

EE4-26

## SOLUTIONS: ESTIMATION AND FAULT DETECTION

### 1. Solution

- a) By inspection of the block-diagram shown in Fig. 1.1 of the text of the exam paper, we immediately obtain

$$W(s) = \frac{G(s)}{1 + G(s)H(s)} E(s)$$

and

$$Y(s) = [1 + C(s)]U(s)$$

Then

$$Y(s) = [1 + C(s)] \frac{G(s)P(s)}{1 + G(s)H(s)} [R(s) - Y(s)]$$

and, after a few algebraic manipulations, we obtain

$$Y(s) = \frac{[1 + C(s)]G(s)P(s)}{1 + G(s)H(s) + [1 + C(s)]G(s)P(s)}$$

[ 3 marks ]

- b) The block given by the transfer function

$$\frac{W(s)}{E(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{1}{s + 10}$$

can be described by the state equations

$$\begin{cases} \dot{x}_1 = -10x_1 + e \\ w = x_1 \end{cases}$$

whereas the block given by the transfer function

$$\frac{U(s)}{W(s)} = P(s) = \frac{s + 10}{(s + 3)^2} = \frac{s + 10}{s^2 + 6s + 9}$$

can be described by the state equations

$$\begin{cases} \begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -9 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 10 \\ 1 \end{bmatrix} w \\ u = [0 \ 1] \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \end{cases}$$

Now, using

$$e = r - x_4 - u; \quad u = x_3; \quad w = x_1$$

we finally obtain

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -10 & 0 & -1 & -1 \\ 10 & 0 & -9 & 0 \\ 1 & 1 & -6 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r \\ u = [0 \ 0 \ 0 \ 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \end{cases}$$

[ 3 marks ]

- c) The output feedback connection  $e = r - y$  does not modify the observability properties of the overall system. Hence, we remove the feedback connection and focus on the open-loop system from the input  $e$  to the output  $y$ . More specifically, let us consider the state equations of the sub-system from the input  $e$  to the output  $u$ :

$$\begin{cases} \dot{x}_1 = -10x_1 + e \\ \dot{x}_2 = 10x_1 - 9x_3 \\ \dot{x}_3 = x_1 + x_2 - 6x_3 \\ u = x_3 \end{cases} \Rightarrow \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -10 & 0 & 0 \\ 10 & 0 & -9 \\ 1 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e \\ u = [0 \ 0 \ 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{cases}$$

Letting

$$A = \begin{bmatrix} -10 & 0 & 0 \\ 10 & 0 & -9 \\ 1 & 1 & -6 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad C = [0 \ 0 \ 1]$$

the observability matrix is given by

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -6 \\ -6 & -6 & 27 \end{bmatrix}$$

As

$$\det Q = 0$$

we conclude that the sub-system from the input  $e$  to the output  $u$  is not completely observable and hence the overall system is not completely observable as well.

[ 7 marks ]

- d) Exploiting the answer to Question 1c), let us first determine the non-observable poles of the sub-system from the input  $e$  to the output  $u$  (having removed the feedback connection  $e = r - y$ ). To proceed, we determine the observability canonical form. To this end, we determine a basis for  $\ker(Q)$ :

$$Qv = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -6 \\ -6 & -6 & 27 \end{bmatrix} v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Now, we look for a basis  $\{\alpha, \beta\}$  for the orthogonal complement to  $\ker(Q)$ . For example:

$$\alpha = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \quad \beta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Selecting the matrix

$$T = [\alpha | \beta | v] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

and computing the inverse

$$T^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & -1/2 & 0 \end{bmatrix}$$

By setting  $x = Tz$ , we obtain the following equivalent observability canonical form:

$$\begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = T^{-1}ATz + T^{-1}Be = \begin{bmatrix} 0 & -9/2 & 0 \\ 2 & -6 & 0 \\ -10 & 9/2 & -\mathbf{10} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix} e \\ u = CTz = [0 \ 1 \ 0] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \end{cases}$$

Hence, the non-observable pole of the sub-system from the input  $e$  to the output  $u$  (having removed the feedback connection  $e = r - y$ ) is  $p_{n.o.} = -10$  (it corresponds to the eigenvalue  $-10$  highlighted in boldface in the above matrix  $T^{-1}AT$ ).

As the feedback connection  $e = r - y$  does not modify the observability properties of the whole system, the non-observable pole  $p_{n.o.} = -10$  of the sub-system from the input  $e$  to the output  $u$  belongs to the set of non-observable poles of the whole system.

[ 7 marks ]

2. Solution

a) The system can be written in as:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = Ax + Bu = \begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ u = Cx = [1 \ 3] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

The observability matrix is given by

$$Q_x = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -4 & -1 \end{bmatrix}$$

As

$$\det Q_x \neq 0$$

we conclude that the system is completely observable. Hence a full-order state observer can be designed so that the estimation error  $e(t) = x(t) - \hat{x}(t)$  satisfies  $e(t) \rightarrow 0$  for  $t \rightarrow \infty$ .

[ 3 marks ]

b) We have

$$\det(sI - A) = \det \begin{bmatrix} s+1 & -2 \\ 1 & s+1 \end{bmatrix} = s^2 + 2s + 3$$

Then, the matrices  $A_o$  and  $C_o$  of the observer canonical form are

$$A_o = \begin{bmatrix} 0 & -3 \\ 1 & -2 \end{bmatrix}; \quad C_o = [0 \ 1]$$

The observability matrix  $Q_o$  computed on the basis of the pair  $(A_o, C_o)$  is

$$Q_o = \begin{bmatrix} C_o \\ C_o A_o \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

Using  $Q_x$  determined in the answer to Question 2a), the matrix  $T_o$  transforming the original state equations into the observer canonical form

$$\begin{cases} \dot{z} = A_o z + B_o u \\ y = C_o z \end{cases} \quad \text{with} \quad A_o = T_o^{-1} A T_o; B_o = T_o^{-1} B; C_o = C T_o$$

is given by

$$T_o = Q_x^{-1} Q_o = \begin{bmatrix} -3/11 & 5/11 \\ 1/11 & 2/11 \end{bmatrix} \quad \text{and} \quad T_o^{-1} = \begin{bmatrix} -2 & 5 \\ 1 & 3 \end{bmatrix}$$

Now considering

$$L_o = \begin{bmatrix} l_o^{(1)} \\ l_o^{(2)} \end{bmatrix}$$

we have

$$\det[sI - (A_o - L_o C_o)] = s^2 + (-2 + l_o^{(2)})s + (-3 + l_o^{(1)})$$

and by equating this polynomial with the polynomial having the desired observer eigenvalues as roots, that is

$$\alpha_d(s) = (s+2)(s+4) = s^2 + 6s + 8$$

we immediately obtain

$$l_o^{(1)} = 5; \quad l_o^{(2)} = 4$$

Finally, the observer gain  $L$  such that  $F = A - LC$  has the desired eigenvalues  $\lambda_1 = -2, \lambda_2 = -4$  is given by

$$L = T_o L_o = \begin{bmatrix} 5/11 \\ 13/11 \end{bmatrix}$$

[ 8 marks ]

- c) The reduced-order observer provides an estimate  $\hat{x}_2$  of  $x_2$ . The estimate  $\hat{x}_1$  of  $x_1$  will be obtained through the output equation

$$\hat{x}_1 = y - 3\hat{x}_2$$

As

$$C = [1 \ 3]$$

we immediately consider the following change of state vector

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T x = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

leading to the equivalent state equations

$$\begin{cases} \dot{z} = \tilde{A}z + \tilde{B}u \\ y = \tilde{C}z \end{cases}$$

with

$$\tilde{A} = TAT^{-1} = \begin{bmatrix} -4 & 11 \\ -1 & 2 \end{bmatrix}; \tilde{B} = TB = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \tilde{C} = CT^{-1} = [1 \ 0]$$

The reduced order observer takes on the form

$$\begin{cases} \dot{\xi} = f\xi + hy + gu \\ \hat{z}_2 = \xi + ly \end{cases}$$

where  $f = \tilde{f} = -2$ . The observer gain  $l$  is obtained as follows:

$$f = \tilde{a}_{22} - l\tilde{a}_{12} \implies -2 = 2 - 11l \implies l = \frac{4}{11}$$

The scalar  $h$  is computed as

$$fl = h + l\tilde{a}_{11} - \tilde{a}_{21} \implies -\frac{8}{11} = h - \frac{16}{11} + 1 \implies h = -\frac{3}{11}$$

Finally, the scalar  $g$  is computed as

$$g = \tilde{b}_2 - l\tilde{b}_1 = -\frac{4}{11}$$

[ 9 marks ]

3. Solution

- a) The general algebraic Riccati equation is

$$P = F \left[ P - PH^T (V_2 + HPH^T)^{-1} HP \right] F^T + V_1$$

Letting  $F = 1/2$ ,  $H = -1/3$ ,  $V_1 = 1$ ,  $V_2 = 1$ , we have

$$P = \frac{1}{4} \left( P - \frac{\frac{1}{9}}{1 + \frac{1}{9}P} P^2 \right) + 1 \implies P^2 + \frac{31}{5}P - 9 = 0$$

thus obtaining the two solutions

$$\tilde{P}_1 \simeq -7.41 \quad \text{and} \quad \tilde{P}_2 \simeq 1.21$$

Clearly, the only admissible solution is the positive one. Thus  $\tilde{P} = \tilde{P}_2 \simeq 1.21$ . Accordingly:

$$\tilde{K} = F\tilde{P}H^T (V_2 + H\tilde{P}H^T)^{-1} \simeq -0.18$$

[ 4 marks ]

- b) We have

$$\text{var}[x(t) - \hat{x}(t|t-1)] = \tilde{P} \simeq 1.21$$

Let us now compute  $\text{var}[x(t)]$ . The stochastic process  $x(\cdot)$  generated by the system

$$x(t+1) = \frac{1}{2}x(t) + v_1(t)$$

is stationary because  $v_1(\cdot) \sim WGN(0, 1)$  and because the system is asymptotically stable.

Because of the stationarity of  $x(\cdot)$ ,  $\text{var}[x(t)] = \text{var}[x(t-1)]$  and hence

$$\text{var}[x(t)] = \frac{1}{4} \text{var}[x(t)] + 1 \implies \text{var}[x(t)] = \frac{4}{3}$$

Now:

$$\text{var}[x(t) - \hat{x}(t|t-1)] \simeq 1.21 < \frac{4}{3} = \text{var}[x(t)]$$

As can be seen, the Kalman estimator allows to reduce the variance with respect to the a-priori one thanks to the use of the measurements  $y(t)$ .

[ 5 marks ]

- c) The Kalman predictor obeys to the following equations:

$$\begin{cases} \hat{x}(t+1|t) = \frac{1}{2}\hat{x}(t|t-1) + \tilde{K}e(t) \\ \hat{y}(t+1|t) = -\frac{1}{3}\hat{x}(t+1|t) \\ e(t) = y(t) + \frac{1}{3}\hat{x}(t|t-1) \end{cases}$$

and thus

$$\hat{x}(t+1|t) = \frac{1}{2}\hat{x}(t|t-1) + \tilde{K} \left[ y(t) + \frac{1}{3}\hat{x}(t|t-1) \right]$$

where  $\tilde{K} \simeq -0.18$ . The block-diagram of the steady-state one-step ahead Kalman predictor is shown in Fig. 3.1.

[ 3 marks ]

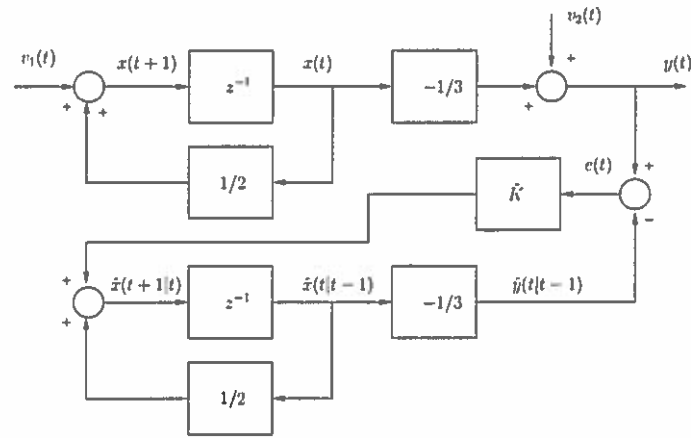


Figure 3.1 Block-diagram of the steady-state one-step ahead Kalman predictor.

- d) Consider the system affected by an *unknown constant fault*  $\tilde{u}$ :

$$\begin{cases} x(t+1) = \frac{1}{2}x(t) + v_1(t) + \tilde{u} \\ y(t) = -\frac{1}{3}x(t) + v_2(t) \end{cases}$$

Clearly, the following equation holds:

$$u(t+1) = u(t), \quad \text{with } u(0) = \tilde{u}$$

We interpret this equation as an *additional* state equation thus getting:

$$\begin{cases} x(t+1) = \frac{1}{2}x(t) + v_1(t) + u(t) \\ u(t+1) = u(t) \\ y(t) = -\frac{1}{3}x(t) + v_2(t) \end{cases}$$

Introducing the new state vector

$$z = \begin{bmatrix} x \\ u \end{bmatrix}$$

we get:

$$\begin{cases} \begin{bmatrix} z_1(t+1) \\ z_2(t+1) \end{bmatrix} = A_z z(t) + B_z v_1(t) = \begin{bmatrix} 1/2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_1(t) \\ y(t) = C_z z(t) = [-1/3 \ 0] \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \end{cases}$$

The observability matrix is given by

$$Q_z = \begin{bmatrix} C_z \\ C_z A_z \end{bmatrix} = \begin{bmatrix} -1/3 & 0 \\ -2/3 & -1/3 \end{bmatrix}$$

As

$$\det Q_z \neq 0$$

we conclude that the "augmented" system is completely observable. Hence a Kalman estimator can be designed so as to simultaneously provide an estimate of the state  $x$  and of the constant fault  $\tilde{u}$ .

[ 8 marks ]

4. Solution

a) The full-order observer takes on the form:

$$\begin{cases} \dot{\hat{x}} = A^{(0)}\hat{x} + B^{(0)}u + L^{(0)}(y - C\hat{x}) \\ \hat{y} = C^{(0)}\hat{x} \end{cases}$$

where

$$A^{(0)} = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}; \quad B^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad C^{(0)} = [1 \ 0]$$

are the matrices of the system described by state equations (4.2) in the text of the exam paper.  $L^{(0)}$  denotes the observer gain matrix to be designed. The pair  $(A^{(0)}, C^{(0)})$  is completely observable as

$$\det \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = 2$$

We let

$$F = A^{(0)} - L^{(0)}C^{(0)} = \begin{bmatrix} -1-l_1 & 2 \\ -l_2 & -3 \end{bmatrix} \quad \text{where} \quad L^{(0)} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

After some algebra, we obtain that by selecting

$$L^{(0)} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

we have  $\det(sI - F) = s^2 + 2s + 1$  and hence  $\lambda_1 = -1, \lambda_2 = -1$ .

Finally

$$e(t) = \mathcal{L}^{-1}[(sI - F)^{-1}] \bar{e} = \mathcal{L}^{-1} \begin{bmatrix} \frac{s+3}{(s+1)^2} & \frac{2}{(s+1)^2} \\ -\frac{2}{(s+1)^2} & \frac{s-1}{(s+1)^2} \end{bmatrix} \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \end{bmatrix}$$

and hence, after some algebra and computing the inverse Laplace transforms, we get

$$\begin{aligned} \varepsilon(t) &= Ce(t) = \mathcal{L}^{-1} \left[ \frac{s+3}{(s+1)^2} \right] \bar{e}_1 - 2\mathcal{L}^{-1} \left[ \frac{1}{(s+1)^2} \right] \bar{e}_2 \\ &= (2t+1)e^{-t}\bar{e}_1 - 2te^{-t}\bar{e}_2, \quad \forall t \in (0, T_0) \end{aligned}$$

| 6 marks |

b) Consider a fault  $f = f_1$  or  $f = f_2$  occurring at some finite time  $T_0$ . Then, the dynamics of the state estimation error after the occurrence of the fault (that is, for  $t \geq T_0$ ) can be written as

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = Fe(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(t)$$

and thus, exploiting the answer to Question 4a), we have

$$\varepsilon(t) = (2t+1)e^{-t}\bar{e}_1 - 2te^{-t}\bar{e}_2 + 2\mathcal{L}^{-1} \left[ \frac{1}{(s+1)^2} F(s) \right]$$

where  $F(s)$  denotes the Laplace transform of the fault function  $f(t)$ . From this relationship, it follows immediately that the residual  $\varepsilon(t)$  does not vanish asymptotically either under the action of fault  $f_1$  or fault  $f_2$ .



Now, as  $|x(0)| \leq 10$ , for  $t < T_0$  we can write

$$|\varepsilon(t)| = |(2t+1)e^{-t}\tilde{e}_1 - 2te^{-t}\tilde{e}_2| \leq (2t+1)e^{-t}|\tilde{e}_1| + 2te^{-t}|\tilde{e}_2| \leq 10(4t+1)e^{-t}$$

Then, letting

$$\tilde{\varepsilon}(t) = 10(4t+1)e^{-t}$$

and observing that  $\tilde{\varepsilon}(t) \rightarrow 0$  for  $t \rightarrow \infty$ , it is immediate to conclude that for  $t \geq T_0$

$$\exists T_d > T_0 : \varepsilon(T_d) > \tilde{\varepsilon}(T_d)$$

[ 6 marks ]

c) Fault  $f_1$  can be rewritten as

$$f_1(t) = \sin(t) \cdot 1(t) = \mathcal{L}^{-1} \left( \frac{1}{s^2 + 1} \right)$$

A state-space description of the above dynamic system generating  $f_1(t)$  is

$$\begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ f_1 = [0 \ 1] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{cases}$$

Hence, we can write the state equation of the whole system affected by the fault  $f_1$  as

$$\begin{cases} \dot{x}_1 = -x_1 + 2x_2 + u \\ \dot{x}_2 = -3x_2 + z_2 \\ \dot{z}_1 = -z_2 \\ \dot{z}_2 = z_1 \\ y = x_1 \end{cases}$$

The observability matrix  $Q_1$  of the system affected by the fault  $f_1$  can thus be computed:

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -8 & 0 & 2 \\ -1 & 26 & 2 & -8 \end{bmatrix}$$

As  $\det(Q_1) = -8$ , the state space description of the faulty system is completely observable. Thus, an asymptotic observer can be designed providing the estimates of the state variables associated with the fault  $f_1$ .

Now, consider fault  $f_2$ . It can be rewritten as

$$f_2(t) = 1(t) = \mathcal{L}^{-1} \left( \frac{1}{s} \right)$$

A state-space description of the above dynamic system generating  $f_1(t)$  is

$$\begin{cases} \dot{z}_1 = 0 \\ f_2 = z_1 \end{cases}$$

Hence, we can write the state equation of the whole system affected by the fault  $f_2$  as

$$\begin{cases} \dot{x}_1 = -x_1 + 2x_2 + u \\ \dot{x}_2 = -3x_2 + z_1 \\ \dot{z}_1 = 0 \\ y = x_1 \end{cases}$$

The observability matrix  $Q_2$  of the system affected by the fault  $f_2$  can thus be computed:

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & -8 & 2 \end{bmatrix}$$

As  $\det(Q_2) = 4$ , the state space description of the faulty system is completely observable. Thus, an asymptotic observer can be designed providing the estimates of the state variables associated with the fault  $f_2$ .

[ 8 marks ]