Afternoon

Properties of Regular Languages

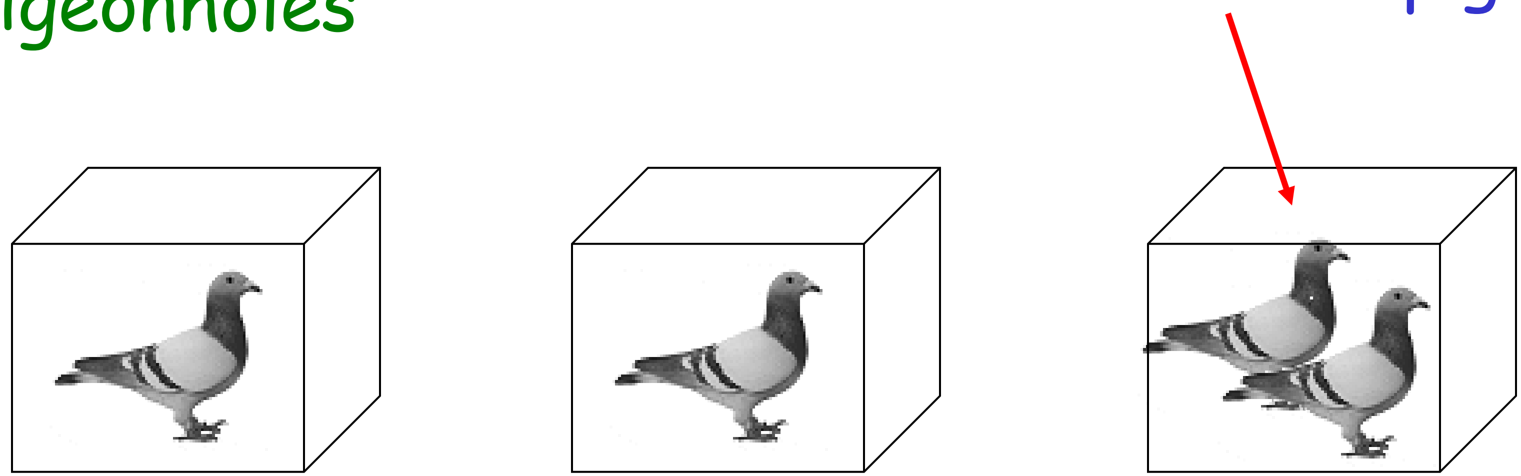
- Pumping lemma
- Closure properties

Pigeonhole Principle

4 pigeons

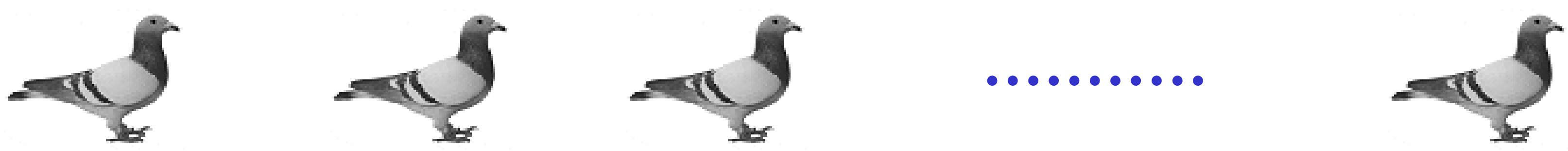
3 pigeonholes

A pigeonhole must
contains at least two pigeons



Pigeonhole Principle

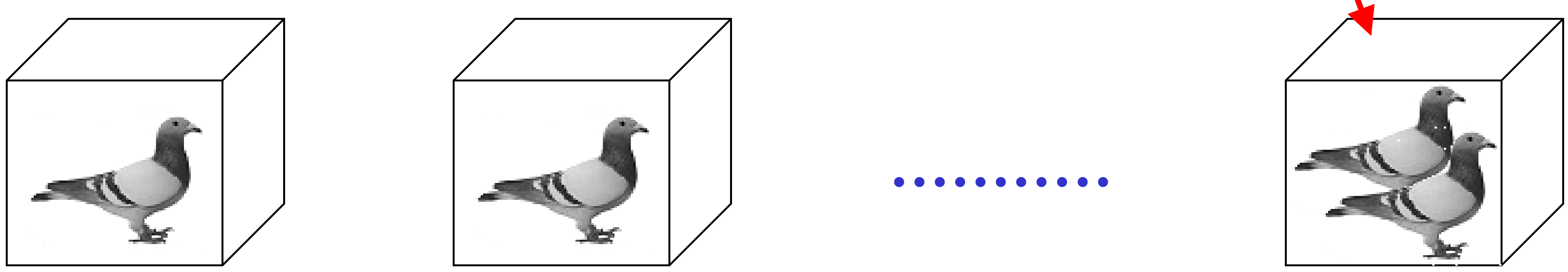
m pigeons



n pigeonholes

There is a pigeonhole with at least 2 pigeons

$$m > n$$



DFA Principle

m symbols

Let $A = (Q, \Sigma, \delta, q_0, F)$, and $n = |Q|$

$$w = a_1 a_2 \ldots \ldots a_m$$

n states

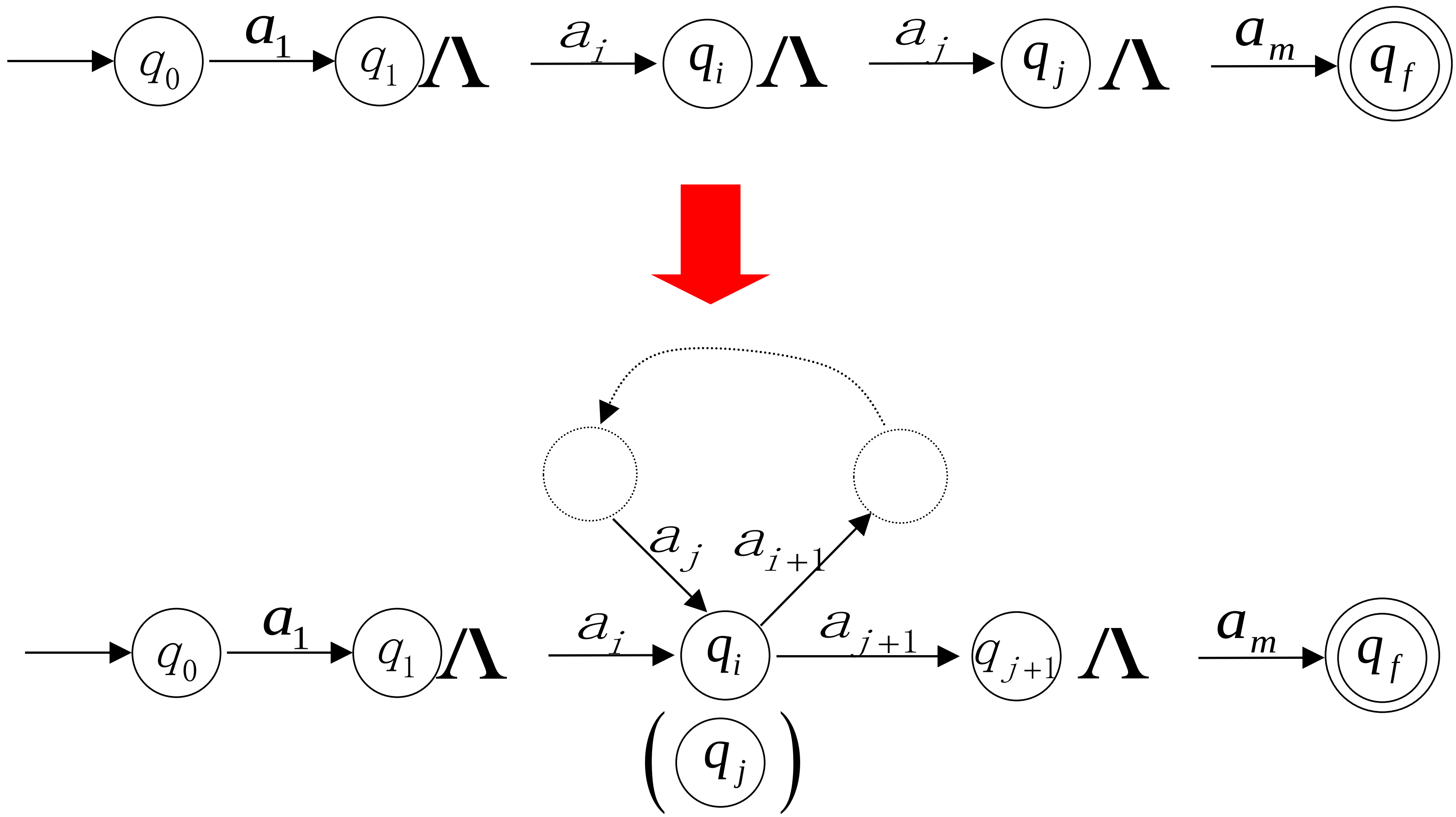
$$m \geq n \qquad a_n \ldots \ldots a_m \,?$$

$q_i : 0 \leq i \leq n-1$

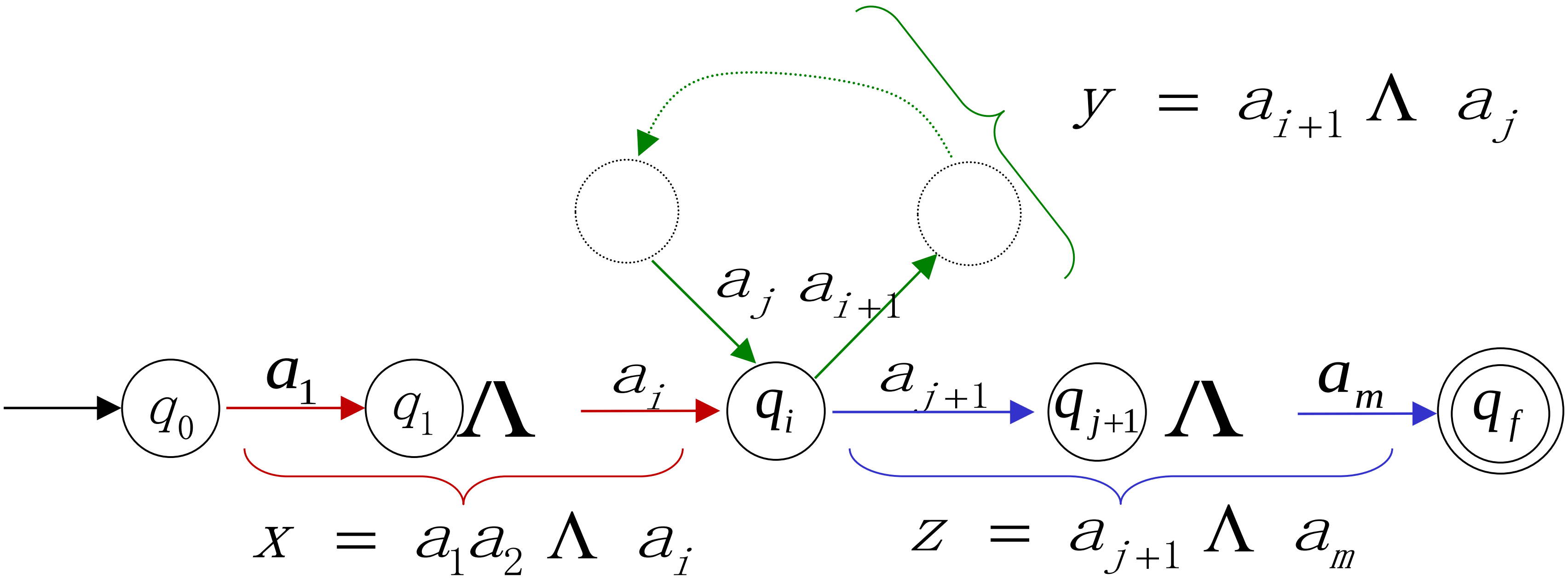
$$\to q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \ldots \xrightarrow{a_{n-2}} q_{n-2} \xrightarrow{a_{n-1}} q_{n-1} \xrightarrow{a_n} ?$$

$$\exists q_i,\ q_j \Rightarrow q_i = q_j,\ 1 \leq i < j \leq n$$

DFA Principle



DFA Principle



$$\Rightarrow w = xyz \begin{cases} |xy| \leq n \\ |y| \geq 1 \text{ or } y \neq \varepsilon \\ xy^kz \in L, \text{ for any } k \geq 0 \end{cases}$$

Pumping lemma

Pumping lemma for regular languages.

Let L be regular. Then

$\exists n, \forall w \in L : |w| \geq n \Rightarrow w=xyz$ such that

- $|xy| \leq n$
- $y \neq \varepsilon$ ($|y|$ >= 1)
- $\forall k \geq 0, xy^k z \in L$

Decidable problem

Is L a regular language ?

Yes

- DFA
- NFA
- ε-NFA
- RegExp

No

- Pumping lemma

Example 1 "NO"

Let $L = \{ 0^n 1^n \mid n \geq 0 \}$. Is L regular ?

Suppose L is regular.

By pumping lemma there exist a constant n, for every $w \in L$, where $|w| \geq n$, w can be broken into three strings, $w=xyz$, such that $|xy| \leq n$, $y \neq \varepsilon$, and $xy^k z \in L$.

Get $w=0^n 1^n \in L$. Then $w=0^n 1^n = xyz$, and $xz=0^{n-|y|} 1^n \in L$.

It derived a contradiction(y contains at least one 0)

So L is not regular.

Example 2 "NO"

$$L = \{vv^R \mid v \in (a,b)^*\}$$

Get w=$a^n b^n b^n a^n \in L$.

for k=0, xz=$a^{n-|y|} b^n b^n a^n \in L$.

Example 3 "NO"

$$L = \{ 0^{n^2} \mid n \geq 0 \}$$

Closure properties

- union : $L \cup M$
- intersection : $L \cap M$
- complement : $\bar{L}$
- difference : $L - M$
- reversal : L^R
- closure(star) : L^*
- concatenation : LM
- homomorphism
- inverse homomorphism

Closure properties

- Union : $L \cup M$

Suppose $L(A)=L$, $L(B)=M$

Let $A= (Q_1,\Sigma_1,\delta_1,q_1,F_1)$, $B= (Q_2,\Sigma_2,\delta_2,q_2,F_2)$

$$C = (Q_1 \cup Q_2 \cup \{q_0\}, \Sigma_1 \cup \Sigma_2, \delta, q_0, F_1 \cup F_2)$$

$$\delta : \delta(q_0, \varepsilon) = \{q_1, q_2\}$$

$$\delta(q, a) = \delta_1(q,a), \forall (q, a) \in Q_1 \times \Sigma_1$$

$$\delta(q, a) = \delta_2(q,a), \forall (q, a) \in Q_2 \times \Sigma_2$$

Then $L(C)= L \cup M$

Closure properties

- Reversal $L^R = \{w^R \mid w \in L\}$

Convert $A(L)$ into $A(L^R)$ by :

- Reverse all the arcs of $A(L)$
- Convert start state of $A(L)$ to accepting state of $A(LR)$
- Create a new state as start state of $A(LR)$ with ε-transitions to all the accepting states of $A(L)$

Closure properties

- Reversal $L^R = \{w^R \mid w \in L\}$

Suppose $L(A)=L$ where A is a DFA

Let $A= (Q_1, \Sigma, \delta_1, q_1, F_1)$, $B = (Q_2, \Sigma, \delta_2\ , q_0, \{q_1\})$

$Q_2 = 2^{Q1} \cup \{q_0\}$ $(q_0 \notin Q_1\ )$

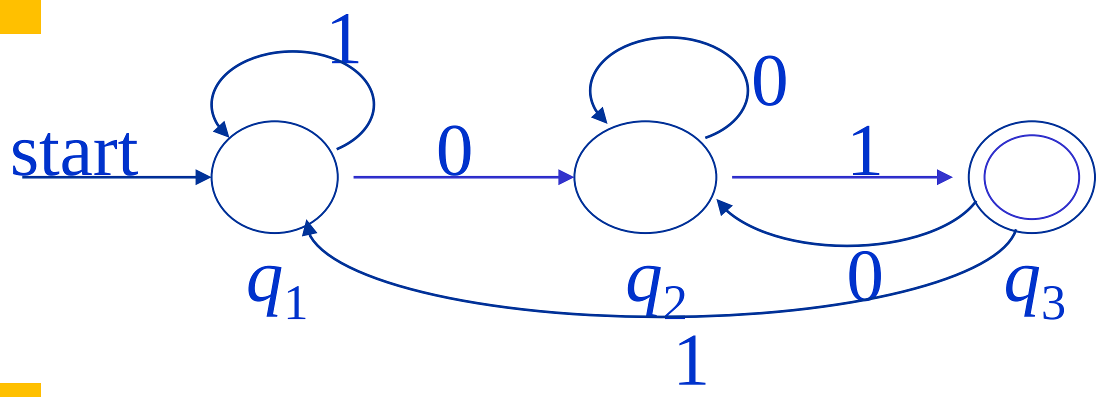
$\delta : \delta\,(q_0, \varepsilon) = \{q \mid q \in F_1)\}$

$\delta_2(q, a) = \{\ p \mid \delta_1(p, a) = q\ \}$

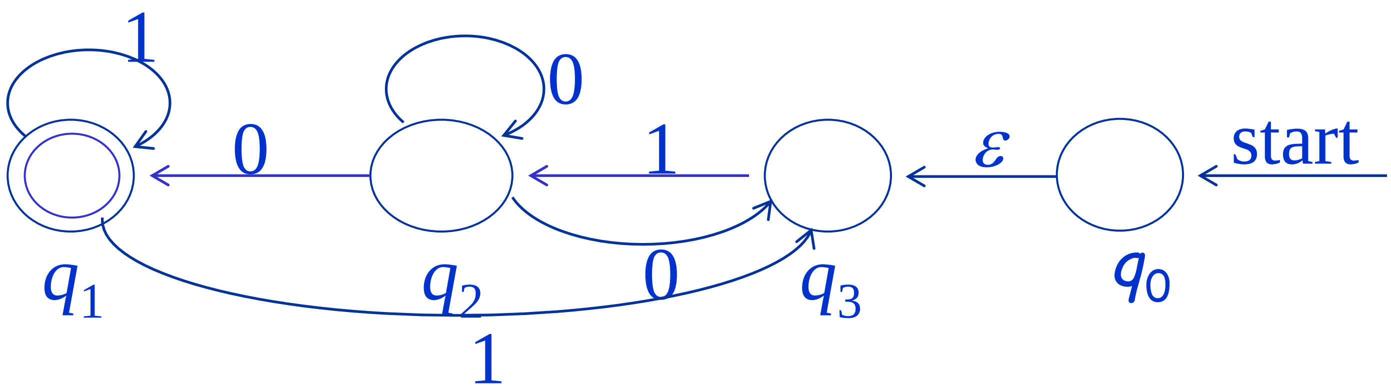
Then $L(B)= L^R$

Example 3 Convert closure

$(0+1)^*01$



$10(0+1)^*$



Closure properties

- Complement

$$\overline{L} = \{w \mid w \in \Sigma^* \text{ and } w \notin L\}$$

Let DFA $A=(Q, \Sigma, \delta, q_0, F)$, and $L(A)=L$

Let DFA $B=(Q, \Sigma, \delta, q_0, S)$, and $S=Q-F$

Then $L(B)= \overline{L}$

Closure properties

- Intersection : $L \cap M$

$$\overline{\overline{L} \cup \overline{M}}$$

Suppose $L(A)=L$, $L(B)=M$

Let $A= (Q_1,\Sigma,\delta_1,q_1,F_1)$, $B= (Q_2,\Sigma,\delta_2,q_2,F_2)$

$$C = (Q_1\times Q_2, \Sigma, \delta, (q_1, q_2), F_1\times F_2)$$

$$\delta : (Q_1\times Q_2) \times\Sigma \to Q_1\times Q_2$$

$$\delta((p, q), a) = (\delta_1(p,a), \delta_2(q,a))$$

Then $L(C)= L \cap M$

Homomorphism

$h: \quad \Sigma^* \rightarrow \Gamma^*$

Let $w = a_1 a_2 \ldots\ldots a_n \in \Sigma^*$, then

$h(w) = h(a_1)h(a_2)\ldots\ldots h(a_n)$

Let $\Sigma = \{0, 1\}$, $\Gamma = \{a, b\}$, $h(0) = ab$, $h(1) = \varepsilon$

$h(0110) = h(0)h(1)h(1)h(0) = ab\varepsilon\varepsilon ab = abab$

$h(L) = \{ h(w) \mid w \text{ is in } L \}$

Homomorphism

Regular language is closed under homomorphism.

Assume r is a *RegExp* and L=L(r).

For any symbol a of r, h(a) is a *RegExp*

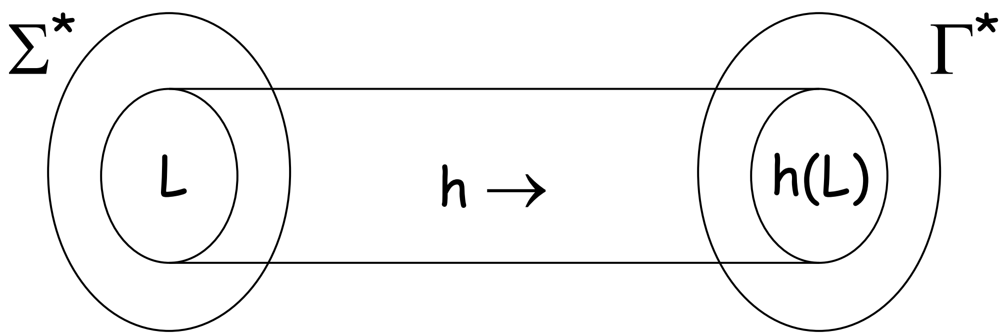
$\Rightarrow$ h(r) is *RegExp*.

$\Rightarrow$ h(L) = L(h(r)) is *RegLang*.

Inverse homomorphism

$$h: \quad \Sigma^* \to \Gamma^*$$

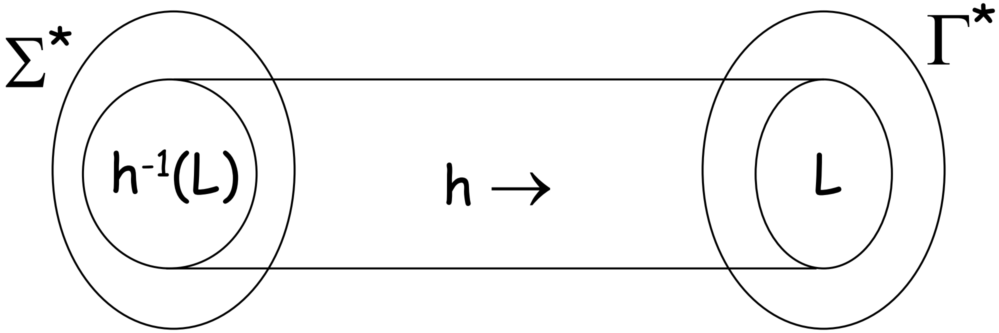
$$h^{-1}(L) = \{ w \mid h(w) \text{ is in } L \}$$



$\forall w \in L \Rightarrow h(w) \in h(L)$

$\forall v \in h(L)$

$\Rightarrow \exists w \in L : h(w) = v$



$\forall w \in h^{-1}(L) \Rightarrow h(w) \in L$

$\forall v \in L \Rightarrow$

$\exists w \in h^{-1}(L) : h(w) = v$?

Example 4 Inverse

Let Σ={ a, b }, Γ={ 0, 1 }, h(a)=01, h(b)=10

Let L = {00,1}* then $h^{-1}(L)$ = ?

L ={ ε,1,00,11,100,001,0000,111,1100,1001, 0011,

10000,00100,00001,000000,1111,11100,11001,... }

h({a,b}*) = {01,10}*

={ε, 01,10,0101,0110,1001,1010,010101,010110,

011001,100101,011010,100110,101001,101010,... }

Example 4

Let Σ={ a, b }, Γ={ 0, 1 }, h(a)=01, h(b)=10

Let L = {00,1}* then h^{-1}(L) = ?

$\{ 00,1 \}^* \cap \{ 01,10 \}^*$

$=\{\varepsilon, 1001, 10011001, 100110011001, \ldots \}$

h(aa)=0101, h(ab)=0110, h(ba)=1001, h(bb)=1010

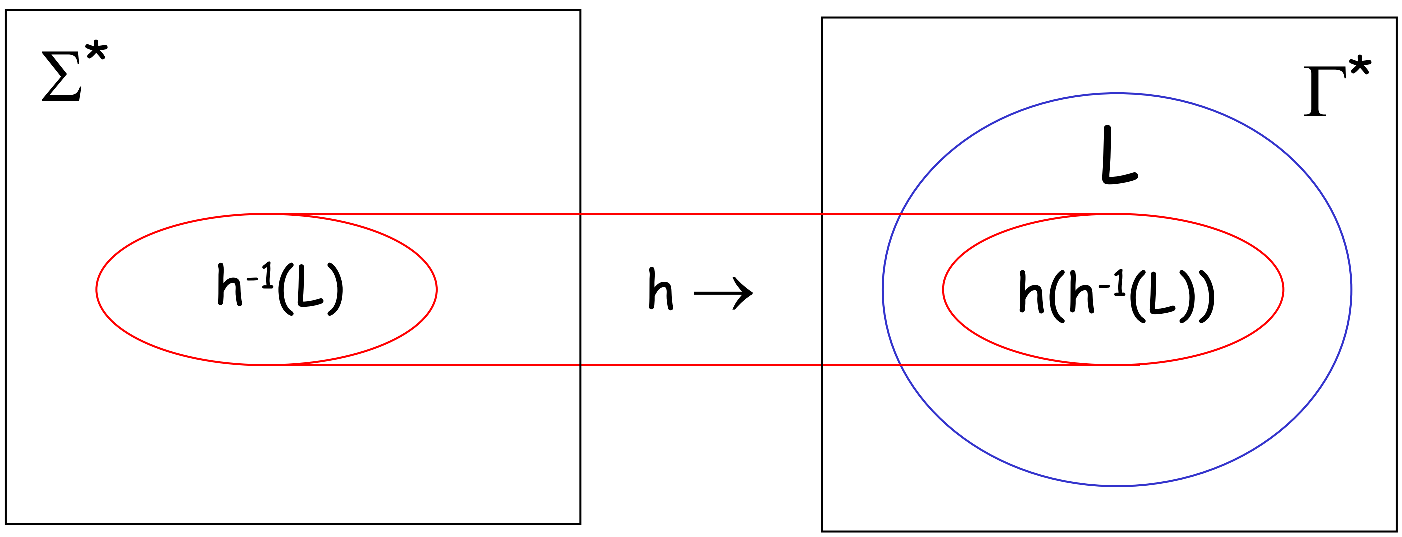
$h(\{ba\}^*) = \{1001\}^* \subset \{ 00,1 \}^*$

$\Rightarrow h^{-1}(L) = \{ba\}^* = \{ w \mid h(w) \in L \}$

Example 4

Let Σ={ a, b }, Γ={ 0, 1 }, h(a)=01, h(b)=10

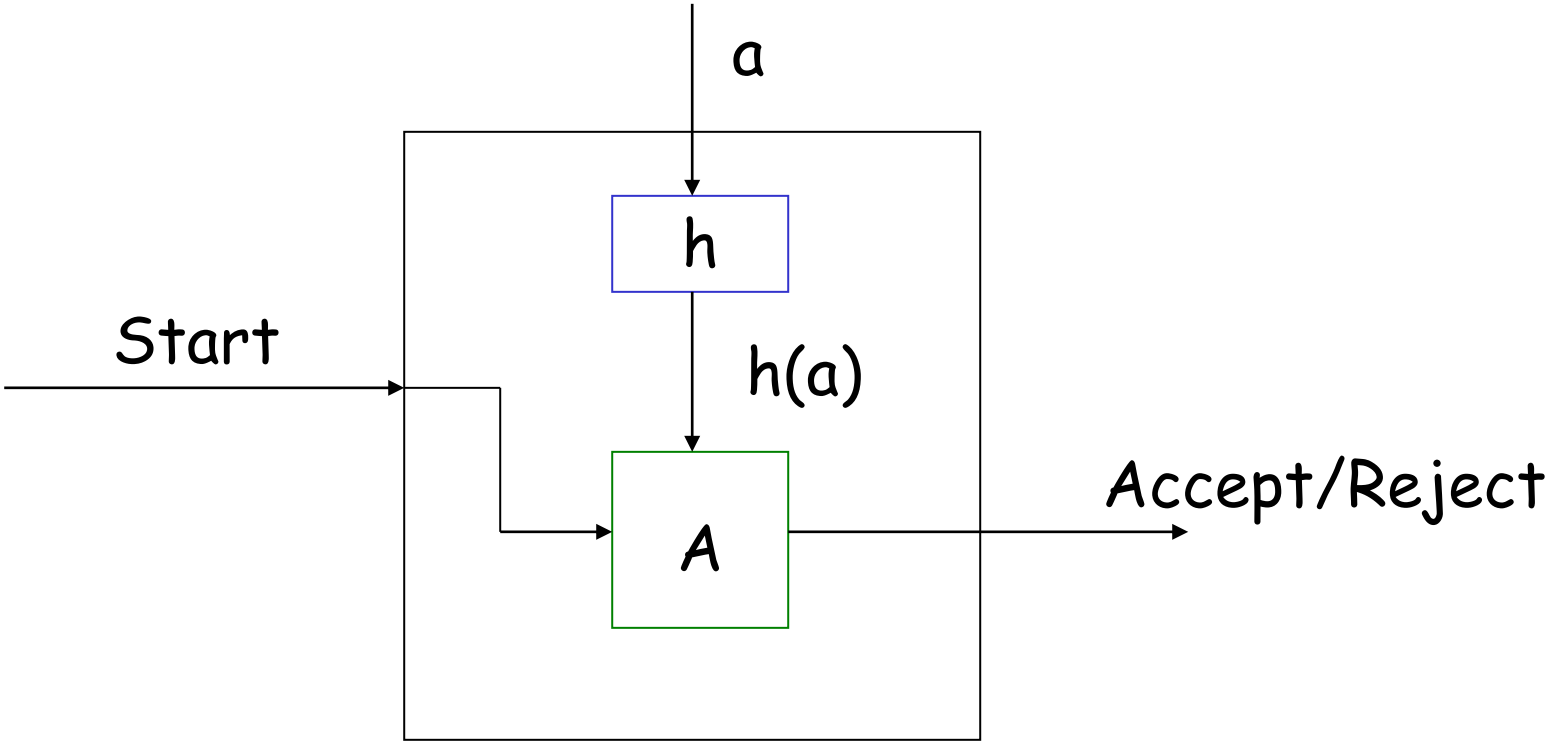
Let $L = \{00,1\}^*$ then $h^{-1}(L) = \{ba\}^*$



$h^{-1}(L)=\{ba\}^*$, $h(h^{-1}(L))=\{1001\}^* \subset L=\{00,1\}^*$

Inverse homomorphism

RegLang is closed under inverse homomorphism.



$$A = (Q, T, \delta, q_0, F), \quad B = (Q, \Sigma, \gamma, q_0, F)$$

$$where \;\; \gamma(q, a) = \hat{\delta}(q, h(a))$$

$$1 \times 8 + 1 = 9$$
$$12 \times 8 + 2 = 98$$
$$123 \times 8 + 3 = 987$$
$$1234 \times 8 + 4 = 9876$$
$$12345 \times 8 + 5 = 98765$$
$$123456 \times 8 + 6 = 987654$$
$$1234567 \times 8 + 7 = 9876543$$
$$12345678 \times 8 + 8 = 98765432$$
$$123456789 \times 8 + 9 = 987654321$$

@壹小寒

Good good study

day day up !