Data Structures and Algorithms [1]

BITS-Pilani K. K. Birla Goa Campus

[1] Material for the presentation taken from Cormen, Leiserson, Rivest and Stein, *Introduction to Algorithms, Third Edition*;

- Which of the following operations will affect the height of a BST?

  SEARCH, MINIMUM, MAXIMUM, PREDECESSOR, SUCCESSOR, INSERT and DELETE

Ch. 13: Red-black Trees

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- Applications of red-black trees :
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  - Implementation of C++ STL : set, map, multiset, multimap.

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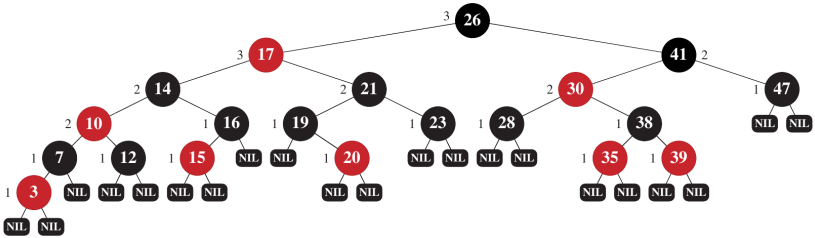
- In an RB Tree, we will use a Sentinel node NIL instead of the NIL value.

Sentinel NIL leaves in Red-black trees



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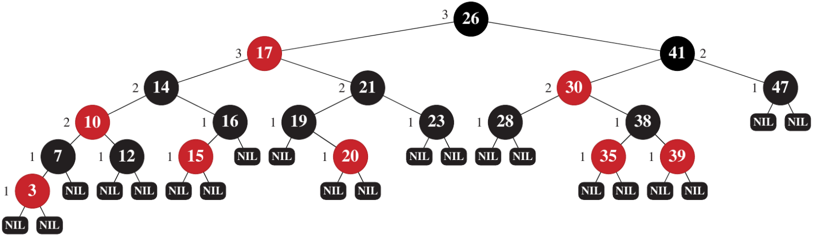

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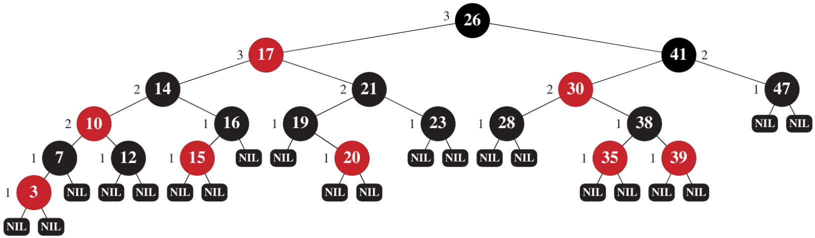

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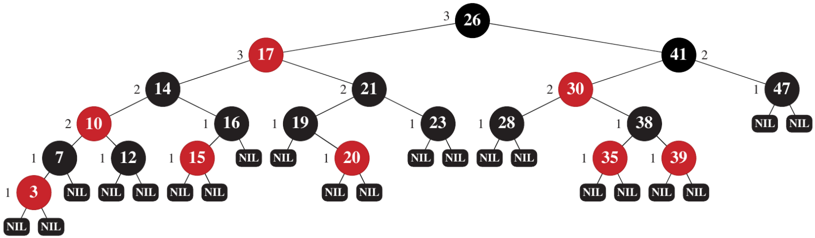
Are all the RB tree properties satisfied?



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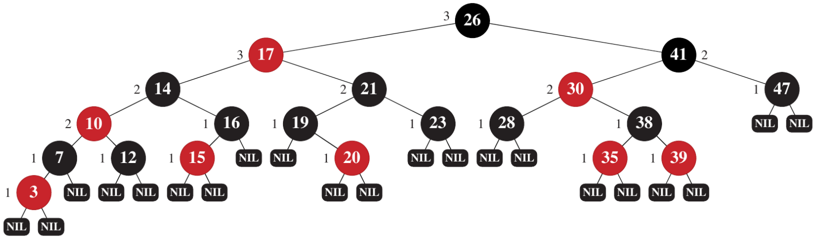

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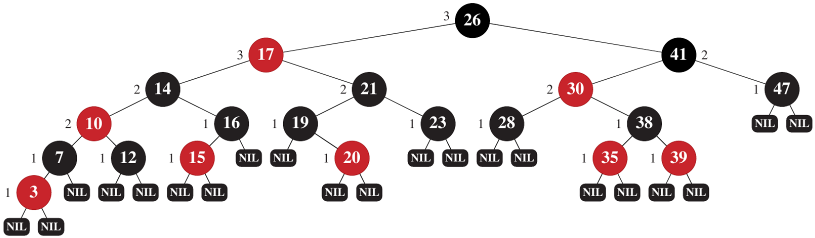

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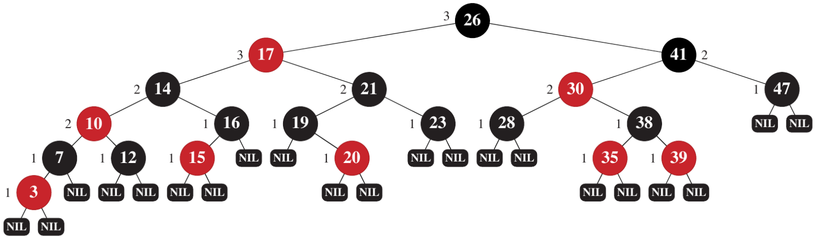
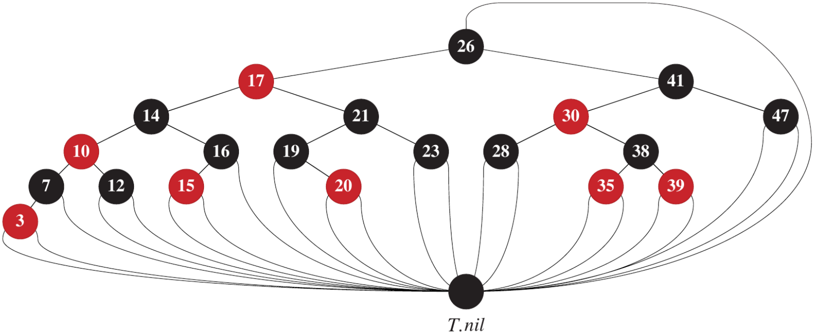


- $T.nil.color = BLACK$; All the other attributes of $T.nil$, can take arbitrary values — because the other values don't matter.

Red-black tree with one sentinel node



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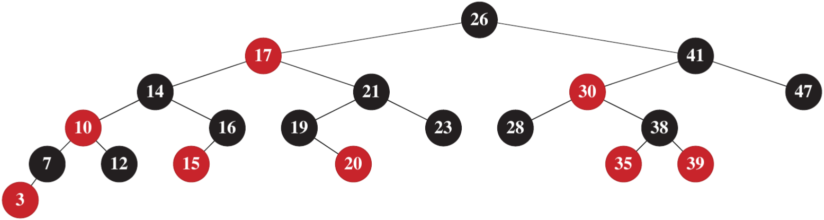
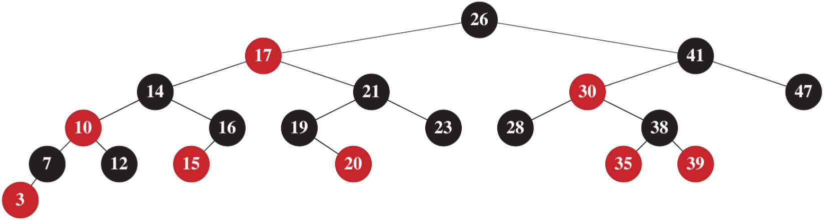


Height and Black height of a node



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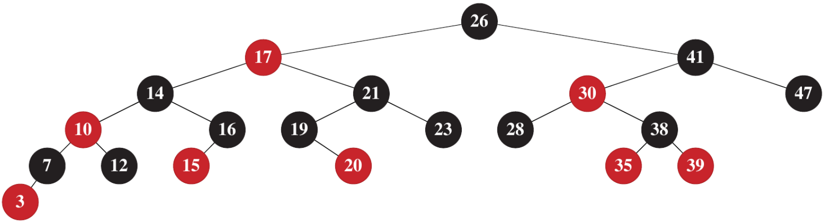

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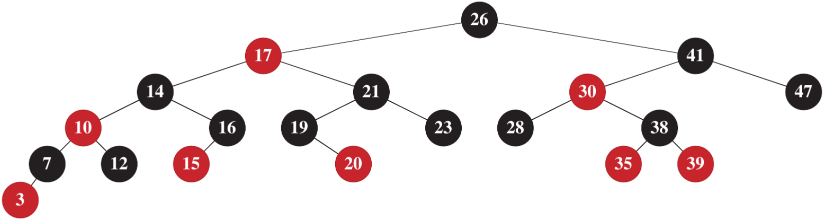

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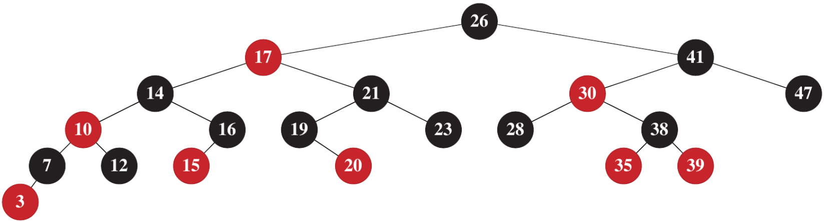

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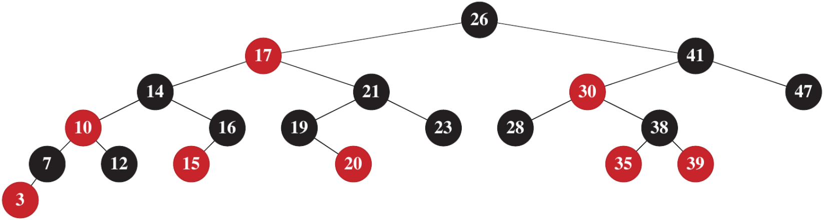

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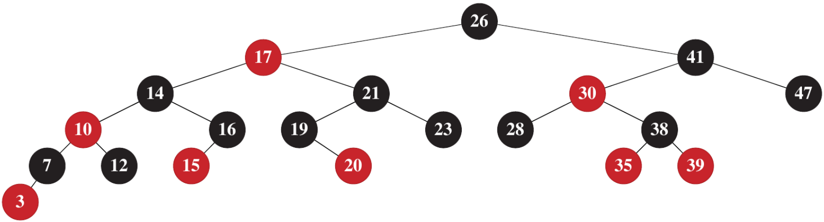

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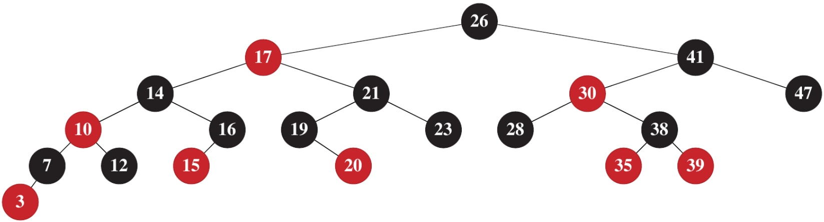

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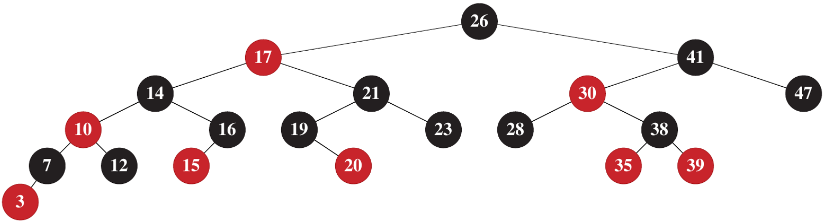

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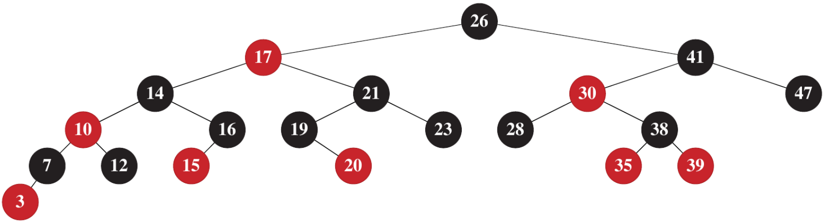

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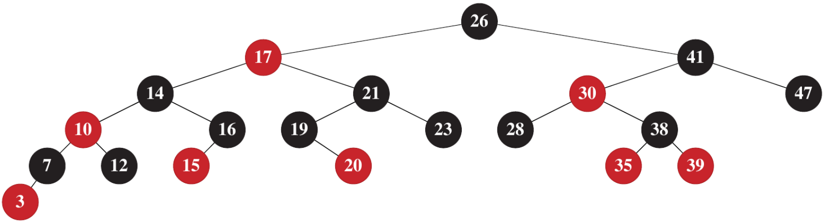

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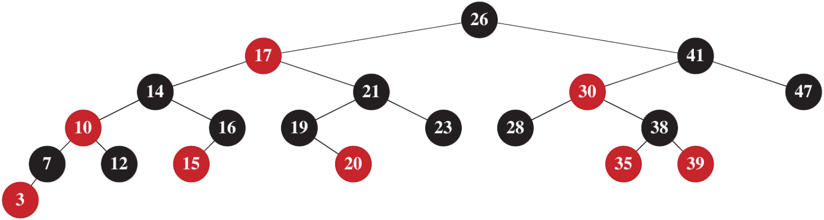

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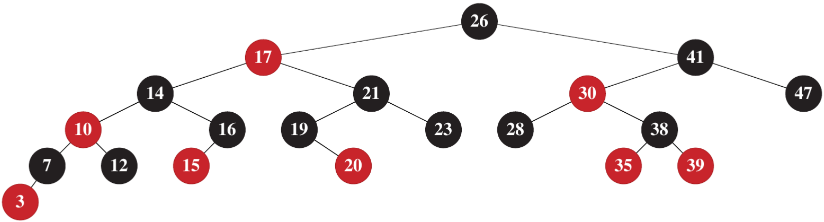

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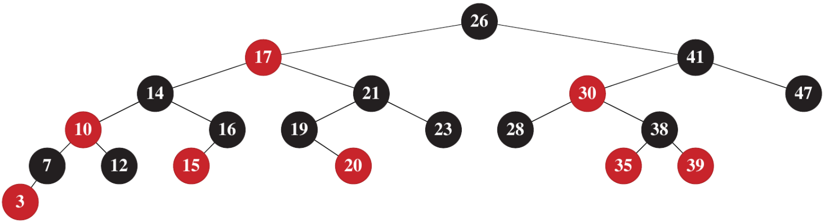

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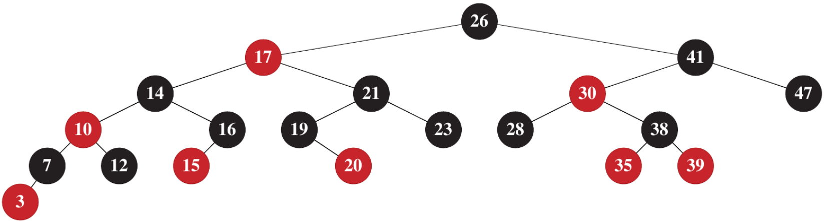

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- Is the following statement true?

  Let x be an internal node. Then black height of a child node of x will be at least $bh(x) - 1$.

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- To prove the above lemma we will first prove that the subtree rooted at any node x contains at least $2^{bh(x)} - 1$ internal nodes.

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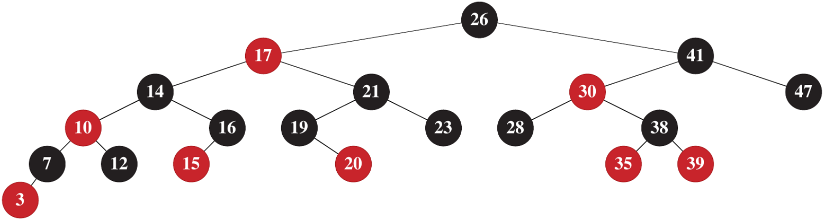

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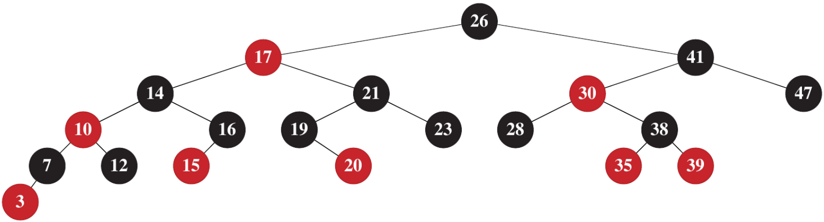

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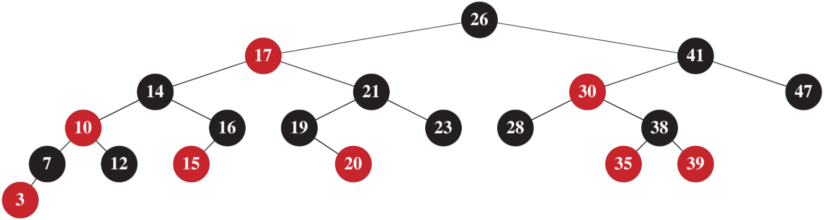

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Lemma 1: A red-black tree with n internal nodes has height at most $2\lg(n+1)$

Lemma 2: A subtree rooted at any node x in a RB tree contains at least $2^{bh(x)} - 1$ internal nodes.

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\lg(n+1) &\geq h/2
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$$

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$$\begin{aligned} n &\geq 2^{bh(T.root)} - 1 \\ &\geq 2^{h/2} - 1 \\ (n+1) &\geq 2^{h/2} \\ \lg(n+1) &\geq h/2 \\ h &\leq 2\lg(n+1) \end{aligned}$$

Implication of **Lemma 1**

- $2\lg(n+1) = O(\lg n)$

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c = 3, &\qquad n_0 = 3
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$$

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- If all the red-black tree properties are satisfied, then $h = O(\lg n)$.

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\end{aligned}
$$

- If all the red-black tree properties are satisfied, then $h = O(\lg n)$.
- If we can ensure that the red-black tree properties are always satisfied, then SEARCH, MINIMUM, MAXIMUM, PREDECESSOR, SUCCESSOR, INSERT and DELETE operations can be performed in $O(\lg n)$ time.

Rotations

- Rotation will be one of the operations that will help us ensure that the red-black tree properties hold after insertions and deletions.

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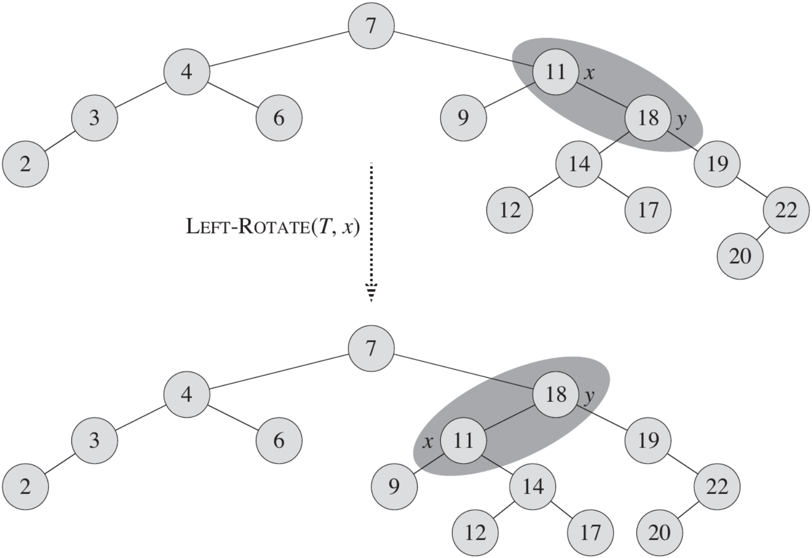
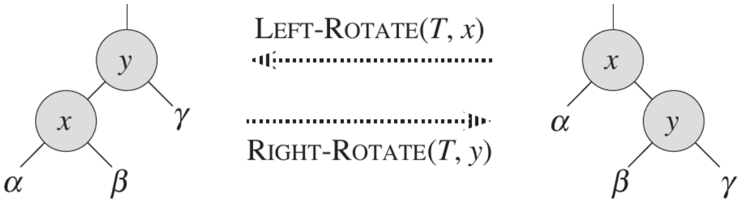
- Rotation will be one of the operations that will help us ensure that the red-black tree properties hold after insertions and deletions.
- Binary search tree property continues to be satisfied after rotation

Rotation: BST property continues to be satisfied



Left and Right rotation



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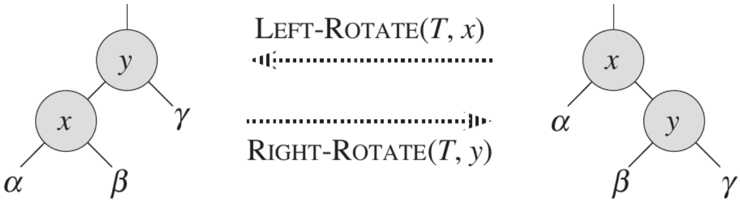


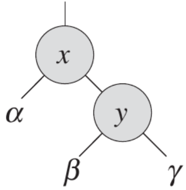
- BST property is maintained by both left and right rotations.

Pseudocode for Left rotate

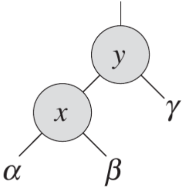
LEFT-ROTATE(T, x)

```
y = x.right
x.right = y.left
```

Before



After

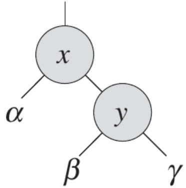


Pseudocode for Left rotate

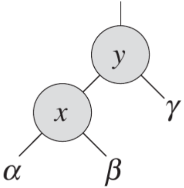
LEFT-ROTATE(T, x)

```
y = x.right
x.right = y.left
if y.left ≠ T.nil
        y.left.p = x
```

Before



After



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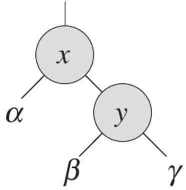

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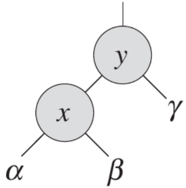
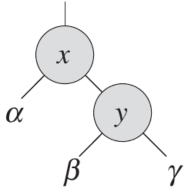


Pseudocode for Left rotate

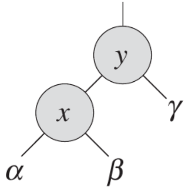
LEFT-ROTATE(T, x)

```
y = x.right
x.right = y.left
if y.left ≠ T.nil
        y.left.p = x
y.p = x.p
if x.p == T.nil
        T.root = y
elseif x == x.p.left
        x.p.left = y
else x.p.right = y
```

Before



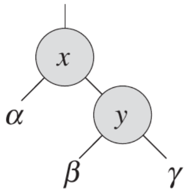
After



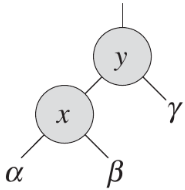
Pseudocode for Left rotate

```
LEFT-ROTATE(T, x)
y = x.right
x.right = y.left
if y.left ≠ T.nil
    y.left.p = x
y.p = x.p
if x.p == T.nil
    T.root = y
elseif x == x.p.left
    x.p.left = y
else x.p.right = y
y.left = x
x.p = y
```

Before



After



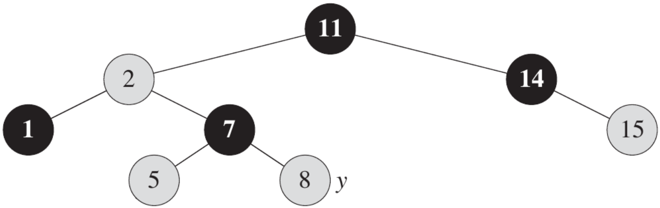
ROTATE procedure

- Running time of left and right rotate operations :

ROTATE procedure

- Running time of left and right rotate operations : $O(1)$

Red-black tree insert



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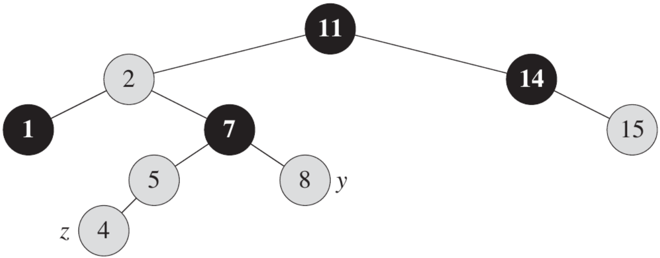
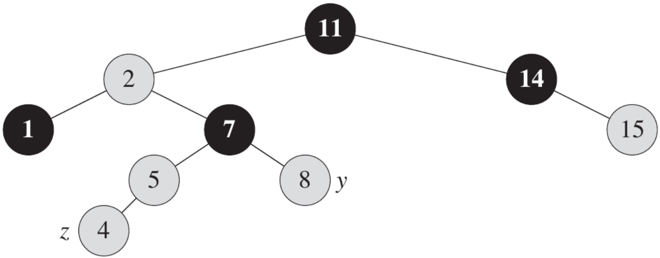
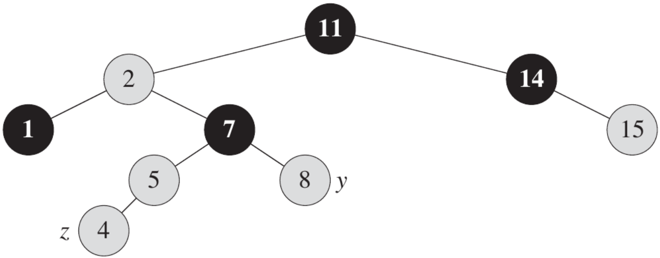


Red-black tree insert



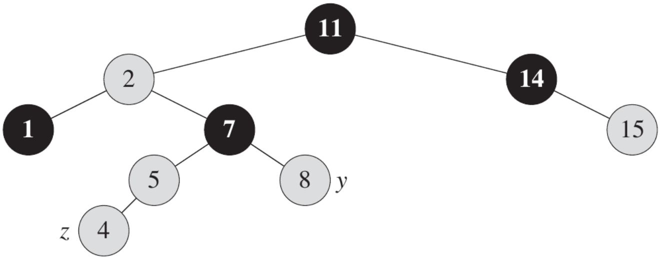
Differences with insert operation in BST:



Differences with insert operation in BST:

1. The new node is inserted as an internal node of the RB tree.

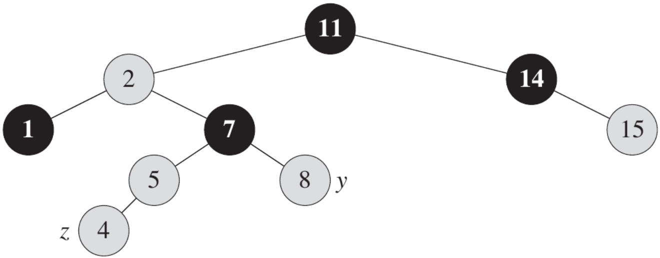
Red-black tree insert



Differences with insert operation in BST:

1. The new node is inserted as an internal node of the RB tree.
2. New node is always colored red.

Red-black tree insert



Differences with insert operation in BST:

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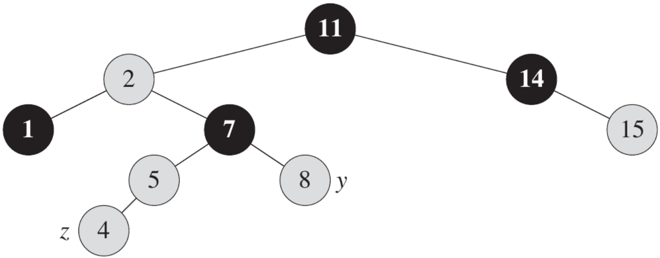
- What RB tree property can get violated as a result of a node insertion?

Red-black tree insert



Differences with insert operation in BST:

1. The new node is inserted as an internal node of the RB tree.
2. New node is always colored red.

- What RB tree property can get violated as a result of a node insertion?
- Either property 2 or property 4 (not both) can get violated.

RB-INSERT : Red-black tree insert

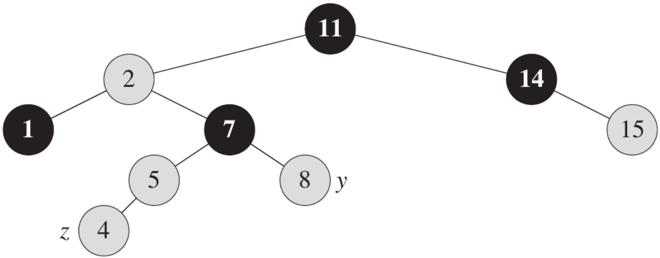
```
RB-INSERT(T, z)
y = T.nil
x = T.root
while x ≠ T.nil
    y = x
    if z.key < x.key
        x = x.left
    else x = x.right
z.p = y
if y == T.nil
    T.root = z
elseif z.key < y.key
    y.left = z
else y.right = z
z.left = T.nil
z.right = T.nil
z.color = RED
RB-INSERT-FIXUP(T, z)
```

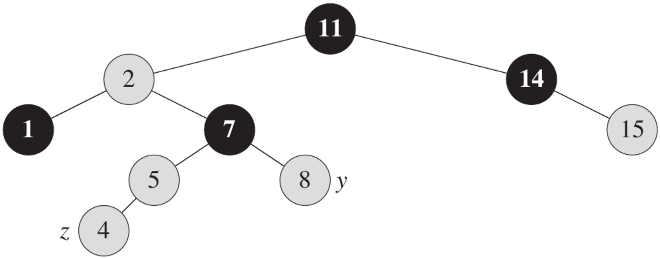
```
TREE-INSERT(T, z)
y = NIL
x = T.root
while x ≠ NIL
    y = x
    if z.key < x.key
        x = x.left
    else x = x.right
z.p = y
if y == NIL
    T.root = z
elseif z.key < y.key
    y.left = z
else y.right = z
```

RB-INSERT-FIXUP : Fix the violated property

```
RB-INSERT-FIXUP(T, z)
while z.p.color == RED
    if z.p == z.p.p.left
        y = z.p.p.right
        if y.color == RED
            z.p.color = BLACK                  // case 1
            y.color = BLACK                    // case 1
            z.p.p.color = RED                  // case 1
            z = z.p.p                          // case 1
        else if z == z.p.right
                z = z.p                        // case 2
                LEFT-ROTATE(T, z)              // case 2
            z.p.color = BLACK                  // case 3
            z.p.p.color = RED                  // case 3
            RIGHT-ROTATE(T, z.p.p)             // case 3
    else (same as then clause
            with "right" and "left" exchanged)
T.root.color = BLACK
```

RB-INSERT-FIXUP : Case 1



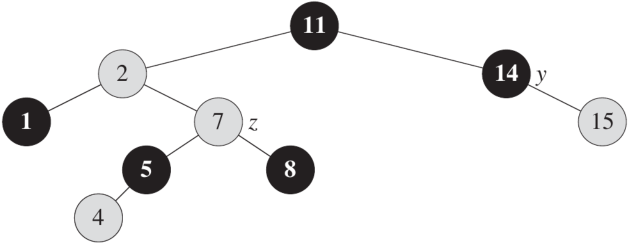


- In case 1, we are assuming that z.p.p is a black node. Why?

RB-INSERT-FIXUP : Fix the violated property

```
RB-INSERT-FIXUP(T, z)
while z.p.color == RED
    if z.p == z.p.p.left
        y = z.p.p.right
        if y.color == RED
            z.p.color = BLACK              // case 1
            y.color = BLACK                // case 1
            z.p.p.color = RED              // case 1
            z = z.p.p                      // case 1
        else if z == z.p.right
                z = z.p                    // case 2
                LEFT-ROTATE(T, z)          // case 2
            z.p.color = BLACK              // case 3
            z.p.p.color = RED              // case 3
            RIGHT-ROTATE(T, z.p.p)         // case 3
    else (same as then clause
            with "right" and "left" exchanged)
T.root.color = BLACK
```

Case 2:



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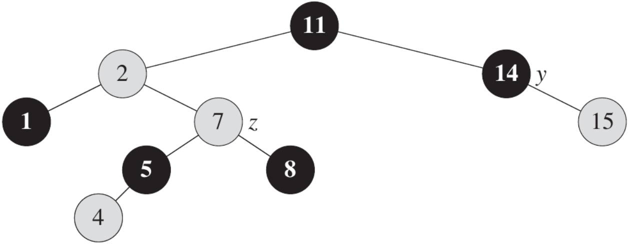
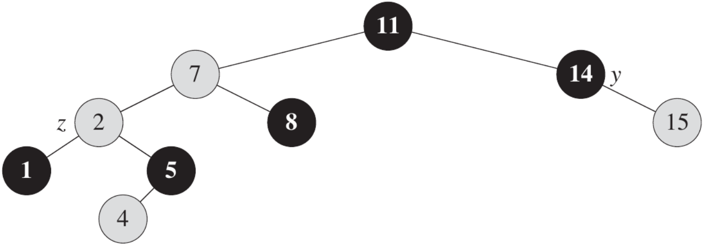


Case 3:



RB-INSERT-FIXUP : Fix the violated property

```
RB-INSERT-FIXUP(T, z)
while z.p.color == RED
    if z.p == z.p.p.left
        y = z.p.p.right
        if y.color == RED
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            y.color = BLACK              // case 1
            z.p.p.color = RED            // case 1
            z = z.p.p                    // case 1
        else if z == z.p.right
                z = z.p                  // case 2
                LEFT-ROTATE(T, z)        // case 2
            z.p.color = BLACK            // case 3
            z.p.p.color = RED            // case 3
            RIGHT-ROTATE(T, z.p.p)       // case 3
    else (same as then clause
            with "right" and "left" exchanged)
T.root.color = BLACK
```
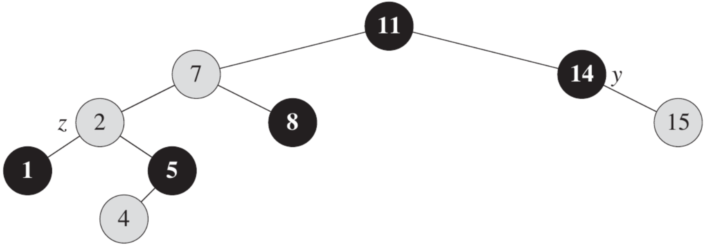
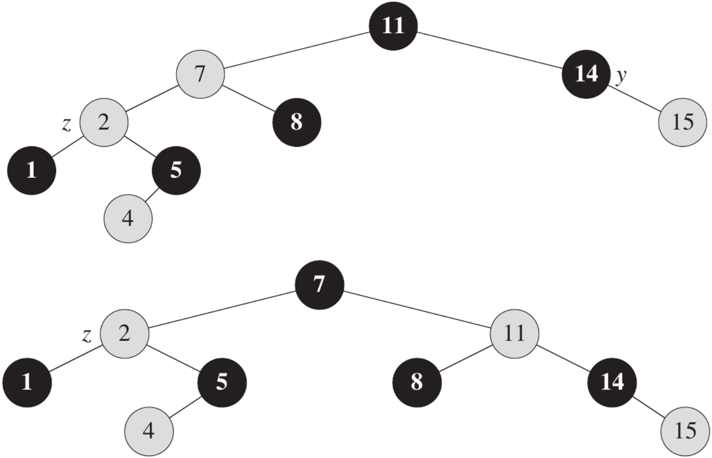

RB-INSERT-FIXUP : Case 3



RB-INSERT-FIXUP : Case 3



RB-INSERT-FIXUP : Fix the violated property

```
RB-INSERT-FIXUP(T, z)
while z.p.color == RED
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            z = z.p.p                              // case 1
        else if z == z.p.right
                z = z.p                            // case 2
                LEFT-ROTATE(T, z)                  // case 2
            z.p.color = BLACK                      // case 3
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    else (same as then clause
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```
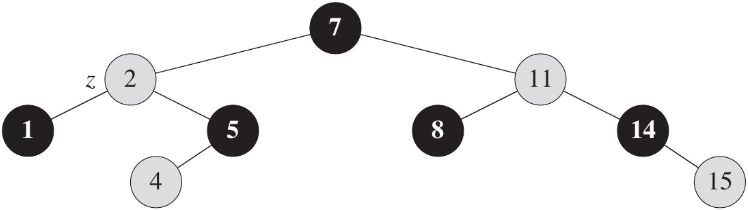
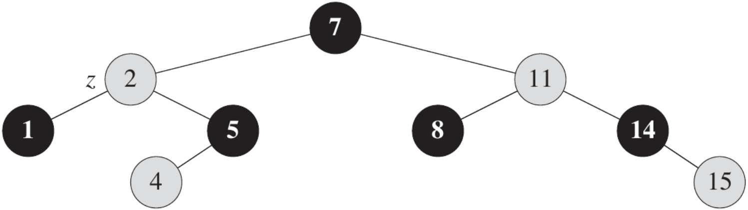

RB-INSERT-FIXUP : All RB tree properties satisfied



RB-INSERT-FIXUP : All RB tree properties satisfied



- Note: Color of the y node helps in differentiating Case 1 from Case 2 and Case 3.

RB-INSERT-FIXUP : Last three cases

```
RB-INSERT-FIXUP(T, z)
while z.p.color == RED
    if z.p == z.p.p.left
        y = z.p.p.right
        if y.color == RED
            z.p.color = BLACK                  // case 1
            y.color = BLACK                    // case 1
            z.p.p.color = RED                  // case 1
            z = z.p.p                          // case 1
        else if z == z.p.right
                z = z.p                        // case 2
                LEFT-ROTATE(T, z)              // case 2
            z.p.color = BLACK                  // case 3
            z.p.p.color = RED                  // case 3
            RIGHT-ROTATE(T, z.p.p)             // case 3
    else (same as then clause
            with "right" and "left" exchanged)
T.root.color = BLACK
```

RB-INSERT-FIXUP

- Each iteration ensures that either only Property 2 or only Property 4 is violated.

RB-Insert-Fixup

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- 6 cases are sufficient to cover all scenarios.

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- In each iteration, the z node moves up the RB tree; and each iteration takes constant time.

RB-INSERT-FIXUP

- Each iteration ensures that either only Property 2 or only Property 4 is violated.
- 6 cases are sufficient to cover all scenarios.
- In each iteration, the z node moves up the RB tree; and each iteration takes constant time.
- RB-INSERT-FIXUP would take $O(\lg n)$ time.

RB-INSERT-FIXUP

- Each iteration ensures that either only Property 2 or only Property 4 is violated.
- 6 cases are sufficient to cover all scenarios.
- In each iteration, the z node moves up the RB tree; and each iteration takes constant time.
- RB-INSERT-FIXUP would take $O(\lg n)$ time.
- Overall running time of RB-INSERT will be $O(\lg n)$.

RB-INSERT-FIXUP

- Each iteration ensures that either only Property 2 or only Property 4 is violated.
- 6 cases are sufficient to cover all scenarios.
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- RB-INSERT-FIXUP would take $O(\lg n)$ time.
- Overall running time of RB-INSERT will be $O(\lg n)$.
- What Binary search tree will we get if we insert keys in the following order : 4, 3, 2, 1 ?

RB-INSERT-FIXUP

- Each iteration ensures that either only Property 2 or only Property 4 is violated.
- 6 cases are sufficient to cover all scenarios.
- In each iteration, the z node moves up the RB tree; and each iteration takes constant time.
- RB-INSERT-FIXUP would take $O(\lg n)$ time.
- Overall running time of RB-INSERT will be $O(\lg n)$.
- What Binary search tree will we get if we insert keys in the following order : 4, 3, 2, 1 ?
- What RB tree will we get for the above example?

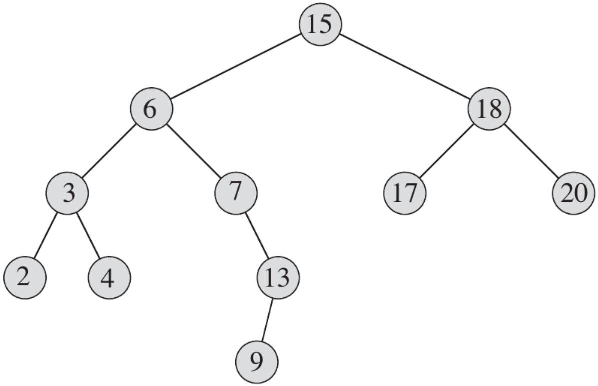
RB-DELETE procedure

- RB-DELETE procedure is similar to the TREE-DELETE procedure.

RB-DELETE procedure

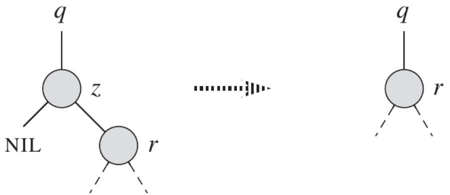
- RB-DELETE procedure is similar to the TREE-DELETE procedure.
- For deleting a node, we need to considered multiple cases.

Tree-Delete

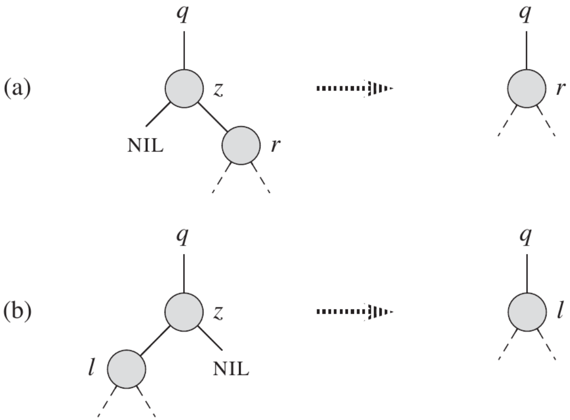


RB-DELETE similar to TREE-DELETE

(a)



RB-DELETE similar to TREE-DELETE



- When node z has two child nodes, we find the successor of node z.

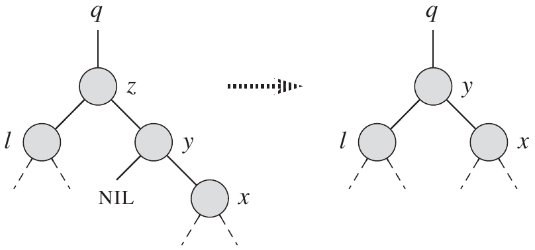
TREE-DELETE : Node z has two child nodes

- When node z has two child nodes, we find the successor of node z.
- We replace node z with its successor.

TREE-DELETE : Node z has two child nodes

(c) Successor of node z is its right child.

(c)



TREE-DELETE : Node z has two child nodes

(d) Successor of node z is not its right child

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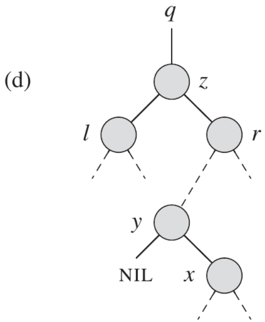

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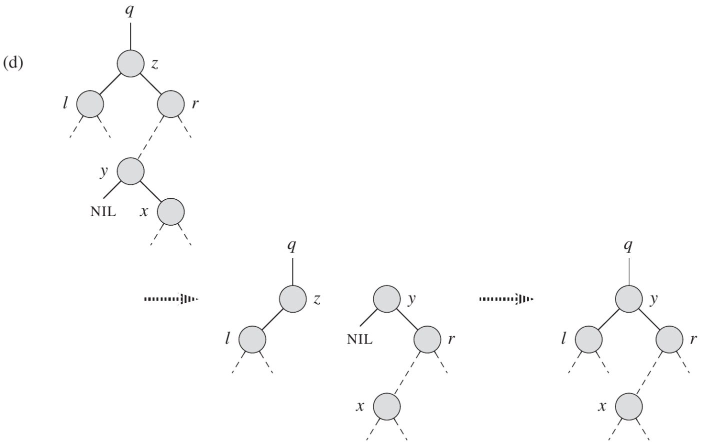
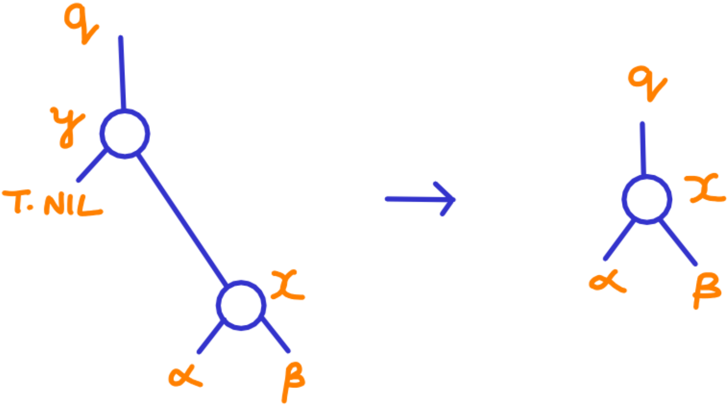


RB-DELETE procedure



RB-DELETE procedure

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RB-DELETE procedure

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- If the node x is "doubly black", then we will try to move the "doubly black" node up the tree.

RB-DELETE procedure

RB-DELETE procedure : Case 2

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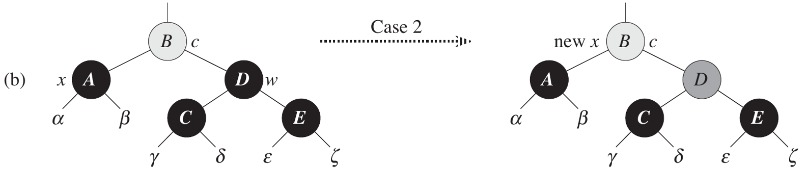
- How are we sure that the sibling of node x is not a leaf node?

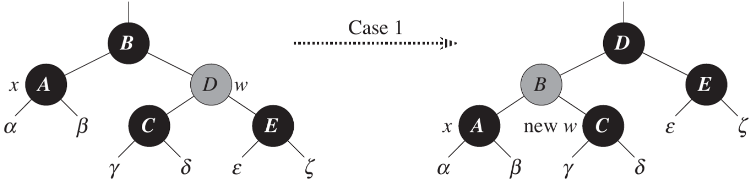
RB-DELETE procedure : Case 2



RB-DELETE procedure : Case 1

RB-Delete procedure : Case 1

(a)



RB-DELETE procedure : Case 4

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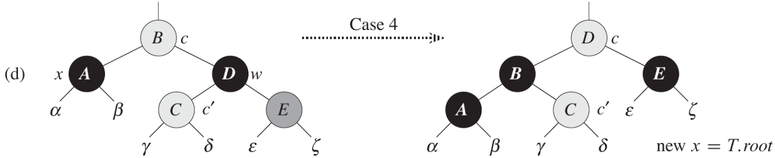

RB-DELETE procedure : Case 3

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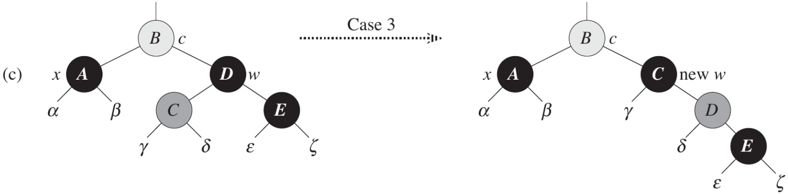
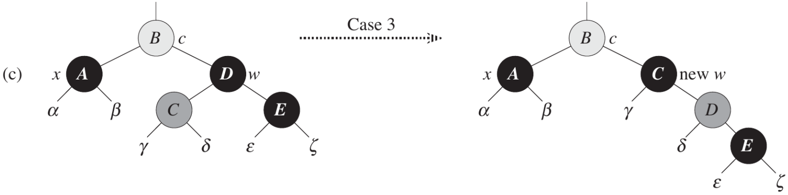


RB-DELETE procedure : Case 3



RB Tree Visualization tool:
`https://www.cs.usfca.edu/~galles/visualization/RedBlack.html`

RB-Delete

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- So, RB-DELETE-FIXUP procedure would take $O(\lg n)$ time in the worst case.
- The overall running time of RB-DELETE procedure will be $O(\lg n)$ time.

TRANSPLANT procedure in RB-DELETE

```
TRANSPLANT(T, u, v)
if u.p == NIL
        T.root = v
elseif u == u.p.left
        u.p.left = v
else u.p.right = v
if v ≠ NIL
        v.p = u.p
```

```
RB-TRANSPLANT(T, u, v)
if u.p == T.nil
        T.root = v
elseif u == u.p.left
        u.p.left = v
else u.p.right = v
v.p = u.p
```

RB-DELETE procedure

Red-black trees: Conclusions

- RB-INSERT and RB-DELETE procedures take $O(\lg n)$ time and maintain all the red-black tree properties.

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- RB-INSERT and RB-DELETE procedures take $O(\lg n)$ time and maintain all the red-black tree properties.
- So, Red-black trees can perform all the common dynamic set operations in $O(\lg n)$ time.