$$\begin{tikzcd}
\mathcal{S}'(\mathbb{R}^m) \arrow[rr] \arrow[dr] & & \mathrm{Hom}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m)) \arrow[dl, "\text{Schwartz kernel Theorem.}"] \\
& \mathcal{S}'(\mathbb{R}^{m+n}) &
\end{tikzcd}$$

FUNCTIONAL ANALYSIS

TOPOLOGICAL VECTOR SPACES, DISTRIBUTIONS AND KERNELS

Lecture Notes

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s \, |\hat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}$$

Unlike typical functional analysis textbooks, this book selects materials that place a greater emphasis on function spaces for documentation. Additionally, it supplements the functional analysis knowledge required in representation theory, particularly concerning self-adjoint forms and representation theory of Lie groups. The scope of functional analysis is by no means limited to partial differential equations.

August 17, 2025

Contents

1. topological Vector Spaces, Spaces of Funct

1.1 Filters, Topological Spaces, Continuous Mappings.

Definition 1.1.1 — Filters. A filter $\mathfrak{F}$ on a set E is a family of subsets of E satisfying three conditions:

- (F_1)The empty set $\emptyset \notin \mathscr{F}$.
- (F_2) $A,B \in \mathscr{F} \Rightarrow A \cap B \in \mathscr{F}$.
- (F_3) $A \in \mathscr{F}, B \supseteq A \Rightarrow B \in \mathscr{F}$.

■ **Example 1.1 — Principal Filter.** The simplest example of a filter is the family of all subsets of E that contain a given non-empty subset $A \subseteq E$. This is called the principal filter generated by A. ■

Definition 1.1.2 — Eventuality Filter. The eventuality filter (or tail filter) associated with a sequence $(x_n)_{n\in\mathbb{N}}$ in E is the family of all subsets $A \subseteq E$ such that:

- (AF) A contains all but a finite number of terms of the sequence.

Definition 1.1.3 — Basis. If a family $\mathscr{B}$ of subsets of E satisfies the following two conditions, it is a basis of a filters $\mathscr{F}$ on E.

- $(BF_1)\mathscr{B} \subseteq \mathscr{F}$.
- (BF_2)Every subset of E belongs to $\mathscr{F}$ contains some subset of E belongs to $\mathscr{B}$.

■ **Example 1.2** The family of all intervals $(-a,a)$ with $a > 0$ is a basis of filter on the straight line,it's a basis of the filter of the neighborhood of 0 in the usual topology . ■

■ **Example 1.3** Let $\mathscr{F}$ be the associated filter with a sequence $S = \{x_1, x_2, \cdots, x_n, \cdots\}$, and $S_n = \{x_n, x_{n+1}, \cdots\} \subseteq S$, then the sequence of subsets $S = S_1 \supseteq S_2 \supseteq \cdots$ is a basis of $\mathscr{F}$. ■

We say a filter $\mathscr{B}$ is finer then $\mathscr{F}$ means that $\mathscr{F} \subseteq \mathscr{B}$.

Definition 1.1.4 — Topology. A topology on the set E is the assignment ,to each $x \in E$, of a filter $\mathscr{N}(x)$ on E, with the following two conditions :

- (N_1)If a set belongs to $\mathscr{N}(x)$, it contains x.

- (N_2)If a set U belongs to $\mathscr{N}(x)$, there is another set V belonging to $\mathscr{N}(x)$, such that given any $y \in V, U \in \mathscr{N}(y)$.

and we call $\mathscr{N}(x)$ is a neighborhood filter of point x.

An open set is a set which is a neighborhood of each one of its points. A subsetof E is closed if its complement is open.

The closure of a set $A \subseteq E$ is the smallest closed set containing A .It will be denoted by $\bar{A}$.

The interior of a set is the largest open set contained in it; if A is the set, its interior will be denoted by A°.

If both B and $A \subseteq B$ are subsets of a topology space E, and $\bar{A} \supseteq B$, A is dense in B.

Easy to check are the basic intersection and union properties about open or closed sets: that the intersection of a finite number of open sets is open (this follows immediately from the fact, itself obvious in virtue of Axiom (F2) ,that the intersection of a finite number of neighborhoods of a point is again a neighborhood of that point) that the union of any number of open sets, be that number finite or infinite, is open (this follows from the fact that the union of a neighborhood of a point with an arbitrary set is a neighborhood of the same point: Axiom (F3) . By going to the complements, one concludes that finite unions of closed sets are closed, arbitrary intersections of closed sets are also closed, etc.

■ **Example 1.4** We have two extremal topologies :

- the trivial topology: every point of E has only one neighborhood, the set E itself
- the discrete topology :given any point $x \in E$, every subset of E contains x is a neighborhood of it, in particular ,$\{x\}$, and constitutes a basis of the filter of neighborhood of x.

■

Let $\mathscr{T}', \mathscr{T}$ be two topology on E. We say $\mathscr{T}'$ is finer than $\mathscr{T}$ if every subset is open in $\mathscr{T}$, and so do $\mathscr{T}'$.We shorten into $\mathscr{T}' \geq \mathscr{T}$.

Given two topologies on the same set, it may very well happen that none is finer than the other. If one is finer than the other, one says sometimes that they are comparable. The discrete topology is finer, on a set E,than any other topology on E the trivial topology is less fine than all the others. Topologies on a set form thus a partially ordered set, having a maximal and a minimal element, respectively the discrete and the trivial topology.

Definition 1.1.5 — Converge. The filter $\mathscr{F}$ converges to the point x if every neighborhood of x belongs to $\mathscr{F}$, in other words if $\mathscr{F}$ is finer than the filter of neighborhoods of x.

A sequence converges to x if and only if the associated filter converges to x.

A filter may converge to several different points.

A filter may not converge.

Definition 1.1.6 — Continuous. In point set topology, a map $f : E \to F$ issaid to be continuous if any one of the following conditions is satisfied :

- (a)give any point x of E and any neighborhood V of the image $f(x) \in F$, the preimage of V:$f^{-1}(V) = \{y \in E, f(y) \in V\}$ is a neighborhood of x.
- the preimage of any open subset of F is an open subset of E.

(a) and (b) is equivalent.

If a sequence $\{x_1, x_2, \cdots\}$ converges in E to a point x, and if f is a continuous function from E to F, then the sequence $\{f(x_1), f(x_2), \cdots\}$converges to $f(x)$ in F.

Let $f: E \to F$ be a mapping from a set E to a set F, and $\mathscr{F}$ bea filter on E. The image $f\mathscr{F}$ of $\mathscr{F}$ under f is defined as the filter with basis $(f\mathscr{F})_0 = \{f(U) \in \mathscr{F} : U \in \mathscr{F}\}$.

Definition 1.1.7 — Continuous at A Point. A function is continuous at a point when for every $V \in \mathscr{F}(f(x)), f^{-1}(x)$belongs to $\mathscr{F}(x)$, or equivalently, $f(\mathscr{F}(x)) \supseteq \mathscr{F}(f(x))$.

If $F = E$, but carries a different topology, and where $f = I$ is the identity mapping of E onto F. The following two properties are equivalent :

- (i)$I: E \to F$ is continuous .
- (ii)the topology of E is finer than the topology of F.

1.2 Vector Spaces .Linear Mappings.

The vector spaces we shall consider will be defined only on one of the two "classical" fields: the field of real numbers, $\mathbb{R}$, or the field of complex numbers , $\mathbb{C}$. We always suppose the field is $\mathbb{C}$ without saying.

Definition 1.2.1 — Vector Space. A vector space E over $\mathbb{C}$ is a system consisted by the three objects (E, A_v, M_s), including a set E and two mappings :

$$A_v : E \times E \to E, (x,y) \mapsto x + y, \tag{1.1}$$

$$M_s : \mathbb{C} \times E \to E, (\lambda, x) \mapsto \lambda x. \tag{1.2}$$

The mapping A_v called vector addition, must have the following properties :

- (associativity):$(x+y)+z = x+(y+z)$.
- (commutativity):$x+y = y+x$.
- (existence of a neutral element):There is an element 0 in E, such that $\forall x, x+0 = x$.
- (existence of an inverse):To every $x \in E$, there is a unique element $-x$ of E, such that $x+(-x) = 0$.

The mapping M_s is called scalar multiplication or multiplication by scalars, and should satisfy the following conditions:

- (i)$1 \cdot x = x$;
- (ii)$0 \cdot x = 0$ (In fact, this axiom is an inference of other axioms.);
- (iii)$\lambda(\mu x) = (\lambda\mu)x$;
- (iv)$(\lambda + \mu)x = \lambda x + \mu x$;
- (v)$\lambda(x+y) = \lambda x + \lambda y$

Definition 1.2.2 — Linear. A mapping $f: E \to F$ of a vector space E into another, F, is called linear if for all $x, y \in E, \lambda, \mu \in \mathbb{C}$,

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y). \tag{1.3}$$

A linear mapping $f: E \to F$ is one-to-one if and only if $f(x) = 0$ implies $x = 0$.

Let E be a vector space and M be a linear subspace of E. For two arbitrary elements $x, y \in E$, the property

$$x - y \in M \tag{1.4}$$

defines an equivalence relation.

Definition 1.2.3 — Quotient Space and Canonical Mapping. E/M is the set of equivalence classes for the relation $x - y \in M$. There is a canonical mapping of E onto E/M:

$$x \mapsto x + M = \{x' + x : x' \in M\} \tag{1.5}$$

It's easy to check :

- if $x \sim y$ mod M, and if $\lambda \in \mathbb{C}$, then $\lambda x \sim \lambda y$ mod M.
- if $x \sim y$ mod M and if $z \in E$, then $x + z \sim y + Z$ mod M

Thus we define vector addition and scalar multiplication in E/M :$(x+M)+(y+M) = (x+y)+M$.

Definition 1.2.4 — Image. The image of f is a subset of F, denoted it by Imf:

$$\mathrm{Im} f = \{y \in F : \exists x \in E, y = f(x)\} \tag{1.6}$$

Definition 1.2.5 — Kernel. The kernel of f is a subset of E, denoted it by Ker$f$:

$$\mathrm{Ker} f = \{x \in E : f(x) = 0\} \tag{1.7}$$

Both Imf and Ker$f$ are linear subspaces of F and E resp..We have the diagram :

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, "\phi"'] & \mathrm{Im}\, f \arrow[r, hook, "i"] & F \\
E/\,\mathrm{Ker}\, f \arrow[ur, "\tilde{f}"'] & &
\end{tikzcd}$$

where i is the natural injection of Imf into F, ϕ is the canonical map of E onto its quotient, $E/$Ker$f$. The mapping f is defined so as to make the diagram commutative, which means that the image of $x \in E$ under f is identical with the image of $\phi(x)$ (i.e., the class of x modulo Ker$f$)under f .

Definition 1.2.6 — Algebraic Dual and Dual. Let E be an arbitrary set (not necessarily a vector space) and F a vector space. Let us denote by $\mathscr{F}(E;F)$ the set of all mappings of E into F. We take :

$$(f+g)(x) = f(x) + g(x), \tag{1.8}$$

$$(\lambda f)(x) = \lambda f(x). \tag{1.9}$$

When E is also a vector space , we will be interested in the linear mappings from E into F. They form a linear subspace of $\mathscr{F}(E;F)$, denoted by $\mathscr{L}(E;F)$. When $F = \mathbb{C}$, $\mathscr{L}(E;F)$ is denoted by E^* and called the algebraic dual of E. When E is a topological vector space, the linear mappings $E \to \mathbb{C}$ which are continuous consist the dual of E, denoted it by E'. E' is not equal to E^* unless some special cases , e.g.when E is finite dimensional. The elements of E^* are referred to as the linear functionals ,or the linear forms on E.

Definition 1.2.7 — Algebraic Transpose. If E,F,G are three vector spaces over $\mathbb{C}$, and $u:E\to F$, $v:F\to G$ two linear mappings, the compose $v\circ u$ defined by

$$(v\circ u)(x)=v(u(x)),x\in E, \tag{1.10}$$

is a linear mapping of F to G. If $G=\mathbb{C}$, v is a linear functional on F,i.e.v is an element x^* of thealgebraic dual F^* of F.the compose $x\circ u$ is a linear functional on E .We obtain thus a mapping $xx\circ u$ of F into E for each given $u\in\mathscr{L}(E;F)$ .This mapping is obviously linear. It is called the algebraic transpose of u we shall denote it by u .As is readily seen, uu is a linear mapping of $\mathscr{L}(E;F)$ into $\mathscr{L}(F;E)$.

1.3 Topological Vector Spaces. Definition.

Definition 1.3.1 — Compatible with the Linear Structure. Let E be a vector space on $\mathbb{C}$ and A_v, M_s be the vector addition and the scalar multiplication in E. A topology $\mathscr{T}$ is said to be compatible with the linear structure of E if A_v and M_s are continuous when we provide E with the topology $\mathscr{T}$, $E\times E$ with the topology $\mathscr{T}\times\mathscr{T}$, and $\mathbb{C}\times E$with the topology $\mathscr{C}\times\mathscr{T}$, where $\mathscr{C}$ is the usual topology in the complex plane.

Definition 1.3.2 — Product Topology. Consider two topology spaces E and F, the product topology on $E\times F$ is the topology with a basis as :

$$U\times V=\{(x',y')\in E\times F: x'\in U, y'\in V\}, \tag{1.11}$$

where U(resp. V) is a neighborhood of x(resp. y) in E(resp. F).

That these rectangles form a basis of filter is trivial they obviously do not form a filter (except in trivial cases), since a set which contains a rectangle does not have to be a rectangle.

The topology $\mathscr{C}$ assigns to each point λ of the complex plane a remarkable basis of neighborhoods, the disks, open or closed, with center at this point (and with positive radius p). When provided with a topology compatible with its linear structure, E becomes a topological vector space, which we shall bbreviate into TVS.

Suppose that E is a TVS. Then its topology is translation invariant, which, the filter $\mathscr{F}(x)$ of the neighborhood of the point x is the family of the set $V+x$, where V varies over the filter of neighborhoods of the neutral element $\mathscr{F}(0)$.

Thus ,we only need to study the filter of neighborhoods of the origin.

Definition 1.3.3 — Absorbing. A subset A of a vector space E is absorbing if as for every $x\in E$, there's a number $c_x>0$, such that as for every $\lambda\in\mathbb{C},|\lambda|\le c_x$, we have$\lambda x\in A$.

A set is absorbing means that it can contains every point in space by proper dilation.

Definition 1.3.4 — Balanced. A subset A of a vector space E is balanced if to every point $x\in A$ and every $\lambda\in\mathbb{C},|\lambda|<1$, we have $\lambda x\in A$.

The only balanced subsets of the complex plane are the open or the closed disks centered at the origin.

Theorem 1.3.1 A filter $\mathscr{F}$ on the vector space E is the filter of neighborhoods of the origin in a topology compatible with linear structure of E if and only if it satisfies the following properties:

- (3.1)The origin belongs to every subset U of $\mathscr{F}$.
- (3.2)As for ever $U \in \mathscr{F}$, there is $V \in \mathscr{F}$ such that $V+V \subseteq U$.
- (3.3)For every $U \in \mathscr{F}$ and for every $\lambda \neq 0 \in \mathbb{C}$, we have $\lambda U \in \mathscr{F}$.
- (3.4)Every $U \in \mathscr{F}$ is absorbing.
- (3.5)Every $U \in \mathscr{F}$ contains some $V \in \mathscr{F}$ which is balanced.

Proposition 1.3.2 There's a basis of neighborhoods of zero in a TVS E which is consisted by closed sets.

Corollary 1.3.3 There is a basis of neighborhoods of 0 in E consisting of closed balanced sets.

Whatever may be the vector space E, the trivial topology is always compatible with the linear structure of E, but the discrete topology doesn't, unless E consists by single point.

■ **Example 1.5** Let $\mathbb{C}[[X]]$ represents the ring of formal power series in one variable X, with complex coefficients, the element in it can be written as :

$$u = u(X) = \sum_{n\geq 0} u_n X^n, u_n \in \mathbb{C}. \tag{1.12}$$

We don't care about it converges or not.If

$$v = \sum_{n\geq 0} v_n X^n, \tag{1.13}$$

we have :

$$u+v = \sum_{n\geq 0} (u_n + v_n) X^n, \tag{1.14}$$

$$uv = \sum_{n,n\geq 0} u_n v_m X^{n+m} = \sum_{n\geq 0} \left(\sum_{m\geq 0} u_{n-m} v_m \right) X^n, \tag{1.15}$$

$$\lambda u = \sum_{n\geq 0} (\lambda u_n) X^n. \tag{1.16}$$

Addition and multiplication make $\mathbb{C}[[X]]$ be a vector space, and scalar multiplication makes it be a algebra.There is a neutral element 1, and it's easy to check u has an inverse if and only if $u_0 \neq 0$.

Let $\mathfrak{M}$ represents the set of the elements don't have a inverse, set $\mathfrak{M}$ is a ideal of $\mathbb{C}[[X]]$, this means that :

- (1)$\mathfrak{M}$ is a vector subspace of $\mathbb{C}[[X]]$.
- (2)for all $u \in \mathfrak{M}$ and all$v \in \mathbb{C}[[X]]$, we have $uv \in \mathfrak{M}$.

It's clear that $\mathfrak{M}$ is the largest proper ideal. For $n > 1$, let $\mathfrak{M}^n$ represents the set of formal power series u such that $u_m = 0$ if $m < n$, we can write as :

$$u = X^{n-1} u_1(X), u_1 \in \mathfrak{M} \tag{1.17}$$

It's obvious that every $\mathfrak{M}^n$ is an ideal of $\mathbb{C}[[X]]$, and as a sequence of sets:

$$\mathfrak{M}^0 = \mathbb{C}[[X]] \supseteq \mathfrak{M}^1 = \mathfrak{M} \supseteq \mathfrak{M}^2 \supseteq \cdots \supseteq \mathfrak{M}^n \supseteq \cdots \tag{1.18}$$

is totally ordered ,it's a basis of filter. Let $\mathscr{F}$ be the filter generated by it, and to a formal power series u, $\mathscr{F}(u)$ represents the filter generated by the basis $u + \mathfrak{M}^n$.

We have

$$\mathfrak{M}^n + \mathfrak{M}^n \subseteq \mathfrak{M}^n. \tag{1.19}$$

$$\mathfrak{M}^p \cdot \mathfrak{M}^q \subseteq \mathfrak{M}^{p+q} \tag{1.20}$$

By above we can proof that the addition and multiplication are continuous, this makes $\mathbb{C}[[X]]$ be a topology ring, but not a topology vector space, the reason is there are neighborhoods of the origin which are not absorbing.

In fact ,if $n > 0$, $\mathfrak{M}^n$ is not absorbing, because there isn't a $\lambda \neq 0$ such that $\lambda \cdot 1 \in \mathfrak{M}^n$, hence ,the scalar multiplication isn't continuous.

Observe furthermore that the $\mathfrak{M}^n$ are open. They are also linear subspaces. ■

Proposition 1.3.4 In a TVS E, if the vector subspace M is open, we have $M = E$.

We can see that every ideal $\mathfrak{M}^n$ is closed, so the basis of neighborhoods of zero consists of sets which are both open and closed.

■ **Example 1.6** There's another topology on $\mathbb{C}[[X]]$, it's compatible with the linear structure of $\mathbb{C}[[X]]$. A formal power series $u = \sum_{n\geq 0} u_n X^n$ said to converge to another formal power series $v = \sum_{n\geq 0} v_n X^n$, if For every n, u_n converges to v_n resp..

The topology defined by the ideals $\mathfrak{M}^n$ is finer than this kind of coefficient-simple-converge topology.

In our case, the basis will be the collection of the following sets of formal power series:

$$V_{m,n} = \left\{ u = \sum_{p\geq 0} u_p X^p \in \mathbb{C}[[X]] : \forall p \leq n, |u_p| \leq \frac{1}{m} \right\}, n = 0, 1, \cdots, m = 1, 2, \cdots. \tag{1.21}$$

For this topology, the ideals $\mathfrak{M}^n$ are closed.

They are not open in view of Proposition 1.3.2. It should also be noted that in the topology of simple convergence of the coefficients, the origin, and therefore each point, has a countable basis of neighborhoods.This property was also valid for the first topology we have defined on $\mathbb{C}[[X]]$. ■

1.4 Hausdorff Topological Vector Spaces. Quotient Topological Vector Spaces. Continuous Linear Mappings.

In this chapter, we always denote by E a TVS over $\mathbb{C}$.

1.4.1 Hausdorff Topological Vector Space.

Definition 1.4.1 — Hausdorff. A topology space X is said to be Hausdorff if given two distinct points x and y of X, there's a neighborhood U of x and a neighborhood V of y such that $U \cap V = \emptyset$.

A very important of Hausdorff topology space is the uniqueness of the limit: A filter on a Hausdorff space converges to one point at most.

In Hausdorff topology space, every set consists of single point is closed.

Proposition 1.4.1 A TVS E is Hausdorff if and only if to every $x \neq 0$, there's a neighborhood U of 0 such that $x \notin U$.

Proposition 1.4.2 In topology vector space E, the intersection N of the neighborhoods of 0 is a vector subspace of E, it's the closure of $\{0\}$

Corollary 1.4.3 A TVS E is Hausdorff if and only if the set $\{0\}$ is closed in E, or equally, the compliment of $\{0\}$ is open.

It's also equal that $N = \{0\}$ or there isn't a point $x \neq 0$ belongs to all neighborhoods of 0.

Proposition 1.4.4 Let f, g be two continuous mappings from topology space X to the Hausdorff topology vector space E, set

$$A = \{x \in X : f(x) = g(x)\} \tag{1.22}$$

is closed. Indeed, A is the preimage of the closed set $\{0\} \subseteq E$ under the mapping $f - g$.

Proposition 1.4.5 Let X, E, f, g be as Proposition 1.4.4, if f equals to g on a dense subset Y of X, then they're equal in X.

1.4.2 Quotient Topology Vector Space.

Definition 1.4.2 — Quotient Topology. Let M is a vector subspace of E, we consider the quotient vector space E/M and the linear canonical mapping $\phi : E \to E/M$. We define a topology on E/M with a canonical way ,it's called as quotient topology on E/M. The filter of neighborhood of the origin in E/M is the image of the filter of neighborhood of the origin in E under ϕ.

An equivalent and more standard definition is that the quotient topology is the finest topology on E/M that makes the canonical map $\phi : E \to E/M$ continuous. That is, a subset $U \subseteq E/M$ is open if and only if its preimage $\phi^{-1}(U)$ is open in E.

Note that ϕ transforms a neighborhood of a point into a neighborhood of a point, but it isn't true in general cases: the preimage of a neighborhood is a neighborhood under a continuous mapping, but to the image there's nothing said.

We can see ϕ is continuous.

The image of a closed set of ϕ may be not a closed set.

■ **Example 1.7** Consider the hyperbola $\{(x_1, x_2) \in \mathbb{R}, x_1 x_2 = 1\}$ inthe plane $\mathbb{R}$. Take one of the coordinate axes as M, and the other equals to E/M, ϕ equals to the orthogonal projection on it.

The hyperbola is closed in $\mathbb{R}$, but the image of it under ϕ is the compliment of a point on a line, it's open. ■

The quotient topology is the finest topology on E/M such that ϕ is continuous.

The quotient topology on E/M is compatible with the linear structure of E/M.

Proposition 1.4.6 Let E be a topology vector space and M is a linear subspace of E, the following two properties are equivalent:

- (a)M is closed.
- (b)E/M is Hausdorff.

Corollary 1.4.7 The TVS $E/\overline{\{0\}}$ is Hausdorff.

The TVS $E/\overline{\{0\}}$ is called as the Hausdorff TVS associated with the TVS E. When E is Hausdorff, ϕ is one-to-one onto, and E is identified with $E/\overline{\{0\}}$.

1.4.3 Continuous Linear Mappings.

Let E, F be two TVS and f is a linear from E to F. We suppose that F is Hausdorff and f is continuous under usual meaning. Then the kernel of f is closed.

Consider where i is the natural injection, ϕ the canonical map, and $\overline{f}$ the unique linear map which makes the diagram commutative:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\phi"'] & \operatorname{Im} f \arrow[r, hook, "i"] & F \\ E/\operatorname{Ker} f \arrow[ur, "\tilde{f}"'] & & \end{tikzcd}$$

Proposition 1.4.8 The mapping $\overline{f}$ is continuous if and only if f is continuous.

Theinverse of f is well defined on Imf but may not be continuous, that's f isn't bicontinuous.

Definition 1.4.3 — Topological Embedding and Isomorphism. Topological Embedding and Isomorphism. A continuous linear map f is called a topological embedding if it is a homeomorphism onto its image, i.e., the inverse map $f^{-1} : \mathrm{Im}(f) \to E$ is continuous. If a map is a topological embedding and also bijective, it is called a topological isomorphism.

he set of continuous linear maps of a TVS E into another TVS F will be denoted by $L(E;F)$ .Of course, it is a subset of $\mathscr{L}(E;F)$, the vector space of linear maps, continuous or not, from E into F.

When $E = \mathbb{C}$, use E' to represent $L(E;F)$, and call it the dual of E (sometimes, the topological dual, aimed at underline the difference between E^* and E').

E' is a linear subspace of E^*. Elements in E' are denoted by x', y'.

Proposition 1.4.9 Let E, F be two TVS, u is a linear mapping of E into F, then the mapping u is continuous if and only if u is continuous at the origin.

1.5 Cauchy Filter.Complete Subsets.Completion.

Definition 1.5.1 — Cauchy Sequence. Let $S = \{x_1, x_2, \cdots\}$ be a sequence and S is a Cauchy sequence if to every neighborhood of 0, There is a integer $n(U)$ such that

$$n, m \geq n(U) \Rightarrow x_m - x_n \in U. \tag{1.23}$$

When the TVS E is the complex plane, the definition agrees with the usual definition.

Let $S_n = \{x_{n+1}, x_{n+2}, \cdots\}$, the definition means that :

$$S_{n(U)} - S_{n(U)} \subseteq U. \tag{1.24}$$

Observing that the S_n form a basis of the filter associated with the sequence S.

Definition 1.5.2 — Cauchy Filter. A filter $\mathscr{F}$ in a TVS E is a Cauchy filter, if to every neighborhood U of 0, there's a subset M belongs to $\mathscr{F}$ such that

$$M - M \subseteq U. \tag{1.25}$$

■ **Example 1.8** Suppose there's a metric $d(x,y)$ defined on $E \times E$ and it defines the topology of E. Choose U such that for some $\varepsilon > 0$, $x - y \in U$ implies $d(x,y) < \varepsilon$. The diameter of a subset M of E is defined as the supermum of $d(x,y)$, when x and y vary over M. So $M - M \subseteq U$ implies that the diameter of M less than ε.

Then we know that a filter $\mathscr{F}$ on a subset $A \subseteq E$ is a Cauchy filter if it contains subsets of A with arbitrary small diameter. ■

Proposition 1.5.1 The filter associated with a Cauchy sequence is a Cauchy filter.

Proposition 1.5.2
- (a)The filter of neighborhoods of a point $x \in E$ is a Cauchy filter.
- (b)A filter finer than a Cauchy filter is a Cauchy filter.
- (c)Every converging filter is Cauchy filter.

Definition 1.5.3 — Complete. A is a subset of E, if every Cauchy filter converges to a point in A, it's said to be complete.

If every Cauchy sequence in A converges to a limit in A,we say it's sequentially complete. Complete always implies sequentially complete. If the space is metrizable, for which the converse is true.

Proposition 1.5.3 In a Hausdorff TVS E, any complete subset is closed.

Proposition 1.5.4 In a complete TVS E, any closed subset is complete.

Definition 1.5.4 — Uniformly Continuous Function. A mapping $f : A \to F$ is said to be uniformly continuous function if for every neighborhood V of 0 in F, there's a neighborhood U of 0, such that for all $x_1, x_2 \in A$

$$x_1 - x_2 \in U \Rightarrow f(x_1) - f(x_2) \in V. \tag{1.26}$$

Every uniformly continuous function is continuous at every point.

Proposition 1.5.5 Every continuous linear mapping from a TVS E into a TVS F is uniformly continuous.

Proposition 1.5.6 Let f be a uniformly continuous function from $A \subseteq E$ into F, then the image of a Cauchy filter on A is a Cauchy filter on F under f.

Theorem 1.5.7 Let E, F be two Hausdorff TVS and A a dense subset of E. f is a uniformly continuous function from A into F. If F is complete ,then there's a unique continuous linear mapping $\overline{f}$ of E into F extending f, that's for all $x \in A$,

$$\overline{f}(x) = f(x). \tag{1.27}$$

Moreover, $\overline{f}$ is uniformly continuous, and if A is a linear subspace and f is linear, then $\overline{f}$ is linear.

Theorem 1.5.8 Let E be a TVS,if E is Hausdorff, then there's a complete Hausdorff TVS $\hat{E}$ and a mapping i of E into $\hat{E}$, with the following properties:
- (a)The mapping i is an isomorphism of E into $\hat{E}$ for the topology structure.
- (b)The image of E under i is dense in $\hat{E}$.

- (c)To every complete Hausdorff TVS F and every continuous linear mapping $f : E \to F$, there's a continuous linear mapping $\hat{f} : \hat{E} \to F$ such that thefollowing diagram is commutative:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "i"] & F \\ \tilde{E} \arrow[ur, "\tilde{f}"] & \end{tikzcd}$$

Furthermore,

- (I)Any other pairs consists of a complete Hausdorff TVS $\hat{E}_1$ and a mapping $i_1 : E \to \hat{E}_1$ such that the properties (a) and (b) hold, is isomorphic to (E, i). This means there's a isomorphism j of $\hat{E}$ into $\hat{E}_1$ such that the following diagram is commutative:

$$\begin{tikzcd} E \arrow[r, "i_1"] \arrow[d, "i"] & \widehat{E}_1 \\ \tilde{E} \arrow[ur, "j"] & \end{tikzcd}$$

- Given F and f as the properties (c), the continuous linear mapping f is unique.

In the sequel, we always identify E with $i(E)$, and regard E as a dense vector subspace of $\hat{E}$.

1.6 Compact Sets

Definition 1.6.1 A topological space X is said to be compact if it's Hausdorff and every open covering $\{\Omega_i\}$ of X contains a finite subcovering. Bygoing to the compliment, we obtain an equivalent definition of compactness: a Hausdorff space is compact if every family of closed sets whose intersection is empty contains a finite subfamily whose intersection is empty.

Compact spaces is subsets of TVS,and with the topology induced by the TVS, we shall call them compact sets.

Proposition 1.6.1 A closed subset of a compact space is compact.

Proposition 1.6.2 Let f is a continuous mapping of a compact space X into a Hausdorff space Y, then $f(X)$ is a compact subset of Y.

Proposition 1.6.3 Let f is a one-to-one continuous mapping of a compact space X into a compact space Y, then f is a homeomorphism, i.e. f^{-1} is also continuous.

Proposition 1.6.4 Let $\mathscr{T}, \mathscr{T}'$ be two topology on the set X. Suppose $\mathscr{T}$ is finer than $\mathscr{T}'$, and that X equipped with $\mathscr{T}$ is compact, then $\mathscr{T} = \mathscr{T}'$

Finite unions and arbitrary intersections of compact sets are compact. In a Hausdorff space X ,every point is compact every converging sequence is compact-provided that we include in it its limit point.

■ **Example 1.9** Consider the real number line $\mathbb{R}^1$, the Borel-Lebesgue-Heine theorem says that the compact subsets of $\mathbb{R}^1$ are those are closed and bounded.

Note also that the Lebesgue measure of a sequence is equal to zero, and that if a set A is measurable, given any $\varepsilon > 0$,there is a compact set $K \subseteq A$ such that the measure of $A \cap K^C$ is ε. Take then the points x ,with $0x1$,which are nonrational they form a set of measure 1, since the rationals, which form a sequence, form a set of measure zero. This means that there are compact sets, contained in the interval $[0, 1]$, which do not contain any rational number and whose Lebesgue measure is arbitrarily close to 1. ■

In the immediate sequel, E is a Hausdorff topological space when it is expressly mentioned, E is a TVS.

Definition 1.6.2 — Accumulation Point. A point x in E is called an accumulation point of the filter $\mathscr{F}$ if it belongs to every closure of set belongs to $\mathscr{F}$.

Let $S = \{x_0, x_1, \cdots\}$ be a sequence,if every neighborhood of x contains a point different from x, then the point x in E is called an accumulation point.

Let $M \in \mathscr{F}_S$ be arbitrary and contain a subsequent of the form

$$S_n = \{x_n, x_{n+1}, \cdots\} \tag{1.28}$$

Then, x is a accumulation point of S if and only if every neighborhood U of x contains some point x_k.

Proposition 1.6.5 If a filter $\mathscr{F}$ converges to a point x, then x is a accumulation point of $\mathscr{F}$.

A filter may have more than one accumulation point.

■ **Example 1.10** Let $\mathscr{F}$ be the filter of all the subsets of E which contains the given subset A of E. Every point in A is a accumulation point of $\mathscr{F}$. ■

Proposition 1.6.6 The following two conditions are equivalent:

- (a)x is a accumulation point of $\mathscr{F}$.
- there's a filter $\mathscr{F}'$, it's finer than $\mathscr{F}$ and the filter of neighborhoods x, $\mathscr{F}(x)$. In other words, there's a filter $\mathscr{F}'$ converges to x, it's finer than $\mathscr{F}$.

Proposition 1.6.7 If a Cauchy filter $\mathscr{F}$ on the TVS E has an accumulation point x, then it converges to x.

Proposition 1.6.8 Let K be a Hausdorff topological space, then the followings are equivalent:

- (a)K is compact.
- (b)every filter on K has at least one accumulation point.

Corollary 1.6.9 A compact subset K of a Hausdorff topological space X is closed.

Corollary 1.6.10 A compact subset of a Hausdorff TVS is complete.

Corollary 1.6.11 Every sequence has an accumulation point in compact Hausdorff space.

Definition 1.6.3 — Relatively Compact. A subset A of a topological space X is relatively compact if the closure of it is compact.

Aconverging sequence (without the limit point) is a relatively compact set.

Definition 1.6.4 — Precompact. A subset A of a Hausdorff TVS E is said to be precompact, if A is relatively compact in the completion $\hat{E}$ of E.

ACauchy sequence in E is precompact it is not necessarily relatively compact. For this would mean that it converges.

■ **Example 1.11** Let Ω is an open subset of $\mathbb{R}^n$ different from $\mathbb{R}^n$, every bounded open subset of Ω is precompact.But the open subset Ω' of Ω is relatively compact if and only if Ω' is bounded and its closure is contained by Ω. ■

A subset K of a Hausdorff TVS E is compact.

Definition 1.6.5 — Ultrafilter. A filter $\mathscr{U}$ is called an ultrafilter, if every filter finer than it on A is identical to $\mathscr{U}$.

Lemma 1.6.12 Let $\mathscr{F}$ be a filter on set A, than there's at least one ultrafilter finer than $\mathscr{F}$ on A.

Lemma 1.6.13 Let A be a topological space, if a ultrafilter $\mathscr{U}$ has an accumulation point in A, then it converges to x.

Lemma 1.6.14 A Hausdorff topological space K is compact if and only if every ultrafilter on K converges.

Proposition 1.6.15 The properties as following of a subset K of a Hausdorff TVS E are equivalent:

- (a)K is precompact.
- (b)given a neighborhood V of 0 in E, there's a finite family consists of $x_1, x_2, \cdots, x_r$ in K, such that the sets $x_i + V$ form a covering of K.

1.7 Locally Convex Spaces. Seminorms.

Definition 1.7.1 — Convex. A subset K of a vector space E is said to be convex if $x, y \in K$ and $\alpha, \beta > 0, \alpha + \beta = 1$ implies :

$$\alpha x + \beta y \in K. \tag{1.29}$$

Definition 1.7.2 — Convex Hull. Let S is a arbitrary subset of E. We call the convex hull of S is the set of all finite combinations of elements in S with nonnegative coefficients satisfies the sum of the coefficients is 1.

Thus a set is convex if it is equal to its own convex hull. And the convex hull of a set S is the smallest convex set containing S .

Arbitrary intersections of convex sets are convex sets. Unions of convex sets are generally not convex. The vector sum of two convex sets is convex. The image and the preimage of a convex set under a linear map is convex.

Proposition 1.7.1 Let E be a TVS, then the closure and the interior of a convex set is convex.

Definition 1.7.3 — Barrel. A subset T of a TVS E is called a barrel if T satisfies the following properties :

- (1)T is absorbing.
- (2)T is balanced.
- (3)T is closed.
- (4)T is convex.

Let U be a neighborhood of 0 in E, we use $T(U)$ to represent the set

$$\bigcup_{\lambda \in \mathbb{C}, |\lambda| \leq 1} \lambda U, \tag{1.30}$$

then $T(U)$ is a barrel.

Definition 1.7.4 — Locally Convex Space. A TVS E is called a locally convex space if there's a basis of neighborhoods consists of convex sets in E

Proposition 1.7.2 There's a basis of neighborhoods of 0 consists of barrels in a locally convex space.

Definition 1.7.5 — Seminorm. A nonnegative function $p : E \to \mathbb{R}$ is called a seminorm if it satisfies the following conditions:

- p is subadditive, i.e. for all $x, y \in E$, $p(x+y) \leq p(x) + p(y)$.
- p is positively homogeneous of degree 1, i.e. for all $x \in E$ and all $\lambda \in \mathbb{C}$, $p(\lambda x) = |\lambda| p(x)$.
- $p(0) = 0$.

Definition 1.7.6 — Norm. A seminorm p on a vector space E is called a norm if:

$$x \in E, p(x) = 0 \Rightarrow x = 0. \tag{1.31}$$

■ **Example 1.12** Let $E = \mathbb{C}^n$, and M be a vector subspace of E. Let $p_M(x) = d(x, M)$ in the usual sense of the distance in $\mathbb{C}^n$.

If $\dim M \geq 1$, then p_M is a seminorm and not a norm, the kernel of p_M is M.

If $M = \{0\}$, p_M is a norm, called Euclidean norm. ■

■ **Example 1.13** Let E be $\mathbb{C}^n$, and $|\cdot|_p$:

$$\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n) \mapsto |\zeta|_p = (|\zeta_1|^p + |\zeta_2|^p + \cdots + |\zeta_n|^p)^{\frac{1}{p}}, 1 \leq p \leq +\infty \tag{1.32}$$

$$\zeta \mapsto |\zeta|_\infty = \sup_{1 \leq j \leq n} |\zeta_j|, p = +\infty \tag{1.33}$$

Observe that $|\zeta|_2$ is Euclidean and Hermitian norm, it will be denoted by $|\zeta|$. ■

■ **Example 1.14** Let E be a vector space, there's a sesquilinear form $B(e, f)$ defined on it:

$$B(e_! + e_2, f) = B(e_1, f) + B(e_2, f); \tag{1.34}$$

$$B(e, f_1 + f_2) = B(e, f_1) + B(e, f_2); \tag{1.35}$$

$$B(\lambda e, f) = \lambda B(e, f); \tag{1.36}$$

$$B(e, \lambda f) = \overline{\lambda} B(e, f); \tag{1.37}$$

and $B(e, f)$ is complex valued.

Suppose B is Hermitian, which means:

$$B(e, f) = \overline{B(f, e)}. \tag{1.38}$$

Then for all $e \in E$, $B(e, e)$ is a real number, and if it's nonnegative, we say B is nonnegative.After that we can prove that:

$$e \mapsto \sqrt{B(e, e)} \tag{1.39}$$

is a seminorm. It's a norm if and only if B is definite positive, which mean that for all $e \neq 0 \in E$, $B(e, e) > 0$. ■

Definition 1.7.7 — Complex Pre-Hilbert Space. A vector space E over $\mathbb{C}$ is called a complex pre-Hilbert space if it is equipped with a Hermitian nonnegative form.

■ **Example 1.15** Let $\mathscr{C}^0(\mathbb{R}^1)$ be the vector space consists of complex valued continuous functions defined on the real line. For every bounded interval $[a,b], -\infty < a < b < +\infty$ and function $f \in \mathscr{C}^0(\mathbb{R}^1)$, we define:

$$\mathscr{P}_{[a,b]}(f) = \sup_{a \le t \le b} |f(t)|. \tag{1.40}$$

Then $\mathscr{P}_{[a,b]}$ is aseminorm.

It's never a norm, because f may not be 0 out of $[a,b]$. ■

■ **Example 1.16** There are some other norms on $\mathscr{C}^0(\mathbb{R}^1)$:

$$f \mapsto |f(0)|. \tag{1.41}$$

$$f \mapsto \left(\int_a^b |f(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}}, 1 \le p < +\infty \tag{1.42}$$

When $p < 1$, it's not subadditive, so it cannot be a seminorm. ■

■ **Example 1.17** Let l_p represent the vector space consists of complex sequences $\{c_0, c_1, \cdots, c_k, \cdots\}$ such that

$$\left(\sum_{k \ge 0} |c_k|^p \right)^{\frac{1}{p}} < +\infty. \tag{1.43}$$

Then the LGH can be seem as a norm, like above, the subadditive holds depends on $p > 1$. We also define l_∞ asthe vector space consists by complex sequences $\{c_0, c_1, \cdots, c_k, \cdots\}$ such that

$$\sup_{j \ge 0} |c_j| < +\infty. \tag{1.44}$$

Then the LGH defines a norm on l_∞. ■

Definition 1.7.8 — the Closed and the Open Unit Semiball. Let E be a vector space and p is a seminorm on it ,then the sets

$$U_p = \{x \in E : p(x) \le 1\}, U_p^\circ = \{x \in E : p(x) < 1\} \tag{1.45}$$

are call respectively the closed and the open unit semiball of p.

Proposition 1.7.3 Let E be a TVS, p is a seminorm on it, then the following conditions are equivalent:

- (a)the open unit semiball of p is open.
- (b)p is continuous at the origin.
- (c)p is continuous at every point.

Proposition 1.7.4 If p is a continuous seminorm on a TVS E, then the closed unit semiball of it is a barrel.

Proposition 1.7.5 Let E be a TVS, T is a barrel in it ,then there's a unique seminorm p on it such that T is the closed unit semiball of p. The seminorm p is continuous if and only if T is a neighborhood of 0.

Corollary 1.7.6 Let E be a locally convex space, the closed unit semiballs of continuous seminorms form a basis of neighborhoods of the origin.

Definition 1.7.9 — Basis of Continuous Seminorm. A family $\mathscr{P}$ of continuous seminorm on a locally convex space E is called a basis of continuous seminorm on E, if for every continuous seminorm q on E, there's a continuous seminorm $p \in \mathscr{P}$ and a constant C, such that for all $x \in E$,

$$q(x) < Cp(x). \tag{1.46}$$

Denote the closed unit semiball of q(resp. p) by U_q(resp. U_p), then it means that

$$C^{-1}U_p \subseteq U_q. \tag{1.47}$$

Proposition 1.7.7 Let $\mathscr{P}$ be the basis of continuous seminorm on the locally convex space E, then the closed unit seminorm λU_p, where p varies over $\mathscr{P}$ and λ over $\mathbb{R}^+$, form a basis of neighborhoods of 0.

Conversely, given a family $\mathscr{B}$ of neighborhoods of 0, consists of barrels. If for every $U \in \mathscr{B}$, λU form a basis of neighborhoods of 0 when λ varies $\mathbb{R}^+$, then the seminorms whose closed unit seminorm is a barrel form a basis of continuous seminorms in E.

A basis of continuous seminorms on a locally convex space E defines the topology (or the TVS structure) of E.For instance, the seminorm $\mathscr{P}_{[a,b]}$ defines the topology of uniformly converge on $\mathscr{C}_{\prime}(\mathbb{R}^1)$.

We say the family $\{p_\alpha\}(\alpha \in A)$ of seminorms on E define the topology of E, needn't be a basis of seminorms. It means that every seminorm p_α is continuous and the family obtained by forming the supermums of finite seminorms is a basis of continuous seminorms.This family consists of the seminorms as :

$$p_{(B)}(x) = \sup_{\alpha \in B} p_\alpha(x) \tag{1.48}$$

where B varies all finite subsets of the set of indices A of $\{p_\alpha\}$.

Forming the supermum of finite seminorms is equivalent to forming the intersection of finite closed unit seminorms of them, and than take the gauge of the intersection.

A seminorm p is the gauge of a set U if U is the closed unit semiball of p.

Proposition 1.7.8 Let E,F be two locally convex space, $f : E \to F$ is a linear mapping if and only iffor every continuous seminorm p, there's a continuous seminorm q on F such that for all $x \in E$,

$$q(f(x)) \leq p(x). \tag{1.49}$$

Corollary 1.7.9 A linear form f is continuous on a locally convex space if and only if there's a seminorm p on F, such that for all $x \in E$,

$$|f(x)| \leq p(x). \tag{1.50}$$

Above proposition and its corollary are often used as: given a basis $\mathscr{P}$(resp. $\mathscr{Q}$) of continuous seminorms on E(resp. F), then the mapping f is continuous if for every $q \in \mathscr{Q}$, there's a seminorm

$p \in \mathscr{P}$ and a constant $C > 0$ such that for all $x \in E$,

$$q(f(x)) \leq Cp(x). \tag{1.51}$$

Suppose the topology of E and F can defined by single seminorm, we use $\| \cdot \|$ to represent them. Then linear mapping $f : E \to F$ is continuous if and only if there's a constant $C > 0$, such that for all $x \in E$,

$$\| f(x) \| \leq C \| x \| . \tag{1.52}$$

In this case, the linear functional f on E is continuous if and only if there's a constant C such that

$$|f(x)| \leq C \| x \| . \tag{1.53}$$

The absolute value on $\mathbb{C}^1$ defines a continuous norm for usual topology, and itself consists a basis of continuous seminorms in $\mathbb{C}^1$.

The Euclidean norm on $\mathbb{C}^n$ defines the topology of $\mathbb{C}^n$.

The Hermitian norm on $\mathbb{R}^n$ defines the topology of $\mathbb{R}^n$.

Proposition 1.7.10 Let E be a locally convex space. Let $\mathscr{P}$ be a basis of continuous seminorm on E. A filter F on E converges to a point x if and only if for every $\varepsilon > 0$ and every seminorm $p \in \mathscr{P}$, there's a subset M of E belongs to $\mathscr{F}$, such that for all $y \in M$,

$$p(x-y) < \varepsilon. \tag{1.54}$$

Corollary 1.7.11 A sequence $\{x_1, x_2, \cdots, x_n\}$ converges to x if and only if for every $\varepsilon > 0$ and every seminorm $p \in \mathscr{P}$, there's an integer $n(p,\varepsilon)$, such that $n > n(p,\varepsilon)$ implies that

$$p(x-x_n) < \varepsilon. \tag{1.55}$$

Proposition 1.7.12 Let E be a locally convex space and M is a linear subspace of E. Let ϕ be the canonical mapping, then

- (1)the topology of the quotient TVS E/M is locally convex.
- (2)If $\mathscr{P}$ is a basis of continuous seminorms oon E, denote by $\dot{\mathscr{P}}$ represents the family of seminorms on E/M

$$\dot{p}(\dot{x}) = \inf_{\phi(x)=\dot{x}} p(x), \dot{x} \in E/M. \tag{1.56}$$

Then $\dot{\mathscr{P}}$ is a basis of continuous seminorms on E/M.

Consider the complex 2-dimension space $\mathbb{C}^2$ and its subspace

$$M = \{(\zeta_1, \zeta_2 \in \mathbb{C}^2) : \zeta_1 = 0\}. \tag{1.57}$$

The quotient E/M can be written as

$$M^0 = \{(\zeta_1, \zeta_2) : \zeta_2 = 0\}, \tag{1.58}$$

and the canonical mapping is identified with

$$(\zeta_1,\zeta_2)\mapsto(\zeta_1,0) \tag{1.59}$$

The Euclidean norm on $\mathbb{C}^2$(resp. M^0) defines the topology on $\mathbb{C}^2$(resp. M^0).

If we called p as the Euclidean norm in $\mathbb{C}^2$ and seem it as a seminorm, we have

$$\dot{p}(\zeta_1,0)=\inf_{\zeta_2\in\mathbb{C}^1}p(\zeta_1,\zeta_2)=\inf_{\zeta_1\in\mathbb{C}^1}\left(|\zeta_1|^2+|\zeta_2|^2\right)^{\frac{1}{2}}=|\zeta_1|. \tag{1.60}$$

This shows that we find as associated seminorm $\dot{p}$ exactly what we would expect to find.

But it's not true that if a family of continuous seminorms defines the topology on E, but it's not a basis of continuous seminorms, then the family of continuous seminorm on E/M obtained by the formula (1.56) must define the topology on E/M.

Example 1.18 $E=\mathbb{C}^2, M=\{z_1,z_2\in\mathbb{C}^2: z_1=z_2\}$ given a counterexample.

When we take $p_1: z=(z_1,z_2)\mapsto|z_1|$ and $p_2: z=(z_1,z_2)\mapsto|z_2|$, this pair of seminorm obviously define the topology of E, but

$$\dot{p}_i(z)=\inf_{\dot{z}=z+M}p_i(z) \tag{1.61}$$

is equal to 0 for $i=1,2$, and all $\dot{z}\in E/M$.

Indeed, every equivalence class $z+M$ intersects both subspaces $z_1=0$ and $z_2=0$.

■

We call kernel of a seminorm p on E is the set of x such that $p(x)=0$,and denote it by $\mathrm{Ker}p$.

$\mathrm{Ker}p$ is a vector subspace of E.If p is continuous ,it's closed, this is because if we seem p as a mapping from E into the real line, it's the preimage of 0.

In locally convex space, the closure of the origin is the intersection

$$\bigcap_p \mathrm{Ker}p \tag{1.62}$$

when p varies all the family of continuous seminorms.

Proposition 1.7.13 In a locally convex space E, the closure of $\{0\}$ is the intersection of the (closed) linear subspace $\mathrm{Ker}p$, when p varies over a basis of continuous seminorms on E.

Thus the Hausdorff space associated with an E is locally convex it is the quotient space

$$E/\left(\bigcap_{p\in\mathscr{P}}\mathrm{Ker}p\right) \tag{1.63}$$

where $\mathscr{P}$ is any basis of continuous seminorm on E.

In particular, suppose E has a basis of continuous seminorm consists by single seminorm p_0, then $E/\mathrm{Ker}p_0$ is the Hausdorff space associated with E, and it's topology can be defined by some seminorm $\dot{p}_0(\dot{x})$ satisfies $\phi(x)=\dot{x}$.

Indeed, notice that the seminorm p_0 is constant along the submanifolds $x+\mathrm{Ker}p_0$.

If a seminorm want to define a topology of a locally convex space personally, and this topology is Hausdorff, this seminorm must be a norm.

Thus $\dot{p}_0$ is a norm. In this case, $E/\mathrm{Ker}p_0$ is called the normed space associated with E.

Proposition 1.7.14 Let E be a locally convex Hausdorff TVS, K is a precompact subset of E, Then the convex hull $\Gamma(K)$ is precompact.

Corollary 1.7.15 If E is complete ,the closed convex hull of a compact subset of E is compact.

The convex hull of a compact cannot be compact, even cannot not be closed.

If the surrounding space E isn't complete, athe closed convex hull of a compact cannot be compact.

1.8 Metrizable Topological Vector Spaces

Definition 1.8.1 — Metrizable. A TVS E is said to be metrizable if it's Hausdorff and there's is a countable basis of neighborhoods of 0.

The topology of a TVS E can be defined by a metric if and only if E is Hausdorff and has a countable basis of neighborhoods of 0.

Definition 1.8.2 — Metric. A metric d on E is a mapping $(x,y) \mapsto d(x,y)$ from $E \times E$ into the nonnegative half real line R_+ with the properties as following:

- (1)$d(x,y) = 0$ if and only if $x = y$.
- (2)$d(x,y) = d(y,x)$ for all $x,y \in E$.
- (3)$d(x,y) + d(y,z) \geq d(x,z)$ for all $x,y,z \in E$.

Then the topology of E is defined by d means that for every $x \in E$, set

$$B_\rho(x) = \{y \in E : d(x,y) \leq \rho\}, \rho \geq 0, \tag{1.64}$$

form a basis of neighborhoods of x.

Definition 1.8.3 If the following conditions holds, metric d is said to be translation invariant:

- (4)$d(x,y) = d(x+z,y+z)$ for all $x,y,z \in E$.

Property (4) is saying that for all $x \in E$ and $\rho > 0$,

$$d(x,y) = d(x-y,0). \tag{1.65}$$

In any metrizabel TVS E, there's a translation invariant metric which defines the topology of E.

The norm on vector space E defines a metric. If we use $\|\cdot\|$ to represent the norm , then the metric is $(x,y) \mapsto \| x-y \|$.

However, the topology of metrizable space cannot always be defined by norm.

Proposition 1.8.1 Let E be a locally convex metrizable TVS, $\{p_1, p_2, \cdots\}$ be a nondecreasing countable basis of continuous seminorms on E, and $\{a_1, a_2, \cdots\}$ be a positive number sequence, such that

$$\sum_{j \geq 1} a_j < +\infty \tag{1.66}$$

Then the following function on $E \times E$:

$$(x,y) \mapsto d(x,y) = \sum_{j \geq 1} a_j \cdot \frac{p_j(x-y)}{1 + p_j(x-y)}, \tag{1.67}$$

is a translation invariant metric on E which is equipped with the topology of E.

We introduce three famous results of the theory of metric spaces which is about the generous case that the TVS E is metrizable but can not be locally convex.

Proposition 1.8.2 A subset K of a metrizable space E is complete if and only if every Cauchy sequence in K converges to a point in K.

In other words, in metrizable spaces, sequentially complete implies complete.

Proposition 1.8.3 A complete metrizable TVS E is a Baire space i.e. it satisfies:

- (B) The union of any countable family of closed sets, none of which has interior points, has no interior points.

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- 1. The union of a sequence of closed sets can not be closed.
- 2. Evening the space is Baire space, the union of a sequence of closed sets can have interior points: take the real line and the usual topology, every point is closed. The set of rational numbers $\mathbb{Q}$ is a union of countable family of closed sets has no interior point. It has no interior point, but the closure of it is the entire real line.
- By taking the compliments, property (B) can be stated equivalently in the following way:
(B')The intersection of any countable family of everywhere dense open sets is an everywhere dense set.Indeed, the complement of a closed set without interior points is an everywhere dense open set.
- There's a TVS isn't Baire space. The LF-spaces is an example.
- There's a Baire space which isn't metrizable and a metrizable space which isn't Baire space (of course, its not complete). There's a nonmetrizable space which isn't Baire space.

Proposition 1.8.4 Set K is compact in a metrizable TVS E, if and only if every sequence in K has a accumulation point in K.

Another useful property of metrizable spaces is the equivalence of continuity with sequential continuity. In the statements below, the mappings f are not supposed to be linear.

Definition 1.8.4 — Sequentially Continuous. A mapping f of topological space E into topological space F is said to be sequentially continuous if for every sequence $\{x_n\}$ converges to x in E, sequence $\{f(x_n)\}$ converges to $f(x)$ in F.

Proposition 1.8.5 A mapping f of metrizable TVS E into TVS F is continuous if and only if it's sequentially continuous.

1.9 Finite Dimension Hausdorff Topological Vector Spaces. Linear subspaces with Finite Codimension. Hyperplans.

Let E be a vector space over the field of complex numbers $\mathbb{C}$, then the following two properties are equivalent:

- (a)E is finite dimensional;
- (b)there's an integer $n \geq 0$ such that there's a one-to-one mapping of E into $\mathbb{C}^n$.

n is called the dimension of E and denote it by $\dim E$. If E is not finite dimensional, we say it's infinite dimensional.

Theorem 1.9.1 Let E be a finite dimensional Hausdorff TVS, then:

- (a)E is isomorphic to $\mathbb{C}^d, d = \dim E$ as a TVS. More precisely, given any basis $(e_1, e_2, \cdots, e_d)$, mapping

$$(x^1, x^2 \cdots, x^d) \in \mathbb{C}^d \mapsto x^1 e_1 + x^2 e_2 + \cdots + x^d e_d \tag{1.68}$$

is am isomorphism of $\mathbb{C}^d$ into E, for the TVS structure.

- (b)Every linear functional on E is continuous.
- (c)Every linear mapping of E into any TVS F is continuous.

Corollary 1.9.2 Every finite Hausdorff TVS is complete.

Corollary 1.9.3 Every finite dimensional linear subspace of Hausdorff TVS is closed.

In virtue of the Heine-Borel-Lebesgue theorem, the closures of bounded open subsets of $\mathbb{C}^d$ are compact thus the origin, and consequently every point of a finite dimensional TVS, has a basis of neighborhoods consisting of compact sets. A topological space with such a property is said to be locally compact (this, for us, implies Hausdorff).

Theorem 1.9.4 A locally compact TVS is finite dimensional.

The codimension of a linear subspace M of a vector space E is the dimension of the quotient space E/M.

Definition 1.9.1 — Hyperplane. A linear subspace with the codimension 1 is called hyperplane.

Let M be the linear subspace with codimension $n < +\infty$ in vector space E. Consider the canonical mapping $\phi : E \to E/M$. If $b_1, b_2, \cdots, b_n$ is a basis of the quotient space E/M, we can lift it into a linear independent set $\{e_1, e_2, \cdots, e_n\}$ of n vectors in E. Let E be the linear subspace spanned by $e_1, e_2, \cdots, e_n$ in E. We claim that

$$E = M \oplus N. \tag{1.69}$$

Proposition 1.9.5 The hyperplane H in the vector space E is the maximal proper linear subspace of E.

Proposition 1.9.6 The hyperplane H in TVS E is either everywhere dense or closed.

Proposition 1.9.7 Let E be a TVS, M be a closed linear subspace with finite codimension in E. Then there's a homomorphism p of E into M such that $p^2 = p$. We have $E = M \oplus \mathrm{Ker} p$.

Definition 1.9.2 — Projection. A linear mapping p of the vector space E into itself is called a projection if satisfies $p^2 = p$.

R

- 1. Let E be a TVS, M is a linear subspace of E. Even suppose M is closed, in general, it's not true that there is must a continuous projection p of E into M.
- 2. Let E be a TVS, A, B are two closed linear subspaces. Suppose E is the direct sum of A and B in algebraic meaning. In general, it's not true that there's must a continuous projection p of E into A. If mapping

$$(x, y) \mapsto x + y \tag{1.70}$$

of $A \times B$ into E is one-to-one, onto and continuous both ways, then call E is the topological direct sum of A and B. Due to the continuity of vector addition, if x and y converge to 0, so do their sum $x+y$.
It's nontrivial that if $x+y$ converges to 0, so do x and y.
If E (supposed to be Hausdorff) is the topological direct sum of two linear subspaces A and B ,they are automatically closed in E .Then, of course, E/A is isomorphic (for the TVS structures) with B .

1.10 Frechet Spaces. Examples.

Definition 1.10.1 — Frechet Space. A Frechet space (or in short, an F-space) is a TVS satisfies:

- (a)it's metrizable (in particular, it's Hausdorff);
- (b)it's complete (thus a Baire space);
- it's locally convex (thus it carries a metric d of the type in proposition 1.8.1).

Any closed subspace of an F-space is an F-space (for the induced topology).
Any product of two F-spaces is an F-space.
The quotient spaceof an F-spacemodule a closed subspace is an F-space.
Finite dimensional Hausdorff space, Hilbert space and Banach space are all F-space.

1.10.1 Example 1.the Space of $\mathscr{C}^k$ Functions on an Open Subset Ω of $\mathbb{R}^n$

The varieties in $\mathbb{R}^n$ is denoted by $x = \{x_1, x_2, \cdots, x_n\}, \xi = \{\xi_1, \xi_2, \cdots, \xi_n\}$, etc. The first-order partial differentitations with respect to the varieties x_j is denoted by $\frac{\partial}{\partial x_j}$.

We shall use the vector $p = (p_1, p_2, \cdots, p_n)$ with nonnegative integers as components as differentitation indices and we call it n-tuples.Thus we write

$$\left(\frac{\partial}{\partial x}\right)^p = \left(\frac{\partial}{\partial x_1}\right)^{p_1} \left(\frac{\partial}{\partial x_2}\right)^{p_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{p_n} \tag{1.71}$$

and use $|p|$ to denote the length of n-tuples p, i.e.

$$|p| = p_1 + p + 2 + \cdots + p_n \tag{1.72}$$

The length $|p|$ is the order of the differentiation operator $\left(\frac{\partial}{\partial x}\right)^p$.

We shall be dealing with the complex valued function $\phi(x)$ of the variables $x = (x_1, x_2, \cdots, x_n)$. This function is defined on some open subset Ω of $\mathbb{R}^n$ and keep to be fixed.

The complex valued f defined on Ω, if it's continuous and all the derivatives with order less than k exists and is continuous in Ω when $k \geq 1$, we call it a $\mathscr{C}^k$ function.

All $\mathscr{C}^k$ functions are $\mathscr{C}^{k-1}$ functions.

If the function f defined on Ω is $\mathscr{C}^k$ function for all integers $k = 01\cdots$, call it a $\mathscr{C}^\infty$ function.

$\mathscr{C}^k$ functions in Ω form a vector spaceon $\mathbb{C}$, we denote it $\mathscr{C}^k(\Omega)$, it's locally convex. We choose a seminorm to equip a topology on it to make it an F-space:

$$|f|_{m,K} = \sup_{|p|<m} \left(\sup_{x \in K} \left| \left(\frac{\partial}{\partial x}\right)^p f(x) \right| \right). \tag{1.73}$$

First, K is any compact subset of Ω.Observe that a continuous function is always bounded on a compact set.Thus if f is a $\mathscr{C}^k$ function, then $|f|_{m,K}$ is finite.

We provided $\mathscr{C}^k(\Omega)$with the topology defined by the seminorm:

$$f \mapsto |f|_{m,K}. \tag{1.74}$$

This kind of topology usually be called $\mathscr{C}^k$ topology or the topology of the function with its derivatives of order $\leq k$ uniformly converges on compact subset.

$\mathscr{C}^k$ topology makes $\mathscr{C}^k(\Omega)$ be a locally convex space (this LC space is Hausdorff space).

Lemma 1.10.1 Let Ω be a open subset of $\mathbb{R}^n$,there's a compact subset sequence $K_1, K_2, \cdots, K_r, \cdots$ holds the following two properties:

- (a)K_j is contained by the interior of K_j for every $j = 1, 2, \cdots$.
- (b)the union of all the set K_j is Ω.

By this, we shall prove that the topology defined on $\mathscr{C}^k(\Omega)$ is metrizable, or equivalently there's a countable basis of continuous seminorms.

1.10.2 Example II. the Space of Holomorphic Functions in an Open Subset Ω of $\mathbb{C}^n$

Let Ω be a open subset of the complex space $\mathbb{C}^n$. The variables in $\mathbb{C}^n$ is denote by $z_1, z_2, \cdots, z_n$. For every $j = 12\cdots, n$, we have $z_j = x_j + \mathrm{i}y, \mathrm{i} = \sqrt{-1}$.

We use $H(\Omega)$ to represent the vector spaceof holomorphic functionin Ω.

Let h be a $\mathscr{C}^1$ function, when we identify $\mathbb{C}^n$ to the real vector space$\mathbb{R}^{2n}$ by the mapping:

$$z = (z_1, z_2, \cdots, z_n) \mapsto (x, y) = (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n). \tag{1.75}$$

Then Ω becomes an open subset of $\mathbb{R}^{2n}$, thus $\mathscr{C}^1$ means that h has continuous derivatives with respect to x_j and y_k. If h satisfies the Cauchy-Riemann equation, we say it is holomorphic:

$$\frac{\partial h}{\partial x_j} + \mathrm{i}\frac{\partial h}{\partial y_j} = 0, j = 1, 2, \cdots, n. \tag{1.76}$$

This defination means f is infinitely differentiable in Ω. Let us write

$$\left(\frac{\partial}{\partial z}\right)^p = \left(\frac{\partial}{\partial z_1}\right)^{p_1}\left(\frac{\partial}{\partial z_2}\right)^{p_2}\cdots\left(\frac{\partial}{\partial z_n}\right)^{p_n}, \tag{1.77}$$

where every differential operator $\frac{\partial}{\partial z_j}$ is defined by

$$\frac{\partial f}{\partial z_j}(x, y) = \frac{1}{2}\left(\frac{\partial f}{\partial x_j}(x, y) - \mathrm{i}\frac{\partial f}{\partial y_j}(x, y)\right). \tag{1.78}$$

Let $z^0 = (z_1^0, z_2^0, \cdots, z_n^0)$ be any point in Ω, consider the polydisk:

$$D(r_1, r_2, \cdots, r_n) = \{z \in \mathbb{C}^n : |z_j - z_j^0| \leq r_j, j = 1, 2, \cdots n\}. \tag{1.79}$$

Suppose it's totally contained in Ω, then if h is holomorphic, for every $p = (p_1, p_2, \cdots, p_n)$, we have

$$\frac{1}{p!}\left(\frac{\partial}{\partial z}\right)^p h(z^0) = \oint_{|z-z_1^0|=r_1, |z-z_2^0|=r_2, \cdots, |z-z_n^0|=r_n} \frac{(2\mathrm{i}\pi)^{-n}h(z)\,\mathrm{d}z_1, \mathrm{d}z_2, \cdots, \mathrm{d}z_n}{(z-z_1^0)^{p_1+1}(z-z_2^0)^{p_2+1}\cdots(z-z_n^0)^{p_n+1}} \tag{1.80}$$

where each integral represents usual complex integration (in the complex plane). Cauchys formula has the immediate consequence that if a sequence of holomorphic functions in Ω converges uniformly on every compact subset of Ω ,then their derivatives of any order also converge uniformly on every compact subset of Ω.

By Cauchy-Riemann equation we can see if h is holomorphic on Ω, then

$$\frac{\partial h}{\partial z_j} = \frac{\partial h}{\partial x_j}, 1 \leq j \leq n; \tag{1.81}$$

$$\frac{\partial h}{\partial z_j} = -\mathrm{i}\frac{\partial h}{\partial y_j}, 1 \leq j \leq n. \tag{1.82}$$

1.10.3 Example III. the Space of Formal Power Series in n Indeterminates

Let we denote by $\mathbb{C}[[X_1, X_2, \cdots, X_n]]$ or shortly $\mathbb{C}[[X]]$, th formal power series with complex coefficients in n letters $X_1, X_2, \cdots, X_n$, that's

$$u = \sum_{p \geq 1} u_p X^p \tag{1.83}$$

where the summation is performed of over all the vectors $p = (p_1, p_2, \cdots, p_n)$, the set of such p is denoted by $\mathbb{N}^n$.

The coefficient u_p is complex number and X^p represents the monomial

$$X_1^{p_1} X_2^{p_2} \cdots X_n^{p_n} \tag{1.84}$$

Can seem u as a sequence $\{u_p\}$ depends on n indices $p_1, p_2, \cdots, p_n$, with no condition whatsoever on the complex numbers which constitute it.

We provide a topology for $\mathbb{C}[[X]]$ by seminorm:

$$|u|_m = \sup_{|p| \leq m} |u_p|, m = 0, 1, \cdots. \tag{1.85}$$

It sometimes is called the topology of simple convergence of the coefficients.

After equipping this topology, $\mathbb{C}[[X]]$ is a locally metrizable space and complete space, hence it's an F-space.

We can provide the discrete topology on $\mathbb{N}^n$: the basis of neighborhoods of a point in $\mathbb{N}^n$ consists of itself. Then a subset is compact of $\mathbb{N}^n$ if and only if it's finite. So the formal power series u can be seemed as a function on $\mathbb{N}^n$: for every $p \in \mathbb{N}^n$, it values the p-th coefficient u_p.

On the discrete space, every function is continuous, thus $\mathbb{C}[[X]]$ can be seemed as the space of all functions or the space of all continuous functions on $\mathbb{N}^n$.

The topology of simple convergence of coefficients is nothing else but the topology of pointwise convergence in $\mathbb{N}^n$, or the topology of uniform convergence on the compact subset of $\mathbb{N}^n$.

1.10.4 Example IV. the Space $\mathscr{Y}$ of $\mathscr{C}^\infty$ Functions in $\mathbb{R}^n$ Rapidly Decreasing at Infinity

It's a space of $\mathscr{C}^\infty$ functions on $\mathbb{R}^n$.

The functional space we consider is denoted by $\mathscr{S}$: the element is complex valued function fm these functions are defined on $\mathbb{R}^n$ and infinitely differentiable, and which have the additional property, regulating their growth (or rather, their decrease) at infinity, that all their derivatives tend to zero at infinity, faster than any power of $\frac{1}{|x|}$, weuse the notation that

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \tag{1.86}$$

This means that any element f in $\mathscr{Y}$, any n-tuple $p = (p_1, p_2, \cdots, p_n) \in \mathbb{N}^n$, and any integer $k \geq 0$,

$$\lim_{|x|\to\infty} |x|^k \left| \left(\frac{\partial}{\partial x}\right)^p f(x) \right| = 0. \tag{1.87}$$

We equip a topology topology defined by seminorm for $\mathscr{Y}$:

$$|f|_{m,k} = \sup_{|p|<m} \left(\sup_{x\in\mathbb{R}^n} \left\{ (1+|x|)^k \left| \left(\frac{\partial}{\partial x}\right)^p f(x) \right| \right\} \right), m,k = 0,1,2,\cdots. \tag{1.88}$$

$\mathscr{Y}$ is metrizable. Notice that $\mathscr{Y}$ is a vector subspace of $\mathscr{C}^\infty(\mathbb{R}^n)$ for the linear structure, but the topology is strictly finer than the topology induced by $\mathscr{C}^\infty$.

Function sequence f_ν converges to 0 if and only if these functions

$$(1+|x|)^k \left(\frac{\partial}{\partial x}\right)^p f_\nu(x) \tag{1.89}$$

uniformly converges to 0 on entire $\mathbb{R}^n$, for every $k = 0, 1, \cdots$ and every $p \in \mathbb{N}^n$. In particular, for every p, the derivatives

$$\left(\frac{\partial}{\partial x}\right)^p f_\nu \tag{1.90}$$

must uniformly converge to 0 in $\mathbb{R}^n$.

We can prove that $\mathscr{Y}$ is complete, so it's an F-space.

The element in $\mathscr{S}$ is called $\mathscr{C}^\infty$ functions rapidly decreasing at infinity. (This implicitly means that also their derivatives are rapidly decreasing at infinity.) When we have to avoid confusion, we shall write $\mathscr{Y}(\mathbb{R}^n)$ instead of $\mathscr{Y}$.

1.11 Normable Spaces. Banach Spaces. Examples.

Definition 1.11.1 — Normable. We say a TVS E is normable if the topology of it can be defined by a norm. That's is there's a norm $\|\cdot\|$, such that the balls

$$B_r = \{x \in E : \| x \| \leq r\}, r > 0, \tag{1.91}$$

consist the basis of neighborhoods of the origin.

Finite dimensional Hausdorff spaceis normable. In generous, infinite dimensional TVS isn't normable. We shall consider two kinds of normable space.

The topology of normable space E can be defined by many kinds of norms. In instance, the topology of $\mathbb{C}^n$ can be defined by anyone of the norms $\|\cdot\|_p$ $(1 \le p \le +\infty)$.

Definition 1.11.2 — Stronger and Equivalent of Norms. Let p, q be two seminorms on vector space$E$. We say p is stronger than q when there's a constant $C > 0$ such that for all $x \in C$

$$q(x) < Cp(x). \tag{1.92}$$

We say they are equivalent if each one is stronger than the other.

If p is stronger than q, than the topology of E defined by p is finer than q, thus if q is a norm, so do p.

Proposition 1.11.1 If p and q define the topology of a normable space E, they are equivalent.

Indeed, the closed unit semiball of p contains that of q, this means that q is stronger than p.

Corollary 1.11.2 Any two norm on a finite dimensional TVS is equivalent.

Indeed, there's a norm on finite dimensional space E making E be a Hausdorff space, thus become a TVS homeomorphic with $\mathbb{C}^{\dim E}$, which means all the norms on E define the same topology.

We use $\mathscr{P}_1^m$ represents a vector spaceof complex coefficient polynomials which is about one indeterminate X, of degree $\le m$. Let $P(X)$ be a polynomial like:

$$P(X) = a_m X^m + a_{m-1} X^{m-1} + \cdots + a_0, a_0, a_1, a_2, \cdots, a_m \in \mathbb{C}^1. \tag{1.93}$$

Consider the following two seminorms:

$$P \mapsto \| P \| = \left(\sum_{0 \le j \le m} |a_j|^2 \right)^{\frac{1}{2}}, \tag{1.94}$$

$$P \mapsto \sup_{t \in \mathbb{R}, |t| < \varepsilon} |P(t)|, \varepsilon > 0. \tag{1.95}$$

They're both norms.

Anormed space is something different from a normable space. A normed space is a pair consisting of a vector space E and a norm on E.

If (E, p) and (F, q) are two normed space, for the structure of topological space, the isomorphism of E into F is a linear isometry of E into F. That's saying there's a linear mapping $u : E \to F$ such that for all $x \in E$,

$$q(u(x)) = p(x). \tag{1.96}$$

Definition 1.11.3 — Banach Space. A complete normed space is called a Banach space (or a B-space).

Due to normed space is metrizable (in particular, it's Hausdorff), B-spaceis a special F-space.They're Baire space.

If M is a closed linear subspace of E, we can equip the quotient spaceE/M with quotient norm, make it a normed space:

$$\dot{p}(\dot{x}) = \inf_{\phi(x)=\dot{x}} p(x) \tag{1.97}$$

where ϕ is canonical mapping. That $\dot{p}$ is a norm and defines the quotient topology on E/M. The normed space $(E/M, \dot{p})$ is the quotient module M of the normed space (E, p).

As for completion, notice that the norm p ostream uniformly continuous, thus there's a unique extension $\hat{p}$ of p into the completion $\hat{E}$.

Due to the topology of $\hat{E}$ is Hausdorff, $\hat{p}$ is a norm, then the normed space $(\hat{E}, \hat{p})$ is a B-space, is called the completion of the normed space (E, p).

■ **Example 1.19 — Finite Dimensional Normed Space.** Since any finite dimensional Hausdorff TVS is complete, finite dimensional normed spaces are Banach spaces. As a matter of fact, they are the only locally compact Banach spaces. ■

■ **Example 1.20 — the Space of Continuous Functions on a Compact Set.** Let K be a compact topological space, there's no algebraic structure on K.

We use $\mathscr{C}(K)$ represents the vector spaceconsists of the complex valued continuous functions defined on K. By considering the maximum of the absolute value of the norms in it, we make $\mathscr{C}(K)$ a normed space:

$$f \mapsto \| f \| = \sup_{x \in K} |f(x)|. \tag{1.98}$$

In this case, $\mathscr{C}(K)$ is a B-space.

■

■ **Example 1.21 — the Space $\mathscr{C}^k(\overline{\Omega}), \Omega$: Bounded Open Subset of $\mathbb{R}^n$.** Let Ω be an open subset of $\mathbb{R}^n$, whose closure $\overline{\Omega}$ is compact and k be a finite nonnegative integer. Consider the subset of $\mathscr{C}^k(\Omega)$ consists by the following functions:

$$\left(\frac{\partial}{\partial x}\right)^p f(x) \tag{1.99}$$

which is continuous function in Ω can be extended to a function which is continuous in $\overline{\Omega}$; It's a vector subspace of $\mathscr{C}^k(\Omega)$. By considering the norm, we make it a normed space:

$$f \mapsto \| f \|_k = \sup_{|p| \le ke} \left(\sup_{x \in \Omega} \left| \left(\frac{\partial}{\partial x}\right)^p f(x) \right| \right). \tag{1.100}$$

Now it's complete.

Theorem 1.11.3 — Minkowski's Inequality. Let p be a real number > 1, and f, g are two complex functions in X, then we have:

$$\left(\int^* |f+g| \, \mathrm{d}x \right)^{\frac{1}{p}} \le \left(\int^* |f|^p \, \mathrm{d}x \right)^{\frac{1}{p}} + \left(\int^* |g|^p \, \mathrm{d}x \right)^{\frac{1}{p}}. \tag{1.101}$$

■ **Example 1.22 — the Space of sequences** l^p $(1 \le p \le +\infty)$. We denote l_p to represent the vector spaceof complex number sequences $\sigma = (\sigma_j)(j = 0, 1, 2, \cdots)$, with the property that:

$$|\sigma|_{l^p} = \left(\sum_{j\ge 0} |\sigma_j|^p\right)^{\frac{1}{p}}, 1 \le p < +\infty, \tag{1.102}$$

$$|\sigma|_{l^\infty} = \sup_{j\ge 0} |\sigma_j|, p = +\infty, \tag{1.103}$$

■

is finite.

Then we know $|\sigma|_{l^p}$ is a seminorm on l^p. In fact, it's a norm, and make l^p be a normed space. This space is complete. ■

■ **Example 1.23 — the Space** L^p $(1 \le p \le +\infty)$. We shall deal with a set X, a positive measure $\mathrm{d}x$ on X is σ-finite. We use $\mathscr{F}^p$ to represent the complex functions on X, such that

$$\int^* |f|^p \,\mathrm{d}x < +\infty. \tag{1.104}$$

■

$f \mapsto \left(\int^* |f|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$ is a seminorm on $\mathscr{F}^p$. Then we use $\mathscr{L}^p$ represent the closure of the linear subspace consists of integrable step-functions in $\mathscr{F}^p$ under the sense of this seminorm. When $p = 1$, $\mathscr{L}^p$ is intergrable function space.

We cna prove that function f belongs to $\mathscr{L}^p$ if and only if $f \in \mathscr{F}^p$ and f is measurable.

For integrable function, we omit the upper star in the integral sign. So we set

$$\| f \|_{L^p} = \left(\int |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}. \tag{1.105}$$

Outside of the special cases, $f \mapsto \| f \|_{L^p}$ isn't a norm on $\mathscr{L}^p$ and we have $\| f \|_{L^p} = 0$ if and only if $f = 0$ almost everywhere. Thus the kernel $\mathscr{N}^p$ of the seminorm $\| \cdot \|_{L^p}$ consists of the elements which vanish almost everywhere, and the associated normed space which is denoted by $L^p = \mathscr{L}^p / \mathscr{N}^p$, is a space of equivalence classes of functions module the relation "$f = g$ almost everywhere". Let $\dot{f} \in L^p$, we define

$$\| \dot{f} \|_{L^p} = \| f \|_{L^p}. \tag{1.106}$$

We usually seem the class $\dot{f}$ as a true function to deal, thus we shall omit the dot and write f instead of $\dot{f}$.

Theorem 1.11.4 — Fischer-Riesz Theorem. Every Cauchy sequence in $\mathscr{L}^p$ converges.

Corollary 1.11.5 L^p is a Banach space.

Now we consider the case that X is an open subset of $\mathbb{R}^n$ and $\mathrm{d}x$ is the induced Lebesgue measure.

Definition 1.11.4 — Support. Let X be a topological space and E be a vector space, f is a mapping of X into E. The closure of the set $\{x \in X : f(x) \neq 0\}$ is called the support of f, we use $\operatorname{Supp} f$ to represent it.

The support of a function f can be defined as the compliment of the (open) set of points $x \in E$ with the following property: f vanishes identically in some neighborhood of x.

Theorem 1.11.6 Let X be a open subset of $\mathbb{R}^n$, if $1 \leq p < +\infty$, the continuous functions with compact support in X consists a dense linear subspace of $\mathscr{L}^p(X)$.

It's more generally true for any Radon measure on a locally compact space.

Corollary 1.11.7 If X is open, then continuous functions with compact support consist a dense subspace of $\mathscr{L}^p(X)$ $(1 \leq p \leq +\infty)$.

Thus in this case $L^p(X)$ can be seemed as a kind of concrete realization of the completion of the continuous functions space equipped with the norm $\|\cdot\|_{L^p}$.

■ **Example 1.24 — the Space L^∞.** Let X be a set and $\mathrm{d}x$ is a positive measure on it. $\mathrm{d}x$ is σ-finite , i.e. X is the union of a sequence of $\mathrm{d}x$ measurable sets.

We use $\mathscr{L}^\infty$ to represent the vector spaceconsists of all the complex valued measurable functions in X, such that there's a finite constant $M \geq 0$ with the following properties:

- There's a subset N of X with measure 0 such that for all $x \in X - N$, we have $f(x) \leq M$.

Weuse $\| f \|_{L^\infty}$ to represent the infimum of all the numbers M with above property, $f \mapsto \| f \|_{L^\infty}$ is a seminorm and it isn't a norm in general. The kernel of it is the set of functions f which vanishes a.e..

The normed space associated with the seminormed space $\mathscr{L}^\infty$ is representedit by L^∞, the norm is represented by $f \mapsto \| f \|_{L^\infty}$.

An element f in $\mathscr{L}^\infty$ (or L^∞) is usually said to be essentially bounded in X (with respect to the measure $\mathrm{d}x$), and $\| f \|_{L^\infty}$ is called the essential supermum. ■

Theorem 1.11.8 Every Cauchy sequence converges in $\mathscr{L}^\infty$.

Corollary 1.11.9 L^∞ is a B-space.

Let's consider that X is an open subset of $\mathbb{R}^n$ and $\mathrm{d}x$ is Lebesgue measure. The space $\mathscr{B}^0(X)$ of the bounded continuous functions on X is a linear subspace of $\mathscr{L}^\infty(X)$. As for $f \in \mathscr{B}^0(X)$, we have

$$\| f \|_{L^\infty} = \sup_{x \in X} |f(x)|. \tag{1.107}$$

This shows that the canonical mapping of $\mathscr{L}^\infty(X)$ into $L^\infty(X)$ induces a isometry of $\mathscr{B}^0(X)$ into$L^\infty(X)$. Now we can see $\mathscr{B}^0(X)$ is a B-space, so the isometry maps it into a closed linear subspace of $L^\infty(X)$ which is not $L^\infty(X)$.

Indeed, there's a $\mathscr{L}^\infty$ functions discontinuous which are not equal almost everywhere to a bounded continuous function.

We can see that the situation with respect to approximation by continuous functions is very different in the case $p = \infty$ from what it is in the case of p finite.

Let E be a normed space and $\|\cdot\|$ is the norm on E. Let E' be the dual of E. If $f \in E'$, then there's a finite constant $C \geq 0$ such that for all $x \in E$,

$$|f(x)| \leq C \| x \| . \tag{1.108}$$

The infimum of the numbers C satisfies above is denoted by $\| f \|$. For any $x \in E$, we have

$$|f(x)| \leq \| f \| \cdot \| x \| \tag{1.109}$$

We could also define $\| f \|$ by either of the following two equalities:

$$\| f \| = \sup_{x \in E, \|x\| \leq 1} |f(x)| \tag{1.110}$$

$$\| f \| = \sup_{x \in E, \|x\| = 1} |f(x)|. \tag{1.111}$$

Thus $\| f \|$ is the minimum supermum of the function $x \mapsto |f(x)|$ on the unit sphere $\{x \in E : \| x \| = 1\}$.

Proposition 1.11.10 Let E be a normed space than $f \mapsto \| f \|$ is a norm on the dual E' of E.

One should be careful not to think that there is always a point x of the unit sphere of E in which $|f(x)| = \| f \|$.

The notion of the norm of a continuous linear functional on a normed space can be immediately generalized to continuous linear maps of a normed space E into another normed space F.

We use $\|\cdot\|$ to represent the norms in E and F, and let $u : E \to F$ is a continuous linear mapping. We know that there's a constant $C \geq 0$, such that for all $x \in E$,

$$\| u(x) \| \leq C \| x \| . \tag{1.112}$$

We define the norm $\| u \|$ of u as the infimum of the above constant C, we have

$$\| u \| = \sup_{x \in E, \|x\| \leq 1} \| u(x) \| = \sup_{x \in E, \|x\| = 1} \| u(x) \| . \tag{1.113}$$

The absolute value of complex numbers has been replaced here by the norm in F.

Let $L(E;F)$ be the vector spaceconsists of all the continuous linear mappings of E into F. Suppose E and F are both normed space, when we seem $L(E,F)$ as a normed space implies that it has the norm defined by (1.113).

Theorem 1.11.11 Let E and F be two normed space and suppose F is complete, then the normed $L(E,F)$ is also complete.

The fact that E is complete or not is unnecessary. Indeed, any continuous linear mapping of E into F can be extended to a continuous linear mapping of $\hat{E}$ into F in a unique way. Thus the mapping of E into $\hat{E}$ defines a isomorphism of $L(E;F)$ into $L(\hat{E};F)$ which keeps the structure of vector space.

This isomorphism is a isometry.

Corollary 1.11.12 Let E be a normed space, then the dual normed space E' of E is a B-space.

1.12 Hilbert Spaces

Definition 1.12.1 — Sesquilinear Form. The sesquilinear form on a vector space$E$ is a mapping $(x,y) \mapsto B(x,y)$ of $E \times E$ into the complex plane $\mathbb{C}$ with the following properties:

- (1)$B(x_1+x_2,y) = B(x_1,y)+B(x_2,y)$;
- (2)$B(x,y_1+y_2) = B(x,y_1)+B(x,y_2)$;
- (3)$B(\lambda x,y) = \lambda B(x,y)$;
- (4)$B(x,\lambda y) = \overline{\lambda} B(x,y)$.

A Hermitian form is the form with the property (1), (2) and (3), and with the additional property:

- (5)$B(x,y) = \overline{B(y,x)}$.

It is then obvious that B must also have Property (4) above and thus be sesquilinear.

Asesquilinear form B on E is said to be nondegenerate if it has the following property:

- (6)If $x \in E$ satisfies that for all $y \in E, B(x,y) = 0$, then $x = 0$. If $y \in E$ satisfies that for all $x \in E, B(x,y) = 0$, then $y = 0$.

1.12.1 Examples in Finite Dimensional Spaces $\mathbb{C}^n$

The usual Hermitian product $(\zeta,\zeta') = \zeta_1\overline{\zeta_1'} + \zeta_2\overline{\zeta_2'} + \cdots + \zeta_n\overline{\zeta_n'}$ is a nondegenerate Hermitian form on $\mathbb{C}^n$.

The form

$$B(\zeta,\zeta') = \zeta_1\overline{\zeta_1'} - \zeta_2\overline{\zeta_2'} \tag{1.114}$$

is a nondegenerate Hermitian form on $\mathbb{C}^2$. If seem it as a form on $\mathbb{C}^n, n > 2$, it's degenerate.

A sesquilinear form is Hermitian if and only if $B(x,x)$ is a real number for all $x \in E$.

We shall essentially be interested, in this chapter, in positive definite forms. These are sequilinear forms which satisfy the following condition:

- (7)For all $x \in E, x \neq 0, B(x,x) > 0$. In particular, positive definite sequilinear forms are Hermitian, they're nondegenerate.

We can also introduce nonnegative sequilinear forms, these are forms which satisfy:

- (8)For all $x \in E, B(x,x) \geq 0$.

A nonnegative sesquilinear is nondegenerate if and only if it's definite positive.

Proposition 1.12.1 — Cauchy-Schwarz Inequality. Let $B(\cdot,\cdot)$ be a nonnegative sequilinear form on E, then for all x and y in E, we have:

$$|B(x,y)|^2 \leq B(x,x)B(y,y). \tag{1.115}$$

Corollary 1.12.2 If B is nonnegative,

$$x \mapsto \sqrt{B(x,x)} \tag{1.116}$$

is a seminorm on E.

If B is definite positive, it becomes a norm.

Definition 1.12.2 — Complex Hausdorff pre-Hilbert Space. The pair consists of a vector space$E$ and a positive definite sequilinear B on it is called a complex Hausdorff pre-Hilbert space.

Let (E,B) be a pre-Hilbert space which isn't Hausdorff and N be a subset of E consist of the vector x such that for all $y\in E$ implies $B(x,y)=0$. This subset is the kernel of the seminorm (1.116). The quotient spaceE/N can then be regarded as a normed space. Notice that if $x,y\in E$ and $z\in N$,

$$B(x+z,y)=B(x,y), \tag{1.117}$$

we see there's a canonical sesquilinear $\dot{B}$: if ϕ is the canonical mapping of E into E/N, we have

$$\dot{B}(\dot{x},\dot{y})=B(x,y),\dot{x}=\phi(x),\dot{y}=\phi(y). \tag{1.118}$$

Then B is positive definite, and the norm of E/N is

$$\dot{x}\mapsto\sqrt{\dot{B}(\dot{x},\dot{x})}. \tag{1.119}$$

We say $(E/N,B)$ is the Hausdorff pre-Hilbert space associated with the pre=Hilbert space (E,B).

Let (E,B) be a pre-Hilbert space, then we can regarded E as a TVS: we consider the topology defined by the seminorm (1.116) on E.

Definition 1.12.3 — Hilbert Space. A complete Hausdorff pre-Hilbert space is called a Hilbert space.

If there's a positive definite sesquilinear $B(\cdot,\cdot)$ such that for all $x\in E$, $\| x \|=\sqrt{B(x,x)}$, we call it a Hilbert norm.

Proposition 1.12.3 A norm $\|\cdot\|$ is a Hilbert norm if and only if for all $x,y\in E$ satisfies the following condition:

$$(\mathbf{HN})\qquad \| x \|^2+\| y \|^2=\frac{1}{2}\left(\| x+y \|^2+\| x-y \|^2\right). \tag{1.120}$$

Now let (E,B) be a Hausdorff pre-Hilbert space and $\|\cdot\|$ is it's norm (1.116). Let $\hat{E}$ be the normed space which is the completion of the normed space $(E,\|\cdot\|)$. By the continuation of the identities, (HN) holds in $\hat{E}$ thus the norm of $\hat{E}$ is a Hilbert norm; Let $\hat{B}$ is the positive definite sesquilinear associated with the norm of $\hat{E}$. It's obvious that $\hat{B}$ extends the B and $(\hat{E},\hat{B})$ is a Hilbert space, it's called the completion of the Hausdorff pre-Hilbert space (E,B).

We can see that $(x,y)\mapsto B(x,y)$ is aseparately continuous function on the product TVS $E\times E$, thus is has a unique extend to the completion of $E\times E$, which is canonical isomorphic to $\hat{E}\times\hat{E}$. This extension is the form $\hat{B}$, and makes $\hat{E}$ be a Hilbert space.

Definition 1.12.4 The anti-dual space of a TVS E is a vector spaceconsists of the continuous mapping of E into the complex plane $\mathbb{C}$, which has the following properties:

- (1)$f(x+y)=f(x)+f(y)$;
- (2)$f(\lambda x)=\lambda f(x)$.

We shall use $\overline{E}'$ to represent the anti-dual space of E; The element of it is called the continuous antilinear forms (or functionals, or sesquilinear forms or functionals).

There's a canonical mapping of E' into $\overline{E}'$, which is one-to-one, onto, and antilinear: for a continuous functional f on E, it assigns the continuous antilinear functional $x \mapsto f(x)$.

Let (E,B) be a pre-Hilbert space, not necessary Hausdorff or complete. Consider the mapping:

$$x \mapsto (y \mapsto B(x,y)). \tag{1.121}$$

It's a mapping of E into the anti-dual space of E. Indeed, for fixed $x \in E$, antilinear functional

$$y \mapsto B(x,y) \tag{1.122}$$

is continuous.

We use $\tilde{x}$ to represent the mapping above (1.122), then the mapping (1.121) can be written as $x \mapsto \tilde{x}$. This latter mapping is one-to-oneif and only if B is nondegenerate, that's to say positive definite.

We call it the canonical mapping of (E,B) into the anti-dual space $\overline{E}'$ of E. It's one-to-oneif and only if E is Hausdorff.

The fundamental theorem of the theory of Hilbert spaces states that it is onto if and only if (E,B) is a Hilbert space (i.e., is Hausdorff and complete). When E is a Hausdorff pre-Hilbert space, we may regard it as a normed space, and we can also regard its dual as a normed space, which moreover is a Banach space.

The fundamental theorem of Hilbert spaces states that the canonical mapping is an isometry of the Hilbert space E onto its anti-dual $\overline{E}'$. This is the theorem that we are now going to prove, and which is often summarized (quite incorrectly) by saying that a Hilbert space is its own dual.

Theorem 1.12.4 Let (E,B) be a Hausdorff pre-Hilbert space, K is a nonempty convex complete subset of E. for every $x \in E$, there's a unique point $x \in E$ such that

$$\| x - x_0 \| = \inf_{y \in K} \| x - y \| . \tag{1.123}$$

Definition 1.12.5 — Orthogonal Projection. The point x_0 in the theorem above is called the orthogonal projection of x in the complete convex set K.

Proposition 1.12.5 Let (E,B) be a pre-Hilbert space, the mapping

$$x \mapsto \tilde{x} : y \mapsto B(x,y), \tag{1.124}$$

is a continuous linear mapping of E into $\overline{E}'$. if and only if E is Hausdorff, that's $B(\cdot,\cdot)$ is nondegenerate, it's an isometry.

Theorem 1.12.6 — Riesz Representation Theorem. Let (E,B) is a Hausdorff pre-Hilbert space. The canonical mapping of E into the anti-dual space $\overline{E}'$ is onto if and only if (E,B) is Hilbert space.

For any point x in E, we consider the orthogonal projection of it on a closed linear subspace M, is denoted by $P_M(x)$. We call P_M is the orthogonal projection on M. Then

$$P_M^2 = P_M. \tag{1.125}$$

The mapping P_M is self-adjoint, that's to say

$$B(P_M x, y) = B(x, P_M y). \tag{1.126}$$

The kernel of P_M is the orthogonal M^0 of M. We have the direct sum decomposition of E:

$$E = M \oplus M^0. \tag{1.127}$$

In a Hilbert space E, a closed subspace always has a supplementary N such that $E = M + N$ and $M \cap N = \{0\}$.

We can take $N = M^0$, orthogonal of M. This feature of Hilbert space is exceptional among TVS and even among B -spaces.

Definition 1.12.6 — Orthonormal Basis. A set of vectors S is called orthonormal if $\| x \| = 1$ for all $x \in S$, and for all satisfy $x \neq y$ implies $B(x, y) = 0$. If the vector spacespanned by S is dense in E, then an orthonormal set of vectors S in the pre-Hilbert space (E, B) is called an orthonormal basis of E.

Theorem 1.12.7 Let S be an orthonormal set in E, V_S is the closure of the linear subspace spanned by S, then the following hold:

- (1)For all $x \in E$

$$\sum_{e \in S} |B(x, e)|^2 \leq \| x \|^2 \quad \textbf{(Bessel's Inequality)} \tag{1.128}$$

- (2)For $x \in E$, the following properties are equivalent: (a)$x \in V_S$; (b)Bessel's inequality gets equal. (c)the series $\sum_{e \in S} B(x, e)e$ converges and we have

$$x = \sum_{e \in S} B(x, e)e. \tag{1.129}$$

- (3)If V_S is complete, then for all $x \in E$, the series $\sum_{e \in S} B(x, e)e$ converges, and we have

$$P_{V_S}(x) = \sum_{e \in S} B(x, e)e \tag{1.130}$$

$$\| P_{V_S}(x) \|^2 = \sum_{e \in S} |B(x, e)|^2 \quad \textbf{(Parseval's Identity)} \tag{1.131}$$

Let S be an arbitrary set and we use $l^2(S)$ to represent the set of the complex valued λ such that

$$\sum_{s \in S} |\lambda(s)|^2 < +\infty \tag{1.132}$$

If a function $\lambda \in l^2(S)$, then it vanishes on the compliment of some countable subset of S (that's the support of it is countable); we provide the discrete topology on S, makes all the subset of S is closed, then the support of a function is the set where it isn't 0.

We take the square root of RHS as the norm of $\lambda \in l^2(S)$.

We can see written this norm $\| \cdot \|$ then it satisfies (HN). Thus it is defined by a sesquilinear, in fact it is

$$(\lambda | \mu) = \sum_{s \in S} \lambda(s)\overline{\mu(s)}. \tag{1.133}$$

We can prove that $l^2(S)$ is complete i.e. it is a Hilbert space.

Theorem 1.12.8 In a Hilbert space(E,B), there's always an orthonormal basis. Furthermore, given any orthonormal subset L of E, there's an orthonormal basis containing L.

Let S be an orthonormal basis of E. For every $x \in E$ there is a complex valued functions defined in S, that is

$$f_x : s \mapsto B(x,s). \tag{1.134}$$

The mapping

$$x \mapsto f_x \tag{1.135}$$

is a linear isometry of E into $l^2(S)$.

An orthonormal set S of E is an orthonormal basis of E if and only if for any $x \in E, B(x,s) = 0$ for all $s \in S$ implies $x = 0$.

1.12.2 Examples of Hilbert Spaces

■ **Example 1.25** the space l^2 of complex sequences $\sigma = (\sigma_n)$ such that

$$\sum_{n\geq 0} |\sigma_n|^2 < +\infty. \tag{1.136}$$

The inner product is

$$(\sigma|\tau) = \sum_{n\geq 0} \sigma_n \overline{\tau_n} \tag{1.137}$$

■

■ **Example 1.26** The space L^2: This is the space of square-integrable functions f with respect to some positive measure $\mathrm{d}x$ on a set X. The inner product is given by

$$(f|g) = \int f(x)\overline{g(x)}\,\mathrm{d}x \tag{1.138}$$

the onrm is therefore

$$\| f \|_{L^2} = \left(\int |f(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}. \tag{1.139}$$

■

■ **Example 1.27** Any finite dimensional space $\mathbb{C}^n$ equipped with usual Hermitian inner product. ■

R The isomorphism of a normed (E,p) into another (F,q) is a linear mapping u of E into F such that for all $x \in E$,

$$q(u(x)) = p(x). \tag{1.140}$$

It's a linear isometry and obviously one-to-one.

Similarly, the isomorphism of pre-Hilbert space (E,B) into another (E_1,B_1) can also be defined as a linear mapping of E into E_1, which is one-to-one, and for all $x,y \in E$,

$$B_1(u(x),u(y)) = B(x,y). \tag{1.141}$$

If now both E and E_1 are Hausdorff, that's say the sesquilinear is positive definite, we can seem them as normed space. It's the same thing that the isomorphism of E into E_1 in the meaning of normed and the meaning of Hausdorff pre-Hilbert space.

R The linear space M of a pre-Hilbert space E is equipped with the structure of pre-Hilbert space naturally: take the restriction $B|_M$ to M of the inner product $(x,y) \mapsto B(x,y)$ which makes E a pre-Hilbert space. If (E,B) is Hausdorff space, so do $(M,B|_M)$. If (E,B) is Hilbert space and M is closed, then $(M,B|_M)$ is also a Hilbert space.

R If (E_1,B_1) and (E_2,B_2) are two pre-Hilbert space, we can consider the sesquilinear form on the vector space$E \times E$ to make it a pre-Hilbert space

$$B((x_1,x_2),(y_1,y_2)) = B_1(x_1,y_1) + B(x_2,y_2). \tag{1.142}$$

The latter is called the product pre-Hilbert space of the two given ones.

1.13 Spaces LF. Examples.

Let E be a vector spaceon $\mathbb{C}$. Suppose E is the union of the increasing sequence of the subspaces $E_n, n = 1,2,\cdots$ and there's a structure of Frechet space, such that the natural injection of E_n into E_{n+1} is isomorphism, this means that the topology of E_n induced by E_{n+1} is the same as the initial given on E_n.

Then we can define the Hausdorff locally convex structure on E in this way: a subset V of E is a neighborhood of 0, if and only if for every $n = 1,2,\cdots, V \cap E_n$ isa neighborhood of 0 in the F-spaceE_n.

Definition 1.13.1 — LF Space. When we provided this topology for E, we say E is a LF space, or equivalently, a countable strict inductive limit of F-space, and the sequence of F-space$\{E_n\}, n = 1,2\cdots$ is a well-defined sequence of E.

The canonical example of an LF space, which motivates the entire definition, is the space of test functions, denoted $D(\Omega)$ or $C_c^\infty(\Omega)$. Let $\{K_j\}_{j=1}^\infty$ be a sequence of compact sets such that K_j is contained in the interior of K_{j+1} for each j, and $\Omega = \bigcup_{j=1}^\infty K_j$.

We can then express the space of test functions as the union $D(\Omega) = \bigcup_{j=1}^\infty C_c^\infty(K_j)$, where $C_c^\infty(K_j)$ is the space of smooth functions with support contained in the compact set K_j. Each space $C_c^\infty(K_j)$ is a Fréchet space.

The LF topology on $D(\Omega)$ is then defined as follows: a convex set $U \subseteq D(\Omega)$ is a neighborhood of 0 if and only if for every j, the intersection $U \cap C_c^\infty(K_j)$ is a neighborhood of 0 in the Fréchet space topology of $C_c^\infty(K_j)$. This construction is crucial for the modern theory of distributions.

A LF spacecan have many, indeed, is infinite sequences of definition.

Let $\{E_n\}$ be a sequence of definition of a LF space$E$. Every E_n is isomorphically embedded in the subsequent ones, $E_{n+1}, E_{n+2}, \cdots$. But we don't know is E_n is isomorphically embedded in E, (the topology of E_n induced by E is the same with the initial ones).

If U is a convex neighborhood of 0 in E, and $V \cap E_n$ must be a neighborhood of 0 in E_n, this means that the topology of E_n induced by E is less fine than the initial ones of E_n.

Lemma 1.13.1 Let E be a locally convex space, U be a convex neighborhood of 0 in E_0, x_0 is a point doesn't belong to U in E. Then there's a convex neighborhood V of 0 in E, it doesn't contain x_0 and satisfies $V \cap E_0 = U$.

Proposition 1.13.2 Let E be a LF space, $\{E_k\}, k = 0, 1, \cdots$ is a sequence of definition of E, F is an arbitrary locally convex TVS and u is a linear mapping of E into F. The mapping u is continuous if and only if for every k, the restriction $u|_{E_k}$ of u on E_k is a continuous linear mapping of E_k into F.

Corollary 1.13.3 A linear form on E is continuous if and only if the restriction of it on every E_k is continuous.

R Unless $E = \lim_{n\to\infty} E_n$ is an F-space, E is never a Baire space. This is precisely what makes such spaces interesting and important in modern analysis. They possess a different, non-metrizable form of completeness (often called "quasi-completeness") which is perfectly suited for defining distributions as continuous linear functionals on them. In the context of distribution theory, the failure to be a Baire space is a feature, not a bug.

Indeed, every E_n is a complete (thus is closed) linear subspace of E, thus E is the countable union of the closed subsets E_n: if E isa Baire space, then there's must a E_{n_0} has a interior point x_0.

As $x \mapsto x - x_0$ is a homeomorphism of E into itself, the origin should be a interior point, too. That's saying that E_{n_0} is a neighborhood of 0. Due to the neighborhood of 0 is absorbing, E_{n_0} should be also absorbing. Due to E_{n_0} is a linear subspace, this means $E_{n_0} = E$ is an F-space.

R Let E be a LF space, $\{E_n\}$ be a sequence of definition of E, M be a closed linear subspace. In general, the topology of M induced by E ostream different from the inductive limit topology of F-space$E_n \cap M$.

Theorem 1.13.4 Any LF spaceis complete.

■ **Example 1.28 — the Space of Polynomials.** Let us denote by $\mathbb{C}[X]$ the vector spaceof polynomials in n letters $X = (X_1, X_2, \cdots, X_n)$ with complex coefficients. The vector spacehas a canonical basis of vectors, that's

$$X^p = X_1^{p_1} X_2^{p_2} \cdots X_n^{p_n}, p = (p_1, p_2, \cdots, p_n) \in \mathbb{N}^n. \tag{1.143}$$

Any polynomial is the finite linear combination of the momomials X^p. The element in $\mathscr{P}_n^m$ is the polynomial with the degree $\leq m$. The degree of the polynomial $P(X)$ is the minimum integer m such that $P(X) \in \mathscr{P}_n^m$, we denote it by $\deg P$. There's just $\binom{m+n}{n}$ monomials X^p such that $|p| \leq m$, that is saying that

$$\dim \mathscr{P}_n^m = \frac{(m+n)!}{m!n!}. \tag{1.144}$$

Let $P(X)$ be a polynomial,

$$P(X) = \sum_{|p| \leq \deg P} c_p X^p \tag{1.145}$$

It's obvious that $P(X)$ can be seemed as a function on $\mathbb{N}^n$. If we provide $\mathbb{N}^n$ with the discrete topology and notice that every set in this topology is closed, we can see that the function corresponding to the polynomial is the function with compact support: A subset of $\mathbb{N}^n$ is compact if and only if it's finite. Any function on $\mathbb{N}^n$ corresponding to the formal power series; We can seem $\mathbb{C}[X]$ is a vector subspace of $\mathbb{C}[[X]]$, even a subalgebra or a subring.

Being a finite dimensional vector space, $\mathscr{P}_n^m$ carries the unique Hausdorff topology, for which it becomes a F-space. Then we can seem $\mathbb{C}[X]$ as the union of the F-space$\mathscr{P}_n^m$ when $m = 0, 1, 2\cdots$.

The topology defined on $\mathbb{C}[X]$ in this way ostream finer than the topology induced by $\mathbb{C}[[X]]$ with the topology of simple convergence of the coefficients. ■

■ **Example 1.29 — Spaces of Test Functions.** Let Ω be a nonempty open subset of $\mathbb{R}^n$, and $F(\Omega)$ represent anyone of the following spaces:

$$\mathscr{C}^k(\Omega), 0 \le k \le +\infty; \quad \mathscr{C}^\infty(\Omega); \quad L^p(\Omega)(1 \le p \le +\infty). \tag{1.146}$$

The first two are F-spaceand the last one is B-space. The space $L^2(\Omega)$ is a Hilbert space.

Let K be a compact subset of Ω, which means it's bounded and closed and the closure of it is contained by Ω. Consider the subset of $F(\Omega)$ and denote it $F_c(K)$, it consists of the functions whose support contained by K.

When $F(\Omega) = \mathscr{C}^0(\Omega)$ and K contains the single point, $F_c(K)$ only contains the zero function. But at any event, $F_c(K)$ is a linear subspace of $F(\Omega)$, and it's easy to see that it's always closed.

Thus be seemed as the subspace of $F(\Omega)$, $F_c(K)$ is a F-space. When $F = L^p$, it's a B-space, and when $F = L^2$, it's a Hilbert space. When $F = \mathscr{C}^k, 0 \le k < +\infty$, it's also right.

In this case, the topology of $F(K)$ can be described by

$$f \mapsto \sup_{|p| \le k} \left(\sup_{x \in \mathbb{R}^n} \left| \left(\frac{\partial}{\partial x} \right)^p f(x) \right| \right). \tag{1.147}$$

When $F = \mathscr{C}^\infty$, $F_c(K)$ is a B-spacedoesn't hold. It's a F-spaceand isn't normable.

When $F(\Omega) = \mathscr{C}^k(\Omega), 0 \le k \le +\infty$, we write $\mathscr{C}^k(K)$ for $F_c(K)$.

When $F(\Omega) = L^p(\Omega), 1 \le p \le +\infty$, we write $L^p(K)$ for $F_c(K)$.

L^p is the same with it representedit by the usual way.

We use $F_c(K)$ to represent the union of the subspace $F_c(K)$ when K varies on the family of the compact subset of Ω in all kinds of ways. It's a vector spaceof $F(\Omega)$ and the subspace consists of all the functions with compact support and belong to $F(\Omega)$.

We provide $F_c(\Omega)$ a topology finer than which is induced by $F(\Omega)$ to make it a *LF* space:

We consider a sequence of compact sets $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_j \subseteq \cdots \subseteq \Omega$, the union of it is Ω. We can choose K_j as the closure of the open subset of Ω, and make K_j contained by the interior of K_{j+1}. Then thw space $F_c(\Omega)$ can be seemed as the union of the subspace $F_c(K_j)$ when $j = 1, 2, \cdots$. Then we can provide the space $F_c(\Omega)$ wtih the topology which is the inductive limit of the topology of the F-space$F_c(K_j)$, then $F_c(\Omega)$ becomes a *LF* space.

When we write $\mathscr{C}_c^k(\Omega), \mathscr{C}_c^\infty(\Omega)$, and $L_c^p(\Omega)$ for $F_c(\Omega)$ when F is meant $\mathscr{C}^k, \mathscr{C}^\infty$, and L^p, respectively. The topology defined on $F_c(\Omega)$ is called the canonical *LF* topology.

The elements in $\mathscr{C}_c^\infty(\Omega)$ is called the test functions. When $\mathscr{C}_c^\infty(\Omega)$ is equipped with the canonical *LF* topology, a distribution in Ω is a continuous linear functional.

■

Proposition 1.13.5 We have the following continuous injections:

$$\mathscr{C}_c^\infty(\Omega) \to \mathscr{C}_c^k(\Omega) \to \mathscr{C}_c^{k-1}(\Omega) \to L_c^\infty(\Omega)(0 < k < +\infty) \tag{1.148}$$

and

$$L_c^\infty(\Omega) \to L_c^{p+1}(\Omega) \to L_c^p(\Omega) \to L_c^1(\Omega)(1 < p < +\infty). \tag{1.149}$$

1.14 Bounded Sets

Let E be a TVS (not necessarily Hausdorff nor locally convex). We wish to generalize the notion of bounded set, familiar to us in finite dimensional spaces or even in normed spaces.

Definition 1.14.1 — Bounded Set. A subset B of a TVS E is said to be bounded, if for every neighborhood U of zero in E, there's a number $\lambda ge0$ such that

$$B \subseteq \lambda U \tag{1.150}$$

We see that the closure of a bounded set is bounded. It is quite obvious that finite sets, bounded subsets (in the usual sense) of finite dimensional spaces, balls with finite radii in normed spaces, are bounded sets. Also obvious are the following properties:

- (1)Finite unions of bounded sets are bounded sets (we recall that any neighborhood of zero contains a balanced one).
- (2)Any subset of a bounded set is a bounded set.

Notice that these properties are, in a sense, dual of the properties of neighborhoods of a point

Definition 1.14.2 A family of bounded subsets $\{B_\alpha\}$ of E is called a basis of bounded subsets of E, if for every bounded subset B of E, there's an index $\alpha \in \Omega$ such that $B \subseteq B_\alpha$.

Abasis of neighborhoods of zero is a family of neighborhoods of 0 such that any given neighborhood of zero contains some neighborhood belonging to the family. A basis of bounded sets is a family of bounded sets such that any given bounded subset of E is contained in some bounded subset belonging to the family.

As we shall see when we study the strong topology on the dual of a TVS, this duality between neighborhoods of zero and bounded sets has important implications.

Proposition 1.14.1 Compact sets are bounded.

In finite dimensional spaces, every bounded set, provided that it is closed, is a compact set. This is not true, in general, in infinite dimensional TVS.

■ **Example 1.30** Let E be a infinite dimensional normed space, if every bounded set is compact in E, in particular, the balls centered at the origin. Then E is must locally compact, but it's impossible, because E is infinite dimensional.

But in a kind of infinite dimensional space called Montel spaces, in which it is true that every closed bounded set is compact.

The space $\mathscr{C}^\infty(\Omega), \mathscr{S}$ and $\mathscr{C}_c^\infty(\Omega)$ are Montel spaces. ■

Corollary 1.14.2 Suppose E is Hausdorff space then the pre-compact subset of E is bounded in E.

Corollary 1.14.3 Suppose E is Hausdorff then the union of a converging sequence and the limit of it in E is a bounded set.

Corollary 1.14.4 Let E be Hausdorff, then any Cauchy sequence in E is bounded set.

The Cauchy filter associated with a Cauchy sequence contains a bounded set. This is not true about a Cauchy filter in general.

Proposition 1.14.5 The image of a bounded set under a continuous linear mapping is a bounded set.

Corollary 1.14.6 Let f be a continuous linear functional on E and B be a bounded subset of E. Then f is bounded on B, that's

$$\sup_{x\in B}|f(x)| < +\infty \tag{1.151}$$

Proposition 1.14.7 Let E be any TVS, then the subset B of E is bounded if and only if every sequence contained in B is bounded.

Any ball in a normed space is a bounded set thus we see that there exist, in normed spaces, sets which are at the same time bounded and neighborhoods of zero. This property is characteristic of normable spaces, at least among Hausdorff locally convex spaces.

Proposition 1.14.8 Let E be a Hausdorff locally convex space. If there's a neighborhood of 0 is bounded in E, then E is normable.

Proposition 1.14.9 Let E be a locally convex space, and the subset B of E is bounded if and only if every seminorm belongs to some basis of continuous seminorms of E is bounded on B.

Proposition 1.14.10 Let E be a LF spaceand $\{E_n\}(n = 0, 1, 2, \cdots)$ is a sequence of definition of E. The subset B of E is bounded in E if and only if for large enough n, B is contained by E_n and B is bounded in that F-spaceE_n.

Corollary 1.14.11 Let E and E_n be as above, sequence $\{x_k\}$ converges in E if and only if it is contained by some subspace E_n and converges in it.

Proposition 1.14.12 Let E be a metrizable space, if the linear mapping of E into the TVS F is bounded, then it's continuous.

Corollary 1.14.13 A bounded linear map of a Frechet space (resp. a normed space, resp. an LF space) into a TVS is continuous.

Definition 1.14.3 — Equicontinuous. Let X be a topological space and F be a TVS, x^0 is a point of X. A set S of mappings of X into F is said to be equicontinuous at the point x^0, if for every neighborhood V of 0 in F, there's a neighborhood $U(x^0)$ of x^0, such that for all $f \in S$:

$$x \in U(x^0) \Rightarrow f(x) - f(x^0) \in V. \tag{1.152}$$

The conditions implies that every $f \in S$ is continuous at x^0.

Now, if X is a compact space, a set S of mappings from X into F which is equicontinuous at every point is uniformly equicontinuous.

In order that such a set of mappings be equicontinuous everywhere, and also uniformly equicontinuous, it is necessary and sufficient that it be equicontinuous at the origin.

Consider the complex valued functions which is unnecessarily linear. The set S of mappings $f : X \mapsto \mathbb{C}$ is equicontinuousif for every $\varepsilon > 0$, there's a neighborhood $U(x^0)$ of x^0 such that for all $f \in S$,

$$x \in U(x^0) \Rightarrow |f(x) - f(x^0)| < \varepsilon. \tag{1.153}$$

Theorem 1.14.14 — ArzelàAscoli Theorem. Let K be a compact metric space and let $C(K)$ be the Banach space of continuous complex-valued functions on K with the supremum norm. A subset $F \subseteq C(K)$ is relatively compact (i.e., its closure $\overline{F}$ is compact) if and only if it is:

1. Equicontinuous, and
2. Pointwise bounded.

R Consider the B-space$\mathscr{C}(K)$ consists of the continuous functions in K, with the topology of uniform convergence, that's to say with the norm:

$$f \mapsto \sup_{x\in K} |f(x)|. \tag{1.154}$$

Ascolistheorem states that a bounded and equicontinuous subset of $\mathscr{C}(K)$ has a compact closure.

Ascolis theorem has a converse: Any subset of $\mathscr{C}(K)$ with the compact closure is bounded and equicontinuous.

We shall suppose K is contained by a bounded open subset Ω of $\mathbb{R}^n$, and consider the space $\mathscr{C}^1(\overline{\Omega})$: it is the space of once continuously differentiable functions in Ω whose derivatives of order 0 and 1 can be extended as continuous functions to the closure $\overline{\Omega}$ of Ω. The space carries the norm:

$$\| f \|_1 = \sup_{x\in\Omega} \left(\sup |f(x)|, \sup_{j=1,2,\cdots,n} \left| \left(\frac{\partial}{\partial x_j} \right) f(x) \right| \right) \tag{1.155}$$

or any equivalent norm, like for instance

$$f \mapsto \sup_{x\in\Omega} \left(|f(x)| + \sum_{1\le j\le n} \left| \left(\frac{\partial}{\partial x_j} \right) f(x) \right| \right). \tag{1.156}$$

In this case, $\mathscr{C}^1(\overline{\Omega})$ is a B-space.

Theorem 1.14.15 — Arzelà's Theorem. Let K be a compact subset of a bounded open subset Ω of $\mathbb{R}^n$. The restriction mapping $f \mapsto f|_K$ which assigns a function on Ω to its restriction on K, transform bounded subsets of $\mathscr{C}^1(\overline{\Omega})$ into relatively compact subsets.

"Relatively compact" means "has a compact closure".

Theorem 1.14.16 Let $[a,b]$ be a closed and bounded interval of the real line, and S is a bounded subset of $\mathscr{C}^1([a,b])$. Then the set S is relatively compact in $\mathscr{C}([a,b])$.

Theorem 1.14.17 Let Ω be an open subset of $\mathbb{R}^n$. If $k<\infty$, then any bounded subset of $\mathscr{C}^{k+1}(\Omega)$ is relatively compact in $\mathscr{C}^k(\Omega)$.

1.15 Approximation Procedures in Spaces of Functions

E' can be seemed as the vector subspace of F' if:

- (a)as a vector space, F can be regarded as a subspace of E
- (b)F is a dense linear subspace of E.
- (c)the topology of F is at least the same fine as the one induced by E.

We begin by studying analytic functions and, as a matter of fact, entire functions of n variables.

The holomorphic functions in the open subset Ω of $\mathbb{C}^n$ form a vector space, which usually be provided the topology of uniform convergence of functions on every compact subset of Ω. This makes it a F-space, we denote it as $H(\Omega)$. When $\Omega=\mathbb{C}^n$, we write it as H.

Theorem 1.15.1 The polynomials are dense in the space of entire functions in $\mathbb{C}^n$. Every entire function f is the limit of its finite Taylor expansion in H:

$$\sum_{|p|\leq m}\frac{1}{p!}f^{(p)}(0)z^p, m=0,1,2,\cdots, \tag{1.157}$$

where

$$p!=p_1!p_2!\cdots p_n!, f^{(p)}=\left(\frac{\partial}{\partial z_1}\right)^{p_1}\left(\frac{\partial}{\partial z_2}\right)^{p_2}\cdots\left(\frac{\partial}{\partial z_n}\right)^{p_n}f, z^p=z_1^{p_1}z_2^{p_2}\cdots z_n^{p_n}. \tag{1.158}$$

When we consider the open set $\Omega\neq\mathbb{C}^n$, it is not always true that an holomorphic function in is the limit of polynomials or of entire functions.

Definition 1.15.1 A open subset Ω of $\mathbb{C}^n$ is called a Runge domain if the restrictions of the entire functions on it is dense in $H(\Omega)$.

Theorem 1.15.2 The open polydisks

$$\{z\in\mathbb{C}^n : |z_j|<R_j\leq+\infty, j=1,2,\cdots,n\} \tag{1.159}$$

are Runge domains.

Lemma 1.15.3 Let f be a continuous functions with compact support in $\mathbb{R}^n$. For each integer $k=1,2,\cdots$, the function

$$f_k(x)=\left(\frac{k}{\sqrt{\pi}}\right)^n\int_{\mathbb{R}^n}\exp\left(-k^2|x-y|^2\right)f(y)\,\mathrm{d}y \tag{1.160}$$

can be extended to the complex of the variables x as an entire function. When $k\to+\infty$, the function f_k converges uniformly to f.

Corollary 1.15.4 Let f be a $\mathscr{C}^m$ function with compact support in $\mathbb{R}^n(0 \leq m \leq +\infty)$. For every differentiation index $p = (p_1, p_2, \cdots, p_n)$ satisfies $|p| < m+1$, when $k \to +\infty$, the function $\left(\frac{\partial}{\partial x}\right)^p f_k$ converges uniformly to $\left(\frac{\partial}{\partial x}\right)^p f$ in $\mathbb{R}^n$.

Corollary 1.15.5 Every function $f \in \mathscr{C}_c^m(\mathbb{R}^n)$ is the limit of a sequence of polynomials in $\mathscr{C}^m(\mathbb{R}^n)$

■ **Example 1.31** The following function is dense everywhere in the space $\mathscr{C}^k(0 \leq k \leq \infty), L^p(1 \leq p < \infty)$:

$$\rho(x) = \begin{cases} a\exp\left(-\frac{1}{1-|x|^2}\right) & |x| < 1; \\ 0 & |x| \geq 1. \end{cases} \tag{1.161}$$

The constant a is defined by

$$a = \left(\int_{|x|<1} \exp\left(-\frac{1}{1-|x|^2}\right) \mathrm{d}x\right)^{-1} \tag{1.162}$$

so what we have

$$\int_{\mathbb{R}^n} \rho(x)\,\mathrm{d}x = 1 \tag{1.163}$$

The function ρ is analytic everywhere in the open ball $\{x : |x| < 1\}$ and exact in the exterior $\{x : |x| > 1\}$ of the ball. As ρ is rotation-invariant, we check the function

$$\begin{cases} \exp\left(-\frac{1}{1-t^2}\right) & |t| \leq 1 \\ 0 & |t| > 1 \end{cases} \tag{1.164}$$

is $\mathscr{C}^\infty$ about $t = 1$. ■

For $\varepsilon > 0$ we set $\rho_\varepsilon(x) = \varepsilon^{-n}\rho\left(\frac{x}{\varepsilon}\right)$.
We have

$$\int_{\mathbb{R}^n} \rho_\varepsilon(x)\,\mathrm{d}x = 1 \tag{1.165}$$

and

$$\operatorname{Supp}\rho_\varepsilon = \{x \in \mathbb{R}^n : |x| \leq \varepsilon\}, \tag{1.166}$$

$$\rho_\varepsilon(0) = \varepsilon^{-n}. \tag{1.167}$$

Lemma 1.15.6 Let f be a continuous function with compact support in $\mathbb{R}^n$. For every $\varepsilon > 0$, the function

$$f_\varepsilon(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-y) f(y)\,\mathrm{d}y \tag{1.168}$$

are $\mathscr{C}_c^\infty$ functions in $\mathbb{R}^n$. Furthermore, the support of f_ε is contained by the neighborhood of order ε of $\operatorname{Supp} f$, that's contained by the set

$$\{x \in \mathbb{R}^n : d(x, \operatorname{Supp} f) \leq \varepsilon\}. \tag{1.169}$$

When $\varepsilon \to 0$, the functions f_ε converges uniformly to f in $\mathbb{R}^n$.

Theorem 1.15.7 Let $0 \leq k \leq +\infty$ be an open set of $\mathbb{R}^n$. Any function $f \in \mathscr{C}^k(\Omega)$ are all the limit of the sequences $\{f_j\}(j = 1, 2, \cdots)$ of three functions with compact support in $\mathscr{C}^\infty$ such that for every compact subset K of Ω, the set $K \cap \operatorname{Supp} f_j$ converges to $K \cap \operatorname{Supp} f$.

A sequence of sets S_j converges to the set S, if for every $\varepsilon > 0$, there's an integer $J(\varepsilon)$, such that for $j \geq J(\varepsilon), S_j$ is contained in the neighborhood of order ε of S, and S is contained by the neighborhood of order ε of S.

Corollary 1.15.8 $\mathscr{C}_c^\infty(\Omega)$ is dense in $\mathscr{C}^k(\Omega)(0 \leq k \leq +\infty)$.

Corollary 1.15.9 $\mathscr{C}_c^\infty(\Omega)$ is sequentially dense in $\mathscr{C}_c^k(\Omega)(0 \leq k \leq +\infty)$.

Corollary 1.15.10 If $1 \leq p < +\infty$, $\mathscr{C}_c^\infty(\Omega)$ is dense in $L^o(\Omega)$.

Corollary 1.15.11 The polynomials form a dense linear subspace of $\mathscr{C}^k(\Omega)(0 \leq k \leq \infty)$.

Theorem 1.15.12 $\mathscr{C}_c^\infty(\mathbb{R}^n)$ is dense in $\mathscr{S}$.

Theorem 1.15.13 The entire analytic functions which belong to $\mathscr{Y}$ is dense in $\mathscr{Y}$.

1.16 Partitions of Unity

The primary utility of a partition of unity is to *localize* global problems. It provides a rigorous way to decompose a global object (like a function or a differential form on a manifold) into a sum of localized pieces, each with small, manageable support. One can then analyze or solve a problem for each local piecewhere the problem might be much simplerand then "patch" the local solutions together to form a global solution. This technique is fundamental in the theory of manifolds and the modern theory of partial differential equations.

If we can represent the function identically equal to one as a sum of $\mathscr{C}^\infty$ functions $\{g^i\}(i \in I)$ with arbitrarily small support, we shall have in our hands the analog representation for arbitrary $\mathscr{C}^k$ functions f by writing $f = \sum_i (g^i f)$.

A family of functions like $\{g^i\}$ is called a partition of unity in $\mathscr{C}^\infty$.

Definition 1.16.1 — Locally Finite. Let A be any topological space, an open covering of A is a family of open subset V^i, whose union is the same as A.

If every point in A has a neighborhood which intersects only a finite number of open sets V^i, then the open covering $\{V^i\}$ is said to be locally finite. If $\{W^j\}$ is another open covering of A and every open W^j is containedit by some open set V^i, it's said that $\{W^j\}$ is finer than $\{V^i\}$.

Theorem 1.16.1 Let Ω be a open subset of $\mathbb{R}^n$ and for every open covering $\{U^i\}(i \in I)$, there's a finer locally finite open covering $\{V^j\}(j \in J)$ of Ω.

R Any locally finite covering of the open set $\Omega \subseteq \mathbb{R}^n$ is countable. Notice that if $\{V^j\}$ is an locally finite open covering of Ω, and if K is any compact subset of Ω, then K only intersects with finite open V^j.

Theorem 1.16.2 Let $\{V^j\}$ be a locally finite open covering of Ω. For every j, there's an open subset W^j of Ω, such that the closure of W^j is contained in V^j, and when j varies, W^j form an open covering of Ω (must be locally finite).

Lemma 1.16.3 Let A be any arbitrary subset of $\mathbb{R}^n$. The function in $\mathbb{R}^n$

$$x \mapsto d(x,A) = \inf_{y \in A} |x-y| \tag{1.170}$$

is continuous.

Definition 1.16.2 — Partition of Unity. Let $\{V^j\}(j \in J)$ be a locally finite open covering of the open set $\Omega \subseteq \mathbb{R}^n$. A set of functions $\{\beta^j\}$ is called a partition of unity of $\mathscr{C}^\infty(\Omega)$ subordinated to covering $\{V^j\}$ if satisfies:

- for each index j, β^j is a $\mathscr{C}^\infty$ function in Ω;
- for each index j, the support of β^j is contained by V^j;
- for each index j, the function β^j is nonnegative everywhere in Ω;
- for all $xin\Omega$

$$\sum_j \beta^j(x) = 1. \tag{1.171}$$

Theorem 1.16.4 Given any locally finite covering of an open subset U in $\mathbb{R}^n$, there's a partition of unity subordinated to this covering.

Corollary 1.16.5 Let $\{U_i\}$ be a locally finite covering of $\mathbb{R}^n$ and f be a function. We can write

$$f = \sum_j f^j \tag{1.172}$$

where $f^j \subseteq \mathscr{C}^k(\Omega)$ and $\operatorname{Supp} f^j \subseteq V^j$ for all j.

Theorem 1.16.6 Let A be a closed subset of $\mathbb{R}^n$ and V be an arbitrary open neighborhood of A. There's a function $f : \mathbb{R}^n \to \mathbb{R}$, which is equip to a in some neighborhood of A and equip to 0 is the compliment of V.

1.17 the Open Mapping Theorem

Proposition 1.17.1 Let X, Y be two topological space. Suppose Y is Hausdorff. Let f is a Continuous mapping of X into Y, then the graph of f in the product topological space $X \times Y$.

Proposition 1.17.2 Let E, F and G be three Hausdorff TVS and j is a one-to-onecontinuous linear mapping of F into G. Let $f : E \to F$ is a linear mapping such that the composed $j \circ f : F \to G$ is continuous, then the graph of f is closed.

Corollary 1.17.3 Let E,F be two Hausdorff TVS, g be a linear mapping of F into E, it's one-to-oneand has a continuous inverse, then the graph of g is closed.

Proposition 1.17.4 Suppose E and F are two TVS with the following properties:

- If G is any closed linear subspace of the product $E\times F$, and u is any continuous linear mapping of G into E, then u is an open mapping.

Under this condition, if f is a linear Mapping of E into F with the closed graph, then f is continuous.

Theorem 1.17.5 — the Open Mapping Theorem. Let E,F be two metrizable and complete TVS. Every continuous linear mapping of E into F is a homomorphism.

Lemma 1.17.6 Let u is a linear mapping of the TVS E into the Baire space F. Given any neighborhood U of 0 in E, the closure $\overline{u(U)}$ of the image $u(U)$ contains a neighborhood of 0 in F.

Lemma 1.17.7 Let u be a continuous linear mapping of a metrizable and complete TVS E into a metrizable (it's unnecessary complete) TVS F, suppose u holds the following property:

- To every number $r>0$, there's a number $\rho>0$, such that for all $x\in E$, we have

$$B_\rho(u(x))\subseteq\overline{u(B_r(x))}\tag{1.173}$$

Then ,if $a>r$, we have for all $x\in E$,

$$B_\rho(u(x))\subseteq u(B_a(x)).\tag{1.174}$$

Corollary 1.17.8 A one-to-one continuous linear map of a metrizable and complete TVS E onto another metrizable and complete TVS F is an isomorphism (i.e., is bicontinuous).

Corollary 1.17.9 Let $\mathscr{T}_1,\mathscr{T}_2$ be two metrizable topologies on the same vector space E, both turning it into a complete TVS. Suppose that one is weaker than the other. Then they are equivalent.

Corollary 1.17.10 Let p,q be two norm on the vector space$E$. Suppose the normed space (E,p) and (E,q) are both B-space, and for some constant $C>0$ and all $x\in E$,

$$p(x)\leq Cq(x).\tag{1.175}$$

Then the norms p and q are equivalent.

Corollary 1.17.11 — the CLosed Graph Theorem. Let E,F be two metrizable TVS, f is a linear mapping of E into F. If the graph of f is closed then f is continuous.

2. Duality. Spaces of Distributions

2.1 The Hahn-Banach Theorem

Theorem 2.1.1 — Analytic form of the Hahn-Banach theorem. Let p be a seminorm on a vector space E, M a linear subspace of E, and f a linear form on M satisfying

$$|f(x)| \leq p(x) \quad \text{for all} \quad x \in M. \tag{2.1}$$

There exists a linear form $\tilde{f}$ on E extending f (i.e., $\tilde{f}(x) = f(x)$ for all $x \in M$) such that

$$|\tilde{f}(x)| \leq p(x) \quad \text{for all} \quad x \in E. \tag{2.2}$$

Theorem 2.1.2 — Geometric form of the Hahn-Banach theorem. Let E be a topological vector space, N a linear subspace of E, and Ω a convex open subset of E with

$$N \cap \Omega = \emptyset. \tag{2.3}$$

There exists a closed hyperplane H in E such that

$$N \subset H \quad \text{and} \quad H \cap \Omega = \emptyset. \tag{2.4}$$

R In Theorem 18.1, E carries no topology, but the seminorm p defines a topology. In Theorem 18.2, E is a TVS requiring neither Hausdorff nor local convexity.

The Hahn-Banach theorem is frequently applied in analysis, particularly to:

1. Approximation problems
2. Existence of solutions to functional equations
3. Separation of convex sets

2.1.1 Approximation Problems:

Let E be a locally convex space, M a closed linear subspace of E, and M_0 a linear subspace of M. The goal is to prove that M_0 is dense in M (i.e., every element of M is a limit of elements from M_0).
Typical example:

- E = function space
- M = solution space of a functional equation (e.g., PDE with constant coefficients)
- M_0 = special solutions (e.g., polynomials, analytic functions, $\mathscr{C}^\infty$ functions)

Corollary 2.1.3 M_0 is dense in M iff every continuous linear form vanishing on M_0 also vanishes on M.

Corollary 2.1.4 Let E be a Hausdorff TVS and $x_0 \neq 0$ in E. There exists a continuous linear form f with $f(x_0) \neq 0$.
(Thus for $E \neq \{0\}$, the dual space is non-trivial.)

Corollary 2.1.5 Let M be a proper closed subspace of an TVS E. There exists a non-zero continuous linear form vanishing on M.

2.1.2 Existence Problems

Let E, F be TVS, and $u : E \to F$ a continuous linear map. The transpose map:

$$ {}^t u : F' \to E', \quad y' \mapsto y' \circ u \tag{2.5}$$

is defined where E', F' are dual spaces. The existence problem is: given $x_0' \in E'$, find $y' \in F'$ solving

$$ {}^t u(y') = x_0'. \tag{2.6}$$

Typical example:

- u = differential operator
- x_0' = Dirac measure
- F' = distribution space

Corollary 2.1.6 If the linear form $\operatorname{Im} u \to \mathbb{R}$, $u(x) \mapsto x_0'(x)$ is continuous under the topology induced by F, then there exists $y' \in F'$ such that

$$ {}^t u(y') = x_0'. \tag{2.7}$$

Corollary 2.1.7 The following are equivalent:

1. $\operatorname{Im} u$ is dense in F
2. ${}^t u : F' \to E'$ is injective

2.1.3 Separation Problems:

Let E be a real TVS. For a closed hyperplane H, the complement of 0 in $E/H \cong \mathbb{R}$ consists of two disjoint open half-lines D_1, D_2. The preimages

$$ \phi^{-1}(D_1) \quad \text{and} \quad \phi^{-1}(D_2) \tag{2.8}$$

under the quotient map $\phi : E \to E/H$ are open half-spaces. Subsets $A, B \subset E$ are:

- **Separated** by H: if in different closed half-spaces
- **Strictly separated**: if in different open half-spaces

Proposition 2.1.8 Let E be a real TVS, A, B disjoint convex sets:

- If A open nonempty and B nonempty, $\exists$ closed hyperplane separating A and B
- If both open, $\exists$ closed hyperplane strictly separating them

Note. Strict separation fails if B not open (even in finite dimensions).

Proposition 2.1.9 Let E be real TVS, A, K disjoint convex sets. If A closed and K compact, $\exists$ closed hyperplane strictly separating them.

Corollary 2.1.10 In Hausdorff TVS over $\mathbb{R}$:

- Every closed convex set equals the intersection of closed half-spaces containing it
- The closure of a linear subspace M equals the intersection of closed hyperplanes containing M

Complex Case

Separation by "sides" is undefined in complex TVS (hyperplane complement connected), but:

Corollary 2.1.11 In Hausdorff TVS over $\mathbb{C}$, the closure of a subspace M equals the intersection of closed hyperplanes containing M.

Proposition 2.1.12 Let E be a complex vector space with two Hausdorff TVS topologies $\mathscr{F}_1, \mathscr{F}_2$ having the same continuous linear forms. For any convex set A, its closures under $\mathscr{F}_1$ and $\mathscr{F}_2$ coincide.

2.2 Topologies on the Dual

Let E be a complex TVS, E' its dual (continuous linear forms $E \to \mathbb{C}$). For $x' \in E'$, $x \in E$, denote the pairing by $\langle x', x \rangle$.

Definition 2.2.1 — Polar set. For $A \subset E$, the polar set is:

$$A^\circ = \left\{ x' \in E' \mid \sup_{a \in A} |\langle x', a \rangle| \leq 1 \right\}. \tag{2.9}$$

Properties:

1. A° is convex balanced in E'
2. $A \subset B \implies B^\circ \subset A^\circ$, and $(\rho A)^\circ = \rho^{-1} A^\circ$ $(\rho > 0)$
$$(A \cup B)^\circ = A^\circ \cap B^\circ \tag{2.10}$$
3. If A is a cone ($x \in A \implies \lambda x \in A\ \forall \lambda > 0$), then:
$$A^\circ = \{ x' \in E' \mid \langle x', x \rangle = 0 \quad \forall x \in A \} \tag{2.11}$$
(When A is a subspace, A° is the orthogonal complement.)

Proposition 2.2.1 If $B \subset E$ is bounded, then B° is absorbing in E'.

Definition 2.2.2 — $\mathfrak{S}$-topology. Let $\mathfrak{S}$ be a family of bounded subsets of E satisfying:

$$(\mathfrak{S}_1) \quad A, B \in \mathfrak{S} \implies \exists C \in \mathfrak{S} : A \cup B \subset C \tag{2.12}$$

$$(\mathfrak{S}_2) \quad A \in \mathfrak{S}, \lambda \in \mathbb{C} \implies \exists B \in \mathfrak{S} : \lambda A \subset B \tag{2.13}$$

The $\mathfrak{S}$-topology on E' has neighborhood basis $\{A^\circ \mid A \in \mathfrak{S}\}$. Denote this space by $E'_{\mathfrak{S}}$.

Proposition 2.2.2 — Convergence characterization. A filter $\mathscr{F}'$ converges to x' in the $\mathfrak{S}$-topology iff

$$\forall A \in \mathfrak{S}, \quad \mathscr{F}' \text{ converges uniformly to } x' \text{ on } A \tag{2.14}$$

i.e.,

$$\forall \varepsilon > 0, \exists M' \in \mathscr{F}' : \sup_{x \in A} |\langle y', x\rangle - \langle x', x\rangle| \leq \varepsilon \quad \forall y' \in M'. \tag{2.15}$$

A basis of neighborhoods is:

$$W_\varepsilon(A) = \{x' \in E' \mid \sup_{x \in A} |\langle x', x\rangle| \leq \varepsilon\}. \tag{2.16}$$

Example I. The weak dual topology, or weak topology on E'

This is the $\mathfrak{S}$-topology corresponding to $\mathfrak{S}$: the family of all finite subsets of E; it is denoted by $\sigma(E', E)$, and E' with this topology is written E'_σ. Continuous linear functionals x' on E' converge weakly to zero if, for each $x \in E$, their values $\langle x', x\rangle$ converge to zero in the complex plane. In other words, the weak topology on E' is the topology of pointwise convergence on E, viewing continuous linear functionals as functions on E. A basis of neighborhoods of E'_σ is the family of sets

$$W_\varepsilon(x_1, \ldots, x_r) = \{x' \in E' \mid |\langle x', x_j\rangle| \leq \varepsilon,\ j = 1, \ldots, r\}, \tag{2.17}$$

where $\{x_1, \ldots, x_r\}$ varies over all finite subsets of E, and $\varepsilon > 0$.

2.2.1 Example II. The topology of convex compact convergence

This is the $\mathfrak{S}$-topology with $\mathfrak{S}$ being the family of all convex compact subsets of E. It is denoted by $\gamma(E', E)$; the dual of E with this topology is written E'_γ.

Example III. The topology of compact convergence

This is the $\mathscr{S}$-topology where $\mathscr{S}$ is the family of all compact subsets of E. It is the topology of uniform convergence on compact subsets of E (or compact convergence); E' with this topology is written E'_c. Note that this topology is not always equivalent to the topology of convex compact convergence. However, when E is a Fréchet space, these two topologies coincide.

Example IV. The strong dual topology, or strong topology on E'

This topology (along with the weak topology) is central. It is defined by taking $\mathscr{S}$ as the family of all bounded subsets of E. A filter in E' converges strongly to zero if it converges uniformly to zero on every bounded subset of E; thus, the strong topology is also called the *topology of bounded convergence*. The dual E' with this topology is called the *strong dual* of E and denoted E_b (where b stands for bounded).

If S_1 and S_2 are two families of bounded subsets of E satisfying (S_1) and (S_2), and if $S_1 \supseteq S_2$, then the S_1-topology is finer than the S_2-topology. In particular, the four topologies on E' satisfy:

$$\sigma(E', E) \leq \gamma(E', E) \leq c(E', E) \leq b(E', E), \tag{2.18}$$

where c and b denote the compact convergence topology and the bounded convergence topology (strong topology), respectively.

Proposition 2.2.3 If the union of sets in the family S is dense in E, then the S-topology on E' is Hausdorff.

Corollary 2.2.4 The weak topology, strong topology, topology of convex compact convergence, and topology of compact convergence on E' are all Hausdorff.

Proposition 2.2.5 Let (E,p) be a normed space and E' its dual. The strong dual topology on E' is defined by the norm

$$p'(x') = \sup_{p(x)=1} |\langle x',x\rangle|. \tag{2.19}$$

Here, p' is the norm on E' whose closed unit ball is the polar of the unit ball in (E,p).

Let E and F be TVSs, and $u : E \to F$ a continuous linear map. Its transpose is defined as

$${}^t u : F' \ni y' \mapsto y' \circ u \in E'. \tag{2.20}$$

Proposition 2.2.6 Let E, F be two TVS, and $\mathfrak{S}$ (resp. $\mathfrak{H}$) a family of bounded subsets of E (resp. F), having Properties $(\mathfrak{S}_1)$, $(\mathfrak{S}_2)$. If for every $A \in \mathfrak{S}$, there exists $B \in \mathfrak{H}$ such that $u(A) \subset B$, then the transpose map

$${}^t u : F'_{\mathfrak{H}} \to E'_{\mathfrak{S}} \tag{2.21}$$

is continuous.

Corollary 2.2.7 The transpose map ${}^t u : F' \to E'$ is continuous when E' and F' carry the weak dual topology, topology of convex compact convergence, topology of compact convergence, or strong dual topology.

The canonical map $E \to E'^*$ sends x to v_x, where $v_x(x') = \langle x',x\rangle$.

Proposition 2.2.8 Let $\mathfrak{S}$ be a family of bounded subsets of E, having Properties $(\mathfrak{S}_1)$, $(\mathfrak{S}_2)$. If $\mathfrak{S}$ covers E, the canonical map $x \mapsto v_x$ sends E into the dual of $E'_{\mathfrak{S}}$, $(E'_{\mathfrak{S}})'$.

Proposition 2.2.9 If E is a locally convex Hausdorff TVS, the canonical map $E \to (E'_{\mathfrak{S}})'$ is injective.

When E' carries the weak topology or topology of convex compact convergence, the canonical map $E \to (E')'$ is surjective. Thus, E can be identified as the dual of its weak dual E'_σ or convex compact convergence dual E'_γ.

2.3 Examples of Duals among L^p Spaces

This chapter presents concrete realizations of duals for some spaces from Part I. Precisely: let E be a TVS of functions, and E' its dual. A realization of E' is a pair (F,j) where F is a vector space and $j : F \to E'$ is a linear map that is injective and surjective. In practice, we seek natural realizations meaningful in analysisideally, when E is a function space, F should also be one, though this is not always possible. For example, the dual of $\mathscr{C}^0(\Omega)$ (continuous functions on $\Omega \subset \mathbb{R}^n$) cannot be naturally realized as a function space. Such cases require extending from functions to measures, distributions, or analytic functionals.

A realization may demand topological consistency: when F is a TVS and j is an isomorphism onto E' (typically with the strong dual topology). For normed or Hilbert spaces, we require F to be a Banach or Hilbert space and j to be an isometric isomorphism. This is standard practice, and key examples will be shown.

Consider Banach spaces E and F. To isometrically identify E' (with dual norm) and F, use a bilinear form $\langle\cdot,\cdot\rangle : E\times F\to\mathbb{C}$ satisfying three conditions. First, *strong continuity*:

$$|\langle e,f\rangle| \le \|e\|_E\|f\|_F \quad \forall e\in E, f\in F. \tag{2.22}$$

This gives $L_f(e)=\langle e,f\rangle\in E'$ and $\|L_f\|_{E'}\le\|f\|_F$. Second, *isometry*: for all $f\in F$ and $\varepsilon>0$, there exists $e\in E$ with

$$\|e\|_E\le 1, \quad |\langle e,f\rangle| \ge \|f\|_F-\varepsilon, \tag{2.23}$$

forcing $\|f\|_F=\|L_f\|_{E'}$. Third, *surjectivity*: for arbitrary $L\in E'$, construct $f\in F$ such that $\langle e,f\rangle = L(e)$ for all $e\in E$. This step varies by space (simple for ℓ^p, requires Lebesgue-Nikodym for L^p).

Let $\mathscr{F}(X,\mathbb{C})$ be complex-valued functions on a set X, with a seminorm p (allowing $+\infty$). Assume p is *increasing*: if f,g are real-valued with $f(x)\ge g(x)\ge 0$ for all $x\in X$, then $p(f)\ge p(g)$. Hölder inequalities hold in this framework and will be applied to two specific seminorms.

Choice 1. Let X be the set of nonnegative integers $\{0,1,2,\dots\}$, so that $\mathscr{F}(X,\mathbb{C})$ is identified with complex sequences $\sigma=(\sigma_j)$. The seminorm is

$$p(\sigma)=\sum_{j=0}^{\infty}|\sigma_j|, \tag{2.24}$$

and the set where p is finite is ℓ^1.

Choice 2. Let X be an open subset of $\mathbb{R}^n$ with seminorm

$$p(f)=\int_X^* |f(x)|\,\mathrm{d}x, \tag{2.25}$$

where $\int_X^*$ denotes the upper Lebesgue integral. The set where $p(f)<+\infty$ is strictly larger than the Lebesgue integrable functions $\mathscr{L}^1$. The latter requires measurability (or equivalence to limits of compactly supported continuous functions under p). When f is integrable, the Lebesgue integral is defined as

$$\int f\,\mathrm{d}x=\int(\operatorname{Re}f)^+\,\mathrm{d}x-\int(\operatorname{Re}f)^-\,\mathrm{d}x+i\int(\operatorname{Im}f)^+\,\mathrm{d}x-i\int(\operatorname{Im}f)^-\,\mathrm{d}x. \tag{2.26}$$

These choices are special cases of measure theory; results generalize to upper integrals with respect to positive measures.

Lemma 2.3.1 — Hölder's inequality. Let X be a set, $\mathscr{F}(X,\mathbb{C})$ complex-valued functions, and p an increasing seminorm (satisfying (20.1)). For all nonnegative functions f,g and numbers $\alpha,\beta>0$ with $\alpha+\beta=1$,

$$p(f^\alpha g^\beta)\le[p(f)]^\alpha[p(g)]^\beta. \tag{2.27}$$

2.3.1 Example I. Duals of Sequence Spaces ℓ^p $(1\le p<+\infty)$

The space ℓ^p consists of complex sequences $\sigma=(z_j)_{j=0}^\infty$ satisfying

$$\|\sigma\|_{\ell^p}=\left(\sum_{j=0}^{\infty}|z_j|^p\right)^{1/p}<+\infty. \tag{2.28}$$

It is a Banach space; ℓ^2 is a Hilbert space. The space ℓ_F of finite sequences (nonzero only for finitely many indices) is dense in each ℓ^p.

Theorem 2.3.2 The bilinear form on $\ell_F \times \ell_F$,

$$\langle \sigma, \tau \rangle = \sum_{j=0}^{\infty} \sigma_j \tau_j, \tag{2.29}$$

extends to $\ell^p \times \ell^{p'}$ where

$$p' = \begin{cases} \frac{p}{p-1} & \text{if } p > 1, \\ +\infty & \text{if } p = 1. \end{cases} \tag{2.30}$$

This extension satisfies Hölder's inequality:

$$|\langle \sigma, \tau \rangle| \le \|\sigma\|_{\ell^p} \|\tau\|_{\ell^{p'}}. \tag{2.31}$$

The map $\tau \mapsto L_\tau$ ($L_\tau(\sigma) = \langle \sigma, \tau \rangle$) is an isometric isomorphism:

$$\ell^{p'} \cong (\ell^p)'. \tag{2.32}$$

When $p = 1$, ℓ^∞ is the space of bounded sequences with norm

$$\|\sigma\|_{\ell^\infty} = \sup_{j \ge 0} |\sigma_j|. \tag{2.33}$$

Example II. Duals of $L^p(\Omega)$ ($1 \le p < +\infty$)

Let $\Omega \subset \mathbb{R}^n$ be open with Lebesgue measure $\mathrm{d}x$. The space $L^p(\Omega)$ consists of measurable functions satisfying

$$\|f\|_{L^p} = \left(\int_\Omega |f(x)|^p \,\mathrm{d}x \right)^{1/p} < +\infty, \tag{2.34}$$

equipped with the Banach space norm (L^2 is Hilbert). The dense subspace $\mathscr{C}_c^0(\Omega)$ (continuous compactly supported functions) replaces finite sequences.

The bilinear form on $\mathscr{C}_c^0(\Omega) \times \mathscr{C}_c^0(\Omega)$,

$$\langle f, g \rangle = \int_\Omega f(x) g(x) \,\mathrm{d}x, \tag{2.35}$$

extends to $L^p \times L^{p'}$ for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, satisfying Hölder's inequality:

$$|\langle f, g \rangle| \le \|f\|_{L^p} \|g\|_{L^{p'}}. \tag{2.36}$$

The map $g \mapsto L_g$ ($L_g(f) = \langle f, g \rangle$) embeds $L^{p'}$ continuously into $(L^p)'$. To prove isometry, for given $g \in L^{p'}$, define

$$f(x) = \begin{cases} \overline{g(x)} |g(x)|^{p'-2} & g(x) \neq 0, \\ 0 & \text{otherwise}. \end{cases} \tag{2.37}$$

This satisfies:

$$\|f\|_{L^p} = \|g\|_{L^{p'}}^{p'/p}, \tag{2.38}$$

$$\langle f, g \rangle = \|g\|_{L^{p'}}^{p'}. \tag{2.39}$$

Combining with the dual norm definition:

$$\|g\|_{L^{p'}}^{p'} \leq \|L_g\|_{(L^p)'} \|f\|_{L^p} = \|L_g\|_{(L^p)'} \|g\|_{L^{p'}}^{p'/p}, \tag{2.40}$$

yielding $\|g\|_{L^{p'}} = \|L_g\|_{(L^p)'}$.

For surjectivity (any $L \in (L^p)'$ equals some L_g), use Radon measures: continuous linear functionals $\mu : \mathscr{C}_c(\Omega) \to \mathbb{C}$ (LF topology). Decompose:

$$\mu = \rho^+ - \rho^- + i(\sigma^+ - \sigma^-), \tag{2.41}$$

where $\rho^\pm, \sigma^\pm$ are positive Radon measures. Apply:

Theorem 2.3.3 — Radon-Nikodym for locally absolutely continuous measures. Let μ be a positive Radon measure. The following are equivalent:

(a) *Local absolute continuity*: For every nonnegative $g \in \mathscr{C}_c(\Omega)$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 \leq h \leq g \quad \text{and} \quad \int h \, \mathrm{d}x \leq \delta \quad \Longrightarrow \quad \langle \mu, h \rangle \leq \varepsilon. \tag{2.42}$$

(b) *Integral representation*: There exists a locally integrable function $f \geq 0$ a.e. (i.e., $\int_K |f| \, \mathrm{d}x < \infty$ for every compact $K \subset \Omega$) satisfying

$$\langle \mu, \varphi \rangle = \int_\Omega \varphi(x) f(x) \, \mathrm{d}x, \quad \forall \varphi \in \mathscr{C}_c(\Omega). \tag{2.43}$$

This representation and measure decomposition complete the proof of surjectivity.

Let λ be a continuous linear functional on $L^p(\Omega)$. There exists $C > 0$ such that for all $u \in L^p(\Omega)$:

$$|\lambda(u)| \leq C \|u\|_{L^p}. \tag{2.44}$$

For any compact $K \subset \Omega$ and $u \in \mathscr{C}_c^0(\Omega)$ supported in K:

$$|\lambda(u)| \leq C(\mathrm{meas}(K))^{1/p} \sup_{x \in K} |u(x)|, \tag{2.45}$$

where $\mathrm{meas}(K)$ is the Lebesgue measure of K. This shows λ is continuous on $\mathscr{C}_c^0(\Omega)$, defining a Radon measure (still denoted λ).

If λ is positive, then for any nonnegative $g \in \mathscr{C}_c^0(\Omega)$ and $h \in \mathscr{C}_c^0(\Omega)$ with $0 \leq h \leq g$:

$$\|h\|_{L^p} \leq \left(\sup_x |g(x)|^{1-1/p} \right) \left(\int h \, \mathrm{d}x \right)^{1/p}. \tag{2.46}$$

Setting $M = \sup_x |g(x)|^{1-1/p}$ and $\delta = (\varepsilon / CM)^p$, we have:

$$\int h \, \mathrm{d}x \leq \delta \quad \Longrightarrow \quad \lambda(h) \leq \varepsilon. \tag{2.47}$$

By Theorem 20.2, there exists a locally integrable $f \geq 0$ a.e. satisfying:

$$\lambda(\varphi) = \int \varphi(x) f(x) \, \mathrm{d}x, \quad \forall \varphi \in \mathscr{C}_c^0(\Omega). \tag{2.48}$$

Take relatively compact open sets $\Omega_1 \subset \Omega_2 \subset \cdots$ covering Ω and define:

$$f_k(x) = \begin{cases} f(x) & x \in \Omega_k \text{ and } f(x) \leq k, \\ 0 & \text{otherwise.} \end{cases} \tag{2.49}$$

Then f_k is bounded, measurable, and compactly supported. Since $f_k \leq f$:

$$\int \varphi f_k \,\mathrm{d}x \leq \int \varphi f \,\mathrm{d}x, \quad \forall \varphi \geq 0 \in \mathscr{C}_c^0(\Omega). \tag{2.50}$$

For arbitrary $\varphi \in \mathscr{C}_c^0(\Omega)$, combining with (20.11) and (20.12):

$$\left| \int \varphi f_k \,\mathrm{d}x \right| \leq C \|\varphi\|_{L^p}. \tag{2.51}$$

By density in $L^p(\Omega)$:

$$\left| \int u f_k \,\mathrm{d}x \right| \leq C \|u\|_{L^p}, \quad \forall u \in L^p(\Omega). \tag{2.52}$$

Choosing $u(x) = [f_k(x)]^{p'-1}$ when $f_k(x) \neq 0$:

$$\|f_k\|_{L^{p'}} \leq C \quad \Longrightarrow \quad \int [f_k]^{p'} \,\mathrm{d}x \leq C^{p'}. \tag{2.53}$$

Since $(f_k)^{p'}$ increases pointwise to $f^{p'}$, by monotone convergence:

$$\int f^{p'} \,\mathrm{d}x = \lim_{k \to \infty} \int f_k^{p'} \,\mathrm{d}x \leq C^{p'}, \tag{2.54}$$

so $f \in L^{p'}(\Omega)$. Density of $\mathscr{C}_c^0(\Omega)$ in $L^p(\Omega)$ implies:

$$\lambda(u) = \int u f \,\mathrm{d}x, \quad \forall u \in L^p(\Omega). \tag{2.55}$$

For general λ, decompose into four positive Radon measures and apply the above to each. The case $p = 1$ is similar.

Theorem 2.3.4 — Dual of L^p (Theorem 20.3). Let $\Omega \subset \mathbb{R}^n$ be open, $1 \leq p < \infty$, and $p' = p/(p-1)$ if $p > 1$, $p' = +\infty$ if $p = 1$. The bilinear form

$$\langle u, v \rangle = \int u v \,\mathrm{d}x \tag{2.56}$$

extends to $L^p \times L^{p'}$, satisfying Hölder's inequality:

$$|\langle u, v \rangle| \leq \|u\|_{L^p} \|v\|_{L^{p'}}. \tag{2.57}$$

The map $v \mapsto (u \mapsto \langle u, v \rangle)$ is an isometric isomorphism from $L^{p'}(\Omega)$ onto the strong dual of $L^p(\Omega)$.

2.4 Radon Measures. Distributions

Duality theorems show that duals of function spaces cannot always be naturally realized as function spaces. This is advantageousit expands the universe of mathematical objects. If duals were always function spaces, duality theory would only provide functions. This chapter introduces two fundamental new objects:

1. **Radon measures**: Dual of $\mathscr{C}_c^0(\Omega)$ (continuous compactly supported functions).
2. **Distributions**: Dual of $\mathscr{C}_c^\infty(\Omega)$ (smooth compactly supported functions), central to subsequent study.

Radon measures and distributions cannot be naturally interpreted as functions. A third object, *analytic functionals* (dual of holomorphic functions), will be introduced later.

2.4.1 Radon Measures in an Open Subset Ω of $\mathbb{R}^n$

Definition 2.4.1 — Radon Measure. Let $\Omega \subset \mathbb{R}^n$ be open. A Radon measure μ is a linear functional on $\mathscr{C}_c^0(\Omega)$ satisfying: for every compact $K \subset \Omega$, there exists $C(\mu, K) > 0$ such that for all $\varphi \in \mathscr{C}_c^0(\Omega)$ supported in K,

$$|\langle \mu, \varphi \rangle| \leq C(\mu, K) \sup_{x \in \Omega} |\varphi(x)|. \tag{2.58}$$

Note: $\mathscr{C}_c^0(K) \neq \mathscr{C}^0(K)$. For example, constant functions extended by zero outside K are discontinuous in Ω and excluded.

- *Dirac measure* at x^0:

$$\langle \mu, \varphi \rangle = \varphi(x^0) \tag{2.59}$$

- *Lebesgue measure*:

$$\langle \mu, \varphi \rangle = \int_\Omega \varphi(x) dx \tag{2.60}$$

- *Density measure* (for locally integrable g):

$$\langle \mu, \varphi \rangle = \int_\Omega \varphi(x) g(x) dx \tag{2.61}$$

  where g satisfies $\int_K |g| dx < \infty$ for every compact $K \subset \Omega$.
- μ is *real* if $\langle \mu, \varphi \rangle \in \mathbb{R}$ for all real-valued $\varphi \in \mathscr{C}_c^0(\Omega)$.
- μ is *positive* if $\langle \mu, \varphi \rangle \geq 0$ for all nonnegative $\varphi \in \mathscr{C}_c^0(\Omega)$. Examples: Lebesgue measure, density measures with $g \geq 0$ a.e., Dirac measures δ_{x^0} ($x^0 \in \Omega$).

For any Radon measure μ:

- *Complex conjugate*:

$$\langle \bar{\mu}, \varphi \rangle = \overline{\langle \mu, \bar{\varphi} \rangle} \tag{2.62}$$

- *Real and imaginary parts*:

$$\operatorname{Re} \mu = \tfrac{1}{2}(\mu + \bar{\mu}), \quad \operatorname{Im} \mu = \tfrac{1}{2i}(\mu - \bar{\mu}) \tag{2.63}$$

For a density measure $\mu = g(x)dx$:

$$\bar{\mu} = \overline{g(x)}dx \tag{2.64}$$

$$\operatorname{Re}\mu = (\operatorname{Re} g)(x)dx \tag{2.65}$$

$$\operatorname{Im}\mu = (\operatorname{Im} g)(x)dx \tag{2.66}$$

satisfying $\mu = \operatorname{Re}\mu + i\operatorname{Im}\mu$. Key results on Radon measures: a positive linear functional on $\mathscr{C}_c^0(\Omega)$ is a positive Radon measure; every real Radon measure decomposes as:

$$\mu = \mu^+ - \mu^- \tag{2.67}$$

For locally Lebesgue integrable g_1, g_2, the density measures:

$$\varphi \mapsto \int \varphi g_1 dx \quad \text{and} \quad \varphi \mapsto \int \varphi g_2 dx \tag{2.68}$$

are equal iff $g_1 = g_2$ a.e. in Ω. Any complex measure decomposes as:

$$\mu = (\operatorname{Re}\mu)^+ - (\operatorname{Re}\mu)^- + i[(\operatorname{Im}\mu)^+ - (\operatorname{Im}\mu)^-] \tag{2.69}$$

The absolute value of a real Radon measure is:

$$|\mu| = \mu^+ + \mu^- \tag{2.70}$$

Integration theory extends positive measures beyond $\mathscr{C}_c(\Omega)$, but general theory requires specialized references. Radon measures generalize to locally compact spaces, though measures like dx or $g(x)dx$ exist only in specific spaces (e.g., Haar measures on locally compact groups). A function $f : \Omega \to \mathbb{C}$ is locally L^p ($1 \le p \le \infty$) if measurable and for every compact $K \subset \Omega$:

$$\int_K |f(x)|^p dx < +\infty \tag{2.71}$$

Crucially, every locally L^p function is locally integrable. Proof uses Hölder: for $1_K = \operatorname{char}(K)$,

$$\int_K^* |f(x)|dx \le \|1_K f\|_{L^p} \|1_K\|_{L^{p'}} \tag{2.72}$$

with conjugate exponent:

$$p' = \begin{cases} \infty & p = 1, \\ \frac{p}{p-1} & 1 < p < \infty. \end{cases} \tag{2.73}$$

Since $\|1_K\|_{L^{p'}} = (\operatorname{meas}(K))^{1/p'} < \infty$,

$$\int_K^* |f(x)|dx < +\infty \tag{2.74}$$

Locally integrable functions form the maximal class defined by local L^p conditions, fundamental in distribution theory.

Distributions in an Open Subset of $\mathbb{R}^n$

Distributions in an open set $\Omega \subset \mathbb{R}^n$ are continuous linear functionals on the test function space $\mathscr{C}_c^\infty(\Omega)$ (infinitely differentiable compactly supported functions) equipped with the canonical LF topology. A linear functional is a distribution iff equivalent conditions hold:

Theorem 2.4.1 A linear functional L is a distribution iff:

(a) For every compact $K \subset \Omega$, there exist $m \geq 0$ and $C > 0$ such that for all $\varphi \in \mathscr{C}_c^\infty(\Omega)$ supported in K:

$$|L(\varphi)| \leq C \sup_{|p| \leq m} \left(\sup_{x \in \Omega} \left| \left(\frac{\partial}{\partial x} \right)^p \varphi(x) \right| \right) \tag{2.75}$$

(b) If $\{\varphi_k\}$ converges uniformly to zero with all derivatives, and supports contained in a fixed compact K, then $L(\varphi_k) \to 0$.

Distributions arise by restricting continuous linear functionals from a space $E \supset \mathscr{C}_c^\infty(\Omega)$ with locally convex Hausdorff topology coarser than the LF topology. Distinct distributions correspond when $\mathscr{C}_c^\infty(\Omega)$ is dense in E (by Hahn-Banach).

Every Radon measure μ restricts to a distribution T_μ, and distinct measures yield distinct distributions:

$$\mu_1 \neq \mu_2 \implies T_{\mu_1} \neq T_{\mu_2} \tag{2.76}$$

Thus T_μ is identified with μ, and such distributions are called Radon measures.

Theorem 2.4.2 A distribution T is a Radon measure iff equivalent:

(a) $\varphi \mapsto \langle T, \varphi \rangle$ is continuous on $\mathscr{C}_c^0(\Omega)$ (induced topology)

(b) For every compact $K \subset \Omega$, there exists $C > 0$ such that for all $\varphi \in \mathscr{C}_c^\infty(\Omega)$ supported in K:

$$|\langle T, \varphi \rangle| \leq C \sup_x |\varphi(x)| \tag{2.77}$$

(c) If $\{\varphi_k\}$ converges uniformly to zero with supports in a fixed compact K, then $\langle T, \varphi_k \rangle \to 0$.

Crucially, locally integrable functions f define Radon measure distributions:

$$T_f(\varphi) = \int \varphi(x) f(x) dx \tag{2.78}$$

Two functions define the same distribution iff equal almost everywhere. Thus T_f is identified with the equivalence class of f (modulo a.e. equality), and such distributions are called **functions**.

When a distribution comes from a function, we transfer function terminology: call it C^k, L^p, polynomial, exponential, analytic, etc., if a representative satisfies the property.

The true significance of distribution theory lies in its extension beyond Radon measures. If all distributions were Radon measures, the theory would be trivial. In reality, distribution space contains far more objects, evident when introducing differential operators: differentiating Radon measures produces distributions that are generally not Radon measures.

Many functions in analysis are not locally integrable and cannot define distributions via standard integration, e.g.:

$$t \mapsto t^{-k} \quad (k = 1, 2, \ldots) \tag{2.79}$$

The issue is not measurability (it is smooth away from zero) but divergence:

$$\int_0^1 t^{-k} dt = +\infty \quad (k \geq 1) \tag{2.80}$$

However, *pseudofunctions* (e.g., $\mathrm{Pf}\, t^{-k}$) define distributions such that for $\varphi \in \mathscr{C}_c^\infty(\mathbb{R})$ supported away from zero:

$$\langle S, \varphi \rangle = \int \varphi(t) t^{-k} dt \tag{2.81}$$

Crucially, such S is never a function distribution. If a locally integrable f satisfied

$$\langle S, \varphi \rangle = \int \varphi(t) f(t) dt, \tag{2.82}$$

then $f(t) = t^{-k}$ for $t \neq 0$, contradicting non-local-integrability.

Some functions like

$$t \mapsto e^{1/t} \tag{2.83}$$

admit no distributional interpretation: no distribution coincides with this function away from zero.

Notation: The space of distributions on open $\Omega \subset \mathbb{R}^n$ is denoted $\mathscr{D}'(\Omega)$.

2.5 More Duals: Polynomials and Formal Power Series. Analytic Functionals

2.5.1 Polynomials and Formal Power Series

Let $\mathscr{P}_n$ be the vector space of all polynomials in n indeterminates with complex coefficients. This space is often denoted by $C[X_1, \dots, X_n]$. We modify momentarily the notation in order to shorten it, and also in order to emphasize that we are interested not in the ring structure of the set of polynomials, but in its topological vector space structure. For $k = 0, 1, 2, \dots$, let $\mathscr{P}_n^k$ be the vector subspace of $\mathscr{P}_n$ consisting of the polynomials of degree $\leq k$; each $\mathscr{P}_n^k$ is finite dimensional, in fact its dimension is easy to compute: it is equal to

$$\frac{(k+n)}{k!n!}. \tag{2.84}$$

We provide $\mathscr{P}_n$ with the locally convex topology which is the inductive limit of the topologies of the Hausdorff finite dimensional spaces $\mathscr{P}_n^k$, $k = 0, 1, \dots$.

On the other hand, we consider the vector space $\mathscr{Q}_n$ of formal power series in n indeterminates, which is usually denoted by $C[[X_1, \dots, X_n]]$. We provide $\mathscr{Q}_n$ with the topology of convergence of each coefficient. This topology is defined by the sequence of seminorms:

$$u = \sum_{p \in \mathbb{N}^n} u_p X^p \mapsto \sup_{|p| \leq k} |u_p|, \quad k = 0, 1, \dots \tag{2.85}$$

This topology turns $\mathscr{Q}_n$ into a Fréchet space.

Now, there is a natural duality between polynomials and formal power series, which can be expressed by the bracket

$$\langle P, u \rangle = \sum_{p \in \mathbb{N}^n} P_p u_p, \tag{2.86}$$

where

$$P = \sum_p P_p X^p, \quad u = \sum_p u_p X^p. \tag{2.87}$$

It should be remembered that all coefficients P_p, except possibly a finite number of them, are equal to zero; this gives a meaning to the bracket $\langle P, u \rangle$. The main result, in the present context, is the following one:

Theorem 2.5.1 1. The map

$$u \mapsto (P \mapsto \langle P, u \rangle) \tag{2.88}$$

is an isomorphism for the structures of topological vector spaces of the Fréchet space of formal power series $\mathscr{Q}_n$ onto the strong dual of the LF-space of polynomials, $\mathscr{P}_n$.

The map

$$P \mapsto (u \mapsto \langle P, u \rangle) \tag{2.89}$$

is an isomorphism of $\mathscr{P}_n$ onto the strong dual of $\mathscr{Q}_n$.

If we forget about the multiplicative structure of the sets $\mathscr{P}_n$ and $\mathscr{Q}_n$, we can regard them as sets of functions with complex values and domain of definition $\mathbb{N}^n$: this simply means that we identify a polynomial or a power series with the collection of its coefficients: instead of writing $u = \sum_{p \in \mathbb{N}^n} u_p X^p$, we write $u = (u_p)_{p \in \mathbb{N}^n}$; then $\mathscr{Q}_n$ turns out to be the space of all complex functions on $\mathbb{N}^n$ and $\mathscr{P}_n$ the space of those functions which vanish outside a finite set. Needless to say, this is the same as identifying $\mathscr{Q}_n$ with the space of arbitrary complex sequences depending on n indices, and $\mathscr{P}_n$ with the space of finite complex sequences. We may also write

$$\mathscr{Q}_n = \prod_{p \in \mathbb{N}^n} \mathbf{C}_p, \quad \mathbf{C}_p \cong \mathbb{C}, \text{ the complex plane.} \tag{2.90}$$

Then $\mathscr{P}_n$ can be regarded as the *direct sum* of the $\mathbf{C}_p$'s. As a matter of fact, the topology of simple convergence of the coefficients on $\mathscr{Q}_n$ is precisely the product topology of the $\mathbf{C}_p$'s, etc. Let us observe that the LF-space $\mathscr{P}_n$, which is canonically isomorphic with the strong dual of $\mathscr{Q}_n$, is continuously embedded in $\mathscr{Q}_n$, and is dense in $\mathscr{Q}_n$.

2.5.2 Analytic Functionals in an Open Subset Ω of $\mathbb{C}^n$

We denote by $H(\Omega)$ the space of holomorphic functions in Ω, equipped with the topology induced by any one of the spaces $\mathscr{C}^k(\Omega)$ $(0 \le k \le +\infty)$ when $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$. For instance, we may consider that $H(\Omega)$ carries the topology of uniform convergence on compact subsets of Ω, i.e., the topology induced by $\mathscr{C}^0(\Omega)$.

Definition 2.5.1 The dual of $H(\Omega)$ is denoted by $H'(\Omega)$; its elements are called *analytic functionals* in Ω.

Observing that $H(\Omega)$ is isomorphically embedded in $\mathscr{C}^0(\Omega)$, we see, in virtue of the Hahn-Banach theorem, that any continuous linear functional L on $H(\Omega)$ can be extended as a continuous linear functional $\bar{L}$ on $\mathscr{C}^0(\Omega)$, which, in turn, by restriction to $\mathscr{C}_c^0(\Omega)$, defines a Radon measure μ in Ω. As $\mathscr{C}_c^0(\Omega)$ is dense in $\mathscr{C}^0(\Omega)$, this Radon measure μ is uniquely determined by $\bar{L}$; but as $\bar{L}$ is not uniquely determined by L (except when $\Omega = \emptyset$), neither is μ. This is easy to understand; for let ϕ_0 be a continuous function with compact support in Ω such that there is a $\mathscr{C}^1$ function with compact support in Ω, ϕ, satisfying the equation

$$\phi_0 = \partial\phi/\partial\bar{z}_1 = \frac{1}{2}\left(\frac{\partial\phi}{\partial x_1} + i\frac{\partial\phi}{\partial y_1}\right), \quad i = (-1)^{1/2}. \tag{2.91}$$

Then, if the Radon measure μ defines an analytic functional L in Ω, the Radon measure $\mu + \phi_0$ defines the same one, L. This simply follows from the fact that the analytic functional defined by

ϕ_0 is equal to zero; this analytic functional is simply

$$H(\Omega) \ni h \mapsto \iint h(x+iy)\phi_0(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.92}$$

It is well defined, as ϕ_0 is continuous with compact support in Ω. By integration by parts, we see immediately that

$$\iint h(x+iy)\phi_0(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint h(x+iy)\frac{\partial \phi}{\partial \bar{z}_1}(x,y)\,\mathrm{d}x\,\mathrm{d}y = -\iint \frac{\partial h}{\partial \bar{z}_1}(x+iy)\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{2.93}$$

In general, that is to say when Ω is an arbitrary open subset of $\mathbb{C}^n$, there is no natural way of interpreting as functions the analytic functionals in Ω. This is however possible when Ω is of a very simple type, for instance when Ω is a polydisk, as we are now going to show.

Notation 2.1. *Let $K_1,\dots,K_n$ be n numbers, $0 < K_j \le +\infty$, for $j = 1,\dots,n$. We denote by $\Delta(K_1,\dots,K_n)$, or simply by $\Delta(K)$, the open polydisk*

$$\{z \in \mathbb{C}^n : |z_1| < K_1,\dots,|z_n| < K_n\}. \tag{2.94}$$

Notation 2.2. *Let $K_1,\dots,K_n$ be n nonnegative finite numbers. We denote by $Exp(K_1,\dots,K_n)$, or simply by $Exp(K)$, the space of entire functions of exponential type $(K_1,\dots,K_n)$, i.e., the space of the entire functions f in $\mathbb{C}^n$ such that there is a constant $A(f) > 0$ such that*

$$|f(z)|\exp(-K_1|z_1| - \cdots - K_n|z_n|) \le A(f) \tag{2.95}$$

for all $z \in \mathbb{C}^n$. If $f \in Exp(K)$, the inf of the constants $A(f)$ can be taken as the norm of f in $Exp(K)$; that it is indeed a norm is easy to check. It induces on $Exp(K)$ a topology which is strictly finer than the one induced by $H(\mathbb{C}^n)$, the space of entire functions. Also observe that, if $K'_j \le K_j$, $j = 1,\dots,n$, we have $Exp(K') \subset Exp(K)$ (the two spaces are regarded here, as subsets of $H(\mathbb{C}^n)$).

Notation 2.3. *Let $K_1,\dots,K_n$ be n numbers, $0 < K_j \le +\infty$ for $j = 1,\dots,n$. We denote by $\mathring{E}xp(K)$ the union of the spaces $Exp(K')$ for all $K' = (K'_1,\dots,K'_n)$ such that $K'_1 < K_1,\dots,K'_n < K_n$.*

We shall not put any topology on the vector space $\mathring{\mathrm{E}}\mathrm{xp}(K)$. Our main result is then the following:

Theorem 2.5.2 Let $K_1,\dots,K_n$ be n positive numbers, some or all of which may be infinite. Given any function $f \in \mathring{\mathrm{E}}\mathrm{xp}(K)$, the linear functional on the space $\mathscr{P}_n$ of polynomials in n indeterminates with complex coefficients (viewed as polynomials functions on $\mathbb{C}^n$, i.e., polynomials in $z_1,\dots,z_n$),

$$P = \sum_p p_p z^p \mapsto \langle f, P\rangle = \sum_p p_p f^{(p)}(0), \tag{2.96}$$

can be extended, in a unique way, as a continuous linear functional on $H(\Delta(K))$, i.e., as an analytic functional in the open polydisk $\Delta(K)$, μ_f. Furthermore, the mapping $f \mapsto \mu_f$ is an isomorphism (for the structures of vector spaces) of $\mathring{\mathrm{E}}\mathrm{xp}(K)$ onto the dual of $H(\Delta(K))$, $H'(\Delta(K))$. The inverse mapping is given by the formula

$$f(\zeta) = \langle \mu_f, e^{\langle z,\zeta\rangle}\rangle, \quad \zeta \in \mathbb{C}^n, \quad \langle z,\zeta\rangle = z_1\zeta_1 + \cdots + z_n\zeta_n, \tag{2.97}$$

where μ_f operates on functions of $z \in \Delta(K)$.

Let now Ω be an arbitrary open subset of $\mathbb{C}^n$; if h is an entire function, the restriction of h to Ω belongs obviously to $H(\Omega)$. Given any analytic functional in Ω, we may consider its value on h, $\langle \mu, h \rangle$. Then it is evident that $h \mapsto \langle \mu, h \rangle$ is a continuous linear functional on $H(\mathbb{C}^n)$, i.e., an analytic functional in $\mathbb{C}^n$ (sometimes called simply an *analytic functional*). If μ_1, μ_2 are two analytic functionals in Ω, it may happen that $\langle \mu_1, h \rangle = \langle \mu_2, h \rangle$ for all $h \in H(\mathbb{C}^n)$, without this being true for all $h \in H(\Omega)$, i.e., without $\mu_1 = \mu_2$ being true. In view of the Hahn-Banach theorem, this will happen whenever the restriction to Ω of entire functions does not form a dense subspace of $H(\Omega)$, in other words, whenever Ω is not a Runge domain. We recall that polydisks are Runge domains. Thus, the space of analytic functionals in $\mathbb{C}^n$, $H'(\mathbb{C}^n)$, can be canonically identified with a linear subspace of $H'(\Omega)$ (disregarding now the question of the topologies) if and only if Ω is a Runge domain.

At any event, we may use the following terminology:

Definition 2.5.2 We say that an analytic functional μ in $\mathbb{C}^n$ is *carried by an open set* $\Omega \subset \mathbb{C}^n$ if there is a relatively compact open subset U of Ω and a constant $C > 0$ such that, for all entire functions h in $\mathbb{C}^n$,

$$|\langle \mu, h \rangle| \le C \sup_{z \in U} |h(z)|. \tag{2.98}$$

In other words, μ is carried by Ω if the linear form $h \mapsto \langle \mu, h \rangle$ defined on the restriction to Ω of the entire functions can be extended as a continuous linear form to the whole of $H(\Omega)$. Furthermore, this extension of μ is unique if and only if Ω is a Runge domain.

Definition 2.5.3 Let μ be an analytic functional in $\mathbb{C}^n$; the function of $\zeta \in \mathbb{C}^n$,

$$\langle \mu, e^{\langle z, \zeta \rangle} \rangle, \tag{2.99}$$

will be called the *Fourier-Borel transform* of μ and denoted by $\hat{\mu}$. Some authors call $\hat{\mu}$ the *Fourier-Laplace transform* of μ.

With these definitions, we may restate a theorem in the following way:

Theorem 2.5.3 The Fourier-Borel transformation is a linear isomorphism of the space of analytic functionals in $\mathbb{C}^n$ onto the space of entire functions of exponential type in $\mathbb{C}^n$.

For every n-tuple of numbers $K_1, \dots, K_n$ such that $0 < K_j \le +\infty$ ($j = 1, \dots, n$), the analytic functional μ is carried by the open polydisk

$$\{z \in \mathbb{C}^n : |z_1| < K_1, \dots, |z_n| < K_n\} \tag{2.100}$$

if and only if there are positive numbers A, ε such that, for all $\zeta \in \mathbb{C}^n$,

$$|\hat{\mu}(\zeta)| \le A \exp((K_1 - \varepsilon)|\zeta_1| + \cdots + (K_n - \varepsilon)|\zeta_n|). \tag{2.101}$$

2.6 Transpose of a Continuous Linear Map

Let E,F be two TVS, and u a continuous linear map of E into F. Let y' be a continuous linear form on F, which we may regard as a continuous linear map of F into $\mathbb{C}$. We are in the situation described by the sequence

$$E \xrightarrow{u} F \xrightarrow{y'} \mathbb{C}. \tag{2.102}$$

We may form the compose $y' \circ u$, which is a continuous linear form of E into $\mathbb{C}$, that is to say a continuous linear form on E. Thus we end up with a mapping

$$y' \mapsto y' \circ u \tag{2.103}$$

of the dual F' of F into the dual E' of E. This mapping is called the **transpose** of u, and will always be denoted by ${}^t u$ in this book. If x is an element of E, by using the brackets for expressing the duality between E and E', F and F', respectively, we see that

$$(y' \circ u)(x) = \langle y', u(x) \rangle. \tag{2.104}$$

As $y' \circ u$ is defined to be ${}^t u(y')$, we have the **transposition formula**:

$$\langle y', u(x) \rangle = \langle {}^t u(y'), x \rangle. \tag{2.105}$$

The notion of transpose of a continuous linear map plays a central role in what follows. The reason for this is that important properties of the mapping u itself can be translated, under favorable circumstances, into properties of its transpose. As an example, let us mention the following property: we assume now, as we shall do from now on, that E and F are locally convex (so that we can apply the Hahn-Banach theorem); then the image of u is dense, i.e., $u(E)$ is dense in F, if and only if ${}^t u : F' \to E'$ is one-to-one. Another reason for the importance of the notion of transpose lies in the fact that it enables us to extend the basic operations of analysis (differentiation, multiplication by functions, regularizing convolutions, Fourier transformation, etc.) to the new objects which have been introduced by taking into consideration the duals of the spaces of functions.

For instance, as immediately seen, the multiplication by a given $\mathscr{C}^\infty$ function ψ defines a continuous linear map of $\mathscr{C}^\infty$ into itself; therefore, by transposition, it defines a continuous linear map of the space of distributions $\mathscr{D}'$ into itself, which may be taken as definition of the multiplication of distributions by the function ψ. In the last example, it can be seen that, when the distribution to be multiplied by ψ is a locally integrable function f, its product by ψ (defined by transposition) is equal to the ordinary product ψf. This means that we have indeed extended the operation of multiplication from functions to distributions. A similar procedure is followedwith a twistwhen differentiation of distributions is defined. Another important example is Fourier transformation; it is easy to check that it is an isomorphism of the space $\mathscr{S}$ of rapidly decreasing $\mathscr{C}^\infty$ functions onto itself; its transpose is then an isomorphism of the dual $\mathscr{S}'$ of $\mathscr{S}$ onto itself. This transpose can then be taken as a definition of the Fourier transformation in $\mathscr{S}'$; on the other hand, $\mathscr{S}'$ can be regarded, in a canonical way, as a vector space of distributions, this is to say, as a linear subspace of $\mathscr{D}'$. We will have thus extended Fourier transformation to a class of distributions (it will be shown that this definition coincides with known ones in the cases where a classical theory of Fourier transformation already exists, for instance when the distributions are L^2 functions). These are only few examples among many which bear witness to the importance of the notion of transpose. We shall study them, and several more, soon, after a few general considerations about transposes.

We begin by a few remarks which do not involve any topology on the dual.

Proposition 2.6.1 If $u : E \to F$ is an isomorphism of E onto F (for the TVS structures), then the transpose of u, ${}^t u : F' \to E'$ is an isomorphism (for the vector space structures) of F' onto E'.

Indeed, let v be the inverse of u; $v : F \to E$. The transpose of u and the one of v are inverse of each other (this means ${}^t v \circ {}^t u =$ identity of F', ${}^t u \circ {}^t v =$ identity of E'). But a map has an inverse (in the sense just explained) if and only if it is one-to-one and onto.

Proposition 2.6.2 Let E, F, and $u : E \to F$ a continuous linear map. Then we have

$$\mathrm{Ker}\, {}^t u = (\mathrm{Im}\, u)^0. \tag{2.106}$$