Data Structures and Algorithms [1]

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[1] Material for the presentation taken from Cormen, Leiserson, Rivest and Stein, *Introduction to Algorithms, Third Edition*;

Part II Sorting and Order Statistics

- Record : Collection of data
- Key : Value to be sorted
- Satellite data
- If satellite data is large, we permute an array of pointers to the records.

Sorting algorithms

- *In place* sorting : If at any time only a constant number of elements are stored outside the array.

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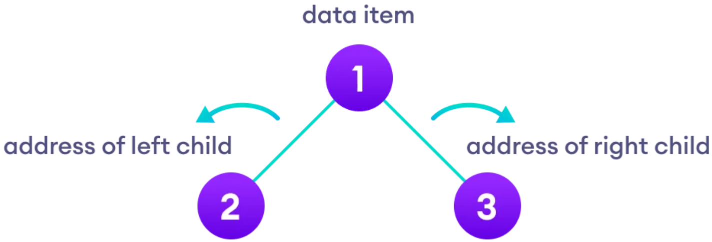
- Ch 6 : Heapsort that uses a data structure called heap.
- Heapsort sorts n elements *in place* in $O(n \lg n)$ time.

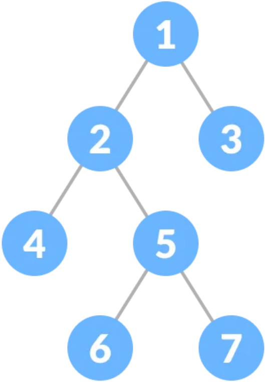
Binary tree

- Binary tree is a tree data structure where each node can have at most two child nodes : left child node and right child node.



Full Binary tree

- Each node is either a leaf node or has two child nodes.



Complete Binary tree

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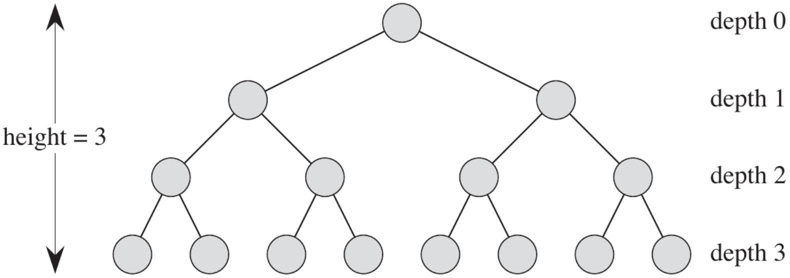
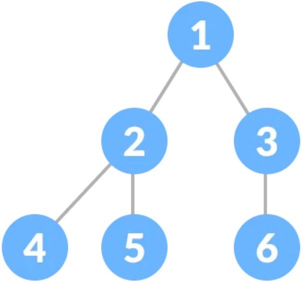


Figure B.8 A complete binary tree of height 3 with 8 leaves and 7 internal nodes.

Heap data structure : a complete binary tree



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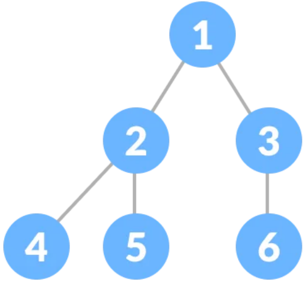

- The last level is not completely filled.

Heap : nearly a complete binary tree

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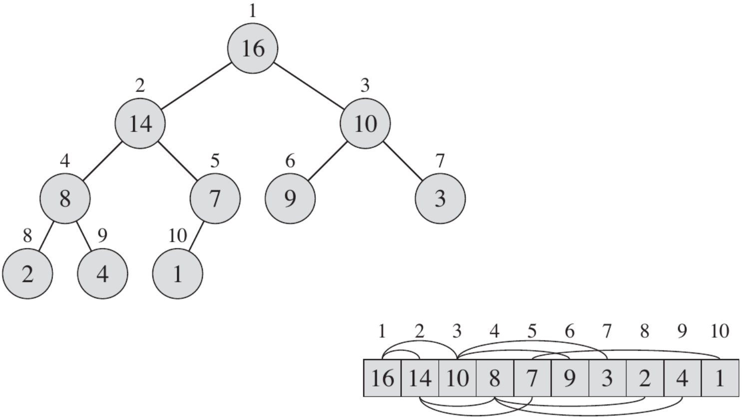

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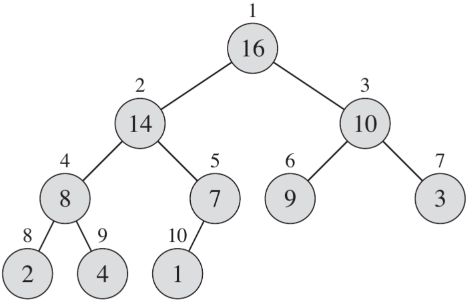

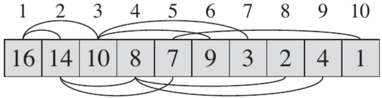

- *A.length*,

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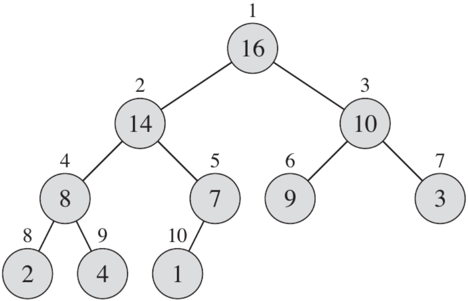

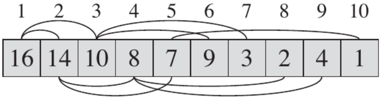

- *A.length*, *A.heap-size*,

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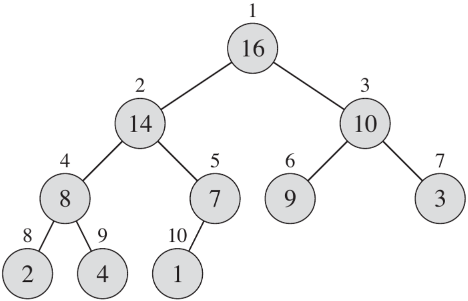
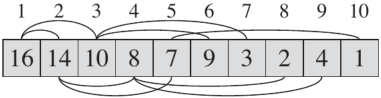
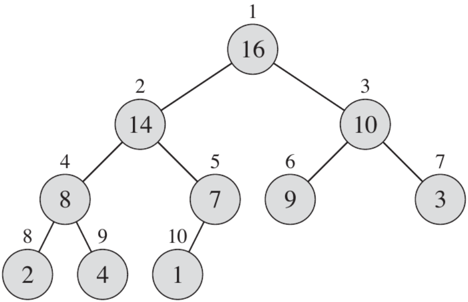




- *A.length*, *A.heap-size*, Root : *A[1]*

Heap : parent, left child, right child



PARENT(i)
return $\lfloor i/2 \rfloor$

LEFT(i)
return $2i$

RIGHT(i)
return $2i + 1$

1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

Binary Heaps

- Either max-heaps or min-heaps

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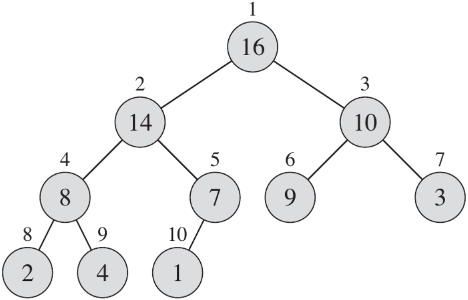

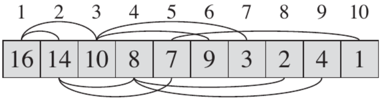

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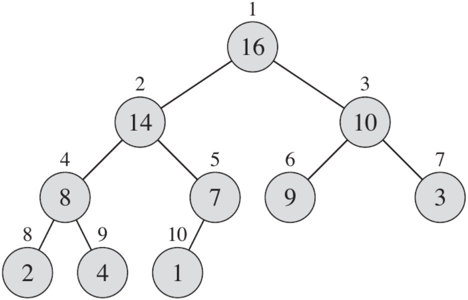

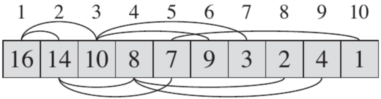

- $A[1]$ contains the maximum element

Binary Heaps

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Binary Heaps

- Min heap
- Min-heap property: $A[PARENT(i)] \leq A[i]$
- $A[1]$ will contain the smallest element

Heap

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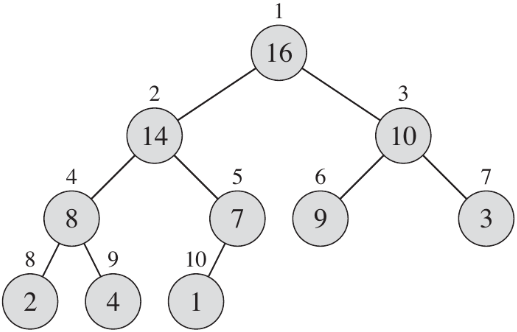

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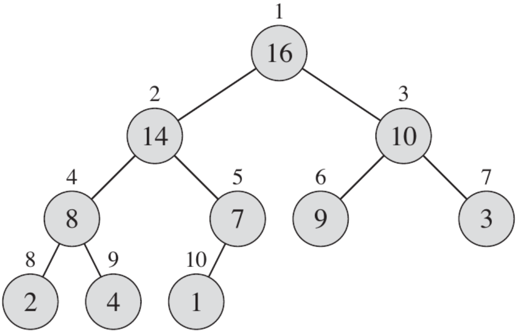

- **Height** of a heap is the height of its root.

Height of a heap

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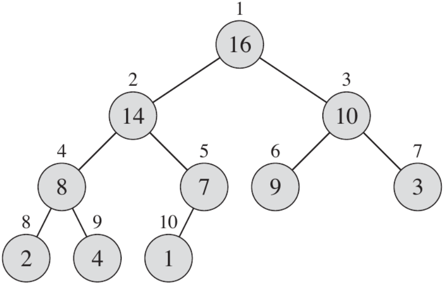

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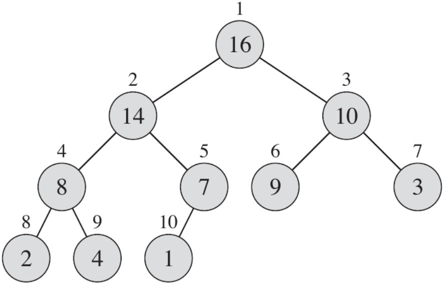

$$2^h$$

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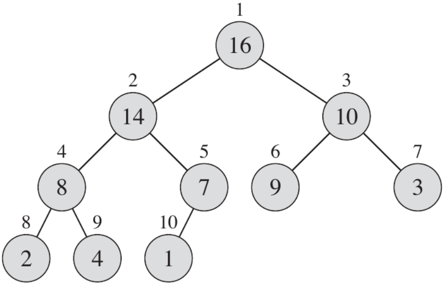

$$2^h \leq n$$

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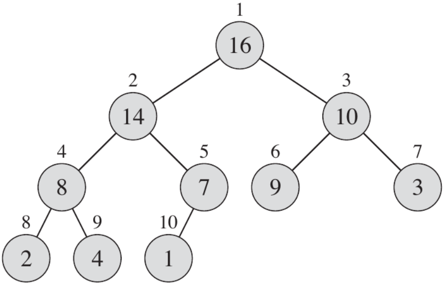

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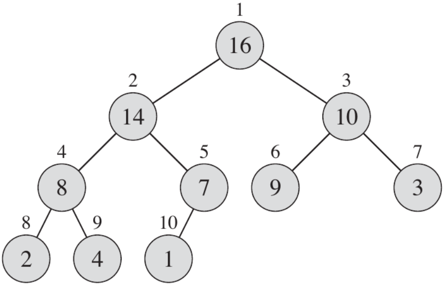

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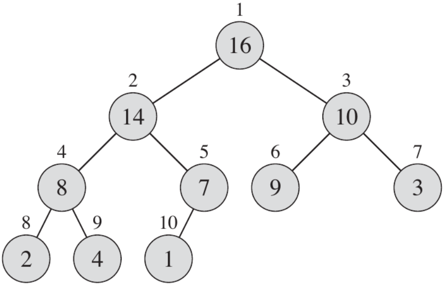

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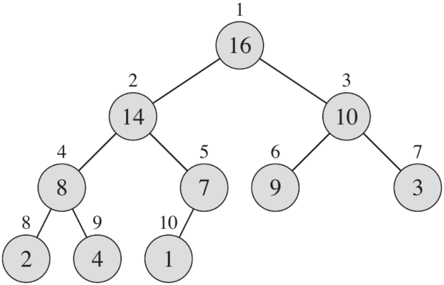

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MAX-HEAPIFY: Maintaining the heap property

- MAX-HEAPIFY(A,i) : binary trees rooted at LEFT(i) and RIGHT(i) satisfy max-heap property.

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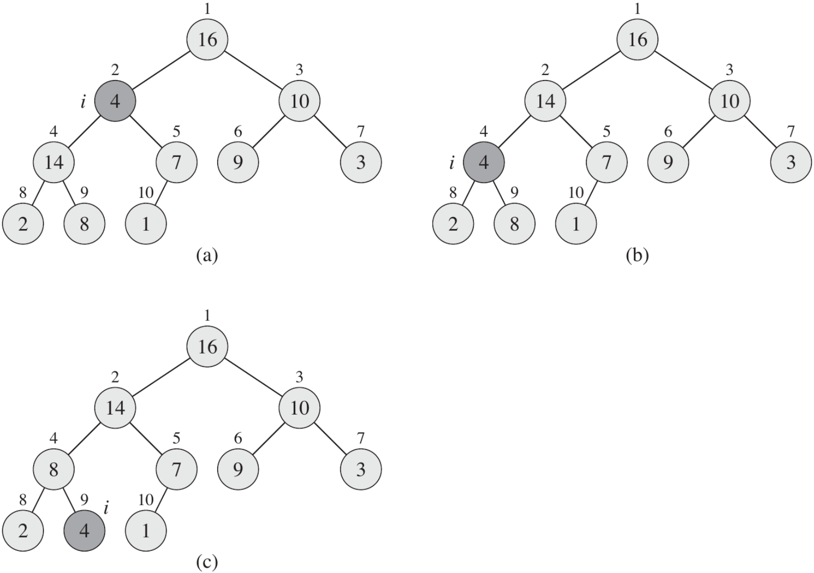

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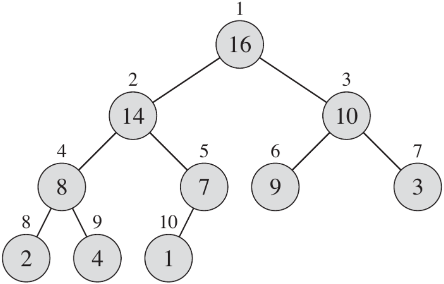

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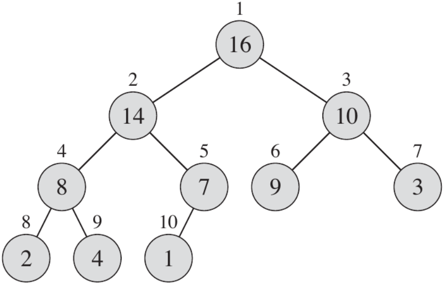

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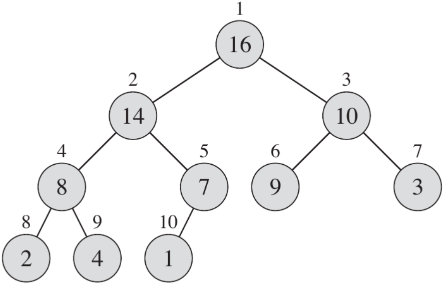

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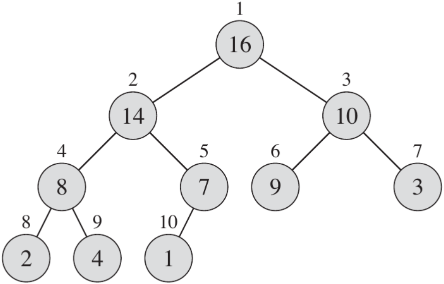

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$$\text{Maximum size of child subtree} < \frac{2n}{3}$$

MAX-HEAPIFY: Maintaining the heap property

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MAX-HEAPIFY(A, i)
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\text{Soln. } & \Theta(n^{\log_b a} \lg^{k+1} n) = \Theta(\lg n) \\
T(n) &= O(\lg n)
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BUILD-MAX-HEAP : Building a heap

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A									
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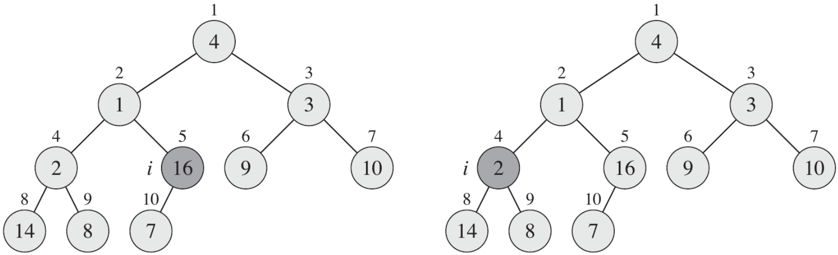

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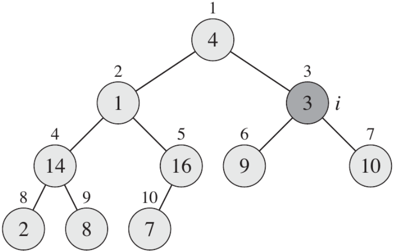
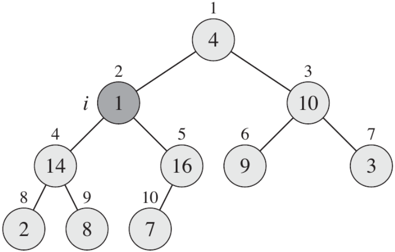


(c)



(d)

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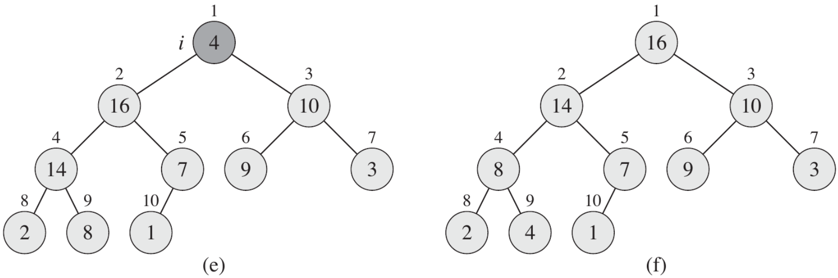

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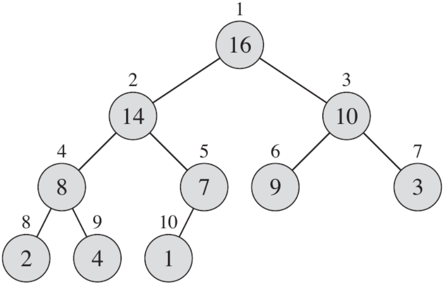
Running time: $O(n \lg n)$ { Not asymptotically tight }

Nodes with height $h \leq \left\lceil \frac{n}{2^{h+1}} \right\rceil$



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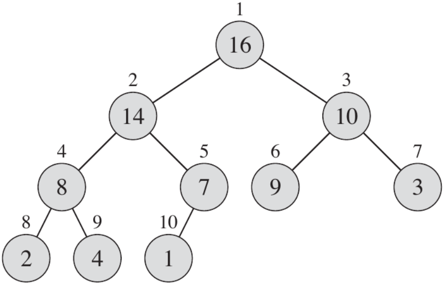

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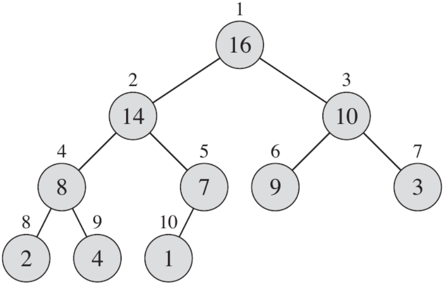

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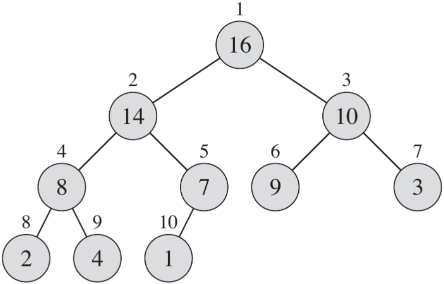

- There are at most $\left\lceil \frac{n}{2^{h+1}} \right\rceil$ nodes having a height h.

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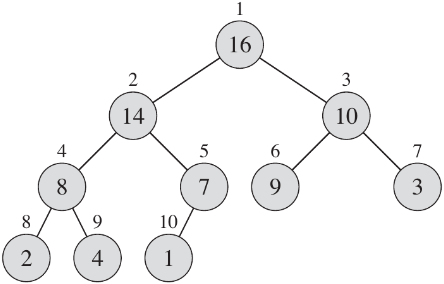

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$$h = 0, \qquad \left\lceil \frac{10}{2^{0+1}} \right\rceil = 5;$$

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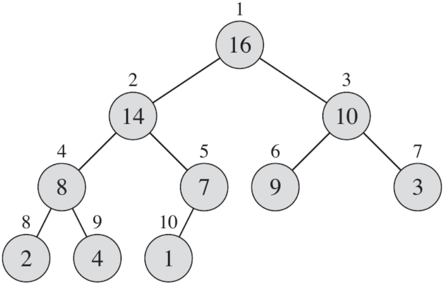

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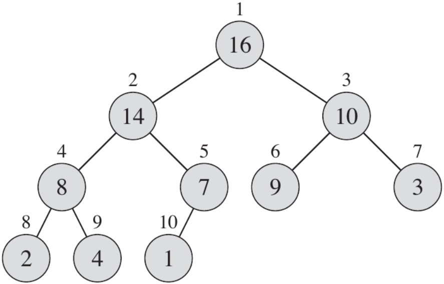

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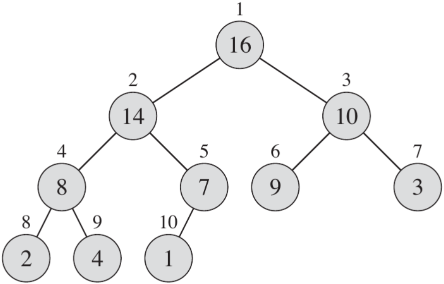

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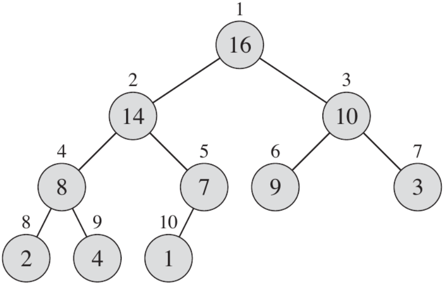

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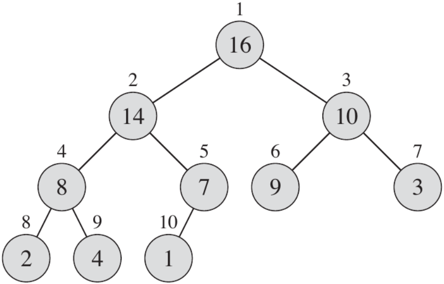

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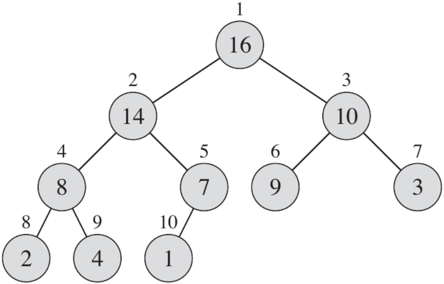

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Nodes with height $h \leq \left\lceil \frac{n}{2^{h+1}} \right\rceil$

- Proof by Induction

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$$
\begin{aligned}
k_h = k'_{h-1} &\leq \left\lceil \frac{n'}{2^{(h-1)+1}} \right\rceil = \left\lceil \frac{\lfloor (n/2) \rfloor}{2^{(h-1)+1}} \right\rceil \\
&\leq \left\lceil \frac{(n/2)}{2^h} \right\rceil = \left\lceil \frac{n}{2^{h+1}} \right\rceil
\end{aligned}
$$

BUILD-MAX-HEAP : Running time

```
BUILD-MAX-HEAP(A)
    A.heap-size = A.length
    for i = ⌊A.length/2⌋ downto 1
        MAX-HEAPIFY(A, i)
```

BUILD-MAX-HEAP : Running time

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- $0 < x < 1$, $\quad 1 + x + x^2 + \dots$

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- Differentiating both sides w.r.t x

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

- Multiplying x on both sides

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

Heap

- We can construct a max heap (or a min heap) from an unordered array in $O(n)$ time.

```
HEAPSORT(A)
BUILD-MAX-HEAP(A)
for i = A.length downto 2
    exchange A[1] with A[i]
    A.heap-size = A.heap-size - 1
    MAX-HEAPIFY(A, 1)
```

Heapsort

```
HEAPSORT(A)
BUILD-MAX-HEAP(A)
for i = A.length downto 2
    exchange A[1] with A[i]
    A.heap-size = A.heap-size − 1
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- Running time : $O(n \lg n)$

Heapsort

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- Running time : $O(n \lg n)$
- Operation (P. 161)

Priority Queues

- Heap data structure can be used to construct efficient priority queue

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Priority Queues

- Heap data structure can be used to construct efficient priority queue
- Job scheduling
- Two forms : max-priority queue and min priority-queue
- Max-priority queue operations:
    1. Insert(S,x)
    2. Maximum(S)
    3. Extract-Max(S)
    4. Increase-Key(S,x,k)

Priority Queues

HEAP-MAXIMUM(A)

1 **return** $A[1]$

Priority Queues

```
HEAP-MAXIMUM(A)
    return A[1]
```

```
HEAP-EXTRACT-MAX(A)
    if A.heap-size < 1
        error "heap underflow"
    max = A[1]
    A[1] = A[A.heap-size]
    A.heap-size = A.heap-size - 1
    MAX-HEAPIFY(A, 1)
    return max
```

Priority Queues

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return A[1]
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A.heap-size = A.heap-size - 1
MAX-HEAPIFY(A, 1)
return max
```
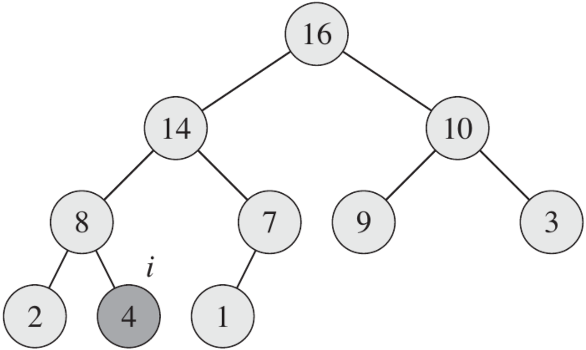

Running time of Heap-Extract-Max : $O(\lg n)$

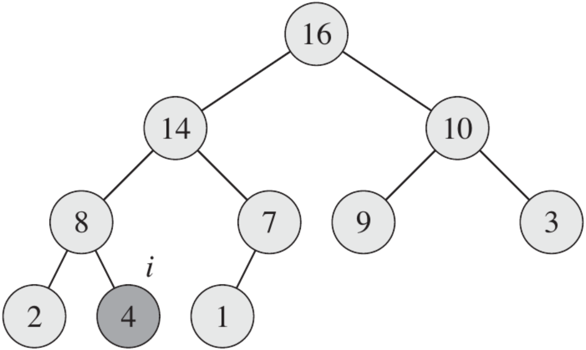
Heap-Increase-Key

```
HEAP-INCREASE-KEY(A, i, key)
if key < A[i]
        error "new key is smaller than current key"
A[i] = key
while i > 1 and A[PARENT(i)] < A[i]
        exchange A[i] with A[PARENT(i)]
        i = PARENT(i)
```

Heap-Increase-Key

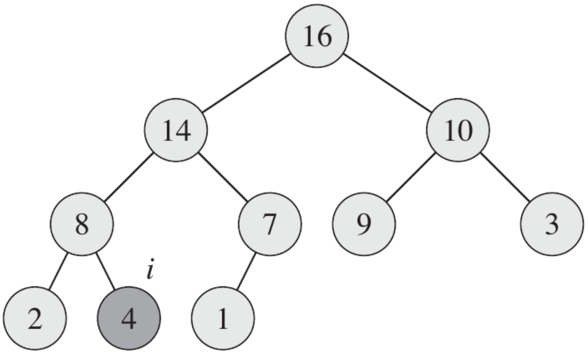


Heap-Increase-Key



- Suppose we increase the value at index i to 15.

Heap-Increase-Key



- Suppose we increase the value at index i to 15.
- Running time of Heap-Increase-Key : $O(\lg n)$

Max-Heap-Insert

MAX-HEAP-INSERT(A, key)

$A.heap\text{-}size = A.heap\text{-}size + 1$

$A[A.heap\text{-}size] = -\infty$

HEAP-INCREASE-KEY(A, $A.heap\text{-}size$, key)

Max-Heap-Insert

MAX-HEAP-INSERT(A, key)

1 $A.heap\text{-}size = A.heap\text{-}size + 1$
2 $A[A.heap\text{-}size] = -\infty$
3 HEAP-INCREASE-KEY$(A, A.heap\text{-}size, key)$

Running time of Heap-Increase-Key : $O(\lg n)$

Priority Queue

- Using a heap, all the basic operations can be performed in $O(\lg n)$ time.