

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

EEE/ISE PART I: MEng, BEng and ACGI

COMMUNICATIONS 1

Corrected Copy

Friday, 12 June 10:00 am

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Q1 is compulsory.

Answer Q1 and any two of questions 2-4.

Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	P.L. Dragotti, P.L. Dragotti
	Second Marker(s) :	M.K. Gurcan, M.K. Gurcan

Special Information for the Invigilators: none

Information for Candidates

Some Fourier Transforms

$$\cos \omega_0 t \quad \Longleftrightarrow \quad \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$\text{rect}\left(\frac{t}{\tau}\right) \quad \Longleftrightarrow \quad \tau \text{sinc}\left(\frac{\omega\tau}{2}\right)$$

$$\frac{W}{\pi} \text{sinc}(Wt) \quad \Longleftrightarrow \quad \text{rect}\left(\frac{\omega}{2W}\right)$$

$$\frac{\alpha^2}{2\pi} \text{sinc}^2\left(\frac{\alpha t}{2}\right) \quad \Longleftrightarrow \quad \Delta\left(\frac{\omega}{\alpha}\right)$$

where

$$\Delta(\omega) = \begin{cases} 1 - |\omega|, & |\omega| \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

The Scaling Property of the Fourier Transform

$$g(\alpha t) \Longleftrightarrow \frac{1}{|\alpha|} G\left(\frac{\omega}{\alpha}\right).$$

Time-Shifting Property of the Fourier Transform

$$g(t - t_0) \Longleftrightarrow G(\omega)e^{-j\omega t_0}$$

Time differentiation

$$\frac{d^n g}{dt^n} \Longleftrightarrow (j\omega)^n G(\omega)$$

Some useful trigonometric identities

$$\cos x \cos y = \frac{1}{2} \cos(x - y) + \frac{1}{2} \cos(x + y).$$

$$\sin x \sin y = \frac{1}{2} \cos(x - y) - \frac{1}{2} \cos(x + y)$$

$$\sin x \cos y = \frac{1}{2} \sin(x - y) + \frac{1}{2} \sin(x + y)$$

The Questions

1. This question is compulsory.

- (a) Consider the following two signals: $x_1(t) = \Delta(t-1)$ and $x_2(t) = \Delta(t-2)$, where $\Delta(t)$ is given by

$$\Delta(t) = \begin{cases} 1 - |t| & \text{for } |t| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the signals are also sketched in Figure 1.

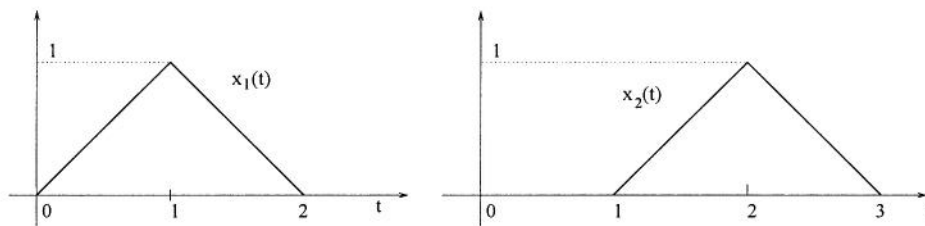


Figure 1: The two signals $x_1(t)$ and $x_2(t)$.

- i. Compute the energy of $x_1(t)$. [4]
- ii. Compute the energy of $x_2(t)$. [4]
- iii. Compute the energy of $x_1(t) + x_2(t)$. [4]

Question 1 continues on next page

(b) The Fourier transform of the triangular pulse $x(t)$ in Figure 2(a) is

$$X(\omega) = \frac{1}{\omega^2}(e^{j\omega} - j\omega e^{j\omega} - 1).$$

Using this information, the scaling property and the time-shifting property, find the Fourier transform of the signal $y(t)$ shown in Figure 2(b). Notice that $y(t)$ is real and even, so you expect $Y(\omega)$ to be real and even as well.

[4]

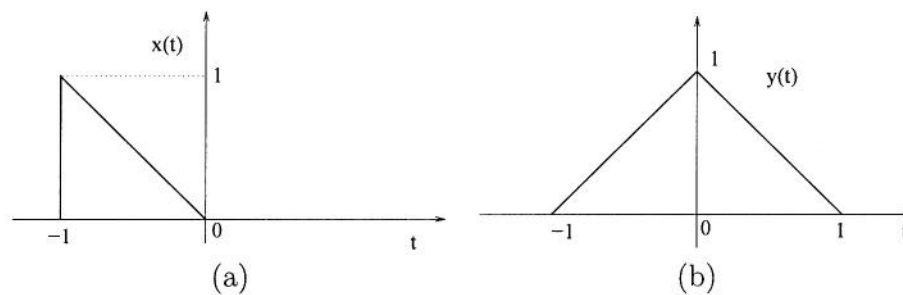


Figure 2: The two signals $x(t)$ and $y(t)$.

(c) Consider the linear time-invariant system $h(t)$ where the input $x(t)$ and output $y(t)$ are related by the following linear differential equation:

$$\frac{dy(t)}{dt} + y(t) = x(t).$$

i. Find the transfer function of $h(t)$. Recall that the transfer function is defined as $H(\omega) = Y(\omega)/X(\omega)$.

[4]

ii. Assume that the Power Spectral Density (PSD) of the input is $S_x(\omega) = \text{rect}(\omega/2)$. Compute the power of $x(t)$.

[4]

iii. Compute the power of the output signal $y(t)$.

[4]

Question 1 continues on next page

(d) Consider the following full-AM signal:

$$\varphi(t) = (A + m(t)) \cos \omega_c t,$$

where $m(t) = \frac{t}{1+t^2}$. The modulation index is $\mu = 0.5$. Find the amplitude A .

[4]

(e) Sketch the PM and FM waves for the modulating signal shown in Figure 3. The constants k_f and k_p are $2\pi 10^5$ and 10π respectively and the carrier frequency is $f_c = 100\text{MHz}$

[4]

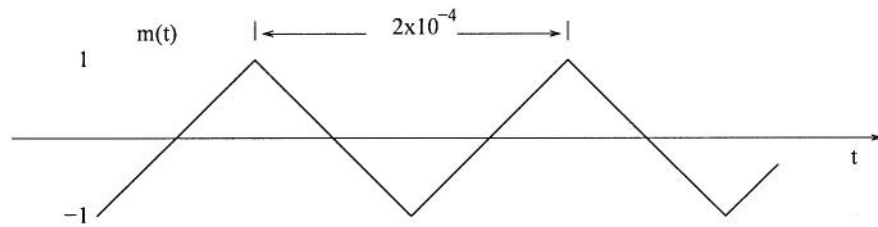


Figure 3: The modulating signal $m(t)$.

(f) A $50\ \Omega$ transmission line is connected to a $100\ \Omega$ line with a matched termination. A sine wave of 1 V amplitude propagating in the former is incident on the junction. Find

i. The voltage reflection coefficient k_v .

[2]

ii. The fraction of the incident power which is transmitted into the second line.

[2]

2. Consider the FM signal

$$\varphi(t) = 10 \cos[2\pi f_c t + k_f \int_{-\infty}^t x(\alpha) d\alpha].$$

Assume that $k_f = 100\pi$, $f_c = 1\text{MHz}$ and that the modulating signal is given by

$$x(t) = 100\text{sinc}(1000\pi t).$$

(a) Determine the frequency deviation Δf .

[7]

(b) Using Carson's rule, determine the bandwidth of $\varphi(t)$.

[7]

(c) The signal $\varphi(t)$ is fed to the 'Armstrong Modulator' shown in Figure 4. The output of the non-linear device is $g(t) = \varphi(t) + \varphi^2(t) + \varphi^3(t)$. The bandpass filter $H(\omega)$ is shown in Figure 5.

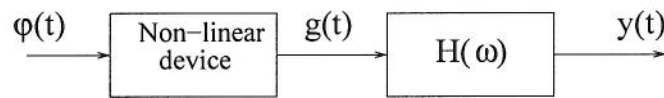


Figure 4: The 'Armstrong modulator'.

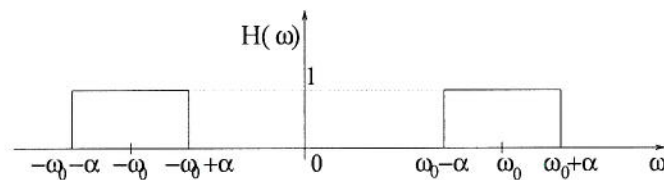


Figure 5: The band-pass filter $H(\omega)$.

Question 2 continues on next page

Determine the bandwidth α and centre frequency ω_0 of the bandpass filter $H(\omega)$ so that the resulting FM signal $y(t)$ has standard deviation $\Delta f = 10000\text{Hz}$. Use Carson's rule to calculate the bandwidth of the FM signals.

[8]

- (d) Assume now that the signal $\varphi(t)$ is fed to the modulator shown in Figure 6. Find the frequency ω_3 , the bandwidth and centre frequency of the two bandpass filters $H_1(\omega)$ and $H_2(\omega)$ that would give an FM signal $y(t)$ with carrier $f_c = 4.5\text{MHz}$ and frequency deviation $\Delta f = 15000\text{Hz}$. The output of the non-linear device is $g(t) = z(t) + z^2(t) + z^3(t)$. Use Carson's rule to calculate the bandwidth of the FM signals.

[8]

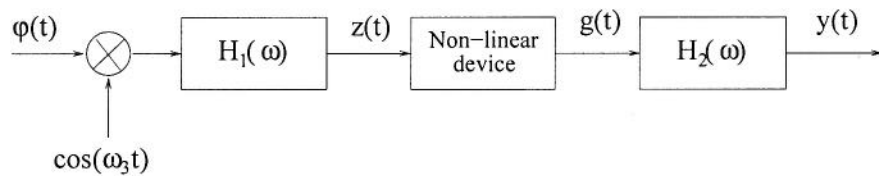


Figure 6: FM modulator.

3. The signals $x_1(t) = \frac{300}{\pi} \text{sinc}(300t)$ and $x_2(t) = 3 \cos 60t + 2 \cos 100t$ are applied at the input of the two ideal low-pass filters $H_1(\omega) = \text{rect}(\omega/100)$ and $H_2(\omega) = \text{rect}(\omega/140)$ (see also Figure 7). The outputs of these two filters are multiplied to obtain the signal $y(t) = y_1(t)y_2(t)$.

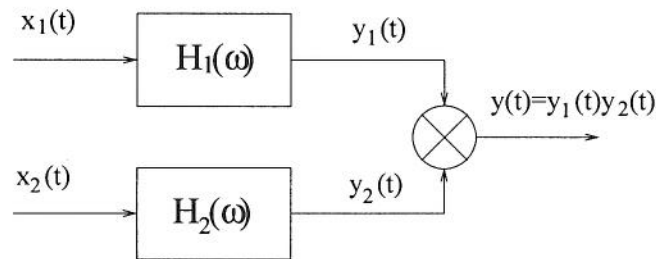


Figure 7: Filtering of $x_1(t)$ and $x_2(t)$.

- (a) Sketch and dimension the Fourier transform of $x_1(t)$ [6]
- (b) Sketch and dimension the Fourier transform of $x_2(t)$ [6]
- (c) Sketch and dimension the Fourier transform of $y_1(t)$ [6]
- (d) Sketch and dimension the Fourier transform of $y_2(t)$ [6]
- (e) Sketch and dimension the Fourier transform of $y(t)$ [3]
- (f) Write the exact time-domain expression of $y(t)$. [3]

4. Consider the demodulator shown in Figure 8, where $y_1(t) = x^2(t)$, $y_3(t) = \sqrt{y_2(t)}$ and the frequency response of the filter $H(\omega)$ is

$$H(\omega) = \begin{cases} 1 & \text{for } |\omega| \leq 6 \text{ rad/s} \\ 0 & \text{otherwise} \end{cases}$$

Assume that $x(t) = m(t) \cos(\omega_c t) + m(t) \sin(\omega_c t)$, where $m(t) = A - \text{sinc}(3t)$ and $\omega_c = 100$.

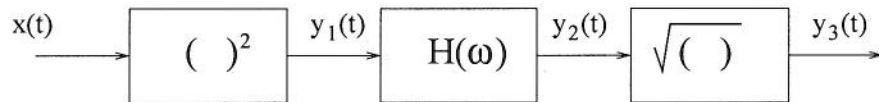


Figure 8: Demodulator.

- (a) Write the exact expression of $y_1(t)$. [6]
- (b) Write the exact expression of $y_2(t)$. [6]
- (c) Write the exact expression of $y_3(t)$. [6]
- (d) Find the minimum value of A that guarantees that $y_3(t) = m(t)$. [6]
- (e) Now assume that $A = 2$, determine the minimum value of ω_c that guarantees that $y_3(t) = m(t)$. Justify your answer. [6]

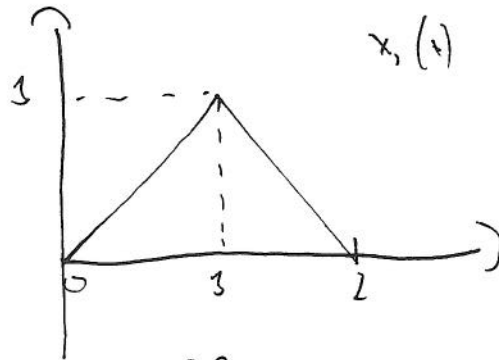
SOLUTIONS

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QUESTION 1

E1.6 Communications I

2009



$$\begin{aligned} \text{i)} \quad E_{x_1} &= \int_{-\infty}^{\infty} x_1^2(t) dt = \int_0^2 x_1^2(t) dt = 2 \int_0^1 t^2 dt = 2 \left. \frac{t^3}{3} \right|_0^1 = \\ &= \frac{2}{3} \end{aligned}$$

$$\text{ii)} \quad E_{x_2} = E_{x_1} = \frac{2}{3} \quad \text{SINCE A SHIFT IN TIME DOES}$$

NOT CHANGE THE ENERGY OF A SIGNAL

$$\text{iii)} \quad E_{x_1+x_2} = E_{x_1} + E_{x_2} + 2 \int_{-\infty}^{\infty} x_1(t) x_2(t) dt ;$$

$$\int_{-\infty}^{\infty} x_1(t) x_2(t) dt = \int_1^2 x_1(t) x_2(t) dt = \int_0^1 t(1-t) dt =$$

$$\int_0^1 t \, dt - \int_1^2 t^2 \, dt = \left. \frac{t^2}{2} \right|_0^1 - \left. \frac{t^3}{3} \right|_1^2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$E_{x_1, x_2} = E_{x_1} + E_{x_2} + 2 \int_{-\infty}^{\infty} x_1(t) x_2(t) \, dt = \frac{2}{3} + \frac{2}{3} + 2 \cdot \frac{1}{6} = \frac{5}{3}$$

b)

$$y(t) = x(t-1) + x(-(t-1))$$

$$x(t-1) \Leftrightarrow X(\omega) e^{-j\omega}$$

$$x(-(t-1)) \Leftrightarrow X(-\omega) e^{j\omega}$$

$$Y(\omega) = X(\omega) e^{-j\omega} + X(-\omega) e^{j\omega} =$$

$$= \frac{1}{\omega^2} \left(1 - j\omega - e^{-j\omega} + 1 + j\omega - e^{j\omega} \right)$$

$$= \frac{1}{\omega^2} \left(2 - e^{j\omega} - e^{-j\omega} \right) = 4 \frac{\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}} \right)^2}{(j\frac{\omega}{2})^2} =$$

$$= \left(\text{sinc} \frac{\omega}{2} \right)^2$$

- c) i. WE TAKE THE FOURIER TRANSFORM ON BOTH SIDES:

$$j\omega Y(\omega) + Y(\omega) = X(\omega)$$

\Downarrow

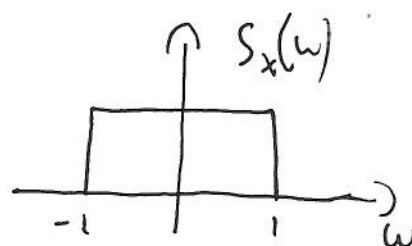
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1+j\omega}$$

ii.

BY DEFINITION OF PSD WE KNOW THAT:

$$P_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) d\omega$$

$$S_x(\omega) = \text{RECT}\left(\frac{\omega}{2}\right) \rightarrow$$



THUS

$$P_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{RECT}\left(\frac{\omega}{2}\right) d\omega = \frac{1}{2\pi} \int_{-1}^1 d\omega = \frac{1}{\pi}$$

iii.

$$P_y = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_y(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 S_x(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-1}^1 \frac{1}{1+\omega^2} d\omega = \frac{1}{2\pi} \left[\tan^{-1} \omega \right]_{-1}^1 = \frac{1}{2\pi} \cdot \frac{\pi}{2} = \frac{1}{4}$$

a)

4

$\mu = \frac{m_p}{A}$. WE NEED TO FIND $m_p = \max |m(t)|$

$$\frac{d m(t)}{d t} = \frac{1+t^2 - 6(2t)}{(1+t^2)^2} = 0 \Rightarrow t = \pm 1$$

THUS $m_p = \frac{1}{2}$

$$A = \frac{1/2}{1/2} = 1$$

2)

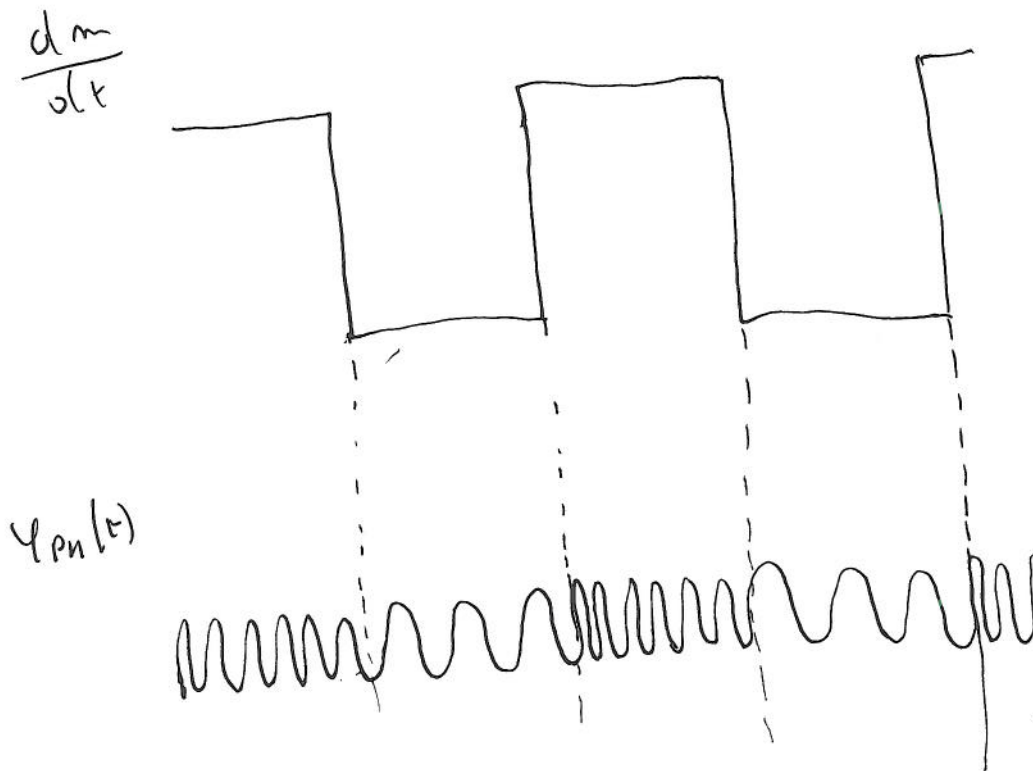
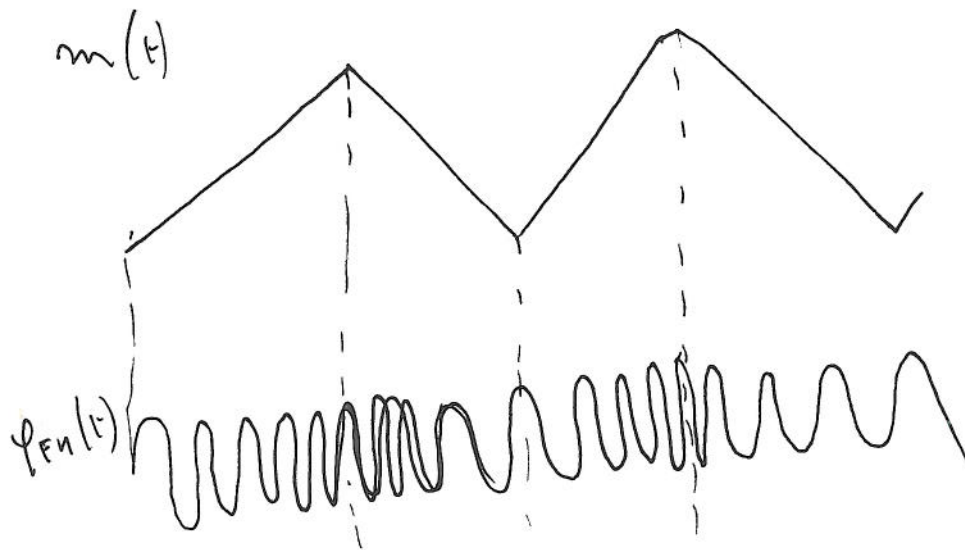
IN BOTH CASES THE MINIMUM AND MAXIMUM INSTANTANEOUS FREQUENCIES ARE :

$$(f_i)_{\min} = 99.9 \text{ MHz}$$

$$(f_i)_{\max} = 100.1 \text{ MHz}$$

- IN THE FM CASE THE INSTANTANEOUS FREQUENCY DEPENDS LINEARLY ON $m(t)$.
- IN THE PM CASE IT DEPENDS ON $\frac{d m(t)}{d t}$

THUS:



$$b) \text{ i. } K_V = \frac{z_L - z_0}{z_L + z_0} = \frac{100 - 50}{100 + 50} = \frac{1}{3}$$

$$\text{ii. } W_p = 1 - |K_V|^2 = 1 - \frac{1}{9} = \frac{8}{9}$$

$$W_{\text{ob}} T_p = \frac{V_L I_L}{V_t I_t} = 1 - W_p = \frac{1}{9}$$

2.

$$(a) \quad \Delta f = \frac{k_f x_p}{2\pi}$$

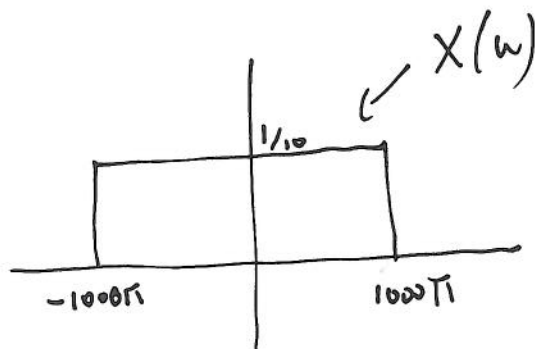
$$x_p = 100, \quad k_f = 100\pi$$

THUS WE HAVE

$$\Delta f = \frac{100\pi \cdot 100}{2\pi} = 5000 \text{ Hz}$$

$$(b) \quad B_{FM} = 2(\Delta f + B)$$

$$100 \text{ sinc } 1000\pi t \Leftrightarrow \frac{1}{10} \text{RECT}\left(\frac{\omega}{2000\pi}\right)$$



$$B = \frac{1000\pi}{2\pi} = 500 \text{ Hz}$$

$$B_{FM} = 2(5000 + 500) = 11000 \text{ Hz}$$

(c)

$$g(t) = \varphi(t) + \varphi^2(t) + \varphi^3(t) =$$

$$= 10 \left[a_1 \cos \left(2\pi f_c t + K_f \int x(z) dz \right) + a_2 \cos \left(4\pi f_c t + 2K_f \int x(z) dz \right) + a_3 \cos \left(6\pi f_c t + 3K_f \int x(z) dz \right) \right] \quad (1)$$

WE WANT $\Delta f = 10000 \text{ Hz}$, WE

THUS NEED TO KEEP THE SECOND TERM OF (1). THE BANDWIDTH OF THIS

SECOND TERM IS

$$B_{FM} = 2(10000 + 500) = 21000 \text{ Hz}$$

THUS

$$\frac{\omega_p}{2\pi} = 2 \cdot \frac{\omega_c}{2\pi} = 2 \cdot 10500 \text{ Hz} = 21000 \text{ Hz}$$

$$\frac{\Delta}{2\pi} = 21000 \text{ Hz} / 2 \quad H_f = 10500 \text{ Hz}$$

(d) WE NOW NEED TO KEEP THE THIRD TERM IN (1) TO OBTAIN

$$\Delta f = 15000 \text{ Hz}.$$

HOWEVER THIS GIVE US $\Delta f_c = 3 \text{ MHz}$.

WE THUS CHOOSE $\omega_3 = 2\pi \cdot 5 \cdot 10^5 \text{ rad/s}$

$$\Rightarrow f_3 = 5 \cdot 10^5 \text{ Hz}$$

$$\varphi(t) \cos \omega_1 t = \frac{10}{2} \cos \left((\omega_c - \omega_3) t + \psi_f \left(\int x(\lambda) d\lambda \right) \right) + \frac{10}{2} \cos \left((\omega_c + \omega_3) t + \psi_f \left(\int x(\lambda) d\lambda \right) \right)$$

NOW $f_c + f_3 = 1.5 \text{ MHz}$ WHICH IS THE FREQUENCY WE NEED.

$$B_{\varphi(t)} = 11000 \text{ Hz}$$

SO $H_1(\omega)$ HAS CENTRAL FREQUENCY

$$\Rightarrow f_1 = 1.5 \text{ MHz}$$

AND BANDWIDTH $B = 11000 \text{ Hz}$ (BANDWIDTH OF $\varphi(t)$)

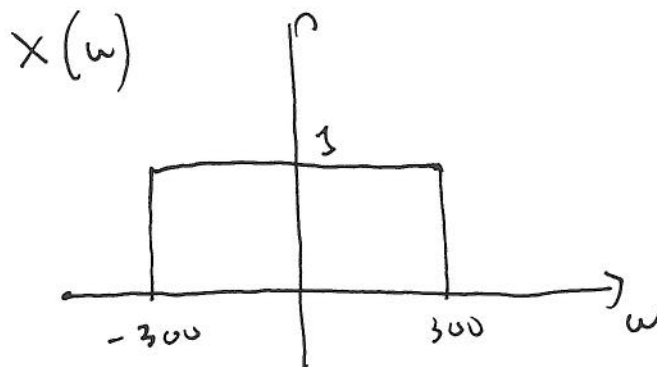
$H_2(\omega)$ HAS CENTRAL FREQUENCY $f_2 = 4.5 \text{ MHz}$

AND BANDWIDTH $B_{\varphi} = 2(15000 + 500) = 46000 \text{ Hz}$

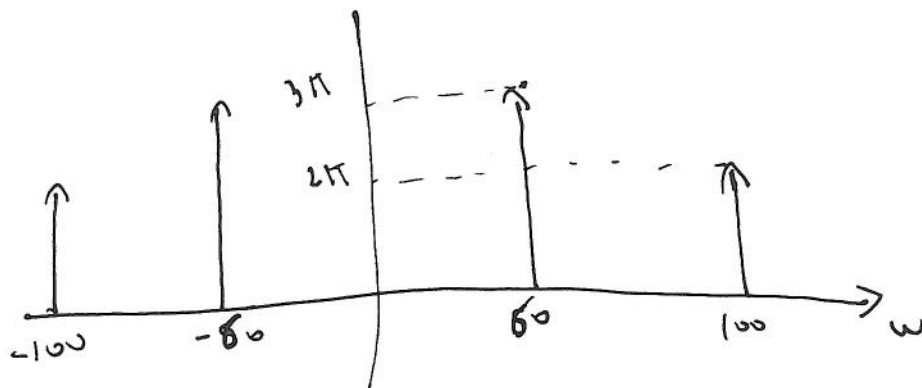
3.

$$(a) \frac{2}{\pi} \text{sinc } 2t \Leftrightarrow \text{RECT}\left(\frac{\omega}{22}\right)$$

$$\frac{300}{\pi} \text{sinc } 300t \Leftrightarrow \text{RECT}\left(\frac{\omega}{600}\right)$$

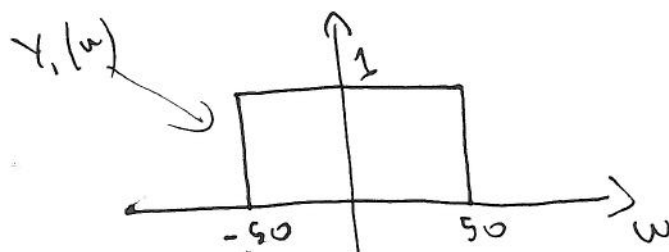


(b)



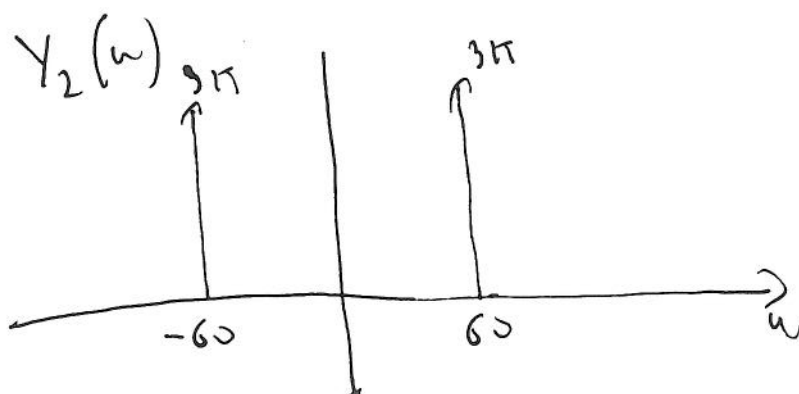
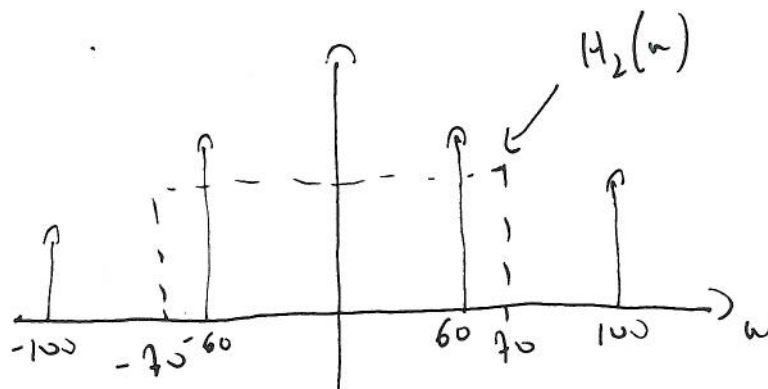
(c)

$$Y_1(\omega) = H_1(\omega) = \text{RECT}\left(\frac{\omega}{100}\right)$$



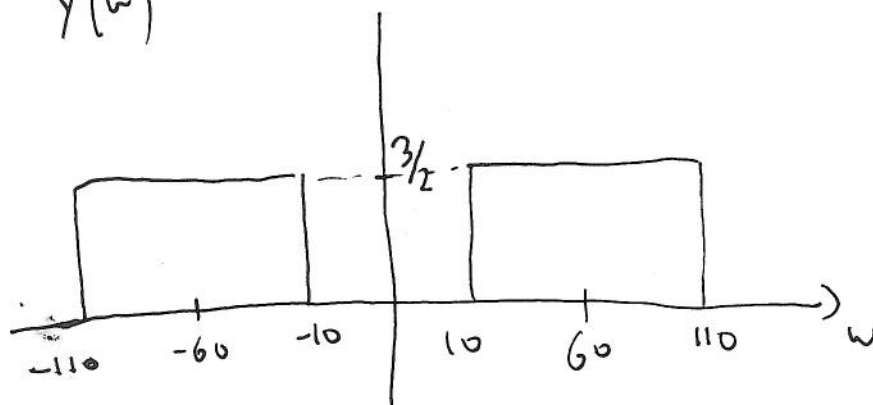
(d)

10



(e)

i.) $Y(\omega)$



i.i. $y(t) = 3 \cdot \frac{50}{\pi} \text{sinc } 50t \cdot \cos 60t = \frac{150}{\pi} \text{sinc } 50t \cdot \cos 60t$

4.

$$\begin{aligned}
 (a) \quad y_1(t) &= m^2(t) (\cos^2 \omega_c t + \sin^2 \omega_c t + 2 \sin \omega_c t \cos \omega_c t) = \\
 &= m^2(t) + 2m^2(t) \sin \omega_c t \cos \omega_c t = \\
 &= m^2(t) + m^2(t) \sin 2\omega_c t
 \end{aligned}$$

$$(b) \quad y_2(t) = m^2(t)$$

$$(c) \quad y_3(t) = \sqrt{m^2(t)} = |m(t)|$$

(d) NOTICE THE SYSTEM BEHAVES LIKE ~~AN~~ AN ENVELOPE DETECTOR.

THUS $y_3(t) = m(t)$ WHEN $m(t) \geq 0 \quad \forall t$,

THIS HAPPENS WHEN $A \geq 1$. MINIMUM VALUE, ~~OF~~ $A = 1$.

(e) THE BANDWIDTH OF $m^2(t)$ IS 6 RAD/S
 THE MINIMUM VALUE OF $\omega_c = 12$ RAD/S.
 IN THIS WAY THE TWO SPECTRA
 $m^2(t)$ AND $m^2(t) \cos \omega_c t$ DO NOT
 OVERLAP

