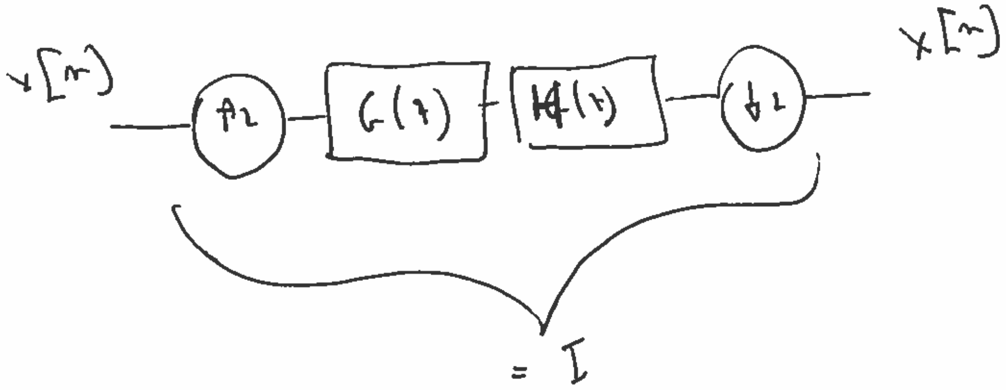
ANSWER

EE4-45 Wavelets & Applications

QUESTION 1

(a) THE SYSTEM IS IDEMPOTENT IF AND ONLY IF



THIS IMPLIES:

$$\frac{1}{2} X(z)\left[G(z^{1/2})H(z^{1/2}) + G(-z^{1/2})H(-z^{1/2})\right] = X(z)$$

$$\Rightarrow \quad H(z)\,G(z) + G(-z)H(-z) = 2 \qquad (*)$$

(b) SINCE WE ARE ASSUMING $H(z) = G(z^{-1})$ CONDITION (*) BECOMES

$$G(z)\,G(z^{-1}) + G(-z)\,G(-z^{-1}) = 2$$

$G(z)$ HAS ~~...~~ 4-TAPS AND TWO ZEROS AT $z=-1$

THIS IMPLIES

$$G(z) = a(1+z)^2 (a_0 z + b_0)$$

DEFINE

$$P(z) = G(z)\,G(z^{-1}) = (1+z)^2 (1+z^{-1})^2 (a z^{-1} + b + a z)$$

WE FIND a AND b USING

$$P(z) + P(-z) = 2$$

$$P(z) = \big(a z^{-3} + b z^{-2} + a z^{-1} + 4a z^{-2} + 4b z^{-1} + 4a + 6a z^{-1}$$
$$+ 6b + 6a z + 4a + 4b z + 4a z^2 + a z + b z^2 + a z^3\big)$$

AND USING $P(z) + P(-z) = 2$

WE HAVE

$$\begin{cases} b + 4a = 0 \\ 8a + 6b = 1 \end{cases} \Rightarrow \begin{aligned} a &= -1/16 \\ b &= 1/4 \end{aligned}$$

SO $P(z) = \frac{1}{16}(1+z)^2 (1+z^{-1})^2 (-z^{-1} + 4 - z)$

~~ROOTS OF~~ $\cancel{-x^2 + 4x - 1 = 0} \Rightarrow$

$$(-z^{-1} + 4 - z) = (z - \alpha)(z^{-1} - \alpha)$$

WITH $\alpha = 2 \pm \sqrt{3}$

THUS

$$G(z) = \frac{1}{4}\left(1+z^{-1}\right)^2\left(z^{-1} - a\right)$$

$$H(z) = \frac{1}{4}\left(1+z\right)^2\left(z - a\right)$$

WITH $\quad a = 2 \pm \sqrt{3}$

(b) ~~THE~~ BECAUSE OF ORTHOGONALITY

THE WAVELET BRANCH

IS

$$G_1(z) = -z^{-1}\,G_0\left(z^{-1}\right)$$

$$H_1(z) = G_1\left(z^{-1}\right)$$

QUESTION 2

(a) $$P(z) = \frac{1}{2\sqrt{2}} (1+z)^2 (1+z^{-1})^2 B(z)$$

IF $B(z) = a \rightarrow P(z) + P(-z) \neq 2$

SO LET'S TRY

$$B(z) = (a z^{-1} + b + a z)$$

$$P(z) = \frac{1}{2\sqrt{2}} (1 + 2z + z^2)(1 + 2z^{-1} + z^{-2})(a z^{-1} + b + a z)$$

$$= \Big[a z^{-3} + (b + 4a) z^{-2} + (7a + 4b) z^{-1} + (8a + 6b) + (7a + 4b) z + (b + 4a) z^2 + a z^3\Big] / 2\sqrt{2}$$

THE HALF-BAND CONDITION IMPLIES THAT:

$$\left.\begin{aligned} & b + 4a = 0 \\ & \frac{1}{2\sqrt{2}} (8a + 6b) = 1 \end{aligned}\right\} \Rightarrow \quad a = -\frac{2\sqrt{2}}{16} \qquad b = \frac{2\sqrt{2}}{4}$$

$$P(z) = \frac{1}{16} (1+z)^2 (1+z^{-1})^2 (-z^{-1} + 4 - z)$$

AND

$$H_0(z) = \frac{1}{4\sqrt{2}} (1+z)(1+z^{-1})(-z + 4 - z^{-1})$$

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(b) $H_1(z) = z\, G_0(-z) = \frac{1}{2\sqrt{2}}\, z(1-z)(1-z^{-1})$

$G_1(z) = z^{-1} H_0(-z) = \frac{z^{-1}}{4\sqrt{2}} (1-z)(1-z^{-1})(z+4+z^{-1})$

(c) ROOTS OF z^2+4z+1 ARE

$z_0 = -2-\sqrt{3}$ AND $z_1 = -2+\sqrt{3}$

NOTE THAT $z_0 = 1/z_1$

THUS

$$P(z) = \frac{1}{16a} (1+z)^2 (1+z^{-1})^2 (1 - z_0 z^{-1})(1 - z_0 z)$$

WITH $a = (-2+\sqrt{3})$

THUS

$$G_0(z) = \frac{1}{4\sqrt{a}} (1+z)^2 (1 - z_0 z^{-1})$$

$H_0(z) = G_0(z^{-1})$

$G_1(z) = -z^{-1} G_0(-z^{-1})$

$H_1(z) = G_1(z^{-1})$

(d) WE NEED $H_1(z)$ TO HAVE 3 ZEROS AT ~~ZERO~~ $\omega=0$ $\Rightarrow$ $G_0(z)$ MUST HAVE 3 ZEROS AT $\omega=\pi$.

SOLUTION

$$G_0(z) = \frac{1}{4\sqrt{2}}(1+z^{-1})^2(1+z)$$

$$H_0(z) = \frac{\sqrt{2}}{4}(1+z)(-z+4-z^{-1})$$

$$H_1(z) = z\,G_0(-z) = \frac{1}{4\sqrt{2}}z(1-z^{-1})^2(1-z) \quad \left[\begin{array}{c}\text{3 ZEROS}\\ \text{AT } \omega=0\end{array}\right]$$

$$G_1(z) = z^{-1}H_0(-z)$$

QUESTION 3

(a) LET'S START WITH $\tilde{\varphi}_1(t)$

i. WE WANT

$$\langle \tilde{\varphi}_1(t), \varphi_j(t) \rangle = \delta_{1,j} \qquad j = 1, 2, 3.$$

SINCE

$$\tilde{\varphi}_1(t) = \sum_{k=1}^{3} \alpha_{1,k}\, \varphi_k(t)$$

WE HAVE

$$\sum_{k=1}^{3} \alpha_{1,k} \langle \varphi_k(t), \varphi_j(t) \rangle = \delta_{1,j} \qquad j = 1, 2, 3.$$

WE NOTE THAT

$$\langle \varphi_1(t), \varphi_1(t) \rangle = \frac{2}{3}$$

$$\langle \varphi_2(t), \varphi_1(t) \rangle = \frac{1}{6}$$

$$\langle \varphi_3(t), \varphi_1(t) \rangle = \frac{1}{6}$$

BECAUSE OF CIRCULAR SYMMETRY WE CONCLUDE THAT

$$\langle \varphi_i(t), \varphi_j(t) \rangle = \begin{cases} \frac{2}{3} & i = j \\ \frac{1}{3} & i \neq j \end{cases}$$

CONSEQUENTLY THE SYSTEM OF EQUATIONS:

$$\sum_{k=1}^{3} \alpha_{1,k} \langle \varphi_k(t), \varphi_j(t) \rangle = \delta_{1,j} \qquad j=1,2,3$$

REDUCES TO:

$$\begin{cases} \frac{2}{3}\alpha_1 + \frac{\alpha_2}{6} + \frac{\alpha_3}{6} = 1 \\ \frac{\alpha_1}{6} + \frac{2}{3}\alpha_2 + \frac{\alpha_3}{6} = 0 \\ \frac{\alpha_1}{6} + \frac{\alpha_2}{6} + \frac{2}{3}\alpha_3 = 0 \end{cases}$$

WITH SOLUTION:

$$\alpha_1 = \frac{5}{3} \quad \alpha_2 = -\frac{1}{3}, \quad \alpha_3 = -\frac{1}{3}.$$

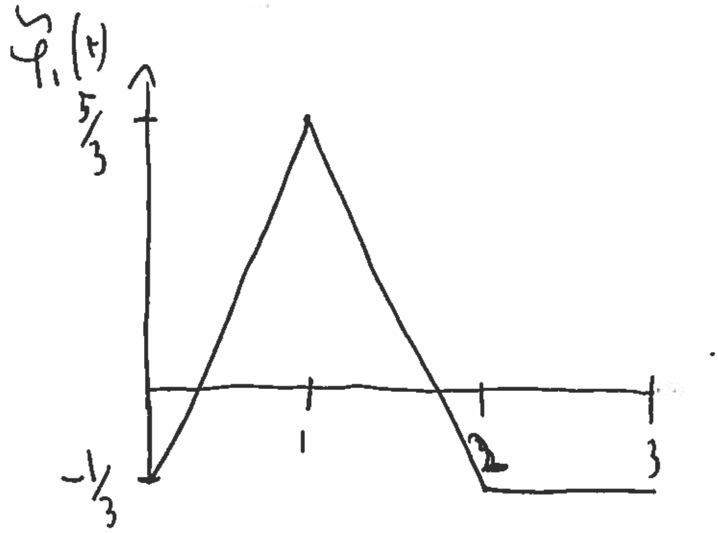
THIS IMPLIES

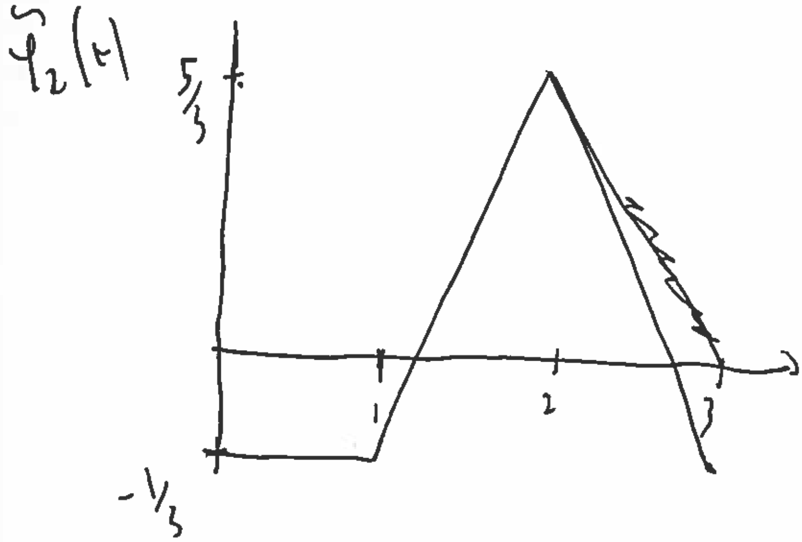
$$\tilde{\varphi}_1(t) = \frac{5}{3}\varphi_1(t) - \frac{1}{3}\varphi_2(t) - \frac{1}{3}\varphi_3(t).$$

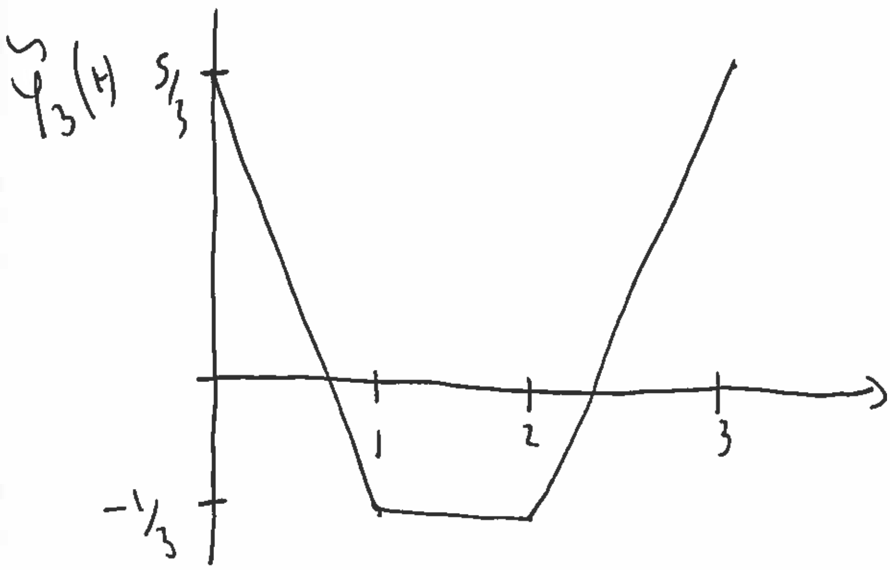
$\tilde{\varphi}_2(t)$ AND $\tilde{\varphi}_3(t)$ ARE OBTAINED BY CIRCULAR SHIFT BY ONE OF $\tilde{\varphi}_1(t)$

ii.

THE SKETCH OF THE THREE FUNCTIONS IS AS FOLLOWS



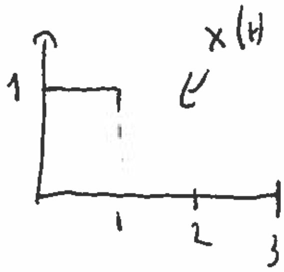




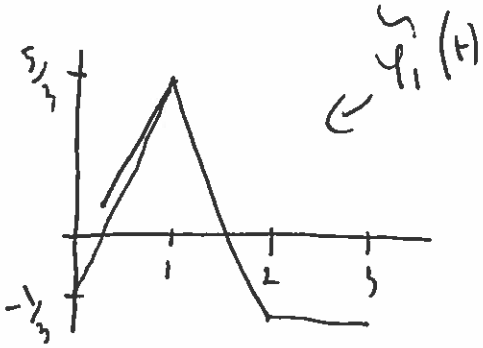
(b)

i. $\langle x(t), \tilde{\varphi}_1(t) \rangle$

SINCE



AND



WE HAVE THAT

$$\langle x(t), \tilde{\varphi}_1(t) \rangle = \int_0^1 \left(2t - \frac{1}{3}\right) dt = 1 - \frac{1}{3} = \frac{2}{3}$$

SIMILARLY

$$\langle x(t), \tilde{\varphi}_2(t) = -\frac{1}{3} \int_0^1 dt = -\frac{1}{3}$$

AND

$$\langle x(t), \tilde{\varphi}_3(t) \rangle = \int_0^1 \left(-2t + \frac{5}{3}\right) dt = -1 + \frac{5}{3} = \frac{2}{3}$$

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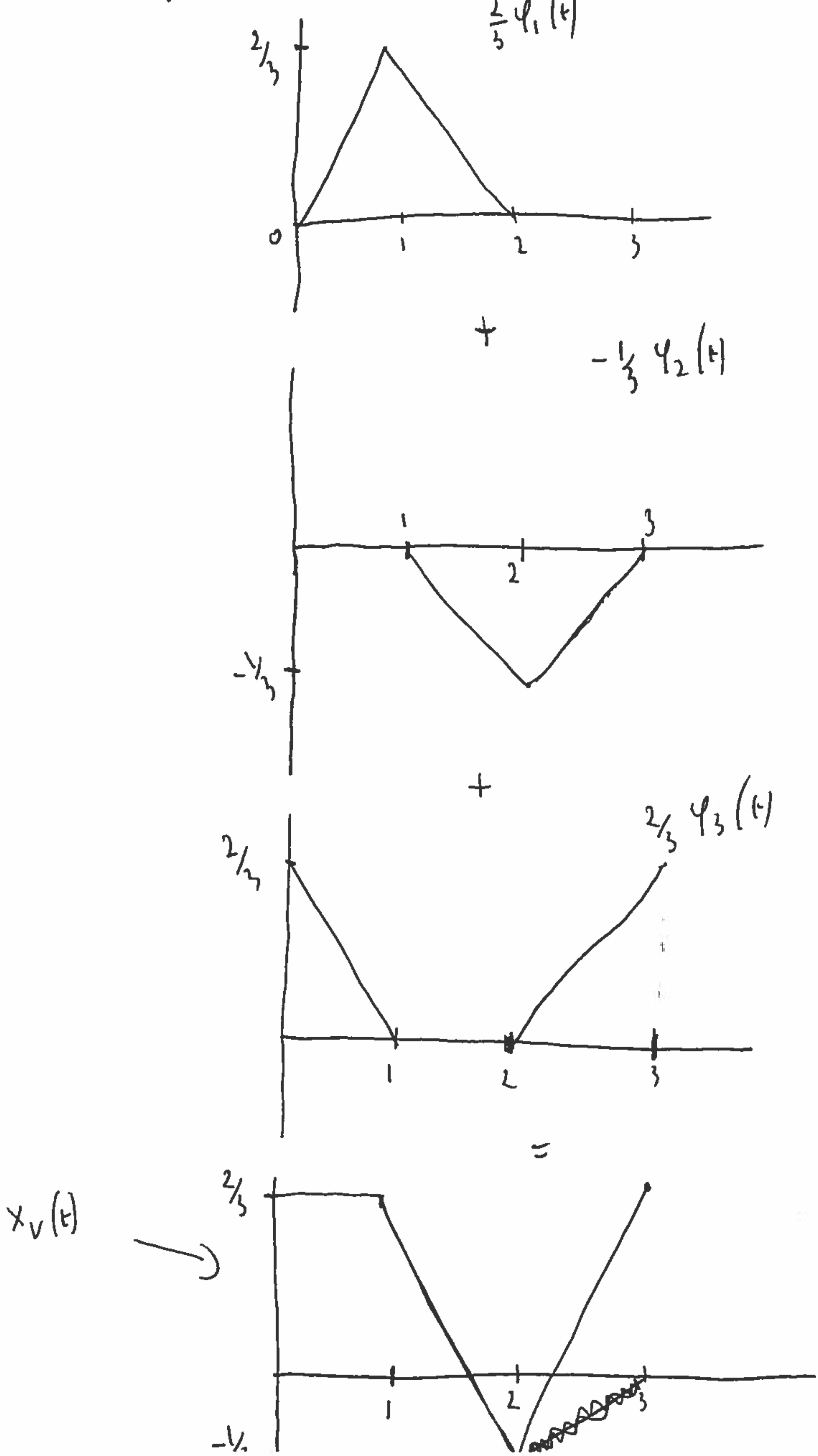
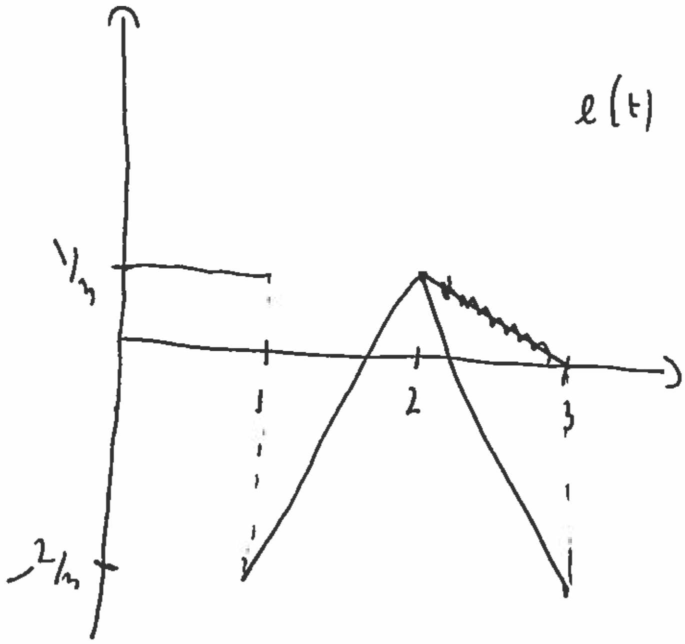
ii. GRAPHICALLY

$x_v(t) =$



(iii)

$$e(t) = x(t) - x_V(t)$$



$e(t) \perp V$ IIF $\langle e(t), \varphi_i(t) \rangle = 0 \quad i = 1, 2, 3.$

$$\langle e(t), \varphi_1(t) \rangle = \frac{1}{3} \int_0^1 t \, dt + \int_1^2 \left(t - \frac{5}{3}\right)(2-t) \, dt$$

$$= \frac{1}{6} - \frac{1}{6} = 0$$

FOR SIMMETRY $\langle e(t), \varphi_3(t) \rangle = 0$

AND

$$\langle e(t), \varphi_2(t) \rangle = 2 \int_1^2 (t-1)\left(t - \frac{2}{3} - 1\right) dt = \int_0^1 t\left(t - \frac{2}{3}\right) dt = 0$$

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QUESTION 4

i)

$$g_0[n] = \frac{1}{4\sqrt{2}}\left(\delta_n + 3\delta_{n-1} + 3\delta_{n-2} + \delta_{n-3}\right)$$

THUS

$$G_0(z) = \frac{1}{4\sqrt{2}}\left(1+z^{-1}\right)^3$$

AND

$$G_0\left(e^{jw}\right)\Big|_{w=0} = \frac{1}{\sqrt{2}} \quad \text{AND} \quad G_0\left(e^{jw}\right)\Big|_{w=\pi} = 0$$

THE TWO NECESSARY CONDITIONS FOR THE LIMIT TO EXIST ARE SATISFIED

ii) MOREOVER,

IF WE DENOTE WITH $M_0(w) = \frac{G_0\left(e^{jw}\right)}{\sqrt{2}}$

WE HAVE THAT

$$M_0(w) = \left(\frac{1+e^{-jw}}{2}\right)^N R(w)$$

WITH $B = \sup R(w) = 1$

SINCE $B < 2^{N-2} = 2$

WE KNOW THAT A SUFFICIENT CONDITION FOR $\varphi(t)$ TO CONVERGE TO A ~~CONTINUOUS~~ FUNCTION ~~IS~~ ~~SATISFIED~~ CONTINUOUS WITH ITS FIRST ORDER DERIVATIVE IS SATISFIED.

iii)

THE ANALYSIS WAVELET TRANSFORM IS

$$\tilde{\psi}(t) = \sum_n h_0^*[n]\, \tilde{\varphi}(2t-n)$$

WITH

$$h_0[n] = (-1)^n g_0[1-n] \iff \underbrace{H_0(t) = -t\, G_0(-t)}_{\text{NOTE THAT BOTH } h_0[n] \text{ AND } g_0[n] \text{ ARE SYMMETRIC}}$$

$H_0(t)$ HAS THREE ZEROS AT $\omega=0 \Rightarrow \tilde{\psi}(t)$ HAS THREE VANISHING MOMENTS.

(b)

$$
\left| \langle f, \psi_{m,n} \rangle \right| = \left| \underbrace{\langle p_{t_0}(t), \psi_{m,n}(t) \rangle}_{=0} + \langle \epsilon(t), \psi_{m,n}(t) \rangle \right|
$$

$$
\overset{(a)}{\leq} K 2^{-m/2} \int_{-\infty}^{\infty} |t - t_0|^{\alpha} \psi(2^{-m}t - n) dt
$$

$$
= K 2^{m/2} \int_{-\infty}^{\infty} |x 2^m + n 2^m - t_0|^{\alpha} \psi(x) dx
$$

THE VANISHING MOMENTS PROPERTY OF $\psi(t)$

$$
\overset{(b)}{\leq} K C 2^{m(\alpha+1/2)} \underbrace{\int_{-\infty}^{\infty} (|x| + |C|)^{\alpha} \psi(x) dx}_{=A}
$$

$$
= C_1 2^{m(\alpha+1/2)}
$$

where (a) follows from [illegible] and (b) from the fact that we are in the cone of influence of t_0 and therefore $|n2^m - t_0| \leq C2^m$.

(c) $$\left| \langle f(t), \psi_{m,n}(t) \rangle \right| = \left| \int_{-\infty}^{\infty} \frac{1}{\sqrt{2^m}} \psi(2^{-m}t - n)\, \delta(t - t_0)\, dt \right|$$

$$= \frac{1}{\sqrt{2^m}} \left| \psi(2^{-m} t_0 - n) \right|$$ WHEN

$\psi_{m,n}(t)$ IS IN THE CONE OF INFLUENCE OF t_0.

DENOTING WITH $B = \max |\psi(t)|$

WE HAVE THAT

$$\left| \langle f, \psi_{m,n} \rangle \right| \leq B\, 2^{-\frac{m}{2}}.$$

SO THE COEFFICIENTS INCREASE FOR LARGE NEGATIVE m RATHER THEN DECREASING AS IN (b).