{\alpha_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell\}$.
- **Positive Roots** Φ^+: $\{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le \ell\}$.
- **Highest Root** $\tilde{\alpha}$: $\varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$.

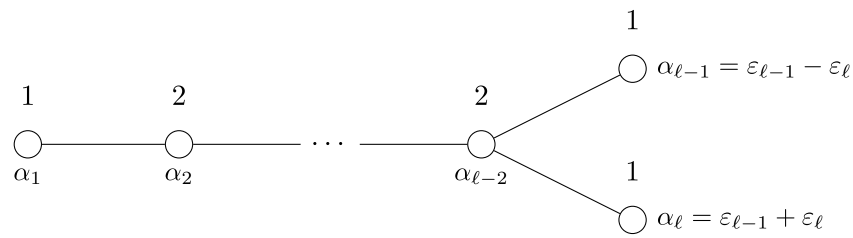


Table 2.1: Root System Data of the Classical Lie Algebras

Type	**Lie Algebra** $\mathfrak{g}$	**Simple Roots** Δ	**Positive Roots** Φ^+	**Highest Root** $\tilde{\alpha}$
A_l $(l \ge 1)$	$\mathfrak{sl}(l+1, \mathbb{C})$	$\varepsilon_i - \varepsilon_{i+1}$	$\varepsilon_i - \varepsilon_j$ $(i < j)$	$\varepsilon_1 - \varepsilon_{l+1}$
B_l $(l \ge 2)$	$\mathfrak{so}(2l+1, \mathbb{C})$	$\varepsilon_i - \varepsilon_{i+1}$ $(i < l)$, ε_l	$\varepsilon_i \pm \varepsilon_j$ $(i < j)$, ε_i	$\varepsilon_1 + \varepsilon_2$
C_l $(l \ge 3)$	$\mathfrak{sp}(2l, \mathbb{C})$	$\varepsilon_i - \varepsilon_{i+1}$ $(i < l)$, $2\varepsilon_l$	$\varepsilon_i \pm \varepsilon_j$ $(i < j)$, $2\varepsilon_i$	$2\varepsilon_1$
D_l $(l \ge 4)$	$\mathfrak{so}(2l, \mathbb{C})$	$\varepsilon_i - \varepsilon_{i+1}$ $(i < l)$, $\varepsilon_{l-1} + \varepsilon_l$	$\varepsilon_i \pm \varepsilon_j$ $(i < j)$	$\varepsilon_1 + \varepsilon_2$

2.4.4 Irreducibility of the Adjoint Representation

Theorem 2.4.5 Let G be one of the groups $\mathbf{SL}(\mathbb{C}^{l+1})$, $\mathbf{Sp}(\mathbb{C}^{2l})$, $\mathbf{SO}(\mathbb{C}^{2l+1})$ with $l \geq 1$, or $\mathbf{SO}(\mathbb{C}^{2l})$ with $l \geq 3$. Then the adjoint representation of G is irreducible.

R For any Lie algebra $\mathfrak{g}$, the subspaces of $\mathfrak{g}$ that are invariant under $\mathrm{ad}(\mathfrak{g})$ are the *ideals* of $\mathfrak{g}$. A Lie algebra is called *simple* if it is not abelian and it has no proper ideals. (By this definition the one-dimensional Lie algebra is not simple, even though it has no proper ideals.) The classical Lie algebras occurring in Theorem 2.4.13 are thus simple. Note that their Dynkin diagrams are connected graphs.

R A Lie algebra is called *semisimple* if it is a direct sum of simple Lie algebras. The low-dimensional orthogonal Lie algebras excluded from Theorem 2.4.11 and Theorem 2.4.13 are $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$, which is semisimple (with a disconnected Dynkin diagram), and $\mathfrak{so}(2,\mathbb{C}) \cong \mathfrak{gl}(1,\mathbb{C})$, which is abelian (and has no roots).

2.5 Semisimple Lie Algebras

2.5.1 Solvable Lie Algebras

Lemma 2.5.1 Let V be a finite - dimensional complex vector space and $A \in \mathrm{End}(V)$. Suppose there exist $X_i, Y_i \in \mathrm{End}(V)$ such that $A = \sum_{i=1}^{k} [X_i, Y_i]$ and $[A, X_i] = 0$ for all i. Then A is nilpotent.

Definition 2.5.1 — Completely Reducible. A finite - dimensional representation (π, V) of a Lie algebra $\mathfrak{g}$ is called completely reducible if for every $\mathfrak{g}$ - invariant subspace $W \subset V$, there exists a $\mathfrak{g}$ - invariant complementary subspace U such that $W \cap U = \{0\}$ and $V = W \oplus U$.

Theorem 2.5.2 Let V be a finite - dimensional complex vector space, $\mathfrak{g}$ be a Lie subalgebra of $\mathrm{End}(V)$, and V be a completely reducible representation of $\mathfrak{g}$. Let $\mathfrak{z} = \{X \in \mathfrak{g} : [X, Y] = 0, \forall Y \in \mathfrak{g}\}$ be the center of $\mathfrak{g}$. Then:

1. Every $A \in \mathfrak{z}$ is a semisimple linear transformation.
2. $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{z} = 0$.
3. $\mathfrak{g}/\mathfrak{z}$ has no non - zero abelian ideal.

Definition 2.5.2 — Derived Algebra. For a Lie algebra $\mathfrak{g}$, we define the derived algebra $\mathcal{D}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ and recursively define $\mathcal{D}^{k+1}(\mathfrak{g}) = \mathcal{D}(\mathcal{D}^k(\mathfrak{g}))$ for $k = 1, 2, \ldots$. By induction on k, one can show that $\mathcal{D}^k(\mathfrak{g})$ is invariant under all derivations of $\mathfrak{g}$. In particular, for each k, $\mathcal{D}^k(\mathfrak{g})$ is an ideal in $\mathfrak{g}$ and $\mathcal{D}^k(\mathfrak{g})/\mathcal{D}^{k+1}(\mathfrak{g})$ is abelian.

Definition 2.5.3 — Solvable Lie Algebra. $\mathfrak{g}$ is solvable if there is an integer $k \geq 1$ s.t. $\mathcal{D}^{||}\mathfrak{g} = /$

It is clear from the definition that a Lie subalgebra of a solvable Lie algebra is also solvable. Also, if $\pi : \mathfrak{g} \to \mathfrak{h}$ is a surjective Lie algebra homomorphism, then $\pi(\mathcal{D}^k(\mathfrak{g})) = \mathcal{D}^k(\mathfrak{h})$. Hence the solvability of $\mathfrak{g}$ implies the solvability of $\mathfrak{h}$. Furthermore, if $\mathfrak{g}$ is a nonzero solvable Lie algebra and we choose k such that $\mathcal{D}^k(\mathfrak{g}) \neq 0$ and $\mathcal{D}^{k+1}(\mathfrak{g}) = 0$, then $\mathcal{D}^k(\mathfrak{g})$ is an abelian ideal in $\mathfrak{g}$ that is invariant under all derivations of $\mathfrak{g}$.

R The archetypical example of a solvable Lie algebra is the $n \times n$ upper - triangular matrices $\mathfrak{b}_n$. Indeed, we have $\mathcal{D}(\mathfrak{b}_n) = \mathfrak{n}_n^+$, the Lie algebra of $n \times n$ upper - triangular matrices with zeros on the main diagonal. If $\mathfrak{n}_{n,r}^+$ is the Lie subalgebra of $\mathfrak{n}_n^+$ consisting of matrices $X = [x_{ij}]$ such that $x_{ij} = 0$ for $j - i \leq r - 1$, then $\mathfrak{n}_n^+ = \mathfrak{n}_{n,1}^+$ and $[\mathfrak{n}_n^+, \mathfrak{n}_{n,r}^+] \subseteq \mathfrak{n}_{n,r+1}^+$ for $r = 1, 2, \cdots$. Hence $\mathcal{D}^k(\mathfrak{b}_n) \subseteq \mathfrak{n}_{n,k}^+$, and so $\mathcal{D}^k(\mathfrak{b}_n) = 0$ for $k > n$.

Corollary 2.5.3 Suppose $\mathfrak{g} \subset \mathrm{End}(V)$ is a solvable Lie algebra and that V is completely reducible as a $\mathfrak{g}$ - module. Then $\mathfrak{g}$ is abelian. In particular, if V is an irreducible $\mathfrak{g}$ - module, then $\dim V = 1$.

Theorem 2.5.4 Let V be a finite - dimensional complex vector space. Let $\mathfrak{g} \subset \mathrm{End}(V)$ be a Lie subalgebra such that $\mathrm{tr}(XY) = 0$ for all $X, Y \in \mathfrak{g}$. Then $\mathfrak{g}$ is solvable.

Recall that a finite - dimensional Lie algebra is simple if it is not abelian and has no proper ideals.

Corollary 2.5.5 Let $\mathfrak{g}$ be a Lie subalgebra of $\mathrm{End}(V)$ that has no nonzero abelian ideals. Then the bilinear form $\mathrm{tr}(XY)$ on $\mathfrak{g}$ is nondegenerate, and $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ (Lie algebra direct sum), where each $\mathfrak{g}_i$ is a simple Lie algebra.

Corollary 2.5.6 Assume:

- V is a finite-dimensional complex vector space
- $\mathfrak{g}$ is a Lie algebra of linear transformations on V
- V is a completely reducible representation of $\mathfrak{g}$ (i.e., V can be decomposed into a direct sum of irreducible subrepresentations)

Then:

1. The derived algebra of $\mathfrak{g}$ (generated by all commutators) is semisimple
2. $\mathfrak{g}$ can be decomposed as the direct sum of its derived algebra and center: $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}$

Definition 2.5.4 — Killing Form. For a Lie algebra $\mathfrak{g}$, the Killing form B is defined as:

$$B(X, Y) = \mathrm{tr}(\mathrm{ad}X \circ \mathrm{ad}Y) \tag{2.60}$$

where:

- $\mathrm{ad}X$ and $\mathrm{ad}Y$ are linear transformations on $\mathfrak{g}$
- $\mathrm{ad}X(Y) = [X, Y]$ (i.e., bracket operation with X)

Theorem 2.5.7 A Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form B is nondegenerate.

Corollary 2.5.8 Suppose $\mathfrak{g}$ is a semisimple Lie algebra and D is a derivation of $\mathfrak{g}$ (i.e., $D \in \mathrm{Der}(\mathfrak{g})$). Then an element X can be found in $\mathfrak{g}$ such that the derivation D is equivalent to the adjoint map $\mathrm{ad}X$. That is, for any element Y in $\mathfrak{g}$, $D(Y) = [X, Y]$.

For the subsequent result, we require the following formula, which holds for any elements Y and Z in a Lie algebra $\mathfrak{g}$, any derivation $D \in \mathrm{Der}(\mathfrak{g})$, and any scalars λ and μ:

$$(D - (\lambda + \mu))^k [Y, Z] = \sum_r \binom{k}{r} [(D - \lambda)^r Y, (D - \mu)^{k-r} Z] \tag{2.61}$$

(The proof of this formula is accomplished by induction on k, making use of the properties of derivations and the inclusion - exclusion identity for binomial coefficients.)

Corollary 2.5.9 Let $\mathfrak{g}$ be a semisimple Lie algebra. If $X \in \mathfrak{g}$ and $\mathrm{ad}X$ has an additive Jordan decomposition $\mathrm{ad}X = S + N$ in $\mathrm{End}(\mathfrak{g})$ (where S is semisimple, N is nilpotent, and $[S, N] = 0$). Then elements X_s and X_n exist in $\mathfrak{g}$ such that $\mathrm{ad}X_s = S$ and $\mathrm{ad}X_n = N$.

2.5.2 Root Space Decomposition

In this section, we aim to demonstrate that every semisimple Lie algebra possesses a root space decomposition with the properties established in Section 2.4 for the Lie algebras of classical groups. We commence with the following generalization of a well - known property of nilpotent linear transformations in the context of Lie algebras:

Theorem 2.5.10 — Engel. Let V be a non - zero finite - dimensional vector space and $\mathfrak{g} \subset \mathrm{End}(V)$ be a Lie algebra. Assume that for every $X \in \mathfrak{g}$, X is a nilpotent linear transformation. Then there exists a non - zero vector $v_0 \in V$ such that $Xv_0 = 0$ for all $X \in \mathfrak{g}$.

Corollary 2.5.11 There exists a basis for V where the elements of $\mathfrak{g}$ are represented by strictly upper - triangular matrices.

Corollary 2.5.12 Suppose $\mathfrak{g}$ is a semisimple Lie algebra. Then there is a non - zero element $X \in \mathfrak{g}$ for which $\mathrm{ad}X$ is semisimple.

For the remainder of this section, let $\mathfrak{g}$ be a semisimple Lie algebra. We define a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ as a toral subalgebra if $\mathrm{ad}X$ is semisimple for every $X \in \mathfrak{h}$. Corollary 2.5.16 implies the existence of non - zero toral subalgebras.

Lemma 2.5.13 Let $\mathfrak{h}$ be a toral subalgebra. Then $[\mathfrak{h}, \mathfrak{h}] = 0$.

We call a toral subalgebra $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra if it has the maximum dimension among all toral subalgebras of $\mathfrak{g}$. We know that $\mathfrak{g}$ contains non - zero Cartan subalgebras and that Cartan subalgebras are abelian. We fix a Cartan subalgebra $\mathfrak{h}$. For $\lambda \in \mathfrak{h}^*$, we define:

$$\mathfrak{g}_\lambda = \{Y \in \mathfrak{g} : [X, Y] = \langle \lambda, X \rangle Y \text{ for all } X \in \mathfrak{h}\} \tag{2.62}$$

In particular, $\mathfrak{g}_0 = \{Y \in \mathfrak{g} : [X, Y] = 0 \text{ for all } X \in \mathfrak{h}\}$ is the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. Let $\Phi \subset \mathfrak{g}^* \setminus \{0\}$ be the set of λ such that $\mathfrak{g}_\lambda \neq 0$. We refer to Φ as the set of roots of $\mathfrak{h}$ on $\mathfrak{g}$.

Since the mutually commuting linear transformations ad$X$ (for $X \in \mathfrak{h}$) are semisimple, there is a root space decomposition:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Phi} \mathfrak{g}_\lambda \tag{2.63}$$

Let B denote the Killing form of $\mathfrak{g}$. Using similar arguments as in Sections 2.4.1 and 2.4.2 for classical groups (but now using B instead of the trace form in the defining representation of a classical group), we have:

1. $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$;
2. $B(\mathfrak{g}_\lambda, \mathfrak{g}_\mu) = 0$ if $\lambda + \mu \neq 0$;
3. The restriction of B to $\mathfrak{g}_0 \times \mathfrak{g}_0$ is non - degenerate;
4. If $\lambda \in \Phi$, then $-\lambda \in \Phi$ and the restriction of B to $\mathfrak{g}_\lambda \times \mathfrak{g}_{-\lambda}$ is non - degenerate.

Proposition 2.5.14 A Cartan algebra is its own centralizer in $\mathfrak{g}$, that is, $\mathfrak{h} = \mathfrak{g}_0$.

Corollary 2.5.15 Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \Phi} \mathfrak{g}_\lambda \tag{2.64}$$

Consequently, if $Y \in \mathfrak{g}$ and $[Y, \mathfrak{h}] \subset \mathfrak{h}$, then $Y \in \mathfrak{h}$. In particular, $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$.

Since the form B is non - degenerate on $\mathfrak{h} \times \mathfrak{h}$, it induces a bilinear form on $\mathfrak{h}^*$, denoted by (α, β).

Theorem 2.5.16 The roots and root spaces satisfy the following properties:

1. Φ spans $\mathfrak{h}^*$.
2. If $\alpha \in \Phi$, then $\dim[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = 1$ and there is a unique element $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\langle \alpha, h_\alpha \rangle = 2$ (we call h_α the coroot of α).
3. If $\alpha \in \Phi$ and $c \in \mathbb{C}$, then $c\alpha \in \Phi$ if and only if $c = \pm 1$. Also, $\dim \mathfrak{g}_\alpha = 1$.
4. Let $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$. Let p be the largest integer $j \geq 0$ such that $\beta + j\alpha \in \Phi$ and q be the largest integer $k \geq 0$ such that $\beta - k\alpha \in \Phi$. Then $\langle \beta, h_\alpha \rangle = q - p \in \mathbb{Z}$, and $\beta + r\alpha \in \Phi$ for all integers r with $-q \leq r \leq p$. Hence $\beta - \langle \beta, h_\alpha \rangle \alpha \in \Phi$.
5. If $\alpha, \beta \in \Phi$ and $\alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

2.5.3 Geometry of Root Systems

Let $\mathfrak{g}$ be a semisimple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ and let Φ be the set of roots of $\mathfrak{h}$ on $\mathfrak{g}$. For $\alpha \in \Phi$, there exists a TDS - triple $\{e_\alpha, f_\alpha, h_\alpha\}$ with $\langle \alpha, h_\alpha \rangle = 2$. Define $\check{\alpha} = n_\alpha \alpha$, where $n_\alpha = B(e_\alpha, f_\alpha) \in \mathbb{Z} \setminus \{0\}$. Then h_α corresponds to $\check{\alpha}$ under the isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$ given by the Killing form B, and we call $\check{\alpha}$ the coroot of α.

Due to the complete reducibility of representations of $\mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{g}$ decomposes into the direct sum of irreducible representations under the adjoint action of $\mathfrak{s}(\alpha) = \mathrm{Span}\{e_\alpha, f_\alpha, h_\alpha\}$. From Proposition 2.3.3 and Theorem 2.3.6, for any finite - dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$, e_α and f_α act as integer matrices with respect to a suitable basis. Hence, the trace of $\mathrm{ad}(e_\alpha)\mathrm{ad}(f_\alpha)$ is an integer.

Since Span$\Phi = \mathfrak{h}^*$, we can choose a basis $\{\alpha_1, \cdots, \alpha_l\}$ for $\mathfrak{h}^*$ consisting of roots. Let $H_i = h_{\alpha_i}$, and from (2.49), $\{H_1, \cdots, H_l\}$ is a basis for $\mathfrak{h}$. Define $\mathfrak{h}_\mathbb{Q} = \mathrm{Span}_\mathbb{Q}\{H_1, \cdots, H_l\}$,

$\mathfrak{h}_{\mathbb{Q}}^* = \mathrm{Span}_{\mathbb{Q}}\{\alpha_1, \cdots, \alpha_l\}$, where $\mathbb{Q}$ is the field of rational numbers.

Lemma 2.5.17 Each root $\alpha \in \Phi$ lies in $\mathfrak{h}_{\mathbb{Q}}^*$, and the element h_α is in $\mathfrak{h}_{\mathbb{Q}}$. Let $a, b \in \mathfrak{h}_{\mathbb{Q}}$, then $B(a, b) \in \mathbb{Q}$, and $B(a, a) > 0$ if $a \neq 0$.

Corollary 2.5.18 Let $\mathfrak{h}_{\mathbb{R}}$ be the real span of $\{h_\alpha : \alpha \in \Phi\}$ and $\mathfrak{h}_{\mathbb{R}}^*$ be the real span of the roots. Then the Killing form is real - valued and positive definite on $\mathfrak{h}_{\mathbb{R}}$. Furthermore, $\mathfrak{h}_{\mathbb{R}} \cong \mathfrak{h}_{\mathbb{R}}^*$ under the Killing - form duality.

Let $E = \mathfrak{h}_{\mathbb{R}}^*$ with the bilinear form $(\cdot, \cdot)$ defined by the dual of the Killing form. By Corollary 2.5.22, E is an l - dimensional real Euclidean vector space. We have $\Phi \subset E$, and the coroots are related to the roots by $\check{\alpha} = \frac{2}{(\alpha,\alpha)}\alpha$ for $\alpha \in \Phi$. Let $\check{\Phi} = \{\check{\alpha} : \alpha \in \Phi\}$ be the set of coroots. Then $(\beta, \check{\alpha}) \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$ by (2.48).

An element $h \in E$ is called regular if $(\alpha, h) \neq 0$ for all $\alpha \in \Phi$. Since the set $\bigcup_{\alpha \in \Phi}\{h \in E : (\alpha, h) = 0\}$ is a finite union of hyperplanes, regular elements exist. Fix a regular element h_0 and define $\Phi^+ = \{\alpha \in \Phi : (\alpha, h_0) > 0\}$. Then $\Phi = \Phi^+ \cup (-\Phi^+)$. We call the elements of Φ^+ positive roots. A positive root α is called indecomposable if there do not exist $\beta, \gamma \in \Phi^+$ such that $\alpha = \beta + \gamma$ (these definitions depend on the choice of h_0).

Proposition 2.5.19 Let Δ be the set of indecomposable positive roots.

1. Δ is a basis for the vector space E.
2. Every positive root is a linear combination of elements of Δ with non - negative integer coefficients.
3. If $\beta \in \Phi^+ \setminus \Delta$, then there exists $\alpha \in \Delta$ such that $\beta - \alpha \in \Phi^+$.
4. If $\alpha, \beta \in \Delta$, then the α - root string through β is $\beta, \beta + \alpha, \cdots, \beta + p\alpha$, where $p = -(\beta, \check{\alpha}) \geq 0$.

We call the elements of Δ simple roots (relative to the choice of Φ^+). Fix an enumeration $\alpha_1, \cdots, \alpha_l$ of Δ and write $E_i = e_{\alpha_i}$, $F_i = f_{\alpha_i}$, and $H_i = h_{\alpha_i}$ for the elements of the TDS - triple associated with α_i. Define the Cartan integers $C_{ij} = \langle \alpha_j, H_i \rangle$ and the $l \times l$ Cartan matrix $C = [C_{ij}]$. Note that $C_{ii} = 2$ and $C_{ij} \leq 0$ for $i \neq j$.

Theorem 2.5.20 The simple root vectors $\{E_1, \cdots, E_l, F_1, \cdots, F_l\}$ generate $\mathfrak{g}$. They satisfy the relations $[E_i, F_j] = 0$ for $i \neq j$ and $[H_i, H_j] = 0$, where $H_i = [E_i, F_i]$. They also satisfy the following relations determined by the Cartan matrix:

$$[H_i, E_j] = C_{ij}E_j, \quad [H_i, F_j] = -C_{ij}F_j \tag{2.65}$$

$$\mathrm{ad}(E_i)^{-C_{ij}+1}E_j = 0 \quad (i \neq j) \tag{2.66}$$

$$\mathrm{ad}(F_i)^{-C_{ij}+1}F_j = 0 \quad (i \neq j) \tag{2.67}$$

Corollary 2.5.21 Define $\mathfrak{n}^+ = \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- = \sum_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}$. Then:

- $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are Lie subalgebras of $\mathfrak{g}$ that are invariant under the adjoint action of $\mathfrak{h}$ (i.e., under $\mathrm{ad}\mathfrak{h}$).
- The Lie algebra $\mathfrak{g}$ can be decomposed as $\mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$.

- Additionally, $\mathfrak{n}^+$ is generated by the set $\{E_1, \dots, E_l\}$, and $\mathfrak{n}^-$ is generated by the set $\{F_1, \dots, F_l\}$.

R We define the height of a root (relative to the system of positive roots) in the same way as for the Lie algebras of classical groups: For a root $\sum_i c_i\alpha_i$, its height $\mathrm{ht}(\sum_i c_i\alpha_i) = \sum_i c_i$ (where the coefficients c_i are integers of the same sign). Then:
- $\mathfrak{n}^- = \sum_{\mathrm{ht}(\alpha)<0} \mathfrak{g}_\alpha$
- $\mathfrak{n}^+ = \sum_{\mathrm{ht}(\alpha)>0} \mathfrak{g}_\alpha$

Let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^+$. Then $\mathfrak{b}$ is a maximal solvable subalgebra of $\mathfrak{g}$, which we call a Borel subalgebra.

The set Δ of simple roots is called decomposable if it can be partitioned into non - empty disjoint subsets $\Delta_1 \cup \Delta_2$ such that $\Delta_1 \perp \Delta_2$ with respect to the inner product on the vector space E. Otherwise, Δ is called indecomposable.

Theorem 2.5.22 A semisimple Lie algebra $\mathfrak{g}$ is a simple Lie algebra if and only if the set Δ of simple roots is indecomposable.

2.5.4 Conjugacy of Cartan Subalgebras

Our previous results about the semisimple Lie algebra $\mathfrak{g}$ were based on the choice of a particular Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Now we show that this choice is actually irrelevant, which is a generalization of Corollary 2.1.8.

If $X \in \mathfrak{g}$ is nilpotent, then $\mathrm{ad}X$ is a nilpotent derivation of $\mathfrak{g}$, and $\exp(\mathrm{ad}X)$ is a Lie algebra automorphism of $\mathfrak{g}$, called an elementary automorphism. By Proposition 1.3.14, it satisfies:

$$\mathrm{ad}(\exp(\mathrm{ad}X)Y) = \exp(\mathrm{ad}X)\mathrm{ad}Y\exp(-\mathrm{ad}X) \quad \text{for } Y \in \mathfrak{g} \tag{2.68}$$

Let $\mathrm{Int}(\mathfrak{g})$ be the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by these elementary automorphisms.

Theorem 2.5.23 Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be Cartan subalgebras of $\mathfrak{g}$. Then there exists an automorphism $\varphi \in \mathrm{Int}(\mathfrak{g})$ such that $\varphi(\mathfrak{h}_1) = \mathfrak{h}_2$.

Lemma 2.5.24 Suppose $Z \in \mathfrak{b}$ is semisimple. Write $Z = H + Y$, where $H \in \mathfrak{h}$ and $Y \in \mathfrak{n}^+$. Then $\dim\ker(\mathrm{ad}Z) = \dim\ker(\mathrm{ad}H) \geq \dim\mathfrak{h}$, with equality if and only if H is regular.

Lemma 2.5.25 Let $H \in \mathfrak{h}$ be regular. For $X \in \mathfrak{n}^+$, define $f(X) = \exp(\mathrm{ad}X)H - H$. Then f is a polynomial map from $\mathfrak{n}^+$ to $\mathfrak{n}^+$.

Corollary 2.5.26 Suppose $Z \in \mathfrak{b}$ is semisimple and $\dim\ker(\mathrm{ad}Z) = \dim\mathfrak{h}$. Then there exist $X \in \mathfrak{n}^+$ and a regular element $H \in \mathfrak{h}$ such that $\exp(\mathrm{ad}X)H = Z$.

We now present the key result related to two Borel subalgebras.

Lemma 2.5.27 Suppose $\mathfrak{b}_i = \mathfrak{h}_i + \mathfrak{n}_i$ are Borel subalgebras of $\mathfrak{g}$, $i = 1, 2$. Then $\mathfrak{b}_1 = \mathfrak{b}_1 \cap \mathfrak{b}_2 + \mathfrak{n}_1$.

R Let $Z \in \mathfrak{g}$ be a semisimple element. If $\dim\ker(\mathrm{ad}Z)$ attains the smallest possible value among all elements of $\mathfrak{g}$, we call Z regular. From Theorem 2.5.28, we know that this minimum dimension is the rank of $\mathfrak{g}$. Moreover, if Z is regular, then $\ker(\mathrm{ad}Z)$ is a Cartan subalgebra of $\mathfrak{g}$, and all Cartan subalgebras can be obtained in this way.

3. Highest-Weight Theory

3.1 Roots and Weights

In the representation theory of classical groups, when restricting a representation to a maximal torus H, the representation decomposes into a direct sum of weight spaces. These weight spaces are permuted under the action of the Weyl group W_G. As a finite group, W_G plays a central role in representation theory. For each type of classical group, the structure of its W_G is explicitly known. Specifically, W_G acts faithfully on the dual space $\mathfrak{h}^*$ of the Lie algebra of the maximal torus, and is generated by reflections in root hyperplanes. This theoretical framework also applies to the study of general semisimple Lie algebras.

3.1.1 Weyl Group

Let G be a connected classical group and $H \subset G$ its maximal torus. Define the normalizer of H:

$$\mathrm{Norm}_G(H) = \{g \in G : ghg^{-1} \in H \text{ for all } h \in H\} \tag{3.1}$$

The corresponding Weyl group is defined as $W_G = \mathrm{Norm}_G(H)/H$. Since all maximal tori in G are conjugate, W_G is uniquely determined by G as an abstract group. This group acts as automorphisms of H via conjugation. Subsequent studies will show that problems in the representation theory and invariant theory of G can be reduced to studying W_G-invariant functions on H. The effectiveness of this approach lies in the fact that W_G is a finite group with known structure (typically symmetric groups or their extensions).

The Two Definitions of the Weyl Group

It is important to emphasize that the Weyl group has two fundamental definitions. The first, as presented above, is group-theoretic: $W_G = \mathrm{Norm}_G(H)/H$.

For a semisimple Lie algebra $\mathfrak{g}$ with root system Φ, there is a second, more abstract definition: the Weyl group W is the finite group of orthogonal transformations on $\mathfrak{h}^*$ generated by the set of all **root reflections** $\{s_\alpha \mid \alpha \in \Phi\}$.

A non-trivial and deep theorem in Lie theory states that these two definitions are equivalent: the group of cosets $\mathrm{Norm}_G(H)/H$ acting on $\mathfrak{h}^*$ is precisely the group generated by the root reflections. Explicitly mentioning this equivalence helps to bridge the gap between the concrete, group-theoretic viewpoint and the abstract, Lie-algebraic framework that follows.

R [Equivalence of the Weyl Group Definitions] It is also important to emphasize that the equivalence of the two definitions of the Weyl group—the group-theoretic definition $W(G,H) := N_G(H)/H$ and the geometric one as the group generated by root reflections—is a deep and highly non-trivial theorem. This theorem establishes a fundamental bridge between the global topology and structure of the Lie group G and the combinatorial structure of the root system of its Lie algebra $\mathfrak{g}$.

Due to the commutativity of H, there exists a natural homomorphism $\varphi : W_G \to \mathrm{Aut}(H)$ defined by $\varphi(sH)h = shs^{-1}$. This induces an action of W_G on the character group $\mathcal{K}(H)$: for any $\theta \in \mathcal{K}(H)$, define

$$s \cdot \theta(h) = \theta(s^{-1}hs), \quad h \in H \tag{3.2}$$

(Note that the right-hand side depends only on the coset sH). When the character is expressed in exponential form $\theta = e^\lambda$ (where $\lambda \in P(G)$), this action can be written as:

$$s \cdot e^\lambda = e^{s\cdot\lambda}, \quad \langle s \cdot \lambda, x\rangle = \langle \lambda, \mathrm{Ad}(s)^{-1}x\rangle \quad (x \in \mathfrak{h}) \tag{3.3}$$

This defines a representation of W_G on $\mathfrak{h}^*$.

Theorem 3.1.1 W_G is a finite group and its representation on $\mathfrak{h}^*$ is faithful.

For each classical group G, we now describe its Weyl group W_G. Following the method in Theorem 3.1.1's proof, we embed W_G into the symmetric group $\mathfrak{S}_n$. For any permutation $\sigma \in \mathfrak{S}_n$, let $s_\sigma \in \mathbf{GL}(n,\mathbb{C})$ be the permutation matrix satisfying $s_\sigma e_i = e_{\sigma(i)}$, representing the standard action of $\mathfrak{S}_n$ on $\mathbb{C}^n$.

When $G = \mathbf{GL}(n,\mathbb{C})$, the maximal torus H consists of all $n \times n$ diagonal matrices. Clearly $s_\sigma \in \mathrm{Norm}_G(H)$ for any $\sigma \in \mathfrak{S}_n$. By Theorem 3.1.1, every coset in W_G has the form $s_\sigma H$, hence $W_G \cong \mathfrak{S}_n$. The action on coordinate functions is:

$$\sigma \cdot x_i = x_{\sigma^{-1}(i)} \tag{3.4}$$

For $G = \mathrm{SL}(n,\mathbb{C})$, H contains diagonal matrices with determinant 1. Given $\sigma \in \mathfrak{S}_n$, we choose $\lambda_i \in \mathbb{C}^\times$ such that the transformation s defined in (3.2) satisfies $\det(s) = 1$. Since any permutation decomposes into transpositions (e.g., $(1,\ldots,k) = (1,k)\cdots(1,2)$), it suffices to verify the case when σ is a transposition $i \leftrightarrow j$. Taking $\lambda_j = -1$ and $\lambda_k = 1$ for $k \neq j$ ensures $\det(s) = 1$. Thus $W_G \cong \mathfrak{S}_n$, though note this isomorphism requires specific normalizer elements - the full permutation matrices don't form a subgroup of G.

For $G = \mathrm{Sp}(\mathbb{C}^{2l},\Omega)$, let $s_l \in \mathrm{GL}(l,\mathbb{C})$ correspond to the permutation $(1,l)(2,l-1)\cdots$. For $\sigma \in \mathfrak{S}_l$, define:

$$\pi(\sigma) = \begin{bmatrix} s_\sigma & 0 \\ 0 & s_l s_\sigma s_l \end{bmatrix} \in \mathrm{Norm}_G(H) \tag{3.5}$$

with $\pi(\sigma) \in H$ iff $\sigma = 1$, yielding an injection $\bar{\pi} : \mathfrak{S}_l \hookrightarrow W_G$.

For transpositions $(i, 2l+1-i)$ in $\mathfrak{S}_{2l}$, define $\tau_i \in \mathrm{GL}(2l, \mathbb{C})$ by:

$$\tau_i e_i = e_{-i}, \quad \tau_i e_{-i} = -e_i, \quad \tau_i e_k = e_k \ (k \neq \pm i) \tag{3.6}$$

For any subset $F \subset \{1, \ldots, l\}$, $\tau_F = \prod_{i \in F} \tau_i$ generates an abelian subgroup $T_l \cong (\mathbb{Z}/2\mathbb{Z})^l$ with action:

$$x_i \mapsto x_i^{-1} \ (i \in F), \quad x_j \mapsto x_j \ (j \notin F) \tag{3.7}$$

and conjugation relation $\pi(\sigma)\tau_F\pi(\sigma)^{-1} = \tau_{\sigma F)}$.

Lemma 3.1.2 For $G = \mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$, $T_l \triangleleft W_G$ is normal and W_G is the semidirect product of T_l with $\bar{\pi}(\mathfrak{S}_l)$. The group action on $\mathcal{O}[H]$ is:

$$x_i \mapsto (x_{\sigma(i)})^{\pm 1} \tag{3.8}$$

where σ is a permutation and ± 1 are optional exponents.

Consider the odd-dimensional special orthogonal group $G = \mathrm{SO}(\mathbb{C}^{2l+1}, B)$ with symmetric form B as in (2.9). For any permutation $\sigma \in \mathfrak{S}_l$, define the block matrix:

$$\varphi(\sigma) = \begin{bmatrix} s_\sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s_l s_\sigma s_l \end{bmatrix} \in \mathrm{Norm}_G(H) \tag{3.9}$$

where $\varphi(\sigma) \in H$ holds if and only if $\sigma = 1$. This yields an injective homomorphism $\Phi : \mathfrak{S}_l \hookrightarrow W_G$.

Following the symplectic group construction, define index correspondence $e_{-i} = e_{2l+2-i}$ $(i = 1, \ldots, l+1)$. For transpositions $(i, 2l+2-i)$ in $\mathfrak{S}_{2l+1}$, define $\gamma_i \in \mathrm{GL}(2l+1, \mathbb{C})$ by:

$$\gamma_i e_i = e_{-i}, \quad \gamma_i e_{-i} = e_i, \quad \gamma_i e_0 = -e_0 \tag{3.10}$$

with $\gamma_i e_k = e_k$ for $k \neq i, 0, -i$. Setting γ_i to act as -1 on e_{l+1} ensures $\det \gamma_i = 1$, hence $\gamma_i \in \mathrm{Norm}_G(H)$. These satisfy $\gamma_i^2 \in H$ and $\gamma_i\gamma_j = \gamma_j\gamma_i$ for $1 \leq i, j \leq l$.

For any subset $F \subset \{1, \ldots, l\}$, the product $\gamma_F = \prod_{i \in F} \gamma_i$ generates an abelian subgroup $T_l \cong (\mathbb{Z}/2\mathbb{Z})^l$ via its H-cosets. The action on coordinate functions $\mathcal{O}[H]$ coincides with the τ_F action in the symplectic case.

Lemma 3.1.3 For $G = \mathrm{SO}(\mathbb{C}^{2l+1}, B)$, the subgroup $T_l \triangleleft W_G$ is normal, and W_G is the semidirect product of T_l with $\Phi(\mathfrak{S}_l)$. The Weyl group action on coordinates is:

$$x_i \mapsto (x_{\sigma(i)})^{\pm 1} \quad (i = 1, \ldots, l) \tag{3.11}$$

where σ is a permutation and ± 1 are optional exponents.

3.1.2 Root Reflections

For classical groups $G = \mathrm{Sp}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$ $(n \geq 2)$, or $\mathrm{SO}(n, \mathbb{C})$ $(n \geq 3)$, given root system $\Phi \subset \mathfrak{h}^*$ and simple roots $\Delta \subset \Phi^+$, the root reflection $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ is defined by:

$$s_\alpha(\beta) = \beta - \langle \beta, h_\alpha \rangle \alpha \quad (\beta \in \mathfrak{h}^*) \tag{3.12}$$

This operator acts as:

$$s_\alpha(\beta) = \begin{cases} -\beta & \text{if } \beta \in \mathbb{C}\alpha \\ \beta & \text{if } \langle \beta, h_\alpha \rangle = 0 \end{cases} \tag{3.13}$$

Geometrically, it represents reflection across the hyperplane $(h_\alpha)^\perp$. Using the inner product on $\mathfrak{h}_R^*$, the formula becomes:

$$s_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \tag{3.14}$$

Lemma 3.1.4 For $W = \mathrm{Norm}_G(H)/H$:

1. Every root reflection s_α corresponds to some $w \in W$
2. $W \cdot \Delta = \Phi$
3. W is generated by simple reflections $\{s_\alpha : \alpha \in \Delta\}$
4. If $w\Phi^+ = \Phi^+$ then $w = 1$
5. $\exists! w_0 \in W$ such that $w_0\Phi^+ = -\Phi^+$

For a semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ and positive roots Φ^+, let $\Delta = \{\alpha_1, ..., \alpha_l\}$ be simple roots.

Definition 3.1.1 — Fundamental Weights. The fundamental weights $\{\varpi_1, ..., \varpi_l\}$ satisfy:

$$\langle \varpi_i, \hat{\alpha}_j \rangle = \delta_{ij} \tag{3.15}$$

Simple reflections act on them via:

$$s_{\alpha_i}\varpi_j = \varpi_j - \delta_{ij}\alpha_i \tag{3.16}$$

Definition 3.1.2 — Weyl Group. The Weyl group $W(\mathfrak{g}, \mathfrak{h})$ is the orthogonal transformation group generated by root reflections. It is finite and consistent with classical group cases.

Theorem 3.1.5 For a semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, root system Φ, simple roots $\Delta \subset \Phi$, and Weyl group W, all properties (1)-(5) hold.

Definition 3.1.3 — Positive Weyl Chamber. In the Euclidean space $\mathfrak{h}_{\mathbb{R}}^*$, the closed convex cone:

$$C = \{\mu \in \mathfrak{h}_{\mathbb{R}}^* : (\mu, \alpha_i) \geq 0 \text{ for } i = 1, ..., l\} \tag{3.17}$$

is called the *positive Weyl chamber* relative to Φ^+. For $\mu = \sum_{i=1}^{l} c_i\varpi_i$, the coefficients satisfy:

$$c_i = (\mu, \hat{\alpha}_i) = \frac{2(\mu, \alpha_i)}{(\alpha_i, \alpha_i)} \tag{3.18}$$

Thus $\mu \in C$ if and only if all $c_i \geq 0$.

The dual cone is defined as:

$$C^* = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : (\lambda, \varpi_i) \geq 0 \text{ for } i = 1, ..., l\} \tag{3.19}$$

For $\lambda = \sum_{i=1}^{l} d_i \alpha_i$, the coefficients satisfy:

$$d_i = \frac{2(\lambda, \varpi_i)}{(\alpha_i, \alpha_i)} \tag{3.20}$$

Hence $\lambda \in C^*$ iff all $d_i \geq 0$.

Proposition 3.1.6 The positive Weyl chamber C is a fundamental domain for W-action on $\mathfrak{h}_{\mathbb{R}}^*$:

1. For any $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$, $\exists \mu \in C$ and $w \in W$ such that $w \cdot \lambda = \mu$
2. μ is uniquely determined
3. When λ is regular, w is also unique

3.1.3 Weight Lattice

Definition 3.1.4 — Reductive. A complex Lie algebra $\mathfrak{g}$ is *reductive* if it decomposes as:

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_1 \tag{3.21}$$

where $\mathfrak{z}(\mathfrak{g})$ is the center and $\mathfrak{g}_1$ is semisimple. In this case $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ (since semisimple algebras equal their derived algebras). Example: $\mathfrak{gl}(n, \mathbb{C}) = \mathbb{C}I \oplus \mathfrak{sl}(n, \mathbb{C})$.

For a reductive $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h} = \mathfrak{z}(\mathfrak{g}) + \mathfrak{h}_0$, define:

$$P(\mathfrak{g}) = \{\mu \in \mathfrak{h}^* : \langle \mu, h_\alpha \rangle \in \mathbb{Z}, \forall \alpha \in \Phi\}, \quad Q(\mathfrak{g}) = \mathbb{Z}\Phi \tag{3.22}$$

with $Q(\mathfrak{g}) \subset P(\mathfrak{g})$. These depend only on $\mathfrak{g}$ by Cartan subalgebra conjugacy.

Theorem 3.1.7 For a finite-dimensional representation (π, V), the weight spaces:

$$V(\mu) = \{v \in V : \pi(Y)v = \langle \mu, Y \rangle v, \forall Y \in \mathfrak{h}\} \tag{3.23}$$

satisfy $V = \bigoplus_{\mu \in \mathcal{X}(V)} V(\mu)$ when $\mathfrak{z}(\mathfrak{g})$ acts semisimply, with $\mathcal{X}(V) \subset P(\mathfrak{g})$.

Fix positive roots Φ^+ and simple roots $\Delta = \{\alpha_1, ..., \alpha_l\}$. Then:

$$P(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})^* \oplus \bigoplus_{i=1}^{l} \mathbb{Z}\varpi_i \tag{3.24}$$

where ϖ_i are fundamental weights (vanishing on $\mathfrak{z}(\mathfrak{g})$). For semisimple $\mathfrak{g}$, $P(\mathfrak{g})$ is a free abelian group of rank l.

We now give the fundamental weights for each type of classical group in terms of the weights $\{\varepsilon_i\}$ (see the formulas in Section 2.4.3 giving H_i in terms of the diagonal matrices $E_i = e_{ii}$).

Type A ($\mathfrak{g} = \mathfrak{sl}(l+1, \mathbb{C})$)

Since $H_i = E_i - E_{i+1}$, we have

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{l+1}(\varepsilon_1 + \cdots + \varepsilon_{l+1}) \quad \text{for } 1 \leq i \leq l. \tag{3.25}$$

Type B ($\mathfrak{g} = \mathfrak{so}(2l+1, \mathbb{C})$)

Here $H_i = E_i - E_{i+1} + E_{-i-1} - E_{-i}$ for $1 \le i \le l-1$ and $H_l = 2E_l - 2E_{-l}$, so we have

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad \text{for } 1 \le i \le l-1, \tag{3.26}$$

and

$$\varpi_l = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l). \tag{3.27}$$

Type C ($\mathfrak{g} = \mathfrak{sp}(l, \mathbb{C})$)

In this case $H_l = E_l - E_{-l}$ and for $1 \le i \le l-1$ we have $H_i = E_i - E_{i+1} + E_{-i-1} - E_{-i}$. Thus

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad \text{for } 1 \le i \le l. \tag{3.28}$$

Type D ($\mathfrak{g} = \mathfrak{so}(2l, \mathbb{C})$ with $l \ge 2$)

For $1 \le i \le l-1$ we have

$$H_i = E_i - E_{i+1} + E_{-i-1} - E_{-i} \quad \text{and} \quad H_l = E_{l-1} + E_l - E_{-l} - E_{-l+1}. \tag{3.29}$$

A direct calculation shows that

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad \text{for } 1 \le i \le l-2, \tag{3.30}$$

and

$$\varpi_{l-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{l-1} - \varepsilon_l), \quad \varpi_l = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{l-1} + \varepsilon_l). \tag{3.31}$$

Let G be a connected classical group. Since the functionals ε_i are weights of the defining representation of G, we have $\varepsilon_i \in P(\mathfrak{g})$ for $i = 1, \ldots, l$. Thus $P(G) \subset P(\mathfrak{g})$ by (2.21).

For $\mathfrak{g}$ of type A or C all the fundamental weights are in $P(G)$, so $P(G) = P(\mathfrak{g})$ when $G = \mathbf{SL}(n, \mathbb{C})$ or $\mathbf{Sp}(n, \mathbb{C})$.

However, for $G = \mathbf{SO}(2l+1, \mathbb{C})$ we have

$$\varpi_i \in P(G) \quad \text{for } 1 \le i \le l-1, \quad 2\varpi_l \in P(G), \tag{3.32}$$

but $\varpi_l \notin P(G)$.

For $G = \mathbf{SO}(2l, \mathbb{C})$ we have

$$\varpi_i \in P(G) \quad \text{for } 1 \le i \le l-2, \quad m\varpi_{l-1} + n\varpi_l \in P(G) \quad \text{if } m + n \in 2\mathbb{Z}, \tag{3.33}$$

but ϖ_{l-1} and ϖ_l are not in $P(G)$. Therefore

$$P(\mathfrak{g})/P(G) \cong \mathbb{Z}/2\mathbb{Z} \quad \text{when } G = \mathbf{SO}(n, \mathbb{C}). \tag{3.34}$$

This means that for the orthogonal groups in odd (resp. even) dimensions there is no single-valued character χ on the maximal torus whose differential is ϖ_l (resp. ϖ_{l-1} or ϖ_l). We will resolve this difficulty in Chapter 6 with the construction of the groups $\mathbf{Spin}(n, \mathbb{C})$ and the *spin representations*.

[Weight Lattice vs. Character Lattice] One of the most profound concepts in representation theory is the distinction between the weight lattice of a Lie algebra, $P(\mathfrak{g})$, and the character lattice (or weight lattice) of a Lie group, $P(G)$. For a group such as $G = \mathbf{SO}(n, \mathbb{C})$, the character lattice $P(G)$ is a **proper sublattice** of the weight lattice $P(\mathfrak{g})$ of its Lie algebra $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$. The critical implications are:

1. **Classification by Algebra vs. Group:** The weight lattice $P(\mathfrak{g})$ classifies all possible finite-dimensional irreducible representations of the *Lie algebra* $\mathfrak{g}$.
2. **The Integration Condition:** A finite-dimensional representation of the Lie algebra $\mathfrak{g}$ can be "integrated" or "exponentiated" to a true, single-valued representation of the *Lie group* G if and only if all of its weights lie in the character lattice $P(G)$.
3. **Existence of Non-Group Representations:** The fact that $P(G) \subsetneq P(\mathfrak{g})$ for $G = \mathbf{SO}(n, \mathbb{C})$ means there exist irreducible representations of the Lie algebra $\mathfrak{so}(n, \mathbb{C})$ that are **not** single-valued representations of the group $\mathbf{SO}(n, \mathbb{C})$. These are the celebrated **spin representations**.
4. **Topological Origin:** For $n \geq 3$, the group $\mathbf{SO}(n, \mathbb{C})$ is not simply connected. A path in the group corresponding to a non-contractible loop can result in a representation matrix that does not return to the identity, but to its negative. This gives rise to so-called "double-valued representations".
5. **Resolution via Covering Group:** This issue is resolved by passing to the universal covering group of $\mathbf{SO}(n, \mathbb{C})$, which is the **Spin group**, $Spin(n, \mathbb{C})$. On $Spin(n, \mathbb{C})$, all finite-dimensional representations of the Lie algebra $\mathfrak{so}(n, \mathbb{C})$, including the spin representations, become bona fide single-valued group representations.

One of the most profound concepts in representation theory is the distinction between the weight lattice of a Lie algebra, $P(\mathfrak{g})$, and the character lattice (or weight lattice) of a Lie group, $P(G)$. For a group such as $G = \mathbf{SO}(n, \mathbb{C})$, the character lattice $P(G)$ is a **proper sublattice** of the weight lattice $P(\mathfrak{g})$ of its Lie algebra $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$. The critical implications are:

1. **Classification by Algebra vs. Group:** The weight lattice $P(\mathfrak{g})$ classifies all possible finite-dimensional irreducible representations of the *Lie algebra* $\mathfrak{g}$.
2. **The Integration Condition:** A finite-dimensional representation of the Lie algebra $\mathfrak{g}$ can be "integrated" or "exponentiated" to a true, single-valued representation of the *Lie group* G if and only if all of its weights lie in the character lattice $P(G)$.
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3.1.4 Dominant Weights

Definition 3.1.5 — Dominant Integral Weights. Fix a positive root system Φ^+. Define:

$$P_{++}(\mathfrak{g}) = \{\lambda \in \mathfrak{h}^* : \langle \lambda, H_i \rangle \in \mathbb{N}, i = 1, ..., l\} \tag{3.35}$$

where $\mathbb{N} = \{0, 1, 2, ...\}$. From (3.15) we have:

$$P_{++}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})^* \oplus \mathbb{N}\varpi_1 \oplus \cdots \oplus \mathbb{N}\varpi_l \tag{3.36}$$

$\lambda \in P_{++}(\mathfrak{g})$ is *regular* if $\langle \lambda, H_i \rangle > 0$ for all i.

For connected classical group G:

$$P_{++}(G) = P(G) \cap P_{++}(\mathfrak{g}) \tag{3.37}$$

By Section 3.1.3 formulas:
- $\mathbf{SL}(n, \mathbb{C})$ and $\mathbf{Sp}(n, \mathbb{C})$ satisfy $P_{++}(G) = P_{++}(\mathfrak{g})$
- Orthogonal cases:

■ **Example 3.1 Type A_2 ($\mathfrak{sl}(3, \mathbb{C})$):**
- Simple roots: equal length, 120° angle
- Hexagonal symmetry, dominant weights in 60° Weyl chamber
- Unique dominant root: $\alpha_1 + \alpha_2 = \varpi_1 + \varpi_2$

■

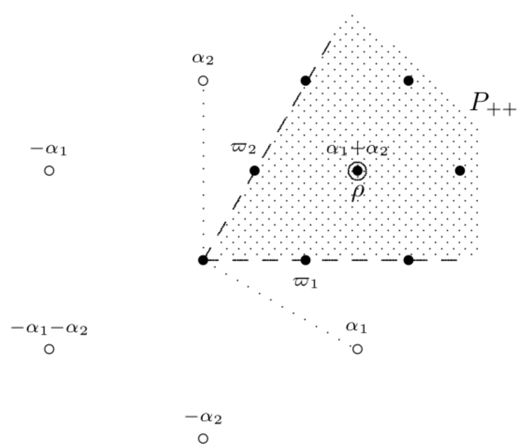


Figure 3.1: Roots and dominant weights for SL$(3, \mathbb{C})$.

■ **Example 3.2 Type C_2 ($\mathfrak{sp}(2, \mathbb{C})$):**
- Root length ratio $\sqrt{2} : 2$, 135° angle
- Square symmetry, dominant weights in 45° chamber
- Dominant roots: $\alpha_1 + \alpha_2$ and $2\alpha_1 + \alpha_2$

■

Proposition 3.1.8 1. For $\mathbf{SO}(2l + 1, \mathbb{C})$, $P_{++}(G)$ consists of weights:

$$n_1\varpi_1 + \cdots + n_{l-1}\varpi_{l-1} + n_l(2\varpi_l) \quad (n_i \in \mathbb{N}) \tag{3.38}$$

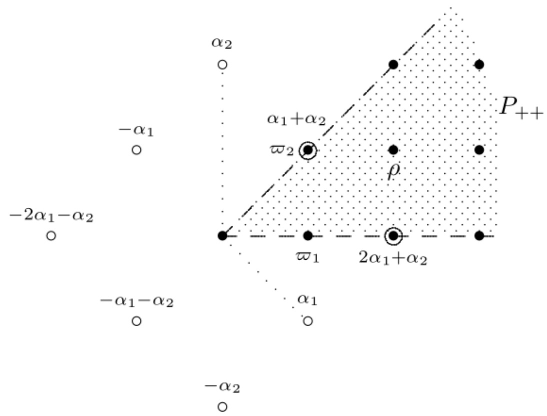


Figure 3.2: Roots and dominant weights for $\mathrm{Sp}(4, \mathbb{C})$.

2. For $\mathbf{SO}(2l, \mathbb{C})$ $(l \geq 2)$, $P_{++}(G)$ consists of:

$$n_1\varpi_1 + \cdots + n_{l-2}\varpi_{l-2} + n_{l-1}(2\varpi_{l-1}) + n_l(2\varpi_l) + n_{l+1}(\varpi_{l-1} + \varpi_l) \tag{3.39}$$

with $n_i \in \mathbb{N}$

■ **Example 3.3 Special Case ($\mathfrak{so}(4, \mathbb{C})$):**

- Orthogonal simple roots, disconnected Dynkin diagram
- Isomorphic to $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
- Dominant weights in $90°$ chamber

■

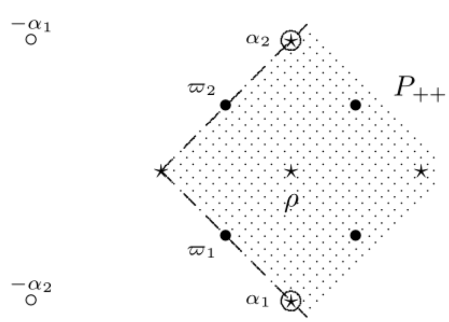


Figure 3.3: Roots and dominant weights for $\mathfrak{g} = \mathfrak{so}(4, \mathbb{C})$.

Proposition 3.1.9 Any $\lambda \in P(\mathfrak{g})$ transforms uniquely to dominant $\mu \in P_{++}(\mathfrak{g})$ under W. When μ is regular, $|W \cdot \mu| = |W|$.

1. $\mathbf{GL}(n, \mathbb{C})/\mathbf{SL}(n, \mathbb{C})$: $\mu = \sum k_i\varepsilon_i$ with $k_1 \geq \cdots \geq k_n$, $k_i - k_{i+1} \in \mathbb{Z}$
2. $\mathbf{SO}(2l + 1, \mathbb{C})$: $\mu = \sum k_i\varepsilon_i$ with $k_1 \geq \cdots \geq k_l \geq 0$, $2k_i, k_i - k_j \in \mathbb{Z}$
3. $\mathbf{Sp}(l, \mathbb{C})$: As above but $k_i \in \mathbb{Z}$

4. $\mathbf{SO}(2l, \mathbb{C})$: $\mu = \sum k_i \varepsilon_i$ with $k_1 \geq \cdots \geq |k_l|$, $2k_i, k_i - k_j \in \mathbb{Z}$

Lemma 3.1.10 Define $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$, then:

- $\rho = \varpi_1 + \cdots + \varpi_l$
- Explicit formulas:

$$A_l : \rho = \sum_{i=1}^{l+1} \frac{l+2-2i}{2}\varepsilon_i \tag{3.40}$$

$$B_l : \rho = \sum_{i=1}^{l} (l - i + \frac{1}{2})\varepsilon_i \tag{3.41}$$

$$C_l : \rho = \sum_{i=1}^{l} (l + 1 - i)\varepsilon_i \tag{3.42}$$

$$D_l : \rho = \sum_{i=1}^{l-1} (l - i)\varepsilon_i \tag{3.43}$$

Table 3.1: Root Systems of the Classical Lie Algebras

Type	**Simple Roots (Δ)**	**Positive Roots (Φ^+)**	**Highest Root ($\tilde{\alpha}$)**
A_l ($l \geq 1$)	$\epsilon_i - \epsilon_{i+1}$ ($1 \leq i \leq l$)	$\epsilon_i - \epsilon_j$ ($1 \leq i < j \leq l+1$)	$\epsilon_1 - \epsilon_{l+1}$
B_l ($l \geq 2$)	$\epsilon_i - \epsilon_{i+1}$ ($1 \leq i < l$) ϵ_l	$\epsilon_i \pm \epsilon_j$ ($i < j$) ϵ_i ($1 \leq i \leq l$)	$\epsilon_1 + \epsilon_2$
C_l ($l \geq 3$)	$\epsilon_i - \epsilon_{i+1}$ ($1 \leq i < l$) $2\epsilon_l$	$\epsilon_i \pm \epsilon_j$ ($i < j$) $2\epsilon_i$ ($1 \leq i \leq l$)	$2\epsilon_1$
D_l ($l \geq 4$)	$\epsilon_i - \epsilon_{i+1}$ ($1 \leq i < l$) $\epsilon_{l-1} + \epsilon_l$	$\epsilon_i \pm \epsilon_j$ ($1 \leq i < j \leq l$)	$\epsilon_1 + \epsilon_2$

3.2 Irreducible Representations

The theorem of the highest weight states that for semisimple Lie algebras, every irreducible representation is uniquely determined by its highest weight. This weight occurs with multiplicity one and serves as a complete invariant for the representation, analogous to how eigenvalues classify eigenvectors in linear algebra. The result generalizes the special case of $\mathfrak{sl}(2, \mathbb{C})$ discussed earlier.

3.2.1 Theorem of the Highest Weight

Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. The root system Φ describes the possible weight shifts under the algebra's action, where we select positive roots Φ^+ to establish directionality. The positive and negative root spaces are defined respectively as:

$$\mathfrak{n}^+ = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}^- = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha}. \tag{3.44}$$

These spaces act as raising and lowering operators that modify weights in representations.

Given a representation (π, V), the weight space $V(\lambda)$ consists of all vectors v satisfying $\pi(h)v = \langle\lambda, h\rangle v$ for every $h \in \mathfrak{h}$. The set $\mathcal{X}(V)$ collects all weights with non-trivial weight spaces. When root vectors from $\mathfrak{g}_\alpha$ act on $V(\lambda)$, they produce vectors in $V(\lambda+\alpha)$, demonstrating how weights transform under the algebra's operations.

We introduce the root order $\prec$ on weights where $\mu \prec \lambda$ means μ can be obtained from λ by subtracting positive roots. This partial order allows us to compare weights and identify maximal elements, with the highest weight being the most significant in this hierarchy.

Definition 3.2.1 — Highest-Weight Representations. A representation V is called a highest-weight representation if there exists a non-zero vector $v_0 \in V$ and weight λ such that:

- $\mathfrak{n}^+ v_0 = 0$ (v_0 is annihilated by raising operators)
- $hv_0 = \langle\lambda, h\rangle v_0$ for all $h \in \mathfrak{h}$ (v_0 has weight λ)
- $V = \mathbf{U}(\mathfrak{g})v_0$ (v_0 generates the entire representation)

Here $\mathbf{U}(\mathfrak{g})$ denotes the universal enveloping algebra, which allows us to apply all possible algebraic operations to v_0.

Theorem 3.2.1 — The Theorem of the Highest Weight. Let $\mathfrak{g}$ be a complex semisimple Lie algebra.

1. **Existence and Uniqueness:** For every dominant integral weight $\lambda \in P_+$, there exists an irreducible finite-dimensional representation $L(\lambda)$ with highest weight λ. This representation is unique up to isomorphism.
2. **Classification:** Every irreducible finite-dimensional representation of $\mathfrak{g}$ is isomorphic to $L(\lambda)$ for some unique dominant integral weight $\lambda \in P_+$.

Thus, the set of dominant integral weights P_+ provides a complete classification of the irreducible finite-dimensional representations of $\mathfrak{g}$.

Lemma 3.2.2 Every highest-weight representation decomposes as:

$$V = \mathbb{C}v_0 \oplus \bigoplus_{\mu \prec \lambda} V(\mu), \tag{3.45}$$

where $\mathbb{C}v_0$ is the one-dimensional highest weight space and the remaining terms are weight spaces for lower weights. Crucially:

- $\dim V(\lambda) = 1$ (uniqueness of highest weight vector)
- All $V(\mu)$ are finite-dimensional
- λ is maximal in $\mathcal{X}(V)$ under $\prec$

The highest weight λ serves as a complete invariant for the irreducible representation, with the representation space constructed by applying lowering operators to the highest weight vector v_0. This structure theorem provides both a classification scheme and explicit construction method for representations of semisimple Lie algebras.

Corollary 3.2.3 For a nonzero finite-dimensional irreducible representation (π, V) of a semisimple Lie algebra $\mathfrak{g}$, the maximal weight $\lambda \in \mathcal{X}(V)$ from Corollary 3.2.3 is called the **highest weight of** V, denoted λ_V. This weight completely characterizes the representation up to isomorphism.

The universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ allows us to study representations through module theory. We define a $\mathbf{U}(\mathfrak{g})$-module structure on the dual space $\mathbf{U}(\mathfrak{g})^*$ via:

$$(g \cdot f)(u) = f(ug) \quad \text{for } g, u \in \mathbf{U}(\mathfrak{g}), f \in \mathbf{U}(\mathfrak{g})^*. \tag{3.46}$$

This action satisfies the composition property:

$$g \cdot (g' \cdot f) = (gg') \cdot f, \tag{3.47}$$

making $\mathbf{U}(\mathfrak{g})^*$ a representation space where $\mathbf{U}(\mathfrak{g})$ plays the role analogous to an algebraic group G, and $\mathbf{U}(\mathfrak{g})^*$ corresponds to regular functions on G.

Theorem 3.2.4 For any $\lambda \in \mathfrak{h}^*$:

1. There exists an irreducible highest-weight representation (σ, L^λ) with highest weight λ
2. Any irreducible highest-weight representation with weight λ is isomorphic to L^λ

The space L^λ is generally infinite-dimensional, with finite-dimensionality requiring special conditions on λ.

Theorem 3.2.5 The irreducible highest weight representation L^λ is finite-dimensional if and only if λ is dominant integral.

This establishes the fundamental correspondence:

$$\begin{Bmatrix} \text{Dominant integral} \\ \text{weights } \lambda \in \mathfrak{h}^* \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \text{Finite-dimensional irreducible} \\ \text{highest-weight representations} \end{Bmatrix} \tag{3.48}$$

- The highest weight λ_V serves as a complete invariant for finite-dimensional irreducible representations
- The universal enveloping algebra provides a powerful tool for constructing representations
- Dominant integral weights characterize when infinite-dimensional theory reduces to the finite-dimensional case

3.2.2 Weights of Irreducible Representations

Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. We consider:

- The root system $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$
- Positive roots Φ^+ with simple roots $\alpha_1, \ldots, \alpha_l$
- Coroots $H_i = h_{\alpha_i}$
- Root lattice Q_+ generated by Φ^+

For each root $\alpha \in \Phi$, we have:

- Root reflection on $\mathfrak{h}$: $s_\alpha Y = Y - \langle \alpha, Y \rangle h_\alpha$
- Dual action on $\mathfrak{h}^*$: $s_\alpha \beta = \beta - \langle \beta, h_\alpha \rangle \alpha$

The **Weyl group** W is the finite group generated by these reflections. It preserves:

- The inner product (α, β) from the Killing form
- The norm $\|\alpha\|^2 = (\alpha, \alpha)$

Corollary 3.2.6 For a finite-dimensional representation (π, V):

$$V = \bigoplus_{\mu \in \mathcal{K}(V)} V(\mu) \tag{3.49}$$

Given a TDS triple $\{e_\alpha, f_\alpha, h_\alpha\}$ and operators:

$$E = \pi(e_\alpha) \tag{3.50}$$
$$F = \pi(f_\alpha) \tag{3.51}$$
$$\tau_\alpha = \exp(E)\exp(-F)\exp(E) \in \mathrm{GL}(V) \tag{3.52}$$

We have key properties:
1. $\tau_\alpha \pi(Y) \tau_\alpha^{-1} = \pi(s_\alpha Y)$ for $Y \in \mathfrak{h}$
2. $\tau_\alpha V(\mu) = V(s_\alpha \mu)$
3. $\dim V(\mu) = \dim V(s \cdot \mu)$ for all $s \in W$

- The weight set $\mathcal{K}(V)$ is **Φ-saturated**:

$$\lambda \in \mathcal{K}(V) \Rightarrow \lambda - k\alpha \in \mathcal{K}(V) \text{ for } 0 \le k \le \langle \lambda, h_\alpha \rangle \tag{3.53}$$

- A weight λ is **Φ-extreme** if:

$$\lambda + \alpha \notin \mathcal{K}(V) \text{ or } \lambda - \alpha \notin \mathcal{K}(V) \text{ for all } \alpha \in \Phi \tag{3.54}$$

For an irreducible module V with highest weight λ:
1. $\mathcal{K}(V)$ is the minimal Φ-saturated set containing λ
2. The W-orbit of λ is exactly the Φ-extreme weights

R The transformation τ_α comes from the matrix identity in $\mathrm{SL}(2, \mathbb{C})$:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3.55}$$

which represents a Weyl group element.

Lemma 3.2.7 Let (π, V) be a finite-dimensional representation of $\mathfrak{g}$. For any weight $\lambda \in \mathcal{X}(V)$ and root $\alpha \in \Phi$, the weight $\lambda - k\alpha$ belongs to $\mathcal{X}(V)$ for all integers $0 \le k \le \langle \lambda, h_\alpha \rangle$, where h_α is the coroot corresponding to α.

Proposition 3.2.8 For a finite-dimensional irreducible $\mathfrak{g}$-module V with highest weight λ:
1. $\mathcal{K}(V)$ is the minimal Φ-saturated subset of $P(\mathfrak{g})$ containing λ
2. The Weyl group orbit $W \cdot \lambda$ consists exactly of the Φ-extreme weights in $\mathcal{K}(V)$

Proposition 3.2.9 If V is any finite-dimensional representation of $\mathfrak{g}$ and $\mu \in P_{++}(\mathfrak{g})$ satisfies $\mu \prec \nu$ for some $\nu \in \mathcal{K}(V)$, then $\mu \in \mathcal{K}(V)$.

Corollary 3.2.10 For the irreducible module L^λ, the dominant weights are precisely those $\mu \in P_{++}(\mathfrak{g})$ with $\mu \preceq \lambda$. The complete weight set is obtained by:

- Taking all $\beta \in Q_+$ with $\|\lambda - \beta\| \leq \|\lambda\|$
- Checking non-negativity of $\mu = \lambda - \beta$ in the fundamental weight basis
- Taking the union of all W-orbits of such dominant weights

■ **Example 3.4** For $L^{\varpi_1+2\varpi_2}$:

- Highest weight λ has full 6-element Weyl group orbit
- Subdominant weights: $\lambda - \alpha_2$ and $\lambda - \alpha_1 - \alpha_2$ (each with 3-element orbits)
- Multiplicity pattern: Outer weights have multiplicity 1, inner weights have multiplicity 2
- Total dimension: 15

■

R [Terminology for Saturated Sets] The term "Φ-saturated" is sometimes used, but the standard and more common terminology is simply **saturated**. A set of weights is saturated if for any weight μ in the set, and any root $\alpha \in \Phi$, the entire string of weights $\mu - k\alpha, \ldots, \mu, \ldots, \mu + q\alpha$ is also contained in the set.

R [Context for Example 3.4] To make the example discussing the representation $L(\varpi_1 + 2\varpi_2)$ fully concrete, the specific rank-2 Lie algebra should be identified. For instance, is it a representation of $A_2 = \mathfrak{sl}(3, \mathbb{C})$, $B_2 = \mathfrak{so}(5, \mathbb{C})$, or $C_2 = \mathfrak{sp}(4, \mathbb{C})$? Specifying the algebra is necessary to properly interpret the fundamental weights ϖ_i and the structure of the representation.

3.2.3 Lowest Weights and Dual Representations

When we replace the system of positive roots Φ^+ with $-\Phi^+$, several structures change accordingly. The subalgebra $\mathfrak{n}^+$ becomes $\mathfrak{n}^-$, and the Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^+$ transforms into its opposite $\overline{\mathfrak{b}} = \mathfrak{h} + \mathfrak{n}^-$. This opposite Borel subalgebra plays a crucial role in the theory.

Corollary 3.2.11 For any irreducible finite-dimensional $\mathfrak{g}$-module V, there exists a unique lowest weight μ contained in $-P_{++}(\mathfrak{g})$. This weight is characterized by the property that $\mu \preceq \nu$ holds for all weights ν in $\mathcal{K}(V)$. The corresponding weight space $V(\mu)$ is one-dimensional, and any nonzero vector in this space is called a lowest-weight vector.

Theorem 3.2.12 Let (π, V) be an irreducible finite-dimensional $\mathfrak{g}$-module with highest weight λ, and let (π^*, V^*) denote its dual module. Then the lowest weight of V is given by $w_0(\lambda)$, where w_0 is the unique element of the Weyl group W satisfying $w_0\Phi^+ = -\Phi^+$. Furthermore, the dual module V^* has highest weight $-w_0(\lambda)$ and lowest weight $-\lambda$.

3.2.4 Symplectic and Orthogonal Representations

Theorem 3.2.13 Let (π, V) be an irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight λ. There exists a nonzero $\mathfrak{g}$-invariant bilinear form on V if and only if $-w_0\lambda = \lambda$. When this condition holds:

- The form is nonsingular
- Unique up to scalar multiples
- Either symmetric or skew-symmetric

An irreducible representation (π, V) admitting such a form is called *symplectic* if the form is skew-symmetric and *orthogonal* if symmetric. The classification depends on the highest weight:

Lemma 3.2.14 For $G = \mathrm{SL}(2, \mathbb{C})$ with $(m+1)$-dimensional irreducible representation (π, V):

- π is symplectic when m is odd
- π is orthogonal when m is even

This extends to general semisimple Lie algebras through a special $\mathfrak{sl}_2$-triple construction. Let $\{E_i, F_i, H_i\}$ be a TDS triple with $E_i \in \mathfrak{g}_{\alpha_i}$, $F_i \in \mathfrak{g}_{-\alpha_i}$. Define:

$$e^0 = \sum_{i=1}^{l} E_i \tag{3.56}$$

$$f^0 = \sum_{i=1}^{l} c_i F_i \tag{3.57}$$

$$h^0 = \sum_{i=1}^{l} H_i \tag{3.58}$$

Then $\{e^0, f^0, h^0\}$ forms a TDS triple in $\mathfrak{g}$, yielding:

Theorem 3.2.15 For irreducible (π, V) with highest weight $\lambda \neq 0$ satisfying $-w_0\lambda = \lambda$, set $m = \langle \lambda, h^0 \rangle$. Then:

- m is a positive integer
- π is symplectic when m is odd
- π is orthogonal when m is even

3.3 Reductivity of Classical Groups

In this section, we give two proofs of the complete reducibility of regular representations of a classical group G. The first proof is algebraic and applies to any semisimple Lie algebra; the key tool is the Casimir operator, which will also play an important role in Chapter 7. The second proof is analytic and uses integration on a compact real form of G.

These two proofs are not merely alternatives; they beautifully showcase two distinct yet equally powerful techniques central to Lie theory. The algebraic method highlights the power of internal algebraic structures (the universal enveloping algebra and its center), while the analytic method, known as Weyl's Unitary Trick, demonstrates the profound connection between the algebraic properties of complex Lie algebras and the topological properties of their compact real forms. Including both provides a richer and more complete perspective.

3.3.1 The Two Approaches to Complete Reducibility

Before delving into the technical details, we outline the philosophical and logical strategy behind each proof.

R [Logic of the Algebraic Proof via Casimir Operator] The core idea of this proof is to use an element that "sees" the entire structure of the representation and allows us to construct the necessary invariant projections.

1. We construct the **Casimir operator** C_π, an element of the universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ that commutes with every element of $\mathfrak{g}$ under the representation π.
2. By **Schur's Lemma**, C_π must act as a scalar multiple of the identity, $c_\lambda \cdot \mathrm{Id}$, on any irreducible representation L^λ. The eigenvalue formula, $c_\lambda = (\lambda+\rho, \lambda+\rho) - (\rho, \rho)$, shows that this scalar depends on the highest weight and is non-zero for any non-trivial representation.
3. If a representation V has a subrepresentation W, we can find a projection $p : V \to W$. While p is not necessarily $\mathfrak{g}$-invariant, we can "average" it using the Casimir operator to construct a new map $p_0 : V \to W$ which *is* $\mathfrak{g}$-invariant.
4. The kernel of this $\mathfrak{g}$-invariant projection, $\ker(p_0)$, is a $\mathfrak{g}$-invariant complement to W, so $V = W \oplus \ker(p_0)$. Thus, V is completely reducible.

R [Logic of the Analytic Proof via Weyl's Unitary Trick] This elegant argument leverages analysis and topology to prove a purely algebraic result.

1. A fundamental theorem states that any complex semisimple Lie algebra $\mathfrak{g}$ has a **compact real form** $\mathfrak{k}$ (e.g., $\mathfrak{su}(n)$ for $\mathfrak{sl}(n, \mathbb{C})$).
2. A representation of $\mathfrak{g}$ can be restricted to a representation of the compact Lie group K corresponding to $\mathfrak{k}$.
3. On any compact group K, one can define an inner product that is invariant under the action of K. This is done by taking any arbitrary inner product $(\cdot, \cdot)$ and averaging it over the group using the unique normalized **Haar measure** dk:
$$\langle v, w \rangle_K := \int_K (\pi(k)v, \pi(k)w) dk \tag{3.59}$$
4. A K-invariant inner product is also $\mathfrak{k}$-invariant, and by complexification ($\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$), it is also $\mathfrak{g}$-invariant.
5. Any representation that possesses an invariant inner product is completely reducible. If W is a $\mathfrak{g}$-invariant subspace, its orthogonal complement $W^\perp$ is also $\mathfrak{g}$-invariant, giving the desired decomposition $V = W \oplus W^\perp$.

3.3.2 Reductive Groups

A rational representation (ρ, V) of a linear algebraic group G is **completely reducible** if for every G-invariant subspace $W \subset V$, there exists a complementary G-invariant subspace U such that $V = W \oplus U$.

Equivalently, in matrix terms, for any basis $\{w_1, \ldots, w_p\}$ of W, there exists $\{u_1, \ldots, u_q\}$ such that the matrix representation takes the block-diagonal form:
$$\rho(g) = \begin{bmatrix} \sigma(g) & 0 \\ 0 & \tau(g) \end{bmatrix} \tag{3.60}$$
where $\sigma = \rho|_W$ and $\tau = \rho|_U$.

Definition 3.3.1 A linear algebraic group G is **reductive** if every rational representation (ρ, V) of G is completely reducible. A representation is completely reducible if for every G-invariant subspace $W \subset V$, there exists a complementary G-invariant subspace U such that $V = W \oplus U$.

Lemma 3.3.1 Let (ρ, V) be a completely reducible rational representation of G. For any invariant subspace $W \subset V$, both the restricted representation (σ, W) and the quotient representation $(\pi, V/W)$ are completely reducible.

Proposition 3.3.2 For a rational representation (ρ, V) of G, the following are equivalent:

1. (ρ, V) is completely reducible
2. V decomposes as a direct sum $V_1 \oplus \cdots \oplus V_d$ of irreducible G-submodules
3. V is spanned by irreducible G-submodules $V_1 + \cdots + V_d$

Corollary 3.3.3 The direct sum $(\rho \oplus \sigma, V \oplus W)$ of two completely reducible regular representations is completely reducible.

Proposition 3.3.4 Let $H \subset G$ be linear algebraic groups where H is reductive and has finite index in G. Then G is reductive.

R These methods extend to modules over $\mathbb{C}$-algebras, with appropriate modifications as discussed in Section 4.1.4.

Corollary 3.3.5 — Maschke's Theorem. Every finite group G is reductive.

Significance of the Casimir Eigenvalue

The Casimir operator and its eigenvalue formula are key tools in the theory of semisimple Lie algebras, particularly for proving complete reducibility. The logic is as follows:

- By its construction, the Casimir operator C_π commutes with the action of the entire Lie algebra, i.e., $[C_\pi, \pi(\mathfrak{g})] = 0$.
- According to Schur's Lemma, any operator that commutes with all operators in an irreducible representation must act as a scalar multiple of the identity. Therefore, on any irreducible representation L^λ, C_π acts as a scalar.
- The formula $C_\pi = [(\lambda + \rho, \lambda + \rho) - (\rho, \rho)] \cdot \mathrm{Id}$ explicitly provides this scalar value.

Crucially, this eigenvalue depends directly on the highest weight λ. If two irreducible representations have different highest weights, their Casimir eigenvalues will be different. This provides a powerful method to distinguish and classify irreducible representations, and it plays a vital role in the algebraic proof of complete reducibility.

3.3.3 Casimir Operator

Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$, root system Φ, and positive roots Φ^+. The Killing form B induces an inner product $(\cdot, \cdot)$ on $\mathfrak{h}^*_{\mathbb{R}}$.

For any basis $\{X_i\}$ of $\mathfrak{g}$ with B-dual basis $\{Y_i\}$, and representation (π, V), the **Casimir operator** is defined as:

$$C_\pi = \sum_i \pi(X_i)\pi(Y_i) \tag{3.61}$$

Lemma 3.3.6 The Casimir operator C_π is well-defined (independent of basis choice) and commutes with $\pi(\mathfrak{g})$.

Lemma 3.3.7 For a highest-weight representation (π, V) with highest weight λ and $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$, the Casimir operator acts as a scalar:

$$C_\pi v = \big[(\lambda+\rho, \lambda+\rho) - (\rho,\rho)\big]v \quad \text{for all } v \in V \tag{3.62}$$

Proposition 3.3.8 Every finite-dimensional highest-weight $\mathfrak{g}$-module with highest weight λ satisfies:

1. λ is dominant integral
2. The module is irreducible
3. It is isomorphic to L^λ

R Unlike the finite-dimensional case, infinite-dimensional highest-weight modules may be reducible even when the highest weight is dominant integral.

3.3.4 Algebraic Proof of Complete Reducibility

We now come to the main result of this section.

Theorem 3.3.9 Every classical group G is reductive.

Theorem 3.3.10 Let $\mathfrak{g}$ be a semisimple Lie algebra. Then every finite-dimensional $\mathfrak{g}$-module V is completely reducible.

Lemma 3.3.11 Let λ and μ be dominant integral weights. If a finite-dimensional $\mathfrak{g}$-module V contains a submodule $Z \cong L^\lambda$ with $V/Z \cong L^\mu$, then $V \cong L^\lambda \oplus L^\mu$.

Corollary 3.3.12 For a finite-dimensional $\mathfrak{g}$-module V, the following are equivalent:

1. V is irreducible
2. $\dim V^{\mathfrak{n}^+} = 1$ (where $V^{\mathfrak{n}^+} = \{v \in V \mid X \cdot v = 0 \text{ for all } X \in \mathfrak{n}^+\}$)

3.3.5 Weyl's Unitary Trick

We now present an analytical proof of the complete reducibility of representations for semisimple Lie algebras. This celebrated method, known as **Weyl's Unitary Trick**, provides a powerful bridge between the algebraic problems of complex Lie algebras and the analytic properties of compact Lie groups.

The argument proceeds as follows:

1. **Existence of a Compact Real Form**: A fundamental theorem states that every complex semisimple Lie algebra $\mathfrak{g}$ has a **compact real form** $\mathfrak{k}$. This means $\mathfrak{k}$ is a real Lie subalgebra of $\mathfrak{g}$ whose corresponding connected Lie group K is compact, and $\mathfrak{g}$ can be reconstructed from $\mathfrak{k}$ as its complexification:

$$\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k} \tag{3.63}$$

2. **Restriction to the Compact Form**: Let (π, V) be any finite-dimensional representation of the complex Lie algebra $\mathfrak{g}$. We can restrict this representation to the subalgebra $\mathfrak{k}$. This gives us a representation of $\mathfrak{k}$ on V, which in turn corresponds to a representation of the compact Lie group K.

3. **Constructing an Invariant Inner Product**: Since K is a compact group, it possesses a unique normalized Haar measure dk. This allows us to construct a K-invariant Hermitian inner product on V. Starting with any arbitrary inner product $(\cdot,\cdot)$ on V, we define a new inner product $\langle\cdot,\cdot\rangle$ by averaging over the group K:

$$\langle v, w\rangle = \int_K (\pi(k)v, \pi(k)w)\ \mathrm{d}\,k \tag{3.64}$$

By construction, this new inner product is invariant under the action of K.
4. **Implication for the Lie Algebra**: The invariance under the group K implies invariance under its Lie algebra $\mathfrak{k}$. Because $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$, this invariance extends to the entire complex Lie algebra $\mathfrak{g}$. Thus, we have found a $\mathfrak{g}$-invariant inner product on V.
5. **Conclusion of Complete Reducibility**: The existence of a $\mathfrak{g}$-invariant inner product guarantees complete reducibility. If $W \subseteq V$ is a $\mathfrak{g}$-invariant subspace, its orthogonal complement $W^\perp$ with respect to the invariant inner product $\langle\cdot,\cdot\rangle$ is also a $\mathfrak{g}$-invariant subspace, and $V = W \oplus W^\perp$.

This elegant argument beautifully connects two different mathematical worlds: it solves a purely algebraic problem (the complete reducibility of $\mathfrak{g}$-modules) by leveraging the tools of analysis (integration on compact manifolds).

Corollary 3.3.13 All classical groups are reductive.

Proof. The Lie algebras of the classical groups, such as $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$, and $\mathfrak{sp}(n,\mathbb{C})$, are semisimple. Therefore, they possess compact real forms (e.g., $\mathfrak{su}(n)$ is the compact real form of $\mathfrak{sl}(n,\mathbb{C})$). By Weyl's Unitary Trick, all their finite-dimensional representations are completely reducible, which implies that the groups themselves are reductive. ■

R [Logic of the Algebraic Proof via Casimir Operator] The core idea of this proof is to use an element that "sees" the entire structure of the representation.

1. Construct the **Casimir operator** C, an element of the universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ that commutes with every element of $\mathfrak{g}$.
2. By **Schur's Lemma**, C must act as a scalar multiple of the identity, $c_\pi \cdot \mathrm{Id}$, on any irreducible representation π. The eigenvalue c_π is non-zero for any non-trivial representation.
3. If a representation V has a subrepresentation W, we can find a projection $p : V \to W$. While p is not necessarily $\mathfrak{g}$-invariant, we can average it using the Casimir operators of V and W to construct a new map $p_0 : V \to W$ which *is* $\mathfrak{g}$-invariant.
4. The kernel of this $\mathfrak{g}$-invariant projection, $\ker(p_0)$, is a $\mathfrak{g}$-invariant complement to W, so $V = W \oplus \ker(p_0)$. Thus, V is completely reducible.

R [Logic of the Analytic Proof via Weyl's Unitary Trick] This elegant argument leverages analysis and topology to prove a purely algebraic result.

1. Any complex semisimple Lie algebra $\mathfrak{g}$ has a **compact real form** $\mathfrak{k}$.
2. A representation of $\mathfrak{g}$ restricts to a representation of the compact Lie group K corresponding to $\mathfrak{k}$.

3. On any compact group K, one can define an inner product that is invariant under the action of K. This is done by taking any arbitrary inner product $\langle \cdot, \cdot \rangle$ and averaging it over the group using the **Haar measure** μ:

$$\langle v, w \rangle_K := \int_K \langle g \cdot v, g \cdot w \rangle \, \mathrm{d}\, \mu(g) \tag{3.65}$$

4. A K-invariant inner product is also $\mathfrak{k}$-invariant, and by complexification, it is $\mathfrak{g}$-invariant.
5. Any representation that possesses an invariant inner product is completely reducible. If W is a $\mathfrak{g}$-invariant subspace, its orthogonal complement $W^\perp$ is also $\mathfrak{g}$-invariant, giving $V = W \oplus W^\perp$.

4. Algebraic Groups and Homogeneous Spac

4.1 General Properties of Linear Algebraic Groups

We now develop the theory of linear algebraic groups and their homogeneous spaces, as a preparation for the geometric approach to representations and invariant theory.

4.1.1 Algebraic Groups as Affine Varieties

Let V be a finite-dimensional complex vector space. We view $\mathbf{GL}(V)$ as the principal open set $\{g \in \mathrm{End}(V) : \det(g) \neq 0\}$ in the vector space $\mathrm{End}(V)$, and we give $\mathbf{GL}(V)$ the Zariski topology.

A subgroup $G \subset \mathbf{GL}(V)$ is a linear algebraic group if G is a closed subset of $\mathbf{GL}(V)$, relative to the Zariski topology. To see that this agrees with the definition in, we observe that the Zariski-closed subsets of $\mathbf{GL}(V)$ are defined by equations of the form

$$f(x_{11}(g), x_{12}(g), \ldots, x_{nn}(g), \det(g)^{-1}) = 0 \tag{4.1}$$

, where f is a polynomial in $n^2 + 1$ variables. Since $\det(g) \neq 0$, we can multiply this equation by $\det(g)^k$ for a suitably large k to obtain a polynomial equation in the matrix coefficients of g.

The Lie algebra of the general linear group $\mathbf{GL}(V)$, denoted $\mathfrak{gl}(V)$ or $\mathrm{Lie}(\mathbf{GL}(V))$, is the vector space End(V) equipped with the commutator bracket $[A, B] = AB - BA$. The Lie algebra of a linear algebraic group $G \subset \mathbf{GL}(V)$, denoted $\mathfrak{g}$ or $\mathfrak{g}$, is the Lie subalgebra of $\mathfrak{gl}(V)$ consisting of all elements $X \in \mathfrak{gl}(V)$ whose corresponding left-invariant vector fields on $\mathbf{GL}(V)$ are tangent to G at every point. This is equivalent to the condition that these vector fields leave the vanishing ideal $\mathcal{I}_G \subset \mathcal{O}[\mathbf{GL}(V)]$ invariant. Any such vector field X_A induces a derivation on the coordinate algebra $\mathcal{O}[G] = \mathcal{O}[\mathbf{GL}(V)]/\mathcal{I}_G$.

Theorem 4.1.1 Let G be a linear algebraic group. For every $g \in G$ the map $A \mapsto (X_A)_g$ is a linear isomorphism from $\mathrm{Lie}(G)$ onto $T(G)_g$. Hence G is a smooth algebraic set and $\dim \mathrm{Lie}(G) = \dim G$.

Every affine algebraic set has a unique decomposition into irreducible components. For the case of a linear algebraic group, this decomposition can be described as follows:

Theorem 4.1.2 Let G be a linear algebraic group. Then G contains a unique subgroup G° that is closed, irreducible, and of finite index in G. Furthermore, G° is a normal subgroup and its cosets in G are both the irreducible components and the connected components of G relative to the Zariski topology.

An algebraic group G is defined to be *connected* if the ring $\mathcal{O}[G]$ has no zero divisors. Here are two other equivalent definitions.

Corollary 4.1.3 Let G be a linear algebraic group. The following are equivalent:
1. G is a connected topological space in the Zariski topology.
2. G is irreducible as an affine algebraic set.
3. The ring $\mathcal{O}[G]$ has no zero divisors.

4.1.2 Subgroups and Homomorphisms

Let $G \subset \mathbf{GL}(n, \mathbb{C})$ be a linear algebraic group. An *algebraic subgroup* of G is a Zariski-closed subset $H \subset G$ that is also a subgroup. The definition of a linear algebraic group implies that an algebraic subgroup H of G in this sense is also a linear algebraic group as defined there. Furthermore, the inclusion map $\iota : H \subset G$ is regular and $\mathcal{O}[H] \cong \mathcal{O}[G]/\mathcal{I}_H$. Here

$$\mathcal{I}_H = \{f \in \mathcal{O}[G] : f|_H = 0\}. \tag{4.2}$$

We used this definition of $\mathcal{I}_H$ initially when $G = \mathbf{GL}(n, \mathbb{C})$. Since the regular functions on G are the restrictions to G of the regular functions on $\mathbf{GL}(n, \mathbb{C})$, the definition of the ideal $\mathcal{I}_H$ is unambiguous as long as the ambient group G is understood.

Lemma 4.1.4 Let K be a subgroup of G. Then the closure $\overline{K}$ of K in the Zariski topology is a group, and hence $\overline{K}$ is an algebraic subgroup of G. Furthermore, if K contains a nonempty open subset of $\overline{K}$ then K is closed in the Zariski topology.

Regular homomorphisms of algebraic groups always have the following desirable properties:

Theorem 4.1.5 Let $\varphi : G \longrightarrow H$ be a regular homomorphism of linear algebraic groups. Then:
1. The kernel $\mathrm{Ker}(\varphi)$ is a closed normal subgroup of G.
2. The image $\varphi(G)$ is a closed subgroup of H. Hence $\varphi(G)$ is an algebraic group.

Discussion. The statement that the image $\varphi(G)$ is closed is a deep and non-trivial result due to Chevalley. It marks a fundamental distinction between the theory of algebraic groups and that of Lie groups. For Lie groups, a continuous homomorphism does not necessarily have a closed image (e.g., the irrational-slope line in a torus). This property ensures that quotients and images in the algebraic category are well-behaved, which is foundational for the entire theory. ■

Corollary 4.1.6 Let $\varphi : G \longrightarrow H$ be a regular homomorphism of linear algebraic groups. Set $K = \varphi(G)$. Let $u : K \longrightarrow H$ be the inclusion map and let $\psi : G \longrightarrow K$ be the homomorphism φ, viewed as having image K. Then u is regular and injective, ψ is regular and surjective, and φ factors as $\varphi = u \circ \psi$.

The factorization in Corollary is described by the following diagram:

$$\begin{tikzcd} G \arrow[r, "\psi"] \arrow[dr, "\varphi"'] & K \arrow[d, "u"] \\ & H \end{tikzcd}$$

Corollary 4.1.7 Let $\varphi : G \longrightarrow H$ be a regular homomorphism of linear algebraic groups. Assume that G and H are connected in the Zariski topology and that $\mathrm{d}\,\varphi : \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(H)$ is surjective. Then $\varphi(G) = H$. In particular, if G is a closed subgroup of H and $\dim G = \dim H$, then $G = H$.

Proposition 4.1.8 Let G and H be algebraic subgroups of $\mathbf{GL}(n, \mathbb{C})$. Then the algebraic group $G \cap H$ has Lie algebra $\mathrm{Lie}(G) \cap \mathrm{Lie}(H)$.

Theorem 4.1.9 Suppose G is a connected algebraic group with Lie algebra $\mathfrak{g}$. Let (π, V) be a regular representation of G. If $W \subset V$ is a linear subspace such that $\mathrm{d}\,\pi(X)W \subset W$ for all $X \in \mathfrak{g}$ then $\pi(g)W \subset W$ for all $g \in G$. Hence if $(\mathrm{d}\,\pi, V)$ is a completely reducible representation of $\mathfrak{g}$, then (π, V) is a completely reducible representation of G.

Proposition 4.1.10 Let G be a connected linear algebraic group with Lie algebra $\mathfrak{g}$. Suppose $\sigma : G \to \mathbf{GL}(n, \mathbb{C})$ is a regular representation and $H \subset \mathbf{GL}(n, \mathbb{C})$ is a linear algebraic subgroup with Lie algebra $\mathfrak{h}$ such that $\mathrm{d}\,\sigma(\mathfrak{g}) \subset \mathfrak{h}$. Then $\sigma(G) \subset H$.

4.1.3 Group Structures on Affine Varieties

In the foundational definition of a linear algebraic group G, the group operations (multiplication and inversion) are typically inherited from an ambient embedding into $\mathbf{GL}(n, \mathbb{C})$. The following seminal result demonstrates that this embedding is non-essential, revealing an intrinsic characterization analogous to the abstract definition of Lie groups:

Theorem 4.1.11 Let X be an affine algebraic set equipped with a group structure such that:

- The multiplication map $(x, y) \mapsto xy$
- The inversion map $x \mapsto x^{-1}$

are regular morphisms. Then there exist a linear algebraic group G and a group isomorphism $\Phi : X \longrightarrow G$ that is simultaneously an isomorphism of affine algebraic sets.

4.1.4 Quotient Groups

The construction of quotient spaces G/H for linear algebraic groups presents fundamental challenges. When H is a *normal algebraic subgroup*, the quotient inherits an algebraic group structure, whereas for non-normal subgroups the theory requires more general algebraic spaces (deferred for later treatment). In both cases, the following representation-theoretic machinery provides the essential foundation:

Theorem 4.1.12 Let G be a linear algebraic group and $H \subset G$ an algebraic subgroup.

1. There exist a regular representation (π, V) of G and a 1D subspace $V_0 \subset V$ such that:

$$H = \{g \in G : \pi(g)V_0 = V_0\}, \quad \mathrm{Lie}(H) = \{X \in \mathrm{Lie}(G) : d\,\pi(X)V_0 \subset V_0\} \tag{4.3}$$

2. If H is normal in G, then there exists a regular representation (φ, W) with:

$$H = \mathrm{Ker}(\varphi), \quad \mathrm{Lie}(H) = \mathrm{Ker}(d\,\varphi) \tag{4.4}$$

Lemma 4.1.13 Let $M \subset \mathbb{C}^n$ be a d-dimensional subspace, π the representation of $\mathbf{GL}(n, \mathbb{C})$ on $\bigwedge^d \mathbb{C}^n$, and $N = \bigwedge^d M$.

1. If $g \in \mathbf{GL}(n, \mathbb{C})$ satisfies $\pi(g)N = N$, then $gM = M$.
2. If $X \in \mathbf{GL}(n, \mathbb{C})$ satisfies $d\,\pi(X)N \subset N$, then $XM \subset M$.

Algebraic Structure on Quotient Groups

For connected G with normal algebraic subgroup H:

- Choose a regular representation $(\varphi, W)$with $Ker(\varphi) = H$.
- Define algebraic group structure via isomorphism $\mu : G/H \xrightarrow{\sim} \varphi(G) \subset \mathbf{GL}(W)$
- Coordinate ring: $\mathcal{O}[G/H] = \mu^* \mathcal{O}[\varphi(G)]$

$$\begin{tikzcd}
G \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & G/H \arrow[d, dashed, "\mu"] \\
& \varphi(G)
\end{tikzcd}$$

Theorem 4.1.14 Let G, K, M be algebraic groups with G connected. For regular homomorphisms $\psi : G \to K$ (surjective) and $\varphi : G \to M$ with $\mathrm{Ker}(\psi) \subset \mathrm{Ker}(\varphi)$, the induced map $\mu : K \to M$ satisfying $\varphi = \mu \circ \psi$ is regular.

Corollary 4.1.15 For connected G and K, a bijective regular homomorphism $\psi : G \to K$ has regular inverse, hence is an isomorphism of algebraic groups.

Theorem 4.1.16 Let G be connected and $H \triangleleft G$ normal. For any rational representation φ with $\mathrm{Ker}(\varphi) = H$:

1. The algebraic group structure on G/H is independent of φ
2. The quotient map $\pi : G \to G/H$ is regular
3. The pullback map π^* induces an isomorphism of coordinate algebras

$$\pi^* : \mathcal{O}[G/H] \xrightarrow{\sim} \mathcal{O}[G]^H \tag{4.5}$$

where $\mathcal{O}[G]^H = \{f \in \mathcal{O}[G] \mid f(gh) = f(g) \text{ for all } g \in G, h \in H\}$ is the algebra of regular functions on G that are invariant under the right action of H.

Corollary 4.1.17 For an exact sequence of connected algebraic groups $H \xrightarrow{\varphi} G \xrightarrow{\psi} K$ (i.e., $\mathrm{Im}(\varphi) = \mathrm{Ker}(\psi)$), the induced Lie algebra sequence

$$\mathrm{Lie}(H) \xrightarrow{\mathrm{d}\,\varphi} \mathrm{Lie}(G) \xrightarrow{\mathrm{d}\,\psi} \mathrm{Lie}(K) \tag{4.6}$$

is exact: $\mathrm{d}\,\varphi$ injective, $\mathrm{Im}(\mathrm{d}\,\varphi) = \mathrm{Ker}(\mathrm{d}\,\psi)$, $\mathrm{d}\,\psi$ surjective.

4.2 Structure of Algebraic Groups

4.2.1 Commutative Algebraic Groups

We now determine the structure of commutative linear algebraic groups.

Theorem 4.2.1 Suppose $G \subset \mathbf{GL}(V)$ is a commutative algebraic group.

1. The set G_s of semisimple elements and the set G_u of unipotent elements are subgroups of G
2. There exists a basis for V such that the matrix of $g \in G$ is upper triangular and the semisimple component g_s is $\mathrm{diag}(g_{11}, \dots, g_{nn})$
3. G_s is closed in G and consists of diagonal matrices relative to this basis
4. The map $g \mapsto (g_s, g_u)$ is an isomorphism of algebraic groups $G \cong G_s \times G_u$

We now determine commutative groups with semisimple elements. By the previous theorem, every such group embeds into some D_n.

Theorem 4.2.2 Suppose $G \subset D_n$ is a closed subgroup.

1. The identity component G° is a torus
2. There is a finite subgroup $F \subset G$ such that $G^\circ \cap F = \{1\}$ and $G = G^\circ \cdot F$

Foundations for Further Structure

We complete the structure theory with three fundamental components:

- **Levi decomposition**: Splits a Lie algebra into semisimple subalgebra and solvable ideal
- **Unipotent radical**: $R_u(G)$ defined for algebraic groups
- **Reductive groups**: Defined by $R_u(G) = \{1\}$
- **Connectivity**: Connected linear algebraic groups are connected as Lie groups

4.2.2 Unipotent Radical

The complete reducibility (semisimplicity) of an invertible linear transformation is expressed by the triviality of the unipotent factor in its multiplicative Jordan decomposition. Our goal in this section is to obtain an analogous characterization of reductive algebraic groups. We call an algebraic group unipotent if all its elements are unipotent.

Theorem 4.2.3 — Engel's Theorem. Let G be an algebraic group and $U \subset G$ a normal subgroup of unipotent elements. Suppose (ρ, V) is a regular representation of G. Then there exists a G-invariant flag:

$$V = V_1 \supset V_2 \supset \cdots \supset V_r \supset V_{r+1} = \{0\} \quad (V_i \neq V_{i+1}) \tag{4.7}$$

such that $(\rho(u) - I)V_i \subset V_{i+1}$ for all $u \in U$ and $i = 1, \ldots, r$. In particular:

- $V_r = V^U \neq 0$ (V^U fixed points by U)
- If ρ is irreducible, then $V = V^U$ and $\rho(U) = \{I\}$

Unipotent Radical

For any linear algebraic group G, define its *unipotent radical*:

$$R_u(G) = \bigcup_{U \subset G} U \quad \text{(union over normal unipotent subgroups)} \tag{4.8}$$

Lemma 4.2.4 $R_u(G)$ is a closed normal unipotent subgroup.

Theorem 4.2.5 Let G be a linear algebraic group. If (ρ_1, V_1) and (ρ_2, V_2) are two completely reducible rational representations of G, then the tensor product representation $(\rho_1 \otimes \rho_2, V_1 \otimes V_2)$ is also completely reducible.

Corollary 4.2.6 If $G \subset \mathbf{GL}(n, \mathbb{C})$ is an algebraic subgroup whose action on $\mathbb{C}^n$ is completely reducible, then G is reductive.

Theorem 4.2.7 A linear algebraic group G is reductive if and only if $R_u(G) = \{1\}$.

Equivalent Characterization of Reductive Groups

An alternative and powerful characterization of reductivity is as follows: A connected linear algebraic group G is **reductive** if and only if its **radical**, $R(G)$, is a torus.

Explanation

The radical $R(G)$ is the maximal connected normal solvable subgroup of G. By the structure theorem for solvable groups, $R(G)$ can be written as a semidirect product $T \ltimes R_u(R(G))$, where T is a torus. For $R(G)$ to be a torus itself, its unipotent radical $R_u(R(G))$ must be trivial. Since the unipotent radical of the entire group, $R_u(G)$, is a characteristic subgroup of $R(G)$, the condition $R_u(R(G)) = \{1\}$ is equivalent to $R_u(G) = \{1\}$. This equivalence provides a deep connection between the structure of a group and its maximal solvable subgroup.

Corollary 4.2.8 Let $U = R_u(G)$. Then G/U is reductive.

4.2.3 Connected Algebraic Groups and Lie Groups

Recall that an algebraic group has a unique Lie group structure such that the regular functions are smooth. The purpose of this section is to prove the following result:

Theorem 4.2.9 If G is a connected linear algebraic group, then G is connected in the Lie group topology.

Let $N = R_u(G)$ (the unipotent radical of G). Then N is connected in both the Zariski topology and the Lie group topology. Thus to prove the theorem it suffices to consider the case $N = \{1\}$. That is, we may assume G is reductive.

Let Z be the center of G and Z° the identity component of Z in the Zariski topology. Then Z° is an algebraic torus, isomorphic to a product of subgroups isomorphic to $\mathbf{GL}(1,\mathbb{C})$. Hence Z° is connected in the Lie group topology. Applying the connectivity lemma again, we need only prove the theorem when $Z^\circ = \{1\}$.

The remainder of the proof follows the same argument as in a previous key theorem, provided we establish:

Theorem 4.2.10 Let G be a Zariski-connected reductive linear algebraic group with finite center. Then G is generated by its unipotent elements.

4.2.4 Simply Connected Semisimple Groups

Algebraically Simply Connected Groups

Let $\mathfrak{g}$ be a semisimple Lie algebra. We construct a linear algebraic group $\widetilde{G}$ that is algebraically simply connected with Lie algebra $\mathfrak{g}$.

Adjoint Group

The automorphism group $\mathrm{Aut}(\mathfrak{g})$ is a linear algebraic group. Define the *adjoint group* $G = \mathrm{Aut}(\mathfrak{g})^\circ$.

Proposition 4.2.11 The Lie algebra of G is $\mathrm{Der}(\mathfrak{g})$, and the adjoint representation

$$\mathrm{ad} : \mathfrak{g} \xrightarrow{\sim} \mathrm{Der}(\mathfrak{g}) \tag{4.9}$$

is a Lie algebra isomorphism.

The adjoint group may not be simply connected (e.g., $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ gives $G \cong \mathbf{SO}(3,\mathbb{C})$). The simply connected cover $\widetilde{G}$ exists (e.g., $\mathbf{SL}(2,\mathbb{C})$ for $\mathfrak{sl}_2$).

Construction via Highest Weight

Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and positive roots Φ^+. Let:

- Simple roots $\Delta = \{\alpha_1, \dots, \alpha_l\}$ ($l = \dim \mathfrak{h}$)
- Fundamental weights $\{\varpi_1, \dots, \varpi_l\}$
- Dominant weights $P_{++} = \sum_{i=1}^{l} \mathbb{Z}_{\geq 0} \varpi_i$

For $\lambda \in P_{++}$, let $(\rho^\lambda, V^\lambda)$ be the irreducible representation with highest weight λ. For each simple root α_j:

- Take subalgebra $\mathfrak{s}_j \cong \mathfrak{sl}(2,\mathbb{C})$
- Construct representation $(\pi_j^\lambda, V^\lambda)$ of $\mathbf{SL}(2,\mathbb{C})$ with $\mathrm{d}\,\pi_j^\lambda = \rho^\lambda|_{\mathfrak{s}_j}$
- Define $G_j^\lambda = \pi_j^\lambda(\mathbf{SL}(2,\mathbb{C}))$

Lemma 4.2.12 Let $G^\lambda \subset \mathbf{SL}(V^\lambda)$ be generated by $G_1^\lambda, \dots, G_l^\lambda$.

1. G^λ is connected algebraic subgroup
2. For $\mathfrak{g}$ simple, $\rho^\lambda : \mathfrak{g} \xrightarrow{\sim} \mathrm{Lie}(G^\lambda)$

Simply Connected Group for Simple Algebras

For $\mathfrak{g}$ simple:

- Set $G_j = G^{\varpi_j}$ (fundamental weights)
- Let G be adjoint group

- Induce homomorphisms $\varphi_j : G_j \to G$
- Define algebraic group:

$$\Gamma = \{(g_1, \dots, g_l, g) : \varphi_j(g_j) = g,\ \forall j\} \subset G_1 \times \cdots \times G_l \times G \tag{4.10}$$

- Take identity component $\widetilde{G} = \Gamma^\circ$

Theorem 4.2.13 For $\mathfrak{g}$ simple, $\widetilde{G}$ is algebraically simply connected with $\mathrm{Lie}(\widetilde{G}) \cong \mathfrak{g}$.

Semisimple Case and Center Structure

Corollary 4.2.14 For semisimple $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ ($\mathfrak{g}_j$ simple):
- Let $\widetilde{G}_j$ be simply connected group for $\mathfrak{g}_j$
- Then $\widetilde{G} = \widetilde{G}_1 \times \cdots \times \widetilde{G}_r$ is algebraically simply connected for $\mathfrak{g}$

Structure of the Center

Let $\mathfrak{g}$ be a simple Lie algebra, let $\tilde{G}$ be the corresponding algebraically simply connected group, and let G_{ad} be the adjoint group. All other connected groups with Lie algebra $\mathfrak{g}$ are of the form $\tilde{G}/D$, where D is a subgroup of the center $Z(\tilde{G})$.

The center $Z(\tilde{G})$ is a finite abelian group. It is canonically isomorphic to the quotient of the weight lattice P by the root lattice Q:

$$Z(\tilde{G}) \cong P/Q \tag{4.11}$$

The order of the center, $|Z(\tilde{G})| = |P/Q|$, is the index of the root lattice within the weight lattice. This index is related to the determinant of the Cartan matrix C. For simply-laced algebras (types A, D, E), we have $|P/Q| = \det(C)$.

The relationship between the simply connected group $\tilde{G}$ and the adjoint group G_{ad} is given by $G_{\mathrm{ad}} \cong \tilde{G}/Z(\tilde{G})$. A crucial fact is that for any semisimple Lie algebra $\mathfrak{g}$, the center of the adjoint group G_{ad} is always trivial.

The correct statement relates the center of $\tilde{G}$ to the connectivity of G_{ad}: the adjoint group G_{ad} is itself algebraically simply connected if and only if the center $Z(\tilde{G})$ is trivial. This occurs precisely when the weight and root lattices coincide ($P = Q$). This is true for simple Lie algebras of the following types:

$$\mathfrak{g} \text{ is of type } C_n, G_2, F_4, \text{ or } E_8. \tag{4.12}$$

For these types, $\tilde{G} \cong G_{\mathrm{ad}}$. For all other types (e.g., A_n, D_n, E_6, E_7), the center $Z(\tilde{G})$ is non-trivial, and thus G_{ad} is not simply connected, even though its own center remains trivial.

4.3 Homogeneous Spaces

For a reductive algebraic group G, homogeneous spaces have rich geometric structure. Key examples:
- **Flag manifolds** G/B: B Borel subgroup (upper-triangular matrices in suitable $G \hookrightarrow \mathbf{GL}(n, \mathbb{C})$)
- **Symmetric spaces** G/K: K fixed-point set of involution of G

We realize G/B as projective algebraic set and classify symmetric spaces for classical groups via affine models.

4.3.1 G-Spaces and Orbits

Algebraic Group Actions

Let M be quasiprojective algebraic set (see Appendix). An *algebraic action* of linear algebraic group G on M is a regular map

$$\alpha : G \times M \to M, \quad (g, m) \mapsto g \cdot m \tag{4.13}$$

satisfying:

$$g \cdot (h \cdot m) = (gh) \cdot m, \quad 1 \cdot m = m \quad (\forall g, h \in G,\ m \in M) \tag{4.14}$$

(Note $G \times M$ quasiprojective by Section A.4.2.) We often write gm for $g \cdot m$.

Theorem 4.3.1 — Orbit-Stabilizer Structure. For any $x \in M$:
- Stabilizer $G_x = \{g \in G : g \cdot x = x\}$ is algebraic subgroup
- Orbit $G \cdot x$ is smooth quasiprojective subset of M

Corollary 4.3.2 There exists $x \in M$ such that $G \cdot x$ is closed in M.

Quotient Space Construction

Theorem 4.3.3 — Characterization of Quotients. Let H be closed subgroup of linear algebraic group G.
1. **Realization**: $\exists$ regular action of G on $\mathbb{P}^n$ and $x_0 \in \mathbb{P}^n$ such that:
   - $H =$ stabilizer of x_0
   - Map $g \mapsto g \cdot x_0$ bijects $G/H \xrightarrow{\sim} G \cdot x_0$
   - Endows G/H with quasialgebraic variety structure (unique up to regular isomorphism)
2. **Regular quotient**: Projection $G \to G/H$ is regular map
3. **Universal property**: If G acts algebraically on quasiprojective M and $H \subset G_x$ (stabilizer of x), then

$$gH \mapsto g \cdot x : G/H \to G \cdot x \tag{4.15}$$

   is regular map.

4.3.2 Flag Manifolds

Grassmannians and Flag Varieties

Let V be a finite-dimensional complex vector space, $\bigwedge^p V$ its p-th exterior power. Elements are called p-vectors. For $u \in \bigwedge^p V$, define the linear map

$$T_u : V \to \bigwedge^{p+1} V, \quad T_u v = u \wedge v. \tag{4.16}$$

The kernel $V(u) = \{v \in V : u \wedge v = 0\}$ is the annihilator. Nonzero p-vectors of form $v_1 \wedge \cdots \wedge v_p$ ($v_i \in V$) are decomposable.

Lemma 4.3.4 Let $\dim V = n$.
1. For $0 \neq u \in \bigwedge^p V$, $\dim V(u) \leq p$ and $\mathrm{Rank}(T_u) \geq n - p$. Equality iff u decomposable.

2. If $u = v_1 \wedge \cdots \wedge v_p$ decomposable, $V(u) = \mathrm{Span}\{v_1, \ldots, v_p\}$. If $V(u) = V(w)$, then $w = cu$ $(c \in \mathbb{C}^\times)$, so $V(u)$ determines $[u] \in \mathbb{P}(\bigwedge^p V)$.
3. For $0 < p < l < n$, $0 \neq u \in \bigwedge^p V$, $0 \neq w \in \bigwedge^l V$ decomposable, $V(u) \subset V(w)$ iff $\mathrm{Rank}(T_u \oplus T_w) = n - p$.

Grassmannian Manifold

Denote by $\mathrm{Grass}_p(V)$ the set of p-dimensional subspaces. By Lemma 11.3.4(2), it identifies with decomposable p-vectors in $\mathbb{P}(\bigwedge^p V)$.

Proposition 4.3.5 $\mathrm{Grass}_p(V)$ is an irreducible projective algebraic set.

Plücker Coordinates

Take $V = \mathbb{C}^n$. Let $X \subset M_{n,p}$ be $n \times p$ matrices of rank p. p-dimensional subspaces correspond to column spaces. Since $x = yg$ for $g \in \mathbf{GL}(p, \mathbb{C})$ iff same column space, $\mathrm{Grass}_p(V) = X/\sim$ with $x \sim y$ iff $x = yg$.

For increasing p-tuple $J = (i_1, \ldots, i_p)$ $(1 \leq i_1 < \cdots < i_p \leq n)$, define minor

$$\xi_J(x) = \det \begin{bmatrix} x_{i_1 1} & \cdots & x_{i_1 p} \\ \vdots & \ddots & \vdots \\ x_{i_p 1} & \cdots & x_{i_p p} \end{bmatrix}. \tag{4.17}$$

Set $X_J = \{x \in M_{n,p} : \xi_J(x) \neq 0\}$. These cover X. Under right multiplication, $\xi_J(xg) = \xi_J(x) \det g$, so ξ_J ratios are rational functions on $\mathrm{Grass}_p(V)$.

Each $x \in X_J$ equivalent to matrix in affine patch

$$A_J = \{x \in M_{n,p} : x_{i_r s} = \delta_{rs} \text{ for } r, s = 1, \ldots, p\}. \tag{4.18}$$

On A_J, $\xi_J = 1$, and entries $\{x_{rs} : r \notin J\}$ restrict to Plücker coordinates. For $J = (1, 2, \ldots, p)$, $x \in A_J$ is

$$x = \begin{bmatrix} I_p \\ Y \end{bmatrix}, \quad Y \in M_{n-p,p}. \tag{4.19}$$

For an index tuple $L = (1, \ldots, \hat{s}, \ldots, p, r)$, where an index $s \in \{1, \ldots, p\}$ is omitted and an index $r \in \{p+1, \ldots, n\}$ is added, the corresponding Plücker coordinate on this affine patch is given by $\xi_L(x) = \pm y_{r-p,s}$, where $Y = (y_{ij})$ is the $(n-p) \times p$ matrix block. Since there are $p(n-p)$ such free entries, we have $\dim \mathrm{Grass}_p(\mathbb{C}^n) = p(n-p)$.

Isotropic Subspaces

Let ω be bilinear form (symmetric/skew-symmetric). Subspace $W \subset V$ isotropic if $\omega(x, y) = 0$ $\forall x, y \in W$. Isotropic Grassmannian $\mathcal{J}_p(V)$ is isotropic subspaces in $\mathrm{Grass}_p(V)$. Fixing basis, ω via matrix Γ, then W isotropic iff $x^\top \Gamma x = 0$ for basis matrix x of W. On each A_J, quadratic in coordinates, so $\mathcal{J}_p(V)$ projective algebraic.

Flag Varieties

For integers $0 < p_1 < \cdots < p_k < n$, set $\mathbf{p} = (p_1, \ldots, p_k)$. Flag manifold $\mathrm{Flag}_{\mathbf{p}}(V)$ consists of chains $V_1 \subset \cdots \subset V_k \subset V$ with $\dim V_i = p_i$. It embeds into

$$\mathrm{Grass}_{\mathbf{p}}(V) = \prod_{i=1}^{k} \mathrm{Grass}_{p_i}(V). \tag{4.20}$$

By Lemma 11.3.4(3), $\mathrm{Flag}_{\mathbf{p}}(V)$ closed in $\mathrm{Grass}_{\mathbf{p}}(V)$.

Group Actions and Borel Subgroups

$\mathbf{GL}(V)$ acts on $\mathrm{Grass}_{\mathbf{p}}(V)$. Fix basis $\{e_1, \ldots, e_n\}$, set $V_i = \mathrm{Span}\{e_1, \ldots, e_{p_i}\}$. Then $\mathrm{Flag}_{\mathbf{p}}(V)$ is orbit of $x_{\mathbf{p}} = (V_1, \ldots, V_k)$. Stabilizer $P_{\mathbf{p}}$ is block upper-triangular matrices:

$$\begin{bmatrix} A_1 & * & \cdots & * \\ & A_2 & \cdots & * \\ & & \ddots & \vdots \\ 0 & & & A_{k+1} \end{bmatrix}, \quad A_i \in \mathbf{GL}(m_i, \mathbb{C}) \tag{4.21}$$

with $m_1 = p_1$, $m_2 = p_2 - p_1$, $\ldots$, $m_{k+1} = n - p_k$.

Let $G \subset \mathbf{GL}(n, \mathbb{C})$ classical group, H diagonal subgroup. Set $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, $\mathfrak{h} = \mathrm{Lie}(H)$, $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ (strictly upper-triangular), N_n^+ upper-triangular unipotent matrices.

Theorem 4.3.6 For connected classical G, there exists projective algebraic set X_G with algebraic transitive G-action such that:

1. $\exists x_0 \in X_G$ with stabilizer B satisfying $\mathrm{Lie}(B) = \mathfrak{b}$
2. $B = H \cdot N^+$ semidirect, N^+ connected unipotent normal in B
3. $\mathrm{Lie}(N^+) = \mathfrak{n}^+$ and $N^+ = G \cap N_n^+$

4.3.3 Involutions and Symmetric Spaces

Symmetric Spaces as Affine Algebraic Sets

Let G be connected linear algebraic group, θ involutive automorphism ($\theta^2 = \mathrm{id}$). Its differential (also θ) is Lie algebra automorphism with $\theta^2 = I$. Set $K = G^\theta$ (fixed-point set). We embed G/K into G as affine algebraic subset.

Define θ-twisted conjugation action:

$$g \star y = gy\theta(g)^{-1} \quad (g, y \in G) \tag{4.22}$$

This satisfies $(g \star (h \star y)) = (gh) \star y$, defining a group action. Set

$$Q = \{y \in G : \theta(y) = y^{-1}\} \tag{4.23}$$

Then Q is algebraic subset of G, and $G \star Q = Q$ (since $\theta(g \star y) = (g \star y)^{-1}$).

Theorem 4.3.7 The θ-twisted action of G is transitive on each irreducible component of Q. Hence Q is finite union of closed θ-twisted G-orbits.

Corollary 4.3.8 The orbit of identity $P = G \star 1 = \{g\theta(g)^{-1} : g \in G\}$ satisfies:

- P is closed irreducible subset of G
- As G-space (under θ-twisted action), $P \cong G/K$

This realizes the symmetric space G/K concretely as affine algebraic set $P \subset G$.

4.3.4 Involutions of Classical Groups

Classification of Involutions for Classical Groups

Let $G \subset \mathbf{GL}(n, \mathbb{C})$ be connected classical group with $\mathrm{Lie}(G)$ simple. The involutions θ and symmetric spaces G/K are classified by three geometric structures:

1. Nondegenerate bilinear forms (symmetric/skew-symmetric)
2. Polarizations $\mathbb{C}^n = V_+ \oplus V_-$ with $V_\pm$ totally isotropic
3. Orthogonal decompositions $\mathbb{C}^n = V_+ \oplus V_-$ w.r.t. nondegenerate form

This gives seven symmetric space types:

- **Case (i)**: $G = \mathbf{SL}(n, \mathbb{C})$, K preserves form (2 subtypes)
- **Case (ii)**: G preserves form (or $\mathbf{SL}(n, \mathbb{C})$ if form zero), K preserves decomposition (3 subtypes)
- **Case (iii)**: G preserves form, K preserves decomposition (2 subtypes)

Proposition 4.3.9 — Automorphism Characterization. Let σ be regular automorphism of classical G:

1. $G = \mathbf{SL}(n, \mathbb{C})$: $\exists s \in G$ s.t. $\sigma(g) = sgs^{-1}$ or $\sigma(g) = s(g^T)^{-1}s^{-1}$
2. $G = \mathbf{Sp}(n, \mathbb{C})$: $\exists s \in G$ s.t. $\sigma(g) = sgs^{-1}$
3. $G = \mathbf{SO}(n, \mathbb{C})$ ($n \neq 2, 4$): $\exists s \in \mathbf{O}(n, \mathbb{C})$ s.t. $\sigma(g) = sgs^{-1}$

Theorem 4.3.10 — Involution Classification. Let θ be involution of classical G (Lie(G) simple). Modulo G-conjugation:

1. $G = \mathbf{SL}(n, \mathbb{C})$:
   (a) $\theta(x) = T(x^T)^{-1}T^T$ ($T \in G$, $T^T = T$)
      - Symmetric nondegenerate form $B(u, v) = u^T T v$
      - Determined by $T^T = T$

   (b) $\theta(x) = T(x^T)^{-1}T^T$ ($T \in G$, $T^T = -T$)
      - Skew-symmetric nondegenerate form
      - Determined by $T^T = -T$

   (c) $\theta(x) = JxJ^{-1}$ ($J \in \mathbf{GL}(n, \mathbb{C})$, $J^2 = I_n$)
      - Decomposition $\mathbb{C}^n = V_+ \oplus V_-$ ($V_\pm = \{v : Jv = \pm v\}$)
      - Determined by $\dim V_+$
2. $G = \mathbf{SO}(V, \omega)$ **or** $\mathbf{Sp}(V, \omega)$:
   (a) $\theta(x) = JxJ^{-1}$ (J preserves ω, $J^2 = I$)
      - Orthogonal decomposition $V = V_+ \oplus V_-$ ($V_\pm = \{v : Jv = \pm v\}$)
      - $\omega|_{V_\pm}$ nondegenerate
      - Determined by $\dim V_+$

   (b) $\theta(x) = JxJ^{-1}$ (J preserves ω, $J^2 = -I$)
      - Decomposition $V = V_+ \oplus V_-$ ($V_\pm = \{v : Jv = \pm iv\}$)
      - $\omega|_{V_\pm} = 0$, V_+ dual to V_- via ω
      - Unique determination

4.4 Classical Involutive Symmetric Spaces over $\mathbb{C}$

This section provides a corrected and consistent classification of the classical involutive symmetric spaces over the complex field $\mathbb{C}$. The primary object of study is a pair (G, K), where G is a classical complex Lie group and $K = G^\theta$ is the fixed-point subgroup of an involutive automorphism θ of G. It is crucial to distinguish these from Riemannian symmetric spaces, where G is a real Lie group and K is a maximal compact subgroup.

Table 4.1: Standard Involutions for Classical Irreducible Affine Symmetric Spaces over $\mathbb{C}$

Type	Symmetric Space G/K	Involution $\theta(g)$	Notes
AI	$\mathbf{SL}(n,\mathbb{C})/$ $\mathbf{SO}(n,\mathbb{C})$	$(g^t)^{-1}$	K consists of $g \in G$ satisfying $g^t g = I$.
AII	$\mathbf{SL}(2n,\mathbb{C})/$ $\mathbf{Sp}(2n,\mathbb{C})$	$T(g^t)^{-1}T^{-1}$	$T = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. K consists of $g \in G$ satisfying $g^t T g = T$.
AIII	$\mathbf{SL}(p+q)/$ $\mathbf{S}(\mathbf{GL}(p) \times \mathbf{GL}(q))$	$I_{p,q} g I_{p,q}^{-1}$	$I_{p,q} = \mathbf{diag}(I_p, -I_q)$. K consists of block-diagonal matrices in G.
BDI	$\mathbf{SO}(p+q)/$ $\mathbf{S}(\mathbf{O}(p) \times \mathbf{O}(q))$	$I_{p,q} g I_{p,q}^{-1}$	$I_{p,q} \in \mathbf{O}(p+q)$. K consists of block-diagonal matrices in G.
DIII	$\mathbf{SO}(2n)/$ $\mathbf{GL}(n)$	JgJ^{-1}	$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \mathbf{SO}(2n)$. K consists of matrices in G that commute with J.
CI	$\mathbf{Sp}(2n)/$ $\mathbf{GL}(n)$	JgJ^{-1}	Here J satisfies $J^2 = I$ and $J^t \Omega J = -\Omega$. The map $g \mapsto JgJ^{-1}$ is an outer automorphism of $\mathbf{Sp}(2n,\mathbb{C})$. Its fixed-point subgroup is isomorphic to $\mathbf{GL}(n,\mathbb{C})$. The involution in the manuscript is incorrect.
CII	$\mathbf{Sp}(2(p+q))/$ $(\mathbf{Sp}(2p) \times \mathbf{Sp}(2q))$	$I'_{p,q} g (I'_{p,q})^{-1}$	Here G preserves a block-diagonal symplectic form $\Omega = \mathbf{diag}(\Omega_p, \Omega_q)$, and $I'_{p,q}$ is a corresponding block matrix. This non-standard choice of symplectic form must be made explicit.

4.4.1 Type AI: $\mathbf{SL}(n,\mathbb{C})/\mathbf{SO}(n,\mathbb{C})$

Let $G = \mathbf{SL}(n,\mathbb{C})$. The involution is the inverse-transpose map:

$$\theta(g) = (g^t)^{-1} \tag{4.24}$$

The fixed-point subgroup $K = G^\theta$ consists of elements satisfying $g = (g^t)^{-1}$, which is equivalent to $g^t g = I$. These are the special orthogonal matrices.

$$K = \mathbf{SO}(n,\mathbb{C}) \tag{4.25}$$

4.4.2 Type AII: $\mathbf{SL}(2n,\mathbb{C})/\mathbf{Sp}(2n,\mathbb{C})$

Let $G = \mathbf{SL}(2n,\mathbb{C})$ and let T be the standard symplectic matrix:

$$T = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{4.26}$$

The involution is given by:

$$\theta(g) = T(g^t)^{-1}T^{-1} \tag{4.27}$$

An element g is in $K = G^\theta$ if $gTg^t = T$, which is an equivalent condition for $g^tTg = T$. These are the symplectic matrices.

$$K = \mathbf{Sp}(2n,\mathbb{C}) \tag{4.28}$$

4.4.3 Type AIII: $\mathbf{SL}(p+q,\mathbb{C})/\mathbf{S}(\mathbf{GL}(p,\mathbb{C}) \times \mathbf{GL}(q,\mathbb{C}))$

Let $G = \mathbf{SL}(p+q,\mathbb{C})$. The involution is conjugation by the block-diagonal matrix $I_{p,q}$:

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \quad \theta(g) = I_{p,q} g I_{p,q}^{-1} \tag{4.29}$$

The fixed-point subgroup $K = G^\theta$ consists of matrices in G that commute with $I_{p,q}$, which are the block-diagonal matrices with determinant 1.

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mid A \in \mathbf{GL}(p), D \in \mathbf{GL}(q), \det(A)\det(D) = 1 \right\} \cong \mathbf{S}(\mathbf{GL}(p) \times \mathbf{GL}(q)) \tag{4.30}$$

4.4.4 Type BDI: $\mathbf{SO}(p+q,\mathbb{C})/\mathbf{S}(\mathbf{O}(p,\mathbb{C}) \times \mathbf{O}(q,\mathbb{C}))$

Let $G = \mathbf{SO}(p+q,\mathbb{C})$, preserving the standard symmetric form $B = I_{p+q}$. The involution is the same as in Type AIII. Since $I_{p,q}$ is orthogonal, $\theta(g) = I_{p,q} g I_{p,q}^{-1}$ is an automorphism of G.

$$K = G \cap (\mathbf{O}(p,\mathbb{C}) \times \mathbf{O}(q,\mathbb{C})) \cong \mathbf{S}(\mathbf{O}(p,\mathbb{C}) \times \mathbf{O}(q,\mathbb{C})) \tag{4.31}$$

4.4.5 Type DIII: $\mathbf{SO}(2n,\mathbb{C})/\mathbf{GL}(n,\mathbb{C})$

Let $G = \mathbf{SO}(2n,\mathbb{C})$. Let J be a complex structure that is also an orthogonal matrix, for example:

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \mathbf{SO}(2n,\mathbb{C}) \tag{4.32}$$

The involution is conjugation by J: $\theta(g) = JgJ^{-1}$. The fixed-point subgroup $K = G^\theta$ consists of matrices in $\mathbf{SO}(2n,\mathbb{C})$ that commute with J, which gives a standard embedding of $\mathbf{GL}(n,\mathbb{C})$ in $\mathbf{SO}(2n,\mathbb{C})$.

$$K \cong \mathbf{GL}(n,\mathbb{C}) \tag{4.33}$$

4.4.6 Type CI: $\mathbf{Sp}(2n,\mathbb{C})/\mathbf{GL}(n,\mathbb{C})$

Let $G = \mathbf{Sp}(2n,\mathbb{C})$, preserving the symplectic form $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. The involution here is an **outer automorphism**. It can be defined by $\theta(g) = JgJ^{-1}$, where J is an invertible matrix satisfying $J^2 = I$ and, critically, $J^t\Omega J = -\Omega$. A suitable choice is:

$$J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \tag{4.34}$$

This map θ sends G to itself but is not an inner automorphism. The fixed-point subgroup $K = G^\theta$ consists of symplectic matrices commuting with J. This subgroup is isomorphic to $\mathbf{GL}(n,\mathbb{C})$.

$$K \cong \mathbf{GL}(n,\mathbb{C}) \tag{4.35}$$

Note: The choice of $J = \mathbf{diag}(iI_n, -iI_n)$ in the original manuscript is incorrect as that J does not preserve $\mathbf{Sp}(2n,\mathbb{C})$ and is related to the compact real form $\mathbf{Sp}(n)$, not the complex group.

4.4.7 Type CII: $\mathbf{Sp}(2(p+q),\mathbb{C})/(\mathbf{Sp}(2p,\mathbb{C}) \times \mathbf{Sp}(2q,\mathbb{C}))$

Let $G = \mathbf{Sp}(2(p+q),\mathbb{C})$. This case requires careful specification of the symplectic form. Let us choose a **non-standard, block-diagonal** symplectic form:

$$\Omega = \mathbf{diag}(\Omega_p, \Omega_q) = \begin{pmatrix} \Omega_p & 0 \\ 0 & \Omega_q \end{pmatrix}, \quad \text{where } \Omega_k = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix} \tag{4.36}$$

With respect to this form Ω, we can define the involution by conjugation with:

$$I'_{p,q} = \begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & -I_q \end{pmatrix} \quad \text{(a re-ordering of } \mathbf{diag}(I_{2p}, -I_{2q})) \tag{4.37}$$

(The precise form of $I'_{p,q}$ depends on the ordering of basis vectors for Ω). The fixed-point subgroup $K = G^\theta$ preserves the decomposition of the vector space into two symplectic subspaces of dimension $2p$ and $2q$.

$$K \cong \mathbf{Sp}(2p,\mathbb{C}) \times \mathbf{Sp}(2q,\mathbb{C}) \tag{4.38}$$

4.5 Borel Subgroups

With the flag manifolds available as a tool, we return to the structure theory of affine algebraic groups. We show that a Borel subgroup B is the unique maximal connected solvable subgroup (up to conjugacy) and that the conjugates of B cover G. An important consequence is that the centralizer of any torus in G is connected.

4.5.1 Solvable Groups

Solvable Groups

Let G be an abstract group. It is called *solvable* if there exists a chain of subgroups:

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_d \supseteq G_{d+1} = \{1\} \tag{4.39}$$

with G_{i+1} normal in G_i and G_i/G_{i+1} commutative for $i = 0, 1, \ldots, d$.

The *commutator subgroup* $\mathcal{D}(G)$ is generated by the set of commutators $\{xyx^{-1}y^{-1} : x, y \in G\}$. If G_1 is normal in G, then G/G_1 is commutative iff $G_1 \supseteq \mathcal{D}(G)$. Hence G is solvable iff $G \neq \mathcal{D}(G)$ and $\mathcal{D}(G)$ is solvable. Define the *derived series* $\{D^n(G)\}$ by:

$$D^0(G) = G, \quad D^{n+1}(G) = D(D^n(G)). \tag{4.40}$$

Then G is solvable iff $D^{n+1}(G) = \{1\}$ for some n, and the minimal such n is the *solvable length*.

Examples and Properties

The archetypical solvable group is B_n (upper-triangular invertible matrices). Note that the unipotent group N_n^+ (identity diagonal) is normal in B_n, and $B_n/N_n^+ \cong$ diagonal matrices (commutative). Define $N_{n,r}^+$ as the subgroup with zeros on diagonals 2 through r. Then $N_{n,r}^+ \triangleleft B_n$ and $N_{n,r}^+/N_{n,r+1}^+$ is abelian. Any isotropy group of a full flag in $\mathbb{C}^n$ is conjugate to B_n, hence solvable.

Subgroups of solvable groups are solvable. For example, if $G \subseteq \mathbf{GL}(n, \mathbb{C})$ is connected classical, the subgroup B (from earlier) is contained in a flag isotropy group, hence solvable.

Proposition 4.5.1 For a connected linear algebraic group G, the commutator subgroup $\mathcal{D}(G)$ is closed (Zariski topology) and connected.

4.5.2 Lie–Kolchin Theorem

Theorem 4.5.2 Let G be a connected solvable linear algebraic group and $N = \mathrm{Rad}_u(G)$. Then there exists a torus $T \subset G$ such that:

- $G = T \cdot N$
- $G \cong T \ltimes N$ (semidirect product) as a group
- $G \cong T \times N$ as an affine variety
- If $x \in G$ is semisimple and commutes with T, then $x \in T$ (so T is maximal)

Theorem 4.5.3 Let $G = T \cdot N$ be as above and $g \in G$ semisimple. Then:

1. g is conjugate under N to an element of T
2. $\mathrm{Cent}_G(g)$ is connected
3. For $b = tn \in G$ ($t \in T, n \in N$), there exist $u, v \in N$ such that $vbv^{-1} = tu$ and $tu = ut$

Corollary 4.5.4 Let G be connected solvable and $A \subset G$ a torus. Then:
1. There exists $s \in G$ with $sAs^{-1} \subset T$ (and $sAs^{-1} = T$ if A is maximal)
2. $\mathrm{Cent}_G(A)$ is connected
3. For semisimple $x \in \mathrm{Cent}_G(A)$, there exists a torus $S \subset \mathrm{Cent}_G(A)$ containing $A \cup \{x\}$

4.5.3 Conjugacy of Borel Subgroups

Borel Subgroups

A *Borel subgroup* of an algebraic group G is a maximal connected solvable subgroup.

Theorem 4.5.5 Let G be a connected linear algebraic group. Then:
- G contains a Borel subgroup B
- All other Borel subgroups are conjugate to B
- The homogeneous space G/B is a projective variety
- If $S \subseteq G$ is connected solvable and G/S is projective, then S is a Borel subgroup

To prove this, we use:

Theorem 4.5.6 — Borel Fixed Point. Let S be a connected solvable group acting algebraically on a projective variety X. Then there exists $x_0 \in X$ fixed by all $s \in S$.

Example

Let G be a connected classical group and B the solvable subgroup from earlier theorems. Then $X = G/B$ is projective, so B is a Borel subgroup.

4.5.4 Centralizer of a Torus

Theorem 4.5.7 Let G be a connected linear algebraic group and B a fixed Borel subgroup. Then:

$$G = \bigcup_{x \in G} xBx^{-1}. \tag{4.41}$$

Thus every element of G is contained in some Borel subgroup.

Theorem 4.5.8 Let G be a connected linear algebraic group and $A \subseteq G$ a torus. Then:
1. The centralizer $\mathrm{Cent}_G(A)$ is connected.
2. For any semisimple $x \in \mathrm{Cent}_G(A)$, there exists a torus $S \subseteq \mathrm{Cent}_G(A)$ containing $A \cup \{x\}$.

This result enables Lie algebra methods for studying centralizers.

4.5.5 Weyl Group and Regular Semisimple Conjugacy Classes

Let G be a connected linear algebraic group with semisimple Lie algebra $\mathfrak{g}$. Fix a maximal torus H (Lie algebra $\mathfrak{h}$) and compact real form U. Define the *Weyl group*:

$$W_G = \mathrm{Norm}_G(H)/H. \tag{4.42}$$

The adjoint representation restricts to a faithful action of W_G on:

- The Cartan subalgebra $\mathfrak{h}$
- The character group $X(H)$
- The root system $\Phi(\mathfrak{g}, \mathfrak{h})$

In particular, W_G is finite.

Theorem 4.5.9 The action of W_G on $\mathfrak{h}$ coincides with the algebraic Weyl group $W(\mathfrak{g}, \mathfrak{h})$. Every coset in W_G has a representative in U.

Corollary 4.5.10 The natural inclusion $\mathrm{Norm}_U(T)/T \hookrightarrow \mathrm{Norm}_G(H)/H$ is an isomorphism.

Regular Semisimple Elements

Define $G' = \{ghg^{-1} : g \in G,\ h \in H'\}$ (the *regular semisimple elements*), where $H' \subset H$ consists of elements with distinct eigenvalues. Fix positive roots Φ^+ and Borel subgroup $B = HN^+$. Extend characters $\alpha \in X(H)$ to B by $b^\alpha = h^\alpha$ for $b = hn$.
Lemma 4.5.11 For $b \in B$, $b \in G'$ iff $b^\alpha \neq 1$ for all $\alpha \in \Phi$. Thus $B \cap G' = H'N^+$ is Zariski open and dense in B.

Theorem 4.5.12 The set G' is Zariski open and dense in G.

Strongly Regular Elements

Define $G'' = \{ghg^{-1} : g \in G,\ h \in H''\}$ (strongly regular semisimple elements). Then G'' is also Zariski open and dense in G.

4.6 Further Properties of Real Forms

We now turn to the structure of a reductive algebraic group G as a real Lie group. We study conjugations and involutive automorphisms of G, and we obtain the polar decomposition of G relative to a compact real form U.

4.6.1 Groups with a Compact Real Form

Reductive Groups and Compact Real Forms

Theorem 4.6.1 A connected linear algebraic group is *reductive* iff it has a compact real form.

Theorem 4.6.2 Let G be connected reductive with compact real form τ_0. Let σ be either a complex conjugation or an involutive automorphism acting trivially on the identity component of $Z(G)$. Then there exists $g \in G$ such that $\tau(x) = g\tau_0(x)g^{-1}$ satisfies $\tau\sigma = \sigma\tau$.

Lemma 4.6.3 Assume G reductive with finite center. Let U be the compact real form and $\mathfrak{u} = \mathrm{Lie}(U)$. Then $\mathrm{tr}(\mathrm{Ad}(X)^2) < 0$ for all nonzero $X \in \mathfrak{u}$.

Corollary 4.6.4 If G is reductive with finite center and U_1, U_2 are compact real forms, then $\exists\, g \in G$ such that $gU_1g^{-1} = U_2$.

Hermitian Matrix Tools

For $A \in M_n(\mathbb{C})$, define the *Hermitian adjoint* $A^* = \bar{A}^T$. A matrix is *positive definite* if $A^* = A$ and $(Az, z) > 0$ for all $z \neq 0$.

Lemma 4.6.5 Let $A^* = A$. If a polynomial f satisfies $f(\exp(mA)) = 0$ for all $m \in \mathbb{N}$, then $f(\exp(tA)) = 0$ for all $t \in \mathbb{R}$.

For positive definite A, there exists an orthonormal basis $\{f_i\}$ with $Af_i = \lambda_i f_i$ ($\lambda_i > 0$). Define $\log A$ by $(\log A)f_i = (\log \lambda_i)f_i$. Then $(\log A)^* = \log A$ and $A = \exp(\log A)$.

Lemma 4.6.6 The map $\Psi : \mathrm{Herm}_n \to \Omega_n$ given by $\Psi(X) = \exp X$ is a diffeomorphism, where $\mathrm{Herm}_n = \{X \in M_n(\mathbb{C}) : X^* = X\}$ and $\Omega_n = \{\text{positive definite matrices}\}$.

For $A \in \Omega_n$ and $s \in \mathbb{R}$, define $A^s = \exp(s \log A)$.

4.6.2 Polar Decomposition by a Compact Form

Lemma 4.6.7 Let G be connected reductive with compact real form U (corresponding to conjugation τ). Then there exists a regular homomorphism $\Psi : G \to \mathbf{GL}(n, \mathbb{C})$ such that:

- Ψ is an isomorphism onto its image
- $\Psi(\tau(g)) = (\Psi(g)^*)^{-1}$ for all $g \in G$

Theorem 4.6.8 The map $\Phi : U \times \mathfrak{u} \to G$ defined by $\Phi(u, X) = u \exp(\mathrm{i}\, X)$ is a diffeomorphism, where U is the compact real form and $\mathfrak{u} = \mathrm{Lie}(U)$. In particular, U is connected.

Theorem 4.6.9 Let G be connected reductive with conjugation τ (compact real form U), and θ an involutive automorphism satisfying $\theta\tau = \tau\theta$. Set $K = \{g \in G : \theta(g) = g\}$ and $K_0 = K \cap U$. Then:

1. K is reductive
2. K_0 is a compact real form of K and Zariski dense in K

4.7 Gauss Decomposition

The final structural result for Chapter 12 is the *Gauss decomposition* of G relative to a diagonal torus $A \subset G$: Any element admits a factorization

$$g = n^+ \cdot a \cdot n^- \tag{4.43}$$

where

- n^+ is block upper-triangular unipotent
- a is block-diagonal
- n^- is block lower-triangular unipotent

with block sizes determined by the weight multiplicities of A. The set of such decomposable elements is Zariski dense in G. This decomposition extends to real forms of G when A is split relative to the real form.

4.7.1 Gauss Decomposition of $\mathbf{GL}(n, \mathbb{C})$

Let V be a finite-dimensional complex vector space and $T \subset \mathbf{GL}(V)$ an algebraic torus. Let $\mathcal{X}(T)$ be the group of rational characters of T. By Proposition, there exists a finite set

$\Sigma \subset \mathcal{X}(T)$ such that:

$$V = \bigoplus_{\chi \in \Sigma} V(\chi), \quad \text{where } V(\chi) = \{v \in V : tv = \chi(t)v \ \forall t \in T\}. \tag{4.44}$$

There is a subset $S = \{\chi_1, \dots, \chi_m\} \subset \Sigma$ such that the map $\Psi : T \to (\mathbb{C}^\times)^m$ given by $\Psi(t) = [\chi_1(t), \dots, \chi_m(t)]$ is a regular isomorphism. Hence every $\chi \in \mathcal{X}(T)$ is uniquely expressed as:

$$\chi = \chi_1^{p_1} \cdots \chi_m^{p_m} \quad \text{with } p_i \in \mathbb{Z}. \tag{4.45}$$

Set $\Phi = \{\chi\nu^{-1} : \chi, \nu \in \Sigma, \ \chi \neq \nu\}$. Then:

$$\mathrm{End}(V) = \mathrm{End}_T(V) \oplus \bigoplus_{\lambda \in \Phi} \mathrm{End}(V)(\lambda), \tag{4.46}$$

where $\mathrm{End}(V)(\lambda) = \{\Lambda \in \mathrm{End}(V) : t\Lambda t^{-1} = \lambda(t)\Lambda \ \forall t \in T\}$ and $\mathrm{End}_T(V)$ is the commutant of T.

Order $\mathcal{X}(T)$ lexicographically relative to the decomposition. Enumerate $\Sigma = \{\nu_1, \dots, \nu_r\}$ with $\nu_i > \nu_j$ if $i < j$. Set $\dim V = n$ and $m_i = \dim V(\nu_i)$. Choose a basis $\{e_1, \dots, e_n\}$ such that:

$$\{e_{k+1}, \dots, e_{k+m_i}\} \subset V(\nu_i) \qquad \left(k = \sum_{j=1}^{i-1} m_j\right). \tag{4.47}$$

Using this basis, identify $\mathrm{End}(V)$ with $M_n(\mathbb{C})$ and $\mathbf{GL}(V)$ with $\mathbf{GL}(n, \mathbb{C})$. Define $L = \{g \in \mathbf{GL}(n, \mathbb{C}) : gtg^{-1} = t \ \forall t \in T\}$, a linear algebraic subgroup with Lie algebra $\mathfrak{l} = \mathrm{End}_T(V)$. In this basis:

1. L consists of block-diagonal matrices $\mathrm{diag}[g_1, \dots, g_r]$ with $g_i \in M_{m_i}(\mathbb{C})$
2. For $\chi > 1$, the weight space $M_n(\mathbb{C})(\chi)$ is contained in block upper-triangular matrices:

$$\begin{bmatrix} 0_1 & * & \cdots & * \\ & 0_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & 0_r \end{bmatrix} \tag{4.48}$$

3. For $\chi < 1$, $M_n(\mathbb{C})(\chi)$ is contained in block lower-triangular matrices:

$$\begin{bmatrix} 0_1 & & & \\ * & 0_2 & & \\ \vdots & \ddots & \ddots & \\ * & \cdots & * & 0_r \end{bmatrix} \tag{4.49}$$

Define unipotent groups:

$$V^+ = \left\{ \begin{bmatrix} I_1 & * & \cdots & * \\ & I_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & I_r \end{bmatrix} \right\}, \quad V^- = \left\{ \begin{bmatrix} I_1 & & & \\ * & I_2 & & \\ \vdots & \ddots & \ddots & \\ * & \cdots & * & I_r \end{bmatrix} \right\} \tag{4.50}$$

where I_j is the $m_j \times m_j$ identity matrix.

Lemma 4.7.1 — Gauss Decomposition. Set $\Omega = V^- L V^+$. Let $\Delta_i(g)$ be the upper-left minor of size $m_1 + \cdots + m_i$. Then:

$$\Omega = \{g \in \mathbf{GL}(n, \mathbb{C}) : \Delta_i(g) \neq 0 \text{ for } i = 1, \ldots, r-1\}. \tag{4.51}$$

There exist rational maps $\gamma_- : \mathbf{GL}(n, \mathbb{C}) \to V^-$, $\gamma_0 : \mathbf{GL}(n, \mathbb{C}) \to L$, $\gamma_+ : \mathbf{GL}(n, \mathbb{C}) \to V^+$ such that for $g \in \Omega$:

$$g = \gamma_-(g)\gamma_0(g)\gamma_+(g) \tag{4.52}$$

uniquely.

4.7.2 Gauss Decomposition of an Algebraic Group

Let G be a connected linear algebraic group and $T \subset G$ an algebraic torus. Assume G is a Zariski-closed subgroup of $\mathbf{GL}(n, \mathbb{C})$. Define:

$$M = L \cap G = \{g \in G : gtg^{-1} = t \ \forall t \in T\} \quad \text{(centralizer of } T) \tag{4.53}$$

$$N^{\pm} = G \cap V^{\pm} \quad \text{(unipotent subgroups)} \tag{4.54}$$

Then:

- M is connected (by Theorem 11.4.10)
- $N^{\pm}$ are unipotent, hence connected

Set $\Omega = V^- L V^+$. Since $I \in \Omega$, $\Omega \cap G$ is Zariski open and dense in G (by Lemma 11.6.1).

Theorem 4.7.2 If $g \in G \cap \Omega$, then the Gauss decomposition factors satisfy:

$$\gamma_0(g) \in M, \quad \gamma_+(g) \in N^+, \quad \gamma_-(g) \in N^- \tag{4.55}$$

Relation to Bruhat and Iwasawa Decompositions

To place the Gauss decomposition in a broader context, it is helpful to relate it to other fundamental structural results for algebraic groups, namely the Bruhat and Iwasawa decompositions. The **Bruhat decomposition** provides a stratification of the group G into disjoint double cosets $G = \coprod_{w \in W} BwB$, where B is a Borel subgroup and W is the Weyl group.

The Gauss decomposition $G \supset \Omega = V^- L V^+$ is not merely a computational tool; it provides a concrete matrix realization for the unique open dense stratum in this decomposition, known as the **big cell**. This cell corresponds to the longest element $w_0 \in W$, and it can be shown that the set Bw_0B is isomorphic to the variety $V^- L V^+$. This is a crucial insight: the Gauss decomposition offers an explicit coordinate system for the generic, open dense part of the group. This explicit realization is indispensable for many applications in representation theory and algebraic geometry, where understanding the structure of generic elements is paramount. Furthermore, the concept is related to the **Iwasawa decomposition** $G_0 = K_0 A_0 N_0$ for a real form G_0, which provides a factorization into compact, abelian, and nilpotent components and is fundamental to the analysis of real Lie groups.

4.7.3 Gauss Decomposition for Real Forms

σ-Split Tori and Gauss Decomposition

Let G be a linear algebraic group with conjugation σ, and $G_0 = \{g \in G : \sigma(g) = g\}$ the real form. Let $A \subset G$ be an algebraic torus with $\sigma(A) = A$, inducing a real form $A_0 = \{a \in A : \sigma(a) = a\}$.

Definition 4.7.1 — Tori over $\mathbb{R}$ and Split Tori. Let G be a linear algebraic group with a conjugation σ. An algebraic torus $A \subset G$ is said to be **defined over** $\mathbb{R}$ if it is stable under σ, i.e., $\sigma(A) = A$. For any rational character $\chi \in X(A)$, the character χ^σ defined by $\chi^\sigma(a) = \chi(\sigma(a))$ is also in $X(A)$. It can be shown that for any torus defined over $\mathbb{R}$, we have $\chi^\sigma = \overline{\chi}$, where $\overline{\chi}$ is the complex conjugate character. This is equivalent to the identity $\chi(\sigma(a)) = \overline{\chi(a)}$ for all $a \in A$.

A torus A defined over $\mathbb{R}$ is called **split over** $\mathbb{R}$ (or $\mathbb{R}$**-split**) if the conjugation acts trivially on its character group, that is:

$$\chi^\sigma = \chi \quad \text{for all } \chi \in X(A). \tag{4.56}$$

This is a stronger condition which implies that all characters are real-valued on the real points of the torus, $A_0 = \{a \in A \mid \sigma(a) = a\}$. The Gauss decomposition for real forms requires the torus to be $\mathbb{R}$-split.

Lemma 4.7.3 If A is a σ-split torus in G, there exists a regular homomorphism $\varphi : G \to \mathbf{GL}(n, \mathbb{C})$ such that:

1. φ is an isomorphism of G onto its image.
2. The conjugation σ is intertwined with entry-wise complex conjugation via φ:

$$\varphi(\sigma(g)) = \overline{\varphi(g)} \tag{4.57}$$

3. The image of the torus A is contained in the group of diagonal matrices: $\varphi(A) \subset D_n$.

Gauss Decomposition for Real Forms

To formulate the Gauss decomposition for a real form G_0, we require a special type of torus. Let G be a connected linear algebraic group with conjugation σ, and let $A \subset G$ be a torus that is **split over** $\mathbb{R}$ (an $\mathbb{R}$-split torus), as defined above. Define the centralizer and its real points:

$$M = \{g \in G : gag^{-1} = a \ \forall a \in A\}, \quad M_0 = M \cap G_0. \tag{4.58}$$

By the properties of $\mathbb{R}$-split tori, we can choose an embedding $G \subset \mathbf{GL}(n, \mathbb{C})$ such that $\sigma(g) = \bar{g}$ and A is a subgroup of the diagonal matrices D_n. We define $N^\pm$ as the block unipotent subgroups corresponding to the weight space decomposition of A, as in the complex case. Since the torus A is $\mathbb{R}$-split, the condition $\chi^\sigma = \chi$ holds for all its characters. This ensures that the weight spaces are stable under σ, which implies $\sigma(N^\pm) = N^\pm$. We can therefore define the real subgroups $N_0^\pm = N^\pm \cap G_0$.

5. Character Formulas and Branching Laws

5.1 Irreducible Representations of Classical Groups

5.1.1 Skew Duality for Classical Groups

Let V be a finite-dimensional complex vector space and $\bigwedge^\bullet V$ its exterior algebra. For $v \in V$ and $v^* \in V^*$, define operators:

- *Exterior product*: $\varepsilon(v)u = v \wedge u$ for $u \in \bigwedge^\bullet V$
- *Interior product*:

$$\iota(v^*)(v_1 \wedge \cdots \wedge v_k) = \sum_{j=1}^{k} (-1)^{j-1} \langle v^*, v_j \rangle \, v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_k \tag{5.1}$$

  (where $\widehat{v_j}$ denotes omission).

These operators satisfy:

- Degree shift: $\varepsilon(v) : \bigwedge^p V \to \bigwedge^{p+1} V$, $\iota(v^*) : \bigwedge^p V \to \bigwedge^{p-1} V$
- Antiderivation: $\iota(v^*)(w \wedge u) = (\iota(v^*)w) \wedge u + (-1)^k w \wedge (\iota(v^*)u)$ for $w \in \bigwedge^k V$

Define the anticommutator $\{a, b\} = ab + ba$. The operators satisfy canonical anticommutation relations:

$$\{\varepsilon(x), \varepsilon(y)\} = 0, \quad \{\iota(x^*), \iota(y^*)\} = 0, \quad \{\varepsilon(x), \iota(x^*)\} = \langle x^*, x \rangle \, I \tag{5.2}$$

for $x, y \in V$, $x^*, y^* \in V^*$.

On $\bigwedge^\bullet V^*$, define operators $\varepsilon(v^*)$ and $\iota(v)$ satisfying:

$$\varepsilon(v^*) = \iota(v^*)^\top, \quad \iota(v) = \varepsilon(v)^\top. \tag{5.3}$$

The representation ρ of $\mathbf{GL}(V)$ on $\bigwedge^\bullet V$ is:

$$\rho(g)(v_1 \wedge \cdots \wedge v_p) = gv_1 \wedge \cdots \wedge gv_p. \tag{5.4}$$

It satisfies:

$$\rho(g)\varepsilon(v)\rho(g^{-1}) = \varepsilon(gv), \quad \rho(g)\iota(v^*)\rho(g^{-1}) = \iota((g^\top)^{-1}v^*). \tag{5.5}$$

The *skew Euler operator* is defined as:

$$E = \sum_{j=1}^{d} \varepsilon(f_j)\iota(f_j^*), \tag{5.6}$$

where $d = \dim V$ and $\{f_j\}$, $\{f_j^*\}$ are dual bases. Then:

- E commutes with $\mathbf{GL}(V)$
- E acts by k on $\bigwedge^k V$
- Commutation relations:

$$[E, \varepsilon(v)] = \varepsilon(v), \quad [E, \iota(v^*)] = -\iota(v^*) \tag{5.7}$$

For an algebraic subgroup $G \subset \mathbf{GL}(V)$, the projection $Q_k : \bigwedge^\bullet V \to \bigwedge^k V$ commutes with G. Thus:

$$\mathrm{End}_G(\bigwedge^\bullet V) = \bigoplus_{0 \le l,k \le d} \hom_G(\bigwedge^l V, \bigwedge^k V). \tag{5.8}$$

Let $\mathcal{F}(V)$ be the tensor algebra. The projection $P : \mathcal{F}(V) \to \bigwedge^\bullet V$ is:

$$Pu = \frac{1}{m!} \sum_{s \in \mathfrak{S}_m} \mathrm{sgn}(s)\sigma_m(s)u \quad \text{for } u \in V^{\otimes m}. \tag{5.9}$$

Then:

$$\hom_G(\bigwedge^l V, \bigwedge^k V) = \{PRP : R \in \hom_G(V^{\otimes l}, V^{\otimes k})\}. \tag{5.10}$$

We use this and the FFT to find generators for $\mathrm{End}_G(\bigwedge^\bullet V)$ when G is classical.

General Linear Group

Theorem 5.1.1 Let $G = \mathbf{GL}(V)$. Then $\mathrm{End}_G(\bigwedge V)$ is generated by the skew Euler operator E.

Corollary 5.1.2 In the decomposition $\bigwedge V = \bigoplus_{k=0}^{\dim V} \bigwedge^k V$, each summand is an irreducible and mutually inequivalent $\mathbf{GL}(V)$-module.

Orthogonal Group (Symmetric Form)

Let Ω be symmetric and $G = \mathbf{O}(V, \Omega)$.

Theorem 5.1.3 $\mathrm{End}_G(\bigwedge V)$ is generated by the skew Euler operator E.

Corollary 5.1.4 In the decomposition $\bigwedge V = \bigoplus_{k=0}^{\dim V} \bigwedge^k V$, each summand is an irreducible and mutually inequivalent $\mathbf{O}(V,\Omega)$-module.

Symplectic Group (Skew-Symmetric Form)

Let $\dim V = 2n$ and Ω skew-symmetric. Set $G = \mathbf{Sp}(V,\Omega)$. Define operators:

$$X = -\frac{1}{2}PC^*P, \quad Y = \frac{1}{2}PCP \tag{5.11}$$

where C and C^* are defined via Ω, and P is the tensor algebra projection.

Lemma 5.1.5 The operators satisfy:

$$[Y,\epsilon(v)] = 0, \quad [X,\iota(v^*)] = 0 \tag{5.12}$$

$$[Y,\iota(v^\sharp)] = \epsilon(v), \quad [X,\epsilon(v)] = \iota(v^\sharp) \tag{5.13}$$

$$[E,Y] = 2Y, \quad [E,X] = -2X, \quad [Y,X] = E - nI \tag{5.14}$$

for all $v \in V$, $v^* \in V^*$, and $v^\sharp \in V^*$ defined by $\langle v^\sharp, w\rangle = \Omega(v,w)$.

Define $\mathfrak{g}' = \mathrm{Span}\{X, Y, E - nI\}$. By the lemma, $\mathfrak{g}' \cong \mathfrak{sl}_2(\mathbb{C})$.

Theorem 5.1.6 The commutant $\mathrm{End}_G(\bigwedge V)$ is generated by $\mathfrak{g}'$.

Corollary 5.1.7 There is a canonical decomposition:

$$\bigwedge V \cong \bigoplus_{k=0}^{n} F^{(n-k)} \otimes \mathcal{H}^k \tag{5.15}$$

as a $(G,\mathfrak{g}')$-module, where:

- $F^{(k)}$ is the irreducible $\mathfrak{g}'$-module of dimension $k+1$
- $\mathcal{H}^k$ is an irreducible G-module and $\mathcal{H}^k \ncong \mathcal{H}^l$ for $k \neq l$

Lemma 5.1.8 The space $\hom_G(\bigwedge^l V, \bigwedge^k V)$ is spanned by:

1. YQ with $Q \in \hom_G(\bigwedge^l V, \bigwedge^{k-2} V)$
2. QX with $Q \in \hom_G(\bigwedge^{l-2} V, \bigwedge^k V)$
3. $\sum_{p=1}^{2n} \varepsilon(f_p) Q \iota(f_p^*)$ with $Q \in \hom_G(\bigwedge^{l-1} V, \bigwedge^{k-1} V)$

By the induction hypothesis (since $(k-2)+l < 2r$, etc.), the operators Q are in the algebra generated by X, Y, E. Thus the operators of types (1)-(3) are also in this algebra. Hence $\mathrm{End}_G(\bigwedge V)$ is generated by $\mathfrak{g}'$.

5.1.2 Fundamental Representations

Let G be a connected classical group with semisimple Lie algebra $\mathfrak{g}$. An irreducible rational representation of G is uniquely determined by the corresponding irreducible finite-dimensional representation of $\mathfrak{g}$. There is a bijection between irreducible representations of $\mathfrak{g}$ and the set $P_{++}(\mathfrak{g})$ of dominant integral weights (equivalent representations identified, positive roots fixed).

Elements of $P_{++}(\mathfrak{g})$ are $n_1\varpi_1 + \cdots + n_l\varpi_l$ ($n_i \in \mathbb{N}$), where $\varpi_1, \ldots, \varpi_l$ are fundamental weights. An irreducible representation with highest weight ϖ_k is called a fundamental representation. Explicit models for these representations and the actions of $\mathfrak{g}$ and G are constructed below.

Special Linear Group

Theorem 5.1.9 For $G = SL(n, \mathbb{C})$, the representation θ_r on the r-th exterior power $\bigwedge^r \mathbb{C}^n$ is irreducible with highest weight ϖ_r $(1 \le r \le n)$.

R For $r = n$, $\bigwedge^n \mathbb{C}^n$ is one-dimensional and θ_n is the trivial representation of $SL(n, \mathbb{C})$.

Special Orthogonal Group

Let B be the symmetric form on $\mathbb{C}^n$ and let $G = \mathbf{O}(\mathbb{C}^n, B)$. Let $G^\circ = \mathbf{SO}(\mathbb{C}^n, B)$ (the identity component of G).

Theorem 5.1.10 Let σ_r denote the representation of G on $\wedge^r \mathbb{C}^n$ for $1 \le r \le n$ as associated with the defining representation σ_1 on $\mathbb{C}^n$.

1. Let $n = 2l + 1 \ge 3$ be odd.
   If $1 \le r \le l$, then $(\sigma_r, \wedge^r \mathbb{C}^n)$ is an irreducible representation of G° with highest weight ϖ_r for $r \le l - 1$ and highest weight $2\varpi_l$ for $r = l$.
2. Let $n = 2l \ge 4$ be even.
   (a) If $1 \le r \le l - 1$, then $(\sigma_r, \wedge^r \mathbb{C}^n)$ is an irreducible representation of G° with highest weight ϖ_r for $r \le l - 2$ and highest weight $\varpi_{l-1} + \varpi_l$ for $r = l - 1$.
   (b) The space $\wedge^l \mathbb{C}^n$ is irreducible under the action of G. As a module for G° it decomposes into the sum of two irreducible representations with highest weights $2\varpi_{l-1}$ and $2\varpi_l$.

Symplectic Group

Let $G = \mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$, where Ω is a nondegenerate skew-symmetric form. We will use the theorem of the highest weight to identify the isotypic components in the decomposition of $\wedge \mathbb{C}^{2l}$ under the action of G. Let $\theta \in (\wedge^2 V)^G$ be the G-invariant skew 2-tensor corresponding to Ω. Let Y be the operator of exterior multiplication by $(1/2)\theta$, and let $X = -Y^*$ (the adjoint operator relative to the skew-bilinear form on $\wedge V$ obtained from Ω). Set $H = Il - E$, where E is the skew Euler operator. Then we have the commutation relations

$$[H, X] = 2X \quad , \quad [H, Y] = -2Y \quad , \quad [X, Y] = H, \tag{5.16}$$

Set $\mathfrak{g}' = \mathrm{Span}\{X, Y, H\}$. Then $\mathfrak{g}' \cong \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{g}'$ generates the commuting algebra $\mathrm{End}_G(\wedge V)$. From formula we can view X as a skew-symmetric Laplace operator. This motivates the following terminology:

Definition 5.1.1 An element $u \in \wedge \mathbb{C}^{2l}$ is Ω-harmonic if $Xu = 0$. The space of Ω-harmonic elements is denoted by $\mathcal{H}(\wedge \mathbb{C}^{2l}, \Omega)$.

Since $X : \wedge^p \mathbb{C}^{2l} \longrightarrow \wedge^{p-2} \mathbb{C}^{2l}$ shifts degree by two, an element u is Ω-harmonic if and only if each homogeneous component of u is Ω-harmonic. Thus

$$\mathcal{H}(\wedge \mathbb{C}^{2l}, \Omega) = \bigoplus_{p \ge 0} \mathcal{H}(\wedge^p \mathbb{C}^{2l}, \Omega)\,, \tag{5.17}$$

where $\mathcal{H}(\wedge^p \mathbb{C}^{2l}, \Omega) = \{u \in \wedge^p \mathbb{C}^{2l} : Xu = 0\}$. Because X commutes with G, this space is invariant under G.

Theorem 5.1.11 Let $G = \mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$ and $\mathfrak{g}' = \mathrm{Span}\{X, Y, H\}$ as above.

1. If $p > l$ then $\mathcal{H}(\wedge^p \mathbb{C}^{2l}, \Omega) = 0$.
2. Let $F^{(k)}$ be the irreducible $\mathfrak{g}'$-module of dimension $k+1$. Then
$$\wedge \mathbb{C}^{2l} \cong \bigoplus_{p=0}^{l} \left\{ F^{(l-p)} \otimes \mathcal{H}(\wedge^p \mathbb{C}^{2l}, \Omega) \right\} \tag{5.18}$$
as a $(\mathfrak{g}', G)$-module.
3. If $1 \le p \le l$, then $\mathcal{H}(\wedge^p \mathbb{C}^{2l}, \Omega)$ is an irreducible G-module with highest weight ϖ_p.

Corollary 5.1.12 The map $\mathbb{C}[t] \otimes \mathcal{H}(\wedge \mathbb{C}^{2l}, \Omega) \longrightarrow \wedge \mathbb{C}^{2l}$ given by $f(t) \otimes u \mapsto f(\theta) \wedge u$ (exterior multiplication) is a G-module isomorphism. Thus
$$\wedge^k \mathbb{C}^{2l} = \bigoplus_{p=0}^{\lfloor k/2 \rfloor} \theta^p \wedge \mathcal{H}(\wedge^{k-2p} \mathbb{C}^{2l}, \Omega)\,. \tag{5.19}$$
Hence $\wedge^k \mathbb{C}^{2l}$ is a multiplicity-free G-module and has highest weights ϖ_{k-2p} for $0 \le p \le \lfloor k/2 \rfloor$ (where $\varpi_0 = 0$).

Corollary 5.1.13 The space $\mathcal{H}(\wedge^p \mathbb{C}^{2l}, \Omega)$ has dimension $\binom{2l}{p} - \binom{2l}{p-2}$ for $p = 1, \ldots, l$.

Proposition 5.1.14 The space $\mathcal{H}(\wedge^p \mathbb{C}^{2l}, \Omega)$ is spanned by the isotropic p-vectors for $p = 1, \ldots, l$.

Theorem 5.1.15 Let B be the symmetric form on $\mathbb{C}^n$ and let $G = \mathbf{O}(\mathbb{C}^n, B)$. Let $G^\circ = \mathbf{SO}(\mathbb{C}^n, B)$ (the identity component of G). Let σ_r denote the representation of G on $\wedge^r \mathbb{C}^n$ for $1 \le r \le n$ as associated with the defining representation σ_1 on $\mathbb{C}^n$.

1. Let $n = 2l + 1 \ge 3$ be odd.
   If $1 \le r \le l$, then $(\sigma_r, \wedge^r \mathbb{C}^n)$ is an irreducible representation of G° with highest weight ϖ_r for $r \le l-1$ and highest weight $2\varpi_l$ for $r = l$.
2. Let $n = 2l \ge 4$ be even.
   (a) If $1 \le r \le l-1$, then $(\sigma_r, \wedge^r \mathbb{C}^n)$ is an irreducible representation of G° with highest weight ϖ_r for $r \le l-2$ and highest weight $\varpi_{l-1} + \varpi_l$ for $r = l-1$.
   (b) The space $\wedge^l \mathbb{C}^n$ is irreducible under the action of G. As a module for G° it decomposes into the sum of two irreducible representations with highest weights $2\varpi_{l-1}$ and $2\varpi_l$.

5.1.3 Cartan Product

Using skew duality we have constructed the fundamental representations of a connected classical group G whose Lie algebra is semisimple (with three exceptions in the case of the orthogonal groups). Now we obtain more irreducible representations by decomposing tensor products of representations already constructed.

Given finite-dimensional representations (ρ, U) and (σ, V) of G we can form the tensor product $(\rho \otimes \sigma, U \otimes V)$ of these representations. The weight spaces of $\rho \otimes \sigma$ are

$$(U \otimes V)(\nu) = \sum_{\lambda+\mu=\nu} U(\lambda) \otimes V(\mu)\,. \tag{5.20}$$

In particular,

$$\dim(U \otimes V)(\nu) = \sum_{\lambda+\mu=\nu} \dim U(\lambda) \dim V(\mu)\,. \tag{5.21}$$

Decomposing $U \otimes V$ into isotypic components for G and determining the multiplicities of each component is a more difficult problem that we shall treat in later chapters with the aid of the Weyl character formula. However, when ρ and σ are irreducible, then by the theorem of the highest weight we can identify a particular irreducible component that occurs with multiplicity one in the tensor product.

Proposition 5.1.16 Let $\mathfrak{g}$ be a semisimple Lie algebra. Let (π^λ, V^λ) and (π^μ, V^μ) be finite-dimensional irreducible representations of $\mathfrak{g}$ with highest weights $\lambda, \mu \in P_{++}(\mathfrak{g})$.

1. Fix highest-weight vectors $v_\lambda \in V^\lambda$ and $v_\mu \in V^\mu$. Then the $\mathfrak{g}$-cyclic subspace $U \subset V^\lambda \otimes V^\mu$ generated by $v_\lambda \otimes v_\mu$ is an irreducible $\mathfrak{g}$-module with highest weight $\lambda + \mu$.
2. If ν occurs as the highest weight of a $\mathfrak{g}$-submodule of $V^\lambda \otimes V^\mu$ then $\nu \le \lambda + \mu$.
3. The irreducible representation $(\pi^{\lambda+\mu}, V^{\lambda+\mu})$ occurs with multiplicity one in $V^\lambda \otimes V^\mu$.

We call the submodule U in (1) of the proposition the Cartan product of the representations (π^λ, V^λ) and (π^μ, V^μ).

Corollary 5.1.17 Let G be the group $\mathbf{SL}(V)$ or $\mathbf{Sp}(V)$ with $\dim V \ge 2$, or $\mathbf{SO}(V)$ with $\dim V \ge 3$. If π^λ and π^μ are differentials of irreducible regular representations of G, then the Cartan product of π^λ and π^μ is the differential of an irreducible regular representation of G with highest weight $\lambda + \mu$. Hence the set of highest weights of irreducible regular G-modules is closed under addition.

Theorem 5.1.18 Let G be the group $\mathbf{SL}(V)$ or $\mathbf{Sp}(V)$ with $\dim V \ge 2$, or $\mathbf{SO}(V)$ with $\dim V \ge 3$. For every dominant weight $\mu \in P_{++}(G)$ there exists an integer k such that $V^{\otimes k}$ contains an irreducible G-module with highest weight μ. Hence every irreducible regular representation of G occurs in the tensor algebra of V.

5.1.4 Irreducible Representations of $\mathbf{GL}(V)$

We now extend the theorem of the highest weight to the group $G = \mathbf{GL}(n, \mathbb{C})$. Recall that $P_{++}(G)$ consists of all weights

$$\mu = m_1\varepsilon_1 + \cdots + m_n\varepsilon_n\,, \quad \text{with} \quad m_1 \ge \cdots \ge m_n \quad \text{and} \quad m_i \in \mathbb{Z}\,. \tag{5.22}$$

Define the dominant weights

$$\lambda_i = \varepsilon_1 + \cdots + \varepsilon_i \quad \text{for} \quad i = 1, \dots, n\,. \tag{5.23}$$

Note that the restriction of λ_i to the diagonal matrices of trace zero is the fundamental weight ϖ_i of $\mathfrak{sl}(n, \mathbb{C})$ for $i = 1, \dots, n-1$. If μ is given by (5.60) then

$$\mu = (m_1 - m_2)\lambda_1 + (m_2 - m_3)\lambda_2 + \cdots + (m_{n-1} - m_n)\lambda_{n-1} + m_n\lambda_n\,.$$

Hence the elements of $P_{++}(G)$ can also be written uniquely as

$$\mu = k_1\lambda_1 + \cdots + k_n\lambda_n\,, \quad \text{with} \quad k_1 \geq 0, \ldots, k_{n-1} \geq 0 \quad \text{and} \quad k_i \in \mathbb{Z}\,.$$

The restriction of μ to the diagonal matrices of trace zero is the weight

$$\mu_0 = (m_1 - m_2)\varpi_1 + (m_2 - m_3)\varpi_2 + \cdots + (m_{n-1} - m_n)\varpi_{n-1}\,. \tag{5.24}$$

Theorem 5.1.19 Let $G = \mathbf{GL}(n,\mathbb{C})$ and let μ be given by (5.60). Then there exists a unique irreducible rational representation (π_μ^μ, F_μ^μ) of G such that the following hold:

1. The restriction of π_μ^μ to $\mathbf{SL}(n,\mathbb{C})$ has highest weight μ_0 given by (5.62).
2. The element zI_n of G (for $z \in \mathbb{C}^\times$) acts by $z^{m_1+\cdots+m_n}I$ on F_μ^μ.

Define $\pi_\mu^{**}(g) = \pi_\mu^\mu((g^t)^{-1})$ for $g \in G$. Then the representation $(\pi_\mu^{**}, F_\mu^\mu)$ is equivalent to the dual representation $((\pi_\mu^\mu)^*, (F_\mu^\mu)^*)$.

5.1.5 Irreducible Representations $\mathbf{O}(V)$

We now determine the irreducible regular representations of the full orthogonal group in terms of the irreducible representations of the special orthogonal group.

We use the following notation: Let B be the symmetric bilinear form on $\mathbb{C}^n$. Let $G = \mathbf{O}(\mathbb{C}^n, B)$, so that $G^\circ = \mathbf{SO}(\mathbb{C}^n, B)$. Let H be the diagonal subgroup of G, $H^\circ = H \cap G^\circ$, and N^+ be the subgroup of upper-triangular unipotent matrices in G. Let (π^λ, V^λ) be the irreducible representation of G° with highest weight $\lambda \in P_{++}(G^\circ)$.

When $n = 2l+1$ is odd, then $\det(-I) = -1$, and we have $G^\circ \times \mathbb{Z}_2 \cong G$ (direct product). In this case $H \cong H^\circ \times \{\pm I\}$. If (ρ, W) is an irreducible representation of G, then $\rho(-I) = \varepsilon I$ with $\varepsilon = \pm 1$ by Schur's lemma, since $-I$ is in the center of G and $\rho(-I)^2 = 1$. Hence the restriction of ρ to G° is still irreducible, so $\dim W^{N^+} = 1$. The action of H on W^{N^+} is by some character $\chi_{\lambda,\varepsilon}(h,a) = \varepsilon h^\lambda$ for $h \in H^\circ$, where $\varepsilon = \pm$ and

$$\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_l\varepsilon_l\,, \quad \text{with} \quad \lambda_1 \geq \cdots \geq \lambda_l \geq 0\,,$$

is the weight for the action $\mathfrak{h}$ on W^{N^+}. Furthermore, $\rho|_{G^\circ}$ is equivalent to (π^λ, V^λ). We set $V^{\lambda,\varepsilon} = V^\lambda$ and extend π^λ to a representation $\pi^{\lambda,\varepsilon}$ of G by $\pi^{\lambda,\varepsilon}(-I) = \varepsilon I$. Clearly, $\pi^{\lambda,\varepsilon} \cong \rho$. Conversely, we can start with π^λ and extend it in two ways to obtain irreducible representations $\pi^{\lambda,\pm}$ of G in which $-I$ acts by $\pm I$. Thus we have classified the irreducible representations of G in this case as follows:

Theorem 5.1.20 The irreducible regular representations of $G = O(n,\mathbb{C})$ for n odd are of the form $(\pi^{\lambda,\varepsilon}, V^{\lambda,\varepsilon})$, where λ is the highest weight for the action of G°, $\varepsilon = \pm$, and $-I \in G$ acts by εI.

Theorem 5.1.21 Let $n \geq 4$ be even. The irreducible regular representations (σ, W) of $\mathbf{O}(n, \mathbb{C})$ are of the following two types:

1. Suppose $\dim W^{N^+} = 1$ and $\mathfrak{h}$ acts by the weight λ on W^{N^+}. Then g_0 acts on this space by εI $(\varepsilon = \pm)$ and one has $(\sigma, W) \cong (\pi^{\lambda,\varepsilon}, V^{\lambda,\varepsilon})$.
2. Suppose $\dim W^{N^+} = 2$. Then $\mathfrak{h}$ has two distinct weights λ and $g_0 \cdot \lambda$ on W^{N^+}, and one has $(\sigma, W) \cong (\rho^{\lambda}, V^{\lambda})$.

5.2 Character and Dimension Formulas

The central result of this chapter is the celebrated Weyl character formula for irreducible representations of a connected semisimple algebraic group G. We give two (logically independent) proofs of this formula. The first is algebraic and uses the theorem of the highest weight, some arguments involving invariant regular functions, and the Casimir operator. The second is Weyl's original analytic proof based on his integral formula for the compact real form of G.

We begin with a statement of the character formula and derive some of its immediate consequences: the Weyl dimension formula, formulas for inner and outer multiplicities, and character formulas for the commutant of G in a regular G-module. These character formulas will be used to obtain branching laws and to express the characters of the symmetric group in terms of the characters of the general linear group.

5.2.1 Weyl Character Formula

Let G be a connected reductive linear algebraic group with Lie algebra $\mathfrak{g}$. We assume that $\mathfrak{g}$ is semisimple and that G is algebraically simply connected. This excludes $\mathbf{GL}(n, \mathbb{C})$ and $\mathbf{SO}(2, \mathbb{C}) \cong \mathbf{GL}(1, \mathbb{C})$; we will use the results of a previous section to take care of this case at the end of this section.

Fix a maximal algebraic torus H of G with Lie algebra $\mathfrak{h}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and we let $\Phi \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{h}$ on $\mathfrak{g}$. We fix a set Φ^+ of positive roots. Let $P = P(\mathfrak{g}) \subset \mathfrak{h}^*$ be the weight lattice and let $P_{++} \subset P$ be the dominant weights relative to Φ^+. Let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \varpi_1 + \cdots + \varpi_l \,,$$

where ϖ_i are the fundamental weights of $\mathfrak{g}$. The character $h \mapsto h^{\rho}$ is well defined on H, since the weight lattice of g coincides with the weight lattice of $\mathfrak{g}$. We define the *Weyl function*

$$\Delta_G = e^{\rho} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) \,.$$

Here we view the exponentials e^{λ}, for $\lambda \in P$, as elements of the group algebra $A[P]$ of the additive group P of weights; addition of exponents corresponds to the convolution multiplication in the group algebra under this identification. Thus Δ_G is an element of $A[P]$. We can also view Δ_G as a function on H.

When $G = \mathbf{SL}(n, \mathbb{C})$ we write $\Delta_G = \Delta_n$. Since e^{ϖ_i} is the character $h \mapsto x_1 x_2 \cdots x_i$ (where $h = \mathrm{diag}[x_1, \ldots, x_n]$), we see that e^{ρ} is the character $h \mapsto x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$. Since the positive roots give the characters $h \mapsto x_i x_j^{-1}$ for $1 \le i < j \le n$, we have

$$\Delta_n(h) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \prod_{1 \le i < j \le n} (1 - x_j x_i^{-1}) = \prod_{1 \le i < j \le n} (x_i - x_j) .$$

For the other classical groups, Δ_G is given as follows. Let $n = 2l$ and $h = \mathrm{diag}[x_1, \ldots, x_l, x_l^{-1}, \ldots, x_1^{-1}]$. Then we calculate that

$$\Delta_{\mathbf{SO}(2l)}(h) = \prod_{1 \le i < j \le l} (x_i + x_i^{-1} - x_j - x_j^{-1})$$

and

$$\Delta_{\mathbf{Sp}(l)}(h) = \prod_{1 \le i < j \le l} (x_i + x_i^{-1} - x_j - x_j^{-1}) \prod_{k=1}^{l} (x_k - x_k^{-1}) .$$

For $n = 2l + 1$ and $h = \mathrm{diag}[x_1, \ldots, x_l, 1, x_l^{-1}, \ldots, x_1^{-1}]$ in $\mathbf{SO}(C^{2l+1}, B)$ we have

$$\Delta_{\mathbf{SO}(2l+1)}(h) = \prod_{1 \le i < j \le l} (x_i + x_i^{-1} - x_j - x_j^{-1}) \prod_{k=1}^{l} (x_k^{1/2} - x_k^{-1/2}) .$$

Recall that the Weyl group W is equal to $\mathrm{Norm}_G(H)/H$. The adjoint representation of G gives a faithful representation σ of W on $\mathfrak{h}^*$, and we define

$$\mathrm{sgn}(s) = \det(\sigma(s)) .$$

Since W is generated by reflections, we have $\mathrm{sgn}(s) = \pm 1$. The function Δ_G is skew-symmetric:

$$\Delta_G(s h s^{-1}) = \mathrm{sgn}(s) \Delta_G(h) \quad \text{for } h \in H .$$

Indeed, if we write Δ_G (viewed as a formal exponential sum) as

$$\Delta_G = \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2}) ,$$

then the reflection given by a simple root α_i changes the sign of the factor involving α_i and permutes the other factors. Since these reflections generate W, this implies the skew-symmetric property. Of course, this property can also be verified case by case from the formulas above.

A finite-dimensional $\mathfrak{g}$-module V decomposes as a direct sum of $\mathfrak{h}$ weight spaces $V(\mu)$, with $\mu \in P(\mathfrak{g})$. We write

$$\mathrm{ch}(V) = \sum_{\mu \in P} \dim V(\mu) e^{\mu}$$

as an element in the group algebra $A[P]$. We may also view $\mathrm{ch}(V)$ as a regular function on H because G is algebraically simply connected. There is a regular representation π of G on V whose differential is the given representation of $\mathfrak{g}$, and

$$\mathrm{ch}(V)(h) = \mathrm{tr}(\pi(h)) \quad \text{for } h \in H .$$

Theorem 5.2.1 — Weyl Character Formula. Let $\lambda \in P_{++}$ and let V^λ be the finite-dimensional irreducible G-module with highest weight λ. Then

$$\Delta_G \cdot \mathrm{ch}(V^\lambda) = \sum_{s \in W} \mathrm{sgn}(s) e^{s(\lambda+\rho)} .$$

In the Weyl character formula, the character, which is invariant under the action of W, is expressed as a ratio of functions that are skew-symmetric under the action of W (for $G = \mathbf{SL}(2,\mathbb{C})$ it is just the formula for the sum of a finite geometric series). Later in this chapter we shall give two proofs of this fundamental result: an algebraic proof that uses the Casimir operator and the theorem of the highest weight, and an analytic proof (Weyl's original proof). Both proofs require rather lengthy developments of preliminary results. At this point we derive some immediate consequences of the character formula.

We first extend the formula to include the case $G = \mathbf{GL}(n,\mathbb{C})$. Let

$$\mu = m_1\varepsilon_1 + \cdots + m_n\varepsilon_n \quad \text{with} \quad m_1 \geq m_2 \geq \cdots \geq m_n \quad \text{and} \quad m_i \in \mathbb{Z} .$$

Let (π^μ_μ, F^μ_μ) be the irreducible representation of $\mathbf{GL}(n,\mathbb{C})$ associated with μ. Define

$$\rho_n = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1}$$

as an element in the weight lattice for $\mathbf{GL}(n,\mathbb{C})$. Define the function $\Delta_n(h)$, for $h = \mathrm{diag}[x_1, \ldots, x_n]$, by the same formula as for $\mathbf{SL}(n,\mathbb{C})$. Note that when $G = \mathbf{GL}(n,\mathbb{C})$, then the Weyl denominator formula is the product expansion

$$\prod_{1 \leq i < j \leq n} (x_i - x_j) = \det V(h)$$

of the Vandermonde determinant

$$V(h) = \begin{bmatrix} x_1^{n-1} & x_1^{n-2} & \ldots & 1 \\ x_2^{n-1} & x_2^{n-2} & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \ldots & 1 \end{bmatrix} .$$

Corollary 5.2.2 Let $h = \mathrm{diag}[x_1, \ldots, x_n]$. Then

$$\begin{aligned} \Delta_n(h)\mathrm{tr}(\pi^\mu_\mu(h)) &= \sum_{s \in \mathfrak{S}_n} \mathrm{sgn}(s) h^{s(\mu+\rho_n)} \\ &= \det \begin{bmatrix} x_1^{m_1+n-1} & x_1^{m_2+n-2} & \ldots & x_1^{m_n} \\ x_2^{m_1+n-1} & x_2^{m_2+n-2} & \ldots & x_2^{m_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{m_1+n-1} & x_n^{m_2+n-2} & \ldots & x_n^{m_n} \end{bmatrix} . \end{aligned}$$

We now draw some consequences of the Weyl character formula in general.

Corollary 5.2.3 — Weyl Denominator Formula. The Weyl function is the skew-symmetrization of the character e^{ρ} of H:

$$\Delta_G = \sum_{s \in W} \operatorname{sgn}(s) e^{s \cdot \rho}\,.$$

Let $\lambda \in P_{++}$. For $\mu \in P$ we write $m_\lambda(\mu) = \dim V^\lambda(\mu)$ (the multiplicity of the weight μ in V^λ).

Corollary 5.2.4 Suppose $\mu \in P$. If $\mu + \rho = t \cdot (\lambda + \rho)$ for some $t \in W$, then

$$\sum_{s \in W} \operatorname{sgn}(s) m_\lambda(\mu + \rho - s \cdot \rho) = \operatorname{sgn}(t)\,.$$

Otherwise, the sum on the left is zero. In particular, if $\mu = \lambda$ the sum is 1, while if $\mu \in P_{++}$ and $\mu \neq \lambda$, then the sum is zero.

R The steps in the proof just given are reversible, so the previous corollaries imply the Weyl character formula.

Let (σ, F) be a finite-dimensional regular representation of G. Then F decomposes as a direct sum of irreducible representations. The number of times that a particular irreducible module V^λ appears in the decomposition is the *outer multiplicity* $\operatorname{mult}_F(V^\lambda)$.

Corollary 5.2.5 The outer multiplicity of V^λ is the skew-symmetrization over $s \in W$ of the multiplicities of the weights $\lambda + \rho - s \cdot \rho$:

$$\operatorname{mult}_F(V^\lambda) = \sum_{s \in W} \operatorname{sgn}(s) \dim F(\lambda + \rho - s \cdot \rho)\,.$$

Corollary 5.2.6 Let $\mu, \nu \in P_{++}$. The tensor product $V^\mu \otimes V^\nu$ decomposes with multiplicities

$$\operatorname{mult}_{V^\mu \otimes V^\nu}(V^\lambda) = \sum_{t \in W} \operatorname{sgn}(t) m_\mu(\lambda + \rho - t \cdot (\nu + \rho))\,.$$

5.2.2 Weyl Dimension Formula

We now obtain a formula for the dimension of the irreducible G-module V^λ, which is the value of $\operatorname{ch}(V^\lambda)$ at 1. The Weyl character formula expresses this character as a ratio of two functions on the maximal torus H, each of which vanishes at 1 (by skew-symmetry), so we must apply l'Hospital's rule to obtain $\dim V^\lambda$. This can be carried out algebraically as follows.

We define a linear functional $\varepsilon : \mathcal{A}[\mathfrak{h}^*] \longrightarrow \mathbb{C}$ by

$$\varepsilon\left(\sum_\beta c_\beta \mathbf{e}^\beta\right) = \sum_\beta c_\beta. \tag{5.25}$$

To motivate this definition, we note that if we set $\varphi = \sum_{\beta \in P} c_\beta \mathbf{e}^\beta$ and consider φ as a function on H, then $\varepsilon(\varphi) = \varphi(1)$. For $s \in W$ and $f \in \mathcal{A}[\mathfrak{h}^*]$ we have

$$\varepsilon(s \cdot f) = \varepsilon(f), \tag{5.26}$$

where we define $s \cdot \mathbf{e}^\beta = \mathbf{e}^{s(\beta)}$ for $\beta \in \mathfrak{h}^*$.

Fix a W-invariant symmetric bilinear form (α, β) on $\mathfrak{h}^*$, as in Section 2.4.2. For $\alpha \in \mathfrak{h}^*$ define a derivation ∂_α on $\mathcal{A}[\mathfrak{h}^*]$ by $\partial_\alpha(\mathbf{e}^\beta) = (\alpha, \beta)\mathbf{e}^\beta$. Then

$$s \cdot (\partial_\alpha f) = \partial_{s(\alpha)}(s \cdot f) \tag{5.27}$$

for $s \in W$ and $f \in \mathcal{A}[\mathfrak{h}^*]$. Define the differential operator

$$D = \prod_{\alpha \in \Phi^+} \partial_\alpha. \tag{5.28}$$

We claim that

$$s \cdot (Df) = \operatorname{sgn}(s) D(s \cdot f). \tag{5.29}$$

Indeed, if s is a reflection for a simple root, then we see that s changes the sign of exactly one factor in D and permutes the other factors (see Lemma 3.1.21). Since W is generated by simple reflections, this property holds.

Let $\lambda \in P_{++}$ and define

$$A_{\lambda+\rho} = \sum_{s \in W} \operatorname{sgn}(s) \mathbf{e}^{s(\lambda+\rho)} \tag{5.30}$$

(the numerator in the Weyl character formula). From the above claim we have

$$D \cdot A_{\lambda+\rho} = \sum_{s \in W} s \cdot \left(D \cdot \mathbf{e}^{\lambda+\rho}\right) = \left\{ \prod_{\alpha \in \Phi^+} (\rho + \lambda, \alpha) \right\} \sum_{s \in W} \mathbf{e}^{s(\rho+\lambda)}. \tag{5.31}$$

Now for $\lambda = 0$ we have $A_\rho = \Delta_G$ by the Weyl denominator formula. Hence

$$D(\Delta_G) = \left\{ \prod_{\alpha \in \Phi^+} (\rho, \alpha) \right\} \sum_{s \in W} \mathbf{e}^{s \cdot \rho}. \tag{5.32}$$

Thus we obtain $\varepsilon(D \cdot A_{\lambda+\rho}) = |W| \prod_{\alpha \in \Phi^+} (\lambda + \rho, \alpha)$. Applying the Weyl character formula and using the above result, we see that

$$\varepsilon\left(D(\operatorname{ch}(V^\lambda)\Delta_G)\right) = \varepsilon(DA_{\lambda+\rho}) = |W| \prod_{\alpha \in \Phi^+} (\lambda + \rho, \alpha). \tag{5.33}$$

Lemma 5.2.7 If $f \in \mathcal{A}[\mathfrak{h}^*]$, then

$$\varepsilon(D(f\Delta_G)) = \varepsilon(fD(\Delta_G)). \tag{5.34}$$

Theorem 5.2.8 (Weyl Dimension Formula). The dimension of V^λ is a polynomial of degree $|\Phi^+|$ in λ:

$$\dim V^\lambda = \prod_{\alpha\in\Phi^+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}. \tag{5.35}$$

Examples

Type A

Let $G = \mathbf{SL}(n,\mathbb{C})$. If $\lambda \in P_{++}$ then λ is the restriction to H of the weight $\lambda_1\varepsilon_1+\cdots+\lambda_{n-1}\varepsilon_{n-1}$ with $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0$, where the λ_i are integers. Setting $\lambda_n = 0$, we have

$$(\rho, \varepsilon_i - \varepsilon_j) = j - i, \quad (\lambda+\rho, \varepsilon_i - \varepsilon_j) = \lambda_i - \lambda_j + j - i. \tag{5.36}$$

Thus from the Weyl dimension formula we get

$$\dim V^\lambda = \prod_{1\leq i<j\leq n} \frac{\lambda_i - \lambda_j + j - i}{j-i}. \tag{5.37}$$

For example, the representation V^ρ has dimension $2^{n(n-1)/2}$. For $n = 3$ it happens to be the adjoint representation, but for $n \geq 4$ it is much bigger than the adjoint representation.

Types B and C

Let $G = \mathbf{Spin}(2n+1,\mathbb{C})$ or $\mathbf{Sp}(n,\mathbb{C})$. Then $\rho = \rho_1\varepsilon_1 + \cdots + \rho_n\varepsilon_n$ with

$$\rho_i = \begin{cases} n-i+(1/2) & \text{for } G = \mathbf{Spin}(2n+1,\mathbb{C}), \\ n-i+1 & \text{for } G = \mathbf{Sp}(n,\mathbb{C}). \end{cases} \tag{5.38}$$

Let $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ be a dominant integral weight for G. Then we calculate from the Weyl dimension formula that

$$\dim V^\lambda = \prod_{1\leq i<j\leq n} \frac{(\lambda_i+\rho_i)^2 - (\lambda_j+\rho_j)^2}{\rho_i^2 - \rho_j^2} \prod_{1\leq i\leq n} \frac{\lambda_i+\rho_i}{\rho_i}. \tag{5.39}$$

For example, for the smallest regular dominant weight ρ we have $\dim V^\rho = 2^{n^2}$. Note that for the orthogonal group (type B) V^ρ is a representation of $\mathbf{Spin}(2n+1,\mathbb{C})$ but not of $\mathbf{SO}(2n+1,\mathbb{C})$, since ρ is only half-integral.

Type D

Let $G = \mathbf{Spin}(2n,\mathbb{C})$. Then $\rho = \rho_1\varepsilon_1 + \cdots + \rho_n\varepsilon_n$ with $\rho_i = n - i$. Let $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$ with $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n|$ be a dominant integral weight for G. Then we calculate from the Weyl dimension formula that

$$\dim V^\lambda = \prod_{1\leq i<j\leq n} \frac{(\lambda_i+\rho_i)^2 - (\lambda_j+\rho_j)^2}{\rho_i^2 - \rho_j^2}. \tag{5.40}$$

In this case, for the smallest regular dominant weight ρ we have $\dim V^\rho = 2^{n^2-n}$. Since ρ is integral, V^ρ is a single-valued representation of $\mathbf{SO}(2n,\mathbb{C})$.

5.2.3 Commutant Character Formulas

Assume that G is semisimple and algebraically simply connected. For $\lambda \in P_{++}$ let (π^λ, V^λ) be the irreducible representation of G with highest weight λ. Let $N^+ \subset G$ be the unipotent subgroup with Lie algebra $\mathfrak{n}^+$. Denote the character of π^λ by

$$\varphi_\lambda(g) = \mathrm{tr}(\pi^\lambda(g)) \quad \text{for } g \in G. \tag{5.41}$$

Thus φ_λ is a regular invariant function on G (it is constant on conjugacy classes).

Let (π, F) be a rational representation of G. Let $\mathcal{B} = \mathbf{End}_G(F)$ be the commuting algebra for the G action on F. Since G is reductive, the duality theorem implies that F decomposes as

$$F \cong \bigoplus_{\lambda \in \mathbf{Spec}(\pi)} V^\lambda \otimes E^\lambda, \tag{5.42}$$

where $\mathbf{Spec}(\pi) \subset P_{++}$ (the G - spectrum of π) is the set of highest weights of the irreducible G representations occurring in F, and E^λ is an irreducible representation of $\mathcal{B}$. Here $g \in G$ acts by $\pi^\lambda(g) \otimes 1$ and $b \in \mathcal{B}$ acts by $1 \otimes \sigma^\lambda(b)$ on the summands in (7.19), where $(\sigma^\lambda, E^\lambda)$ is an irreducible representation of $\mathcal{B}$. We denote the character of the $\mathcal{B}$ - module E^λ by χ_λ:

$$\chi_\lambda(b) = \mathrm{tr}(\sigma^\lambda(b)). \tag{5.43}$$

Let F^{N^+} be the space of N^+ - fixed vectors in F (since $N^+ = \exp(\mathfrak{n}^+)$, a vector is fixed under N^+ if and only if it is annihilated by $\mathfrak{n}^+$). Then F^{N^+} is invariant under $\mathcal{B}$, and the weight - space decomposition

$$F^{N^+} = \bigoplus_{\lambda \in \mathbf{Spec}(\pi)} F^{N^+}(\lambda) \tag{5.44}$$

is also invariant under $\mathcal{B}$. We have $E^\lambda \cong F^{N^+}(\lambda)$ as a $\mathcal{B}$ - module for $\lambda \in \mathbf{Spec}(\pi)$. Hence

$$\chi_\lambda(b) = \mathrm{tr}\left(b|_{F^{N^+}(\lambda)} \right). \tag{5.45}$$

This formula for the character χ_λ is not very useful, however. Although the full weight space $F(\lambda)$ is often easy to determine, finding a basis for the N^+ - fixed vectors of a given weight is generally difficult. We now use the Weyl character formula to obtain two formulas for the character χ_λ that involve only the full H - weight spaces in F.

Theorem 5.2.9 For $\lambda \in P_{++}$ and $b \in \mathcal{B}$ one has

$$\chi_\lambda(b) = \text{coefficient of } x^{\lambda+\rho} \text{ in } \Delta_G(x)\mathrm{tr}_F(\pi(x)b) \tag{5.46}$$

(where $x \in H$).

We now give the second character formula.

Theorem 5.2.10 For $\lambda \in P_{++}$ and $b \in \mathcal{B}$ one has

$$\chi_\lambda(b) = \sum_{s\in W} \operatorname{sgn}(s)\operatorname{tr}_{F(\lambda+\rho-s\cdot\rho)}(b). \tag{5.47}$$

In particular,

$$\dim E^\lambda = \sum_{s\in W} \operatorname{sgn}(s) \dim F(\lambda+\rho-s\cdot\rho). \tag{5.48}$$

5.3 Branching for Classical Groups

Since each classical group G fits into a descending family of classical groups, the irreducible representations of G can be studied inductively. This gives rise to the *branching problem*: Given a pair $G \supset H$ of reductive groups and an irreducible representation π of G, find the decomposition of $\pi|_H$ into irreducible representations. In this chapter we solve this problem for the pairs $\mathbf{GL}(n,\mathbb{C}) \supset \mathbf{GL}(n-1,\mathbb{C})$, $\mathbf{Spin}(n,\mathbb{C}) \supset \mathbf{Spin}(n-1,\mathbb{C})$, and $\mathbf{Sp}(n,\mathbb{C}) \supset \mathbf{Sp}(n-1,\mathbb{C})$. We show that the representations occurring in $\pi|_H$ are characterized by a simple interlacing condition for their highest weights. For the first and second pairs the representation $\pi|_H$ is multiplicity - free; in the symplectic case the multiplicities are given in terms of the highest weights by a product formula. We prove all these results by a general formula due to Kostant that expresses branching multiplicities as an alternating sum over the Weyl group of G of a suitable partition function. In each case we show that this alternating sum can be expressed as a determinant. The explicit evaluation of the determinant then gives the branching law.

Let G be a reductive linear algebraic group and $H \subset G$ a reductive algebraic subgroup. When an irreducible regular representation of G is restricted to H, it is no longer irreducible, in general, but decomposes into a sum of irreducible H modules. A *branching law* from G to H is a description of the H - irreducible representations and their multiplicities that occur in the decomposition of any irreducible representation of G.

Now assume that G and H are connected. We have already seen that the irreducible representations of G and H are parameterized by their highest weights, so a branching law can be stated entirely in terms of these parameters. By the conjugacy of maximal tori (cf. Section 2.1.2 for the classical groups, and Section 11.4.5 for a general reductive group), we may choose maximal tori T_G in G and T_H in H such that $T_H \subset T_G$. Let λ and μ be dominant integral weights for G and H, respectively, and let V^λ and V^μ be the corresponding irreducible G - module and H - module. Set

$$m(\lambda,\mu) = \mathbf{mult}_{V^\lambda}(V^\mu). \tag{5.49}$$

A $G \to H$ *branching law* is an explicit description of the multiplicity function $m(\lambda,\mu)$, with λ ranging over all dominant weights of T_G. In case the multiplicities are always either 0 or 1 we say that the branching is *multiplicity - free*.

5.3.1 Statement of Branching Laws

We now state some branching laws for the classical groups that we will prove later in the chapter. Denote by $\mathbb{Z}^n_{++}$ the set of all integer n-tuples $\lambda = [\lambda_1,\dots,\lambda_n]$ such that

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n. \tag{5.50}$$

Let $\mathbb{N}^n_{++} \subset \mathbb{Z}^n_{++}$ be the subset of all such weakly decreasing n-tuples with $\lambda_n \geq 0$. For $\lambda \in \mathbb{N}^n$ let $|\lambda| = \sum_{i=1}^n \lambda_i$.

Let $G = \mathbf{GL}(n,\mathbb{C})$. Take $H \cong \mathbf{GL}(n-1,\mathbb{C})$ as the subgroup of matrices $\begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}$, where $y \in \mathbf{GL}(n-1,\mathbb{C})$. For $\lambda = [\lambda_1,\ldots,\lambda_n] \in \mathbb{Z}^n_{++}$ let $(\pi^\lambda_n, F^\lambda_n)$ be the irreducible representation of G with highest weight $\sum_{i=1}^n \lambda_i\varepsilon_i$. Let $\mu = [\mu_1,\ldots,\mu_{n-1}] \in \mathbb{Z}^{n-1}_{++}$. We say that μ interlaces λ if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n. \tag{5.51}$$

Theorem 5.3.1 The branching from $\mathbf{GL}(n,\mathbb{C})$ to $\mathbf{GL}(n-1,\mathbb{C})$ is multiplicity-free. The multiplicity $m(\lambda,\mu)$ is 1 if and only if μ interlaces λ.

An easy consequence of this result is the following branching law from $G = \mathbf{GL}(n,\mathbb{C})$ to $H = \mathbf{GL}(n-1,\mathbb{C}) \times \mathbf{GL}(1,\mathbb{C})$:

Theorem 5.3.2 Let $\lambda \in \mathbb{N}^n_{++}$. There is a unique decomposition

$$F^\lambda_n = \bigoplus_\mu M^\mu \tag{5.52}$$

under the action of $\mathbf{GL}(n-1,\mathbb{C}) \times \mathbf{GL}(1,\mathbb{C})$, where the sum is over all $\mu \in \mathbb{N}^{n-1}_{++}$ such that μ interlaces λ. Here $M^\mu \cong F^\mu_{n-1}$ as a module for $\mathbf{GL}(n-1,\mathbb{C})$, and $\mathbf{GL}(1,\mathbb{C})$ acts on M^μ by the character $z \mapsto z^{|\lambda|-|\mu|}$.

Next we take $G = \mathbf{Spin}(\mathbb{C}^{2n+1}, B)$ with B as in (2.9). Fix a B-isotropic basis $\{e_0, e_{\pm1},\ldots,e_{\pm n}\}$ as in Section 2.4.1. Let $\pi : G \longrightarrow \mathbf{SO}(\mathbb{C}^{2n+1}, B)$ be the covering map from and set $H = \{g \in G : \pi(g)e_0 = e_0\}$. Then $H \cong \mathbf{Spin}(2n,\mathbb{C})$.

We may identify $\mathfrak{g} = \mathbf{Lie}(G)$ with $\mathfrak{so}(\mathbb{C}^{2n+1}, B)$ in the matrix realization of Section 2.4.1 and $\mathbf{Lie}(H)$ with $\mathfrak{h} = \{X \in \mathfrak{g} : Xe_0 = 0\}$. Let $\varpi_n = [1/2,\ldots,1/2] \in \mathbb{R}^n$ and let $\lambda \in \mathbb{N}^n_{++} + \varepsilon\varpi_n$, where ε is 0 or 1. We say that λ is integral if $\varepsilon = 0$ and half-integral if $\varepsilon = 1$. We identify λ with the dominant weight $\sum_{i=1}^n \lambda_i\varepsilon_i$ for $\mathfrak{g}$ as in Proposition 3.1.20. The half-integral weights are highest weights of representations of G that are not representations of $\mathbf{SO}(\mathbb{C}^{2n+1}, B)$. Likewise, the dominant weights for $\mathfrak{h} = \mathbf{Lie}(H)$ are identified with the n-tuples $\mu = [\mu_1,\ldots,\mu_n]$ such that $[\mu_1,\ldots,\mu_{n-1},|\mu_n|] \in \mathbb{N}^n_{++} + \varepsilon\varpi_n$.

Theorem 5.3.3 The branching from $\mathbf{Spin}(2n+1)$ to $\mathbf{Spin}(2n)$ is multiplicity-free. The multiplicity $m(\lambda,\mu)$ is 1 if and only if λ and μ are both integral or both half-integral and

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n \geq |\mu_n|. \tag{5.53}$$

Let $G = \mathbf{Spin}(2n,\mathbb{C})$. We may identify $\mathfrak{g} = \mathbf{Lie}(G)$ with $\mathfrak{so}(2n,\mathbb{C})$ in the matrix realization of Section 2.4.1. Let $\pi : G \longrightarrow \mathbf{SO}(\mathbb{C}^{2n}, B)$ and set $H = \{g \in G : \pi(g)(e_n + e_{n+1}) = e_n + e_{n+1}\}$. Then $H \cong \mathbf{Spin}(2n-1,\mathbb{C})$ and we identify $\mathbf{Lie}(H)$ with

$$\mathfrak{h} = \{X \in \mathfrak{g} : X(e_n + e_{n+1}) = 0\}. \tag{5.54}$$

Theorem 5.3.4 The branching from $\mathbf{Spin}(2n)$ to $\mathbf{Spin}(2n-1)$ is multiplicity-free. The multiplicity $m(\lambda, \mu)$ is 1 if and only if λ and μ are both integral or both half-integral and

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq |\lambda_n|. \tag{5.55}$$

We now turn to the branching law from $G = \mathbf{Sp}(n, \mathbb{C})$ to $H = \mathbf{Sp}(n-1, \mathbb{C})$, where $n \geq 2$. In this case the restriction is not multiplicity-free. The highest weights that occur satisfy a double interlacing condition and the multiplicities are given by a product formula.

We take G in the matrix form of Section 2.1.2. Let $\{e_{\pm i} : i = 1, \ldots, n\}$ be an isotropic basis for $\mathbb{C}^{2n}$ as in Section 2.4.1 and take $H = \{h \in G : he_{\pm n} = e_{\pm n}\}$. Let $\lambda \in \mathbb{N}^n_{++}$ be identified with a dominant integral weight for G by Proposition 3.1.20 and let $\mu \in \mathbb{N}^{n-1}_{++}$ be a dominant integral weight for H.

Theorem 5.3.5 — ($\mathbf{Sp}(n) \to \mathbf{Sp}(n-1)$ Branching Law). The multiplicity $m(\lambda, \mu)$ is nonzero if and only if

$$\lambda_j \geq \mu_j \geq \lambda_{j+2} \quad \text{for } j = 1, \ldots, n-1 \tag{5.56}$$

(here $\lambda_{n+1} = 0$). When these inequalities are satisfied let

$$x_1 \geq y_1 \geq x_2 \geq y_2 \geq \cdots \geq x_n \geq y_n \tag{5.57}$$

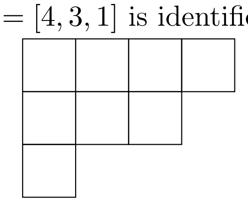
be the nonincreasing rearrangement of $\{\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_{n-1}, 0\}$. Then

$$m(\lambda, \mu) = \prod_{j=1}^{n} (x_j - y_j + 1). \tag{5.58}$$

5.3.2 Branching Patterns and Weight Multiplicities

We can use the $\mathbf{GL}_n \to \mathbf{GL}_{n-1}$ branching law to obtain a canonical basis of weight vectors for the irreducible representations of $\mathbf{GL}(n, \mathbb{C})$ and a combinatorial algorithm for weight multiplicities.

Let $\lambda = [\lambda_1, \ldots, \lambda_n] \in \mathbb{N}^n_{++}$. We shall identify λ with its Ferrers diagram (also called a Young diagram). This diagram consists of p left-justified rows of boxes with λ_i boxes in the ith row. Here p is the largest index i such that $\lambda_i > 0$, and we follow the convention of numbering the rows from the top down, so the longest row occurs at the top. For example, $\lambda = [4, 3, 1]$ is identified with the diagram.



We say that a Ferrers diagram with p rows has depth p (the term length is often used). The total number of boxes in the diagram is $|\lambda| = \sum_i \lambda_i$.

We can describe the branching law in Theorem 8.1.2 in terms of Ferrers diagrams. All diagrams of the highest weights $\mu \in \mathbb{N}^{n-1}_{++}$ that interlace λ are obtained from the diagram of λ as follows:

Box removal rule. Remove all the boxes in the nth row of λ (if there are any). Then remove at most $\lambda_k - \lambda_{k+1}$ boxes from the end of row k, for $k = 1, \ldots, n-1$. We shall indicate this process by putting the integer n in each box of the diagram of λ that is removed to obtain the diagram of μ. For example, if $\lambda = [4, 3, 1] \in \mathbb{N}^3_{++}$, then $\mu = [3, 2]$ interlaces λ. The scheme for obtaining the diagram of μ from the diagram of λ is.

			3
		3	
3			

Note that an element $y = \mathbf{diag}[I_{n-1}, z]$ of $\mathbf{GL}(n-1, \mathbb{C}) \times \mathbf{GL}(1, \mathbb{C})$ acts on the space M^μ by the scalar z^v, where

$$v = |\lambda| - |\mu| \tag{5.59}$$

is the number of boxes containing the integer n.

We can iterate the branching law in Theorem 8.1.2. Let $\mu^{(k)} \in \mathbb{N}^k_{++}$. We say that $\gamma = \{\mu^{(n)}, \mu^{(n-1)}, \ldots, \mu^{(1)}\}$ is an n-fold branching pattern if $\mu^{(k-1)}$ interlaces $\mu^{(k)}$ for $k = n, n-1, \ldots, 2$. Call the Ferrers diagram of $\mu^{(n)}$ the shape of γ. We shall encode a branching pattern by placing integers in the boxes of its shape as follows:

Branching pattern rule. Start with the Ferrers diagram for $\mu^{(n)}$. Write the number n in each box removed from this diagram to obtain the diagram for $\mu^{(n-1)}$. Then repeat the process, writing the number $n-1$ in each box removed from the diagram of $\mu^{(n-1)}$ to obtain $\mu^{(n-2)}$, and so forth, down to the diagram for $\mu^{(1)}$. Then write 1 in the remaining boxes.

This rule fills the shape of γ with numbers from the set $\{1, 2, \ldots, n\}$ (repetitions can occur, and not all numbers need appear). For example, if $\gamma = \{\mu^{(3)}, \mu^{(2)}, \mu^{(1)}\}$ with $\mu^{(3)} = [4, 3, 1], \mu^{(2)} = [3, 2]$, and $\mu^{(1)} = [2]$, then we encode γ by.

1	1	2	3
2	2	3	
3			

Each n-fold branching pattern thus gives rise to a Ferrers diagram of at most n rows, with each box filled with a positive integer $j \leq n$, such that

1. the numbers in each row are nondecreasing from left to right, and
2. the numbers in each column are strictly increasing from top to bottom.

Conversely, any Ferrers diagram of at most n rows with integers $j \leq n$ inserted that satisfy these two conditions comes from a unique n-fold branching pattern with the given diagram as shape. We shall study such numbered Ferrers diagrams (also called semistandard tableaux) in more detail in Chapter 9.

Let T_n be the subgroup of diagonal matrices in $\mathbf{GL}(n, \mathbb{C})$. For $0 \leq k \leq n$ we define $L_{n,k}$ to be the subgroup of $\mathbf{GL}(n, \mathbb{C})$ consisting of all block diagonal matrices

$$g = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \tag{5.60}$$

with $x \in \mathbf{GL}(k, \mathbb{C})$ and $y = \mathbf{diag}[y_1, \ldots, y_{n-k}] \in T_{n-k}$.

Thus we have a decreasing chain of subgroups

$$\mathbf{GL}(n,\mathbb{C}) = L_{n,n} \supset L_{n,n-1} \supset \cdots \supset L_{n,1} = T_n \tag{5.61}$$

connecting $\mathbf{GL}(n,\mathbb{C})$ with its maximal torus T_n.

Proposition 5.3.6 Let $\lambda \in \mathbb{N}^n_{++}$ and let $\gamma = \{\mu^{(n)}, \mu^{(n-1)}, \ldots, \mu^{(1)}\}$ be an n-fold branching pattern of shape λ. There is a unique flag of subspaces $F^\lambda_n \supset M^\gamma_{n-1} \supset \cdots \supset M^\gamma_1$ such that for $1 \le k \le n-1$ the following hold:

1. M^γ_k is invariant and irreducible under $L_{n,k}$.
2. $M^\gamma_k \cong F^{\mu^{(k)}}_k$ as a module for the subgroup $\mathbf{GL}(k,\mathbb{C}) \times I_{n-k}$ of $L_{n,k}$.

The element $\mathbf{diag}[I_k, x_{k+1}, \ldots, x_n] \in L_{n,k}$ acts by the scalar $x^{b_{k+1}}_{k+1} \cdots x^{b_n}_n$ on M^γ_k, where

$$b_j = |\mu^{(j)}| - |\mu^{(j-1)}| \tag{5.62}$$

for $j = 1, \ldots, n$ (with the convention $\mu^{(0)} = 0$).

Corollary 5.3.7 — Gelfand–Cetlin Basis. Let $\lambda \in \mathbb{N}^n_{++}$. The set $\{u_\gamma\}$, where γ ranges over all n-fold branching patterns of shape λ, is a basis for F^λ_n. Hence the weights of F^λ_n are in $\mathbb{N}^n$ and have multiplicities

$$\dim F^\lambda_n(\mu) = \#\,\{n\text{-fold branching patterns of shape } \lambda \text{ and weight } \mu\}\,. \tag{5.63}$$

Example

Let

$$\gamma = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}$$

as above. Then γ has shape $[4,3,1]$ and weight $[2,3,3]$. There is one other threefold branching pattern with the same shape and weight, namely

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}\quad.$$

Hence the weight $[2,3,3]$ has multiplicity 2 in the representation $F^{[4,3,1]}_3$ of $\mathbf{GL}(3,\mathbb{C})$.

6. Representations on Spaces of Regular Funct

6.1 Some General Results

Let G be a reductive algebraic group acting regularly on an affine variety X. The group action induces a representation of G on the coordinate ring $\mathcal{O}[X]$:

$$(\rho_X(g)f)(x) = f(g^{-1}x), \quad f \in \mathcal{O}[X]. \tag{6.1}$$

This representation extends to a locally regular representation of the group algebra $A[G]$: for any finite-dimensional subspace $U \subset \mathcal{O}[X]$, the generated G-invariant space

$$A[G]U = \sum_{g \in G} \rho_X(g)U \tag{6.2}$$

is finite-dimensional, and the G-action on it is regular.

6.1.1 Isotypic Decomposition of $\mathcal{O}[X]$

Assume G is reductive. Let $\hat{G}$ denote the set of equivalence classes of irreducible regular finite-dimensional representations of G. For $\omega \in \hat{G}$, fix a representation (π_ω, V_ω). For a locally regular representation (ρ, E) (e.g., $(\rho_X, \mathcal{O}[X])$):

- **Isotypic subspace** $E_{(\omega)}$: sum of all irreducible subspaces of E with representation equivalent to ω
- **Covariant of type** ω: linear map $T : V_\omega \to E$ intertwining the G-actions

Denote the covariant space by $\mathrm{Hom}_G(\omega, \rho)$.

Proposition 6.1.1 There is an isotypic decomposition:

$$E = \bigoplus_{\omega \in \hat{G}} E_{(\omega)}. \tag{6.3}$$

Furthermore, for each ω, the map $T \otimes v \mapsto T(v)$ gives a G-module isomorphism

$$\mathrm{Hom}_G(\omega, \rho) \otimes V_\omega \cong E_{(\omega)} \tag{6.4}$$

(G acts trivially on $\mathrm{Hom}_G(\omega, \rho)$).

For suitable choices of X, this decomposition is multiplicity-free and provides function-space models for the irreducible regular representations of G.

The subspace $E_{(\omega)}$ is equivalent to a direct sum of irreducible representations in class ω. The number of summands is uniquely determined and called the multiplicity $\mathrm{mult}_p(\omega)$:

$$\mathrm{mult}_p(\omega) = \dim \mathrm{Hom}_G(\omega, p).$$

The representation (p, E) is multiplicity-free if $\mathrm{mult}_p(\omega) \leq 1$ for all $\omega \in \hat{G}$.

Assume G is connected reductive. Fix a Borel subgroup $B = HN^+$. Let $P(G) \subset \mathfrak{h}^*$ be the weight lattice and $P_{++}(G)$ the dominant weights. For $\lambda \in P(G)$, extend the character $h \mapsto h^\lambda$ to B:

$$(hn)^\lambda = h^\lambda, \quad h \in H, n \in N^+.$$

Irreducible regular representations are uniquely determined by their highest weight. The subspace V^{N^+} is one-dimensional. Fix a model (π^λ, V^λ) with highest weight vector v_λ. Define the weight λ subspace:

$$\rho_X(b)f = b^\lambda f, \quad b \in B.$$

The isotypic subspace for $\lambda \in P_{++}(G)$ is spanned by $\rho_X(G)\mathcal{O}[X]^{N^+}(\lambda)$:

This space is isomorphic to $V^\lambda \otimes \mathcal{O}[X]^{N^+}(\lambda)$ with G-action $\pi^\lambda(g) \otimes 1$.

Thus:

$$\mathcal{O}[X] \cong \bigoplus_{\lambda \in P_{++}(G)} V^\lambda \otimes \mathcal{O}[X]^{N^+}(\lambda).$$

The G-multiplicity is $\dim \mathcal{O}[X]^{N^+}(\lambda)$. Under pointwise multiplication:

$$\mathcal{O}[X]^{N^+}(\lambda) \cdot \mathcal{O}[X]^{N^+}(\mu) \subset \mathcal{O}[X]^{N^+}(\lambda + \mu).$$

The G-spectrum

$$S_G(X) = \{\lambda \in P_{++}(G) : \mathcal{O}[X]^{N^+}(\lambda) \neq 0\}$$

is an additive semigroup determining the isotypic decomposition.

6.2 Multiplicity-Free Spaces

6.3 Regular Functions on Symmetric Spaces

6.4 Isotropy Representations of Symmetric Spaces