Advanced Calculus with FE Application: Quiz 4

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Problem 1

(i)

Binomial series (includes the square root for $\alpha = 0.5$)

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \quad \text{for all } |x| < 1 \text{ and all complex } \alpha \tag{1}$$

with generalized binomial coefficients

$$\binom{\alpha}{n} = \prod_{k=1}^{n} \frac{\alpha - k + 1}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}. \tag{2}$$

For $\alpha = 0.5$,

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{0.5}{n} x^n \quad \text{for all } |x| < 1 \tag{3}$$

Or

$$\sqrt{1+x} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \tfrac{7}{256}x^5 + O(x^6) \tag{4}$$

which converges when $|x| < 1$.
Substituting x with $-x^4$, the taylor series expansion of $f_1(x)$ around the point 0 is

$$\begin{aligned} f_1(x) &= \sum_{n=0}^{\infty} \frac{f_1^{(n)}(0)}{n!} x^n \\ &= 1 - \frac{x^4}{2} - \frac{x^8}{8} - \frac{x^{12}}{16} - \frac{5x^{16}}{128} - \frac{7x^{20}}{256} - \frac{21x^{24}}{1024} + O(x^{25}) \\ &= \sum_{n=0}^{\infty} (-1)^n \binom{0.5}{n} x^{4n} \\ &= \sum_{n=0}^{\infty} \frac{(2n)!}{(1-2n)(n!)^2(4^n)} x^{4n} \end{aligned} \tag{5}$$

which converges when $|-x^4| < 1$, or $|x| < 1$. We can also use the formula to calculate, where using stirling's formula

$$a_n = (-1)^n \binom{0.5}{n} \approx -\frac{(-1.5+n)!}{2\sqrt{\pi} n!} \tag{6}$$

The radius of convergence is

$$\begin{aligned} R &= \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} \\ &= \frac{1}{\limsup_{n\to\infty} \left| -\frac{(-1.5+n)!}{2\sqrt{\pi} n!} \right|^{1/n}} \\ &= 1 \end{aligned} \tag{7}$$

(ii)

Exponential function

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + O(x^4) \quad \text{for all } x \tag{8}$$

which converges everywhere.
Substitute x with x^2, the taylor series expansion of e^{-x^2} around the point 0 is

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} \tag{9}$$

which still converges everywhere.
Therefore the taylor series expansion of $f_2(x)$ around the point 0 is

$$\begin{aligned} f_2(x) &= e^2 e^{-x^2} \\ &= \sum_{n=0}^{\infty} \frac{e^2(-x^2)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{e^2(-1)^n}{n!} x^{2n} \end{aligned} \tag{10}$$

which converges everywhere, that is, with the radius of convergence as $R = \infty$.

Problem 2

(i)

The dollar duration of the portfolio is

$$D_\$ = \sum_i D_i B_i = 20 \times 3 + 50 \times 4 = 260 \tag{11}$$

The dollar convexity of the portfolio is

$$C_\$ = \sum_i C_i B_i = 20 \times 18 + 50 \times 20 = 1360 \tag{12}$$

(ii)

The change of portfolio value is

$$\begin{aligned} \Delta B &= -D_\$ \Delta y + \frac{1}{2} C_\$ (\Delta y)^2 \\ &= -260 \times 0.25\% + 0.5 \times 1360 \times (0.25\%)^2 \\ &= -0.64575 \end{aligned} \tag{13}$$

The approximate value of the portfolio is

$$B_{new} = B_{old} + \Delta B = 69.35425 \tag{14}$$

million dollars.

(iii)

Suppose we long x_1 units of the bond with duration 2 and convexity 7, and long x_2 units of the bond with duration 4 and convexity 11, to immunize the portfolio, then

$$\begin{aligned} D_\$ &= \sum_i D_i B_i = 260 + 2x_1 + 4x_2 = 0 \\ C_\$ &= \sum_i D_i C_i = 1360 + 7x_1 + 11x_2 = 0 \end{aligned} \tag{15}$$

which solved as $x_1 = -430, x_2 = 150$, that is

Problem 2 [(iii)] continued on next page. . .

- Short 430 units of the bond with duration 2 and convexity 7
- Long 150 units of the bond with duration 4 and convexity 11