Data Structures and Algorithms [1]

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[1] Material for the presentation taken from Cormen, Leiserson, Rivest and Stein, *Introduction to Algorithms, Fourth Edition*;

- Average case running time : $O(n \lg n)$

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Ch. 7 : Quicksort

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- Average case running time : $O(n \lg n)$
- Constant factor associated with $O(n \lg n)$ is small compared to merge sort. (Best practical choice.)
- Worst case running time : $O(n^2)$
- Uses the divide-and-conquer paradigm

Quicksort

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QUICKSORT(A, p, r)
if p < r
    q = PARTITION(A, p, r)
    QUICKSORT(A, p, q - 1)
    QUICKSORT(A, q + 1, r)
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Quicksort : Partition

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PARTITION(A, p, r)
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i = p - 1
for j = p to r - 1
    if A[j] ≤ x
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exchange A[i + 1] with A[r]
return i + 1
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Quicksort : Partition

- A = [4, 2, 5, 1, 3] , p = 1 , r = 5

PARTITION(A, p, r)

$x = A[r]$
$i = p - 1$
for $j = p$ **to** $r - 1$
    if $A[j] \leq x$
        $i = i + 1$
        exchange $A[i]$ with $A[j]$
exchange $A[i + 1]$ with $A[r]$
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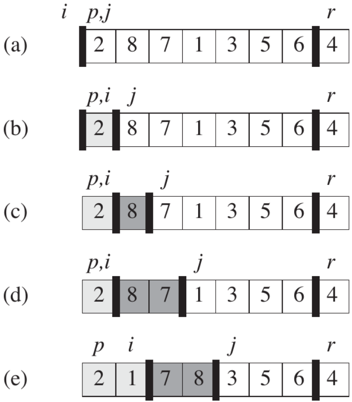

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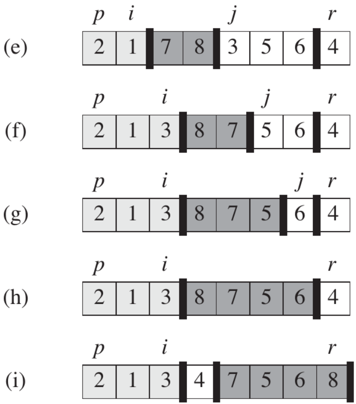
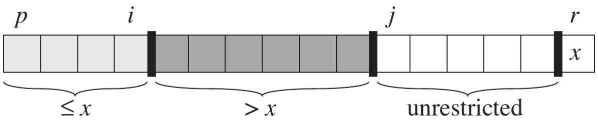


PARTITION procedure: four regions



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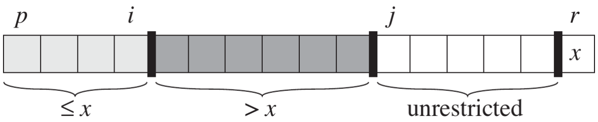


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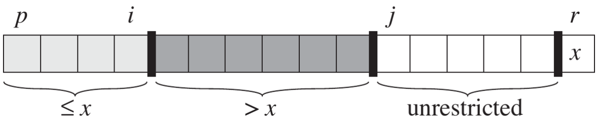


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Running time : $\Theta(n)$

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$[\ 2,\ 5,\ 1,\ 4,\ \underline{3}\ ]$

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[2, $\underline{1}$]   [3]   [4, 5]

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Performance of quicksort

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We can use substitution method : $T(n) = \Theta(n^2)$

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- What would be the average-case performance?

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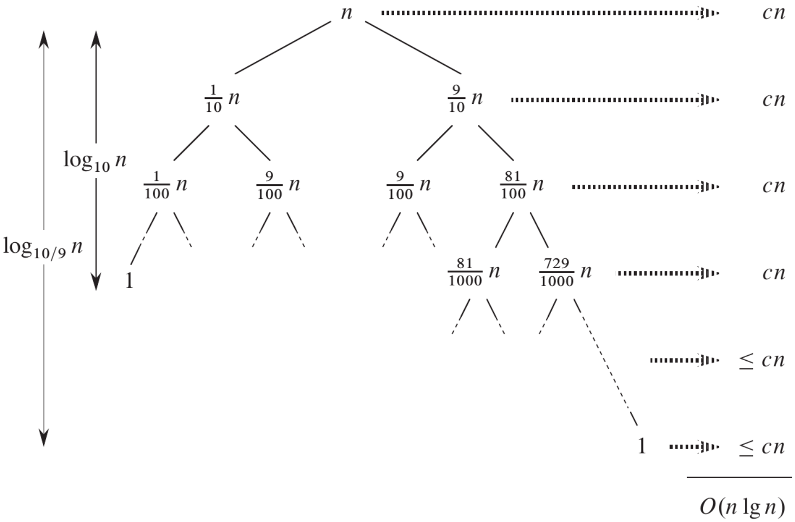
$$T(n) = T(9n/10) + T(n/10) + cn$$

9-to-1 proportional split



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- What if the split was 99-to-1? What would be the running time?
- Any split of the form rn and $(1-r)n$, where $0 < r < 1$ will give a running time of $O(n \lg n)$.
- It is unlikely that at each level PARTITION procedure will give us a highly unbalanced split (assuming a random input array).

Randomized version of quicksort

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- This should lead to the split produced by the PARTITION procedure to be well balanced on average.

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7  exchange A[i + 1] with A[r]
8  return i + 1
```

- How many times can the same element be selected as a pivot during the entire run of the quicksort algorithm?
- Therefore, the partition procedure will be called at most n times.
- So, the overall running time can be bounded by the number of times line 4 of the partition procedure is executed.

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  What is $Z_{3\,6}$?

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- Total number of comparisons X:

  $$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$$

Expected number of comparisons

- Expected value of the total number of comparisons $E[X]$:

$$E[X] = E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right]$$

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&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E\left[X_{ij}\right]
\end{aligned}
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&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E\left[X_{ij}\right] \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr\{z_i \text{ is compared to } z_j\}
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- On the other hand, if z_i is chosen as a pivot before any other element in Z_{ij}, then z_i will be compared with all the elements in Z_{ij}.
- Similarly, if z_j is chosen as a pivot before any other element in Z_{ij}, then z_j will be compared with all the elements in Z_{ij}.

Expected number of comparisons

- In any run of the randomized quicksort algorithm, all elements in the set Z_{ij} are equally likely to be chosen as the first pivot from the set.

Probability that z_i is compared to z_j

$$\Pr\{z_i \text{ is compared to } z_j\} = \Pr\{z_i \text{ or } z_j \text{ is first pivot chosen from } Z_{ij}\}$$

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&= \frac{2}{j-i+1} .
\end{aligned}$$

Expected number of comparisons

Expected value of the total number of comparisons $E[X]$:

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr\{z_i \text{ is compared to } z_j\}$$

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$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1}$$

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&< \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} \\
&= \sum_{i=1}^{n-1} O(\lg n) \qquad \left(\text{Using } \sum_{k=1}^{n} \frac{1}{k} = O(\lg n) \right)
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&< \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} \\
&= \sum_{i=1}^{n-1} O(\lg n) \qquad \left(\text{ Using } \sum_{k=1}^{n} \frac{1}{k} = O(\lg n)\ \right) \\
&= O(n \lg n)
\end{aligned}
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Expected running time of randomized quicksort

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PARTITION(A, p, r)
x = A[r]
i = p - 1
for j = p to r - 1
    if A[j] ≤ x
        i = i + 1
        exchange A[i] with A[j]
exchange A[i + 1] with A[r]
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- Comparison sort

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INSERTION-SORT(A, n)
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    // Insert A[i] into the sorted subarray A[1 : i − 1].
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Quicksort Partition procedure

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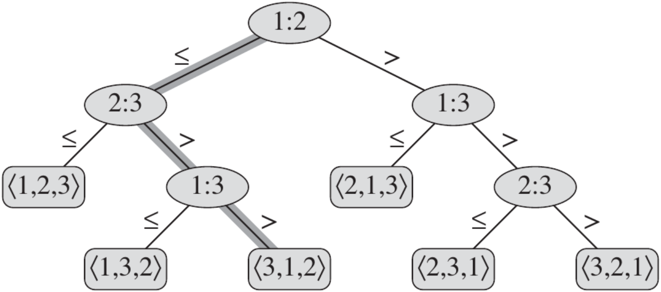
- Is it possible to have a better worst case running time?
- Can we have a sorting algorithm whose worst case running time is $o(n \lg n)$?

Decision tree for comparison sort



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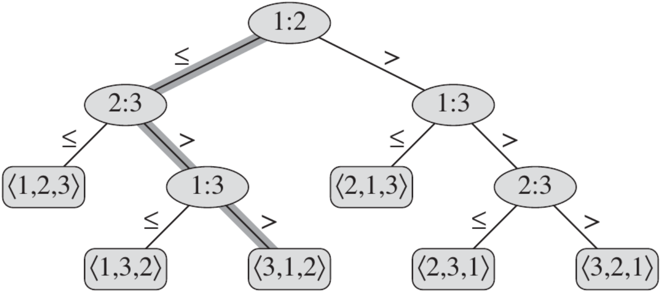

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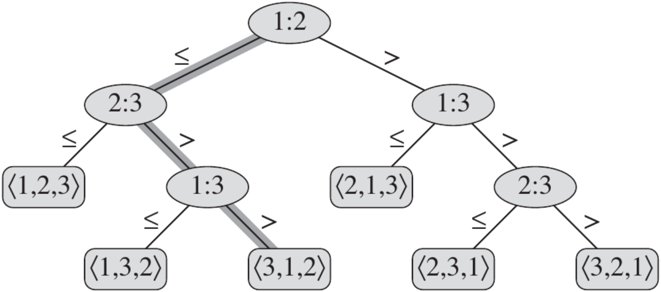

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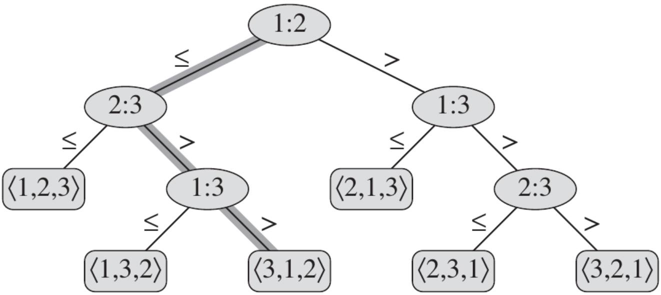
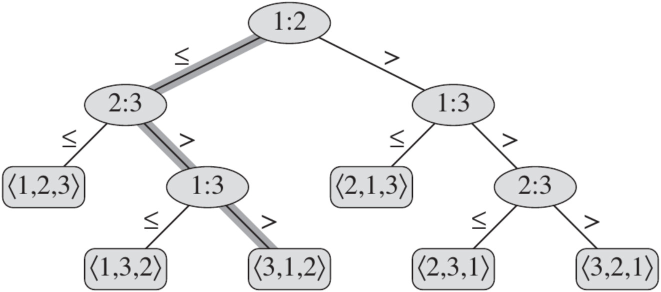


- In the worst case, we must perform at least h comparison operations.
- The height of the decision tree gives a lower bound on the number of comparisons needed in the worst-case.
- Any correct sorting algorithm must be able to produce each of the $n!$ permutations of the n input elements.

Decision tree is a full binary tree



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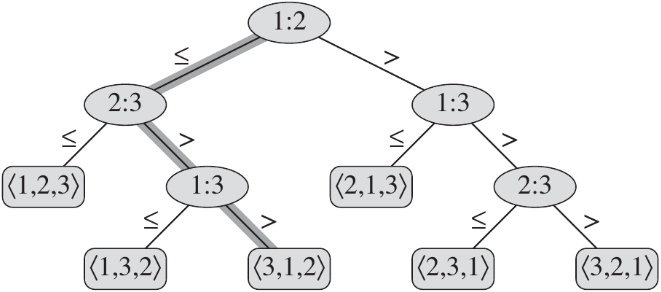

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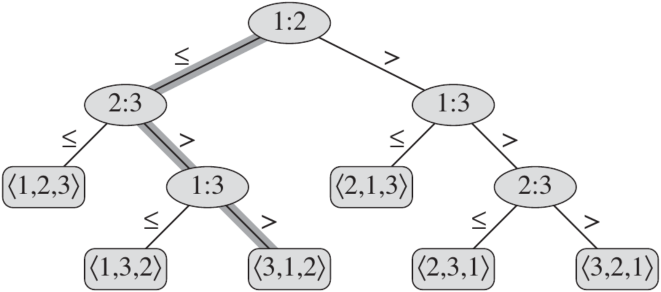

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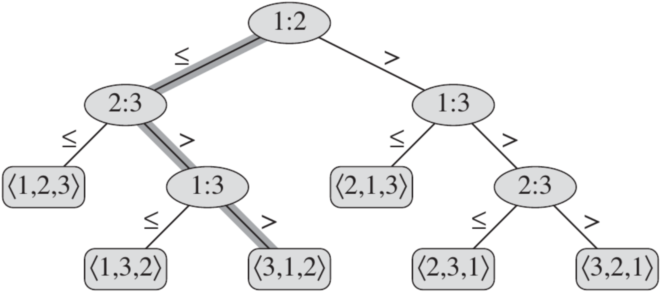

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n! &\leq l \leq 2^h \\
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&= \Omega(n \lg n)
\end{aligned}
$$

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$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta\left(\frac{1}{n}\right)\right)$$

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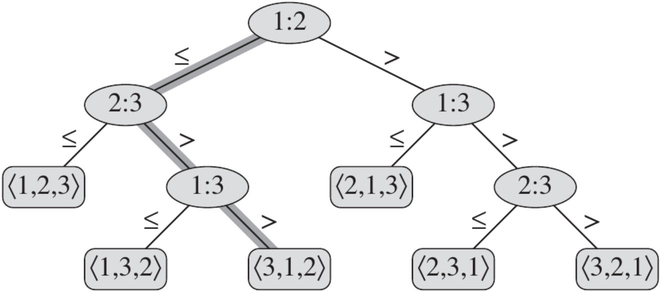

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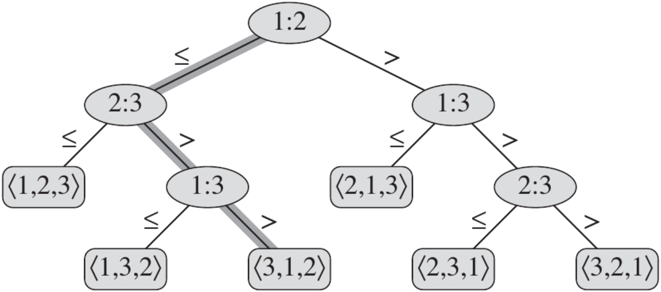

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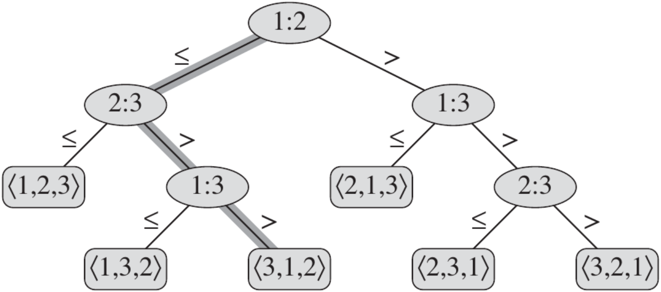

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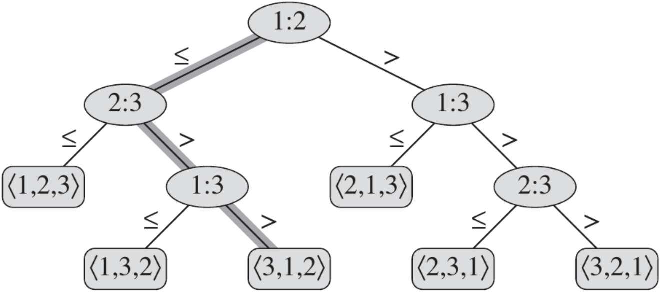

- Any comparison sort algorithm would require $\Omega(n \lg n)$ comparisons in the worst case.
- Any comparison based algorithm would take at least $\Omega(n \lg n)$ time in the worst case for solving the sorting problem.
- Heapsort and Merge sort are asymptotically optimal algorithm for comparison sorting.

Counting sort

- Assumption : Each input element is an *integer* in the range 0 to k for some integer k.

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- Temporary working storage : $C[0 \dots k]$

Counting sort operations

	1	2	3	4	5	6	7	8
A	2	5	3	0	2	3	0	3

	0	1	2	3	4	5
C	2	0	2	3	0	1

(a)

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	1	2	3	4	5	6	7	8
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	0	1	2	3	4	5
C	2	2	4	7	7	8

(b)

Counting sort operations

	1	2	3	4	5	6	7	8
A	2	5	3	0	2	3	0	3

	0	1	2	3	4	5
C	2	0	2	3	0	1

(a)

	0	1	2	3	4	5
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(b)

	1	2	3	4	5	6	7	8
B							3	

	0	1	2	3	4	5
C	2	2	4	6	7	8

(c)

Counting sort operations

	1	2	3	4	5	6	7	8
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	0	1	2	3	4	5
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(b)

	1	2	3	4	5	6	7	8
B							3	

	0	1	2	3	4	5
C	2	2	4	6	7	8

(c)

	1	2	3	4	5	6	7	8
B		0					3	

	0	1	2	3	4	5
C	1	2	4	6	7	8

(d)

Counting sort operations

	1	2	3	4	5	6	7	8
A	2	5	3	0	2	3	0	3

(a)

	1	2	3	4	5	6	7	8
B		0				3	3	

	0	1	2	3	4	5
C	1	2	4	5	7	8

(e)

Counting sort operations

	1	2	3	4	5	6	7	8
A	2	5	3	0	2	3	0	3

(a)

	1	2	3	4	5	6	7	8
B		0				3	3	

	0	1	2	3	4	5
C	1	2	4	5	7	8

(e)

	1	2	3	4	5	6	7	8
B	0	0	2	2	3	3	3	5

(f)

Counting sort

```
COUNTING-SORT(A, B, k)
let C[0..k] be a new array
for i = 0 to k
    C[i] = 0
for j = 1 to A.length
    C[A[j]] = C[A[j]] + 1
// C[i] now contains the number of elements equal to i.
for i = 1 to k
    C[i] = C[i] + C[i − 1]
// C[i] now contains the number of elements less than or equal to i.
for j = A.length downto 1
    B[C[A[j]]] = A[j]
    C[A[j]] = C[A[j]] − 1
```

Running time of Counting sort

- Overall running time $\Theta(k + n)$

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- Overall running time $\Theta(k + n)$
- When $k = O(n)$, the counting sort runs in $\Theta(n)$ time.
- Why running time of Counting sort has a better worst case lower bound?
- Counting sort is **stable**. (What is a stable sorting algorithm?)

Is Insertion sort stable?

```
INSERTION-SORT(A, n)
for i = 2 to n
    key = A[i]
    // Insert A[i] into the sorted subarray A[1 : i − 1].
    j = i − 1
    while j > 0 and A[j] > key
        A[j + 1] = A[j]
        j = j − 1
    A[j + 1] = key
```

Is Quick sort stable?

```
PARTITION(A, p, r)
x = A[r]
i = p - 1
for j = p to r - 1
    if A[j] ≤ x
        i = i + 1
        exchange A[i] with A[j]
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- Stability of sorting algorithms
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