Advanced Calculus with Financial Engineering Applications

Refresher Seminar, Summer 2014

Homework Set 1

Assigned: July 17; Due: July 21, at 6pm

This homework will count as 20% of your final grade.

Options and Arbitrage

(1) Consider a non–dividend–paying asset with bid and ask spot prices $S_{bid} = 44.9$ and $S_{ask} = 45.1$. The bid and ask prices for nine months European options with strike \$47 on this asset are

$$C_{bid} = 3.1; \quad C_{ask} = 3.2;$$
$$P_{bid} = 4.9; \quad P_{ask} = 5.$$

Is there an arbitrage opportunity present, and how would you take advantage of it? Assume that you can deposit and borrow money at 153bps and at 158bps, respectively.

(2) (i) Call options with strikes \$50 and \$55 on the same underlying asset and with the same maturity are trading for \$8 and \$3, respectively (there is no bid–ask spread). Explain why an arbitrage is present, and how would you make a profit?

Is there an arbitrage opportunity if the options above have bid–ask prices of \$7.8 and \$8.2, and \$2.9 and \$3.1, respectively?

(ii) Put with strikes \$60 and \$64 on the same underlying asset and with the same maturity are trading for \$7 and \$11, respectively. Explain why an arbitrage is present, and how would you make a profit?

(3) Create a portfolio with the following payoff at time T:

$$V(T) = \begin{cases} 50 - 2S(T), & \text{if } 0 \leq S(T) < 30; \\ 20 - S(T), & \text{if } 30 \leq S(T) < 50; \\ 2S(T) - 130, & \text{if } 50 \leq S(T), \end{cases}$$

where $S(T)$ is the spot price at time T of a given asset. Use plain vanilla options with maturity T as well as cash positions and positions in the asset itself. Assume, for simplicity, that the asset does not pay dividends and that interest rates are zero.

(4) Assume that the risk–free interest rates are flat at 3%. Consider a non–dividend–paying asset with spot price \$40. The values of the following options on this asset are given below:

Option Type	Strike	Maturity	Value
Put	36	5 months	\$1.2
Call	36	5 months	\$5.5
Put	39	5 months	\$2.4
Call	39	5 months	\$3.5
Put	42	5 months	\$4
Call	42	5 months	\$2.5

(i) Synthesize a long position in a 39–42 bear spread using the 39 strike and the 42 strike put options. Draw the payoff diagram and the P&L diagram at maturity of this position. For what values of the underlying asset at maturity would the bear spread be profitable (i.e., have a positive P&L)?

(ii) Synthesize a long position in a 36–39 bear spread using the 36 strike and the 39 strike call options. Draw the payoff diagram and the P&L diagram at maturity of this position. For what values of the underlying asset at maturity would the bear spread be profitable (i.e., have a positive P&L)?

(iii) Synthesize a long position in a 36–39–42 butterfly spread. Draw the payoff diagram and the P&L diagram at maturity of this position. For what values of the underlying asset at maturity would the butterfly spread be profitable (i.e., have a positive P&L)?

(iv) Draw the payoff diagram and the P&L diagram at maturity of the option combinations below, and indicate for what values of the underlying asset at maturity would the option combinations be profitable (i.e., have a positive P&L):

- a 36–42 strangle;
- a 39 straddle;
- a 36–42 collar;
- a 36–42 risk reversal.

(5) Let $V(S) = C_{BS}(S) - \max(S - K, 0)$ be the premium of the value of a European call option on a non–dividend–paying asset over its intrinsic value $\max(S - K, 0)$, where $C_{BS}(S)$ is the Black–Scholes value of the plain vanilla European call option with strike K and spot price S.

(i) Show that the maximum value of $V(S)$ is obtained at the money, i.e., for $S = K$.

(ii) What is the asymptotic behavior of $V(S)$ as $S \to \infty$?

(iii) Plot $V(S)$ as a function of S.

(iv) Repeat the exercise for a European put option on a non–dividend–paying asset, i.e., find the maximum and the asymptotic behavior (both for $S \searrow 0$ and for $S \to \infty$) of $V(S) = P_{BS}(S) - \max(K - S, 0)$. Also, plot $V(S)$ as a function of S.

Interest Rates and Bonds

(6) (i) Find the yield of a 25 months semiannual coupon bond with coupon rate 3.5%, if the risk–free zero rate curve is

$$r(0,t) = 0.015 + \frac{t}{100 + \sqrt{1+t^2}}.$$

(ii) What are the modified duration and convexity of the bond?

(7) Assume that the continuously compounded zero rate curve is

$$r_c(0,t) = 0.03 + 0.01 \ln\left(1 + \frac{t}{2+t}\right).$$

(i) find the instantaneous interest rate curve;

(ii) compute the corresponding annually compounded zero rate curve;

(iii) compute the corresponding semiannually compounded zero rate curve.

(8) The instantaneous rate curve $r(t)$ is given by

$$r(t) = \frac{0.05}{1+\exp(-(1+t)^2)}.$$

Assume that interest is compounded continuously.

(i) Compute the 3 months, 9 months, and 15 months discount factors with six decimal digits accuracy, and compute the 21 months discount factor with eight decimal digits accuracy, using Simpson's Rule; recall that the discount factor corresponding to time t is

$$\exp\left(-\int_0^t r(\tau)\, d\tau\right).$$

(ii) Find the price of a 21 months year semiannual coupon bond with coupon rate 5% (and face value 100).

(9) A two year semiannual swap exchanges LIBOR for 4% on a notional of \$10 million. The zero rate curve at time 0 is $0.01 + \frac{t}{100}$.

(i) Describe the cash flows of the swap, assuming that the zero rate curve observed in six months is $0.02 + \frac{t}{200}$, the zero rate curve observed in one year is $0.025 + \frac{t}{100}$, and the zero rate curve observed in 18 months is $0.02 + \frac{t}{400}$.

(ii) What is the value of the swap in four months from now, assuming that the zero rate curve four months from now will be $0.025 + \frac{t}{200}$?

Note: All zero rate curves above are semiannually compounded.

(10) The LIBOR zero rate curve is flat at 4.5% (continuously compounded) for the next year. The 18-months semiannual swap rate is 5%. What is the 18-months LIBOR rate (also continuously compounded)?

(11) A financial institution has entered into a 4-year fixed-for-floating semiannual interest rate swap. It has agreed to pay LIBOR in exchange for 5% fixed on a notional of \$100 million. The swap has a remaining life of 16 months. The 6-month LIBOR rate at the last payment date was 5%. If the LIBOR curve is now flat at 4% continuously compounded, what is the value of the swap for the party paying fixed?

Options Hedging

(12) You hold a portfolio made of a long position in 2000 put options with strike price 40 and maturity of three months, on a non–dividend–paying stock with lognormal distribution with volatility 30%, a long position in 400 shares of the same stock, which has spot price \$35, and \$10,000 in cash. Assume that the risk-free rate is constant at 2%.

(i) How much is the portfolio worth?

(ii) How do you adjust the stock position to make the portfolio Delta–neutral?

(iii) The spot prices of the underlying asset at the end of the next four weeks are

$$\$40,\ \$36,\ \$32,\ \$37.$$

You rebalance the hedge at the end of each week, to make the portfolio Delta–neutral. Record the action you take and the portfolio and the action you are taking to do so in a table like the one below ("BH" means before balancing the hedge; "AH" means after balancing the hedge). Recall that cash accumulates interest over each time period.

	Options Position	Asset Position	Cash Position
Week 0 – AH			
Week 1 – BH			
Week 1 – AH			
Week 2 – BH			
Week 2 – AH			
Week 3 – BH			
Week 3 – AH			
Week 4 – BH			
Week 4 – AH			

	Hedging Action
Week 1	
Week 2	
Week 3	
Week 4	

(13) Assume that an asset with spot price 50 paying dividends continuously at rate $q = 0.02$ has lognormal distribution with volatility $\sigma = 0.2$. Assume that the risk–free rates are constant and equal to $r = 0.04$.

Consider a portfolio made of a long position in 1000 six months calls struck at 55 and a long position in 600 six months put options struck at 45.

(i) Delta hedge your portfolio using units of the underlying asset. Keep track of the cash position as well.

(ii) Delta hedge and Gamma hedge your portfolio using units of the underlying asset and by taking a position in a nine months ATM call on the asset. (All options prices are assumed to be equal to the Black–Scholes values.)

(iii) Assume that the asset price jumps at 54 the next day, and that the volatility of the asset also jumps to 0.3. What would have been the change in value of the initial portfolio, what is the change in value of the delta hedged portfolio, and what is the change in value of the delta and gamma hedged portfolio? Explain.

Black–Scholes

(14) In the Black–Scholes framework, compute

$$\frac{\partial C}{\partial K} \quad \text{and} \quad \frac{\partial^2 C}{\partial K^2}.$$

Then, use the Put–Call parity to compute

$$\frac{\partial P}{\partial K} \quad \text{and} \quad \frac{\partial^2 P}{\partial K^2}.$$

(15) Assume that an asset with spot price 50 paying dividends continuously at rate $q = 0.02$ has lognormal distribution with volatility $\sigma = 0.3$. Assume that the risk–free rates are constant and equal to $r = 0.04$. What are deltas and gammas of a long positions in three months straddles on this asset, with strikes 40, 45, 50, 55, and 60, respectively?

(16) Let r be the constant risk–free rate.
Consider the following options with maturity T and strike K on an underlying asset with lognormal distribution with drift μ and volatility σ:

- Asset–or–Nothing Call;
- Cash–or–Nothing Call with cash payout B;
- Asset–or–Nothing Put;
- Cash–or–Nothing Put with cash payout B.

Find the Black–Scholes values, the Delta, Gamma, vega, and Theta of all these options.

(17) Consider a call option with strike K and maturity T, on a lognormally distributed underlying asset with spot price S, volatility σ, and paying dividends continuously at rate q. Assume that the risk–free interest rates are constant equal to r.

Finding the value of the strike price such that the Δ of the call is 0.5 requires solving

$$\Delta(C) \;=\; e^{-qT} N(d_1) \;=\; \frac{1}{2} \tag{1}$$

for K, where $N(x)$ be the cumulative density of the standard normal variable and

$$d_1 \;=\; \frac{\ln\left(\frac{S}{K}\right) \;+\; \left(r - q + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}.$$

Write down the Newton's method recursion for solving (1) for K.

Implied Volatility

(18) A seven months call with strike 27 on an underlying asset with spot price 30 and paying dividends continuously at a 1% rate is worth \$4.5. Assume that the risk free interest rate is constant at 4%.

Compute the implied volatility with six decimal digits accuracy, using the secant method with initial guesses $x_0 = 0.5$ and $x_1 = 0.501$, and Newton's method with initial guess 0.5.

(19) A five months at–the–money call on an underlying asset with spot price 40 paying dividends continuously at a 1% rate is worth \$2.75. Assume that the risk free interest rate is constant at 2.5%.

(i) Compute the implied volatility with six decimal digits accuracy, using the bisection method on the interval $[0.0001, 1]$, the secant method with first initial guess 0.5 and second initial guess 0.49, and Newton's method with initial guess 0.5.

(ii) Let σ_{imp} be the implied volatility previously computed using Newton's method. Use the formula

$$\sigma_{imp,approx} \;\approx\; \frac{\sqrt{2\pi}}{S\sqrt{T}} \, \frac{C \;-\; \frac{(r-q)T}{2} S}{1 - \frac{(r+q)T}{2}},$$

to compute an approximate value $\sigma_{imp,approx}$ for the implied volatility, and compute the relative error

$$\frac{|\sigma_{imp,approx} - \sigma_{imp}|}{\sigma_{imp}}.$$

Risk–neutral valuation

(20) Use risk–neutral valuation to find the value of a log option, i.e., of an option paying

$$\max\left[\ln\left(\frac{S(T)}{K}\right), 0\right]$$

at maturity. The underlying asset has lognormal distribution with volatility σ and pays dividends continuously at rate q.