Dynamic Macroeconomics
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Problem Set 1

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Question 1

$$
\begin{aligned}
y_t &= \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 c_{t-1} + \varepsilon_t, \\
c_t &= \beta_0 + \beta_1 c_{t-1} + \beta_2 c_{t-2} + \beta_3 y_{t-1} + \nu_t
\end{aligned}
$$

where $\xi_t \sim$ i.i.d $\mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$, with $\xi_t = \begin{bmatrix} \varepsilon_t \\ \nu_t \end{bmatrix}$, $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\mathbf{\Sigma} = \begin{bmatrix} \sigma_\varepsilon^2 & 0 \\ 0 & \sigma_\nu^2 \end{bmatrix}$.

a.

Solution. In matrix form, the system can be written as:

$$
\begin{bmatrix} y_t \\ c_t \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix} + \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_3 & \beta_1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ c_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \beta_2 \end{bmatrix} \begin{bmatrix} y_{t-2} \\ c_{t-2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \nu_t \end{bmatrix}
$$

That is, $\mathbf{X}_t = \mathrm{A} + \mathrm{B}\mathbf{X}_{t-1} + \mathrm{C}\mathbf{X}_{t-2} + \mathrm{D}\xi_t$ where:

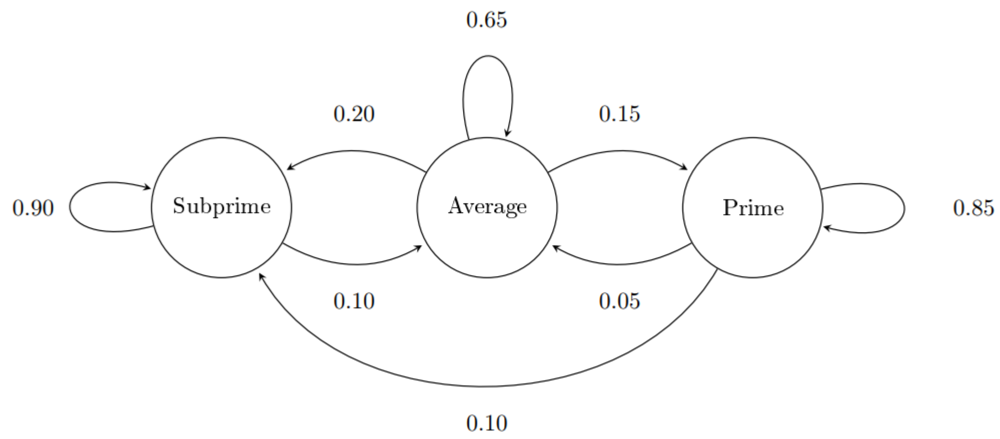
$$
\mathbf{X}_t = \begin{bmatrix} y_t \\ c_t \end{bmatrix}, \quad A = \begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix}, \quad B = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_3 & \beta_1 \end{bmatrix}
$$
$$
C = \begin{bmatrix} 0 & 0 \\ 0 & \beta_2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \xi_t = \begin{bmatrix} \varepsilon_t \\ \nu_t \end{bmatrix}
$$

b.

Solution. To convert to AR(1) form, we write $Z_t = F + GZ_{t-1} + H\xi_t$ where:

$$
Z_t = \begin{bmatrix} y_t \\ c_t \\ c_{t-1} \end{bmatrix}, \quad F = \begin{bmatrix} \alpha_0 \\ \beta_0 \\ 0 \end{bmatrix}, \quad G = \begin{bmatrix} \alpha_1 & \alpha_2 & 0 \\ \beta_3 & \beta_1 & \beta_2 \\ 0 & 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}
$$

Question 2



a.

SOLUTION. There are three states, written in the vector of states:

$$\text{vector of states} = \begin{bmatrix} \text{Subprime} \\ \text{Average} \\ \text{Prime} \end{bmatrix}$$

From the Markov diagram, we can construct the following transition matrix P:

$$\text{P} = \begin{bmatrix} 0.90 & 0.10 & 0.00 \\ 0.20 & 0.65 & 0.15 \\ 0.10 & 0.05 & 0.85 \end{bmatrix}$$

To verify the validity of this transition matrix, we must check two conditions:

1. All entries must be non-negative: Inspection shows all entries are indeed non-negative.

2. Each row must sum to 1:
   Row 1 (Subprime): $0.90 + 0.10 + 0.00 = 1.00$
   Row 2 (Average): $0.20 + 0.65 + 0.15 = 1.00$
   Row 3 (Prime): $0.10 + 0.05 + 0.85 = 1.00$

Since both conditions are satisfied, this is a valid transition matrix.

b.

SOLUTION. The Markov diagram represents how borrowers' credit classifications change over time.

Consider three credit levels:

- Subprime (poor credit)
- Average credit
- Prime (excellent credit)

The arrows in the diagram show possible transitions between these credit levels, with the numbers representing probabilities:

- For borrowers currently in Subprime:
  - 90% chance (0.90) of remaining Subprime
  - 10% chance (0.10) of improving to Average
  - 0% chance (0.00) of directly jumping to Prime
- For borrowers currently in Average:
  - 65% chance (0.65) of remaining Average
  - 20% chance (0.20) of dropping to Subprime
  - 15% chance (0.15) of improving to Prime
- For borrowers currently in Prime:
  - 85% chance (0.85) of maintaining Prime status
  - 5% chance (0.05) of dropping to Average
  - 10% chance (0.10) of dropping to Subprime

This system operates like a board game where your next position depends solely on your current position, not on how you arrived there. The probabilities represent the likelihood of moving to (or remaining in) each credit classification in the next period, based only on the current classification.

Question 3

The AR(1) process:

$$y_t = 0.5 + \gamma_1 y_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim$ i.i.d $\mathcal{N}(0, 1)$

a.

SOLUTION.

Algorithm 1 Rouwenhorst's Method (State Space and Transition Matrix)

Input: N (number of states), μ (constant), ρ (persistence), σ (std. dev.)

Output: $\{z_i\}_{i=1}^{N}, \Pi$

Require: $N > 0$, $|\rho| \neq 1$

$p \leftarrow \frac{1+\rho}{2}$
$q \leftarrow p$
$y_{\min} \leftarrow \mu - \frac{\sigma\sqrt{N-1}}{1-\rho}$

$y_{\max} \leftarrow \mu + \frac{\sigma\sqrt{N-1}}{1-\rho}$

$z_i \leftarrow \text{linspace}(y_{\min}, y_{\max}, N)$

$\Pi_2 \leftarrow \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$

for $i = 3$ to N **do**

$\Pi_i \leftarrow p \begin{bmatrix} \Pi_{i-1} & 0 \\ 0 & 0 \end{bmatrix}$

$\Pi_i \leftarrow \Pi_i + (1-p) \begin{bmatrix} 0 & \Pi_{i-1} \\ 0 & 0 \end{bmatrix}$

$\Pi_i \leftarrow \Pi_i + q \begin{bmatrix} 0 & 0 \\ \Pi_{i-1} & 0 \end{bmatrix}$

$\Pi_i \leftarrow \Pi_i + (1-q) \begin{bmatrix} 0 & 0 \\ 0 & \Pi_{i-1} \end{bmatrix}$

Normalize rows (except first and last) by dividing by 2

end for

Return $\{z_i\}, \Pi$

b.

SOLUTION. Rouwenhorst's discretization results:

The state vectore:

$$\mathbf{z} = \begin{bmatrix} -1.3166 \\ 0.2334 \\ 1.7834 \\ 3.3333 \\ 4.8833 \\ 6.4333 \\ 7.9832 \end{bmatrix}$$

Transition probability matrix:

$$P = \begin{bmatrix} 0.626398 & 0.304734 & 0.061770 & 0.006678 & 0.000406 & 0.000013 & 0.000000 \\ 0.050789 & 0.646988 & 0.257284 & 0.041451 & 0.003350 & 0.000136 & 0.000002 \\ 0.004118 & 0.102914 & 0.659505 & 0.207172 & 0.024925 & 0.001340 & 0.000027 \\ 0.000334 & 0.012435 & 0.155379 & 0.663704 & 0.155379 & 0.012435 & 0.000334 \\ 0.000027 & 0.001340 & 0.024925 & 0.207172 & 0.659505 & 0.102914 & 0.004118 \\ 0.000002 & 0.000136 & 0.003350 & 0.041451 & 0.257284 & 0.646988 & 0.050789 \\ 0.000000 & 0.000013 & 0.000406 & 0.006678 & 0.061770 & 0.304734 & 0.626398 \end{bmatrix}$$

c.

SOLUTION.

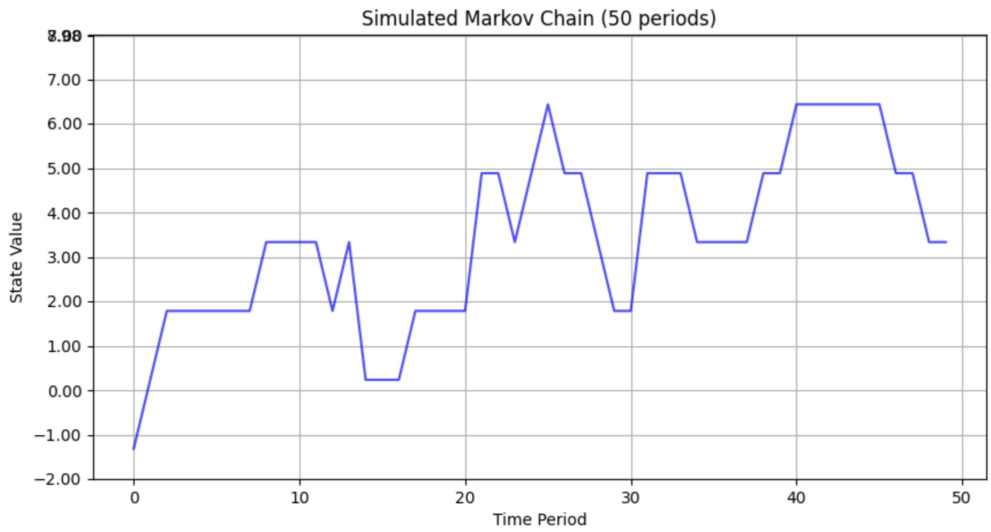


Figure 1: Simulated Markow Chain for $\gamma = 0.85$ (50 periods)

d.

SOLUTION.

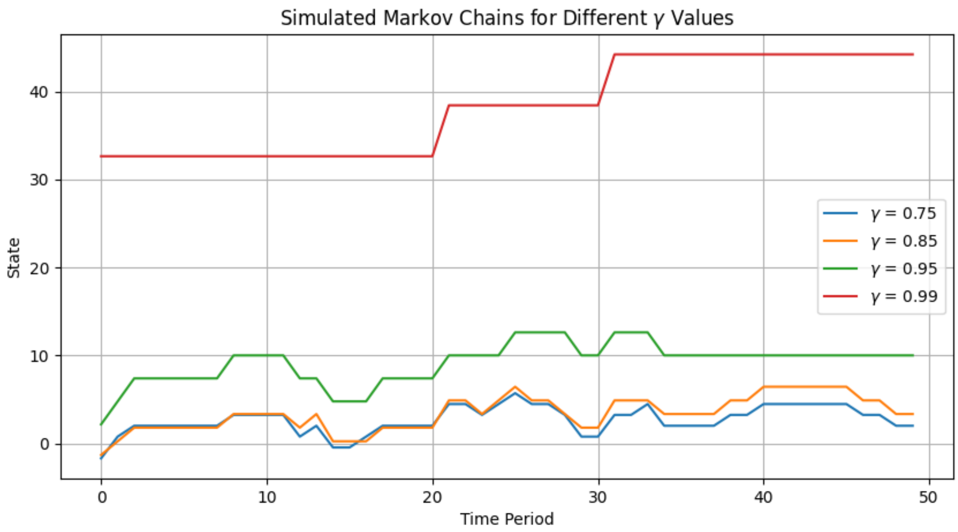


Figure 2: Simulated Markow Chain for different γ values

From figure 2, the state vectors provided for different values of γ (persistence) reveal clear patterns in how the AR(1) process behaves as persistence increases. A low γ results in a quickly mean-reverting process, while a high γ causes deviations to last much longer. When $\gamma = 0.99$, the process closely resembles a random walk, meaning past shocks almost permanently affect the system.

Question 4

a.

SOLUTION. Given:

- Operation cost $c(z_t) = a(z_t) + b(z_t)$
- Cost of a new tank $D_t + c(z_t)$

$$c(z_t) = a(z_t) + b(z_t)$$

where $a(z_t)$ is the maintenance cost and $b(z_t)$ is the breakdown cost.

Since costs are undesirable (higher costs imply lower utility), the per-period utility function should be negative:

For maintaining the existing tank:

$$u(z_t) = -c(z_t) = -\left[a(z_t) + b(z_t)\right]$$

For buying a new tank:

$$u(z_t) = -\left[D_t + c(z_t)\right] = -\left[D_t + a(z_t) + b(z_t)\right]$$

b.

Solution.

The military must repeatedly decide whether to maintain or replace a tank over an infinite time horizon. This recursive framework ensures that the optimal policy dynamically adjusts based on the tank's age and external shocks. The decision is based on cost minimization.

We make the following assumptions to simplify the analysis:

1. z_t (Effective Age) is **representative** for every tank in the military.
2. The costs associated with maintenance and replacement are **identical** for every tank.
3. All tanks have a **maximum longevity of** T, i.e, $t \in [0, T]$. Before reaching T, the government decides between maintenance or replacement based on cost. However, once a tank reaches its maximum longevity T, it must be replaced, and its age resets to 0.
4. Operation cost accumulated through each period if the government decides to maintain the tanks in the next period $a(z_{t+1}) = (1+\delta)^t a(z_t)$: the equation implies increasing maintenance cost over time due to depreciation. As t increases toward T (the tank's lifespan limit), the maintenance cost rises exponentially. This reflects the reality that older equipment requires more frequent and costly repairs.

State variables:

- Effective age z_t: Represents a tank's condition based on years in service and utilization.
  - If the tank is **maintained**, its age evolves as $z_{t+1} = z_t + \Delta t$, provided that $z_t < T$.
  - If the tank reaches $z_t = T$, it must be replaced, and its age resets to **0**.
- Taste shock ε_t: A random external factor affecting the decision in each period, modelled as i.i.d. shocks.

Since utility is driven by cost minimization, the Bellman equation is:

$$V^M(z_t, \varepsilon_t) = a(z_t) + \rho b(z_t) + \beta \mathbb{E}[V(z_{t+1}, \varepsilon_{t+1})]$$

subject to

$$a(z_{t+1}) = (1+\delta)^t a(z_t)$$

where:

- **Replacement Option** ($V^R(z_t, \varepsilon_t)$)

$$V^R(z_t, \varepsilon_t) = D_t + c(0) + \varepsilon_t^R + \beta \mathbb{E}[V(0, \varepsilon_{t+1})]$$

  - D_t = Cost of purchasing a new tank
  - $c(0)$ = Operating cost of a new tank
  - ε_t^R = Taste shock for replacement
  - β = Discount factor

- $\mathbb{E}[V(0, \varepsilon_{t+1})]$ = Expected future value from resetting the tank's age to **0**

- **Maintenance Option ($V^M(z_t, \varepsilon_t)$)**

$$V^M(z_t, \varepsilon_t) = a(z_t) + \rho b(z_t) + \beta \mathbb{E}[V(z_{t+1}, \varepsilon_{t+1})]$$

  - $a(z_t)$ = Maintenance cost at age z_t
  - $\rho b(z_t)$ = Breakdown cost, where $\rho \geqslant 0$ is the shock factor for tank failures. If $\rho = 0$, then $b(z_t) = 0$), meaning no breakdown costs are incurred.
  - $\varepsilon_t^{\text{maintain}}$ = Taste shock for maintenance
  - $\beta \mathbb{E}[V(z_{t+1}, \varepsilon_{t+1})]$ = Expected future cost of keeping the aging tank

Interpretation:

The military chooses the **cost-minimizing strategy** at each period:

- If $a(z_{ti})$ grows too large, it becomes more economical to replace the tank rather than continue repairs. If D_T (replacement cost) is lower than cumulative maintenance costs, the military should replace the tank before reaching T.
- The government will continue maintaining a tank until it reaches its maximum longevity T, at which point it must be replaced, and its age resets to 0.
- **Exogenous factors** ε_t (e.g., budget constraints, geopolitical considerations) can shift this decision over time.

c.

SOLUTION. Transition Probabilities for Effective Age (z_t):

1. If the tank is maintained:
   - The effective age will increase with certainty by some amount that depends on:
     - Current usage intensity
     - Maintenance quality
     - Environmental conditions
   - There may be randomness in how much the effective age increases.
   - Probability is nonzero for transitioning to nearby higher effective ages.
   - Probability is zero for transitioning to lower effective ages or very distant higher ages.
2. If the tank is replaced:
   - The effective age transitions to 0 with certainty (probability = 1).
   - Probability is zero for transitioning to any other age.

Thus, the transition probabilities:

- Depend on both the state (current z_t) and control (maintain/replace decision).

- Are not uniform across possible next-period states.
- Have positive probability mass only for feasible transitions.
- Sum to 1 for each combination of current state and control.

The transitions are deterministic for replacement (always to age 0) but stochastic for maintenance (uncertainty in the aging process).

Question 5

a.

SOLUTION.

Economic growth in Vietnam

Vietnam has a developing mixed socialist-oriented market economy and ranks as the 33rd-largest economy globally by nominal GDP and 26th-largest by purchasing power parity (PPP) (Gabriele, 2006; World Bank, 2023b). It is classified as a lower-middle-income country with a relatively low cost of living (Segall et al., 2002).

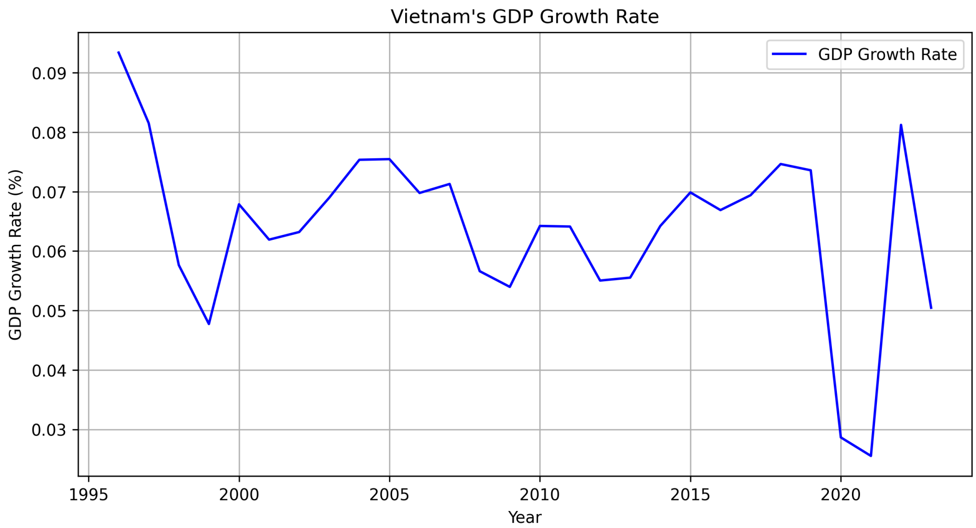


Figure 3: Vietnam's GDP growth rate (%) from 1996-2023 (World Bank, 2023c)

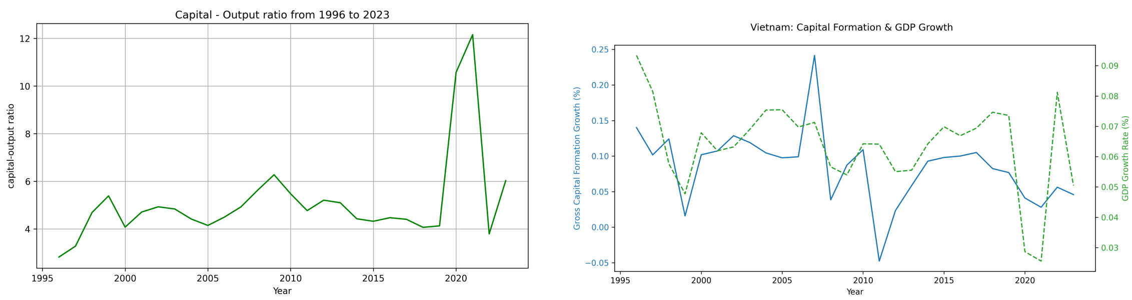
Figure 3 shows Vietnam's GDP growth rate over the course of 27 years from 1996 to 2023. Following the 1997 Asian Financial Crisis, Vietnam adopted a cautious economic policy, prioritizing macroeconomic stability over rapid growth (Staff, 2000; Vuong, 2010). Although the country transitioned toward a market-oriented economy, the government still maintains control over key sectors, including banking, state-owned enterprises, and foreign trade (Wells-Dang, 2024).

In the late 1990s, GDP growth slowed to 5.76% in 1998 and 4.77% in 1999. However, from 2000 onward, Vietnam consistently achieved real GDP growth of at least 5%, averaging 7.1% annually from 2000 to 2004. By 2005, growth reached 8.4%, making Vietnam the second-fastest-growing economy in Asia, behind only China (Vuong and Tran, 2009; Virmani, 2012).

Since 2006, Vietnam's economy has expanded at an annual rate exceeding 7%, positioning it among the world's fastest-growing economies. However, this rapid growth followed a low starting point due to the long-term economic impact of the Vietnam War (1950s–1970s), U.S.-led embargoes, and post-war austerity measures (Woods, 2002; Wells-Dang, 2024).

In the early 2020s, external challenges such as global trade tensions, the COVID-19 pandemic, and deglobalization trends led to a sharp slowdown, with GDP growth falling to 2.55% in 2021. Despite these setbacks, Vietnam remained one of Asia's top-performing economies, demonstrating resilience in the face of economic disruptions (Lee, 2021).

Kaldor's six "remarkable historical constancies" offer key insights into the fundamental relationships between capital accumulation, labor productivity, income distribution, and overall economic output [1]. We use some intuition from Kaldor's about capital/output ratio [2] to interpret the following graph:



(a) (b)

Figure 4: Vietnam's capital-output ratio and capital formation growth. 4a shows the capital-output ratio in Vietnam from 1996-2023 (World Bank, 2023c,d), and 4b presents capital formation growth and GDP growth rate over the same period (World Bank, 2023c,e).

The capital-output ratio (COR) measures how much capital investment is required to produce one additional unit of output (GDP). Vietnam's investment trends, as reflected in Gross Capital Formation Growth (%) and Gross Fixed Capital Formation (% of GDP) in figure 4, indicate a sustained commitment to capital accumulation as a key driver of economic growth. The GFCF-to-GDP ratio remains consistently high, often exceeding 30%, suggesting strong infrastructure development and industrial expansion. However, the fluctuations in capital formation growth highlight periods of rapid investment increases followed by slowdowns. Notably, a sharp decline in gross capital formation growth, such as -4.77%, signals periods of economic contraction or reduced investment efficiency, which could be linked to policy shifts, global market disruptions, or investor uncertainty.

Since integrating into the global economy in the early 1990s, Vietnam has emerged as

[1] Kaldor (1961) pointed out the 6 following' remarkable historical constancies revealed by recent empirical investigations': (1) The shares of national income received by labor and capital are roughly constant over long periods; (2) The rate of growth of the capital stock per worker is roughly constant over long periods; (3) The rate of growth of output per worker is roughly constant over long periods; (4) The capital/output ratio is roughly constant over long periods; (5) The rate of return on investment is roughly constant over long periods; (6) There are appreciable variations (2 to 5 percent) in the rate of growth of labor productivity and of total output among countries.

[2] I calculated the Captial-output ratio by dividing the average annual share of investment in GDP by the average annual growth rate of the GDP.

a prime destination for Foreign Direct Investment (FDI), reflecting its strong economic fundamentals and investor-friendly policies:

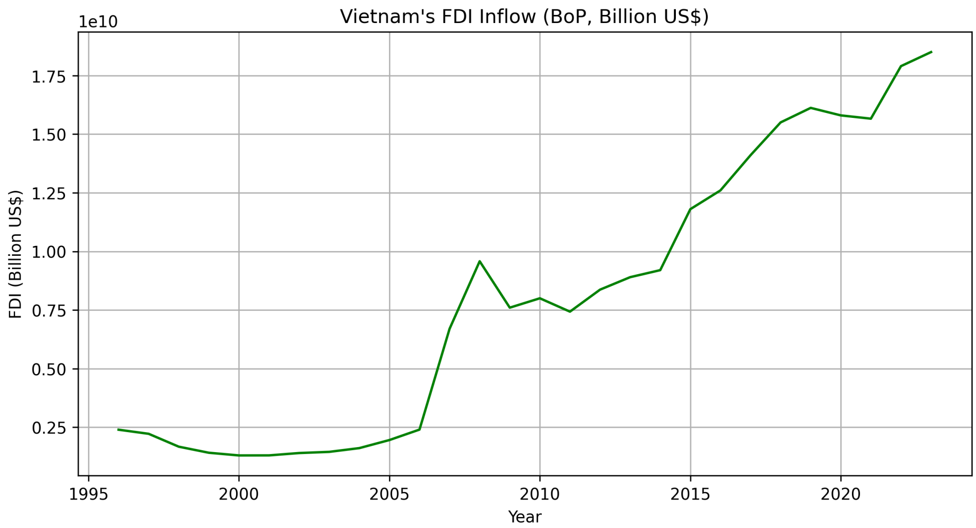


Figure 5: FDI in Vietnam from 1996-2023 (World Bank, 2023a)

By 2015, Vietnam ranked among the top five global destinations for FDI projects, following China, India, Singapore, and Australia, and was fourth in total investment capital, trailing only India, China, and Indonesia (ESCAP, 2015). Over the 2005–2015 period, Vietnam's FDI inflows demonstrated a steady upward trajectory, with an average annual growth rate of 8.1% in project numbers and 13.4% in registered capital. The 2008 surge in FDI reflected global economic expansion before the financial crisis and heightened foreign interest following Vietnam's WTO accession in 2007. Major investments included large-scale projects in petrochemicals, steel, software, and tourism. However, global financial instability led to delays or cancellations of many projects.

According to Le et al. (2021), Vietnam's FDI inflows have continued to grow, reaching $18.5 billion in 2019, despite short-term fluctuations. The manufacturing sector remains the dominant recipient, attracting the bulk of foreign capital, particularly in electronics, textiles, and high-tech industries. By the end of 2015, Asian countries accounted for 67.2% of Vietnam's cumulative FDI, with Japan, South Korea, and Singapore being top investors. The impact of FDI on Vietnam's economic development has been substantial, with its contribution to GDP rising from 15.1% in 2005 to 18% in 2015.

b.

Solution.

As mentioned in (a), Vietnam is a socialist-oriented market economy, which emphasize the strong intervention in the market to *maintain its legitimacy, while keeping intact its absolute political power* (Wells-Dang, 2024). Given the economy's sensitivity to state policies, **fiscal policy** remains a vital tool for sustaining growth, ensuring macroeconomic stability, and fostering long-term development. Alongside fiscal measures, **monetary policy**-administered by the State Bank of Vietnam—plays a crucial role in maintaining economic stability through inflation control and financial system regulation. Associated with these policies are the two key economic variables: government spending through taxation and interest rates.

1. **Interest rates**: Interest rates, as a monetary tool, influence borrowing costs, credit

availability, and inflation control. By adjusting interest rates, the State Bank of Vietnam can stimulate or restrain economic activity to maintain stability. Higher interest rates increase the cost of borrowing, making it more expensive for businesses to invest in capital goods and expansion. This leads to a decline in Gross Fixed Capital Formation (GFCF), slowing down economic growth in the long run. As borrowing becomes more costly, households reduce spending on big-ticket items such as homes, cars, and durable goods. Higher interest rates also encourage saving over consumption, leading to lower aggregate demand.

2. **Government spending through taxation**: Fluctuations in economic growth are largely influenced by two key fiscal factors: changes in government spending and taxation rates. These fiscal disturbances impact labor supply and overall economic activity. Fluctuating tax rates can lead to greater variability in labor supply and consumption, as individuals adjust their balance between market and non-market activities in response to changing incentives. However, the tax revenue collected becomes transfer payments. Assuming that government spending is productive, productive public spending can stimulate economic growth by improving infrastructure, enhancing human capital, and supporting technological progress, thereby increasing overall productivity and long-term economic output. Hence, without specific empirical scenarios or quantitative validation, the overall effect of the variable remains ambiguous.

Hence, policymakers can use interest rates and taxation to either stimulate or stabilize economic growth, depending on macroeconomic conditions.

1. **Simulating growth through expansionary fiscal and monetary policies**:
   - **Expansionary Fiscal Policy - Reducing Tax Rates**: A decrease in corporate and income tax rates allows businesses and individuals to retain more disposable income, leading to higher consumer spending and increased private-sector investment.
   - **Expansionary Monetary Policy - Lowering Interest Rates**: When economic growth slows, central banks, such as the State Bank of Vietnam (SBV), may reduce interest rates to encourage borrowing and investment. Lower interest rates make credit more affordable for businesses and consumers, stimulating spending, increasing aggregate demand, and boosting employment.
2. **Stablizing growth through tightening policies**:
   - **Tightening Fiscal Policy - Increasing Taxation**: When inflation rises or the economy overheats, SBV may increase interest rates to slow down excessive borrowing and spending. Higher interest rates discourage investment and reduce consumer demand, helping to stabilize prices and prevent asset bubbles.
   - **Tightening Monetary Policy - Raising Interest Rates**: Higher tax rates on corporations and households reduce disposable income, which in turn lowers consumption and curbs inflationary pressures. Governments may implement this policy when they need to control excessive demand or reduce budget deficits.

The choice between expansionary and tightening policies depends on whether the economy is experiencing a slowdown or overheating.

c.

SOLUTION.

According to the lecture, the original stochastic growth model is:

$$\begin{aligned}
\max_{\{c_t\}_{t=1}^{\infty},\{k_{t+1}\}_{t=1}^{\infty}} \mathbb{E}_0 \sum_{t=1}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \quad &, \\
\text{s.t. } y_t = c_t + i_t \quad &, \\
y_t = A_t k_t^{\alpha} \quad &, \\
k_{t+1} = (1-\delta)k_t \ + i_t \ &, \\
\log(A_{t+1}) = \mu + \rho\log(A_t) + \varepsilon_{t+1} \quad &, \\
c_t > 0, k_t \geqslant 0 \ &,
\end{aligned}$$

where $|\rho| < 1$, $\varepsilon_{t+1} \sim \mathcal{N}(0, \sigma_\varepsilon^2)$, and the constraints are for $t = 1, 2, ...$,

I introduce an extension from Chapter 5 in Adda and Cooper (2003) - adding government to the model. The extension is referenced from the work of McGrattan (1994), stating that the government will affect the households (firm) consumption decision through distortionary taxation. In McGrattan's, the model economy comprises the government, number of identical firms, and a large number of identical households, whom are infinitely-lived. Government policy in this economy is characterized by sequences of tax rates on capital and labor and a sequence of government consumption and spending. The taxes and spendings are assumed to be exogeneous.

McGrattan provides a stochastic growth model in which disturbance in capital and labour tax rates and government consumption can explain the variance in aggregated consumption, investments, capital stock, and output. For simplicity, we assume that labor supply is inelastic, meaning that workers allocate all their available time to work, with no leisure time [3].

Model setup

We write the three agents' problems in the economy:

1. **The household**: As mentioned above, we know that household preference are given by $\mathrm{U}(c, g)$, where c is private consumption, g is public consumption, and n is labor input.

    - **Bugdet constraint**:

    $$c_t + i_t = \left(1 - \tau_t^k\right) r_t k_t + (1 - \tau_t^n)\, w_t n_t + \delta \tau_t^k k_t + g_t \tag{1}$$

    where in each period the household purchases consumption, c_t, and investment, i_t, goods with after-tax income obtained from renting its capital and labor to a firm. The capital income in time t is $r_t k_t^*$, where r is the price of renting capital and k^* is the capital stock of the household. Labor income in t is $w_t^* n_t$, where w^* is the wage rate. In (1), the prices of consumption and investment goods are normalized to 1. Capital and labor income are taxed at rates τ_t^k and τ_t^n, respectively. The tax rates are assumed to be stochastic, exogenous processes. g_t are lump-sum transfer payments made by the government in period t. The final source of income is depreciation allowances $\delta \tau_t^k k_t$, where $\delta \in (0, 1]$ is the constant rate of capital depreciation (Adda and Cooper, 2003).

[3] Thus, when expressed in intensive form, we set $n = 1$, where n represents the labor input.

- The **utility function** is in CRRA form:

$$\mathrm{U}\left(c,g\right)=\frac{c_t^{1-\sigma}}{1-\sigma}+\frac{g_t^{1-\sigma}}{1-\sigma} \tag{2}$$

2. **The firms**: The production side of the economy follows a constant return to scale Cobb-Douglas production function. We write in intensive form:

$$y_t = A_t k_t^{\alpha} \tag{3}$$

The household owns the technology to convert investment and the current stock to next period capital, which can be expressed as follows:

$$k_{t+1}=\left(1-\delta\right)k_t+i_t \tag{4}$$

This is also the Law of motion equation for capital accumulation, where δ is the capital depreciation rate ($\delta \in (0,1]$). Variation of labor productivity A_t is a source of fluctuation, or shocks in this economy. The evolution of A follows an $AR(1)$ process:

$$\log(A_{t+1})=\mu+\rho\log(A_t)+\varepsilon_{t+1} \tag{5}$$

3. **The government**: The government imposes taxes on the rental income of factors of production to finance a stochastic stream of expenditures. Any revenue that is not allocated to current expenditures is distributed to households as lump-sum transfers. Hence, the real transfers to households at time t are given by:

$$\delta\tau_t^k k_t+g_t=\tau_t^k r_t k_t+\tau_t^n w_t n_t \tag{6}$$

This is essentially the budget constraint of the government. This specification assumes that the government balances its budget each period and never issues debt (McGrattan, 1994).

From the information above, we write the overall optimization problem:

$$\begin{aligned}
\max_{\{c_t\}_{t=1}^{\infty},\{k_{t+1}\}_{t=1}^{\infty}} \quad & \mathbb{E}_0\sum_{t=1}^{\infty}\beta^t\frac{c_t^{1-\sigma}}{1-\sigma}+\frac{g_t^{1-\sigma}}{1-\sigma}\ ,\\
\text{s.t.}\quad & y_t=c_t+i_t\ ,\\
& y_t=A_t k_t^{\alpha},\\
& c_t+i_t=\left(1-\tau_t^k\right)r_t k_t+\left(1-\tau_t^n\right)w_t n_t+\delta\tau_t^k k_t+g_t\\
& g_t=\tau_t^k r_t k_t+\tau_t^n w_t n_t-\delta\tau_t^k k_t\\
& k_{t+1}=(1-\delta)k_t+i_t\ ,\\
& \log(A_{t+1})=\mu+\rho\log(A_t)+\varepsilon_{t+1},\\
& c_t>0, k_t\geqslant 0, n_t=1
\end{aligned} \tag{7}$$

where $|\rho|<1$, $\varepsilon_{t+1}\sim\mathcal{N}(0,\sigma_\varepsilon^2)$, and the constraints are for $t=1,2,...,$

Recursive problem formulation

The recursive formulation, combining the constraints into one budget constraint, is

$$
\begin{aligned}
V_t(k_t, A_t) = \max_{c_t, k_{t+1}} & \frac{c_t^{1-\sigma}}{1-\sigma} + \frac{(1-g_t)^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[V_{t+1}(k_{t+1}, A_{t+1}) \right] \ , \\
\text{s.t. } & k_{t+1} = A_t k_t^\alpha n_t^{1-\alpha} - c_t + (1-\delta) k_t \text{ for } t = 1, 2, ..., \\
& y_t = A_t k_t^\alpha , \\
& c_t + i_t = \left(1 - \tau_t^k\right) r_t k_t + (1 - \tau_t^n) w_t n_t + \delta \tau_t^k k_t + T_t \\
& g_t = \tau_t^k r_t k_t + \tau_t^n w_t n_t - \delta \tau_t^k k_t \\
& k_{t+1} = (1-\delta) k_t + i_t \ , \\
& \log(A_{t+1}) = \mu + \rho \log(A_t) + \varepsilon_{t+1}, \\
& c_t > 0, k_t \geqslant 0, n_t = 1
\end{aligned}
\tag{8}
$$

where $|\rho| < 1$, $\varepsilon_{t+1} \sim \mathcal{N}(0, \sigma_\varepsilon^2)$, and the constraints are for $t = 1, 2, ...,$

d.

SOLUTION.

From (8), we have the recursive problem:

$$
\begin{aligned}
V_t(k_t, A_t) = \max_{c_t, k_{t+1}} & \frac{c_t^{1-\sigma}}{1-\sigma} + \frac{(1-g_t)^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[V_{t+1}(k_{t+1}, A_{t+1}) \right] \ , \\
\text{s.t. } & k_{t+1} = A_t k_t^\alpha - c_t + (1-\delta) k_t \text{ for } t = 1, 2, ..., \\
& y_t = A_t k_t^\alpha , \\
& c_t + i_t = \left(1 - \tau_t^k\right) r_t k_t + (1 - \tau_t^n) w_t n_t + \delta \tau_t^k k_t + g_t \\
& g_t = \tau_t^k r_t k_t + \tau_t^n w_t n_t - \delta \tau_t^k k_t \\
& k_{t+1} = (1-\delta) k_t + i_t \ , \\
& \log(A_{t+1}) = \mu + \rho \log(A_t) + \varepsilon_{t+1}, \\
& c_t > 0, k_t \geqslant 0, n_t = 1
\end{aligned}
\tag{9}
$$

where $|\rho| < 1$, $\varepsilon_{t+1} \sim \mathcal{N}(0, \sigma_\varepsilon^2)$, and the constraints are for $t = 1, 2, ...,$

From (4), we rewrite to have investment i_t:

$$
i_t = k_{t+1} - (1-\delta) k_t \tag{10}
$$

Substitute (10) to the household's budget constraint (1) and rewrite to have the aggregated consumption c_t w.r.t k_{t+1}:

$$
c_t = \left(1 - \tau_t^k\right) r_t k_t + (1 - \tau_t^n) w_t n_t + \delta \tau_t^k k_t - k_{t+1} + (1-\delta) k_t + g_t \tag{11}
$$

Substituting the new budget constraint (11) into the utility function gives

$$
\begin{aligned}
V_t(k_t, A_t) = \max_{k_{t+1}} & \frac{(C - k_{t+1})^{1-\sigma}}{1-\sigma} + \frac{(1-g_t)^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[V_{t+1}(k_{t+1}, A_{t+1}) \right] \ , \\
\text{s.t. } & \log(A_{t+1}) = \mu + \rho \log(A_t) + \varepsilon_{t+1}, \\
& k_t \geqslant 0, n_t = 1
\end{aligned}
\tag{12}
$$

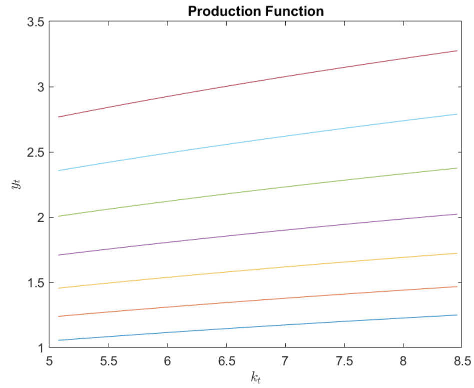
where

$$C = \left(1 - \tau_t^k\right) r_t k_t + \left(1 - \tau_t^n\right) w_t n_t + \delta \tau_t^k k_t - k_{t+1} + (1 - \delta) k_t + g_t,$$

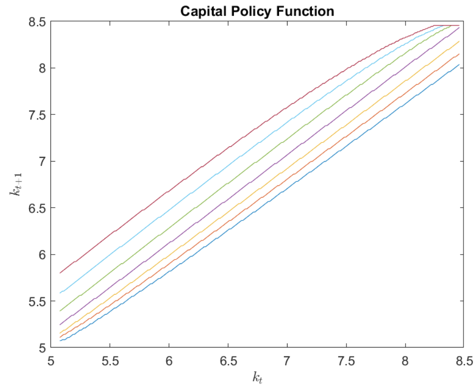
$$|\rho| < 1 \text{ and } \varepsilon_{t+1} \sim \mathcal{N}(0, \sigma_\varepsilon^2)$$

With the provided recursive problem, we numerically solve and simulate the model using MATLAB.

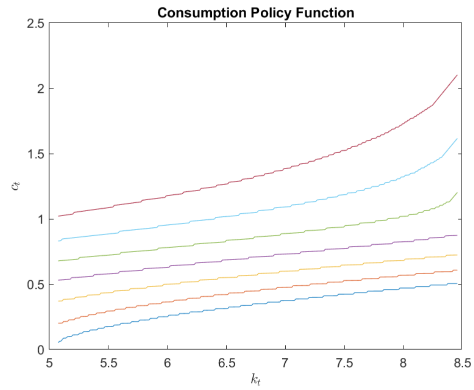
Policy Functions



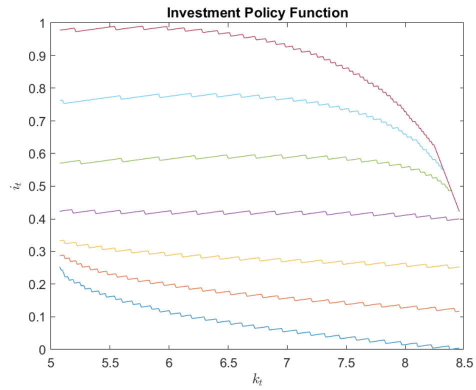
(a) Production Function



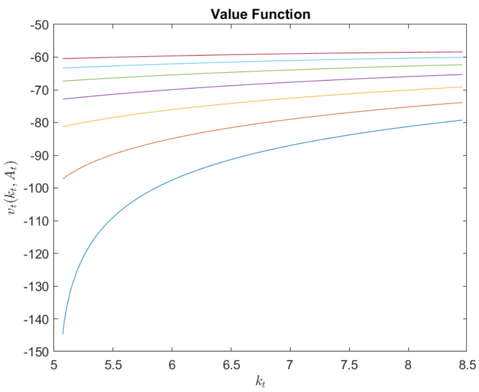
(b) Capital Policy Function



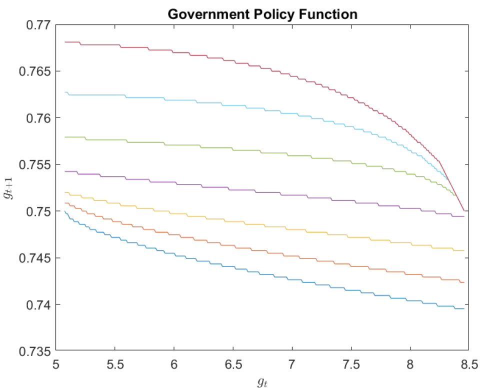
(c) Consumption Policy Function



(d) Investment Policy Function



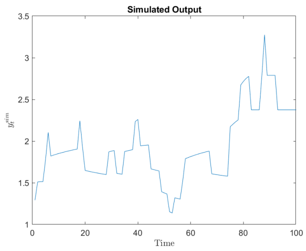
(e) Value Function



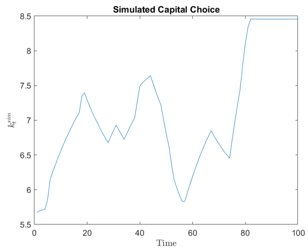
(f) Government Policy Function

Figure 6: Policy Functions

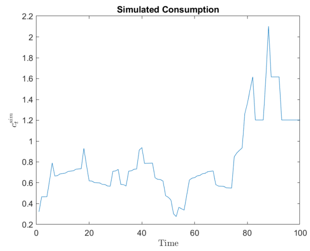
Simulated Data



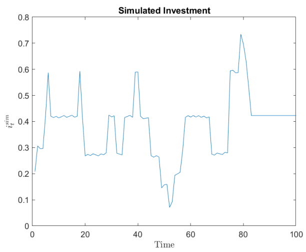
(a) Simulated Output



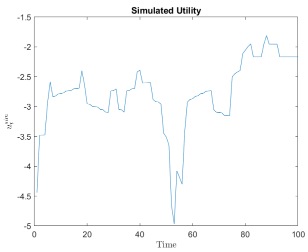
(b) Simulated Capital Choice



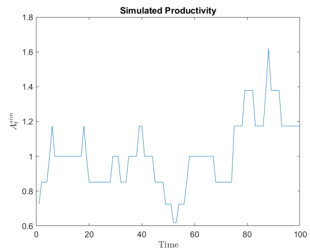
(c) Simulated Consumption



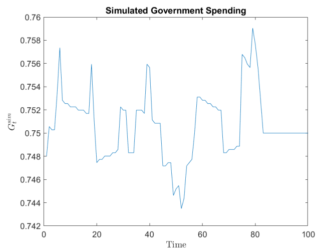
(d) Simulated Investment



(e) Simulated Utility



(f) Simulated Productivity



(g) Simulated Government Spending

Figure 7: Simulated Time Series Plots

Comparison: Simulated vs Actual data of Vietnam from 1996 to 2023

In this section, we first import the dataset of Vietnam used in **(a)** and **(b)** from the designated folder. In the original `data.csv`, we calculate the per capita terms of the macroeconomic indicators, such as GDP per capita, investment per capita, and consumption per capita, adjusted to constant 2015 USD. However, the real-world data for Capital Stock and Government Spending in Vietnam are challenging to measure accurately.

Exploratory Data Analysis

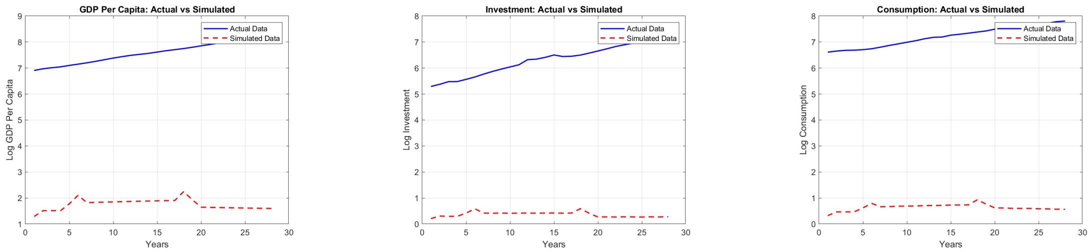
Before plotting the simulated and actual data, we conduct the following pre-processing steps:

1. To make the data comparable with the simulated values, we normalize the values by computing the natural logarithm of key variables.
2. Aligning the Time Horizon: The simulated model generates data over a long horizon, but the actual dataset only covers 28 years. To make the comparisons valid, we

extract the first 28 values from the simulated data to have both datasets share the same time length.

```
%% Extract first 28 values of simulated data
sim_t = 1:28;
sim_ysim = sim.ysim(1:28);
sim_isim = sim.isim(1:28);
sim_csim = sim.csim(1:28);
```

Plot the comparison



(a) GDP Per Capita (b) Investment (c) Consumption

Figure 8: Comparison of Actual vs. Simulated Macroeconomic Indicators

e.

SOLUTION.

Decribe the Conterfactual exercise

Previously in **(b)**, we proposed two policy regimes that the government uses to stimulate or stabilize the economy—fiscal and monetary policies. In our growth model, we focus on the government sector, particularly expansionary and tightening fiscal policies. These policies serve as our counterfactual scenarios.

Expansionary Fiscal Policy

To model an expansionary fiscal policy, we assume that the government reduces tax rates to encourage investment and consumption:

```
par_expansion = par;  % Create a copy of the parameters
par_expansion.tauk = par.tauk * 0.8;  % Reduce tax rate by 20%
par_expansion.taun = par.taun * 0.85; % Reduce income tax rate by 15%

sol_expansion = solve.grow(par_expansion);
sim_expansion = simulate.grow(par_expansion, sol_expansion);
```

Tightening Fiscal Policy

Conversely, a tightening fiscal policy increases tax rates, potentially slowing economic activity:

```
par_tightening = par;  % Copy original parameters
par_tightening.tauk = par.tauk * 1.2;  % Increase tax rate on capital by 20%
par_tightening.taun = par.taun * 1.15; % Increase income tax rate by 15%

sol_tightening = solve.grow(par_tightening);
sim_tightening = simulate.grow(par_tightening, sol_tightening);
```

Comparison of Results

Plot the comparison

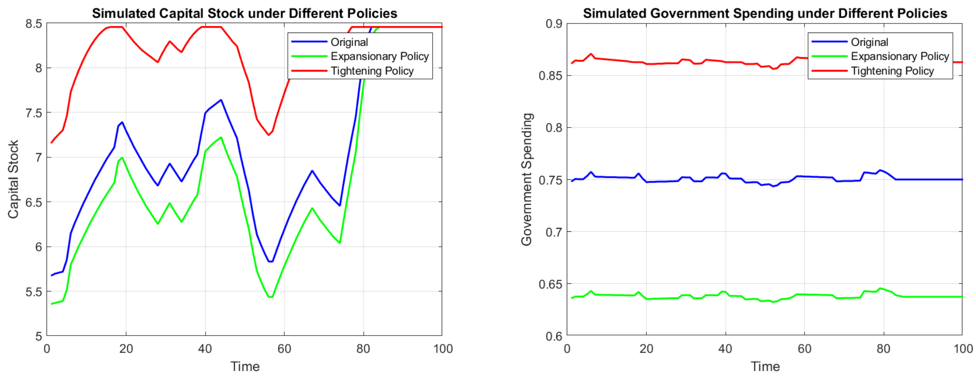


Figure 9: Capital Stock (Left) and Government Spending (Right) under Different Policies

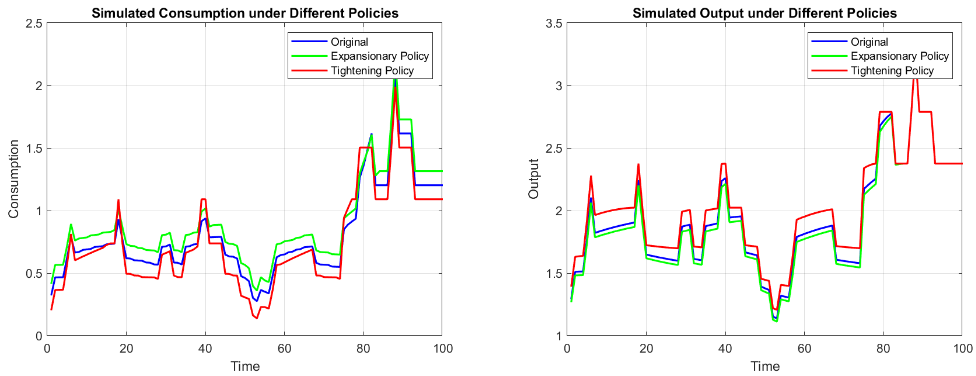


Figure 10: Consumption (Left) and Output (Right) under Different Policies

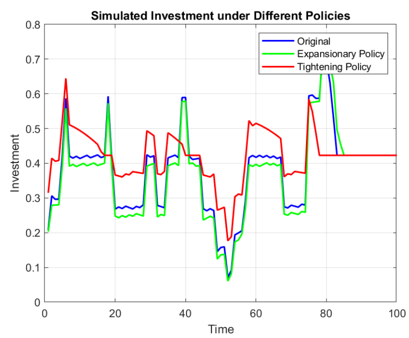


Figure 11: Investment under Different Policies

Statistical Analysis of Simulated Variables

Table 1 presents descriptive statistics of government spending under three different policy scenarios: Original, Expansionary, and Tightening.

Table 1: Statistical Summary of Government Spending under Different Policies

Scenario	Min	Max	Mean	Median	StdDev
Original	0.74349	0.75905	0.75068	0.75000	0.00296
Expansionary	0.63252	0.64565	0.63811	0.63773	0.00248
Tightening	0.85605	0.87065	0.86288	0.86250	0.00228

From the table, we see that Expansionary policy (lower tax rates) reduces government spending compared to the original scenario, while Tightening policy (higher tax rates) increases government spending. These changes align with the description of the policies.

The statistical results for Capital, Output, Consumption, and Investment under three policy regimes (Original, Expansionary, and Tightening) are presented in table 2:

Table 2: Statistical Summary of Economic Variables under Different Policies

Variable	Min	Max	Mean	Median	StdDev
Original Scenario					
Capital	5.673	8.4659	7.0983	6.89	0.8411
Output	1.1796	3.275	1.9283	1.863	0.4257
Consumption	2.0756	3.1022	2.1833	2.7073	0.3812
Investment	0.07093	0.7345	0.3886	0.41719	0.1321
Expansionary Policy					
Capital	5.357	8.4659	6.769	6.714	0.8962
Output	1.114	3.275	1.899	1.8276	0.4392
Consumption	2.1955	3.2147	2.2147	2.6983	0.3956
Investment	0.06081	0.7599	0.3672	0.39456	0.1321
Tightening Policy					
Capital	7.155	8.4569	8.1707	8.292	0.36485
Output	1.2873	3.217	2.107	1.968	0.3923
Consumption	1.7556	3.1987	1.989	2.4623	0.4032
Investment	0.17678	0.64459	0.42846	0.4285	0.073512

Overall, the changes in these variables align well with the expected effects of expansionary and tightening fiscal policies.

- Capital:
  - Expansionary policy reduces capital stock slightly (Min: 5.357, Mean: 6.769) compared to the original scenario (Min: 5.673, Mean: 7.098).
  - Tightening policy leads to an increase in capital (Min: 7.155, Mean: 8.170) as expected, since higher taxes on consumption encourage savings and capital accumulation.

- Output (GDP):
  - Expansionary policy slightly decreases output (Mean: 1.899) compared to the original (Mean: 1.928).
  - Tightening policy significantly increases output (Mean: 2.107) due to higher capital accumulation, consistent with economic expectations.
- Consumption:
  - Expansionary policy leads to higher consumption (Mean: 2.214) compared to the original scenario (Mean: 2.183).
  - Tightening policy reduces consumption (Mean: 1.989), which aligns with expectations since higher taxes discourage spending.
- Investment:
  - Expansionary policy slightly decreases investment (Mean: 0.367) compared to the original (Mean: 0.388).
  - Tightening policy reduces investment even further (Mean: 0.428), suggesting that despite an increase in capital.

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