Afternoon

Properties of CFL

- Pumping lemma for CFL
- Closure properties

Pumping lemma for CFL

$R = \{ S \to 0B,\ B \to 1 \mid 0BC,\ C \to 1 \}$ $\qquad L = \{ 0^n 1^n \mid n \geq 0 \}$

Pumping lemma for CFL

Let L be a CFL . Then there exists some positive integer n such that any $w \in L$ with $|w| \geq n$ can be decomposed as

$$w=uvxyz$$

with

$$|vxy| \leq n$$

and

$$|vy| \geq 1$$

such that

$$uv^ixy^iz \in L$$

for all i=0,1,2,......

Proof

L is a CFL $\Rightarrow$ There is a CFG $G=(V,T,R,S)$ generating L.

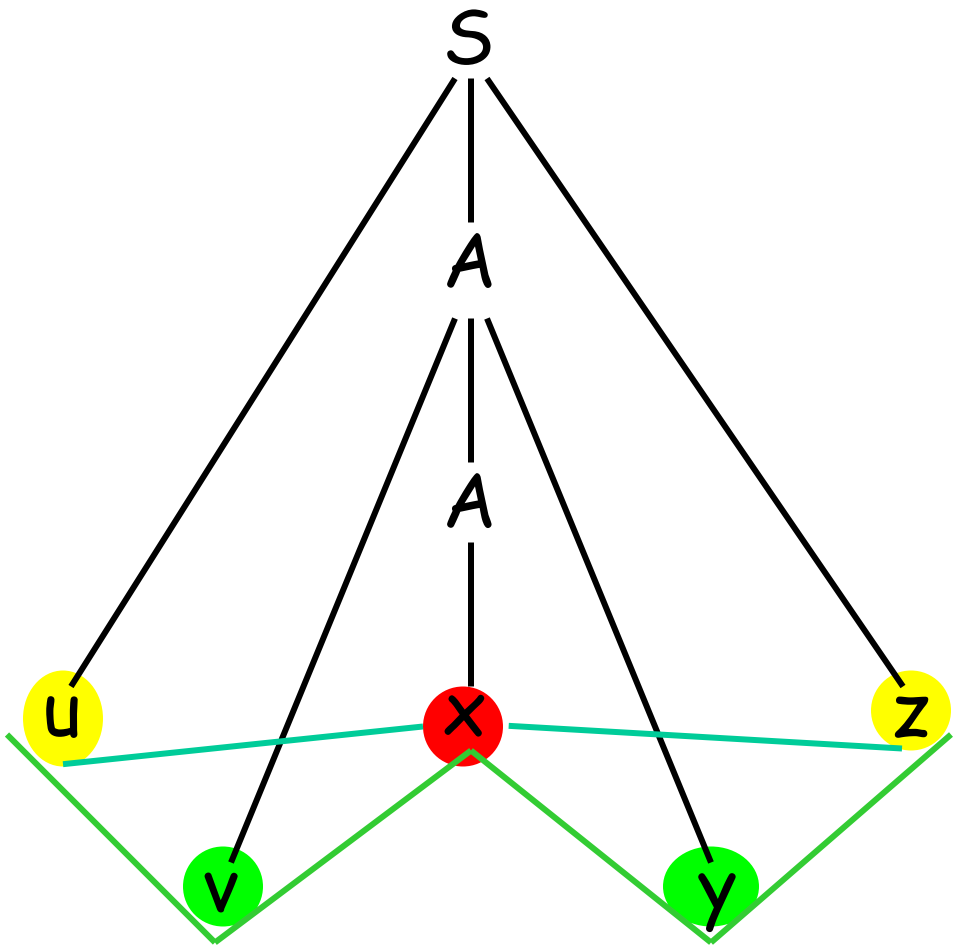
V is finite $\Rightarrow$ $m=|V|$

$|\alpha|$ is finite for all $A\rightarrow\alpha$ $\Rightarrow$ $k=\max\{|\alpha| \text{ for all } A\rightarrow\alpha\}$

Let $n=k^m$

For any $w\in L$ with $|w|\geq n$, there must be some variable A that appears at least two times in the parse tree.

That is : $S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} w$

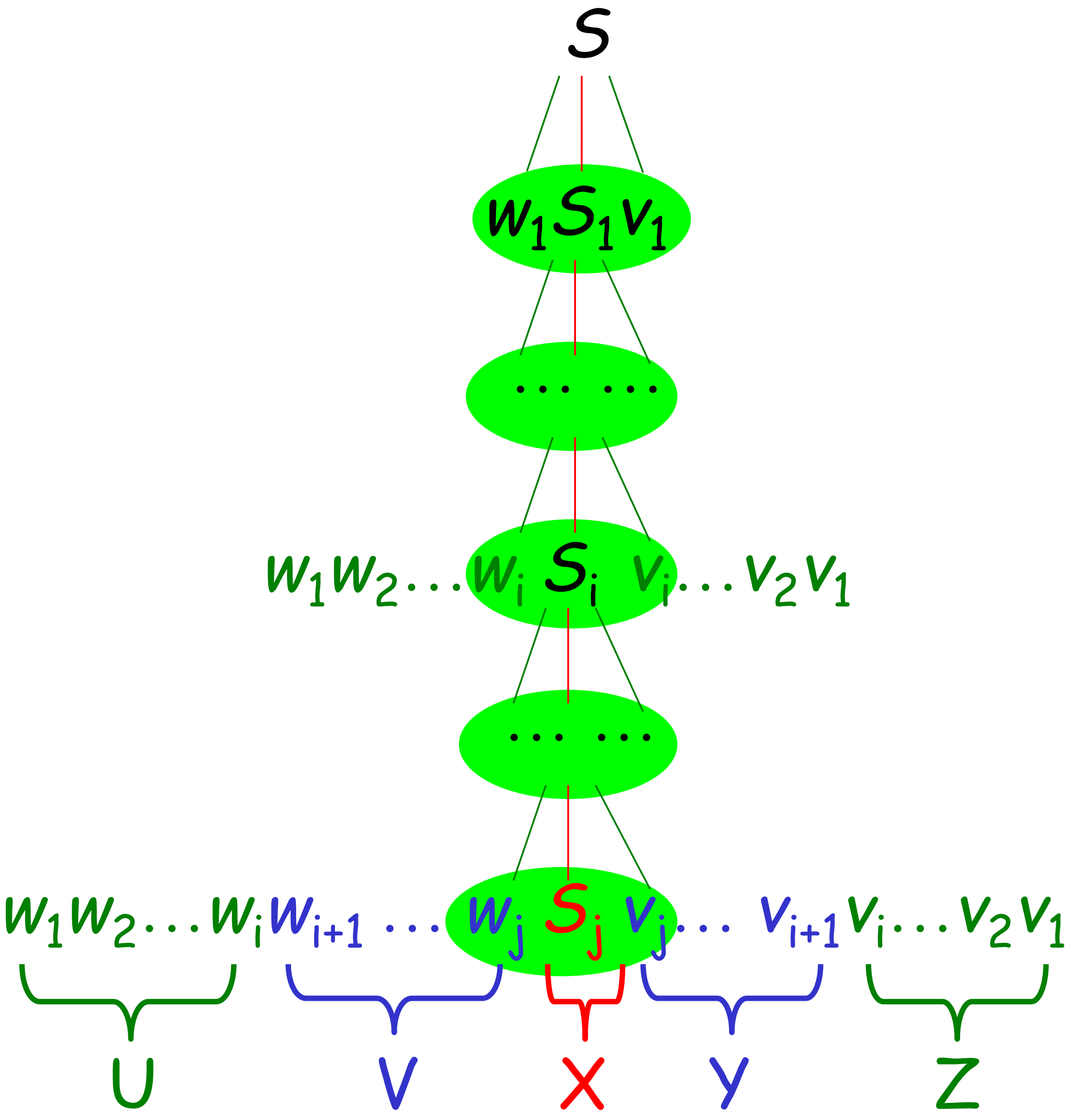


$$S \stackrel{*}{\Rightarrow} uAz \stackrel{*}{\Rightarrow} uvAyz \stackrel{*}{\Rightarrow} w$$

$$
\begin{aligned}
S &\overset{*}{\Longrightarrow} w_1 S_1 v_1 \\
&\overset{*}{\Longrightarrow} w_1 w_2 S_2 v_2 v_1 \\
&\overset{*}{\Longrightarrow} w_1 w_2 w_3 S_3 v_3 v_2 v_1 \\
&\overset{*}{\Longrightarrow} \ldots \ \ldots \\
&\overset{*}{\Longrightarrow} w_1 w_2 \ldots w_m S_m v_m \ldots v_2 v_1
\end{aligned}
$$

where

$$w_1, w_2, \ldots, w_m,\ v_1, v_2, \ldots, v_m \in T^*,\ S_1, S_2, \ldots, S_m \in V$$


S
w1S1v1
... ...
w1w2...wi Si vi...v2v1
... ...
w1w2...wiwi+1 ... wj Sj vj... vi+1vi...v2v1
U
V
X
Y
Z

$$S_i = S_j$$

Example 1 Show L is not CFL

$$L = \{ a^n b^n c^n \mid n \geq 0 \}$$

Example 2 Show L is not CFL

$$L = \{ ww \mid w \in \{0,1\}^* \}$$

Example 2 Show L is not CFL

$$L = \{ 0^n 1^m \mid n=m^2 \}$$

Closure properties

- union : $L \cup M$
- concatenation
- closure(star)
- reversal
- intersection : $L \cap M$
- complement
- difference : $L - M$
- homomorphism
- inverse homomorphism

Closure properties

- Union

  If L_1 and L_2 are CFL , then so is $L_1 \cup L_2$.

Proof

Let $G(L_1)=(V_1,T_1,R_1,S_1)$, $G(L_2)=(V_2,T_2,R_2,S_2)$

Then $G(L_1 \cup L_2)=(V_1 \cup V_2 \cup \{ S \}, T_1 \cup T_2, R, S)$

$R= \{S \to S_1 \mid S_2\} \cup R_1 \cup R_2$

Closure properties

- Concatenation

  If L_1 and L_2 are CFL , then so is $L_1 L_2$.

Proof

Let $G(L_1)=(V_1,T_1,R_1,S_1)$, $G(L_2)=(V_2,T_2,R_2,S_2)$

Then $G(L_1\ L_2)=(V_1 \cup V_2 \cup \{ S \}, T_1 \cup T_2, R, S)$

$$R = \{S \to S_1\ S_2\} \cup R_1 \cup R_2$$

Closure properties

- Star

If L is a CFL , then so is L^* .

Proof

Let $G(L)=(V,T,R,S)$

Then $G(L^*)=(V,T,\{S \rightarrow SS|\varepsilon\} \cup R,S)$

Closure properties

- Reversal

If L is a CFL , then so is L^R .

Proof

Let $G(L)=(V,T,R,S)$

Then $G(L^R)=(V,T, \{A \to \alpha^R | A \to \alpha \in R\},S)$

Closure properties

- Intersection

CFL is not closed under intersection .

Proof

$$L_1 = \{ a^n b^n c^m \mid n \geq 0, m \geq 0 \}$$

$$L_2 = \{ a^n b^m c^m \mid n \geq 0, m \geq 0 \}$$

$$L_1 \cap L_2 = \{ a^n b^n c^n \mid n \geq 0 \}$$

- Intersection

  If L_1 is a CFL and L_2 is a RL , then $L_1 \cap L_2$ is CFL.

Proof

$$P(L_1) = (Q_1, \Sigma_1, \Gamma, \delta_1, q_1, z_0, F_1)$$

$$A(L_2) = (Q_2, \Sigma_2, \delta_2, q_2, F_2)$$

$$P(L_1 \cap L_2) = (Q_1 \times Q_2, \Sigma_1 \cap \Sigma_2, \Gamma, \delta, (q_1, q_2), z_0, F_1 \times F_2)$$

$$\delta((q,p),a,X)=((r,s), \alpha)$$

where $\quad \delta_1(q,a,X)=(r,\alpha)$, $\delta_2(p,a)=s$

Example 4 Show that the language

$$L = \{ 0^n 1^n \mid n \geq 0, n \neq 100 \}$$

is context-free.

Example 5 Show that the language

$$L = \{ w \mid w \in \{a,b,c\}^*, n_a(w)=n_b(w)=n_c(w) \}$$

is not context-free.

MATHEMATICAL PROOFS

A good proof should be:

Clear -- easy to understand

Correct

Here's an example.

Suppose $A \subseteq \{1, 2, \ldots, 2n\}$ with $|A| = n+1$

TRUE or FALSE?
There are always two numbers in A such that one number divides the other number

TRUE

Example: $A \subseteq \{1, 2, 3, 4\}$
1 divides every number.
If 1 isn't in A then $A = \{2,3,4\}$, and 2 divides 4

In writing mathematical proofs, it can be very helpful to provide three levels of detail

- The first level: a short phrase/sentence giving a 'hint' of the proof

(e.g. "Proof by contradiction," "Proof by induction," "Follows from the pigeonhole principle")

- The second level: a short, one paragraph description of the main ideas

- The third level: the full proof (and nothing but)

Level 1

Hint 1 :

THE PIGEONHOLE PRINCIPLE

If you drop n+1 pigeons in n holes then at least one hole will have more than one pigeon

Hint 2 :

Every integer a can be written as $a = 2^k m$, where m is an odd number (k is an integer)

Call m the "odd part" of a

Level 2

Proof Idea

Given $A \subseteq \{1, 2, \ldots, 2n\}$ with $|A| = n+1$

Using the pigeonhole principle,
we'll show there are elements $a_1 \neq a_2$ of A

such that $a_1 = 2^i m$ and $a_2 = 2^k m$
for some odd m and integers i and k

Level 2

Proof

Suppose $A \subseteq \{1, 2, \ldots, 2n\}$ with $|A| = n+1$

Write each element of A in the form $a = 2^k m$ where m is an odd number in $\{1,\ldots, 2n\}$

Observe there are n odd numbers in $\{1, \ldots, 2n\}$

Since $|A| = n+1$, there must be two distinct numbers in A with the same odd part

Let a_1 and a_2 have the same odd part m.
Then $a_1 = 2^i m$ and $a_2 = 2^k m$, so one must divide the other (e.g., if $k > i$ then a_1 divides a_2)

Good good study

day day up !