

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING  
EXAMINATIONS 2010

MSc and EEE/ISE PART IV: MEng and ACGI

**WAVELETS AND APPLICATIONS**

Thursday, 20 May 10:00 am

Time allowed: 3:00 hours

**There are FOUR questions on this paper.**

**Answer THREE questions.**

*All questions carry equal marks.*

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible	First Marker(s) :	P.L. Dragotti
	Second Marker(s) :	K.D. Harris



**Special Information for the Invigilators: NONE**

**Information for Candidates:**

*Sub-sampling by an integer  $N$ :*

$$x_{\downarrow N}[n] \longleftrightarrow \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N}) = \frac{1}{N} \sum_{k=0}^{N-1} X(W_N^k z^{1/N}),$$

where

$$W_N^k = e^{-j2\pi k/N}.$$

*Parseval's identity:*

$$\langle g(t), f^*(t) \rangle = \frac{1}{2\pi} \langle \hat{g}(\omega), \hat{f}^*(\omega) \rangle,$$

where  $\hat{g}(\omega)$  and  $\hat{f}(\omega)$  are the Fourier transforms of  $g(t)$  and  $f(t)$  respectively.

*Poisson summation formula:*

$$\sum_{n=-\infty}^{\infty} f(t-n) = \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k) e^{j2\pi kt}.$$

*Fourier transform pair:*

$$\frac{\sin \pi t}{\pi t} \longleftrightarrow \text{rect}\left(\frac{\omega}{2\pi}\right),$$

where

$$\text{rect}(x) = \begin{cases} 1 & |x| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

## The Questions

1. Consider the oversampled three-channel filter bank shown in Figure 1a. Note that the down-sampling as well as the up-sampling factor is two.

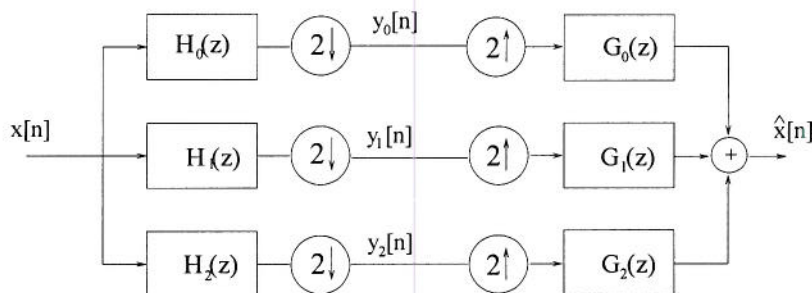


Figure 1a: Three-channel filter bank with down-sampling by 2.

- (a) Express  $\hat{X}(z)$  as a function of  $X(z)$  and the filters. Then, derive the two perfect reconstruction conditions the filters have to satisfy. [7]
- (b) Assume that  $G_0(z)$ ,  $G_1(z)$  and  $G_2(z)$  are the ideal filters shown in Figure 1b, (check carefully the cut-off frequencies). Assume that  $H_i(z) = \alpha G_i(z^{-1})$ , for  $i = 0, 1, 2$ . Here  $\alpha$  is a constant. Sketch and dimension the Fourier transform of  $y_0[n]$ ,  $y_1[n]$ ,  $y_2[n]$  and  $\hat{x}[n]$  assuming that  $x[n]$  has the spectrum shown in Figure 1c. [5]
- (c) Choose the constant  $\alpha$ , so that  $\hat{X}(z) = X(z)$ . [2]

Question 1 continues on the next page

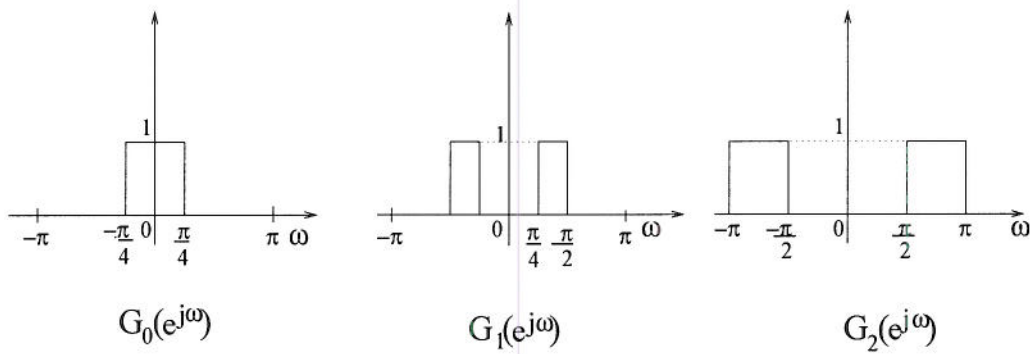


Figure 1b: Fourier transforms of the synthesis filters of Figure 1a.

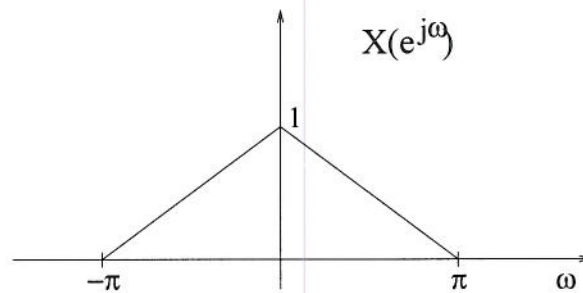


Figure 1c: Spectrum of  $x[n]$ .

(d) Now, the filter bank is iterated on the  $H_0$  branch to form a 2-level decomposition.

i. Draw either the synthesis or the analysis filter bank of the equivalent 5-channel filter bank clearly specifying the transfer functions and downsampling factors.

[3]

ii. If the filters are those shown in Figure 1b, draw the Fourier transform of the equivalent filters of each branch before downsampling.

[3]

2. Consider the two-channel filter bank shown in Figure 2.

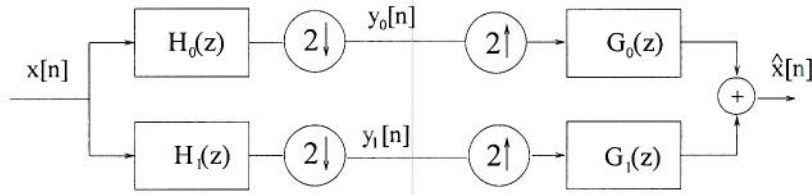


Figure 2: Two-channel filter bank.

- (a) You are first asked to design an orthogonal filter-bank.
- i. Start by designing the shortest possible filter  $G_0(z)$  with a zero at  $\omega = \pi$  and at least one zero at  $\pi/3$ . You want a filter with real-valued coefficients. [Hint: Recall that if the coefficients of a filter  $G(z)$  are real then if  $z_k$  is a complex root of  $G(z)$  so is  $z_k^*$ , where  $*$  denotes the complex conjugate]. [4]
  - ii. Choose  $H_0(z) = G_0(z^{-1})$  and make sure that  $P(z) = H_0(z)G_0(z)$  satisfies the half-band condition:  $P(z) + P(-z) = 2$ . [Hint: you might have to multiply both filters  $H_0(z)$  and  $G_0(z)$  by a proper coefficient  $\alpha$ ]. [4]
  - iii. Design the filters  $H_1(z)$  and  $G_1(z)$  in order to have a perfect reconstruction orthogonal filter-bank. [4]
- (b) You are now willing to design a symmetric/antisymmetric biorthogonal filter bank. Take
- $$P(z) = (z + 1 + z^{-1}) \left( bz^2 + \frac{1}{4}z + \frac{1}{2} + \frac{1}{4}z^{-1} + bz^{-2} \right).$$
- i. Find  $b$  so that  $P(z) + P(-z) = 2$ . [4]
  - ii. Choose  $G_0(z) = (z + 1 + z^{-1})$ . Design the filters  $H_0(z)$ ,  $H_1(z)$  and  $G_1(z)$  in order to have a perfect reconstruction bio-orthogonal filter-bank. [4]

3. *Shannon Multiresolution Analysis.* For  $j \in \mathbb{Z}$ , let  $V_j$  be the space of all finite energy signals  $f$  for which the Fourier transform  $\hat{f}(\omega) = 0$  outside of the interval  $[-2^j\pi, 2^j\pi]$ . We want to show that the function  $\varphi(t) = \frac{\sin \pi t}{\pi t}$  with Fourier transform  $\hat{\varphi}(\omega) = \text{rect}(\omega/2\pi)$  is the scaling function of this multiresolution analysis. We need to show that  $\varphi(t)$  satisfies the three criteria of a valid scaling function. More specifically:

(a) Show that  $\langle \varphi(t - n), \varphi(t - m) \rangle = \delta_{n,m}$ . This is equivalent to showing that  $\{\varphi(t - n)\}_{n \in \mathbb{Z}}$  is an orthonormal basis of the space  $V_0 = \text{span}\{\varphi(t - n)\}_{n \in \mathbb{Z}}$ . [Hint: use Parseval's identity].

[5]

(b) Show that  $\varphi(t)$  satisfies partition of unity:

$$\sum_{n=-\infty}^{\infty} \varphi(t - n) = 1.$$

[5]

(c) Finally, derive the coefficients  $g_0[n]$  that lead to the two-scale equation

$$\varphi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \varphi(2t - n).$$

[5]

(d) Given  $\varphi(t)$  and the two-scale equation, derive the corresponding wavelet  $\psi(t)$ .

[5]

4. Let  $\varphi(t)$  and  $\psi(t)$  be the Haar scaling and wavelet functions, respectively. Let  $V_j$  and  $W_j$  be the spaces generated by  $\varphi_{j,n}(t) = \sqrt{2^{-j}}\varphi(2^{-j}t - n)$ ,  $n \in \mathbb{Z}$  and  $\psi_{j,n}(t) = \sqrt{2^{-j}}\psi(2^{-j}t - n)$ ,  $n \in \mathbb{Z}$ , respectively. Consider the function defined on  $0 \leq t < 2$  given by

$$f(t) = \begin{cases} 0 & 0 \leq t < 1/4 \\ 1 & 1/4 \leq t < 1/2 \\ 0 & 1/2 \leq t < 2. \end{cases}$$

- (a) Decompose  $f(t)$  into its component parts  $W_{-1}$ ,  $W_0$ , and  $V_0$ . In other words, find the coefficients  $c_{0,n}$ ,  $d_{-1,n}$  and  $d_{0,n}$ ,  $n \in \mathbb{Z}$  that leads to the following decomposition

$$f(t) = \sum_{n=0}^1 c_{0,n} \varphi_{0,n}(t) + \sum_{j=-1}^0 \sum_{n=0}^{2^{-j+1}} d_{j,n} \psi_{j,n}(t).$$

[6]

- (b) Verify the Parseval equality. That is, verify that:

$$\|f(t)\|^2 = \sum_n |c_{0,n}|^2 + \sum_{j=-1}^0 \sum_n |d_{j,n}|^2.$$

[6]

- (c) You now want to compress  $f(t)$  using  $R$  bits in total. Assume that  $R$  is large.

- i. Compute an approximated operational distortion-rate curve  $D(R)$  that you obtain by scalar quantizing each of the 8 transform coefficient of the above decomposition and by allocating the same amount of bits to each coefficient. Justify your answer.

[4]

- ii. Devise a non-linear strategy where only the non-zero coefficients are scalar quantized. Derive the new  $D(R)$ . Justify your answer.

[4]



# QUESTIONS

Wavelets & Applications

Solutions 2010

E4-45

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$i=0,1,2$

## QUESTION 1

(a)

$$Y_i(\tau) = \frac{1}{2} H_i(\tau^{1/2}) X(\tau^{1/2}) + \frac{1}{2} H_i(-\tau^{1/2}) X(-\tau^{1/2})$$

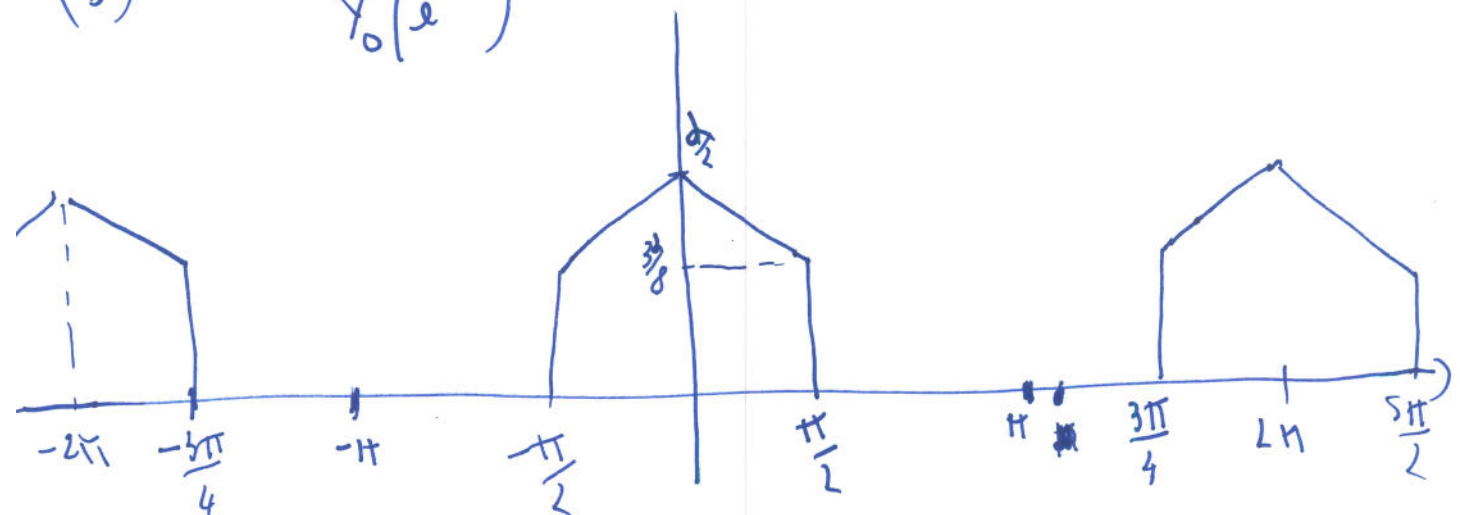
$$\begin{aligned} \hat{X}(\tau) &= \frac{1}{2} X(\tau) (G_0(\tau) H_0(\tau) + G_1(\tau) H_1(\tau) + G_2(\tau) H_2(\tau)) \\ &\quad + \frac{1}{2} X(-\tau) (G_0(\tau) H_0(-\tau) + G_1(\tau) H_1(-\tau) + G_2(\tau) H_2(-\tau)) \end{aligned}$$

PIL CONDITIONS

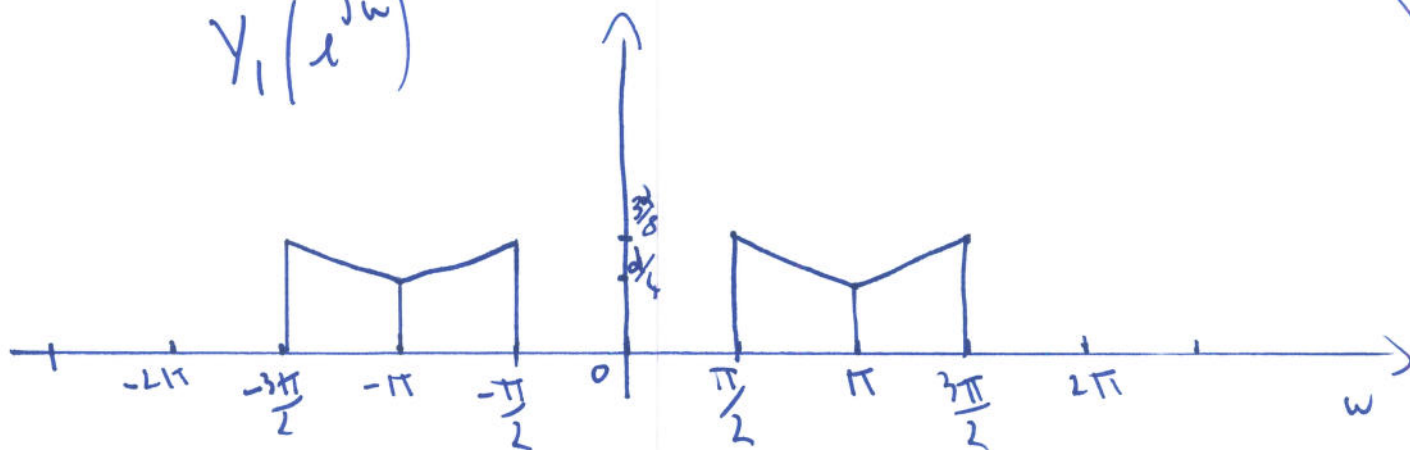
$$\begin{cases} G_0(\tau) H_0(\tau) + G_1(\tau) H_1(\tau) + G_2(\tau) H_2(\tau) = 2 \\ G_0(\tau) H_0(-\tau) + G_1(\tau) H_1(-\tau) + G_2(\tau) H_2(-\tau) = 0 \end{cases}$$

(b)

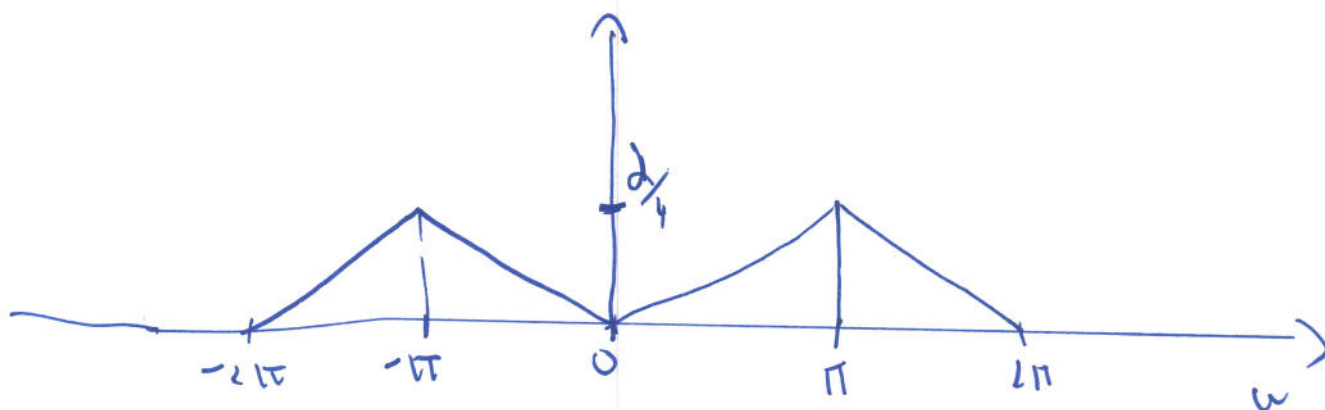
$$Y_0(e^{j\omega})$$



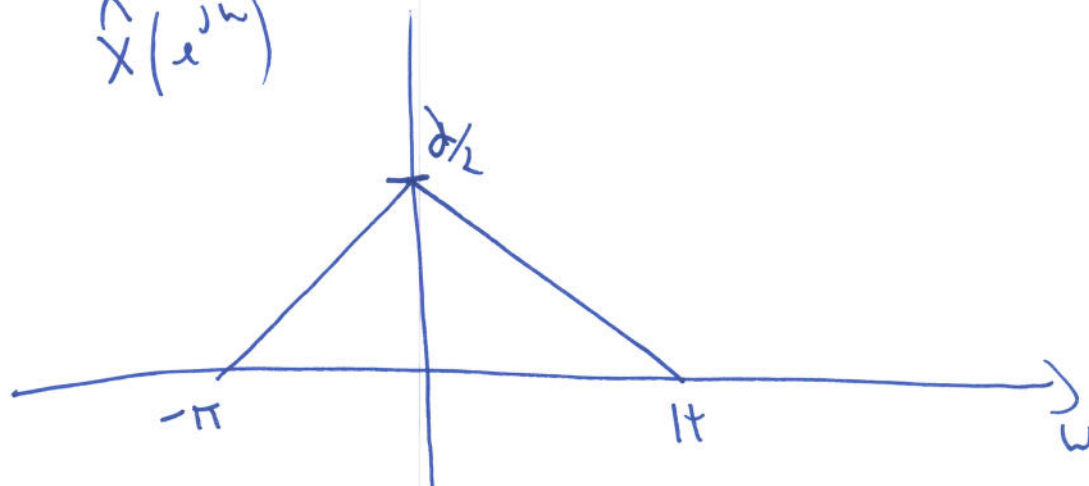
$$Y_1(e^{j\omega})$$



$$Y_2(e^{j\omega})$$



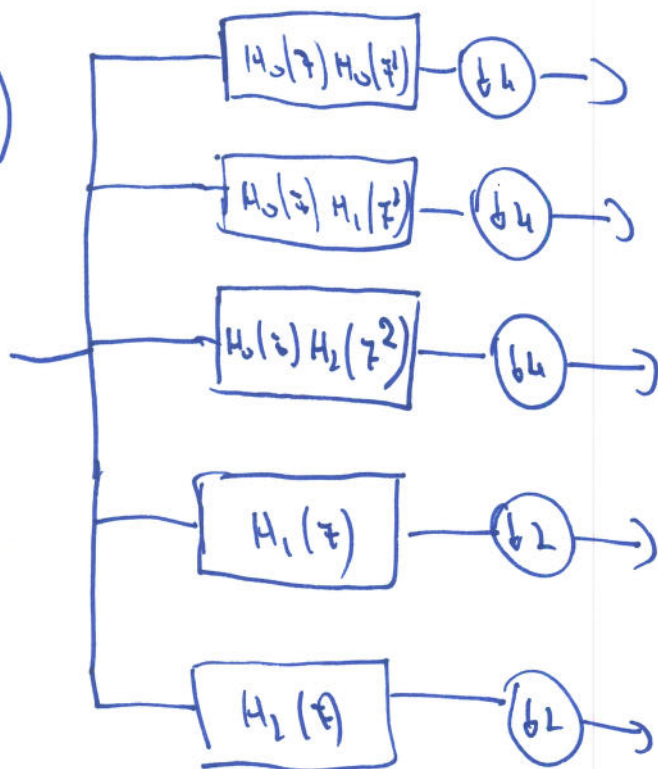
$$\hat{X}(e^{j\omega})$$



c)  $d = 2$  in order to have  $X(z) = \hat{X}(z)$

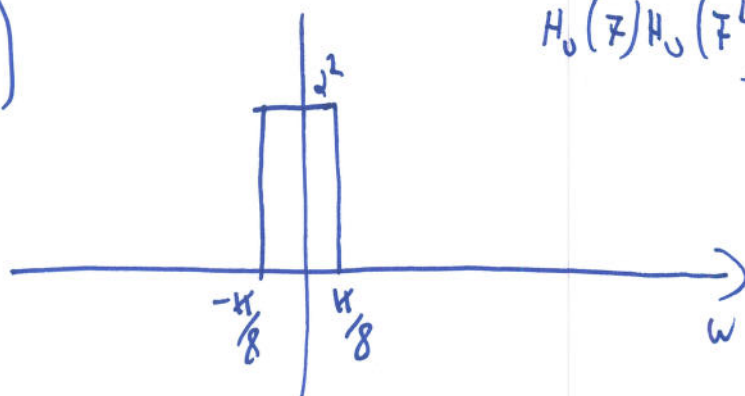
d)

i.)

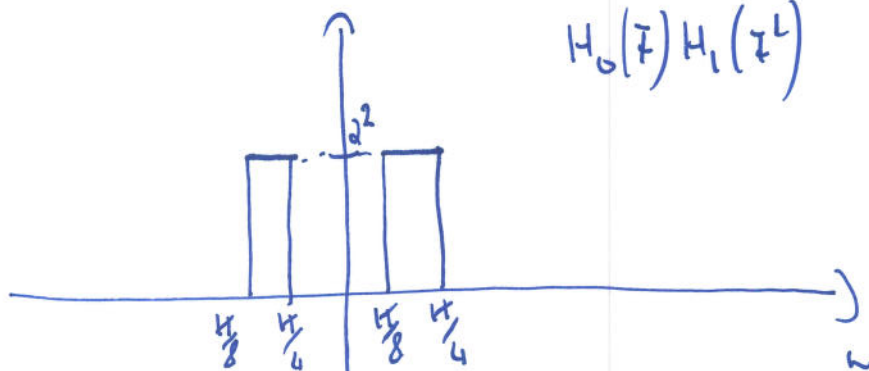


ii.)

$$H_0(z)H_0(z^L)$$



$$H_0(z)H_1(z^L)$$



$$H_0(z)H_2(z^L) = 0$$



$H_1(z)$  and  $H_2(z)$

AS BEFORE

## QUESTION 2

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a) SHORTEST FILTER WITH REAL COEFFICIENTS:

i.)

$$G_0(z) = d(z^{-1} + 1)(z^{-1} - e^{j\frac{\pi}{3}})(z^{-1} - e^{-j\frac{\pi}{3}})$$

$$= d(z^{-1} + 1)(z^{-2} + 2\cos\frac{\pi}{3}z^{-1} + 1)$$

$$= d(z^{-1} + 1)(z^{-2} - z^{-1} + 1)$$

$$= d(1 + z^{-3})$$

$$H_0(z) = G_0(z^{-1}) = d(1 + z^3)$$

ii.)

$$P(z) = (1 + z^{-3})(1 + z^3) = d^2(1 + z^3 + z^{-3} + 1)$$

$$P(z) = P(-z) = 4d^2 = 1 \quad d^2 = \frac{1}{4} \Rightarrow d = \frac{1}{2}$$

$$iii.) \quad G_1(z) = -z^{-1}G_0(z^{-1}) = -\frac{z^{-1}}{\sqrt{2}}(1 - z^3) = \frac{z^{-1}}{\sqrt{2}} - \frac{z^{-1}}{\sqrt{2}}$$

$$H_1(z) = G_1(z^{-1}) = \frac{z^{-2}}{\sqrt{2}} - \frac{z}{\sqrt{2}}$$

QUESTION 2  
b)

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i.) CLEARLY  $p(t) + p(-t) = 2$  WHEN

$$b = -\frac{1}{4}$$

i.i.)

$$H_0(t) = -\frac{1}{4} t^2 + \frac{1}{4} t + \frac{1}{2} + \frac{1}{4} t^{-1} - \frac{1}{4} t^2$$

$$H_1(\bar{t}) = \bar{t} G_0(-\bar{t}) = \bar{t}(-\bar{t} + 1 - \bar{t}^{-1}) = -\bar{t}^2 + \bar{t} - 1$$

$$G_1(\bar{t}) = \bar{t}^{-1} H_0(-\bar{t}) = \bar{t}^{-1} \left( -\frac{1}{4} \bar{t}^2 - \frac{1}{4} \bar{t} + \frac{1}{2} - \frac{1}{4} \bar{t}^{-1} - \frac{1}{4} \bar{t}^2 \right)$$

$$= \left( -\frac{1}{4} \bar{t} - \frac{1}{4} + \frac{1}{2} \bar{t}^{-1} - \frac{1}{4} \bar{t}^{-2} - \frac{1}{4} \bar{t}^{-3} \right)$$

### QUESTION 3

6

(a) PARSEVAL EQUALITY:  $\langle g(t), f^*(t) \rangle = \frac{1}{2\pi} \langle \hat{g}(\omega), \hat{f}^*(\omega) \rangle$ .

THEREFORE, WE HAVE THAT

$$\langle \psi(t-m), \psi(t-m) \rangle = \frac{1}{2\pi} \left\langle \text{RECT}\left(\frac{\omega}{2\pi}\right) e^{-j\omega m}, \text{RECT}\left(\frac{\omega}{2\pi}\right) e^{j\omega m} \right\rangle$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j\omega(m-m)} d\omega = \frac{\sin \pi(m-m)}{\pi(m-m)}$$

$$= \begin{cases} 1 & \text{IF } m=m \\ 0 & \text{OTHERWISE} \end{cases}$$

(b)

USING POISSON SUM FORMULA, WE HAVE THAT

$$\sum_{m=-\infty}^{\infty} \psi(t-m) = \sum_{k=-\infty}^{\infty} \hat{\psi}(2\pi k) e^{j2\pi k t} = \sum_{k=-\infty}^{\infty} \text{RECT}\left(\frac{2\pi k}{2\pi}\right) e^{j2\pi k t}$$



AND CLEARLY

$$\sum_{k=-\infty}^{\infty} \text{RECT}\left(\frac{2\pi k}{2\pi}\right) e^{j2\pi k t} = 1$$

(c)

WE FIRST SHOW THAT

$$g_0[k] = \sqrt{2} \langle \varphi(t), \varphi(2t-k) \rangle.$$

SINCE  $\varphi(t)$  SATISFIES THE TWO-SCALE

$$\text{RELATION } \varphi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \varphi(2t-n), \quad (1)$$

WE HAVE THAT

$$\langle \varphi(t), \varphi(2t-k) \rangle = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \langle \varphi(2t-n), \varphi(2t-k) \rangle$$

WHERE I HAVE USED (1) AND THE LINEARITY OF INNER PRODUCT.

BY REPLACING  $2t = x$  WE HAVE, THAT

$$\begin{aligned} \langle \varphi(t), \varphi(2t-k) \rangle &= \frac{\sqrt{2}}{2} \sum_{n=-\infty}^{\infty} g_0[n] \underbrace{\langle \varphi(x-n), \varphi(x-k) \rangle}_{\delta_{n,k}} = \\ &= \frac{1}{\sqrt{2}} g_0[k], \end{aligned}$$

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WHERE I USED THE FACT THAT  $\varphi(t)$   
IS ORTHOGONAL TO ITS SHIFTS.

THEREFORE

$$g_0[k] = \sqrt{2} \langle \varphi(t), \varphi(2t-k) \rangle$$

USING PARSEVAL WE GET:

$$\begin{aligned} g_0[k] &= \sqrt{2} \langle \varphi(t), \varphi(2t-k) \rangle = \frac{\sqrt{2}}{2\pi} \left\langle \hat{\varphi}(\omega), \frac{1}{2} \hat{\varphi}^*\left(\frac{\omega}{2}\right) e^{j\frac{k\omega}{2}} \right\rangle \\ &= \frac{\sqrt{2}}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} e^{j\frac{k\omega}{2}} d\omega = \frac{\sqrt{2}}{2} \frac{\sin \frac{k\pi}{2}}{\frac{k\pi}{2}} \end{aligned}$$

THEREFORE WE CAN WRITE

$$\varphi(t) = \sum_{m=-\infty}^{\infty} \frac{\sin \frac{m\pi}{2}}{\frac{m\pi}{2}} \varphi(2t-m)$$

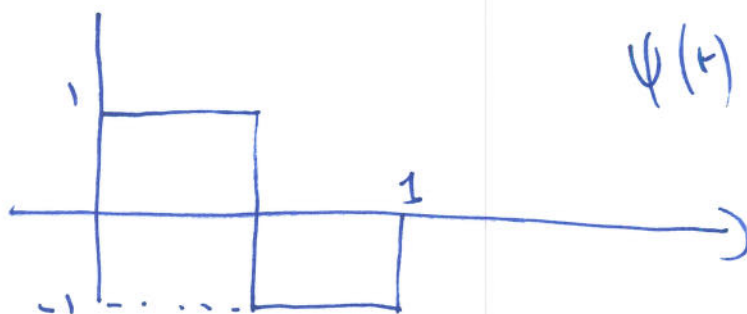
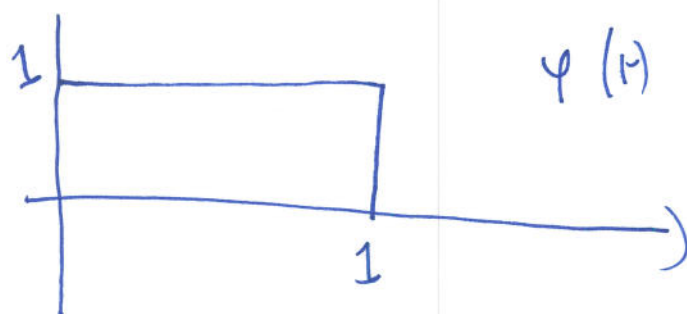
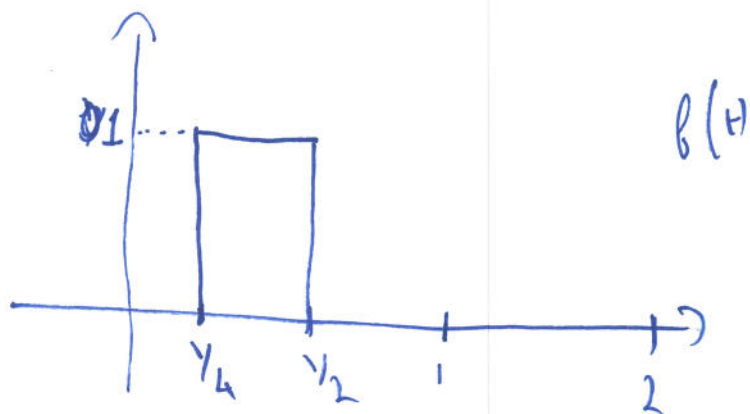
$$\begin{aligned} (d) \quad \psi(t) &= \sqrt{2} \sum_{n=-\infty}^{\infty} (-1)^n g_0[1-n] \varphi(2t-n) \\ &= \sum_{m=-\infty}^{\infty} (-1)^m \frac{\sin \frac{(m-1)\pi}{2}}{\frac{(m-1)\pi}{2}} \varphi(2t-m) \end{aligned}$$



# QUESTION 4

9

(a)



CLEARLY

$$c_{0,n} = \langle f(t), \phi(t-n) \rangle = \begin{cases} 1/4 & \text{For } n=0 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$d_{0,n} = \langle f(t), \psi(t-n) \rangle = \begin{cases} 1/4 & \text{For } n=0 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$d_{-1,m} = \sqrt{2} \langle f(t), \psi(2t-m) \rangle = \begin{cases} -\frac{\sqrt{2}}{4} & m=0 \\ 0 & \text{OTHERWISE} \end{cases}$$

so

$$f(t) = \frac{1}{4} \varphi(t) + \frac{1}{4} \psi(t) \cdot -\frac{\sqrt{2}}{4} \psi_{-1,0}(t).$$

(b)

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{1/4}^{1/2} dt = 1/4$$

$$\|f\|^2 = |c_{0,0}|^2 + |d_{0,0}|^2 + |d_{-1,0}|^2 = \frac{1}{16} + \frac{1}{16} + \frac{2}{16} = \frac{1}{4}.$$

PARSEVAL VERIFIED

(c) i) AT HIGH RATE THE MSE DISTORTION OF A SCALAR QUANTIZER IS

$$D(n) = C 2^{-2R}$$

WHERE  $C$  DEPENDS ON THE STATISTICS OF THE SOURCE AND THE QUANTIZER USED.

BECAUSE THE TOTAL BIT RATE IS  $R$   
AND WE HAVE 8 COEFFICIENTS TO  
QUANTIZE. THEREFORE THE DISTORTION  
RELATED TO EACH OF THEM IS  $D(n) = C 2^{-2R/8}$

~~STRAIGHT~~ BECAUSE OF PARSEVAL IDENTITY WE  
HAVE THAT:

$$D(n) = E[|f - \hat{f}|^2] = E[|c_{0,0} - \hat{c}_{0,0}|^2 + |d_{0,0} - \hat{d}_{0,0}|^2 + |d_{-1,0} - \hat{d}_{-1,0}|^2]$$

WHERE  $\hat{c}_{0,0}$ ,  $\hat{d}_{0,0}$ ,  $\hat{d}_{-1,0}$  ARE THE  
QUANTIZED COEFFICIENTS.

WE THEREFORE HAVE THAT

$$D(n) \approx (C_1 + C_2 + C_3) 2^{-2R/8} = C 2^{-2R/8}$$

i.i. WE CAN ALTERNATIVELY QUANTIZE ONLY  
THE NON-ZERO COEFFICIENTS.

WE NEED 3 BITS TO LOCATE EACH  
NON-ZERO COEFFICIENT. THIS MEANS THAT WE

NEED 9 BITS TO LOCATE THEM.

WE CAN THEN USE THE REMAINING  $n-9$  BITS  
TO QUANTIZE THEM. THIS LEADS TO THE FOLLOWING  
D(n) CURVE:  $D(n) \approx C 2^{-2(n-9)/3}$