Advanced Calculus: Quiz #1

Due on July 15, 2014

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Problem 1

Compute

$$\lim_{x\to\infty} 3x^2 - \sqrt{9x^4 + 12x^2 + 7} \tag{1}$$

Answer

$$\begin{aligned}
\lim_{x\to\infty} 3x^2 - \sqrt{9x^4 + 12x^2 + 7} &= \lim_{x\to\infty} \frac{(3x^2 - \sqrt{9x^4 + 12x^2 + 7})(3x^2 + \sqrt{9x^4 + 12x^2 + 7})}{3x^2 + \sqrt{9x^4 + 12x^2 + 7}} \\
&= \lim_{x\to\infty} \frac{(3x^2)^2 - (\sqrt{9x^4 + 12x^2 + 7})^2}{3x^2 + \sqrt{9x^4 + 12x^2 + 7}} \\
&= \lim_{x\to\infty} \frac{-12x^2 - 7}{3x^2 + \sqrt{9x^4 + 12x^2 + 7}} \\
&= \lim_{x\to\infty} \frac{-12 - \frac{7}{x^2}}{3 + \sqrt{9 + \frac{12}{x^2} + \frac{7}{x^4}}} \\
&= \frac{-12}{3 + \sqrt{9}} \\
&= -2
\end{aligned} \tag{2}$$

Problem 2

Compute

$$\begin{aligned}
&\frac{\partial}{\partial x}(\frac{1}{1+tx})^n \\
&\frac{\partial}{\partial x_j}(\prod_{i=1}^{n}\frac{1}{1+tx_i})
\end{aligned} \tag{3}$$

Answer

$$\begin{aligned}
\frac{\partial}{\partial x}(\frac{1}{1+tx})^n &= \frac{\partial}{\partial x}(1+tx)^{-n} \\
&= (-n)(1+tx)^{-n-1}\frac{\partial}{\partial x}(1+tx) \\
&= -nt(1+tx)^{-n-1}
\end{aligned} \tag{4}$$

$$\begin{aligned}
\frac{\partial}{\partial x_j}(\prod_{i=1}^{n}\frac{1}{1+tx_i}) &= (\prod_{i\neq j}\frac{1}{1+tx_i})\frac{\partial}{\partial x_j}(\frac{1}{1+tx_j}) \\
&= (\prod_{i\neq j}\frac{1}{1+tx_i})(-t(1+tx_j)^{-2}) \\
&= -\frac{t}{1+tx_j}(\prod_{i=1}^{n}\frac{1}{1+tx_i})
\end{aligned} \tag{5}$$

Problem 3

Compute

$$\lim_{x\to 0} \frac{N(x) - \frac{1}{2} - \frac{x}{\sqrt{2\pi}}}{x^3} \tag{6}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy \tag{7}$$

is the cumulative density function of the standard normal variable.

Answer

Since

$$\lim_{x\to 0} N(x) - \frac{1}{2} - \frac{x}{\sqrt{2\pi}} = 0 \tag{8}$$

and

$$\lim_{x\to 0} x^3 = 0 \tag{9}$$

We can use l'Hospital's rule, that is

$$\begin{aligned} \lim_{x\to 0} \frac{N(x) - \frac{1}{2} - \frac{x}{\sqrt{2\pi}}}{x^3} &= \lim_{x\to 0} \frac{N'(x) - \frac{1}{\sqrt{2\pi}}}{3x^2} \\ &= \frac{1}{3\sqrt{2\pi}} \lim_{x\to 0} \frac{e^{x^2/2} - 1}{x^2} \end{aligned} \tag{10}$$

where

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{11}$$

Similarly, using l'Hospital's rule,

$$\begin{aligned} \frac{1}{3\sqrt{2\pi}} \lim_{x\to 0} \frac{e^{x^2/2} - 1}{x^2} &= \frac{1}{3\sqrt{2\pi}} \lim_{x\to 0} \frac{e^{-x^2/2}(-x)}{2x} \\ &= -\frac{1}{6\sqrt{2\pi}} \lim_{x\to 0} e^{-x^2/2} \\ &= -\frac{1}{6\sqrt{2\pi}} \end{aligned} \tag{12}$$

Problem 4

(i) Synthesize a long position in a 3642 bull spread using the 36 strike and the 42 strike put options. Draw the payoff diagram and the P&L diagram at maturity of this position. For what values of the underlying asset at maturity does the position make money?
(ii) Synthesize a long position in a 3642 bull spread using the 36 strike and the 42 strike call options. Draw the payoff diagram and the P&L diagram at maturity of this position. For what values of the underlying asset at maturity does the position make money?

(i)

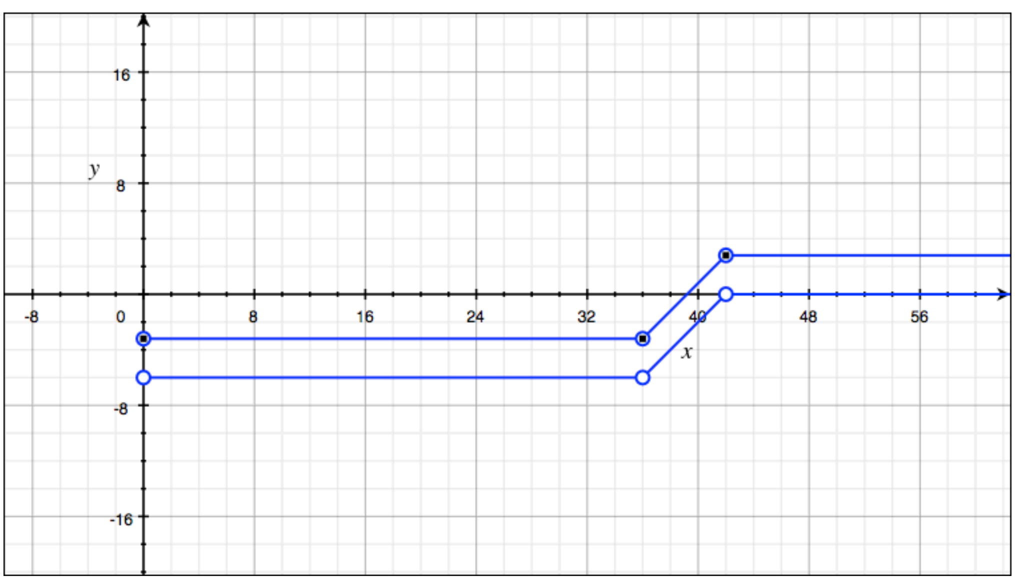
A long position in a 3642 bull spread can be synthesized by long the 36 strike put options and short the 42 strike put options. The payoff of the bull spread is

$$\begin{aligned} V(T) &= \max(36 - S(T), 0) - \max(42 - S(T), 0) \\ &= \begin{cases} -6, & \text{if } S(T) < 36 \\ S(T) - 42, & \text{if } 36 \le S(T) < 42 \\ 0, & \text{if } 42 \le S(T) \end{cases} \end{aligned} \tag{13}$$

The P&L at maturity of this position is

$$\begin{aligned} P\&L(T) &= V(T) + 4 - 1.2 \\ &= \begin{cases} -3.2, & \text{if } S(T) < 36 \\ S(T) - 39.2, & \text{if } 36 \le S(T) < 42 \\ 2.8, & \text{if } 42 \le S(T) \end{cases} \end{aligned} \tag{14}$$

The payoff diagram and the P&L diagram at maturity of this position is



where the hollow points represent payoff and the solid points represent P&L.
For the underlying asset at maturity with $S(T) > 39.2$, the position makes money.

(ii)

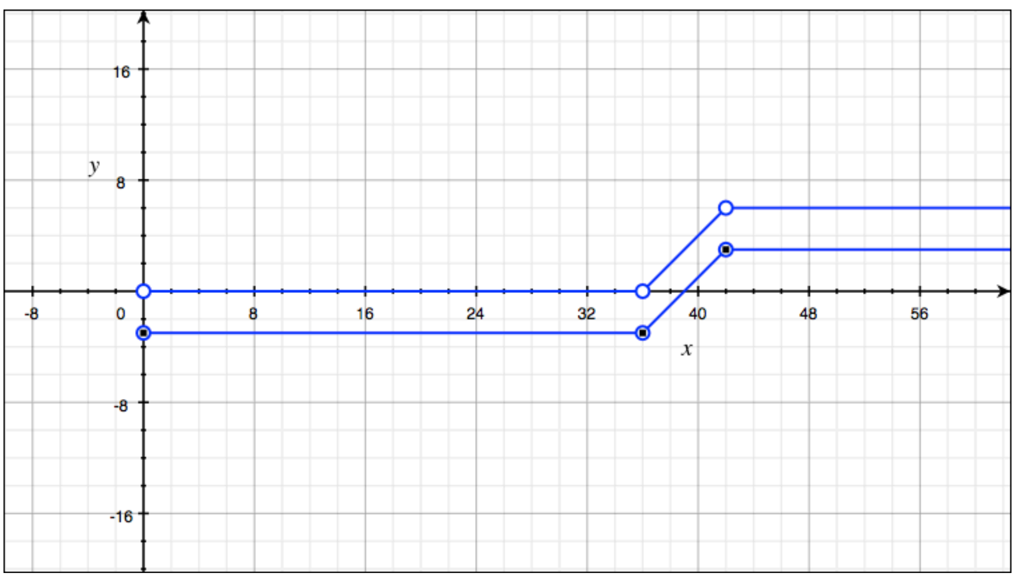
A long position in a 3642 bull spread can be synthesized by long the 36 strike call options and short the 42 strike call options. The payoff of the bull spread is

$$\begin{aligned} V(T) &= \max(S(T) - 36, 0) - \max(S(T) - 42, 0) \\ &= \begin{cases} 0, & \text{if } S(T) < 36 \\ S(T) - 36, & \text{if } 36 \leq S(T) < 42 \\ 6, & \text{if } 42 \leq S(T) \end{cases} \end{aligned} \tag{15}$$

The P&L at maturity of this position is

$$\begin{aligned} P\&L(T) &= V(T) + 5.5 - 2.5 \\ &= \begin{cases} -3, & \text{if } S(T) < 36 \\ S(T) - 39, & \text{if } 36 \leq S(T) < 42 \\ 3, & \text{if } 42 \leq S(T) \end{cases} \end{aligned} \tag{16}$$

The payoff diagram and the P&L diagram at maturity of this position is



where the hollow points represent payoff and the solid points represent P&L.
For the underlying asset at maturity with $S(T) > 39$, the position makes money.

Problem 5

The bid and ask prices for nine months ATM European options on an underlying asset paying 1% dividends continuously and with spot price $60 are

$$\begin{aligned} &C_{bid} = 6; C_{ask} = 6.5; \\ &P_{bid} = 5.5; P_{ask} = 6. \end{aligned} \tag{17}$$

Answer

According to the put-call parity, an arbitrage exists if

$$C_{bid} - P_{ask} > Se^{-qT} - Ke^{-rT} \tag{18}$$

or

$$C_{ask} - P_{bid} < Se^{-qT} - Ke^{rT} \tag{19}$$

However we find that

$$\begin{aligned} C_{bid} - P_{ask} &= 6 - 6 = 0 \\ C_{ask} - P_{bid} &= 6.5 - 5.5 = 1 \\ Se^{-qT} - Ke^{-rT} &= 60(e^{-0.01\times0.75} - e^{-0.01\times0.75}) = 0.887 \in [0, 1] \end{aligned} \tag{20}$$

Therefore there is no arbitrage opportunity present.