Data Structures and Algorithms [1]

BITS-Pilani K. K. Birla Goa Campus

[1] Material for the presentation taken from Cormen, Leiserson, Rivest and Stein, *Introduction to Algorithms, Third Edition*;

Part II Sorting and Order Statistics

- Record : Collection of data
- Key : Value to be sorted
- Satellite data
- If satellite data is large, we permute an array of pointers to the records.

Sorting algorithms

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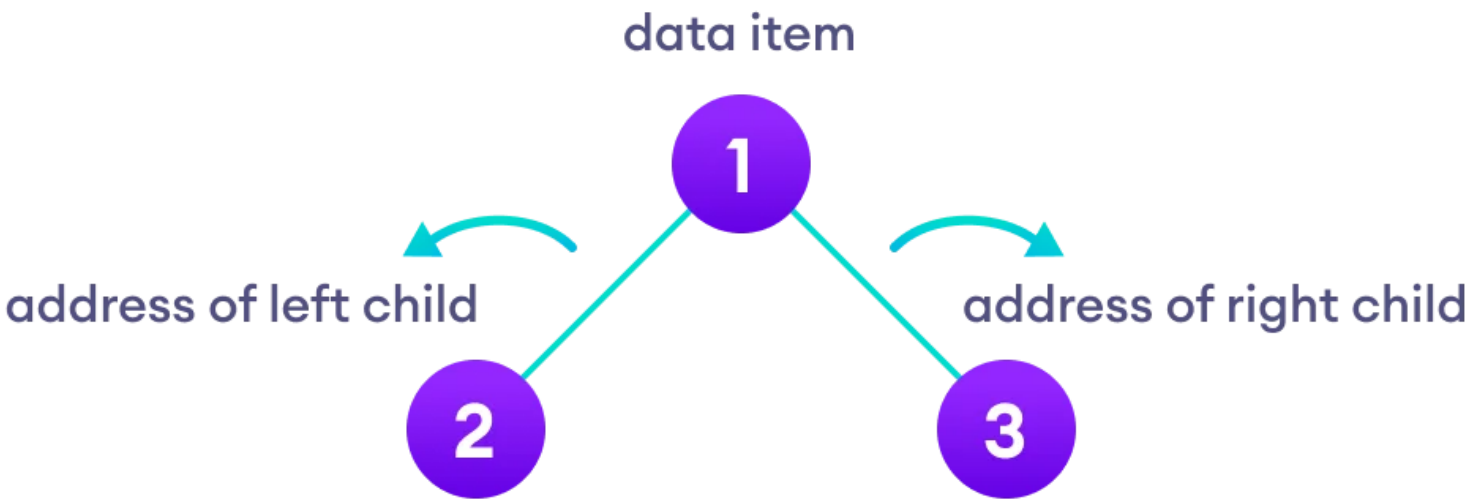
- Ch 6 : Heapsort that uses a data structure called heap.
- Heapsort sorts n elements *in place* in $O(n \lg n)$ time.

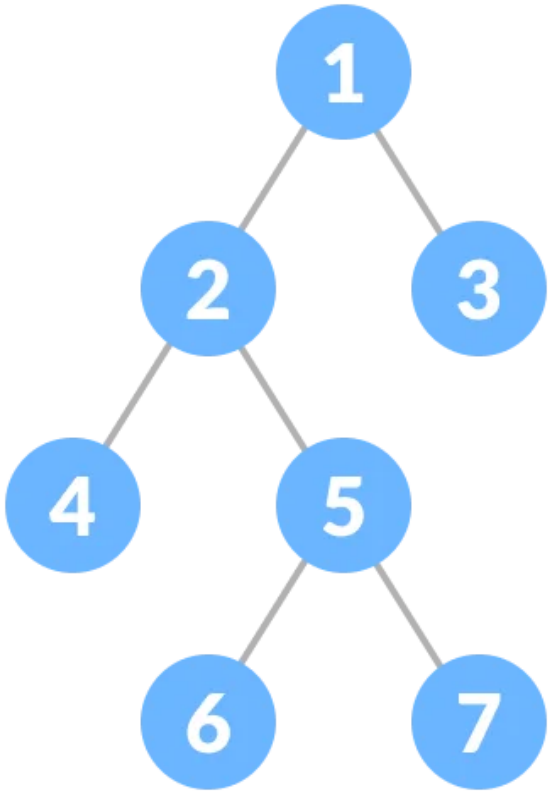
Binary tree

- Binary tree is a tree data structure where each node can have at most two child nodes : left child node and right child node.



Full Binary tree

- Each node is either a leaf node or has two child nodes.



Complete Binary tree

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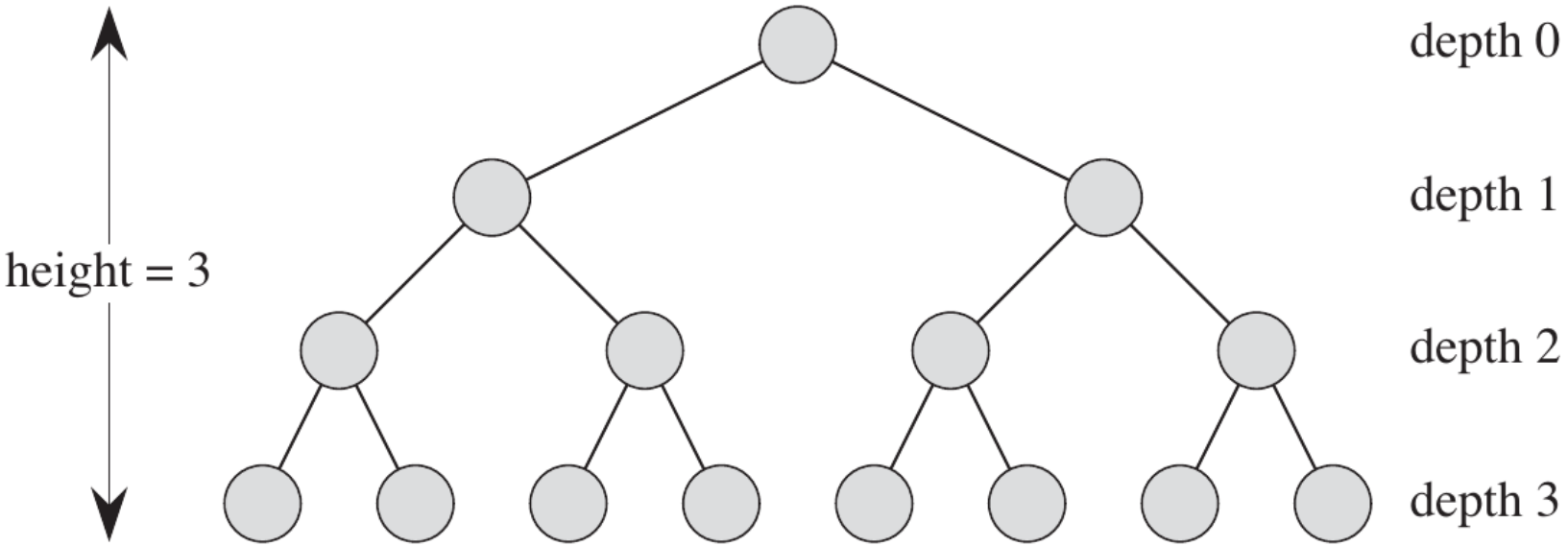
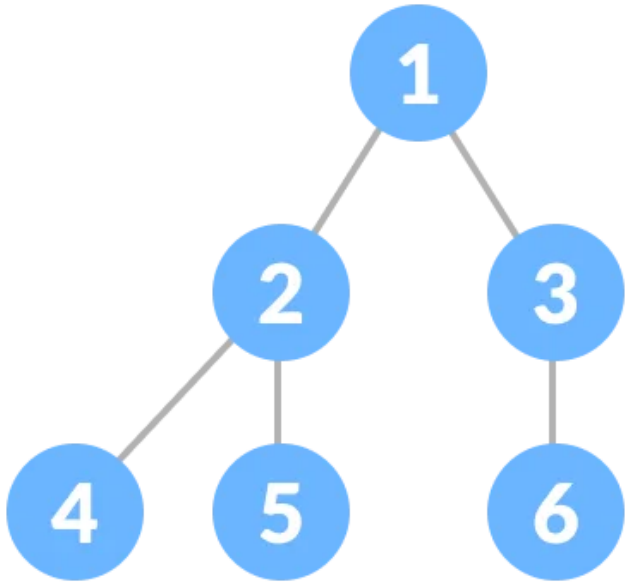


Figure B.8 A complete binary tree of height 3 with 8 leaves and 7 internal nodes.

Heap data structure : a complete binary tree



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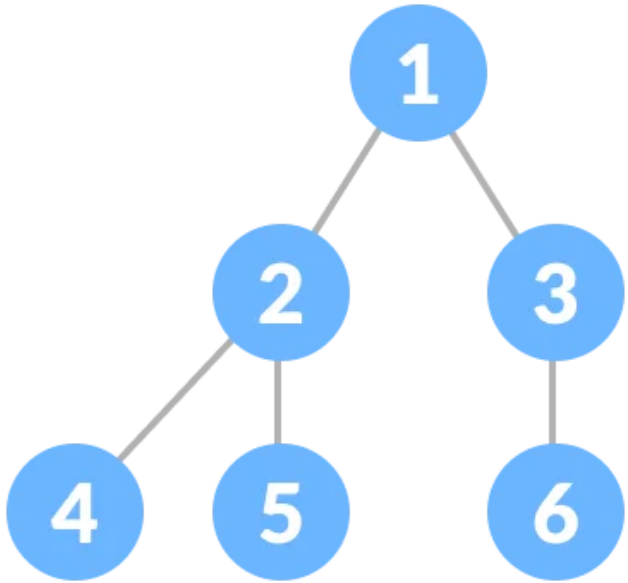

- The last level is not completely filled.

Heap : nearly a complete binary tree

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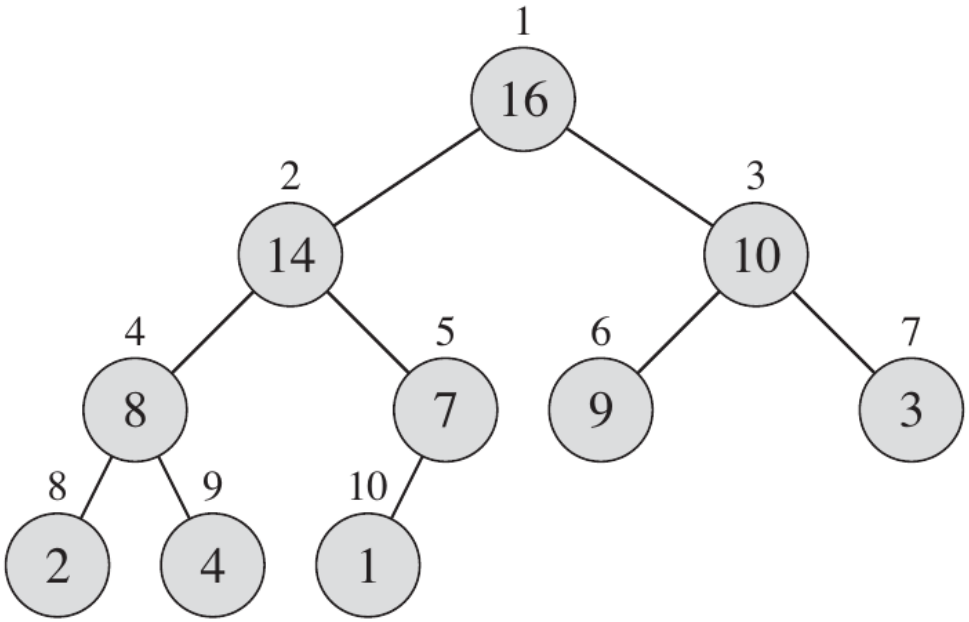

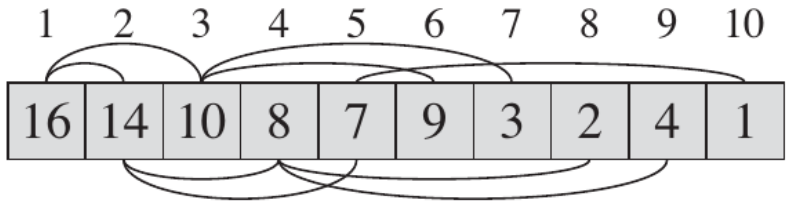

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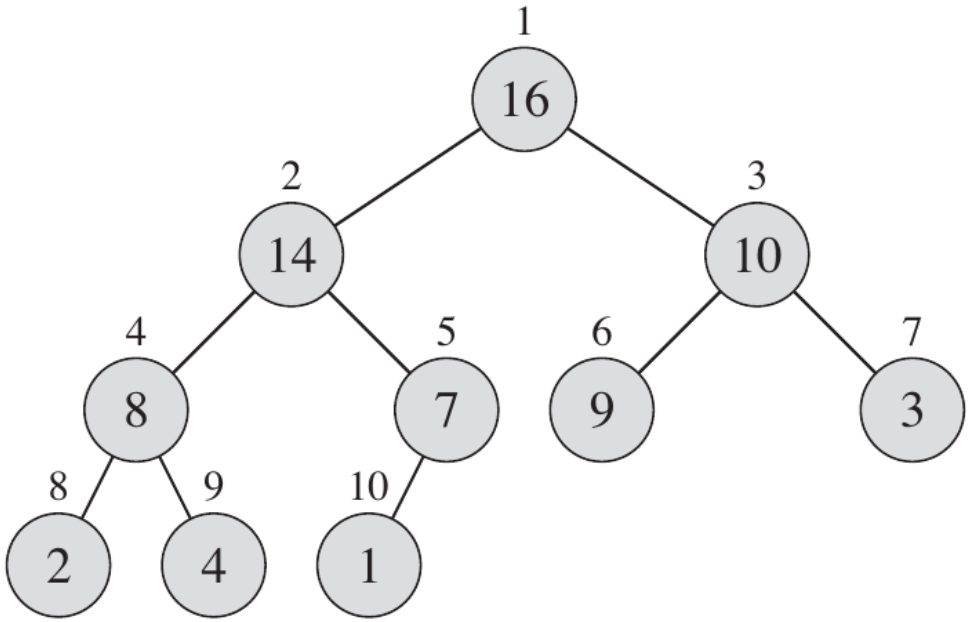

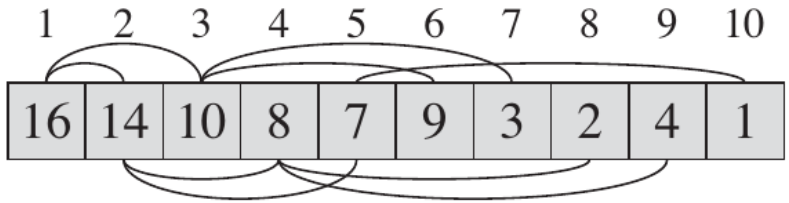

- *A.length*,

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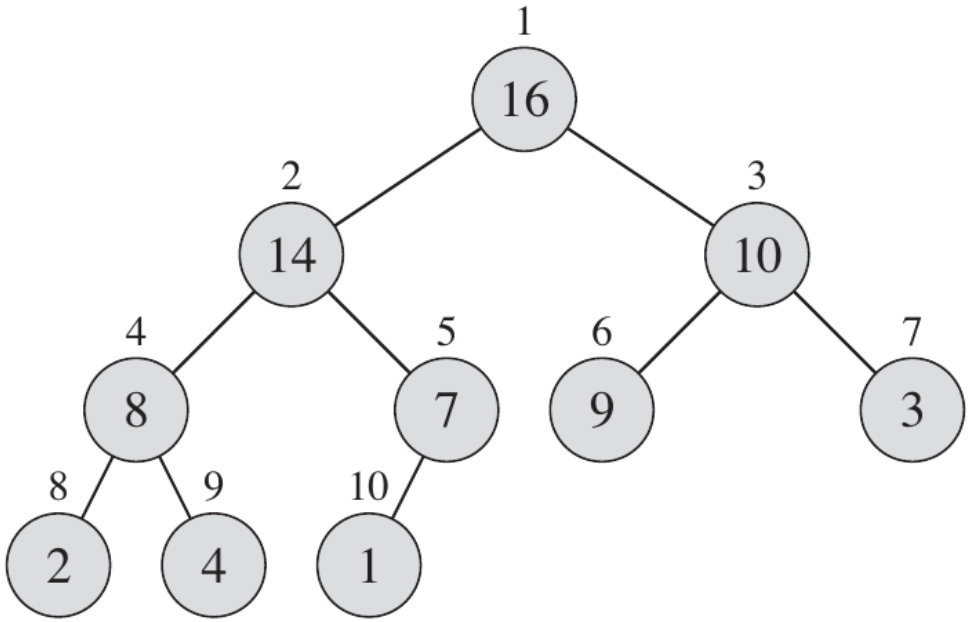

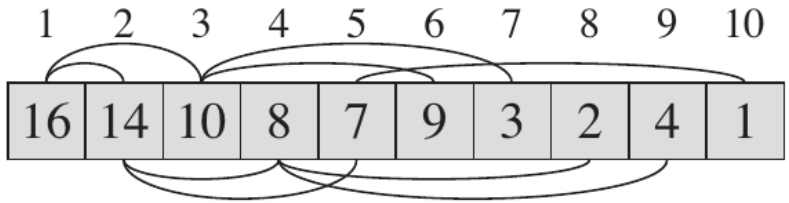

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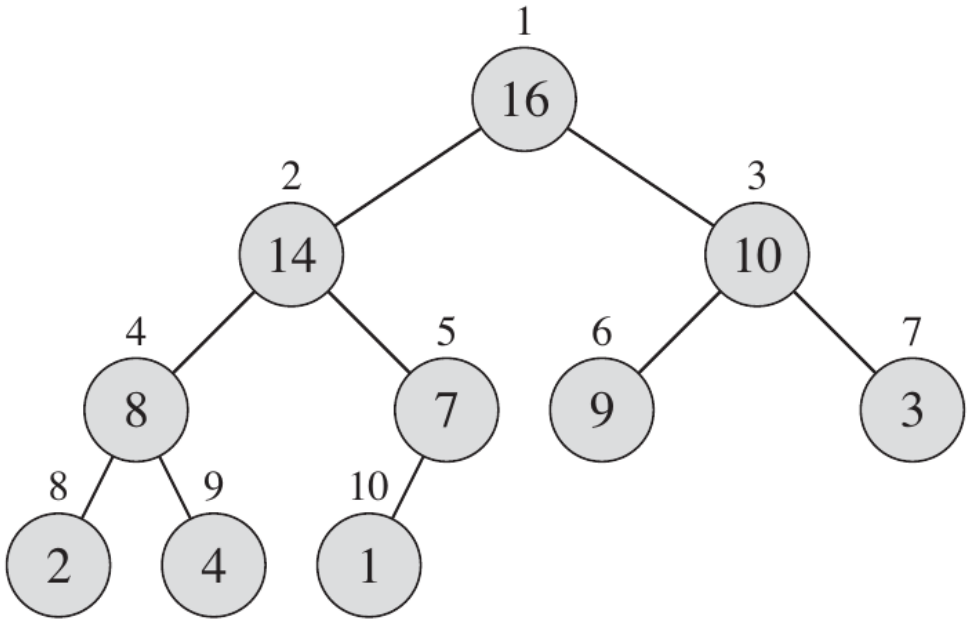
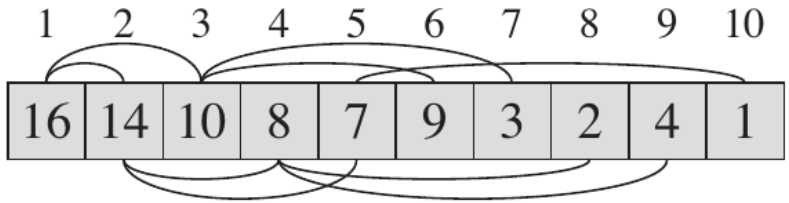
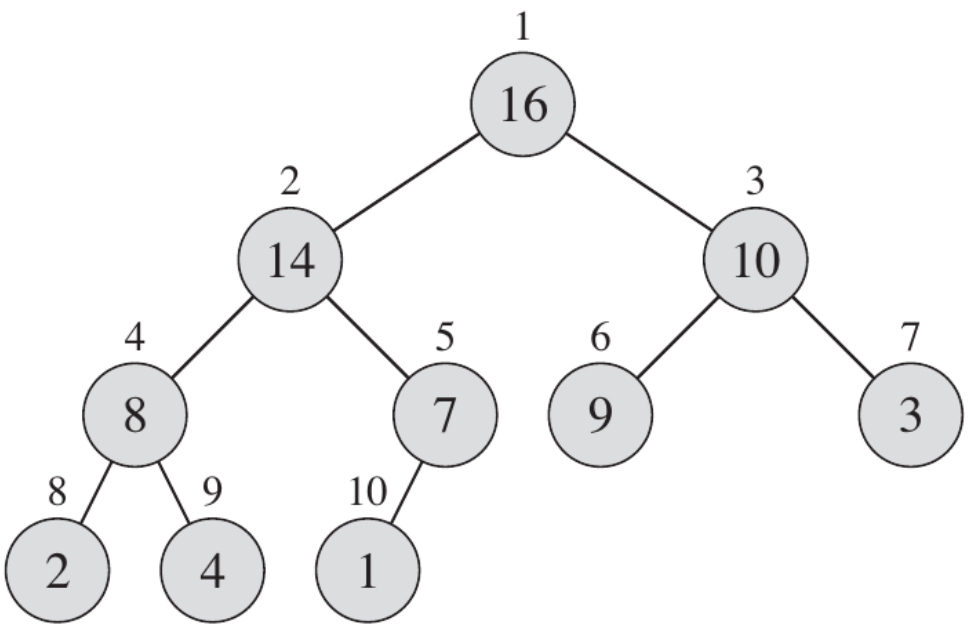




- *A.length*, *A.heap-size*, Root : *A[1]*

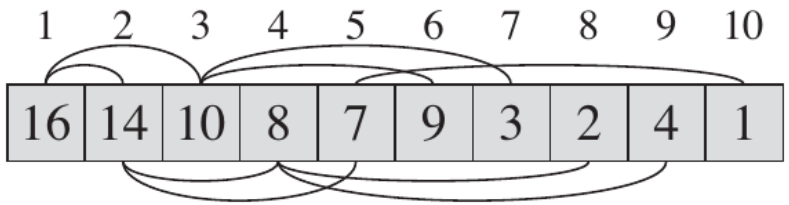
Heap : parent, left child, right child




```
PARENT(i)
1  return ⌊i/2⌋

LEFT(i)
1  return 2i

RIGHT(i)
1  return 2i + 1
```



Binary Heaps

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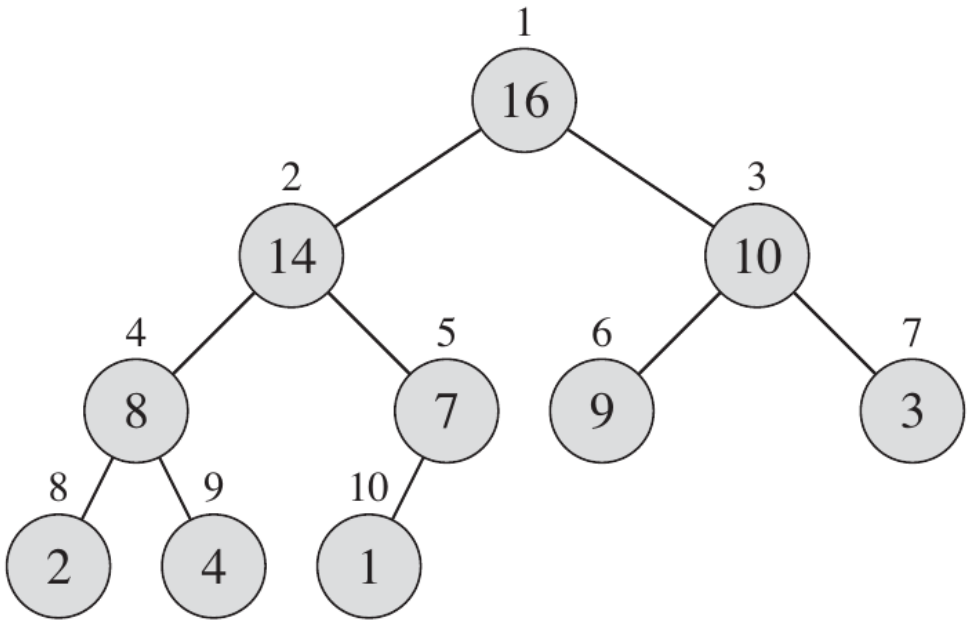

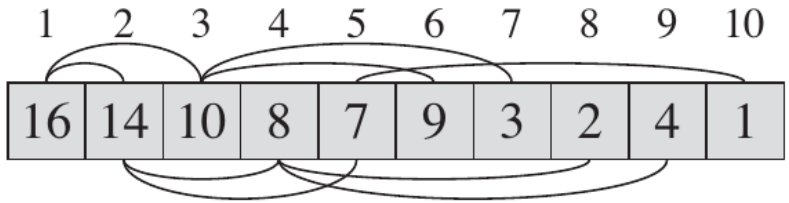

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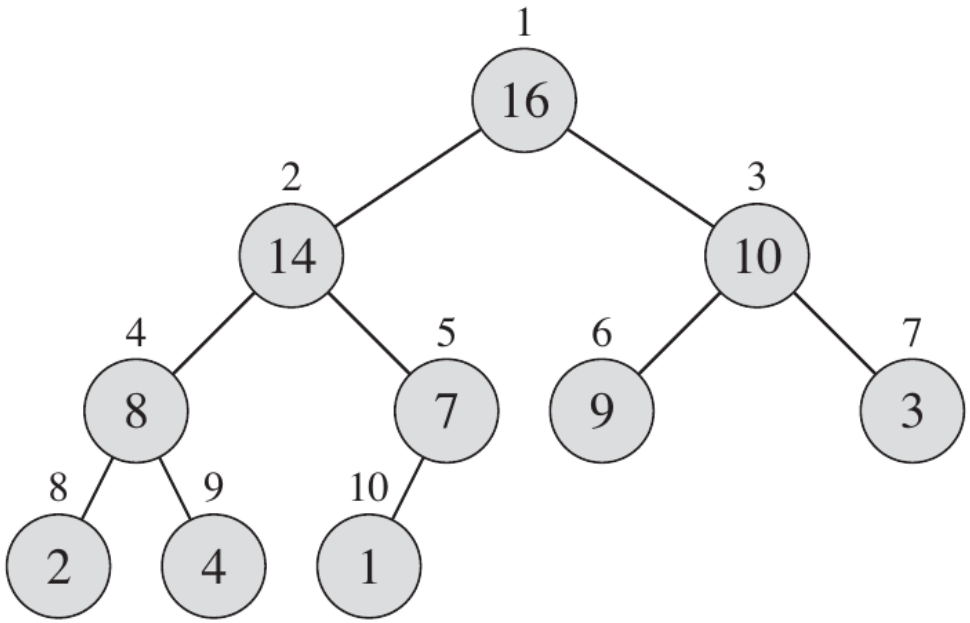

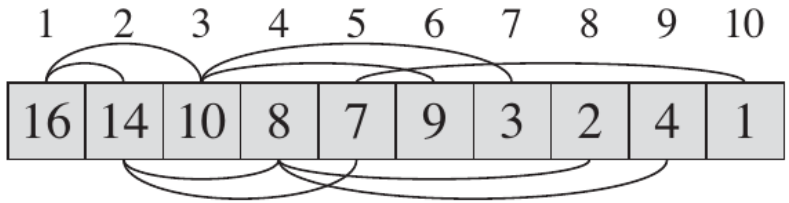

- $A[1]$ contains the maximum element

Binary Heaps

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Binary Heaps

- Min heap
- Min-heap property: $A[PARENT(i)] \leq A[i]$
- $A[1]$ will contain the smallest element

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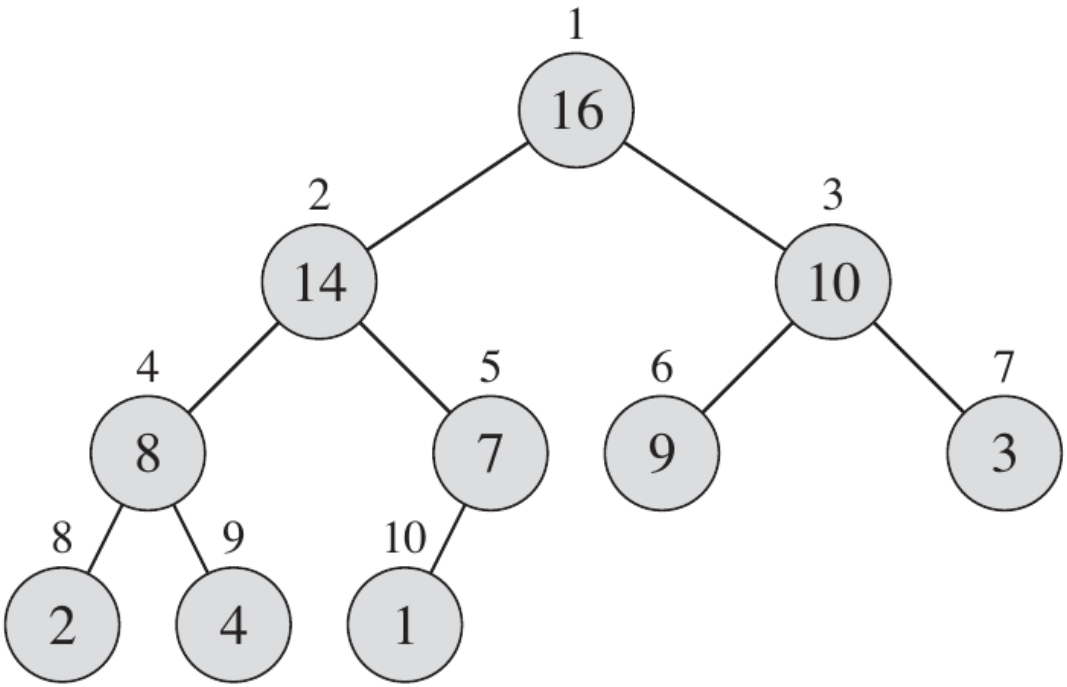

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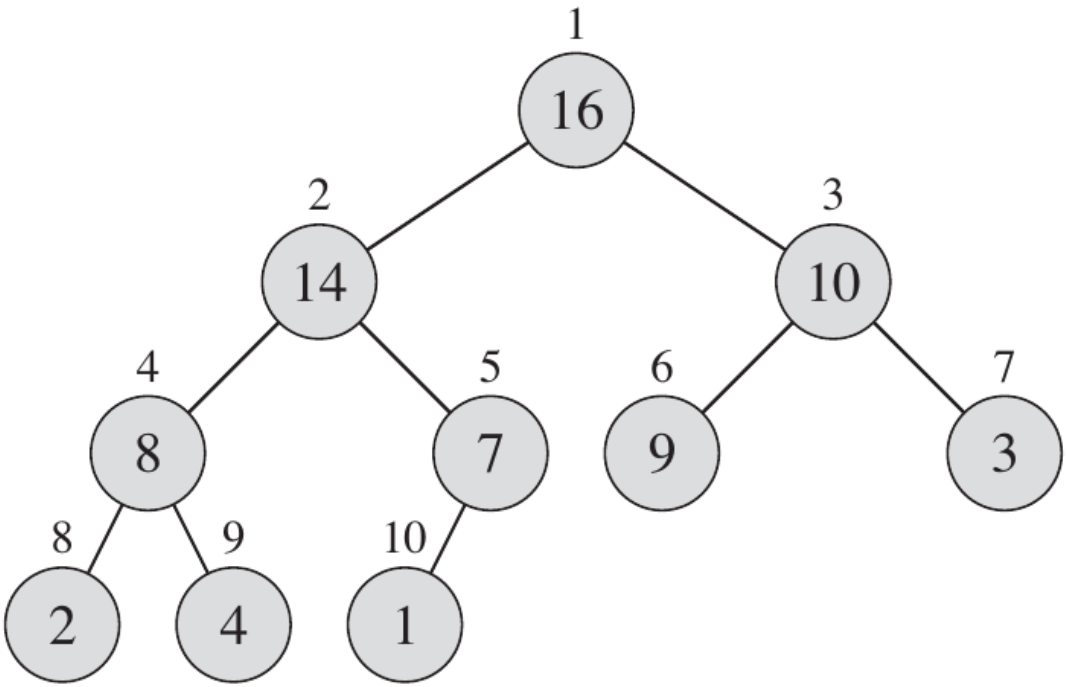

- **Height** of a heap is the height of its root.

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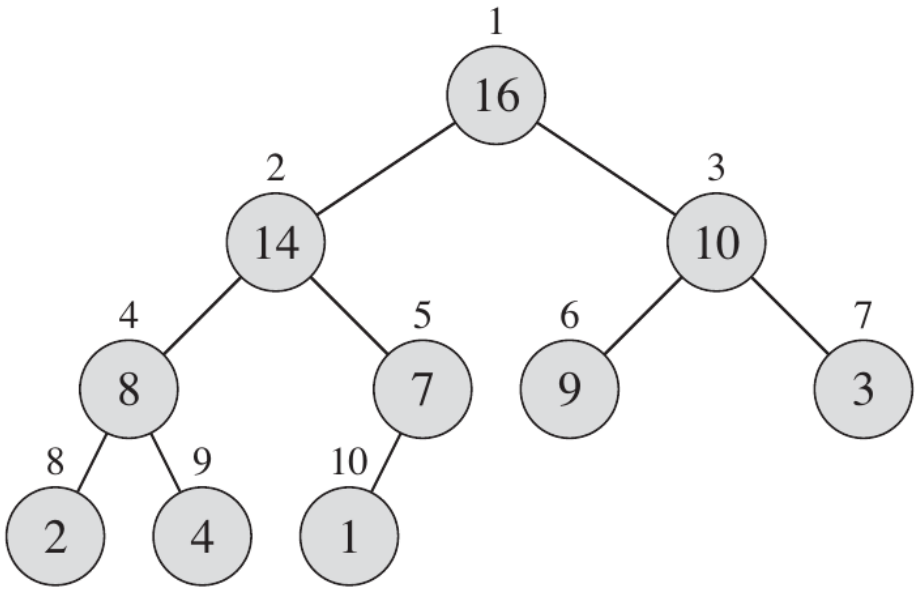

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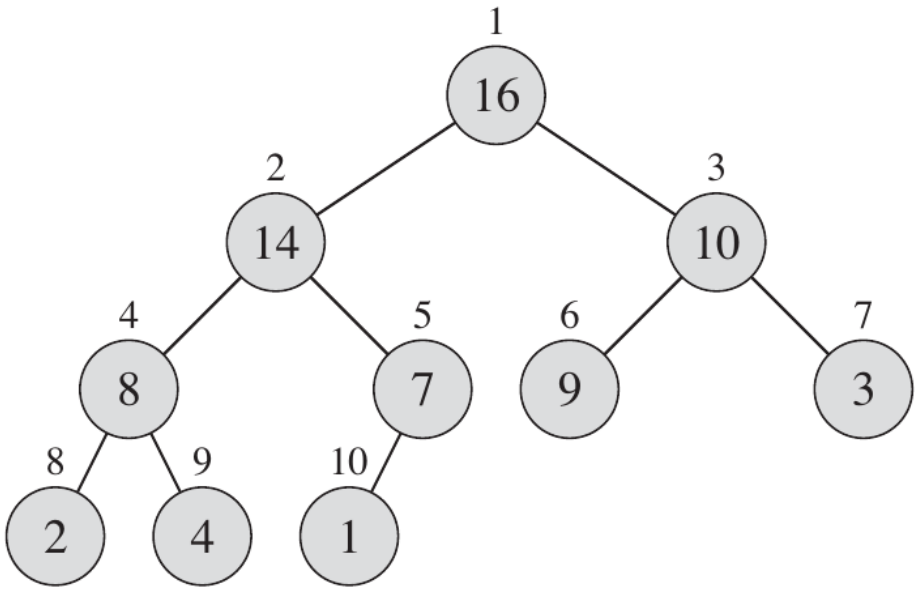

$$2^h$$

Height of a heap

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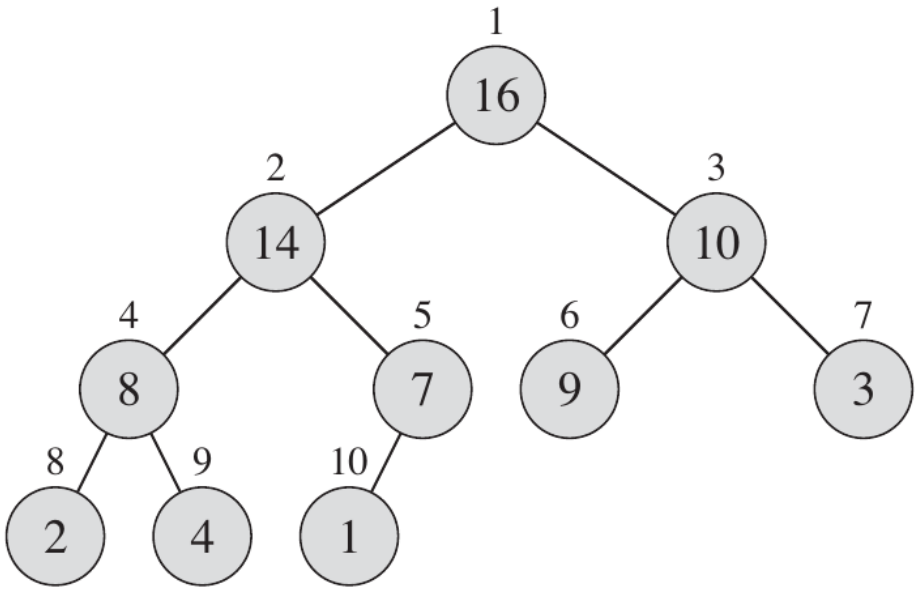

$$2^h \leq n$$

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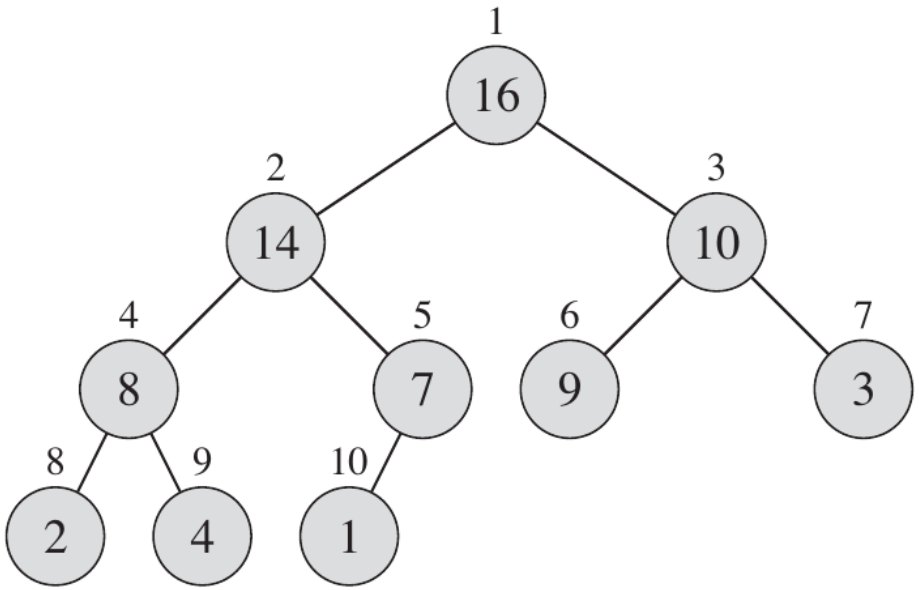

$$2^h \leq n \leq 2^{h+1} - 1$$

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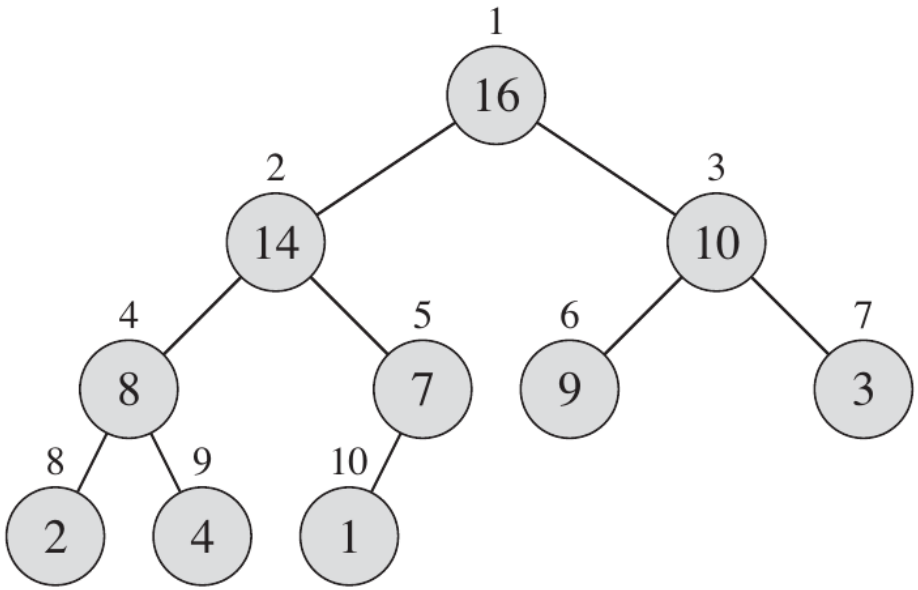

$$2^h \leq n \leq 2^{h+1} - 1$$

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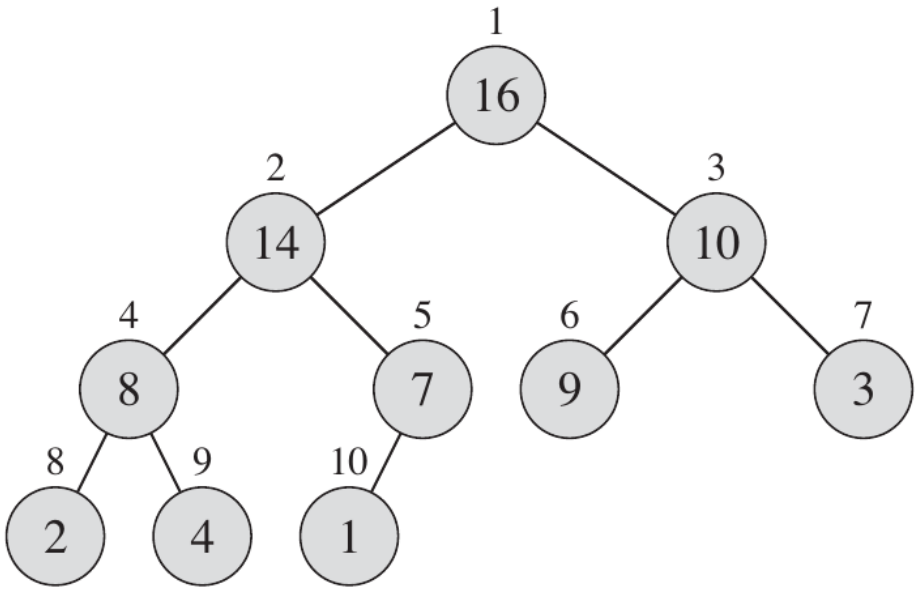

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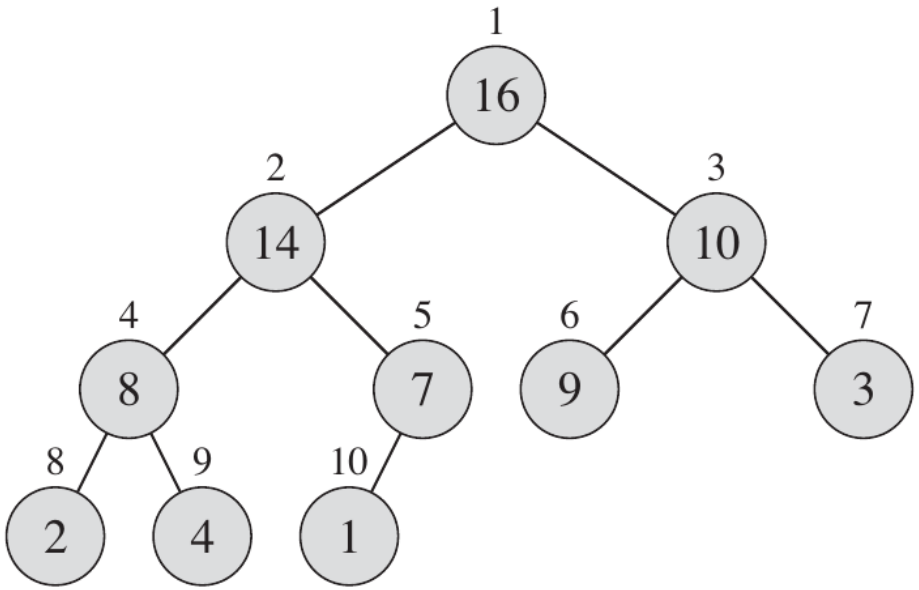

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$$2^h \leq n < 2^{h+1}$$

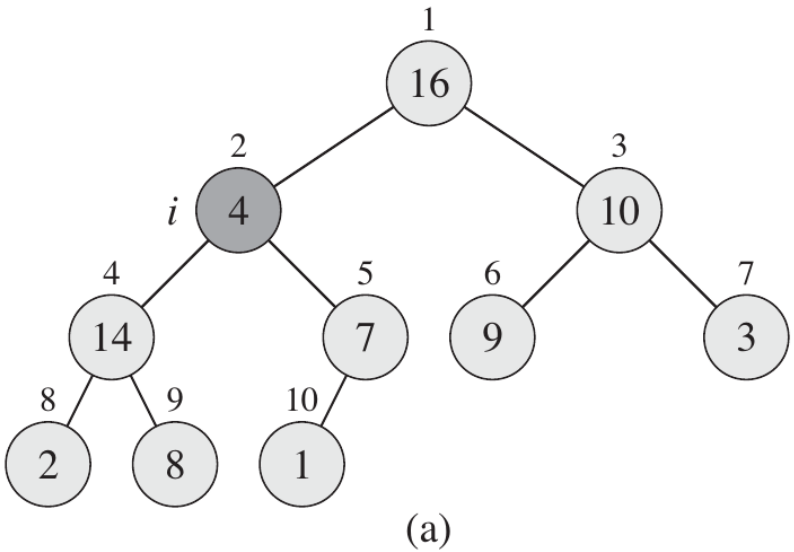
$$h \leq \lg n < (h+1)$$

$$h = \lfloor \lg n \rfloor$$

MAX-HEAPIFY: Maintaining the heap property

- MAX-HEAPIFY(A,i) : binary trees rooted at LEFT(i) and RIGHT(i) satisfy max-heap property.

MAX-HEAPIFY: Maintaining the heap property



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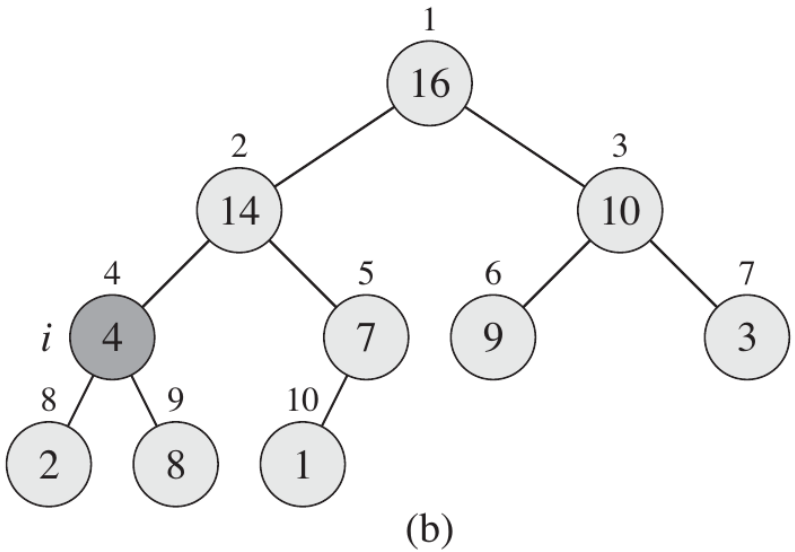

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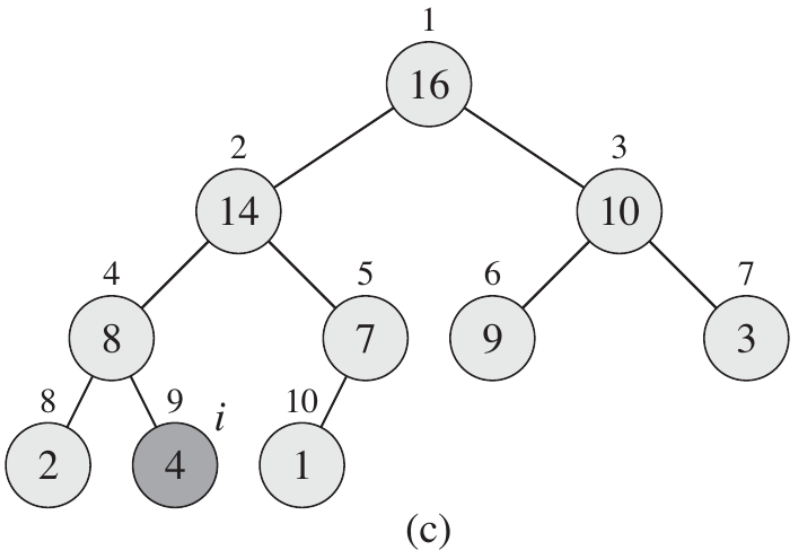

(c)

MAX-HEAPIFY: Maintaining the heap property

```
MAX-HEAPIFY(A, i)
 1  l = LEFT(i)
 2  r = RIGHT(i)
 3  if l ≤ A.heap-size and A[l] > A[i]
 4       largest = l
 5  else largest = i
 6  if r ≤ A.heap-size and A[r] > A[largest]
 7       largest = r
 8  if largest ≠ i
 9       exchange A[i] with A[largest]
10       MAX-HEAPIFY(A, largest)
```

MAX-HEAPIFY: Finding recurrence

- Let the tree rooted at i have n nodes. The child subtree will have a size at most $2n/3$

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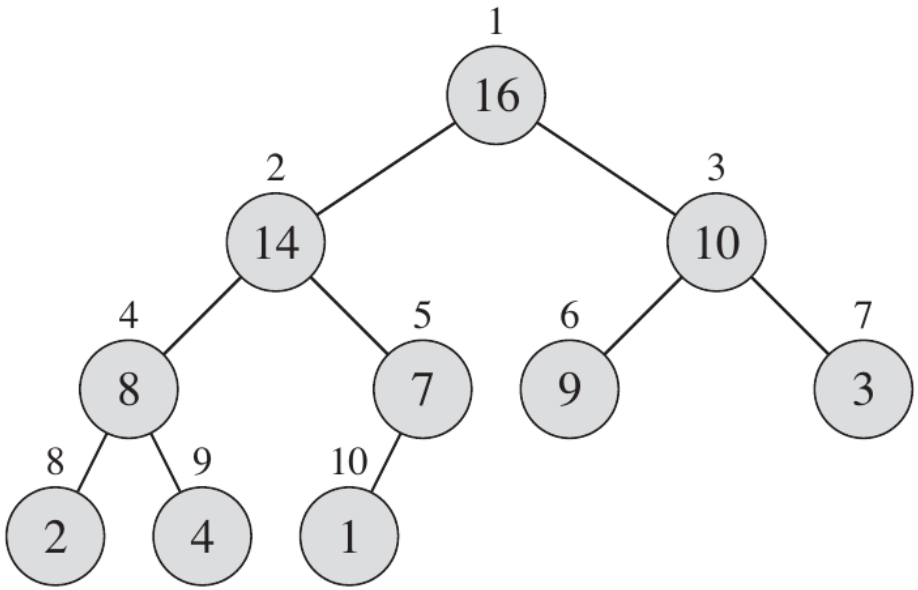

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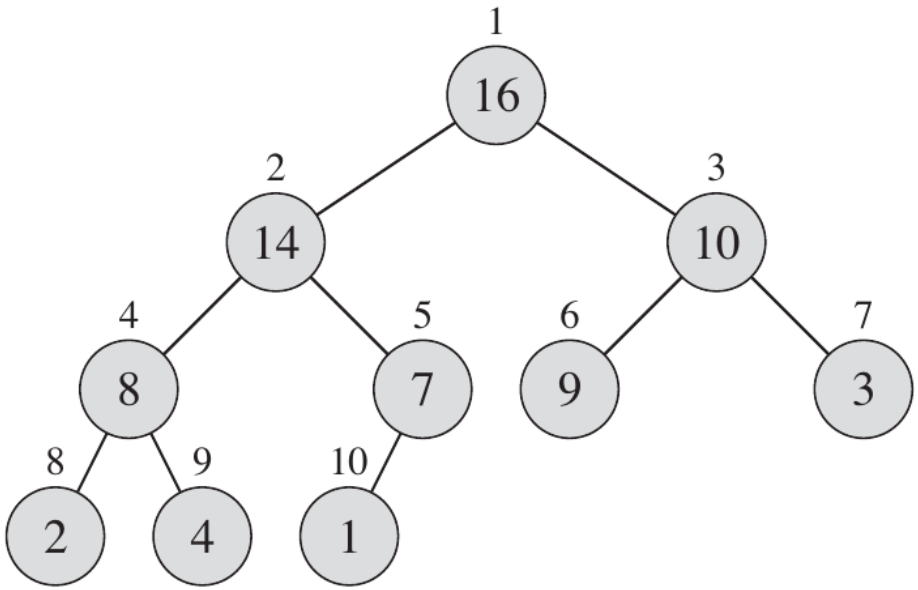

$h =$ height of node i

Maximum size of child subtree $= 2^h - 1$

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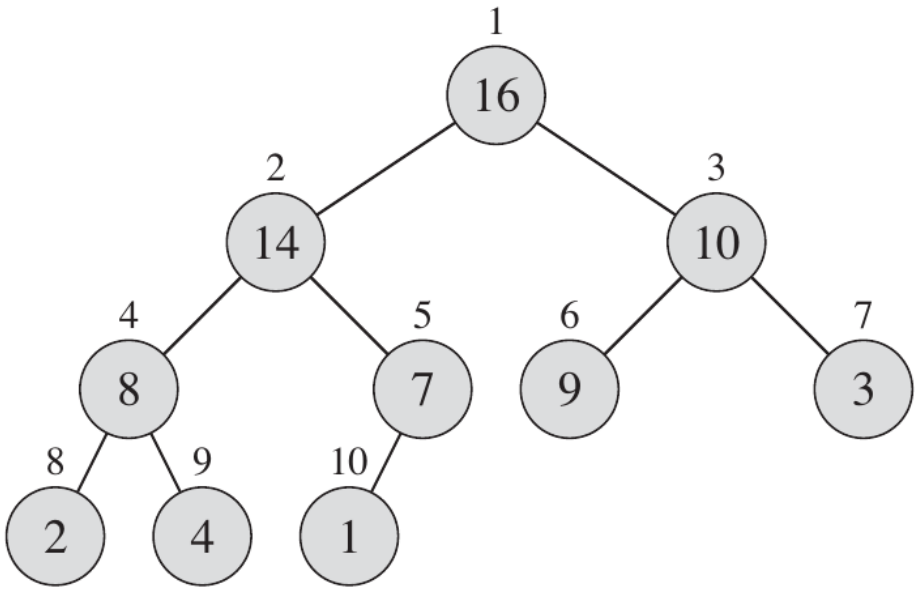

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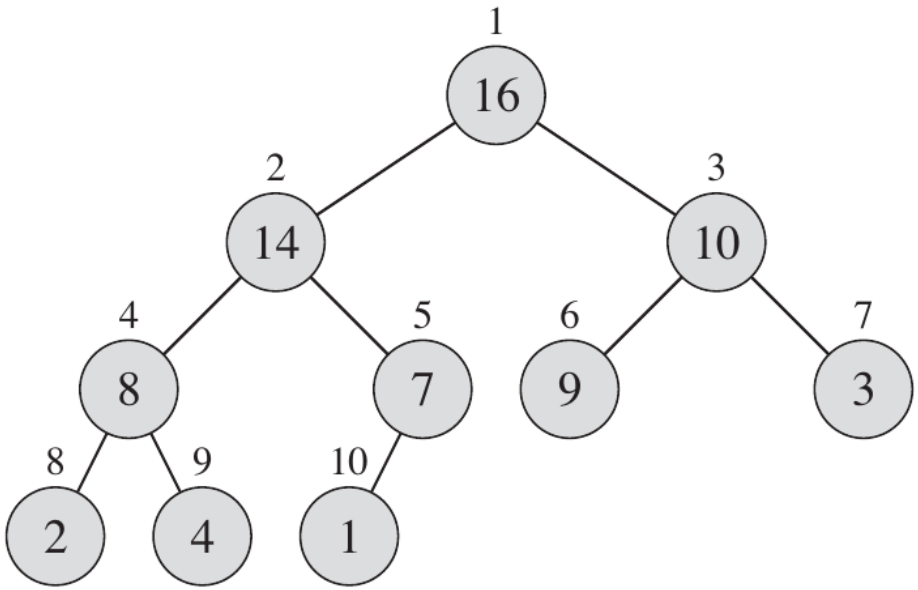

$$h = \text{ height of node } i$$

$$\text{Maximum size of child subtree } = 2^h - 1$$

$$\text{Maximum fraction of nodes } = \frac{2^h - 1}{2^h - 1 + 2^{h-1}}$$

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$$\text{For } 0 < a < b\ , \qquad \frac{a}{b} < \frac{a+1}{b+1}$$

$$\frac{2^h - 1}{2^h - 1 + 2^{h-1}} < \frac{2^h}{2^h + 2^{h-1}}$$

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$$\text{Maximum size of child subtree} < \frac{2n}{3}$$

MAX-HEAPIFY: Maintaining the heap property

```
MAX-HEAPIFY(A, i)
 1  l = LEFT(i)
 2  r = RIGHT(i)
 3  if l ≤ A.heap-size and A[l] > A[i]
 4       largest = l
 5  else largest = i
 6  if r ≤ A.heap-size and A[r] > A[largest]
 7       largest = r
 8  if largest ≠ i
 9       exchange A[i] with A[largest]
10       MAX-HEAPIFY(A, largest)
```

MAX-HEAPIFY: Running time

- $T(n) \leq T(2n/3) + c$

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$$T(n) = O(\lg n)$$

BUILD-MAX-HEAP : Building a heap

- Convert an unordered array into a max-heap using MAX-HEAPIFY

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```
BUILD-MAX-HEAP(A)
1  A.heap-size = A.length
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3      MAX-HEAPIFY(A, i)
```
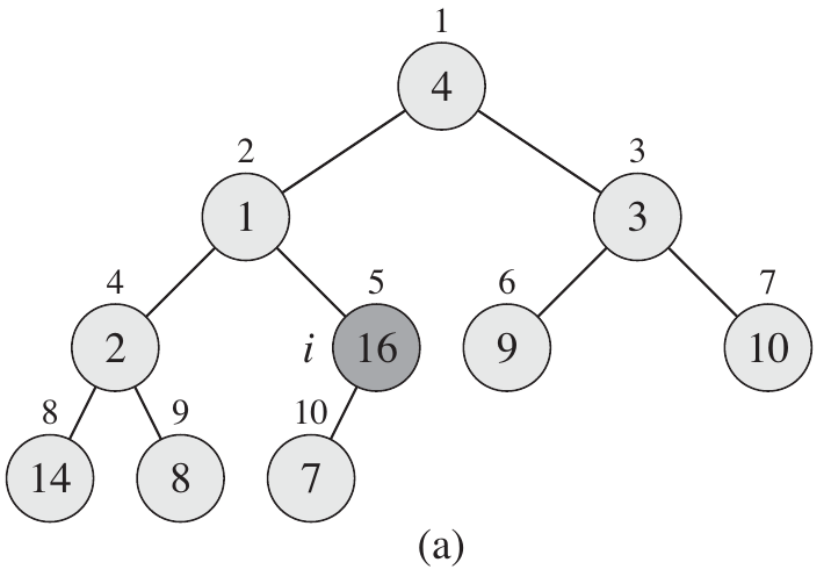
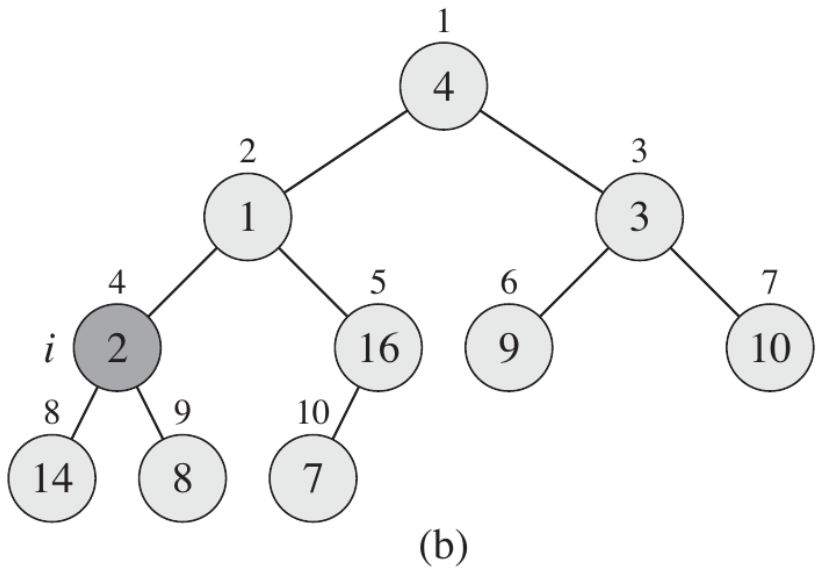

BUILD-MAX-HEAP : Building a heap

A	4	1	3	2	16	9	10	14	8	7

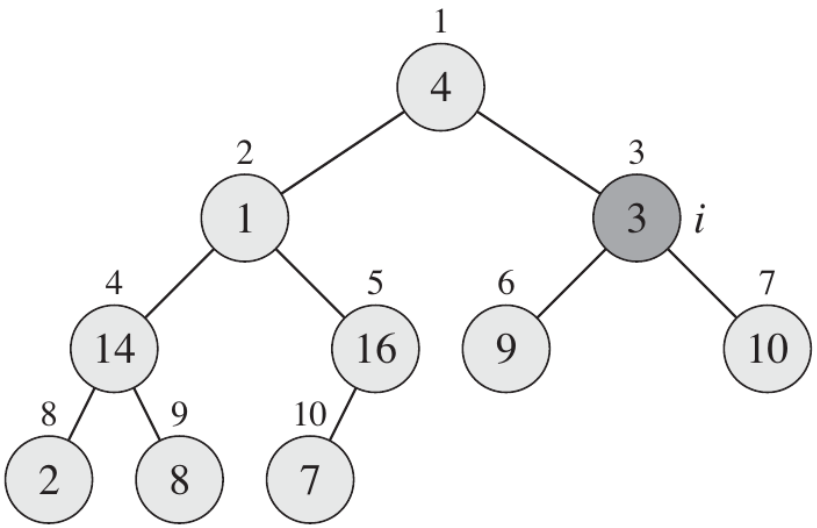


(a)

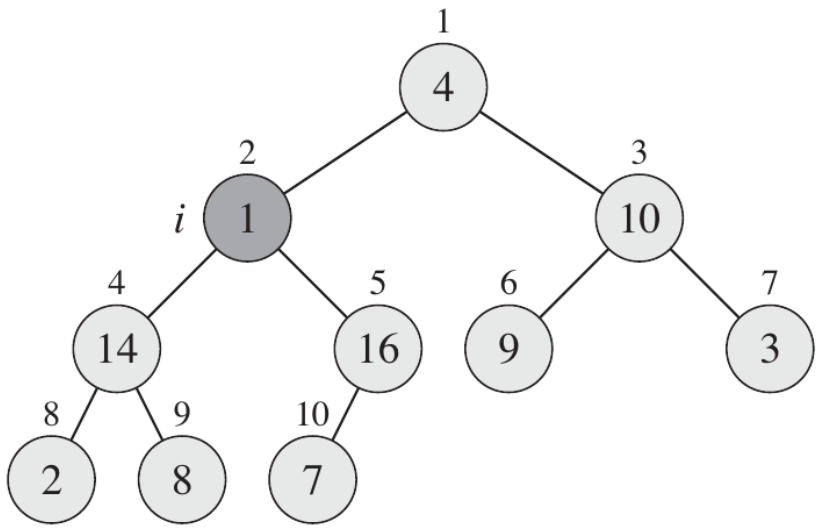


(b)

BUILD-MAX-HEAP : Building a heap

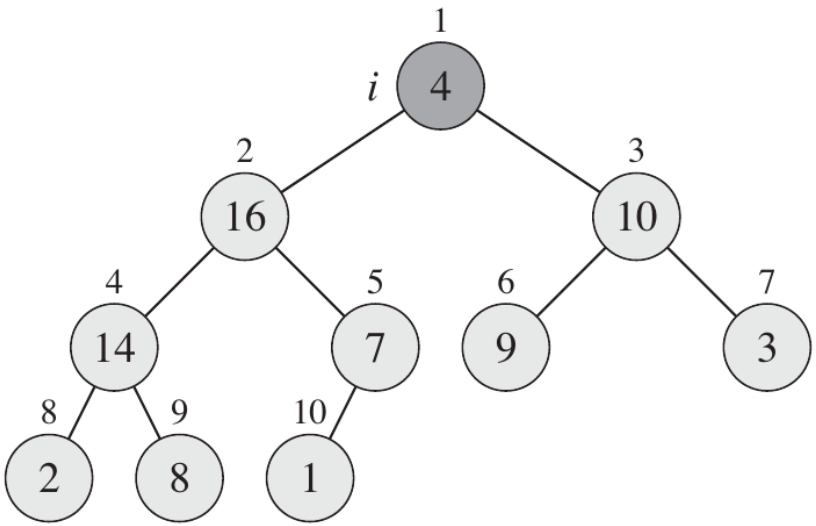


(c)

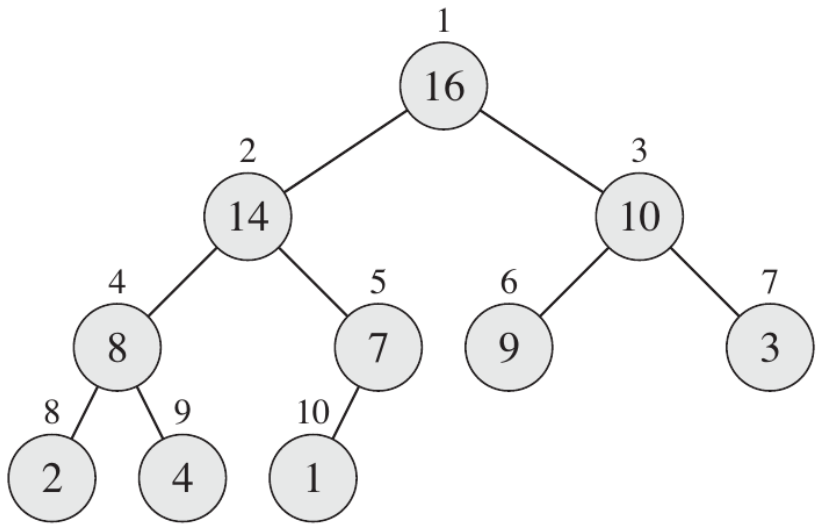


(d)

BUILD-MAX-HEAP : Building a heap



(e)



(f)

BUILD-MAX-HEAP : Running time

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Running time: $O(n \lg n)$

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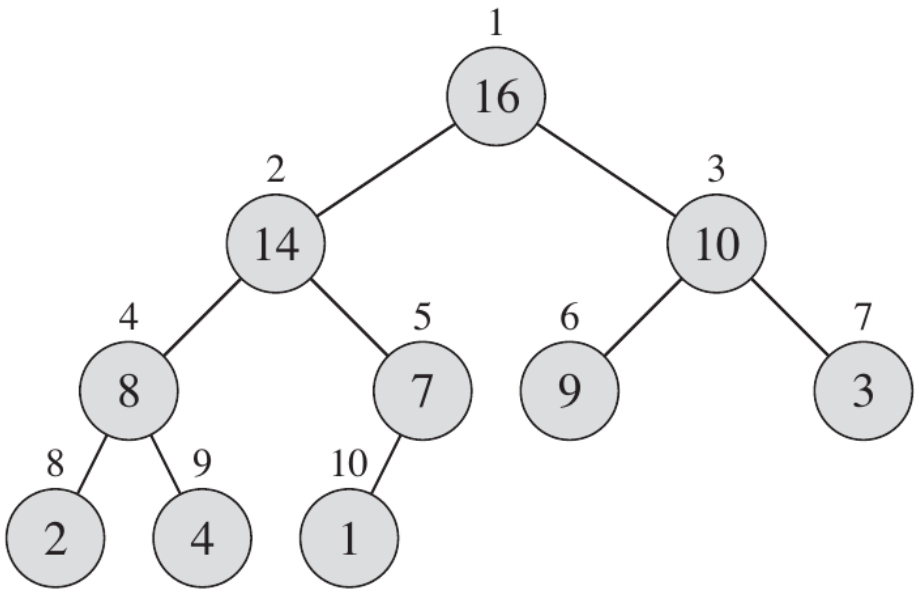
Running time: $O(n \lg n)$ { Not asymptotically tight }

Nodes with height $h \leq \left\lceil \frac{n}{2^{h+1}} \right\rceil$



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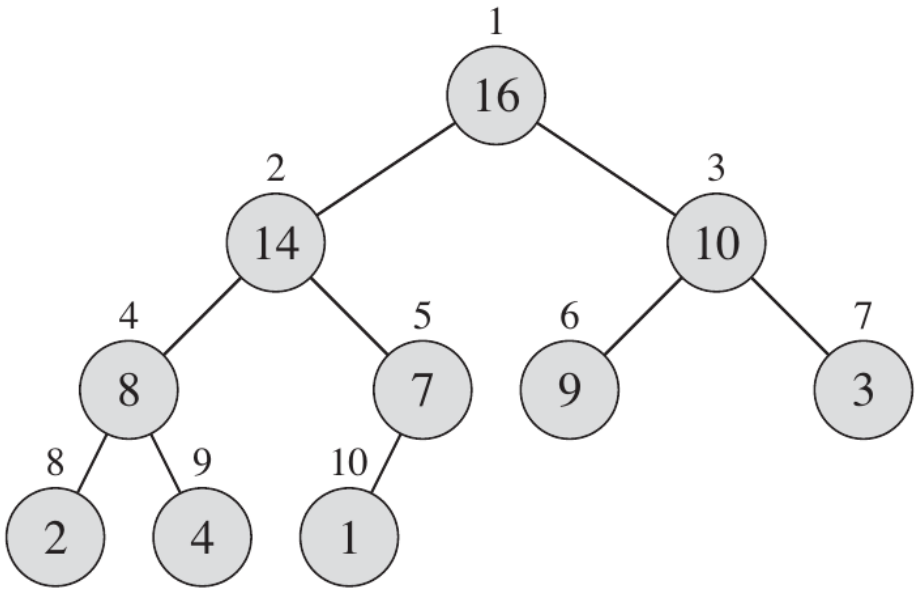

- There are at most $\left\lceil \frac{n}{2^{h+1}} \right\rceil$ nodes having a height h.

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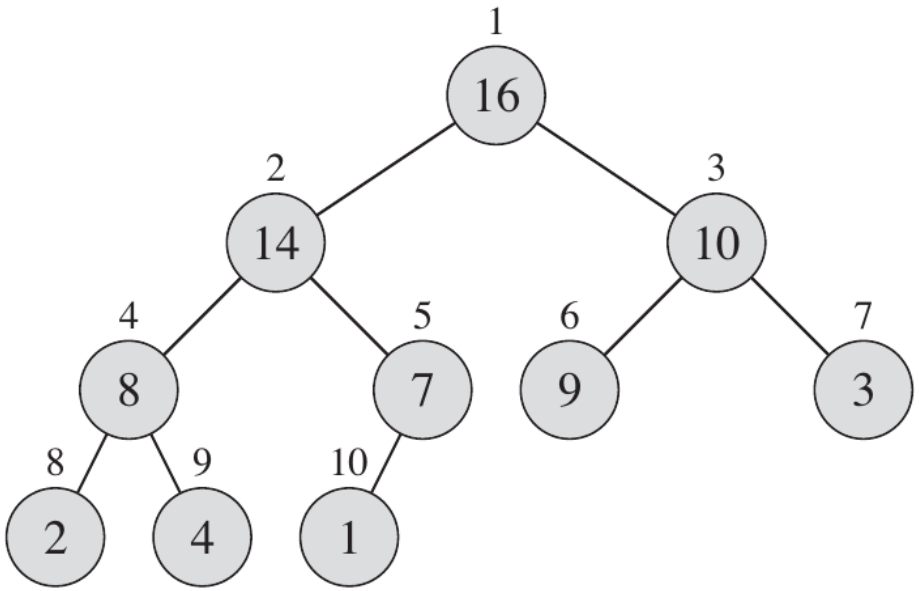

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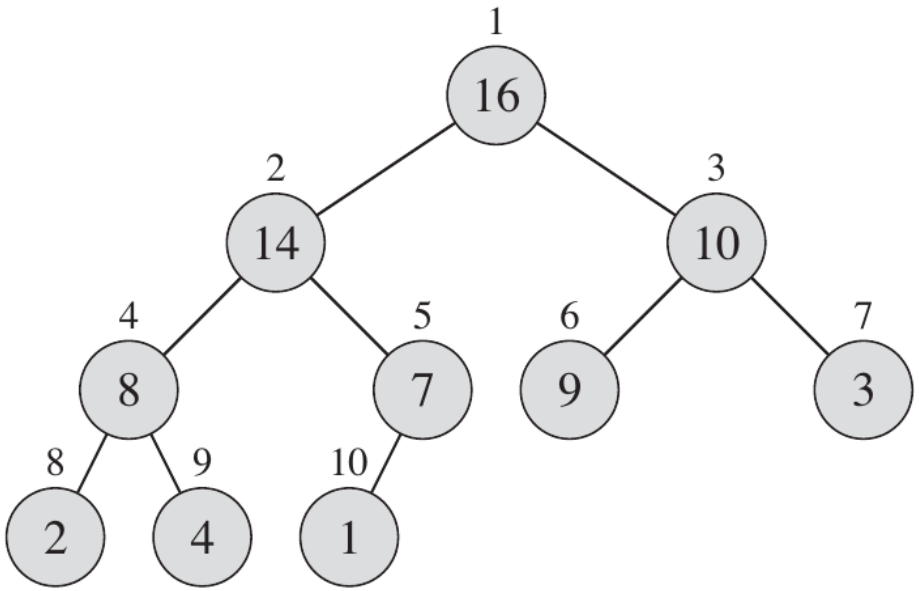
$h = 0$,

Nodes with height $h \leq \left\lceil \frac{n}{2^{h+1}} \right\rceil$



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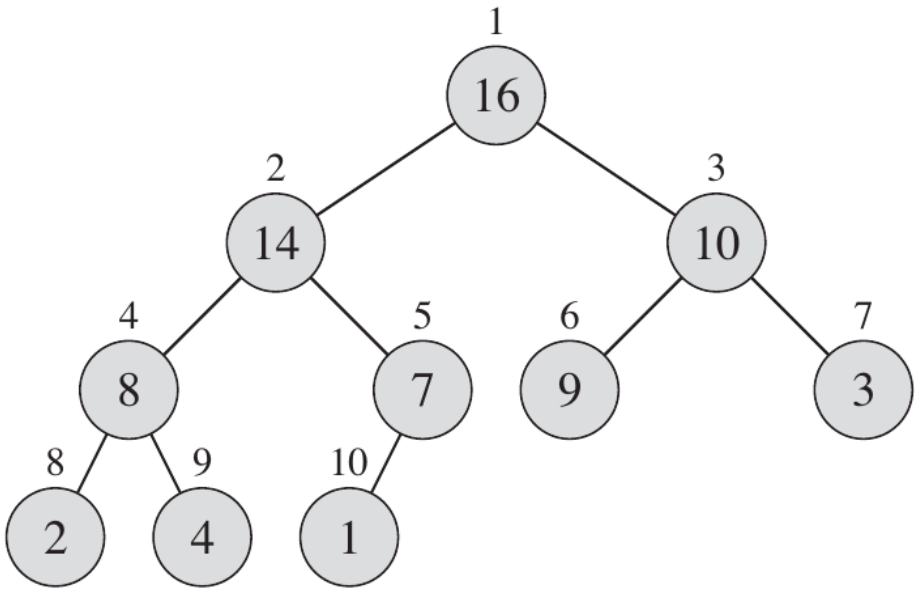

- There are at most $\left\lceil \frac{n}{2^{h+1}} \right\rceil$ nodes having a height h.

$$h = 0, \qquad \left\lceil \frac{10}{2^{0+1}} \right\rceil = 5;$$

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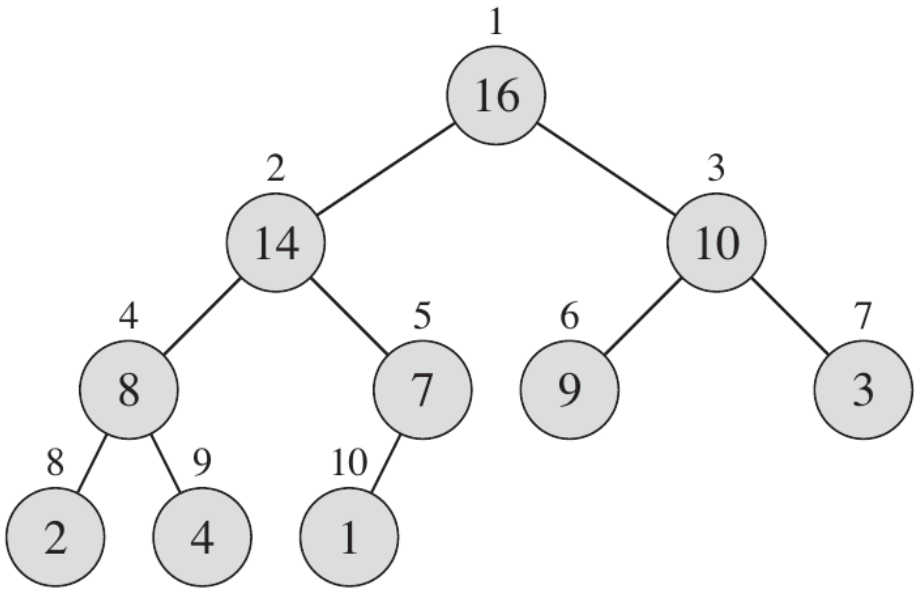

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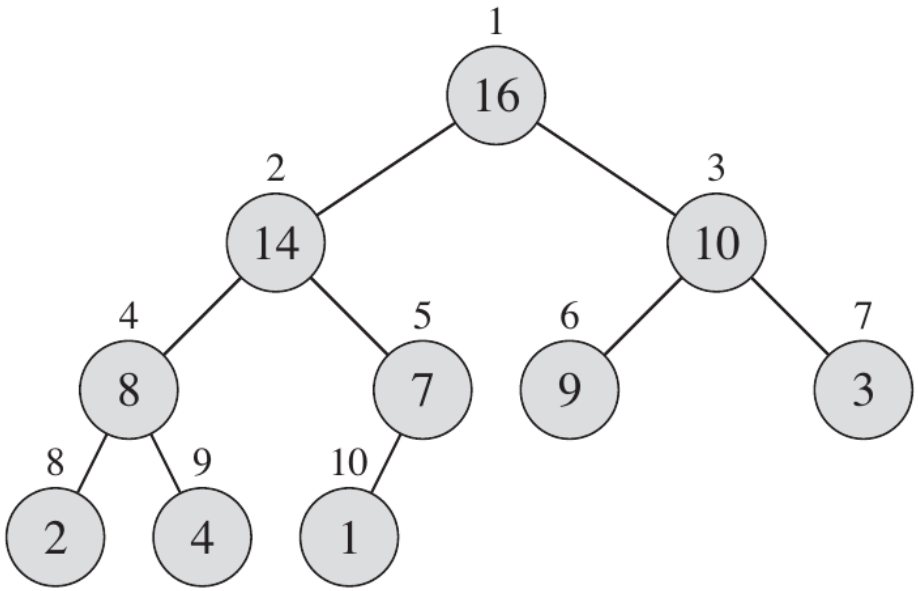

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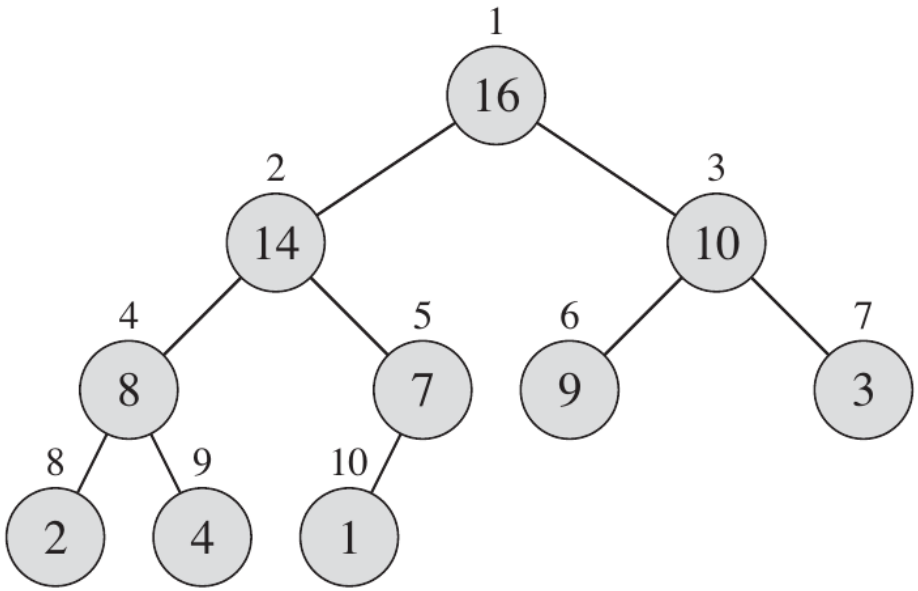

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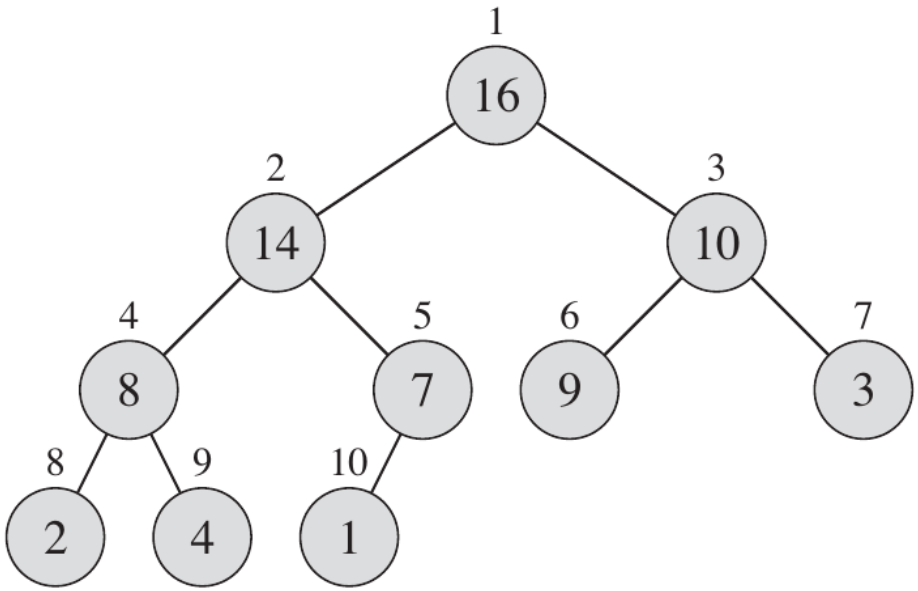

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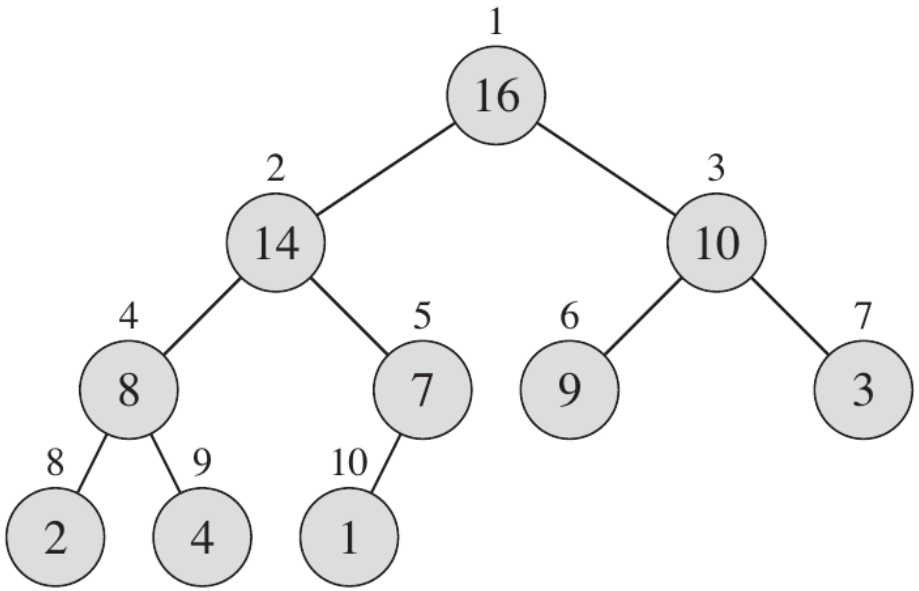

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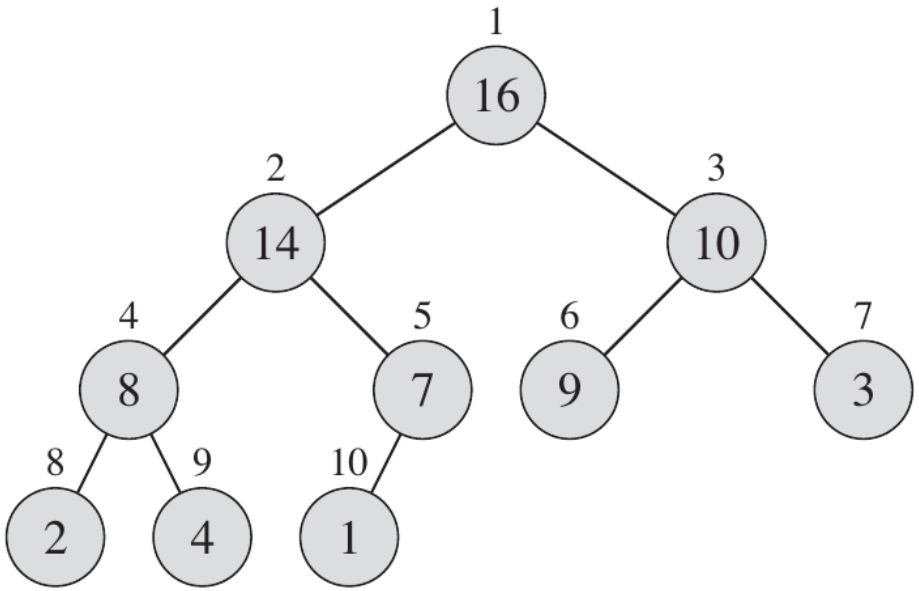

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Nodes with height $h \leq \left\lceil \frac{n}{2^{h+1}} \right\rceil$

- Proof by Induction

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- Inductive step:
  Assume that there are at most $\left\lceil \frac{n}{2^{(h-1)+1}} \right\rceil$ number of nodes of height $h - 1$.

Nodes with height $h \leq \left\lceil \frac{n}{2^{h+1}} \right\rceil$

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  Let k_h be the number of nodes of height h in the binary heap T.

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$$k_h = k'_{h-1}$$

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$$k_h = k'_{h-1} \leq \left\lceil \frac{n'}{2^{(h-1)+1}} \right\rceil = \left\lceil \frac{\lfloor (n/2) \rfloor}{2^{(h-1)+1}} \right\rceil$$

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BUILD-MAX-HEAP : Running time

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BUILD-MAX-HEAP : Running time

BUILD-MAX-HEAP(A)

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$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h)$$

BUILD-MAX-HEAP : Running time

BUILD-MAX-HEAP(A)

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$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right)$$

BUILD-MAX-HEAP : Running time

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2 **for** $i = \lfloor A.length/2 \rfloor$ **downto** 1
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$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right)$$

$$\sum_{k=0}^{\infty} kx^k$$

BUILD-MAX-HEAP : Running time

BUILD-MAX-HEAP(A)

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$$= O(2n) = O(n)$$

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$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

- Multiplying x on both sides

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

Heap

- We can construct a max heap (or a min heap) from an unordered array in $O(n)$ time.

Heapsort

```
HEAPSORT(A)
1  BUILD-MAX-HEAP(A)
2  for i = A.length downto 2
3      exchange A[1] with A[i]
4      A.heap-size = A.heap-size − 1
5      MAX-HEAPIFY(A, 1)
```

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- Running time : $O(n \lg n)$

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- Running time : $O(n \lg n)$
- Operation (P. 161)

Priority Queues

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  3. Extract-Max(S)

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- Heap data structure can be used to construct efficient priority queue
- Job scheduling
- Two forms : max-priority queue and min priority-queue
- Max-priority queue operations:
  1. Insert(S,x)
  2. Maximum(S)
  3. Extract-Max(S)
  4. Increase-Key(S,x,k)

Priority Queues

```
HEAP-MAXIMUM(A)
1  return A[1]
```

Priority Queues

```
HEAP-MAXIMUM(A)
1   return A[1]
```

```
HEAP-EXTRACT-MAX(A)
1   if A.heap-size < 1
2         error "heap underflow"
3   max = A[1]
4   A[1] = A[A.heap-size]
5   A.heap-size = A.heap-size − 1
6   MAX-HEAPIFY(A, 1)
7   return max
```

Priority Queues

```
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1  return A[1]
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3  max = A[1]
4  A[1] = A[A.heap-size]
5  A.heap-size = A.heap-size − 1
6  MAX-HEAPIFY(A, 1)
7  return max
```

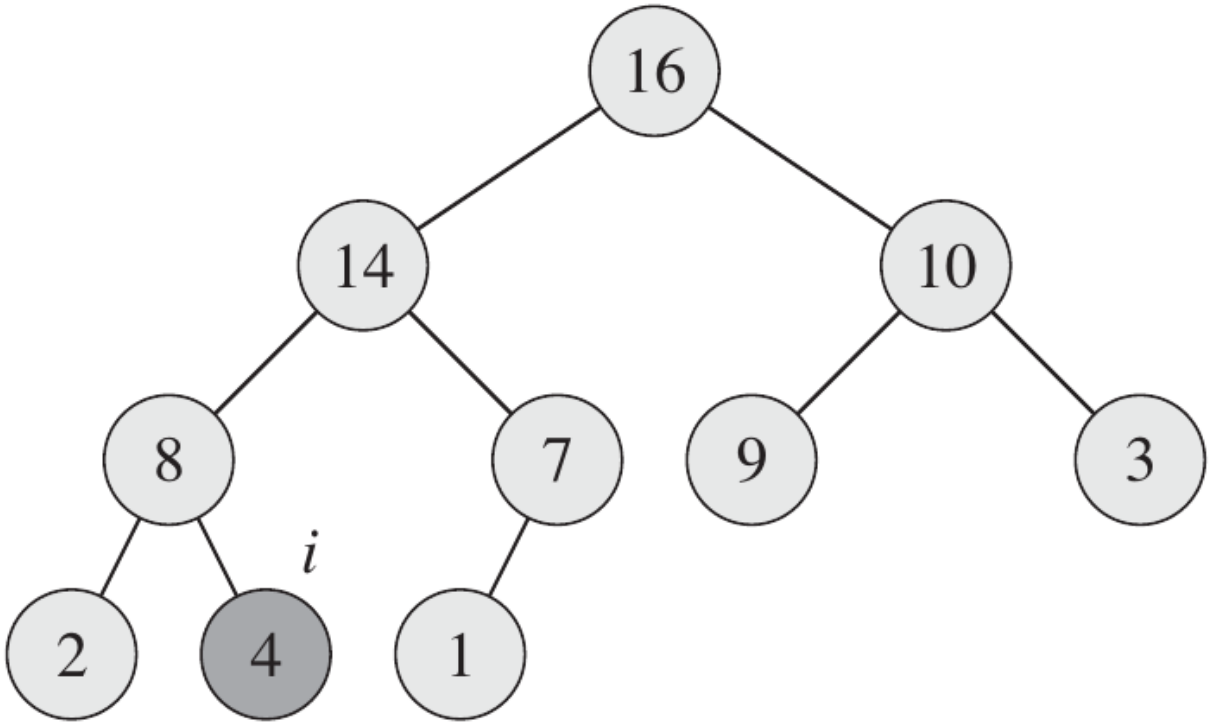
Running time of Heap-Extract-Max : $O(\lg n)$

Heap-Increase-Key

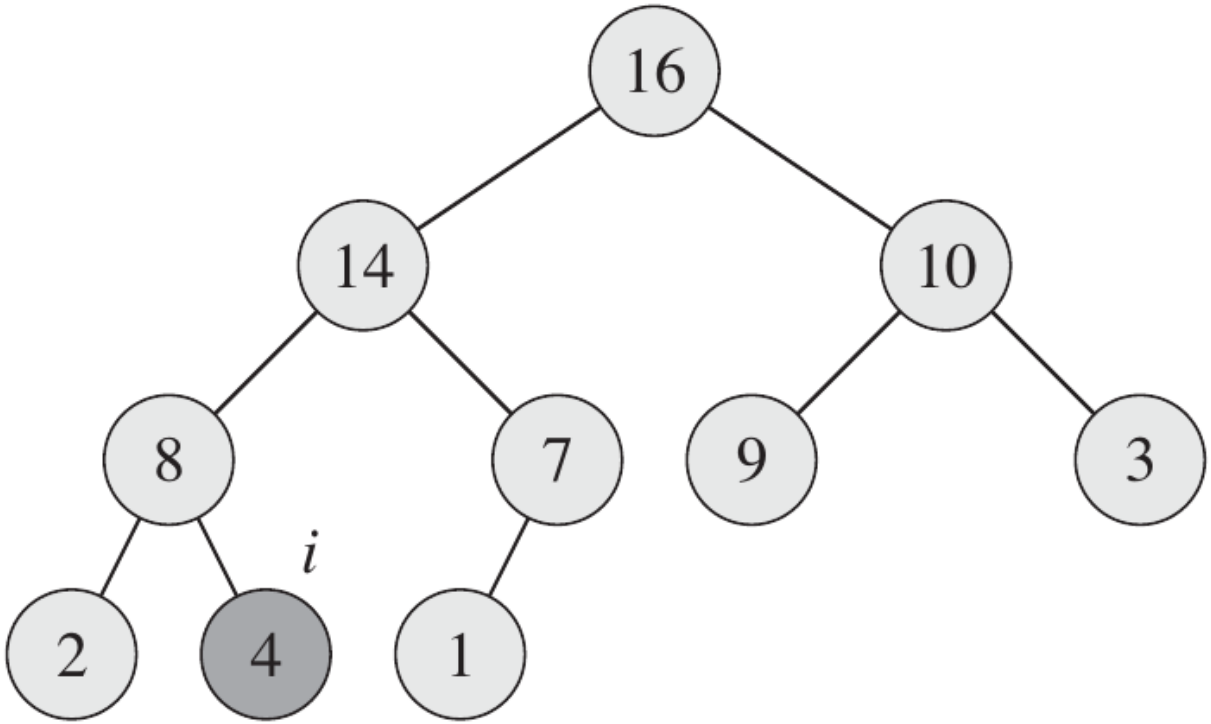
HEAP-INCREASE-KEY(A, i, key)

```
1  if key < A[i]
2      error "new key is smaller than current key"
3  A[i] = key
4  while i > 1 and A[PARENT(i)] < A[i]
5      exchange A[i] with A[PARENT(i)]
6      i = PARENT(i)
```

Heap-Increase-Key

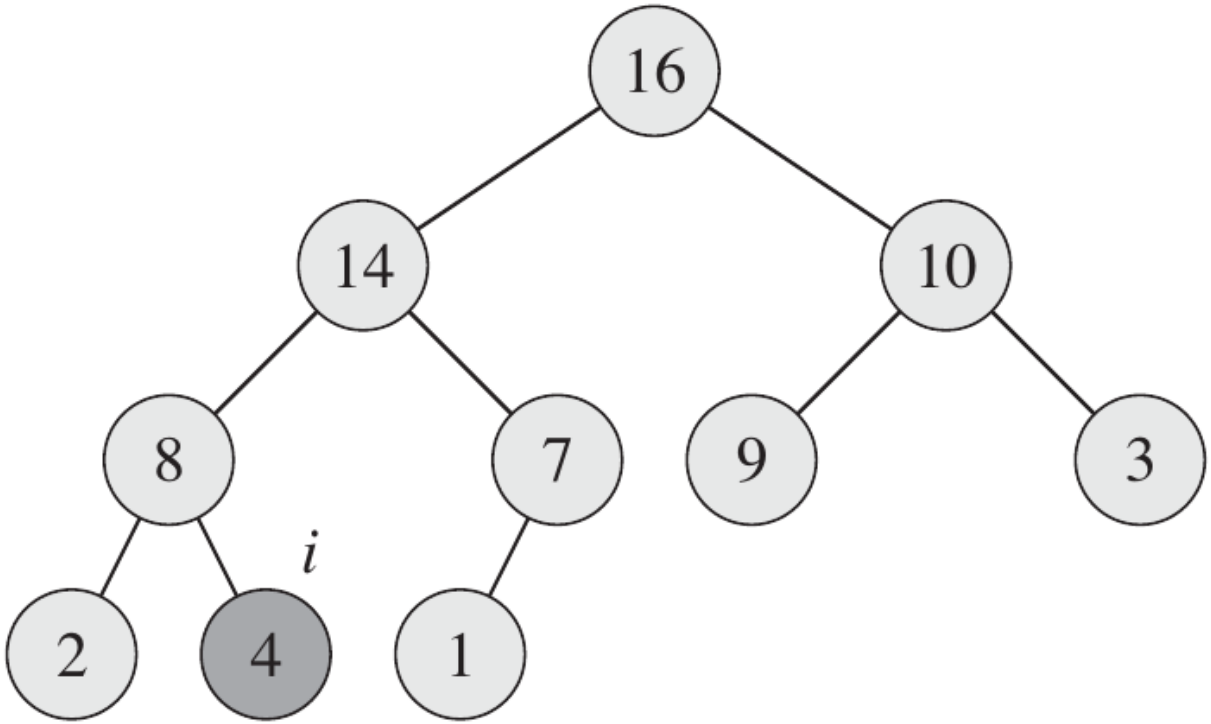


Heap-Increase-Key



- Suppose we increase the value at index i to 15.

Heap-Increase-Key



- Suppose we increase the value at index i to 15.
- Running time of Heap-Increase-Key : $O(\lg n)$

Max-Heap-Insert

MAX-HEAP-INSERT(A, key)

1 $A.heap\text{-}size = A.heap\text{-}size + 1$
2 $A[A.heap\text{-}size] = -\infty$
3 HEAP-INCREASE-KEY$(A, A.heap\text{-}size, key)$

Max-Heap-Insert

MAX-HEAP-INSERT(A, key)

1 $A.heap\text{-}size = A.heap\text{-}size + 1$
2 $A[A.heap\text{-}size] = -\infty$
3 HEAP-INCREASE-KEY$(A, A.heap\text{-}size, key)$

Running time of Heap-Increase-Key : $O(\lg n)$

Priority Queue

- Using a heap, all the basic operations can be performed in $O(\lg n)$ time.