Imperial College
London

[E1.14 (Maths 2) 2011]

B.ENG. AND M.ENG. EXAMINATIONS 2011

PART I : MATHEMATICS 2 (ELECTRICAL ENGINEERING)

Date Thursday 9th June 2011 10.00 - 12.00

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Answer Question 1 and THREE of the remaining FIVE questions.

Answer Section A and Section B in different answerbooks.

Question 1 carries twice the marks of each of the other questions.

A mathematical formulae sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of SIX questions. Ask the invigilator for a replacement if your copy is faulty.]

SECTION A

1. (i) Determine the union and the intersection of the sets

$$A = \{ n \in \mathbb{N} \mid 0 < n \leq 10 \} ,$$

$$B = \{ n \in \mathbb{N} \mid 0 < n < 10 \text{ and } n \text{ prime .} \}$$

(ii) Draw the truth table for

$$(p \to q) \vee p$$

and

$$(p \wedge q) \vee (\overline{p} \wedge \overline{q}) .$$

(iii) Determine the truth value of the following propositions

(a) $\forall x \in \mathbb{R} \quad \exists y \in Q \quad \forall z \in \mathbb{R} \qquad z > xy,$

(b) $\forall x \in \mathbb{R} \quad \exists y \in Q \quad \exists z \in \mathbb{R} \qquad z > xy\,,$

(c) $\exists x \in \mathbb{R} \quad \forall y \in Q \quad \exists z \in \mathbb{R} \qquad z > xy\,,$

(d) $\forall x \in \mathbb{R} \quad \exists y \in Q \quad \forall z \in \mathbb{R} \qquad z^2 > xy\,.$

(iv) Find

$$\int_0^{2\pi} \cos(mx)\,\cos(nx)\,dx$$

for positive integers m and n .

(v) Let $f(x) = \exp(x)$.

Determine $\displaystyle\int_0^{2\pi} f(x)\,\sin(nx)dx$ for $n \in \mathbb{N},\ n > 0.$

(vi) Let $f(x, y) = \exp(y \sin(x))$.
Determine the partial derivatives $f_x(x, y)$, $f_y(x, y)$ $f_{xy}(x, y)$ and $f_{yx(x,y)}$.

(vii) Show that

$$x \frac{\partial}{\partial x} u(x, y) + y \frac{\partial}{\partial y} u(x, y) = u(x, y)$$

for $u(x, y) = x \ln(x/y)$.

(viii) Let $u(x, y) = \cos(y/x)$.

Find $\dfrac{d}{dt} u(x(t), y(t))$ in terms of t for $x(t) = t$ and $\mathrm{y(t)} = \frac{1}{2} t^2$, using partial differentiation.

PLEASE TURN OVER

(ix) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the identity matrix. Find all l so that

$$\det(A - lI) = 0 .$$

(x) Find all vectors $\boldsymbol{r}$ such that

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \boldsymbol{r} = -\boldsymbol{r} .$$

SECTION B

2. (i) Given the sets $S_1 = \{6, 8\}$, $S_2 = \{3, 4\}$, list all the elements of
   - (a) $S_1 \cup S_2$;
   - (b) $S_1 \cap S_2$;
   - (c) $S_1 - S_2$;
   - (d) $P(S_1)$.

   (ii) What is meant by a relation R from a set S to itself?
   What is meant if we say the relation is (i) reflexive, (2) symmetric and (iii) transitive?
   Apply these definitions to the relations below and say which are (i) reflexive, (ii) symmetric and (iii) transitive.

   I "Is a sibling of" on the set of all people (sibling means brother or sister, with both parents in common).

   II "Is the daughter of" on the set of all people.

   III "Is the same sex as" on the set of all people.

   IV "Is greater than" on the set of all integers.

3. Let

$$f(x) = \begin{cases} x - \dfrac{\pi}{2} & \text{for} \quad x \in [\,0,\, \pi) \\[2ex] \dfrac{3\pi}{2} - x & \text{for} \quad x \in [\pi,\, 2\pi) \end{cases}$$

define a periodic function with period 2π.

(i) Determine the coefficients a_n for $n \geq 0$ and b_n for $n > 0$ of the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) .$$

(ii) Express

$$\int_0^{2\pi} f^2(x)\, dx$$

in terms of a_n and b_n and thus determine $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}$.

PLEASE TURN OVER

4. (i) If $f(x,y) = 2\,x\,\tan^{-1}\left(\frac{y}{x}\right) \;+\; y\,\ln(x^2+y^2)$ find

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

and show that

$$x\,\frac{\partial f}{\partial x} \;+\; y\,\frac{\partial f}{\partial y} \;=\; f \;+\; 2\,y\,.$$

(ii) The function $u(x,t)$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial x^2} \;=\; \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\,.$$

New variables p and q are defined by

$$p \;=\; x \;-\; c\,t \;\text{ and }\; q \;=\; x \;+\; c\,t\,.$$

Determine $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial t^2}$ in terms of derivatives with respect to p and q and hence show that the wave equation in the new variables becomes

$$\frac{\partial^2 u}{\partial p\,\partial q} \;=\; 0\,.$$

Hence find the general solution of the wave equation for u in terms of x and t.

5. You are given

$$A \;=\; \begin{pmatrix} -1 & -2 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix}\,.$$

Compute A^2 and A^3 .

Verify $A^3 \;-\; 3\,A \;+\; 2\,I \;=\; 0$.

Find scalars a and b such that

$$A^2 \;+\; a\,A \;+\; b\,I \;=\; 0\,.$$

Hence or otherwise find A^{-1} .

6. (i) Solve the differential equation

$$(x+1)\,\frac{dy}{dx} - x\,y = e^x$$

subject to the condition $y = 0$ when $x = 1$.

(ii) Under what circumstances is

$$P(x,y)\,dx \;+\; Q(x,y)\,dy \;=\; 0$$

exact.
Show that

$$(2\,x\,y + \frac{1}{3}\,y^3)\,dx \;+\; (x^2 + x\,y^2)\,dy \;=\; 0$$

is exact and hence solve the differential equation

$$(2\,x\,y + \frac{1}{3}\,y^3) \;+\; (x^2 + x\,y^2)\,\frac{dy}{dx} \;=\; 0.$$

Note: *Leave the solution in its implicit form.*

END OF PAPER

M A T H E M A T I C S   D E P A R T M E N T

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\,\mathbf{b},\,\mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha\, x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots\ ,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n\frac{x^{2n}}{(2n)!} + \dots\ ,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \dots\ ,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n\frac{x^{n+1}}{(n+1)} + \dots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \, \cos b + \cos a \, \sin b \, ;$$

$$\cos(a+b) = \cos a \, \cos b - \sin a \, \sin b \, .$$

$$\cos iz = \cosh z \, ; \quad \cosh iz = \cos z \, ; \quad \sin iz = i \sinh z \, ; \quad \sinh iz = i \sin z \, .$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}\,g + \ldots + \binom{n}{r} D^r f\,D^{n-r} g + \ldots + D^n f\;g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\quad \epsilon_n(h) = h^{n+1} f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x,\, y)$ about $(a,\, b)$:

$$f(a+h,\, b+k) = f(a,\, b) + [hf_x + kf_y]_{a.b} + 1/2!\,\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a.b} + \ldots$$

(d) Partial differentiation of $f(x,\, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u,\, v)$, $y = y(u,\, v)$, then $f = F(u,\, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let $(a,\, b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D > 0$ and $f_{xx}(a,\, b) < 0$, then $(a,\, b)$ is a maximum;
If $D > 0$ and $f_{xx}(a,\, b) > 0$, then $(a,\, b)$ is a minimum;
If $D < 0$ then $(a,\, b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $\quad I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\, y)dx + Q(x,\, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2\,t/(1+t^2)\,, \quad \cos\theta = (1-t^2)/(1+t^2)\,, \quad d\theta = 2\,dt/(1+t^2)\,.$

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a\,.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - \left[f(x_n)/f'(x_n)\right], \quad n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh\,, \quad y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)\,[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)\,[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let $I_1,\ I_2$ be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15\,,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t, \alpha)$	$(\partial/\partial\alpha)F(s, \alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}, \ (s > 0)$
e^{at}	$1/(s-a), \ (s > a)$	$\sin \omega t$	$\omega/(s^2 + \omega^2), \ (s > 0)$
$\cos \omega t$	$s/(s^2 + \omega^2), \ (s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	$e^{-sT}/s, \ (s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right).$$

EXAMINATION ~~QUESTIONS~~/~~SOLUTIONS~~ 2010-2011

Course EE1(2) 1

Question: Long question (Solutions)

Marks & seen/unseen

Parts

i)

$A \cup B = \{1,2,3,4,5,6,7,8,9,10\}$ 2

$A \cap B = \{2,3,5,7\}$ 2

ii)

p	q	$p \to q$	$(p\to q)\cup p$	$(p\wedge q)\cup(\bar p \wedge \bar q)$	
T	T	T	T	T	1
T	F	F	T	F	1
F	T	T	T	F	1
F	F	T	T	T	1

iii)

a) false 1

b) true 1

c) true 1

d) ~~true~~ false 1

iv)

Double integration by parts or

$$\cos(mx)\cos(nx) = \frac{1}{2}\left(\cos((m+n)x) + \cos((m-n)x)\right)$$ 1

Without loss of generality $n \geq 0$, so integral over $\cos(m+n)x$ always vanishes: 1

$$\int_0^{2\pi} \cos(mx)\cos(nx)\,dx = \frac{1}{2}\int_0^{2\pi}\cos((m-n)x)\,dx$$

$$= \begin{cases} \pi & \text{for } m=n \\ 0 & \text{otherwise} \end{cases}$$ 1 1

v)

$$\int_0^{2\pi} e^x \sin(nx)\,dx = \underbrace{\left[e^x \sin(nx)\right]_0^{2\pi}}_{\to 0} - \int_0^{2\pi} e^x n\cos(nx)\,dx$$ 1

$$= -\left[e^x n\cos(nx)\right]_0^{2\pi} - \int_0^{2\pi} e^x n^2 \sin(nx)\,dx \quad \Rightarrow$$ 1

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2010-2011

Course: EE1 (2) (1)

Question: Long question

Parts		Marks & seen/unseen
	$\int_0^{2\pi} e^x \sin(nx)\,dx = -\frac{n}{1+n^2}\left(e^{2\pi}-1\right)$	2
vi)	$f(x,y) = \exp(y\sin(x))$	
	$f_x = y\cos(x)\exp(y\sin(x))$	1
	$f_y = \sin(x)\exp(y\sin(x))$	1
	$f_{xy} = f_{yx} = \cos(x)\exp(y\sin(x)) + y\sin(x)\cos(x)\exp(y\sin(x))$	2
vii)	$u = x\ln(x/y)$	
	$\left.\begin{array}{l}\partial_x u = \ln(\frac{x}{y})+1 \\ \partial_y u = -\frac{x}{y}\end{array}\right\\} x\partial_x u + y\partial_y u = u + x - x = u$	1, 1 ; 2
viii)	$\frac{d}{dt}u(x,y) = \dot{x}u_x + \dot{y}u_y$	1
	$\dot{x} = 1,\ \dot{y} = t,\ u_x = \frac{\sin(y/x)}{x^2}y,\ u_y = -\frac{\sin(y/x)}{x}$	1, 1
	$\Rightarrow \frac{d}{dt}u = \frac{\sin(y/x)}{x^2}y - t\frac{\sin(y/x)}{x} = -\frac{1}{2}\sin(\frac{1}{2}t)$	1
	Give one mark only for $u = \cos(\frac{1}{2}t)$ and $\frac{d}{dt}u = -\frac{1}{2}\sin(\frac{1}{2}t)$ (i.e. finding the results without using partial derivatives).	

Setter's initials: GP

Checker's initials:

Page number: L2

Examination Solutions 2010-2011
Long question

ix) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

$$A - \lambda \mathbb{1} = \begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} \qquad 1$$

$$\det(A - \lambda \mathbb{1}) = (1-\lambda)(2-\lambda) - 6$$
$$= \lambda^2 - 3\lambda - 4 = 0 \qquad 1$$

Solutions by inspection or otherwise : $\lambda = -1, 4$ 2

x) $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{r} = -\vec{r}$ $\qquad \vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ 1

$x + 2y = -x$ $\qquad 2x + 2y = 0$ $\qquad x = 1 \Rightarrow y = -1$
$3x + 2y = -y$ $\qquad 3x + 3y = 0$ $\qquad 1, 1$

Solution: Any multiple of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. 1

GP

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course EE1(2)
Question 2a		Marks & seen/unseen
Parts	$S_1 = \{6,8\}$, $S_2 = \{3,4\}$	
	(i) $S_1 \cup S_2 = \{6,3,4,8\}$	1
	(ii) $S_1 \cap S_2 = \Phi$	1
	(iii) $S_1 - S_2 = \{6,8\}$	1
	(iv) $P(S_1) = \{\emptyset, \{6\}, \{8\}, \{6,8\}\}$	2
	Setter's initials; Checker's initials GT	Page number

EXAMINATION QUESTIONS/SOLUTIONS 2010-2011

Course: EE I (2) (2)

Question: 2b

Marks & seen/unseen

Parts

1b. A relation on a set S is a subset of $S \times S$. 4

reflexive iff $(a,a) \in R$ for every $a \in A$.

symmetric iff $(a,b) \in R$ whenever $(b,a) \in R$.

transitive iff whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$

	reflexive	symmetric	transitive
I	✗	✓	✓
II	✗	✗	✗
III	✓	✓	✓
IV	✗	✗	✓

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Setter's initials

Checker's initials GP

Page number

EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2010-2011

Course: EE1(2) (3)

Question: Fourier question

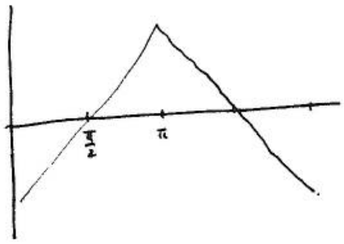
Marks & seen/unseen

Parts

i) $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx$; $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$ 4

Seen similar (throughout)

Graph of $f(x)$



$f(x)$ is even $\Rightarrow b_n = 0$, and $a_0 = 0$

$n>0$

$$\pi a_n = \int_0^{\pi}\left(x-\frac{\pi}{2}\right)\cos(nx)\,dx + \int_{\pi}^{2\pi}\left(\frac{3\pi}{2}-x\right)\cos(nx)\,dx$$

$$= \underbrace{\left[\frac{1}{n}\left(x-\frac{\pi}{2}\right)\sin(nx)\right]_0^{\pi}}_{\to 0} - \int_0^{\pi}\frac{1}{n}\sin(nx)\,dx$$

$$+ \underbrace{\left[\frac{1}{n}\left(\frac{3\pi}{2}-x\right)\sin(nx)\right]_{\pi}^{2\pi}}_{\to 0} + \int_{\pi}^{2\pi}\frac{1}{n}\sin(nx)\,dx$$ 4

$$= \left[\frac{1}{n^2}\cos(nx)\right]_0^{\pi} - \left[\frac{1}{n^2}\cos(nx)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{n^2}\left\{\left(1-(-)^n\right) - \left((-)^n - 1\right)\right\} = \begin{cases} 4/n^2 & \text{for } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

a_0 vanishes, $\frac{3\pi}{2} - x = \frac{\pi}{2} - (x-\pi)$

$$a_n = \frac{-4}{\pi n^2} \text{ for } n \text{ odd, all other coefficients vanish.}$$ 4

ii) Parseval's theorem:

$$\frac{1}{\pi}\int_0^{2\pi} f^2(x)\,dx = \sum_{n=1}^{\infty} a_n^2 + b_n^2 + \frac{a_0^2}{2}$$

$$\frac{1}{\pi}\int_0^{2\pi} f^2(x)\,dx = \frac{4}{\pi}\int_0^{\pi/2} x^2\,dx = \frac{4}{3\pi}\left(\frac{\pi}{2}\right)^3 = \frac{\pi^2}{6}$$ ✓ 3

Setter's initials: GP

Checker's initials

Page number: S1

	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2010-2011	Course EE I (2) (3)
Question		Marks & seen/unseen
Parts	$$\frac{a_0^2}{2}+\sum a_n^2+b_n^2 = \sum_{m=0}^{\infty}\frac{16}{\pi^2(2m+1)^4}$$ $$\Rightarrow \sum_{m=0}^{\infty}\frac{1}{(2m+1)^4} = \frac{\pi^4}{96}$$	2 3 20
	Setter's initials GP Checker's initials	Page number S2

(i) $u = 2x\tan^{-1}\left(\frac{y}{x}\right) + y\ln(x^2+y^2)$

$$\frac{\partial u}{\partial x} = 2\tan^{-1}\left(\frac{y}{x}\right) + \frac{2x}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) + \frac{y\,2x}{x^2+y^2} \quad \leftarrow 4$$

$$= 2\tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial y} = \frac{2x}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} + \ln(x^2+y^2) + \frac{y\,2y}{x^2+y^2}$$

$\leftarrow 4$

$$= 2 + \ln(x^2+y^2)$$

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2x\tan^{-1}\left(\frac{y}{x}\right) + 2y + y\ln(x^2+y^2)$$

2

$$= u + 2y$$

(ii) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} = \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}$ 2

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial t} = c\left(\frac{\partial u}{\partial q} - \frac{\partial u}{\partial p}\right)$$

4

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial p} + \frac{\partial}{\partial q}\right)\left(\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}\right) = \frac{\partial^2 u}{\partial p^2} + 2\frac{\partial^2 u}{\partial p\partial q} + \frac{\partial^2 u}{\partial q^2}$$

$$\frac{\partial^2 u}{\partial t^2} = c\left(\frac{\partial}{\partial q} - \frac{\partial}{\partial p}\right)c\left(\frac{\partial u}{\partial q} - \frac{\partial u}{\partial p}\right) = c^2\left(\frac{\partial^2 u}{\partial p^2} - 2\frac{\partial^2 u}{\partial p\partial q} + \frac{\partial^2 u}{\partial q^2}\right)$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \iff 4\frac{\partial^2 u}{\partial p\partial q} = 0 \qquad 2$$

Thus $u = f(p) + g(q)$ [f and g are arbitrary functions]

2

Thus $u = f(x-ct) + g(x+ct)$

forward travelling wave   backward travelling wave

20

$$A^2 = \begin{pmatrix} -1 & -2 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix}\begin{pmatrix} -1 & -2 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$ 4

$$A^3 = \begin{pmatrix} -1 & -2 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

4

$$= \begin{pmatrix} -5 & -6 & 0 \\ -3 & -2 & 0 \\ -6 & -6 & 1 \end{pmatrix}$$

$$A^3 - 3A + 2\underline{I} = \begin{pmatrix} -5 & -6 & 0 \\ -3 & -2 & 0 \\ -6 & -6 & 1 \end{pmatrix} - 3\begin{pmatrix} -1 & -2 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ 4

$$A^2 + aA + b\underline{I} = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 1 \end{pmatrix} + a\begin{pmatrix} -1 & -2 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

11 Element is $3 - a + b = 0$

12 Element is $2 - 2a = 0$ 4

$\Rightarrow a = 1,\ b = -2$

Check all other elements are then 0

ie $A^2 + A - 2\underline{I} = 0$

Multiply by A^{-1} and manipulate to get $A^{-1} = \frac{1}{2}(A + \underline{I})$

$$= \begin{pmatrix} 0 & -1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

4

$\boxed{20}$

Q6 EE 2nd paper EE I (2) 6

(i) Divide eqn. by $(x+1)$ to get standard form:

$$\frac{dy}{dx} - \frac{x}{x+1}\, y = \frac{e^x}{x+1}$$

Multiply by integrating factor

$$I = \exp\left[-\int \frac{x}{x+1}\, dx\right] = e^{-x}(x+1)$$ 2

to get: $e^{-x}(x+1)\frac{dy}{dx} - x e^{-x} y = 1$

i.e. $\frac{d}{dx}\left[(x+1)e^{-x} y\right] = 1$ 2

i.e. $(x+1)e^{-x} y = x + C$ 2

Condition implies $C = 1$

Thus $y = \frac{x-1}{x+1} e^x$ 2

(ii) Exact if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ 2

$P = 2xy + \frac{1}{3}y^3$, $Q = x^2 + xy^2$

$\frac{\partial P}{\partial y} = 2x + y^2$, $\frac{\partial Q}{\partial x} = 2x + y^2$ 2

Equal thus exact 2

Solution is $F(x,y) = C$

where $\frac{\partial F}{\partial x} = P = 2xy + \frac{1}{3}y^3$ — (1) 2

and $\frac{\partial F}{\partial y} = Q = x^2 + xy^2$ — (2)

(2) $\Rightarrow$ $F = x^2 y + \frac{1}{3} x y^3 + f(x)$ — (3) 2

Subst. (3) in (1) to get

$\frac{df}{dx} = 0 \Rightarrow f(x) = \text{const}$ 2

∴ solution is $x^2 y + \frac{1}{3} x y^3 = C$

$\boxed{20}$