Data Structures and Algorithms [1]

BITS-Pilani K. K. Birla Goa Campus

[1]Material for the presentation taken from Cormen, Leiserson, Rivest and Stein, *Introduction to Algorithms, Third Edition*;

Handout Discussion

My expectations from students

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- Use PYQs only for solving additional problems.

Role of algorithms in computing

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procedure $\textsc{LinearSearch}(A, k)$
    for $i \leftarrow 1$ to length of array A **do**
        if $A[i] = k$ **then**
            return i $\triangleright$ Key found at index i
    return -1 $\triangleright$ Key not found

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- Are we using any algorithm right now?

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- A good algorithm would be *efficient* in terms of computing time and memory that is used.

Efficiency of Algorithms

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  Insertion sort requires 2^{11} (≈ 2000) times more machine-level instructions for solving the same problem!

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- Faster computer A takes more than 20,000 seconds (5.5 hours), whereas slower computer B takes around 1163 seconds ($<$ 20 minutes).

Importance of efficiency

- We should learn how to analyse and design efficient algorithms.

Comparison of running times

- For each function $f(n)$ and time t in the following table, determine the largest size n of a problem that can be solved in time t, assuming that the algorithm to solve the problem takes $f(n)$ microseconds.

	1 second	1 minute	1 hour	1 day	1 month	1 year	1 century
$\lg n$							
$\sqrt{n}$							
n							
$n \lg n$							
n^2							
n^3							
2^n							
$n!$							

Running time of Insertion sort

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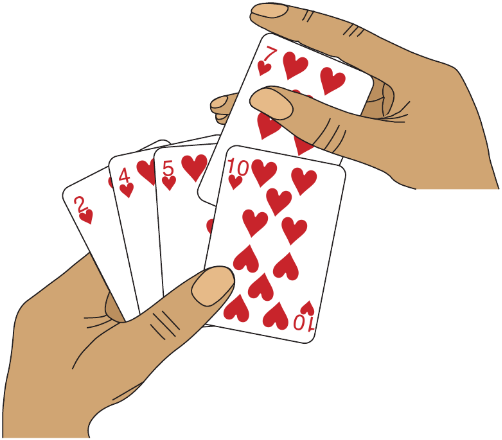
- E.g., Input sequence : $< 31, 41, 59, 26, 41, 58 >$
- Sorting algorithm should give the output: $< 26, 31, 41, 41, 58, 59 >$

Insertion sort

- Main idea: Works the way people sort a hand of playing cards.



Insertion sort

- `https://visualgo.net/en/sorting`

  5,2,4,6,1,3

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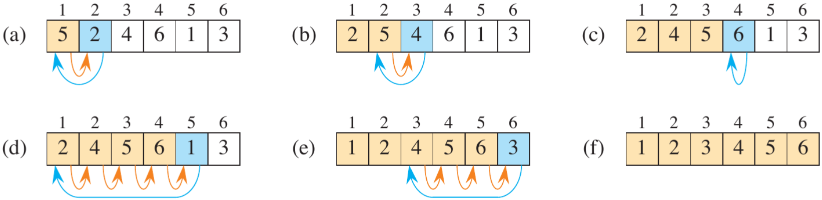
Outline for correctness of Insertion sort



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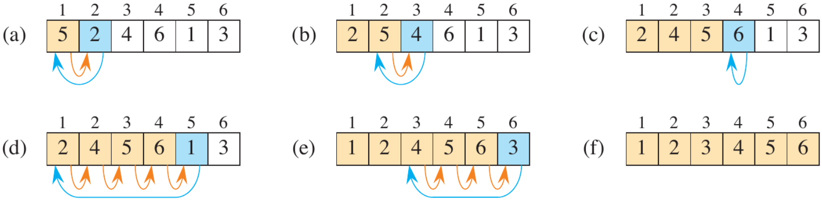

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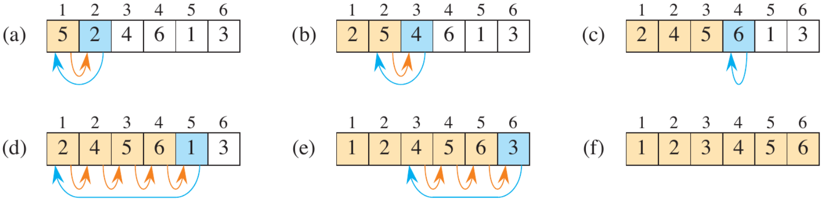

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- We use loop invariant to argue for the correctness of an algorithm.

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- **Initialization, Maintenance, Termination**

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- **Running time** : time needed for the primitive operations or "steps" executed by the Random-access machine (RAM) model for an input of size n.

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- RAM model contains common operations/instructions (add, subtract, load, store, conditional branch, subroutine call etc.)
- Each statement in the pseudocode takes a constant amount of time.

Pseudocode with time costs

INSERTION-SORT(A, n)	*cost*	*times*
for $i = 2$ **to** n	c_1	n
$\quad key = A[i]$	c_2	$n - 1$
$\quad$ // Insert $A[i]$ into the sorted subarray $A[1 : i - 1]$.	0	$n - 1$
$\quad j = i - 1$	c_4	$n - 1$
$\quad$ **while** $j > 0$ and $A[j] > key$	c_5	$\sum_{i=2}^{n} t_i$
$\qquad A[j + 1] = A[j]$	c_6	$\sum_{i=2}^{n}(t_i - 1)$
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$\quad A[j + 1] = key$	c_8	$n - 1$

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$A[j+1] = A[j]$	c_6	$\sum_{i=2}^{n} (t_i - 1)$
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$$
\begin{aligned}
T(n) = {} & c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{i=2}^{n} t_i + c_6 \sum_{i=2}^{n} (t_i - 1) \\
& + c_7 \sum_{i=2}^{n} (t_i - 1) + c_8(n-1) \, .
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Summation of time costs of all the steps

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{i=2}^{n} t_i + c_6 \sum_{i=2}^{n} (t_i - 1) + c_7 \sum_{i=2}^{n} (t_i - 1) + c_8(n-1) .$$

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- The running time is of the form $an + b$, where a and b are constants.
- So, the running time is a **linear function** of n if the input sequence is already sorted (best case scenario).

Summation of time costs in the worst case

$$
\begin{aligned}
T(n) = {} & c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{i=2}^{n} t_i + c_6 \sum_{i=2}^{n} (t_i - 1) \\
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- When will the worst case scenario occur?

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- When will the worst case scenario occur?
- What will be the value of t_i in the worst case scenario?

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- When will the worst case scenario occur?
- What will be the value of t_i in the worst case scenario?

$$\sum_{i=2}^{n} i = \frac{n(n+1)}{2} - 1 \qquad \sum_{i=2}^{n} (i-1) = \frac{n(n-1)}{2}$$

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\end{aligned}
$$

- Worst case running time is of the form $an^2 + bn + c$
- So, the running time is a **quadratic function** of n if the input sequence is sorted in the reverse order (worst case scenario).

Running time estimation

- What will the average case look like?

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  - $t_i = \dfrac{i}{2}$,

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Running time estimation

- What will the average case look like?
  - $t_i = \dfrac{i}{2}$, $T(n)$ will again be a quadratic function of n.
- Very often we will be interested in only the worst-case running time because it gives the upper bound on the running time for *any* input.

Order of growth: Simplifying abstractions

- We will make simplifying abstractions because we are only interested in the rate of growth (order of growth) of the running time for large values of n.

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- We express the above by saying that insertion sort has a worst-case running time of $\Theta(n^2)$.
- Is order of growth of Insertion-sort dependent on how it is implemented?

Problem

2.2-1

Express the function $n^3/1000 - 100n^2 - 100n + 3$ in terms of Θ-notation.

Insertion sort

- Main idea : follow *Incremental* approach

Insertion sort

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INSERTION-SORT(A, n)	*cost*	*times*
for $i = 2$ **to** n	c_1	n
$key = A[i]$	c_2	$n-1$
// Insert $A[i]$ into the sorted subarray $A[1:i-1]$.	0	$n-1$
$j = i - 1$	c_4	$n-1$
while $j > 0$ and $A[j] > key$	c_5	$\sum_{i=2}^{n} t_i$
$A[j+1] = A[j]$	c_6	$\sum_{i=2}^{n}(t_i - 1)$
$j = j - 1$	c_7	$\sum_{i=2}^{n}(t_i - 1)$
$A[j+1] = key$	c_8	$n-1$

The divide-and-conquer approach

- The **merge sort** algorithm follows the divide-and-conquer paradigm

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The divide-and-conquer approach

- The **merge sort** algorithm follows the divide-and-conquer paradigm

  Divide: Divide the n-element sequence to be sorted into two subsequences of n/2 elements each.

  Conquer: Sort the two subsequences recursively using merge sort.

  Combine: Merge the two sorted subsequences to produce the sorted answer.

Combine step in Merge sort

- $MERGE(A, p, q, r)$ procedure.

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- The procedure assumes that the subarrays $A[p..q]$ and $A[q+1..r]$ are sorted.

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- Main idea :

L :	2	4	5	9
R :	3	6	7	8

Combine step in Merge sort

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```
MERGE(A, p, q, r)
n1 = q − p + 1
n2 = r − q
let L[1 .. n1 + 1] and R[1 .. n2 + 1] be new arrays
for i = 1 to n1
    L[i] = A[p + i − 1]
for j = 1 to n2
    R[j] = A[q + j]
L[n1 + 1] = ∞
R[n2 + 1] = ∞
i = 1
j = 1
for k = p to r
    if L[i] ≤ R[j]
        A[k] = L[i]
        i = i + 1
    else A[k] = R[j]
        j = j + 1
```

Merge procedure

- Running time of *MERGE* procedure is $\Theta(n)$

Merge sort

$$
\begin{array}{ll}
\multicolumn{2}{l}{\textsc{Merge-Sort}(A, p, r)} \\
1 & \textbf{if } p < r \\
2 & \quad q = \lfloor (p + r)/2 \rfloor \\
3 & \quad \textsc{Merge-Sort}(A, p, q) \\
4 & \quad \textsc{Merge-Sort}(A, q + 1, r) \\
5 & \quad \textsc{Merge}(A, p, q, r)
\end{array}
$$

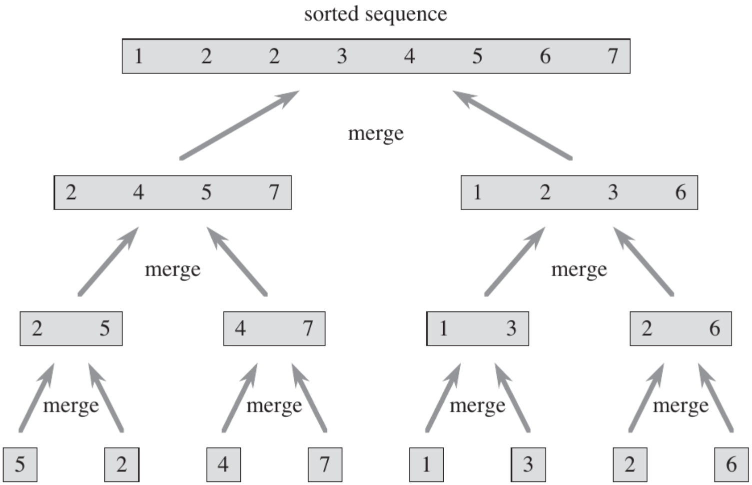
- Initial call $MERGE\text{-}SORT(A, 1, A.length)$

Merge sort operations

$A = [\ 5,\ 2,\ 4,\ 7,\ 1,\ 3,\ 6,\ 2\ ]$

Merge sort operations

A = [5, 2, 4, 7, 1, 3, 6, 2]



Analyzing divide-and-conquer algorithms

- Recurrence equation:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq c \text{ ,} \\ aT(n/b) + D(n) + C(n) & \text{otherwise} \end{cases}$$

Analyzing the merge sort algorithm

- Recurrence for worst-case running time of merge sort:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \, , \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \, . \end{cases}$$

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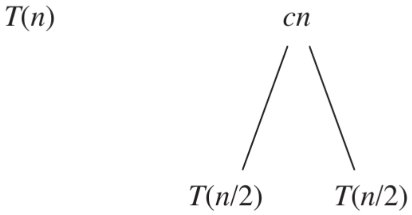
- Recurrence for worst-case running time of merge sort:

$$T(n) = \begin{cases} c & \text{if } n = 1\,, \\ 2T(n/2) + cn & \text{if } n > 1\,, \end{cases}$$

Recursion tree for merge sort



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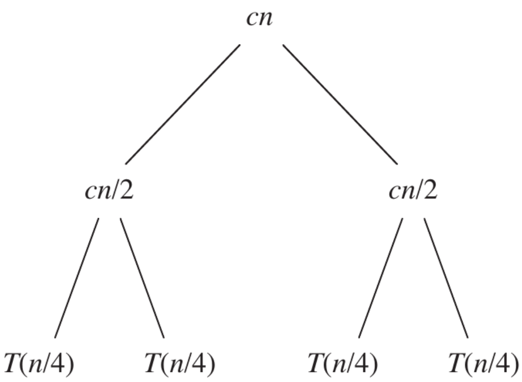

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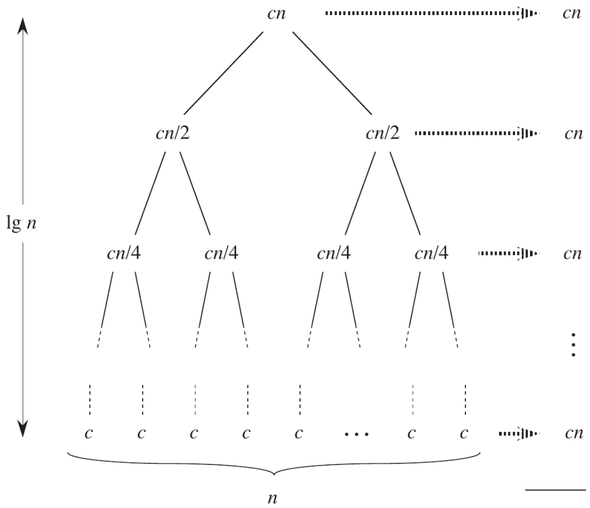

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- The recursion tree will have $\lg n + 1$ levels.
- Worst case running time of merge sort:

$$\begin{aligned} T(n) &= cn \lg n + cn \\ &= \Theta(n \lg n) \end{aligned}$$

- Once input size n is large enough, merge sort ($\Theta(n \lg n)$) will beat insertion sort ($\Theta(n^2)$).

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- We are interested in studying the **asymptotic** efficiency of algorithms (i.e. order of growth when n tends to infinity).
- We use different asymptotic notations (e.g. Θ notation) for describing the efficiency of algrithms.
- When we say running time $T(n) = \Theta(n^2)$, we mean $T(n)$ is a function in the set $\Theta(n^2)$.

Θ notation

$$\Theta(g(n)) = \{ f(n) : \text{there exist positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0 \}$$

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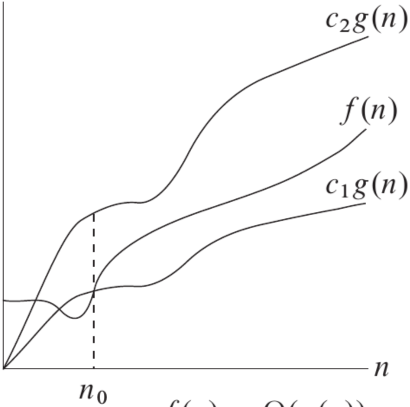

$f(n) = \Theta(g(n))$

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- Eqn. 1 will be true for $c_1 = 1/14$, $c_2 = 1/2$ and $n_0 = 7$. (Other choices were also possible.)

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- In general, if $p(n)$ is a degree d polynomial, then $p(n) \in \Theta(n^d)$.

Caveat: The coefficient of the highest degree term must be positive.

Θ notation

- $2n^2 + \Theta(n) = \Theta(n^2)$

O notation (Big-oh)

- We use O notation to express asymptotic upper bound.

$$O(g(n)) = \{ f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le c g(n) \text{ for all } n \ge n_0 \} .$$

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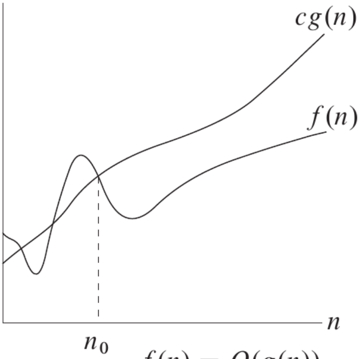

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- Is $2n + 2 \in O(n)$?

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- Is $3n + 2 \in O(n^2)$?
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  If $f(n) = O(g(n))$ then $f(n) = \Theta(g(n))$.

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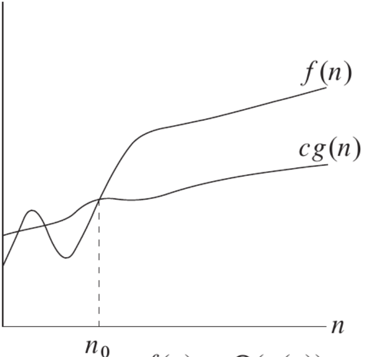

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Theorem 3.1

For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. ■

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Comparing Functions

- Is the following True?

  $f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$

Common Mathematical Functions

- $a^{\log_c b} = b^{\log_c a}$, where $a > 0$ and $b > 0$

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- Go through section 3.2 of the textbook.