Data Structures and Algorithms [1]

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[1] Material for the presentation taken from Cormen, Leiserson, Rivest and Stein, *Introduction to Algorithms, Third Edition*;

Divide the problem into a number of subproblems that are smaller instances of the same problem.

Conquer the subproblems by solving them recursively.

Combine the solutions to the subproblems into the solution for the original problem.

Minimum Element

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1: **procedure** FINDMINIMUM(arr, $left$, $right$)
2: **if** $left = right$ **then**
3: **return** $arr[left]$
4: $mid \leftarrow \lfloor (left + right)/2 \rfloor$

Minimum Element

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procedure $\textsc{findMinimum}(\mathit{arr}, \mathit{left}, \mathit{right})$

  if $\mathit{left} = \mathit{right}$ **then**

    return $\mathit{arr}[\mathit{left}]$

  $\mathit{mid} \leftarrow \lfloor (\mathit{left} + \mathit{right})/2 \rfloor$

  $\mathit{minLeft} \leftarrow \textsc{findMinimum}(\mathit{arr}, \mathit{left}, \mathit{mid})$

  $\mathit{minRight} \leftarrow \textsc{findMinimum}(\mathit{arr}, \mathit{mid} + 1, \mathit{right})$

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```
procedure FINDMINIMUM(arr, left, right)
    if left = right then
        return arr[left]
    mid ← ⌊(left + right)/2⌋
    minLeft ← FINDMINIMUM(arr, left, mid)
    minRight ← FINDMINIMUM(arr, mid + 1, right)
    return min(minLeft, minRight)
```

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(a) Find the recurrence equation for the above algorithm?

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(a) Find the recurrence equation for the above algorithm?

(b) What is the worst case running time of the algorithm?

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```
MATRIX-MULTIPLY(A, B, C, n)
for i = 1 to n
    for j = 1 to n
        for k = 1 to n
            c_ij = c_ij + a_ik · b_kj
```

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- Volker Strassen came up with an algorithm that takes $O(n^{\lg 7})$ time, which is $o(n^{2.81})$. $(\lg 7 = 2.8074)$
- Strassen's algorithm uses a divide-and-conquer approach.

Matrix multiplication: divide-and-conquer

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

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so that we rewrite the equation $C = A \cdot B$ as

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

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$$\begin{aligned} C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21}, \\ C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22}, \\ C_{21} &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21}, \\ C_{22} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22}. \end{aligned}$$

Matrix multiplication: divide-and-conquer

```
MATRIX-MULTIPLY-RECURSIVE(A, B, C, n)
if n == 1
// Base case.
    c11 = c11 + a11 · b11
    return
// Divide.
partition A, B, and C into n/2 × n/2 submatrices
    A11, A12, A21, A22; B11, B12, B21, B22;
    and C11, C12, C21, C22; respectively
// Conquer.
MATRIX-MULTIPLY-RECURSIVE(A11, B11, C11, n/2)
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MATRIX-MULTIPLY-RECURSIVE(A21, B11, C21, n/2)
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MATRIX-MULTIPLY-RECURSIVE(A12, B22, C12, n/2)
MATRIX-MULTIPLY-RECURSIVE(A22, B21, C21, n/2)
MATRIX-MULTIPLY-RECURSIVE(A22, B22, C22, n/2)
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- $T(n) = 8T(n/2) + D(n) + C(n)$

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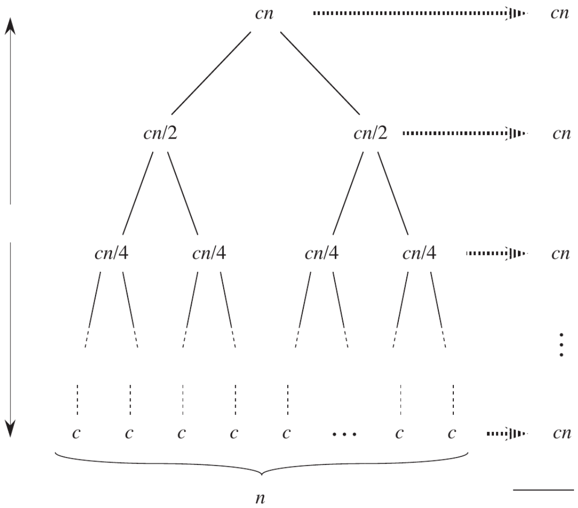
- $T(n) = 8T(n/2) + D(n) + C(n)$
- $T(n) = 8T(n/2) + \Theta(1) + \Theta(1)$

Recursion tree for merge sort



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n	0	1	2	3	4	5	6
8^n	1	8	64	512	4096	32768	262144
7^n	1	7	49	343	2401	16807	117649
2^n	1	2	4	8	16	32	64

Strassen's algorithm : Divide (Step 1)

Divide step involves finding ten $n/2 \times n/2$ matrices (S_i).

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$$
\begin{aligned}
S_1 &= B_{12} - B_{22} , \\
S_2 &= A_{11} + A_{12} , \\
S_3 &= A_{21} + A_{22} , \\
S_4 &= B_{21} - B_{11} , \\
S_5 &= A_{11} + A_{22} , \\
S_6 &= B_{11} + B_{22} , \\
S_7 &= A_{12} - A_{22} , \\
S_8 &= B_{21} + B_{22} , \\
S_9 &= A_{11} - A_{21} , \\
S_{10} &= B_{11} + B_{12} .
\end{aligned}
$$

Strassen's algorithm : Conquer (Step 2)

$$
\begin{aligned}
P_1 &= A_{11} \cdot S_1 \\
P_2 &= S_2 \cdot B_{22} \\
P_3 &= S_3 \cdot B_{11} \\
P_4 &= A_{22} \cdot S_4 \\
P_5 &= S_5 \cdot S_6 \\
P_6 &= S_7 \cdot S_8 \\
P_7 &= S_9 \cdot S_{10}
\end{aligned}
$$

Strassen's algorithm : Conquer (Step 2)

$$
\begin{aligned}
P_1 &= A_{11} \cdot S_1 &&= A_{11} \cdot B_{12} - A_{11} \cdot B_{22} , \\
P_2 &= S_2 \cdot B_{22} &&= A_{11} \cdot B_{22} + A_{12} \cdot B_{22} , \\
P_3 &= S_3 \cdot B_{11} &&= A_{21} \cdot B_{11} + A_{22} \cdot B_{11} , \\
P_4 &= A_{22} \cdot S_4 &&= A_{22} \cdot B_{21} - A_{22} \cdot B_{11} , \\
P_5 &= S_5 \cdot S_6 &&= A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22} , \\
P_6 &= S_7 \cdot S_8 &&= A_{12} \cdot B_{21} + A_{12} \cdot B_{22} - A_{22} \cdot B_{21} - A_{22} \cdot B_{22} , \\
P_7 &= S_9 \cdot S_{10} &&= A_{11} \cdot B_{11} + A_{11} \cdot B_{12} - A_{21} \cdot B_{11} - A_{21} \cdot B_{12} .
\end{aligned}
$$

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Strassen's algorithm : Combine (Step 3)

$$C_{11} = P_5 + P_4 - P_2 + P_6 \, .$$

$$\begin{array}{l}
A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22} \\
\qquad\qquad\qquad\qquad - A_{22} \cdot B_{11} \qquad\qquad\qquad\qquad + A_{22} \cdot B_{21} \\
\qquad\quad - A_{11} \cdot B_{22} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - A_{12} \cdot B_{22} \\
\qquad\qquad\qquad\qquad\qquad\qquad - A_{22} \cdot B_{22} - A_{22} \cdot B_{21} + A_{12} \cdot B_{22} + A_{12} \cdot B_{21} \\
\hline
A_{11} \cdot B_{11} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + A_{12} \cdot B_{21} \, ,
\end{array}$$

Strassen's algorithm : Combine (Step 3)

$$C_{12} = P_1 + P_2 \, ,$$

$$\begin{array}{l}
A_{11} \cdot B_{12} - A_{11} \cdot B_{22} \\
\qquad\qquad\quad + A_{11} \cdot B_{22} + A_{12} \cdot B_{22} \\
\hline
A_{11} \cdot B_{12} \qquad\qquad\qquad\quad + A_{12} \cdot B_{22} \, ,
\end{array}$$

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$$C_{21} = P_3 + P_4$$

$$\begin{array}{rl} & A_{21} \cdot B_{11} + A_{22} \cdot B_{11} \\ & \qquad - A_{22} \cdot B_{11} + A_{22} \cdot B_{21} \\ \hline & A_{21} \cdot B_{11} \qquad\qquad + A_{22} \cdot B_{21} , \end{array}$$

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$$C_{22} = P_5 + P_1 - P_3 - P_7\ ,$$

$$\begin{array}{l} \quad A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22} \\ \qquad\qquad - A_{11} \cdot B_{22} \qquad\qquad\qquad\qquad + A_{11} \cdot B_{12} \\ \qquad\qquad\qquad\qquad - A_{22} \cdot B_{11} \qquad\qquad\qquad\qquad - A_{21} \cdot B_{11} \\ - A_{11} \cdot B_{11} \qquad\qquad\qquad\qquad\qquad - A_{11} \cdot B_{12} + A_{21} \cdot B_{11} + A_{21} \cdot B_{12} \\ \hline \qquad\qquad\qquad\qquad\qquad A_{22} \cdot B_{22} \qquad\qquad\qquad\qquad + A_{21} \cdot B_{12}\ , \end{array}$$

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  $T(n) = O(n^{\lg 7}) = o(n^{2.81})$
- Why should n be a power of 2?

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The above implies that $m^{\lg 7} = \Theta(n^{\lg 7})$.

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  $T(n) \leq dn \lg n$
- We will assume that $T(n/2) \leq d(n/2)\lg(n/2)$. Then show that $T(n) \leq dn \lg n$. (Inductive step)

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$$\begin{aligned} T(n) &= 2T(\lfloor n/2 \rfloor) + n \\ &\le 2d(n/2)\lg(n/2) + n \end{aligned}$$

Substitution method : $T(n) = 2T(\lfloor n/2 \rfloor) + n$

Inductive step : Assume $T(n/2) \leq d(n/2)\lg(n/2)$

$$T(\lfloor n/2 \rfloor) \leq d(\lfloor n/2 \rfloor)\lg(\lfloor n/2 \rfloor) \leq d(n/2)\lg(n/2)$$

$$\begin{aligned} T(n) &= 2T(\lfloor n/2 \rfloor) + n \\ &\leq 2d(n/2)\lg(n/2) + n \\ &= dn(\lg n - \lg 2) + n \end{aligned}$$

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$$\begin{aligned}
T(n) &= 2T(\lfloor n/2 \rfloor) + n \\
&\le 2d(n/2)\lg(n/2) + n \\
&= dn(\lg n - \lg 2) + n \\
&= dn(\lg n - 1) + n \\
&= dn\lg n - dn + n
\end{aligned}$$

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$$\begin{aligned} T(n) &= 2T(\lfloor n/2 \rfloor) + n \\ &\leq 2d(n/2)\lg(n/2) + n \\ &= dn(\lg n - \lg 2) + n \\ &= dn(\lg n - 1) + n \\ &= dn\lg n - dn + n \\ &\leq dn\lg n \text{ , for } d \geq 1 \end{aligned}$$

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\end{aligned}$$

Base case is problematic

$$T(1) = 1$$

Substitution method : $T(n) = 2T(\lfloor n/2 \rfloor) + n$

Inductive step : Assume $T(n/2) \leq d(n/2)\lg(n/2)$

$$T(\lfloor n/2 \rfloor) \leq d(\lfloor n/2 \rfloor)\lg(\lfloor n/2 \rfloor) \leq d(n/2)\lg(n/2)$$

$$\begin{aligned}
T(n) &= 2T(\lfloor n/2 \rfloor) + n \\
&\leq 2d(n/2)\lg(n/2) + n \\
&= dn(\lg n - \lg 2) + n \\
&= dn(\lg n - 1) + n \\
&= dn\lg n - dn + n \\
&\leq dn\lg n \text{ , for } d \geq 1
\end{aligned}$$

Base case is problematic

$$T(1) = 1 \not\leq d\,1\lg 1$$

Substitution method : $T(n) = 2T(\lfloor n/2 \rfloor) + n$

Inductive step : Assume $T(n/2) \leq d(n/2)\lg(n/2)$

$$T(\lfloor n/2 \rfloor) \leq d(\lfloor n/2 \rfloor)\lg(\lfloor n/2 \rfloor) \leq d(n/2)\lg(n/2)$$

$$\begin{aligned}
T(n) &= 2T(\lfloor n/2 \rfloor) + n \\
&\leq 2d(n/2)\lg(n/2) + n \\
&= dn(\lg n - \lg 2) + n \\
&= dn(\lg n - 1) + n \\
&= dn\lg n - dn + n \\
&\leq dn\lg n \text{ , for } d \geq 1
\end{aligned}$$

Base case is problematic

$$T(1) = 1 \not\leq d\,1\lg 1 = 0$$

Substitution method : $T(n) = 2T(\lfloor n/2 \rfloor) + n$

- $T(2) = 2T(\lfloor 2/2 \rfloor) + 2$

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- $T(2) = 2T(\lfloor 2/2 \rfloor) + 2 = 4$

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Substitution method : $T(n) = 2T(\lfloor n/2 \rfloor) + n$

- $T(2) = 2T(\lfloor 2/2 \rfloor) + 2 = 4 \leq d2\lg 2$
- $T(3) = 2T(\lfloor 3/2 \rfloor) + 3$

Substitution method : $T(n) = 2T(\lfloor n/2 \rfloor) + n$

- $T(2) = 2T(\lfloor 2/2 \rfloor) + 2 = 4 \leq d2 \lg 2$
- $T(3) = 2T(\lfloor 3/2 \rfloor) + 3 = 5$

Substitution method : $T(n) = 2T(\lfloor n/2 \rfloor) + n$

- $T(2) = 2T(\lfloor 2/2 \rfloor) + 2 = 4 \leq d2\lg 2$
- $T(3) = 2T(\lfloor 3/2 \rfloor) + 3 = 5 \leq d3\lg 3$

Substitution method : $T(n) = 2T(\lfloor n/2 \rfloor) + n$

- $T(2) = 2T(\lfloor 2/2 \rfloor) + 2 = 4 \leq d2 \lg 2$
- $T(3) = 2T(\lfloor 3/2 \rfloor) + 3 = 5 \leq d3 \lg 3$
- We only need to show that $T(n) \leq dn \lg n$ for $n \geq n_0$, where n_0 need not be 1.

Substitution method : $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$

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- Find a tight upper bound for the recurrence $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$

- Find a tight upper bound for the recurrence $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$
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- We guess $T(n) = O(n)$.

- Find a tight upper bound for the recurrence
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  To prove : $T(n) \leq cn \qquad \forall\, n \geq n_0$

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- Find a tight upper bound for the recurrence $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$
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  To prove : $T(n) \leq cn \qquad \forall\, n \geq n_0$
  Induction step assumption :

$$T(\lfloor n/2 \rfloor) \leq c(\lfloor n/2 \rfloor) \quad , \quad T(\lceil n/2 \rceil) \leq c(\lceil n/2 \rceil)$$

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$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$$

Substitution method : $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$

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$$T(\lfloor n/2 \rfloor) \leq c(\lfloor n/2 \rfloor) \quad , \quad T(\lceil n/2 \rceil) \leq c(\lceil n/2 \rceil)$$

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1 \leq c\lfloor n/2 \rfloor + c\lceil n/2 \rceil + 1$$

Substitution method : $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$

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$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1 \leq c\lfloor n/2 \rfloor + c\lceil n/2 \rceil + 1$$

$$T(n) \leq c\lfloor n/2 \rfloor + c\lceil n/2 \rceil + 1$$

Substitution method : $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$

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- We guess $T(n) = O(n)$.
  To prove : $T(n) \leq cn \qquad \forall\, n \geq n_0$
  Induction step assumption :

$$T(\lfloor n/2 \rfloor) \leq c(\lfloor n/2 \rfloor) \quad , \quad T(\lceil n/2 \rceil) \leq c(\lceil n/2 \rceil)$$

$$\begin{aligned} T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1 &\leq c\lfloor n/2 \rfloor + c\lceil n/2 \rceil + 1 \\ T(n) &\leq c\lfloor n/2 \rfloor + c\lceil n/2 \rceil + 1 \\ &= cn + 1 \end{aligned}$$

Substitution method : $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$

- Find a tight upper bound for the recurrence $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$
- What can be an initial guess for the above question based on recursion tree?
- We guess $T(n) = O(n)$.

  To prove : $T(n) \leq cn \qquad \forall\, n \geq n_0$

  Induction step assumption :

$$T(\lfloor n/2 \rfloor) \leq c(\lfloor n/2 \rfloor) \quad , \quad T(\lceil n/2 \rceil) \leq c(\lceil n/2 \rceil)$$

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1 \leq c\lfloor n/2 \rfloor + c\lceil n/2 \rceil + 1$$

$$T(n) \leq c\lfloor n/2 \rfloor + c\lceil n/2 \rceil + 1$$

$$= cn + 1 \text{ (cannot proceed)}$$

Substitution method : $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$

We need to slightly modify the initial guess for the recurrence $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$.

Substitution method : $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$

We need to slightly modify the initial guess for the recurrence
$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$.
To prove: $T(n) \leq cn - d \qquad \forall\, n \geq n_0$.

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$$T(\lfloor n/2 \rfloor) \leq c(\lfloor n/2 \rfloor) - d$$

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We need to slightly modify the initial guess for the recurrence
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To prove: $T(n) \leq cn - d \qquad \forall\, n \geq n_0$.

$$T(\lfloor n/2 \rfloor) \leq c(\lfloor n/2 \rfloor) - d$$

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$$

Substitution method : $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$

We need to slightly modify the initial guess for the recurrence $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$.
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$$T(\lfloor n/2 \rfloor) \leq c(\lfloor n/2 \rfloor) - d$$

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1 \leq c\lfloor n/2 \rfloor - d + c\lceil n/2 \rceil - d + 1$$

Substitution method : $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$

We need to slightly modify the initial guess for the recurrence $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$.
To prove: $T(n) \leq cn - d \qquad \forall\, n \geq n_0$.

$$\begin{aligned}
T(\lfloor n/2 \rfloor) &\leq c(\lfloor n/2 \rfloor) - d \\
T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1 &\leq c\lfloor n/2 \rfloor - d + c\lceil n/2 \rceil - d + 1 \\
&= cn - 2d + 1
\end{aligned}$$

Substitution method : $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$

We need to slightly modify the initial guess for the recurrence $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$.
To prove: $T(n) \leq cn - d \qquad \forall\, n \geq n_0$.

$$\begin{aligned} T(\lfloor n/2 \rfloor) &\leq c(\lfloor n/2 \rfloor) - d \\ T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1 &\leq c\lfloor n/2 \rfloor - d + c\lceil n/2 \rceil - d + 1 \\ &= cn - 2d + 1 \\ &= cn - d - d + 1 \end{aligned}$$

Substitution method : $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$

We need to slightly modify the initial guess for the recurrence $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$.
To prove: $T(n) \leq cn - d \qquad \forall\, n \geq n_0$.

$$
\begin{aligned}
T(\lfloor n/2 \rfloor) &\leq c(\lfloor n/2 \rfloor) - d \\
T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1 &\leq c\lfloor n/2 \rfloor - d + c\lceil n/2 \rceil - d + 1 \\
&= cn - 2d + 1 \\
&= cn - d - d + 1 \\
&\leq cn - d \text{ , for } d \geq 1
\end{aligned}
$$

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We need to slightly modify the initial guess for the recurrence $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$.
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\begin{aligned}
T(\lfloor n/2 \rfloor) &\leq c(\lfloor n/2 \rfloor) - d \\
T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1 &\leq c\lfloor n/2 \rfloor - d + c\lceil n/2 \rceil - d + 1 \\
&= cn - 2d + 1 \\
&= cn - d - d + 1 \\
&\leq cn - d \text{ , for } d \geq 1 \\
T(1) = 1 \leq cn - d &
\end{aligned}
$$

Substitution method : $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$

We need to slightly modify the initial guess for the recurrence $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$.
To prove: $T(n) \le cn - d \qquad \forall\, n \ge n_0$.

$$
\begin{aligned}
T(\lfloor n/2 \rfloor) &\le c(\lfloor n/2 \rfloor) - d \\
T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1 &\le c\lfloor n/2 \rfloor - d + c\lceil n/2 \rceil - d + 1 \\
&= cn - 2d + 1 \\
&= cn - d - d + 1 \\
&\le cn - d \text{ , for } d \ge 1
\end{aligned}
$$

$$T(1) = 1 \le cn - d \text{ , for } d \ge 1 \text{ and } c \ge d + 1$$

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$$
\begin{aligned}
T(\lfloor n/2 \rfloor) &\leq c(\lfloor n/2 \rfloor) - d \\
T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1 &\leq c\lfloor n/2 \rfloor - d + c\lceil n/2 \rceil - d + 1 \\
&= cn - 2d + 1 \\
&= cn - d - d + 1 \\
&\leq cn - d \text{ , for } d \geq 1
\end{aligned}
$$

$$T(1) = 1 \leq cn - d \text{ , for } d \geq 1 \text{ and } c \geq d + 1$$

For $c = 2,\ d = 1$

Substitution method : $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$

We need to slightly modify the initial guess for the recurrence $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$.
To prove: $T(n) \le cn - d \qquad \forall\, n \ge n_0$.

$$
\begin{aligned}
T(\lfloor n/2 \rfloor) &\le c(\lfloor n/2 \rfloor) - d \\
T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1 &\le c\lfloor n/2 \rfloor - d + c\lceil n/2 \rceil - d + 1 \\
&= cn - 2d + 1 \\
&= cn - d - d + 1 \\
&\le cn - d \text{ , for } d \ge 1 \\
T(1) = 1 \le cn - d &\text{ , for } d \ge 1 \text{ and } c \ge d + 1
\end{aligned}
$$

For $c = 2,\ d = 1 \quad T(n) \le 2n - 1 \quad \forall\, n \ge 1$

Substitution method : $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$

- Instead of showing $T(n) = O(n)$ we have instead shown than $T(n) = O(cn - d)$, which implies $T(n) = O(n)$

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- Find a lower bound for the recurrence $T(n) = T(n-1) + n$

Substitution method : $T(n) = T(n-1) + n$

- Find a lower bound for the recurrence $T(n) = T(n-1) + n$
- What would be a good guess?

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- Find a lower bound for the recurrence $T(n) = T(n-1) + n$
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To show : $T(n) \geq cn^2 \quad \forall\, n \geq n_0$

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Induction step:

$$T(n-1) \geq c(n-1)^2$$

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To show : $T(n) \geq cn^2 \quad \forall\, n \geq n_0$

Induction step:

$$\begin{aligned} T(n-1) &\geq c(n-1)^2 \\ &= c(n^2 + 1 - 2n) = cn^2 + c - 2cn \end{aligned}$$

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Induction step:

$$\begin{aligned} T(n-1) &\geq c(n-1)^2 \\ &= c(n^2 + 1 - 2n) = cn^2 + c - 2cn \\ T(n) &\geq cn^2 + c - 2cn + n \end{aligned}$$

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Induction step:

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Induction step:

$$
\begin{aligned}
T(n-1) &\geq c(n-1)^2 \\
&= c(n^2 + 1 - 2n) = cn^2 + c - 2cn \\
T(n) &\geq cn^2 + c - 2cn + n \text{ (when c=1/2 , R.H.S.} \geq cn^2) \\
&\geq cn^2 \text{ (for c=1/2 and } n \geq 1)
\end{aligned}
$$

Base case:

Substitution method : $T(n) = T(n-1) + n$

- Find a lower bound for the recurrence $T(n) = T(n-1) + n$
- What would be a good guess? $T(n) = \Omega(n^2)$

To show : $T(n) \geq cn^2 \quad \forall\, n \geq n_0$

Induction step:

$$\begin{aligned} T(n-1) &\geq c(n-1)^2 \\ &= c(n^2 + 1 - 2n) = cn^2 + c - 2cn \\ T(n) &\geq cn^2 + c - 2cn + n \text{ (when c=1/2 , R.H.S. } \geq cn^2) \\ &\geq cn^2 \text{ (for c=1/2 and } n \geq 1) \end{aligned}$$

Base case:

$$T(1) \geq cn^2$$

- Recursion tree can be used to generate a good guess for the Substitution method.

Solving recurrences : Recursion tree

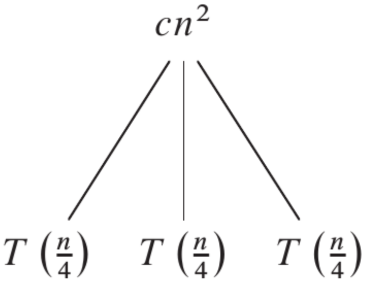
- Recursion tree can be used to generate a good guess for the Substitution method.
- Finding a good guess for $T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$

Solving recurrences : Recursion tree

- Recursion tree can be used to generate a good guess for the Substitution method.
- Finding a good guess for $T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$
- We will ignore the floors and ceilings, because usually they don't affect the asymptotic order of growth.

Recursion tree : $T(n) = 3T(n/4) + cn^2$

$T(n)$



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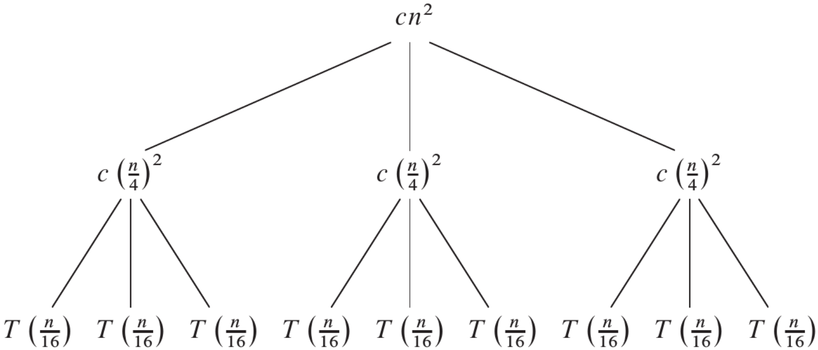

(c)

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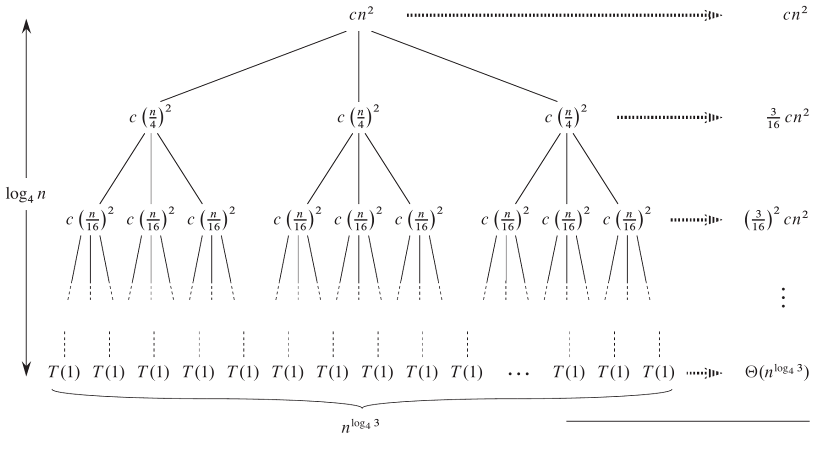

(d)

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$$= \frac{16}{13}cn^2 = O(n^2)$$

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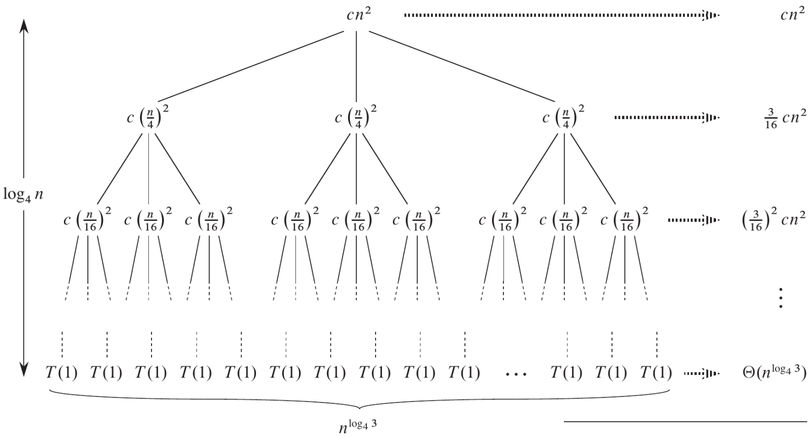

(d)

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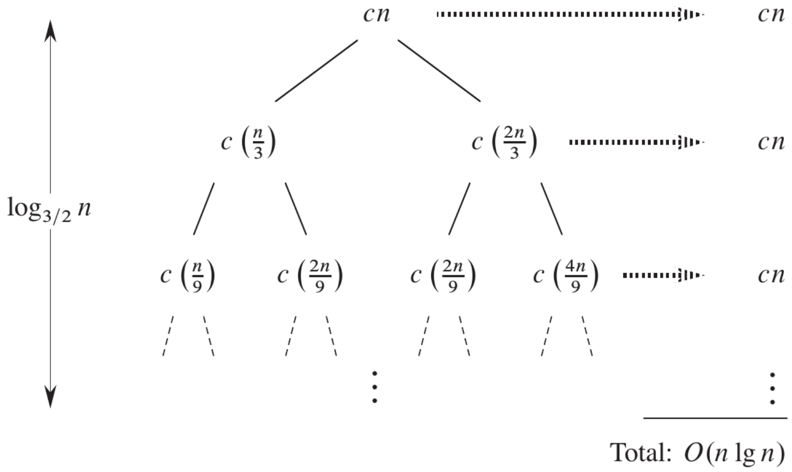
- Then we can use the substitution method to show that our guess is correct.
- To prove : $T(n) \leq dn^2 \quad \forall\, n \geq n_0$

Recursion tree : $T(n) = T(n/3) + T(2n/3) + O(n)$



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- We can again use the substitution method to show the upper bound.

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- For any positive ϵ (however small), is $n^{\epsilon} = \omega(\lg n)$?

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- Using L'Hospital's rule:

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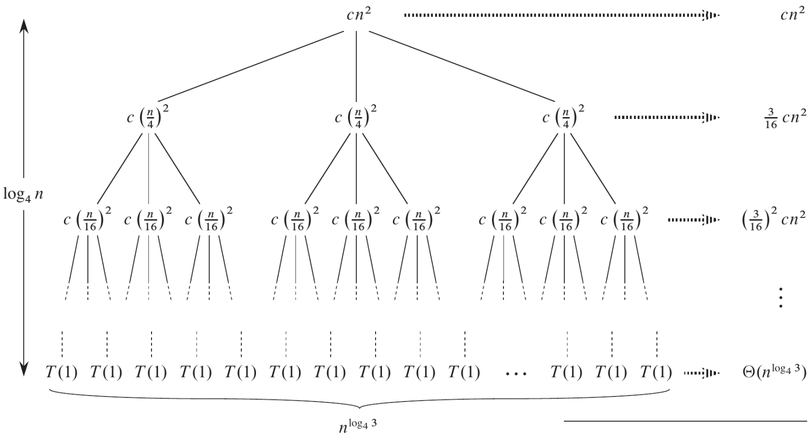
- However, $n^{1+\epsilon} \neq o(n \lg n)$ (n is **not** polynomially smaller)

Master method : $T(n) = 3T(n/4) + cn^2$



(d)

Solving recurrences : Master method

Theorem 4.1 (Master theorem)

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n) ,$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ■

$T(n) = aT(n/b) + f(n)$, $a \geq 1, b > 1$

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
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- $T(n) = 9T(n/3) + n$

$T(n) = aT(n/b) + f(n)$, $a \geq 1, b > 1$

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$$
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f(n) &= \Theta(1) = \Theta(n^{\log_b a}) \\
T(n) &= \Theta(n^{\log_b a} \lg n) \\
&= \Theta(\lg n)
\end{aligned}
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- $T(n) = 3T(n/4) + n \lg n$

$$n^{\log_b a} = n^{\log_4 3}$$

$$f(n) = n \lg n = \Omega(n^{\log_4 3 + \epsilon}) \ ?$$

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- $T(n) = 3T(n/4) + n \lg n$

$$n^{\log_b a} = n^{\log_4 3}$$

$$f(n) = n \lg n = \Omega(n^{\log_4 3 + \epsilon}) \ ?$$

$$a(n/b)\lg(n/b) \leq cn \lg n$$

$T(n) = aT(n/b) + f(n)$, $a \geq 1, b > 1$

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$$\begin{aligned} n^{\log_b a} &= n^{\log_4 3} \\ f(n) &= n \lg n = \Omega(n^{\log_4 3 + \epsilon}) \text{ ?} \\ a(n/b)\lg(n/b) &\leq cn \lg n \\ 3(n/4)\lg(n/4) &\leq (3/4)n \lg n \text{ , for } c = 3/4 \end{aligned}$$

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3(n/4)\lg(n/4) &\leq (3/4)n \lg n \ , \ \text{for } c = 3/4 \\
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\end{aligned}
$$

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$$n^{\log_b a} = n^{\log_2 2} = n$$

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Master method cannot be applied.

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$$n^{\log_b a} = n^{\lg 7} \approx n^{2.81}$$

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$$= O(n^{\lg 7 - \epsilon}) \ ?$$

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\begin{aligned}
n^{\log_b a} &= n^{\lg 7} \approx n^{2.81} \\
f(n) &= \Theta(n^2) = O(n^{\log_b a - \epsilon}) \\
&= O(n^{\lg 7 - \epsilon}) \ ? \\
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Proof of the master theorem

- Section 4.6 will not be a part of our syllabus

- Record : Collection of data

Part II Sorting and Order Statistics

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- If satellite data is large, we permute an array of pointers to the records.

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- *In place* sorting : If at any time only a constant number of elements are stored outside the array.

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- MERGE procedure does not operate in place.
- Ch 6 : Heapsort that uses a data structure called heap.
- Heapsort sorts n elements *in place* in $O(n \lg n)$ time.

Sorting algorithms

Algorithm	Worst-case running time	Average-case/expected running time
Insertion sort	$\Theta(n^2)$	$\Theta(n^2)$
Merge sort	$\Theta(n \lg n)$	$\Theta(n \lg n)$
Heapsort	$O(n \lg n)$	—
Quicksort	$\Theta(n^2)$	$\Theta(n \lg n)$ (expected)
Counting sort	$\Theta(k + n)$	$\Theta(k + n)$
Radix sort	$\Theta(d(n + k))$	$\Theta(d(n + k))$
Bucket sort	$\Theta(n^2)$	$\Theta(n)$ (average-case)

Ch 6 : Heapsort

- Heapsort uses the Heap data structure.

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- Heapsort uses the Heap data structure.
- To understand heap, we need to understand a few terminologies.

Binary tree

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