# Data Structures and Algorithms [1]

BITS-Pilani K. K. Birla Goa Campus

[1] Material for the presentation taken from Cormen, Leiserson, Rivest and Stein, *Introduction to Algorithms, Fourth Edition*;

# Course plan

- Ch. 20, 21, 22.3 (Graph Algorithms)

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- Ch. 12, 13 (Binary Search Tree and Red-Black Tree)

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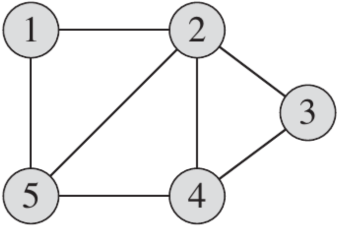
- Ch. 20, 21, 22.3 (Graph Algorithms)
- Ch. 12, 13 (Binary Search Tree and Red-Black Tree)
- Ch. 11 (Hash Tables)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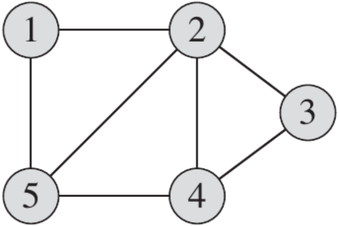
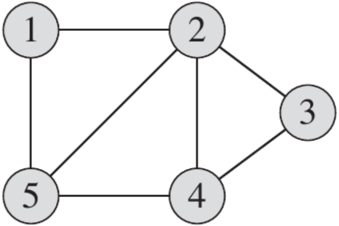


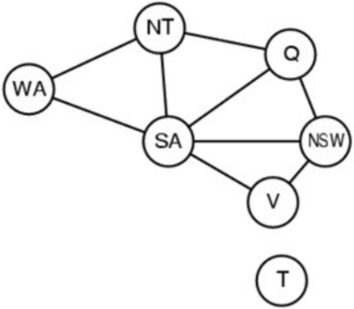

- Graphs are mathematical structures consisting of vertices and edges.




- Graphs are mathematical structures consisting of vertices and edges.
- The input to the graph algorithms will be a graph represented as $G = (V, E)$.

# Map coloring problem

# Vertex coloring problem

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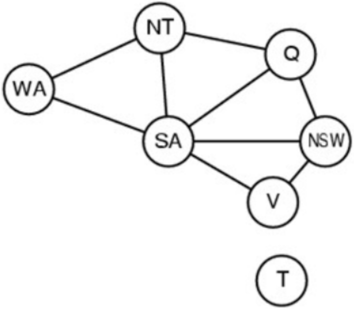

# Module IV: Graph Algorithms

- Ch. 20: Elementary Graph Algorithms

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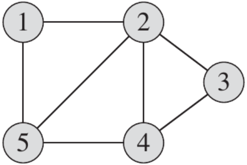
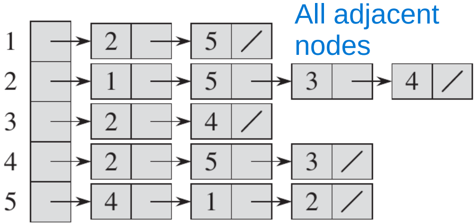
  Breadth-first search, Depth-first search, Topological sort

# Representing undirected graphs

- Adjacency-list representation of undirected graphs



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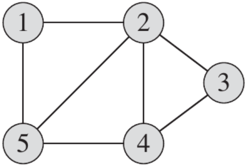
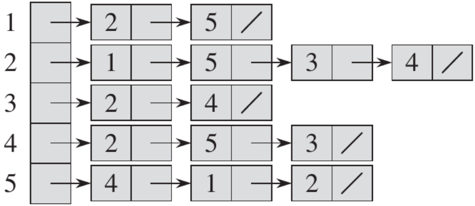
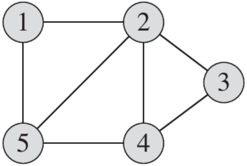
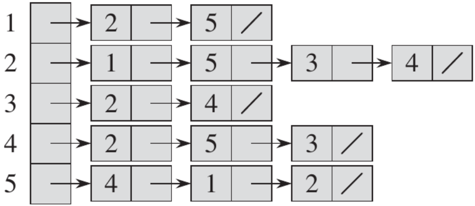


# Representing undirected graphs

- Adjacency-matrix representation of undirected graphs



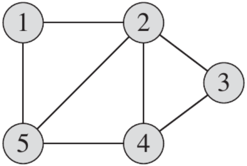



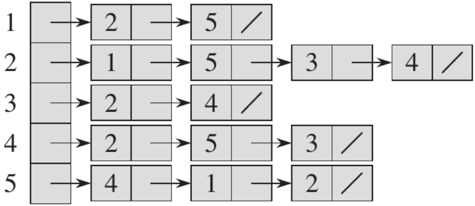

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 0 | 1 |
| 2 | 1 | 0 | 1 | 1 | 1 |
| 3 | 0 | 1 | 0 | 1 | 0 |
| 4 | 0 | 1 | 1 | 0 | 1 |
| 5 | 1 | 1 | 0 | 1 | 0 |

1 if adjacent

# Representing undirected graphs

- Will the Adjacency-matrix for an undirected graph be symmetric? Yes

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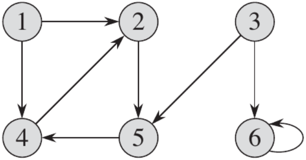
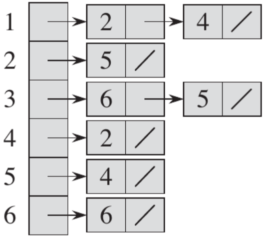
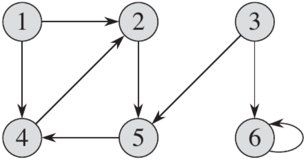
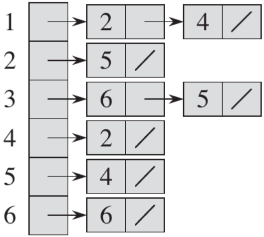



adjacency only 1-way not 2-way

# Representing directed graphs




| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 1 | 0 |
| 3 | 0 | 0 | 0 | 0 | 1 | 1 |
| 4 | 0 | 1 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 1 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 1 |

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- We may use adjacency-list to represent sparse graphs.
- Dense graphs: $|E|$ is close to $|V|^2$.

# Graph Representation : Weighted graphs

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  Weight function $w : E \to \mathbb{R}$.

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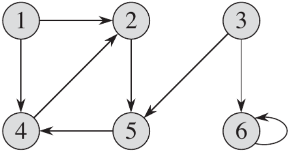
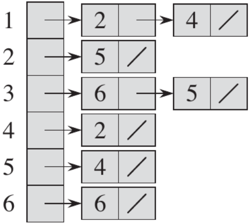


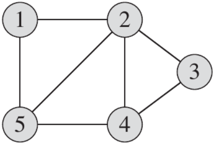



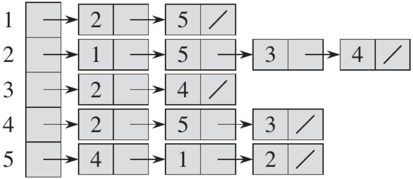

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 1 | 0 |
| 3 | 0 | 0 | 0 | 0 | 1 | 1 |
| 4 | 0 | 1 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 1 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 1 |

https://algorithmtutor.com/images/graph_representation_weighted.png

# Graph Representation

- Can we have an edge which is a self-loop in an undirected graph?

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- Adjacency-list representation requires $\Theta(V+E)$ memory compared to $\Theta(V^2)$ memory for Adjacency matrix.
- Suppose we wish to check whether a given edge $(u, v)$ is present in a graph $G(V, E)$. Which representation will take more time to check this? Adjacency-list or adjacency-matrix?

Matrix will be O(2*V), List will be O(V+E*), where E* would be number of edges for that vertex, ie, degree of that vertex.

Directed graph can double time but pretty much same complexity.

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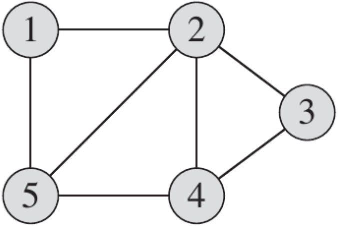
- Operation of Breadth-first search (P. 557)
- Breadth-first search pseudocode (P.556)



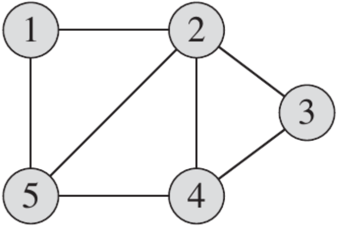

- Suppose we do a Breadth-first search starting from node 5. Which of the following orders of node visits is not possible under Breadth-first search?

a. 5,4,1,2,3

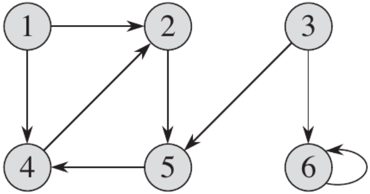
b. 5,1,2,4,3

c. 5,1,2,3,4 c, since 4 is at less dist than 3

# Breadth-first search : Undirected graph




- If we start from node 5, what will be the distance of node 3 found by the Breadth-first search?



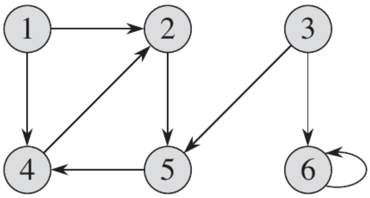

- Suppose we do a Breadth-first search starting from node 1. Which of the following orders of node visits is not possible under Breadth-first search?

a. 1,2,4,5,3,6

b. 1,2,5,4

c. 1,2,4,5 Only c is possible

# Breadth-first search : Directed graph

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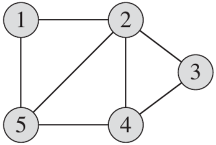
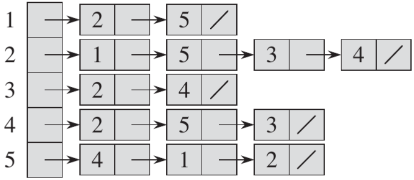
# Breadth-first search



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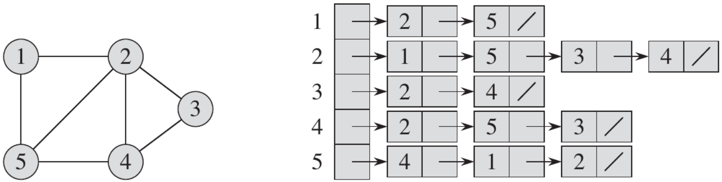


- Running time of BFS is

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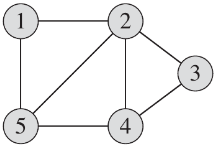
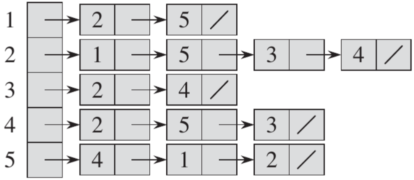
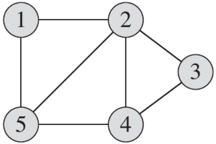
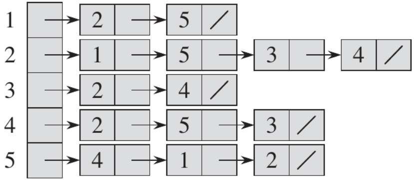



- Running time of BFS is $O(V + E)$.

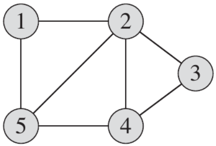
# Breadth-first search

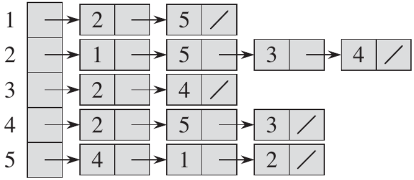



- Running time of BFS is $O(V + E)$. (Aggregate analysis)

# Breadth-first search

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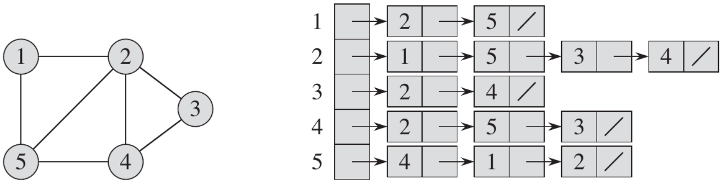
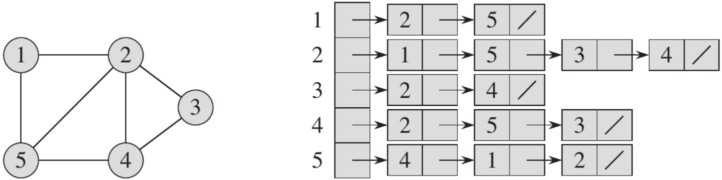



- Running time of BFS is $O(V + E)$. (Aggregate analysis)
- Why is it called Breadth-first search?
- When the BFS algorithm terminates, the attribute $v.d$ contains the shortest path (minimum number of edges) from node $s$ to $v$.

# Predecessor attribute $\pi$




- The attribute $\pi$ stores the parent (predecessor) of each node in the breadth-first tree. We can use $\pi$ attributes to find the shortest path from $s$ to any vertex $v$.

# Breadth-first search

Q. What will be the running time of BFS if we represent its input graph by an adjacency matrix and modify the algorithm to handle this form of graph input?

# Breadth-first tree

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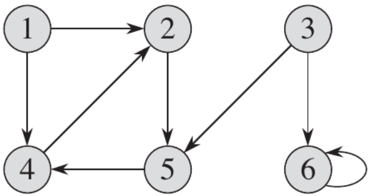
- Predecessor subgraph obtained after breadth-first search is called a breadth-first tree.




- Suppose we do a Breadth-first search starting from node 1. Find the Breadth-First tree $G_\pi = (V_\pi, E_\pi)$ obtained?

Tree consists of parts of graph reachable from given node

# Shortest path in a maze

$$\text{maze} = \begin{bmatrix} S & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & E \end{bmatrix}$$

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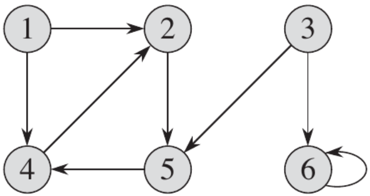
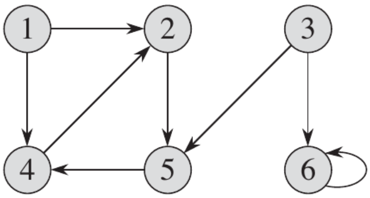
- Discovered and finished timestamps.
- DFS pseudocode (p. 565)




- Suppose that the first node to be discovered is 3. What are the possible finish timestamps for node 4?

# Depth-first search : Directed graph

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- Which node remains white when node 3 is finished?

# Predecessor subgraph

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- Depth-first forest depends on the order in which vertices are visited (Line 5 of $\mathrm{DFS}$ procedure).

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# Predecessor subgraph

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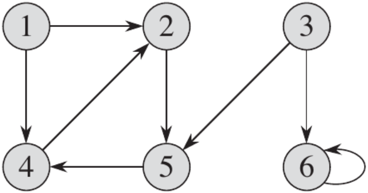
- Running time of $\mathrm{DFS}$ is $\Theta(V + E)$.

Forest is set of trees obtained on performing DFS from various initial nodes

# Depth-first Tree

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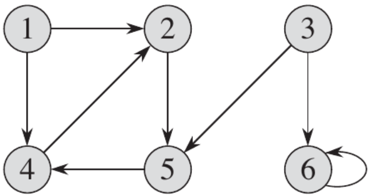
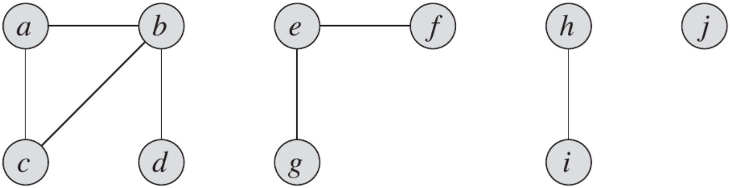


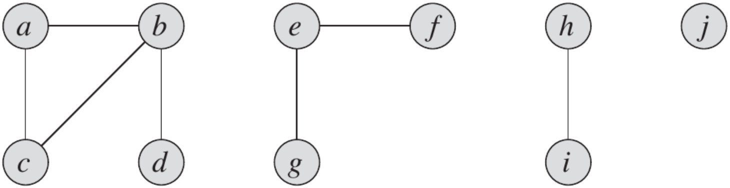

- DFS helps us generate a depth-first forest $G_\pi$.
- Suppose that the first node to be discovered is 3. What will the depth-first forest $G_\pi$ look like?

# Depth-first Forest




- Consider the undirected graph shown above.

# Depth-first Forest

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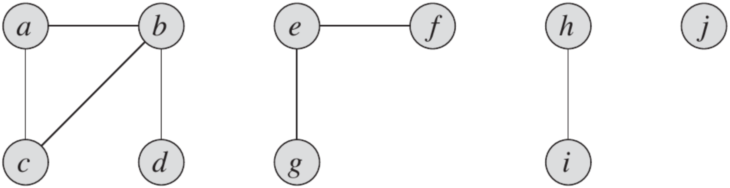
  $|V| = 10$, $|E| = 7$.

# Depth-first Forest




- Consider the undirected graph shown above.
  $|V| = 10$, $|E| = 7$.
- How many trees will the Depth-first forest $G_\pi$ contain?

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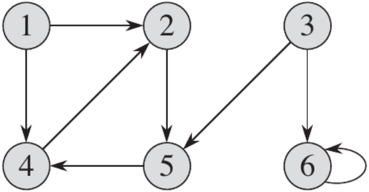

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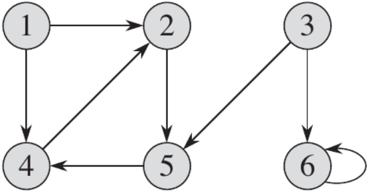


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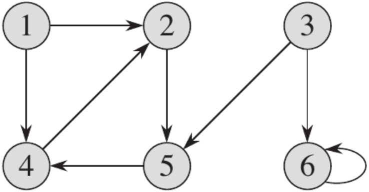


- What will be a back edge in the above graph? 4,2 is back edge
- Back edge helps us detect a cycle in a graph.

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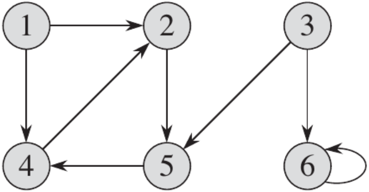

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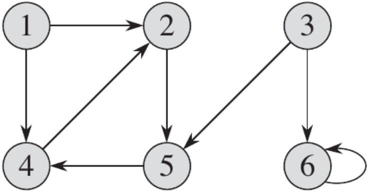


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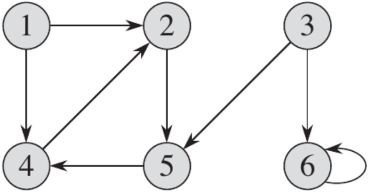


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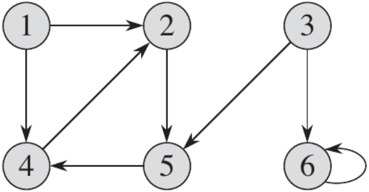


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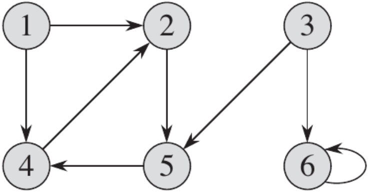


colour implemented by map or hash table

- How can we detect a Forward edge using the *COLOR* attribute?
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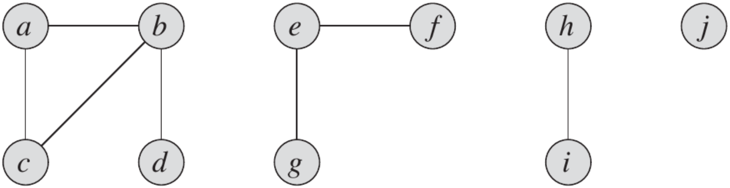
- **Cross edge** : if $v$ is BLACK.

# Connected components of an undirected graph

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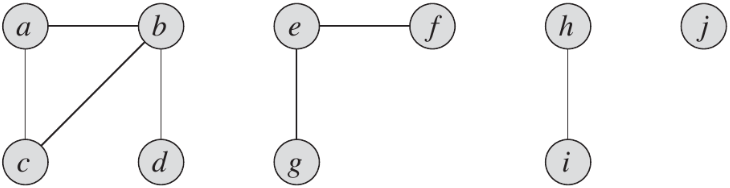


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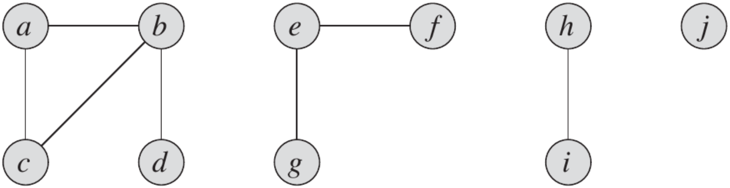


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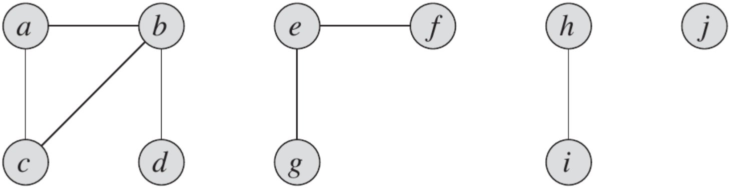
# Counting the connected components

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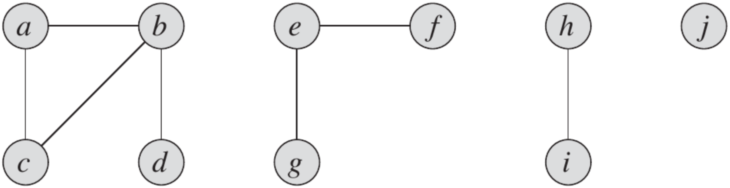


- How many connected components are there in the above graph?
- Can we modify the DFS algorithm to count the number of connected components?

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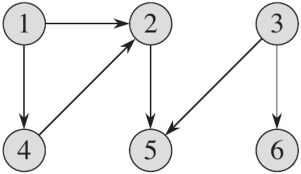

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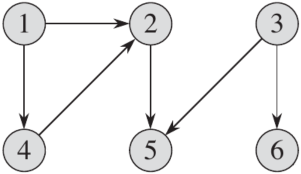


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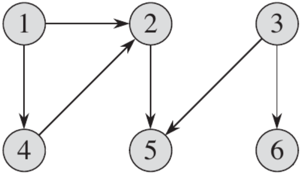


- Topological sort gives us a linear ordering of vertices such that if there is a directed edge $(u, v)$, then $u$ comes before $v$ in the linear ordering.
- Which of the following is a topological sort?

a. 1, 4, 2, 5, 3, 6

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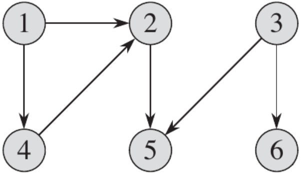


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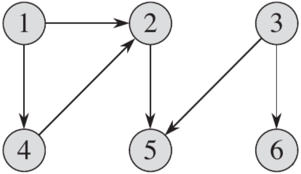


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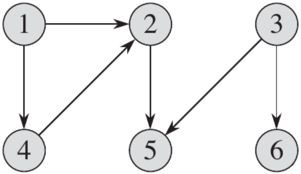


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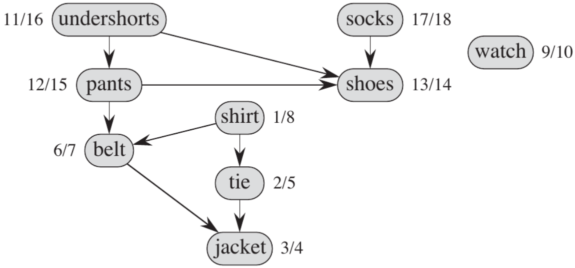
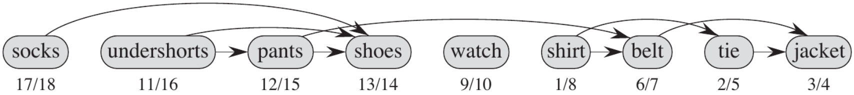
c. 3, 6, 1, 4, 2, 5

# Topological sort example



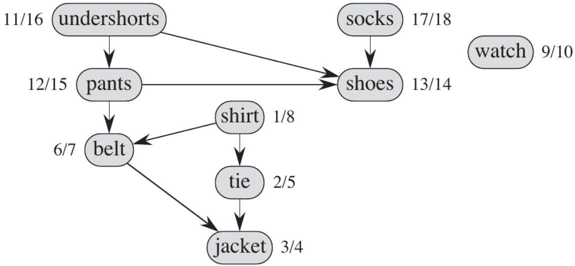



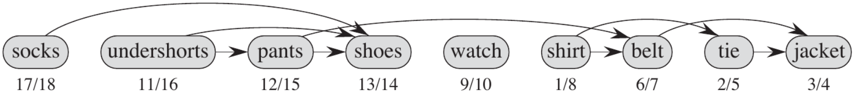
# Topological sort pseudocode

TOPOLOGICAL-SORT$(G)$

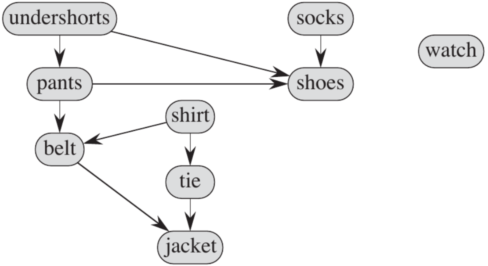
```
call DFS(G) to compute finishing times v.f for each vertex v
as each vertex is finished, insert it onto the front of a linked list
return the linked list of vertices
```

# Topological sort example



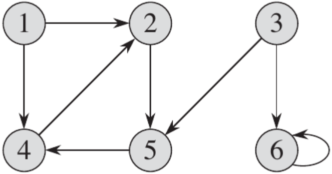
# Topological sort example 2




- Let us say we consider nodes in the following order: Belt, Shoes, Shirt, Undershorts, Watch, Socks

# Topological sort

- Topological sorting should fail if we detect a back edge.



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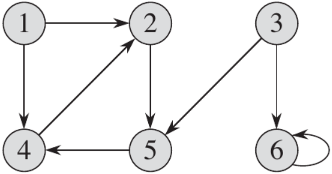


- Why does topological sort algorithm work?

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