EE2-08C - Numerical Analysis of ODEs using Matlab - Coursework 2017

The coursework is due for submission on Monday 13 March 2017 either to me or the UG office room 608, before 4pm. You should hand in a *printed* copy of all your matlab files with thorough comments as well as the plots you generate, clearly labelled. Before 22:00pm on Monday 13 March, you should also submit the pdf of the report as well as the matlab files *separately* on Blackboard. Each group should submit only once: choose a member of your group to submit all items on behalf of the group.

Before embarking on this coursework it is useful to work through the exercises in Example Sheets 1 and 2, covering Euler's method, Heun's method and second-order equations.

1 RL circuit

An high-pass filter takes an input signal V_{in} (for instance, an audio signal) and let only the high-frequency components pass. Usually, high-pass filters are made with capacitors instead of inductors due to the fact that inductors can be more easily manufactured and are generally physically smaller. However, the example of RL circuit we are going to consider is still very important. In fact, RL circuits are usually used in the modeling of real physical systems such as DC motors.

Figure 1: RL circuit

Consider the RL circuit as depicted in Figure 1. For the circuit, we can write the following equations:
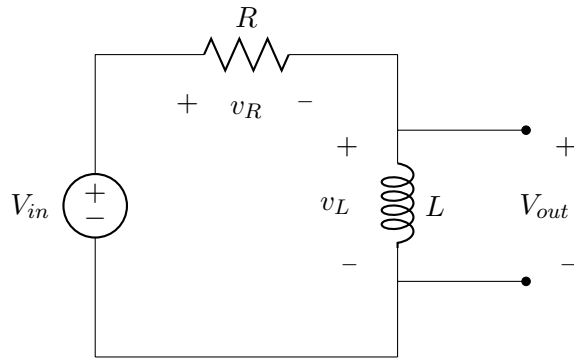
$$
\begin{aligned}
v_L(t) + v_R(t) &= V_{in}(t) \\
L\frac{d}{dt}i_L(t) + R\,i_L(t) &= V_{in}(t) \ .
\end{aligned}
$$

The state is $i_L(t)$. The input is $V_{in}(t)$. The output is $V_{out} = V_{in}(t) - Ri_L(t)$. Assume that the initial current through the inductor at time $t = 0$ is $i_L(0) = 0\,\mathrm{A}$. We want to model a DC motor with inertia $250\,\mu\mathrm{Nm/s}^2$ and maximum torque gain $50\,\mathrm{mN\,m\,A}^{-1}$. The corresponding values for the inductance and resistance are:

$$
\begin{aligned}
R &= 0.5\,\Omega \\
L &= 1.5\,\mathrm{mH} \ .
\end{aligned}
$$

Exercise 1. Take three second-order Runge Kutta methods: Heun's and Midpoint, covered in lectures and Ralston's, which you will need to find in the literature. For each method, write two matlab files, so for Ralston's method, there will be (i) a matlab function called **ralston.m**, implementing the Ralston method to model the RL-circuit, with first line:

```
function [t,vout].  = ralston(?,?,?,...)
```

The arguments for the function ralston.m need to be carefully chosen and one will need to be a function; another argument needs to be the final time t_f. The function should solve the ODE for any values of R, L, t_f and $i_L(0)$, not just those given above. The other file should be a script called **ralston_script.m**,

in which you call ralston.m to model the circuit for different types of input, and plot the output. When you vary the system parameters and input function, give some thought about choosing t_f carefully to allow the features of the output to be seen in the plots, Simulate the system and obtain plots for the following different input signals and observe how the amplitude of the output signal changes:

1. step signal with amplitude $\bar{V}_{in} = 5.5\,\text{V}$;

2. impulsive signal and decay:

$$V_{in} = \bar{V}_{in}\exp\left\{-\frac{t^2}{\tau}\right\} \qquad V_{in} = \bar{V}_{in}\exp\left\{-\frac{t}{\tau}\right\}$$

   with $\bar{V}_{in} = 3.5\,\text{V}$ and $\tau = 160\,(\mu\text{s})^2$, resp. $\tau = 160\,\mu\text{s}$.

3. sine, square, and sawtooth waves with amplitude $\bar{V}_{in} = 4.5\,\text{V}$ and different periods $T = 160\,\mu\text{s}$, $T = 20\,\mu\text{s}$, $T = 450\,\mu\text{s}$, $T = 1000\,\mu\text{s}$. For square and sawtooth wave, try Matlab help: "square-", resp. "sawtooth wave".

For each plot, briefly comment on what you see in terms of the circuit and what you know of its behaviour.

Exercise 2. Write a script called **error_script.m** in which you carry out error analysis given as input a cosine wave of period $T = 150\,\mu\text{s}$ and amplitude $\bar{V}_{in} = 6\,\text{V}$. Use your favourite method to obtain the exact solution of the ODE and compare the numerical solution with the exact solution, obtaining the error as a function of t. Plot the error function for each method. Vary the time step h for a suitable number and range of values, and obtain a log-log plot to show the order of the error. Compare the errors between the three methods, do you notice anything?

2   RLC circuit

An RLC circuit represents a standard example of a harmonic oscillator, namely a device able to resonate to a sinusoidal input signal. Among the many applications for this circuit, there is the tuning of analogue radio receivers. Usually, they are used as band-pass filters.

Consider the RLC circuit as depicted in the Figure. For the circuit, we can write the following equations:
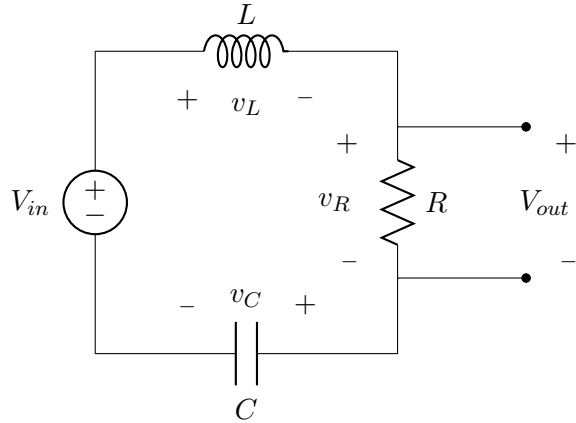
$$v_L(t) + v_R(t) + v_C(t) = V_{in}(t)$$

$$L\frac{d}{dt}i_L(t) + R\,i_L(t) + \frac{1}{C}\int_0^t i_L(t) = V_{in}(t)$$

$$L\frac{d^2}{dt^2}q_C(t) + R\frac{d}{dt}q_C(t) + \frac{1}{C}q_C(t) = V_{in}(t)\,. \quad (1)$$

The state is $q_C(t)$. The input is $V_{in}(t)$. The output is $V_{out} = v_R = R\frac{d}{dt}q_C(t)$. Assume that the capacitor is pre-charged at time $t = 0$ with $q_C(0) = 500\,\text{nC}$. Moreover, no initial current flows through the inductor at time $t = 0$: $i_L(0) = \frac{d}{dt}q_C(0) = 0\,\text{A}$. The values for resistance, capacitance, and inductance are:

$$R = 280\,\Omega\,, \quad C = 4\,\mu\text{F}\,, \quad L = 600\,\text{mH}\,.$$

Exercise 3 Write a matlab script called **RLC_script.m** and a matlab function called **RK4second.m**. In the script you should set up the two coupled first order ODEs in q and q' to solve the RLC second order ODE (1) for q. The script should include calls to **RK4second.m** which should be written to implement the fourth-order *Runge-Kutta 3/8* algorithm, for a **single time-step**, to obtain x_{i+1} and y_{i+1} from x_i, y_i and t_i. The algorithm to be used is the "Runge-Kutta 3/8" method, and you will need to find this in the literature.

The function call should include arguments x_i, y_i and t_i and the function should return x_{i+1} and y_{i+1}.

Once you have the matlab code working, use it to simulate the system, and obtain plots, for different input signals and observe how the amplitude of the output signal changes:

- step signal with amplitude $\bar{V}_{in} = 5\,\text{V}$;

- impulsive signal with decay

$$V_{in} = \bar{V}_{in}\exp\left\{-\frac{t^2}{\tau}\right\}$$

  with $\bar{V}_{in} = 5\,\text{V}$ and $\tau = 3\,(\text{ms})^2$.

- square wave with amplitude $\bar{V}_{in} = 5\,\text{V}$ and different frequencies $f = 5\,\text{Hz}$, $f = 110\,\text{Hz}$, $f = 500\,\text{Hz}$;

- sine wave with amplitude $\bar{V}_{in} = 5\,\text{V}$ and different frequencies $f = 5\,\text{Hz}$, $f = 110\,\text{Hz}$, $f = 500\,\text{Hz}$.

When you vary the system parameters and input function, give some thought about choosing t_f carefully to allow the features of the output to be seen in the plots, For each plot, briefly comment on what you see in terms of the circuit and what you know of its behaviour, including the different output amplitudes