Greedy Algorithms

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Greedy Algorithms

- Optimization problems
  - Dynamic programming, but overkill sometime.
  - Greedy algorithm: being greedy for local optimization with the hope it will lead to a global optimal solution, not always, but in many situations, it works.
- Elements of greedy strategy
  - Determine the optimal substructure
  - Develop the recursive solution
  - Prove one of the optimal choices is the greedy choice yet safe
  - Show that all but one of subproblems are empty after greedy choice
  - Develop a recursive algorithm that implements the greedy strategy
  - Convert the recursive algorithm to an iterative one.

Typical tradition problems with greedy solutions

- Coin changes
  - 25, 10, 5, 1
  - How about 7, 5, 1
- Minimum Spanning Tree
  - Prims algorithm
    - Begin from any node, each time add a new node which is closest to the existing subtree.
  - Kruskals algorithm
    - Sorting the edges by their weights
    - Each time, add the next edge which will not create cycle after added.
- Single source shortest paths: Dijkstra's algorithm
- Huffman coding
- Optimal merge

Applications

- Greedy algorithms for NP-complete problems. For example: greedy coloring for the graph coloring problem.
  - do not consistently find optimum solutions, because they usually do not operate exhaustively on all the data
  - useful because they are quick to think up and often give good approximations to the optimum.
- The theory of matroids, and the more general theory of greedoids, provide whole classes of such algorithms.
- In network routing, using greedy routing, a message is forwarded to the neighboring node which is “closest” to the destination.

Coin Changing

Goal

Given currency denominations: 1,5,10,25,100, devise a method to pay amount to customer using fewest number of coins.

25¢ 5¢ 1¢ 1¢ 1¢ 1¢

Figure: Change 34¢

Coin Changing: Algorithm

Cashier’s algorithm

At each iteration, add coin of the largest value that does not take us past the amount to be paid.

1$ 25¢ 5¢ 1¢

Figure: Change 2.89$

Coin Changing: Algorithm

At each iteration, add coin of the largest value that does not take us past the amount to be paid.

CASHIERS-ALGORITHMS($x, c_1, c_2, \ldots, c_n$)

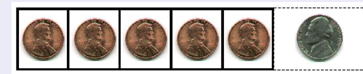
```
SORT n coin denominations so that c1 < c2 < ... < cn
S ← ∅    % set of coins selected
while x > 0
    k ← largest coin denomination ck such that ck ≤ x
    if no such k, return "no solution"
    else
          x ← x − ck
          S ← S ∪ {k}
return S
```

Question

Is cashier's algorithm optimal?

Coin Changing: Properties of optimal solution

Property



Number of pennies ≤ 4.

Proof. Replace 5 pennies with 1 nickel.

Property

Number of nickels ≤ 1.

Property

Number of quarters ≤ 3.

Coin Changing: Properties of optimal solution

Property

Number of nickels + number of dimes ≤ 2.

Proof:

- Replace 3 dimes and 0 nickels with 1 quarter and 1 nickel;
- Replace 2 dimes and 1 nickel with 1 quarter;
- Recall: at most 1 nickel.

k	c_k	All optimal solutions must satisfy	Max value of coins $1, 2, \ldots, k-1$ in any OPT
1	1	$P \leq 4$	-
2	5	$N \leq 1$	4
3	10	$N + D \leq 2$	4+5=9
4	25	$Q \leq 3$	20+4=24
5	100	no limit	75+24=99

Coin Changing: Analysis of Greedy Algorithm

Theorem

Greedy is optimal for U.S. coinage: 1, 5, 10, 25, 100.

Proof: (by induction on x)

- Consider optimal way to change $ck \leq x < c_{k+1}$: greedy takes coin k.
- We claim that any optimal solution must also take coin k.
  - if not, it needs enough coins of type $c_1, \ldots, c_{k-1}$ to add up to x
  - table below indicates no optimal solution can do this
- Problem reduces to coin-changing $x - c_k$ cents, which, by induction, is optimally solved by greedy algorithm.

Coin Changing: Analysis of Greedy Algorithm

Q. Is cashier's algorithm for any set of denominations?

Answer:

- NO. Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500. Counterexample: 140¢
  - Greedy: 140=100+34+1+1+1+1+1+1.
  - Optimal: 140=70+70.
- NO. It may not even lead to a feasible solution if $c_1 > 1$: 7, 8, 9. textcolorblueCounterexample: 15¢
  - Greedy: 15=9+???.
  - Optimal: 15=7+8.

Practicing Problems

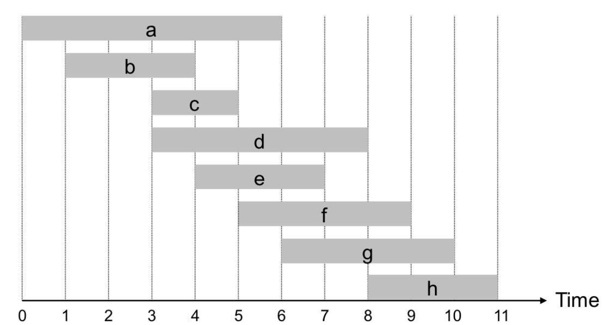
Money Changing

ATM Withdrawal

Interval Scheduling

Description

- Job j starts at s_j and finishes at f_j.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



Interval Scheduling: Greedy Algorithm

Greedy template

Consider jobs in some natural order. Take each job provided it's compatible with the ones already taken.

- [Earliest start time] Consider jobs in ascending order of start time s_j.
- [Earliest finish time] Consider jobs in ascending order of finish time f_j.
- [Shortest interval] Consider jobs in ascending order of interval length $f_j - s_j$.
- [Fewest conflicts] For each job, count the number of conflicting jobs c_j. Schedule in ascending order of conflicts c_j.

Interval Scheduling: Greedy Algorithm

Greedy template

Consider jobs in some natural order. Take each job provided it's compatible with the ones already taken.

Interval Scheduling: earliest-finish-time-first algorithm

EARLIEST-FINISH-TIME-FIRST$(n, s_1, \ldots, s_n, f_1, \ldots, f_n)$

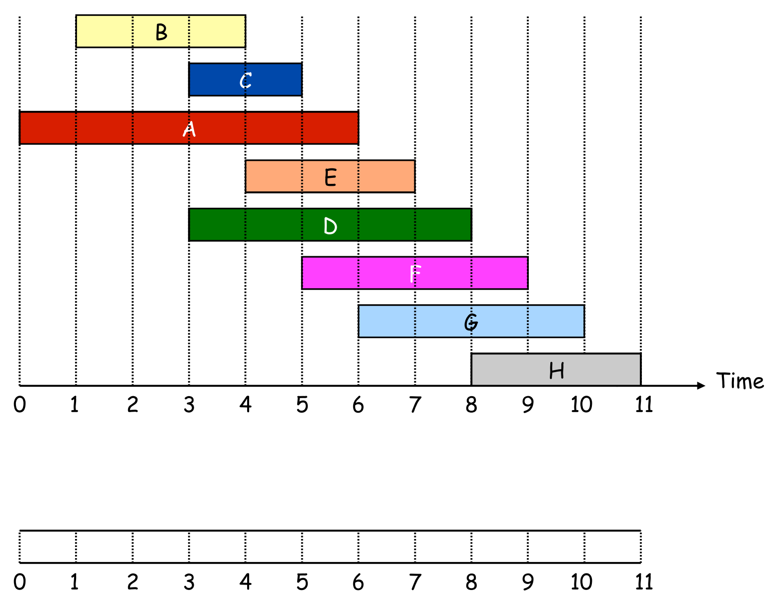
```
SORT jobs by finish time so that f_1 ≤ ... ≤ f_n
A ← ∅    % set of jobs selected
for j=1 to n
    if job j is compatible with A
        A ← A ∪ {j}
return A
```

Proposition

Can implement earliest-finish-time first in $O(n \log n)$ time.

- Keep track of job j* that was added last to A.
- Job j is compatible with A iff $s_j \geq f_{j^*}$.
- Sorting by finish time takes $O(n \log n)$ time.

Interval Scheduling Demo



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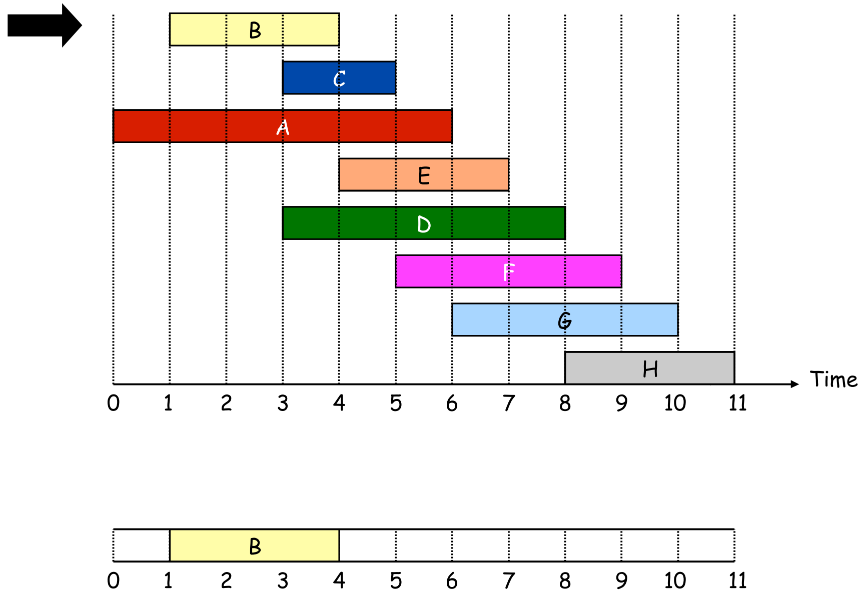

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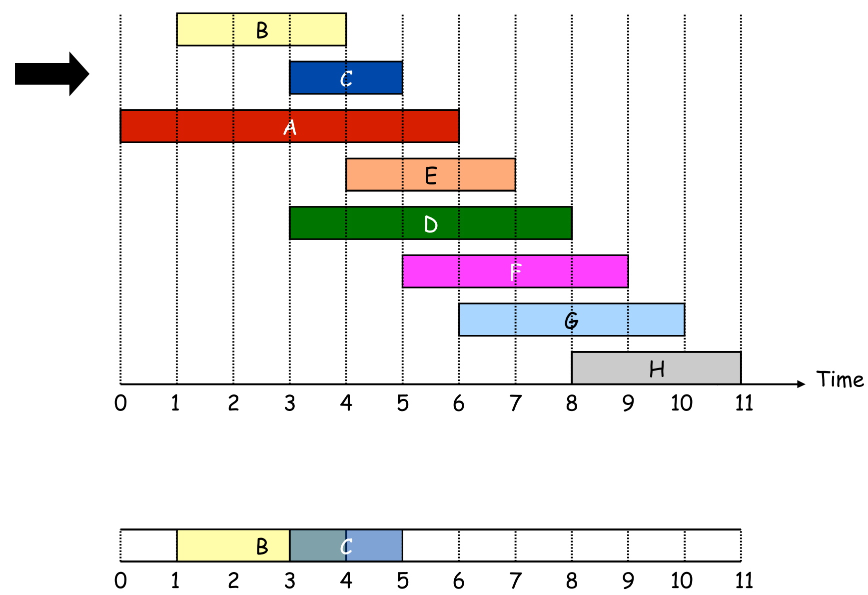

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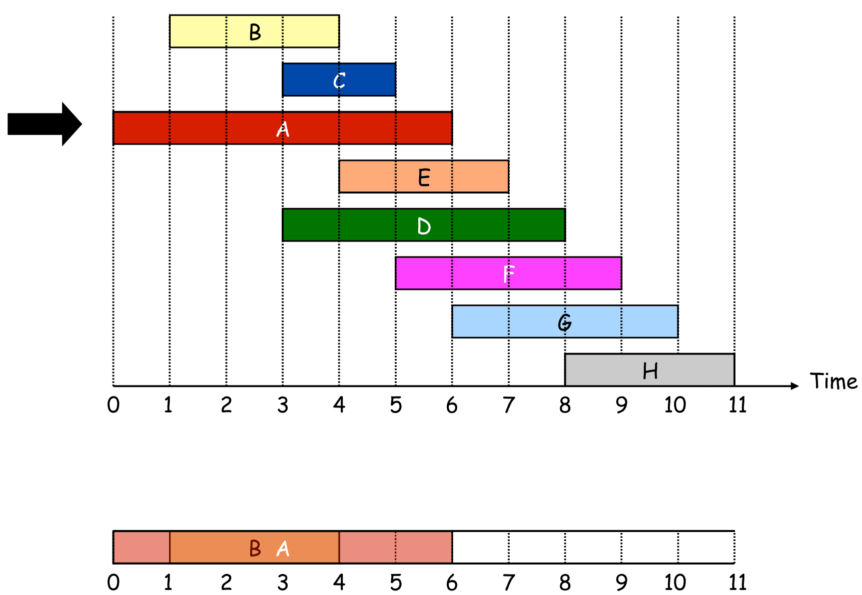

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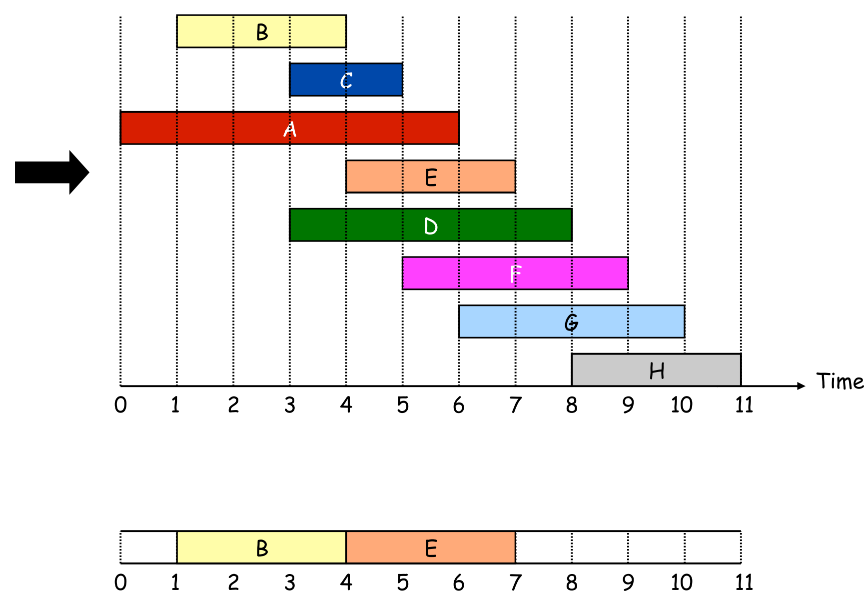

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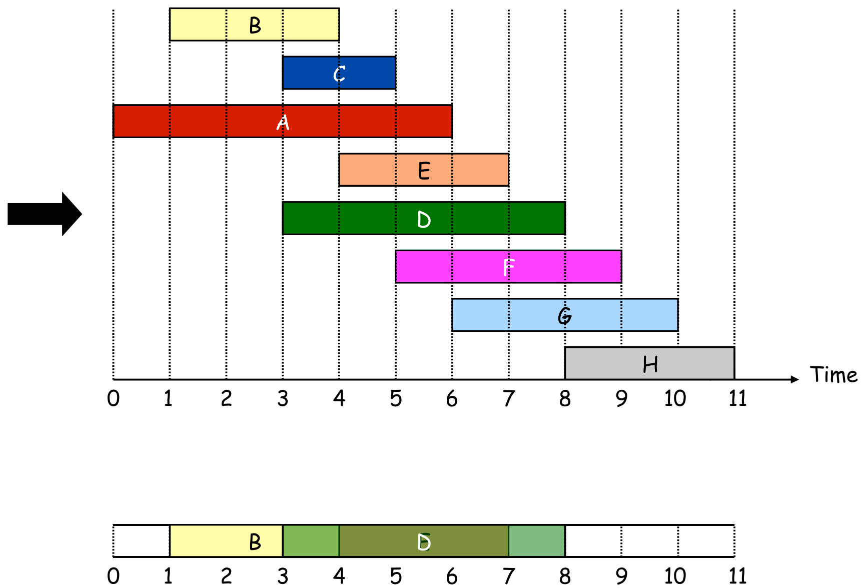

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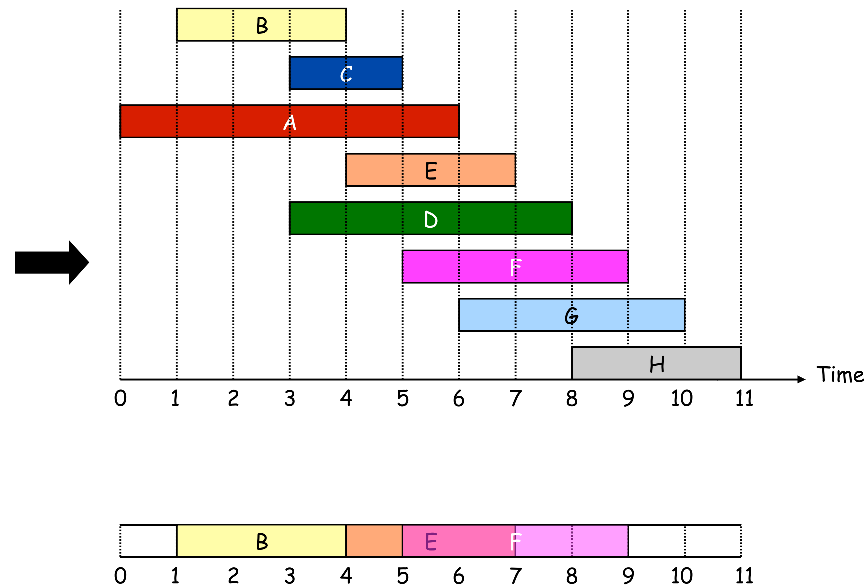

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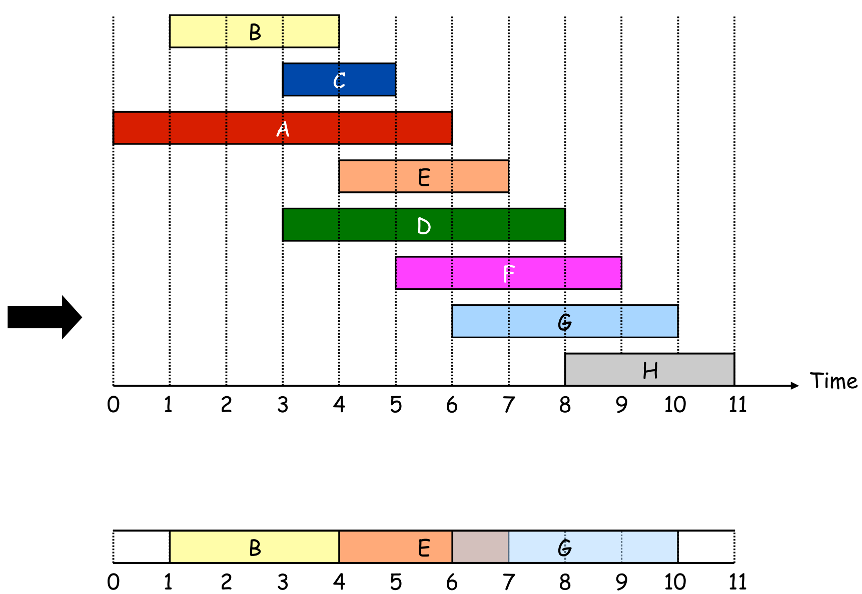

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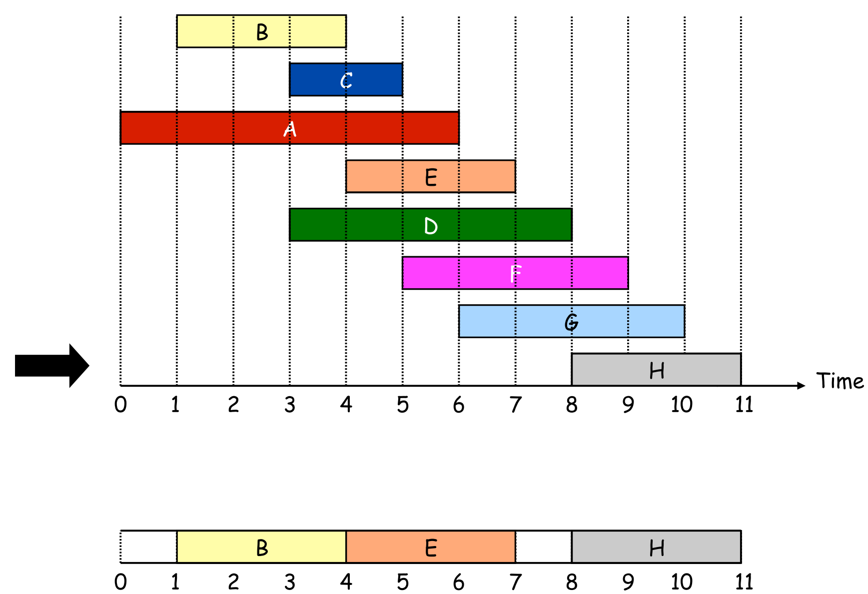


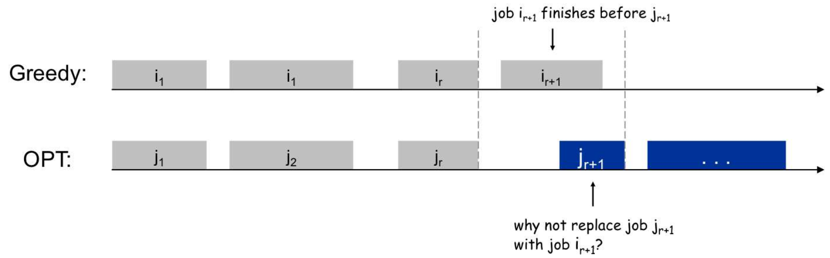
Interval Scheduling: analysis of earliest-finish-time-first algorithm

Theorem

The earliest-finish-time-first algorithm is optimal.

Proof: [by contracdiction]

- Assume greedy is not optimal, and let's see what happens.
- Let $i_1, i_2, \ldots, i_k$ denote set of jobs selected by greedy.
- Let $j_1, j_2, \ldots, j_m$ denote set of jobs in the optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of r.



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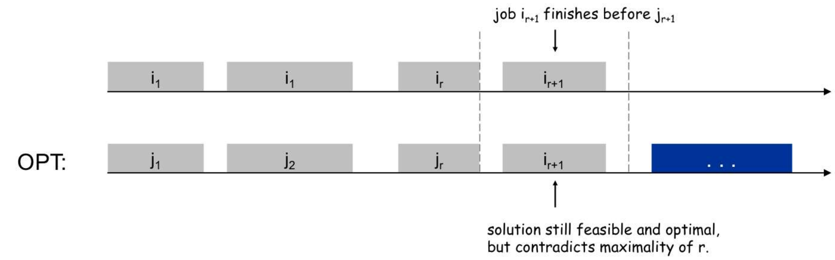

Practicing Problems

Planting Trees