Advanced Calculus with FE Application: Hw 1

Due on July 21, 2014

Advisor: Dan Stefanica

Weiyi Chen

Problem 1

Recall that put-call parity as

$$C - P = Se^{-q(T)} - Ke^{-r(T)} \tag{1}$$

where $K = 47$, $T - t = 9/12 = 0.75$.

Short call, long put

When short a call, long a put and long a share of the underlying asset can make an arbitrage, this requires

$$C_{bid} - P_{ask} > S_{ask} - Ke^{-r_{bid}T} \tag{2}$$

where $r_{bid} = 0.0158$. However

$$\begin{aligned} C_{bid} - P_{ask} &= -1.9 \\ S_{ask} - Ke^{-r_{bid}T} &= -1.3463369325554524 > -1.9 \\ \Delta = -1.9 - (-1.3463369325554524) &\approx -0.553663 < 0 \end{aligned} \tag{3}$$

This cannot make an arbitrage.

Long call, short put

When long a call, short a put and short a share of the underlying asset can make an arbitrage, this requires

$$C_{ask} - P_{bid} < S_{bid} - Ke^{-r_{ask}T} \tag{4}$$

where $r_{ask} = 0.0153$. We find that

$$\begin{aligned} C_{ask} - P_{bid} &= -1.7 \\ S_{bid} - Ke{-}r_{ask}T &= -1.5637575750714845 > -1.7 \\ \Delta = -1.5637575750714845 - (-1.7) &\approx 0.136242 > 0 \end{aligned} \tag{5}$$

so this is possible to make an arbitrage. The portfolio is contructed by longing a call, shorting a put and shorting 1 share of the underlying asset, which cost

$$C_{ask} - P_{bid} - S_{bid} = -46.6 \tag{6}$$

namely generating \$46.6 cash. We will deposit it as part of the portfolio. So the value of the portfolio at maturity is

$$\begin{aligned} V(T) &= \max\{S(T) - K, 0\} - \max\{K - S(T), 0\} - S(T) + 46.6e^{r_{ask}*(T)} \\ &= -K + 46.6e^{r_{ask}*(T)} \\ &= 0.137815 \end{aligned} \tag{7}$$

as an arbitrage.

Problem 2

(i)

An arbitrage exists when short the call option with strike \$50 and long the call option with strike \$55, which benefits

$$K_1 - K_2 = \$8 - \$3 = \$5 \tag{8}$$

The value of the portfolio at maturity is

$$V(T) = -\max\{S(T)-50,0\} + \max\{S(T)-55,0\}$$
$$= \begin{cases} -5, & S(T) > 55 \\ -S(T)+50, & 50 < S(T) \le 55 \\ 0, & 50 \le S(T) \end{cases} \tag{9}$$

Therefore $P\&L(T)$ is

$$P\&L(T) = V(T) + 5e^{rT}$$
$$= \begin{cases} 0, & S(T) > 55 \\ -S(T)+50+5e^{rT}, & 50 < S(T) \le 55 \\ 5e^{rT}, & 50 \le S(T) \end{cases} \tag{10}$$
$$\ge 0$$

which is an arbitrage.
If the options have bidask prices of \$7.8 and \$8.2, and \$2.9 and \$3.1, then there're two ways to make an arbitrage. The first way is the same as above: short the call option with lower strike and long the call option with higher strike,

$$P\&L(T) = V(T) + (7.8-3.1)e^{rT}$$
$$= \begin{cases} -5+4.7e^{rT}, & S(T) > 55 \\ -S(T)+50+4.7e^{rT}, & 50 < S(T) \le 55 \\ 4.7e^{rT}, & 50 \le S(T) \end{cases} \tag{11}$$

which is an arbitrage if and only if r and T satisfies

$$-5+4.7e^{rT} \ge 0 \tag{12}$$

. Otherwise it's not an arbitrage.
The other way is to: long the call option with lower strike and long the call option with higher strike,

$$P\&L(T) = V(T) + (-8.2+2.9)e^{rT}$$
$$= \max\{S(T)-50,0\} - \max\{S(T)-55,0\} - 5.3e^{rT}$$
$$= \begin{cases} 5-5.3e^{rT}, & S(T) > 55 \\ S(T)-50-5.3e^{rT}, & 50 < S(T) \le 55 \\ -5.3e^{rT}, & 50 \le S(T) \end{cases} \tag{13}$$

which is still not an arbitrage.
In conclusion, there exist arbitrage opportunities if and only if

$$-5+4.7e^{rT} \ge 0 \tag{14}$$

(ii)

An arbitrage is present when you long the put with strike \$60 and short the put with strike \$64, the profit and loss statement is

$$\begin{aligned} P\&L(T) &= V(T) + 11 - 7 \\ &= \max\{60 - S(T), 0\} - \max\{64 - S(T)\} + 4 \\ &= \begin{cases} 0, & \text{if } S(T) \le 60 \\ S(T) - 60, & \text{if } 60 < S(T) \le 64 \\ 4, & \text{if } 64 < S(T) \end{cases} \\ &\ge 0 \end{aligned} \tag{15}$$

which is an arbitrage.

Problem 3

First of all, a portfolio with a long position with \$50 cash and a short position in two units of the underlying asset, has value $50 - 2S(T)$ at maturity, when $S(T) < 30$.
To replicate the payoff $20S(T)$ of the portfolio when $30 < S(T) < 50$, note that

$$20 - S(T) = (50 - 2S(T)) + (-30 + S(T)) \tag{16}$$

This is equivalent to adding a long position in one call with strike 30. To replicate the payoff $2S(T)130$ of the portfolio when $S(T) \ge 50$, note that

$$2S(T) - 130 = (20 - S(T)) + (-150 + 3S(T)) = (20 - S(T)) + 3(S(T) - 50) \tag{17}$$

This is equivalent to adding a long position in three calls with strike 50.
Summarizing, the replicating portfolio is made of

- long \$50 cash
- short 2 units of the asset
- long 1 call option with strike $K = 30$ on the asset
- long 3 call options with strike $K = 50$ on the asset

Problem 4

(i)

Synthesize: a short position in the 39 strike put and a long position in the 42 strike put.
Payoff:

$$\begin{aligned} V(T) &= -\max\{39 - S(T), 0\} + \max\{42 - S(T), 0\} \\ &= \begin{cases} 3, & \text{if } S(T) \le 39 \\ 42 - S(T), & \text{if } 39 < S(T) \le 42 \\ 0, & \text{if } 42 < S(T) \end{cases} \end{aligned} \tag{18}$$

Problem 4 [(i)] continued on next page. . .

P&L:

$$
\begin{aligned}
P\&L(T) &= V(T) + (2.4-4)e^{rT} \\
&= \begin{cases} 1.379874, & \text{if } S(T) \le 39 \\ 40.379874 - S(T), & \text{if } 39 < S(T) \le 42 \\ -1.620126, & \text{if } 42 < S(T) \end{cases}
\end{aligned}
\tag{19}
$$

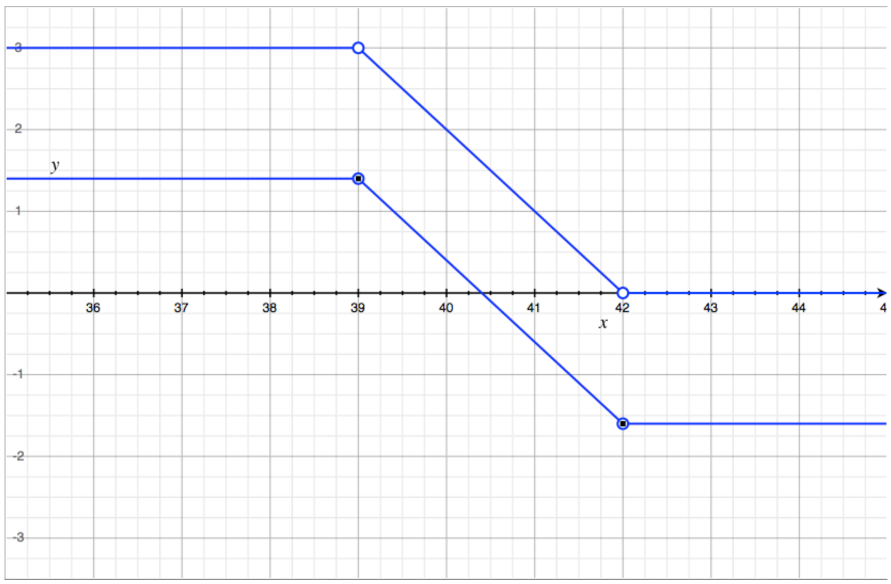

Figure 1: The payoff diagram (hollow points) and the P&L diagram (solid points) of 4(i)

For $S(T) < 40.379874$ would the bear spread be profitable.

(ii)

Synthesize: a short position in the 36 strike call and a long position in the 39 strike call.
Payoff:

$$
\begin{aligned}
V(T) &= -\max\{S(T)-36, 0\} + \max\{S(T)-39, 0\} \\
&= \begin{cases} 0, & \text{if } S(T) \le 36 \\ 36 - S(T), & \text{if } 36 < S(T) <\le 39 \\ -3, & \text{if } 39 < S(T) \end{cases}
\end{aligned}
\tag{20}
$$

P&L:

$$
\begin{aligned}
P\&L(T) &= V(T) + (5.5-3.5)e^{(}rT) \\
&= \begin{cases} 2.025157, & \text{if } S(T) \le 36 \\ 38.025156 - S(T), & \text{if } 36 < S(T) \le 39 \\ -0.974843, & \text{if } 39 < S(T) \end{cases}
\end{aligned}
\tag{21}
$$

For $S(T) < 38.025156$ would the bear spread be profitable.

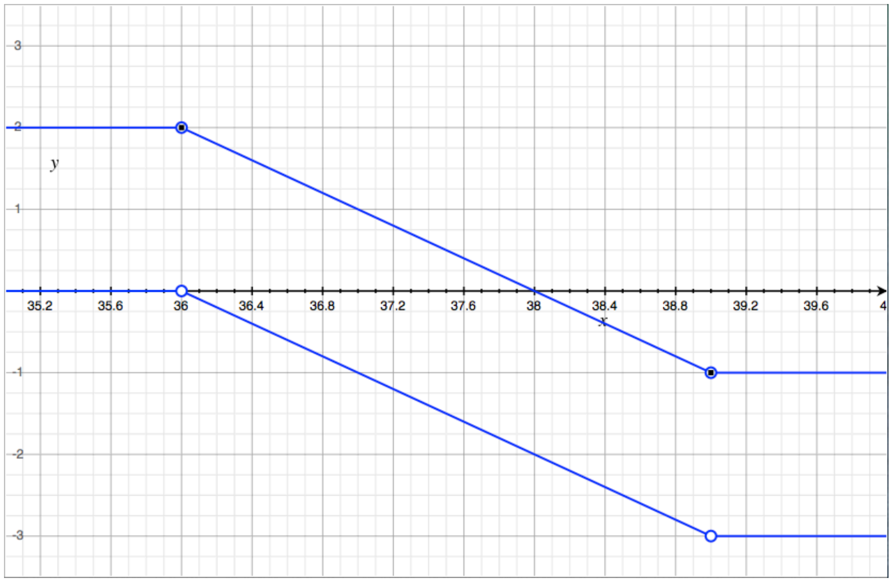

Figure 2: The payoff diagram (hollow points) and the P&L diagram (solid points) of 4(iii)

(iii)

Synthesize: a long position in one unit of the 36 strike call, a short position in two units of the 39 strike call and a long position in one unit of the 42 strike call

Payoff:

$$\begin{aligned} V(T) &= \max\{S(T)-36,0\} - 2\max\{S(T)-39,0\} + \max\{S(T)-42,0\} \\ &= \begin{cases} 0, & \text{if } S(T) \leq 36 \\ S(T)-36, & \text{if } 36 < S(T) \leq 39 \\ 42-S(T), & \text{if } 39 < S(T) \leq 42 \\ 0, & \text{if } 42 < S(T) \end{cases} \end{aligned} \tag{22}$$

P&L:

$$\begin{aligned} P\&L(T) &= V(T) - 5.5 + 2\times 3.5 - 2.5 \\ &= \begin{cases} -1.012578, & \text{if } S(T) \leq 36 \\ S(T)-37.012578, & \text{if } 36 < S(T) \leq 39 \\ 40.987422-S(T), & \text{if } 39 < S(T) \leq 42 \\ -1.012578, & \text{if } 42 < S(T) \end{cases} \end{aligned} \tag{23}$$

For $37.012578 < S(T) < 40.987422$ would the butterfly spread be profitable.

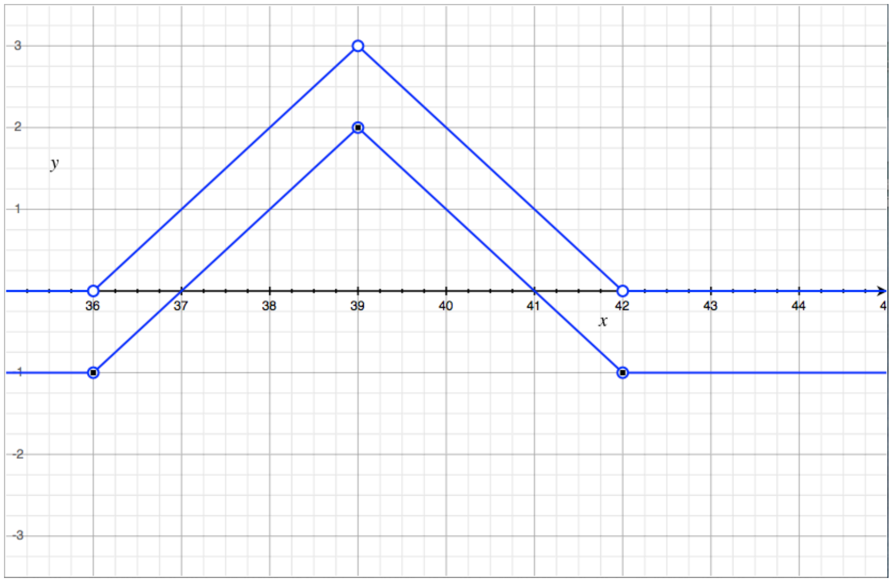

Figure 3: The payoff diagram (hollow points) and the P&L diagram (solid points) of 4(ii)

(iv)

A 3642 strangle: a long position in the 36 strike put and a long position in the 42 strike call
Payoff:

$$V(T) = \max\{36 - S(T), 0\} + \max\{S(T) - 42\}$$
$$= \begin{cases} 36 - S(T), & \text{if } S(T) < 36 \\ 0, & \text{if } 36 \leq S(T) < 42 \\ S(T) - 42, & \text{if } 42 \leq S(T) \end{cases} \tag{24}$$

P&L:

$$P\&L(T) = V(T) - (1.2 + 2.5)e^{rT}$$
$$= \begin{cases} 32.253460 - S(T), & \text{if } S(T) < 36 \\ -3.746540, & \text{if } 36 \leq S(T) < 42 \\ S(T) - 45.746540, & \text{if } 42 \leq S(T) \end{cases} \tag{25}$$

For $S(T) < 32.253460$ or $S(T > 45.746540$ would the strangle be profitable.

A 39 straddle: a long position in th 39 strike put and a long position in the 39 strike call
Payoff:

$$V(T) = \max\{39 - S(T), 0\} + \max\{S(T) - 39, 0\}$$
$$= \begin{cases} 39 - S(T), & \text{if } S(T) \leq 39 \\ S(T) - 39, & \text{if } S(T) > 39 \end{cases} \tag{26}$$

P&L:

$$P\&L(T) = V(T) - (2.4 + 3.5)e^{rT}$$
$$= \begin{cases} 33.025787 - S(T), & \text{if } S(T) \leq 39 \\ S(T) - 44.974213, & \text{if } S(T) > 39 \end{cases} \tag{27}$$

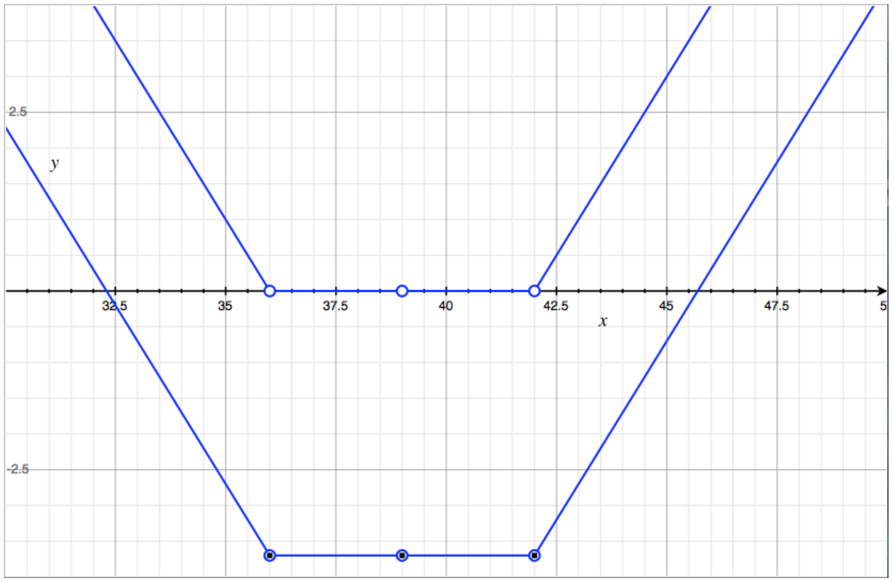

Figure 4: The payoff diagram (hollow points) and the P&L diagram (solid points) of a 36-42 strangle

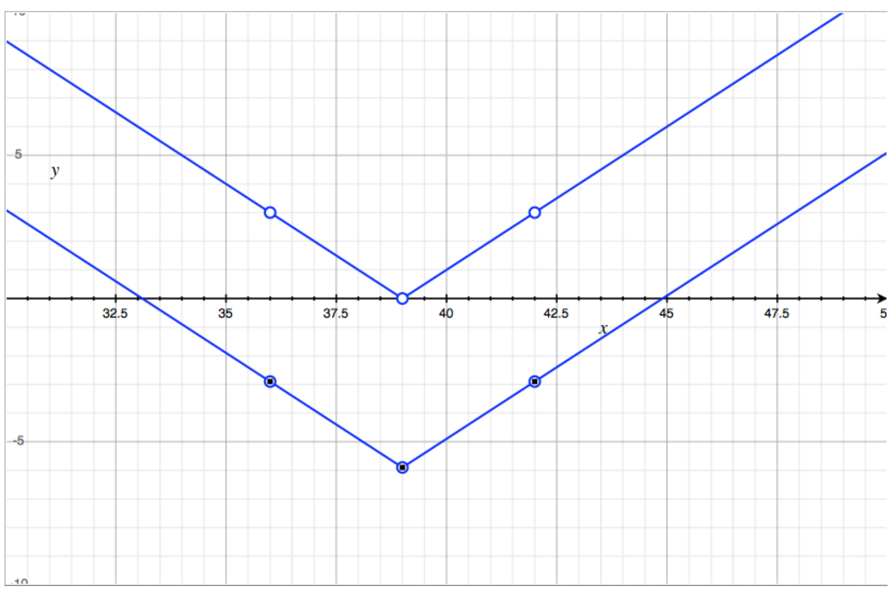

Figure 5: The payoff diagram (hollow points) and the P&L diagram (solid points) of a 39 straddle

For $S(T) < 33.025787$ or $S(T) > 44.974213$ would the 39 straddle be profitable.

A 3642 collar: long the underlying, long a put option at strike price 36, short a call option at strike price 42

Payoff:

$$
\begin{aligned}
V(T) &= S(T) + \max\{36 - S(T), 0\} - \max\{S(T) - 42, 0\} \\
&= \begin{cases} 36, & \text{if } S(T) \leq 36 \\ S(T), & \text{if } 36 < S(T) \leq 42 \\ 42, & \text{if } 42 < S(T) \end{cases}
\end{aligned} \tag{28}
$$

P&L:

$$
\begin{aligned}
P\&L(T) &= V(T) + (-40 - 1.2 + 2.5)e^{rT} \\
&= \begin{cases} -3.186786, & \text{if } S(T) \leq 36 \\ S(T) - 39.186786, & \text{if } 36 < S(T) \leq 42 \\ 2.813214, & \text{if } 42 < S(T) \end{cases}
\end{aligned} \tag{29}
$$

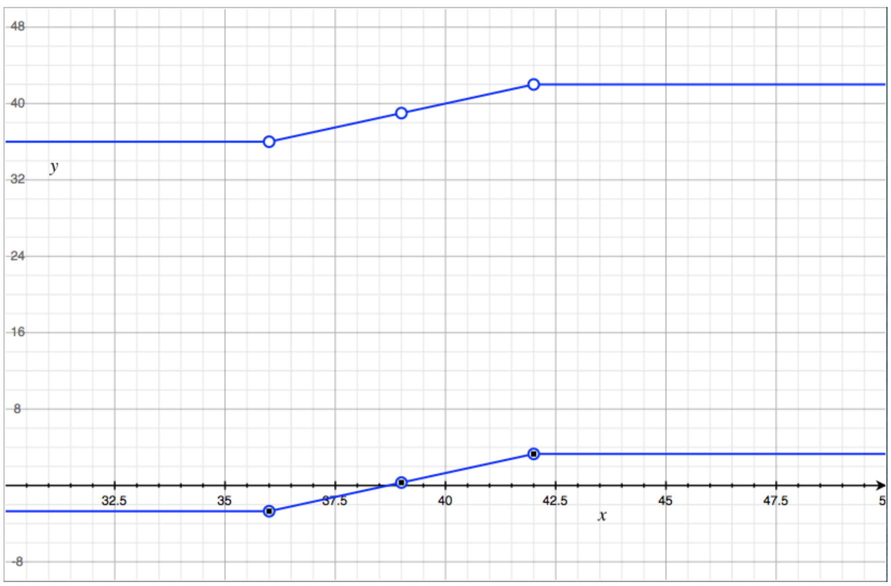

Figure 6: The payoff diagram (hollow points) and the P&L diagram (solid points) of a 36-42 collar

For $S(T) > 39.186786$ would the 36-42 collar be profitable.

A 3642 risk reversal: short a put option at strike price 36, long a call option at strike price 42
Payoff:

$$
\begin{aligned}
V(T) &= -\max\{36 - S(T), 0\} + \max\{S(T) - 42, 0\} \\
&= \begin{cases} S(T) - 36, & \text{if } S(T) \leq 36 \\ 0, & \text{if } 36 < S(T) \leq 42 \\ S(T) - 42, & \text{if } 42 < S(T) \end{cases}
\end{aligned} \tag{30}
$$

P&L:

$$
\begin{aligned}
P\&L(T) &= V(T) + (1.2 - 2.5)e^{rT} \\
&= \begin{cases} S(T) - 37.316352, & \text{if } S(T) \leq 36 \\ -1.316352, & \text{if } 36 < S(T) \leq 42 \\ S(T) - 43.316352, & \text{if } 42 < S(T) \end{cases}
\end{aligned} \tag{31}
$$

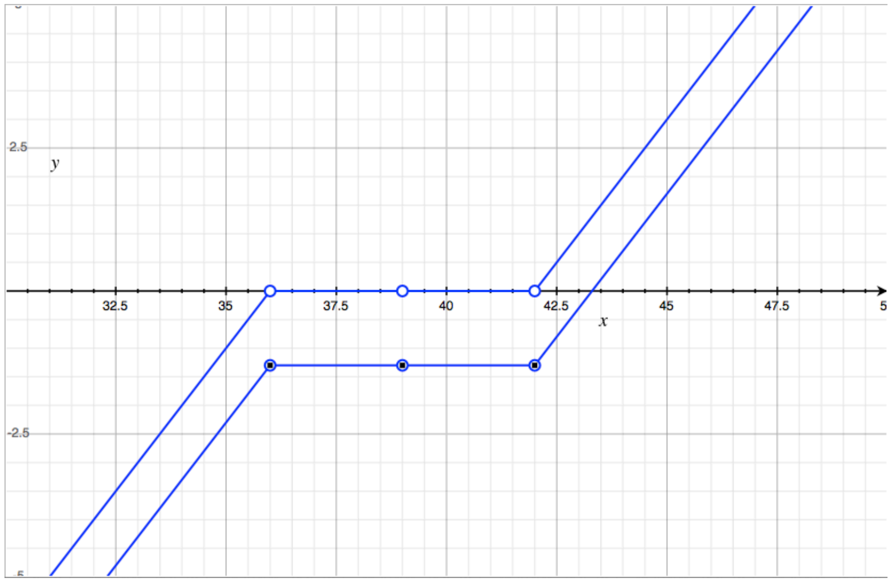

Figure 7: The payoff diagram (hollow points) and the P&L diagram (solid points) of a 36-42 risk reversal

For $S(T) > 43.316352$ would the 36-42 risk reversal be profitable.

Problem 5

(i)

To derive the maximum of $V(S)$,

$$\begin{aligned}\frac{\partial V(S)}{\partial S} &= N(d) - \frac{\partial \max\{S-K,0\}}{S} \\ &= \begin{cases} N(d)-1<0, & \text{if } S>K \\ N(d)\geq 0, & \text{if } S\leq K \end{cases}\end{aligned} \tag{32}$$

where

$$d = \frac{ln(S/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \tag{33}$$

Therefore the maximum value of $V(S)$ is obtained when $S = K$.

(ii)

If $S \to \infty$,

$$d \to \infty, d_2 \to \infty \Rightarrow N(d) = N(d_2) = 1 \tag{34}$$

Therefore,

$$\begin{aligned} V(S) &= C_B S(S) - \max\{S-K,0\} \\ &= (SN(d) - Ke^{-r(T-t)}N(d_2)) - (S-K) \\ &\to (S - Ke^{-r(T-t)}) - (S-K) \\ &= K(1-e^{-r(T-t)}) \end{aligned} \tag{35}$$

Problem 5 [(ii)] continued on next page. . .

We also find from from part(i) that $V'(S) \to 0$ when $S \to \infty$, which indicates convergence, so we can conclude $V(S) \to K(1 - e^{-r(T-t)})$.

(iii)

This is a graph generated by following python code -

```
from math import *
from scipy.stats import norm
import numpy as np
import matplotlib.pyplot as plt

# Black Sholes Function
def BlackSholes(CallPutFlag,S,X,T,r,v):
    d1 = (log(S/X)+(r+v*v/2.)*T)/(v*sqrt(T))
    d2 = d1-v*sqrt(T)
    if CallPutFlag=='c':
        return S*norm.cdf(d1)-X*exp(-r*T)*norm.cdf(d2)
    else:
        return X*exp(-r*T)*norm.cdf(-d2)-S*norm.cdf(-d1)

def Premium(S):
    return BlackSholes(CallPutFlag='c',S=S,X=100.,T=0.5,r=0.05,v=0.3) - max(S-100.,0.)

x = np.linspace(50, 200)
y = [Premium(i) for i in x]
plt.plot(x, y, 'r', linewidth=2)

plt.savefig('5-3.pdf')
```

where I assumed strike as 100, maturity as 6 months, risk rate as 5% and volatilty as 30%.

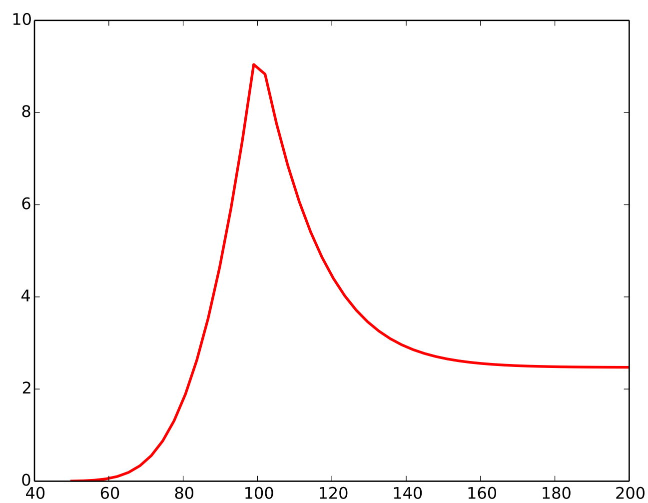

Figure 8: Plot of $V(S)$ as a function of S

(iv)

When it comes to an European put option, we will have the delta as

$$\begin{aligned}\frac{\partial V(S)}{\partial S} &= -N(-d) - \frac{\partial \max\{K-S,0\}}{S} \\ &= \begin{cases} -N(-d)+1>0, & \text{if } S<K \\ -N(-d)\leq 0, & \text{if } S \geq K \end{cases}\end{aligned} \tag{36}$$

Therefore the maximum value of $V(S)$ is obtained when $S = K$.
If $S \to \infty$,

$$d \to \infty, d_2 \to \infty \Rightarrow N(-d) = N(-d_2) = 0 \tag{37}$$

Therefore,

$$\begin{aligned}V(S) &= P_BS(S) - \max\{K-S,0\} \\ &= (-SN(-d) + Ke^{-r(T-t)}N(-d_2)) - 0 \\ &\to (0-0) - 0 \\ &= 0\end{aligned} \tag{38}$$

If $S \to 0$,

$$d \to -\infty, d_2 \to -\infty \Rightarrow N(-d) = N(-d_2) = 1 \tag{39}$$

Therefore,

$$\begin{aligned}V(S) &= P_BS(S) - \max\{K-S,0\} \\ &= (-SN(-d) + Ke^{-r(T-t)}N(-d_2)) - K \\ &\to (0 + Ke^{-r(T-t)}) - K \\ &= K(e^{-r(T-t)} - 1)\end{aligned} \tag{40}$$

The graph generated by similar python codes -

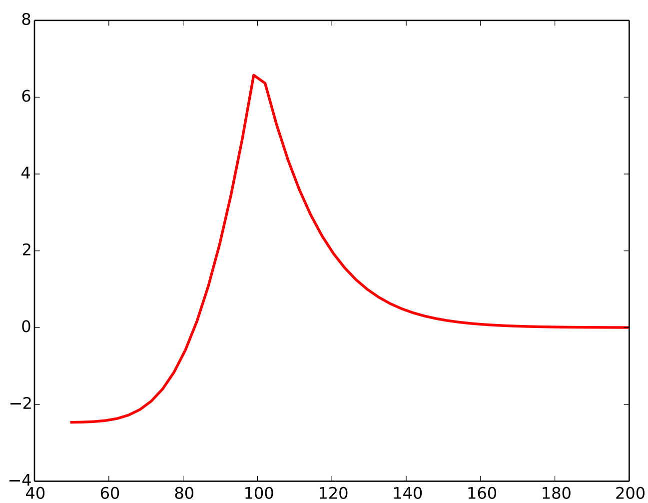

Figure 9: Plot of $V(S)$ as a function of S

Problem 6

(i)

The cash flow of the coupon bound is

$$\begin{aligned} v\_cash\_flow &= [1.75, 1.75, 1.75, 1.75, 101.75] \\ t\_cash\_flow &= [1./12, 7./12, 13./12, 19./12, 25./12] \end{aligned} \tag{41}$$

Given the riskfree zero rate curve, we are able to generate the value of the bond as

$$B = \sum_i v_i e^{-t_i r(0,t_i)} = 101.369304142 \tag{42}$$

The bond value can also be written as the expression of yield,

$$B = \sum_i v_i e^{-y t_i} \tag{43}$$

We solved $y = 0.035126068375$.

(ii)

Given the yield, the modified duration is

$$D = \frac{1}{B} \sum_i v_i t_i e^{-t_i y} = 1.998752969544 \tag{44}$$

And the convexity is

$$C = \frac{1}{B} \sum_i v_i t_i^2 e^{-t_i y} = 4.115468941420 \tag{45}$$

Problem 7

(i)

The instantateous interest rate curve is

$$\begin{aligned} r_c(t) &= \frac{d r_c(0,t) t}{dt} \\ &= t \frac{d r_c(0,t)}{dt} + r_c(0,t) \\ &= 0.01t \times \frac{2+t}{2+2t} \times \frac{(2+t) - t}{(2+t)^2} + r_c(0,t) \\ &= \frac{0.02t}{(2+2t)(2+t)} + 0.03 + 0.01 \ln(1 + \frac{t}{2+t}) \end{aligned} \tag{46}$$

(ii)

Since

$$(1 + \frac{r_m(0,t)}{m})^m = \exp(r_c(0,t)) \tag{47}$$

then for annually compounded zero rate curve,

$$\begin{aligned} r_1(0,t) &= \exp(r_c(0,t)) - 1 \\ &= \exp(0.03) * (1 + \frac{t}{2+t})^{0.01} - 1 \\ &= e^{0.03}(1 + \frac{t}{2+t})^{0.01} - 1 \end{aligned} \tag{48}$$

(iii)

For semiannually compounded zero rate curve,

$$\begin{aligned} r_2(0,t) &= 2(\exp(\frac{r_c(0,t)}{2}) - 1) \\ &= 2(\exp(0.015) * (1 + \frac{t}{2+t})^{0.005} - 1) \\ &= 2e^{0.015}(1 + \frac{t}{2+t})^{0.005} - 2 \end{aligned} \tag{49}$$

Problem 8

(i)

Given the instantaneous rate curve r(t)

$$r(t) = \frac{0.05}{1 + \exp(-(1+t)^2)} \tag{50}$$

we can use simpson's rule to compute the 3 months, 9 months, 15 months and 21 months discount factors as

$$\begin{aligned} disc(3./12) &= \exp(-\int_0^{3/12} r(\tau)d\tau) = 0.990302 \\ disc(9./12) &= \exp(-\int_0^{9/12} r(\tau)d\tau) = 0.968268 \\ disc(15./12) &= \exp(-\int_0^{15/12} r(\tau)d\tau) = 0.944845 \\ disc(21./12) &= \exp(-\int_0^{21/12} r(\tau)d\tau) = 0.92157243 \end{aligned} \tag{51}$$

(ii)

Cash flow of the 21 months year semiannual coupon bond with coupon rate 5% is

$$v\_cash\_flow = [2.5, 2.5, 2.5, 102.5] t\_cash\_flow = [3./12, 9./12, 15./12, 21./12] \tag{52}$$

The price of the 21 months year semiannual coupon bond with coupon rate 5% is

$$B = \sum_i v_i disc(t_i) = 101.719713273 \tag{53}$$

Problem 9

(i)

The cash flows of the fixed leg is

$$\begin{aligned} fixed\_cash\_flow &= [200000.0, 200000.0, 200000.0, 10200000.0] \\ t &= [0.5, 1.0, 1.5, 2.0] \end{aligned} \tag{54}$$

The cash flows of the floating leg is

$$\begin{aligned} Notional \times \frac{r(0,0.5)}{2} &= 75000.0 \\ Notional \times \frac{r(0.5,1.0)}{2} &= 112500.0 \\ Notional \times \frac{r(1.0,1.5)}{2} &= 150000.0 \\ Notional \times \frac{r(1.5,2.0)}{2} &= 10106250.0 \end{aligned} \tag{55}$$

where the units are dollars. Therefore the cash flows of the swap is

$$\begin{aligned} swap\_cash\_flow &= fixed\_cash\_flow - float\_cash\_flow \\ &= [200000.0, 200000.0, 200000.0, 10200000.0] \\ &- [75000.0, 112500.0, 150000.0, 10106250.0] \\ &= [125000.00, 87500.00, 50000.00, 93750.00] \end{aligned} \tag{56}$$

where the units are dollars and their corresponded time is $t = [0.5, 1.0, 1.5, 2.0]$.

(ii)

At $t_1 = 4./12$, the value of the fixed rate bond underlying the swap

$$v_{fixed} = \sum_i fixed\_cash\_flow[i] \times (1 + 0.5r(t_1, t_i))^{-2(t_i - t)} = 38,884,742.17 \tag{57}$$

dollars. For the floating leg, the next payment (2 months later) is decided 4 months ago using zero rate curve at time 0, that is

$$Notional \times \frac{r_0(0,0.5)}{2} = 75,000 \tag{58}$$

million dollars, where $r_0(0, 0.5)$ is 6-month LIBOR observed 4 months ago; 0.075 million dollars is the next interest rate payment that will be paid 2 months from now. The floating rate bond will be worth its par value of 10. million dollars, immediately after the next interest payment of 0.075.
The value of the floating rate bond underlying the swap is

$$v_{float} = (75,000 + 10,000,000)(1 + \frac{r(0, 2./12)}{2})^{-\frac{2./12}{0.5}} = 10,031,991.35 \tag{59}$$

Therefore the value of the swap is

$$v_{swap} = v_{fixed} - v_{float} = 28852750.82 \tag{60}$$

Problem 10

According the question, we have

$$2.5e^{-4.5\%(.5)} + 2.5e^{-4.5\%(1.0)} + 102.5e^{-1.5R} = 100. \tag{61}$$

Therefore the 18-months LIBOR rate is

$$R = 0.049495980179 \approx 4.949598\% \tag{62}$$

Problem 11

The value of the fixed rate bond underlying the swap is

$$2.5e^{-0.04\times\frac{4}{12}} + 2.5e^{-0.04\times\frac{10}{12}} + 102.5e^{-0.04\times\frac{16}{12}} = 102.061481851 \tag{63}$$

The value of the floating rate bond underlying the swap is

$$(2.5 + 100)e^{-0.04\times\frac{4}{12}} = 101.142404085 \tag{64}$$

2.5 equals $.5 \times 5\% \times 100$, where 5% is 6-month LIBOR observed 2 months ago; 2.5 is the next interest rate payment that will be paid 4 months from now. The floating rate bond will be worth its par value of 100 immediately after the next interest payment of 2.5.
Since the firm in question receives floating and pays fixed, the value of the swap is

$$101.142404085 - 102.061481851 = -0.919077766058 \tag{65}$$

where the unit is million dollars, so it's -919077.77 dollars.

Problem 12

(i)

The value of a put options with strike price 40 and maturity of three months, on a non dividend paying stock with lognormal distribution with volatility 30% is

$$P = e^{-r\tau}KN(-d_2) - Se^{-q\tau}N(-d) = 5.4075696734 \tag{66}$$

Therefore the value of the portfolio is

$$V(\Pi) = 2000P + 400S + 10000 = 34815.1393468 \approx 34815.14 \tag{67}$$

(ii)

Delta of the current portfolio is

$$\Delta(\Pi) = 2000\Delta\Pi + 400 = -1165.71252025 \tag{68}$$

To make the portfolio Delta-neutral, we have to long 1166 shares of the same stock.

(iii)

	Options Position	Asset Position	Cash Position
Week 0 - AH:	2000P=10815.14	1566S=54810.00	-30810.00
Week 1 - BH:	2000P=4573.58	1566S=62640.00	-30821.74
Week 1 - AH:	2000P=4573.58	914S=36560.00	-4741.74
Week 2 - BH:	2000P=9306.89	914S=32904.00	-4743.54
Week 2 - AH:	2000P=9306.89	1448S=52128.00	-23967.54
Week 3 - BH:	2000P=15948.01	1448S=46336.00	-23976.67
Week 3 - AH:	2000P=15948.01	1832S=58624.00	-36264.67
Week 4 - BH:	2000P=7921.98	1832S=67784.00	-36278.48
Week 4 - AH:	2000P=7921.98	1319S=48803.00	-17297.48

	Hedging Action
Week 1	sell 652 shares of u.a
Week 2	buy 534 shares of u.a
Week 3	buy 384 shares of u.a
Week 4	sell 513 shares of u.a

Problem 13

(i)

Delta of current portfolio is

$$\Delta(\Pi) = 1000\Delta(C) + 600\Delta(P) = 182.729986558 \tag{69}$$

To delta hedge, we need to sell 183 shares of underlying asset, which will create cash

$$c_1 = 183S = \$9150.0 \tag{70}$$

(ii)

Suppose we will long x_1 units of the underlying asset and x_2 units of the 9-month ATM call to Delta hedge and Gamma hedge,

$$\begin{aligned} \Delta(\Pi) &= 1000\Delta(C) + 600\Delta(P) + x_1 + x_2\Delta(C_{ATM}) = 0 \\ \Gamma(\Pi) &= 1000\Gamma(C) + 600\Gamma(P) + x_2\Gamma(C_{ATM}) = 0 \end{aligned} \tag{71}$$

where

$$\begin{aligned} \Delta(C) &= e^{-q\tau}N(d) = 0.294224765689 \\ \Delta(P) &= -e^{-q\tau}N(-d) = -0.185824631885 \\ \Delta(C_{ATM}) &= 0.560287227854 \\ \Gamma(C) &= e^{-q\tau}\frac{N(d)}{S\sigma\sqrt{\tau}} = 0.0484733742912 \\ \Gamma(P) &= e^{-q\tau}\frac{N(d)}{S\sigma\sqrt{\tau}} = 0.0377099419132 \\ \Gamma(C_{ATM}) &= 0.0447044338962 \end{aligned} \tag{72}$$

Therefore we can derive

$$\begin{aligned} x_1 &= 708.36846442 \approx 708 \\ x_2 &= -1590.43149062 \approx -1590 \end{aligned} \tag{73}$$

That is, we need to long 708 units of the underlying asset and short 1590 units of the 9-month ATM call, this will create cash

$$c_2 = -x_1 S - x_2 C_{ATM} = -29449.1790844 \tag{74}$$

which means cost 29449.18 dollars.

(iii)

For the three portfolios, the original values are

$$\begin{aligned} V(\Pi_0) &= 1000C + 600P = 1687.0368332 \\ V(\Pi_1) &= 1000C + 600P - 183S + c_1 = 1687.0368332 \\ V(\Pi_2) &= 1000C + 600P + 708S - 1590C_{ATM} + c_2 = 1687.0368332 \end{aligned} \tag{75}$$

After 1. day ($\Delta t = 1./252$ year) the asset price jumps at 54, and the volatility of the asset also jumps to 0.3, the values of the three portfolios become

$$\begin{aligned} V(\Pi_0) &= 1000C + 600P = 4917.05250929 \\ V(\Pi_1) &= 1000C + 600P - 183Se^{q\Delta t} + c_1 e^{r\Delta(t)} = 4185.70012914 \\ V(\Pi_2) &= 1000C + 600P + 708Se^{q\Delta t} - 1590C_{ATM} + c_2 e^{r\Delta t} = 1085.56128968 \end{aligned} \tag{76}$$

Therefore the change of the value is

$$\begin{aligned} \Delta V(\Pi_0) &= 3230.01567609 \\ \Delta V(\Pi_1) &= 2498.66329594 \\ \Delta V(\Pi_2) &= -601.475543525 \end{aligned} \tag{77}$$

Problem 14

Recall the Black-Scholes formula

$$C = Se^{-q(T-t)}N(d) - Ke^{-r(T-t)}N(d_2) \tag{78}$$

and the 'magic of Greeks computations', i.e., the fact that

$$Se^{-q(T-t)}N'(d) = Ke^{-r(T-t)}N'(d_2) \tag{79}$$

Then,

$$\begin{aligned} \frac{\partial C}{\partial K} &= Se^{-q(T-t)}N'(d)\frac{\partial d}{\partial K} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial K} - e^{-r(T-t)}N(d_2) \\ &= Se^{-q(T-t)}N'(d)(\frac{\partial d}{\partial K} - \frac{\partial d_2}{\partial K}) - e^{-r(T-t)}N(d_2) \\ &= -e^{-r(T-t)}N(d_2) \end{aligned} \tag{80}$$

since $d - d_2 = \sigma\sqrt{T-t}$ and therefore

$$\frac{\partial d}{\partial d_K} - \frac{\partial d_2}{\partial d_K} = 0 \tag{81}$$

By continously differentiating with respect to K, we obtain that

$$\begin{aligned} \frac{\partial^2 C}{\partial K^2} &= -e^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial K} \\ &= -e^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{d_2^2/2}(-\frac{1}{K\sigma\sqrt{T-t}}) \\ &= \frac{1}{K\sigma\sqrt{2\pi(T-t)}}\exp(-r(T-t) - \frac{d_2^2}{2}) \end{aligned} \tag{82}$$

By differentiating with respect to K the Put-Call parity formula

$$P + Se^{-q(T-t)} - C = Ke^{-r(T-t)} \tag{83}$$

and using former equations, we find that

$$\frac{\partial P}{\partial K} = \frac{\partial C}{\partial K} + e^{-r(T-t)} = e^{-r(T-t)}N(-d_2) \tag{84}$$

By differentiating twice respect to K the Put-Call parity and using former equations, we conclude that

$$\frac{\partial^2 P}{\partial K^2} = \frac{\partial^2 C}{\partial K^2} = \frac{1}{K\sigma\sqrt{2\pi(T-t)}}\exp(-r(T-t) - \frac{d_2^2}{2}) \tag{85}$$

Problem 15

The following python will be able to generate the Deltas and Gammas with different strikes

```python
from mibian import *

spotPrice = 50.
dividendsYield = 0.02 * spotPrice
volatility = 0.3 * 100
riskFreeRate = 0.02 * 100
ls_strike = [40.,45.,50.,55.,60.]
maturity = 3./12 * 365

for f_strike in ls_strike:
    option = Me([spotPrice, f_strike, riskFreeRate, dividendsYield, maturity],
        volatility=volatility)
    straddleDelta = option.putDelta + option.callDelta
    straddleGamma = option.gamma + option.gamma
    print f_strike, "straddleDelta:", round(straddleDelta,6)
    print f_strike, "straddleGamma:", round(straddleGamma,6)
```

The output is

```
40.0 straddleDelta: 0.877461
40.0 straddleGamma: 0.031223
45.0 straddleDelta: 0.560271
45.0 straddleGamma: 0.078248
50.0 straddleDelta: 0.059487
50.0 straddleGamma: 0.105557
55.0 straddleDelta: -0.422676
55.0 straddleGamma: 0.090472
60.0 straddleDelta: -0.742192
60.0 straddleGamma: 0.055242
```

Problem 16

AoN Call

This pays out one unit of asset if the spot is above the strike at maturity. Its value now is given by,

$$\begin{aligned} C &= e^{-r(T-t)}E[S_T I_{S_T>K}] \\ &= e^{-r(T-t)}\int_{Z_0}^{\infty} Se^z\phi(z)dz \\ &= SN(d) \end{aligned} \tag{86}$$

where

$$d = \frac{\ln\frac{S}{K} + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \tag{87}$$

which give us the following PDEs

$$\begin{aligned} \frac{\partial d}{\partial S} &= \frac{1}{S\sigma\sqrt{T-t}} \\ \frac{\partial d}{\partial t} &= \frac{d}{2(T-t)} - \frac{r+\sigma^2/2}{\sigma\sqrt{T-t}} \\ \frac{\partial d}{\partial r} &= \frac{\sqrt{T-t}}{\sigma} \\ \frac{\partial d}{\partial \sigma} &= \sqrt{T-t} - \sigma d \end{aligned} \tag{88}$$

Delta:

$$\begin{aligned} \Delta(C) &= \frac{\partial C}{\partial S} \\ &= N(d) + SN'(d)\frac{\partial d}{\partial S} \\ &= N(d) + S(\frac{1}{\sqrt{2\pi}}e^{-d^2/2})(\frac{1}{\sigma\sqrt{T-t}}\frac{K}{S}\frac{1}{K}) \\ &= N(d) + \frac{e^{-d^2/2}}{\sigma\sqrt{2\pi(T-t)}} \\ &= N(d) + \frac{1}{\sigma\sqrt{T-t}}N'(d) \end{aligned} \tag{89}$$

Gamma:

$$\begin{aligned} \Gamma(C) &= \frac{\partial \Delta(C)}{\partial S} \\ &= N'(d)\frac{\partial d}{\partial S} + \frac{e^{-d^2/2}}{\sigma\sqrt{2\pi(T-t)}}(-d)\frac{\partial d}{\partial S} \\ &= \frac{e^{-d^2/2}}{\sigma\sqrt{2\pi(T-t)}S}(1 - \frac{d}{\sigma\sqrt{T-t}}) \\ &= \frac{N'(d)}{\sigma S\sqrt{T-t}}[1 - \frac{d}{\sigma\sqrt{T-t}}] \end{aligned} \tag{90}$$

Vega:

$$\begin{aligned}
Vege(C) &= \frac{\partial C}{\partial \sigma} \\
&= N'(d)\frac{\partial d}{\partial \sigma} \\
&= N'(d)\frac{\sigma(T-t) - (\ln\frac{S}{K} + (r+\sigma^2/2)(T-t))\sqrt{T-t}}{\sigma^2(T-t)} \\
&= S(\frac{1}{\sqrt{2\pi}}e^{-d^2/2})(\frac{\sigma\sqrt{T-t} - (\ln\frac{S}{K} + (r+\sigma^2/2)(T-t))}{\sigma^2\sqrt{T-t}}) \\
&= S(\sqrt{T-t} - \sigma d)N'(d)
\end{aligned} \tag{91}$$

Theta:

$$Theta(C) = SN'(d)\frac{d}{2(T-t)} - \frac{r+\frac{\sigma^2}{2}}{\sigma\sqrt{T-t}} \tag{92}$$

where $N'(d)$ is the density function of the standard normal distribution and

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \tag{93}$$

AoN Put

Given the definitions of an asset-or-nothing call and put option and referring to the calculated price of the call option the put-call parity gives us the price of the put option as follows:

$$\begin{aligned}
P &= SC \\
&= S - SN(d) \\
&= S(1-N(d)) \\
&= SN(-d)
\end{aligned} \tag{94}$$

Below are the conclusion of greeks calculated in similar ways:

$$\begin{aligned}
\Delta(P) &= N(-d) - \sqrt{1}\sigma\sqrt{T-t}N'(-d) \\
\Gamma(P) &= \Gamma(C_{AoN}) \\
Vega(P) &= -S(\sqrt{T-t} - \sigma d)N'(-d) \\
Theta(P) &= -S\frac{d}{2(T-t)} - \frac{r+\sigma^2/2}{\sigma\sqrt{T-t}}N'(-d)
\end{aligned} \tag{95}$$

CoN Call

If the asset price at maturity is higher than the strike, payoff is B else it is zero. This is also called a bet. This is simple to price because it is the probability of receiving B at maturity, discounted to today

$$C = e^{-r(T-t)}BN(d_2) \tag{96}$$

where

$$d_2 = \frac{\ln\frac{S}{K} + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \tag{97}$$

Below are the conclusion of greeks calculated:

$$\begin{aligned}
\Delta(C) &= B\frac{e^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{T-t}} \\
\Gamma(C) &= -e^{-r(T-t)}B\frac{d_1 N'(d_2)}{\sigma^2 S^2 (T-t)} \\
Vega(C) &= -e^{-r(T-t)}B\frac{N'(d_2)d_1}{\sigma} \\
Theta(C) &= re^{-r(T-t)}BN(d_2) + e^{-r(T-t)}BN'(d_2)(\frac{d_1}{2(T-t)} - \frac{r}{\sigma\sqrt{T-t}})
\end{aligned} \tag{98}$$

CoN Put

Through put-call parity,

$$P = e^{-r(T-t)}B(1 - N(d_2)) \tag{99}$$

Below are the conclusion of greeks calculated:

$$\begin{aligned}
\Delta(P) &= -\Delta(C_{CoN}) \\
\Gamma(P) &= \Gamma(C_{CoN}) \\
Vega(P) &= Vega(C_{CoN}) \\
Theta(C) &= re^{-r(T-t)}B(1 - N(d_2)) - e^{-r(T-t)}BN'(d_2)(\frac{d_1}{2(T-t)} - \frac{r}{\sigma\sqrt{T-t}})
\end{aligned} \tag{100}$$

Problem 17

If using python, there are implemented libraries for newton's method and black-scholes pricing, therefore to realize the problem in the question is quite simple.

```
from scipy.optimize import newton
from mibian import *

def func(strike):
    spotPrice = 50.
    riskFreeRate = 0.02 * 100
    dividendsYield = 0.02 * spotPrice
    maturity = 3./12 * 365
    volatility = 0.3 * 100
    option = Me([spotPrice, strike, riskFreeRate, dividendsYield, maturity], volatility=volatility)
    return option.callDelta - 0.5

print newton(func=func, x0=spotPrice)
```

But if this problem requires my own implementation for newton's method, see follows.

```
def derivative(f):
    def compute(x, dx):
        return (f(x+dx) - f(x))/dx
    return compute
```

Problem 17 continued on next page...

```
def newton(func, x0):
    dx=1e-6
    tolerance=1e-6
    df = derivative(f)
    while True:
        x1 = x - f(x)/df(x, dx)
        t = abs(x1 - x)
        if t < tolerance:
            break
        x = x1
```

Problem 18

Following is the python code to derive implied volatility using the secant method with initial guesses $x_0 = 0.5$ and $x_1 = 0.501$, and Newtons method with initial guess 0.5.

```
from scipy.optimize import newton
from scipy.optimize import secant
from mibian import *

def func(implied_volatility):
    spotPrice = 30.
    strike = 27.
    riskFreeRate = 0.04 * 100
    dividendsYield = 0.01 * spotPrice
    maturity = 7./12 * 365
    option = Me([spotPrice, strike, riskFreeRate, dividendsYield, maturity],
        volatility=implied_volatility)
    return option.callPrice - 4.5

print newton(func=func, x0=0.5)
print secant(func=func, x0=0.5, x1=0.501)
```

Both methods return the implied volatility as 28.0090064420%, or 0.280090064420.

Problem 19

(i)

Python code to compute the implied volatility using the bisection method, the secant method, and Newtons method.

```
from scipy.optimize import bisect
from scipy.optimize import newton
from scipy.optimize import secant
from mibian import *

def func(implied_volatility):
    spotPrice = 40.
    strike = 40.
    riskFreeRate = 0.025 * 100
```

```
    dividendsYield = 0.01 * spotPrice
    maturity = 5./12 * 365
    option = Me([spotPrice, strike, riskFreeRate, dividendsYield, maturity],
        volatility=implied_volatility)
    return option.callPrice - 2.75

print bisect(func, 0.01, 100)
print newton(func, 50)
print secant(func, 50, 49)
```

All three methods return the same answer 25.6903172365%, or 0.256903172365.

(ii)

$$\sigma_{imp,approx} = \frac{\sqrt{2\pi}}{S\sqrt{T}} \frac{C - \frac{(r-q)T}{2}S}{1 - \frac{(r+q)T}{2}} = 0.256710246571 \tag{101}$$

Therefore the relative error is

$$\frac{|\sigma_{imp,approx} - \sigma_{imp}|}{\sigma_{imp}} = 0.000750966958487 \approx 0.000750966958 \tag{102}$$

Problem 20

In other words, this is to compute

$$V(0) = e^{-rT} E_{RN}[\max(\ln(\frac{S(T)}{K}), 0)] \tag{103}$$

Recall that

$$S(T) = S(0)\exp((r - \sigma^2/2)T + \sigma\sqrt{T}Z) \tag{104}$$

And note that

$$S(T) \geq K \Rightarrow Z \geq \frac{\ln(K/S(0)) - (r - \sigma^2/2)T}{\sigma\sqrt{T}} = -d_2 \tag{105}$$

Then,

$$\begin{aligned} V(0) &= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} \ln(\frac{S(T)}{K})e^{-x^2/2}dx \\ &= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} \ln(\frac{S(0)\exp((r-\sigma^2/2)T + \sigma\sqrt{T}x)}{K})e^{-x^2/2}dx \\ &= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} [\ln\frac{S(0)}{K} + (r-\sigma^2/2)T]e^{-x^2/2} + \sigma\sqrt{T}xe^{-x^2/2}dx \\ &= e^{-rT}[\ln\frac{S(0)}{K} + (r-\sigma^2/2)T]N(d_2) + \frac{\sigma\sqrt{T}e^{-rT-d_2^2/2}}{\sqrt{2\pi}} \end{aligned} \tag{106}$$