EE2-10B

Vector Calculus and Electromagnetic Fields 2014 – Solutions

1 a i) Potential functions can be used to represent the spatial variation of the electric potential. With no charges present, the potential is the solution of Laplace's equation $\nabla^2\phi = 0$

[2]

ii) Equipotentials are lines or surfaces of constant potential.

Perfect conductors are equipotentials.

[1]

iii) The gradient of a potential function is a vector representing the local value of the spatial derivative of the potential in each direction, for example as: $\nabla\phi = \partial\phi/\partial x\ \underline{i} + \partial\phi/\partial y\ \underline{j}$

The electric field is related to the potential by $\underline{E} = -\nabla\phi$.

[2]

b) Suppose we have a potential function whose value at (x, y) is $\phi(x, y)$.

The value at a nearby point $\delta\underline{r} = (\delta x, \delta y)$ away is:

$\phi(x + \delta x, y + \delta y) = \phi(x, y) + (\partial\phi/\partial x)\ \delta x + (\partial\phi/\partial y)\ \delta y$, or:

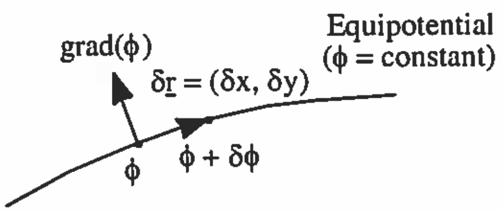
$\phi(x + \delta x, y + \delta y) = \phi(x, y) + \delta\phi$, where $\delta\phi = (\partial\phi/\partial x)\ \delta x + (\partial\phi/\partial y)\ \delta y$

$\delta\phi$ may be written as $\delta\phi = (\partial\phi/\partial x, \partial\phi/\partial y) . (\delta x, \delta y) = \nabla\phi . \delta\underline{r}$

If $\delta\underline{r}$ is restricted to lie on the surface ϕ = constant, then $\delta\phi = 0$.

In this case, 0, so $\nabla\phi$ must be perpendicular to $\delta\underline{r}$.

Thus, $\nabla\phi$ is always perpendicular to equipotentials.



[4]

Since by $\underline{E} = -\nabla\phi$, the electric field is always perpendicular to equipotentials in electrostatics.

[1]

c) If $\phi(x, y) = \exp\{-(x^2 + y^2)\}$, the equipotentials are found as follows:

$\phi(x, y)$ is constant whenever $x^2 + y^2 = \text{const}$

Hence, the equipotentials are circles.

[1]

The gradient is found as follows. The first derivatives of ϕ are:

$\partial\phi/\partial x = -(2x/a^2) \exp\{-(x^2 + y^2)/a^2\}$

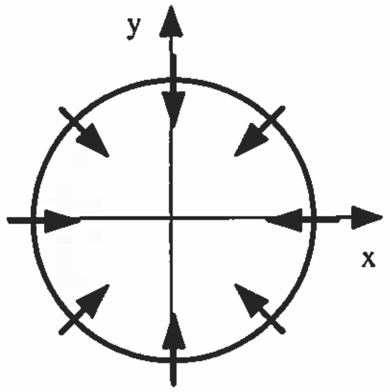
$\partial\phi/\partial y = -(2y/a^2) \exp\{-(x^2 + y^2)/a^2\}$

Hence $\nabla\phi = -(2/a^2) \exp\{-(x^2 + y^2)\} (x \underline{i} + y \underline{j})$

Hence, $\nabla\phi$ is a vector pointing radially inwards.

[1]

Equipotentials and gradient field can be represented graphically thus:



[1]

If $x^2 + y^2$ = const, then $2x + 2y\, dy/dx = 0$

Hence $2x\, dx = -2y\, dy$ and a short section of equipotential is described by $\delta\underline{r} = (dx, -x/y\, dx)$

Hence $\nabla\phi \,.\, \delta\underline{r} = -(2/a^2) \exp\{-(x^2 + y^2)\} (x \underline{i} + 2y \underline{j}) \,.\, (dx, -x/y\, dx)$, or

$\nabla\phi \,.\, \delta\underline{r} = -(2/a^2) \exp\{-(x^2 + y^2)\} \{x\, dx + y\, (-x/y)\, dx\} = 0$

Hence, grad(ϕ) is perpendicular to the equipotentials in this case.

[2]

d) The second derivatives of ϕ are:

$\partial^2\phi/\partial x^2 = (-2/a^2 + 4x^2/a^4). \exp\{-(x^2 + y^2)\}$

$\partial^2\phi/\partial y^2 = (-2/a^2 + 4y^2/a^4) \,.\, \exp\{-(x^2 + y^2)\}$

Hence $\nabla^2\phi = (4/a^4)(x^2 + y^2 - a^2) \exp\{-(x^2 + y^2)\}$

[4]

Since this result is non-zero, ϕ cannot be a valid solution of Laplace's equation.

[1]

e) $\quad {}_0\int^1 {}_0\int^{2x} x^2y\, dy\, dx = {}_0\int^1 {}_0[x^2y^2/2]^{2x}\, dx = {}_0\int^1 2x^4\, dx$

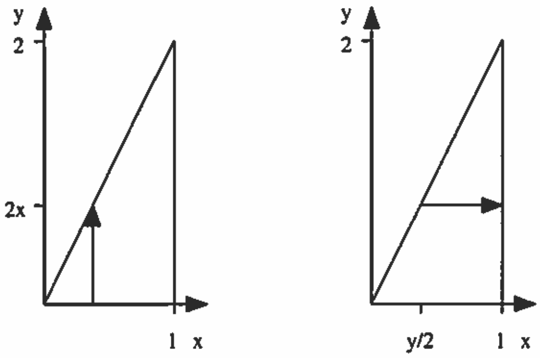
$\quad {}_0\int^1 2x^4\, dx = {}_0[2x^5/5]^1 = 2/5$

[2]

Changing the order of integration:

Previously x ranged from 0 to 1 and y ranged from 0 to 2x.

Now y must range from 0 to 2 and x must range from y/2 to 1.



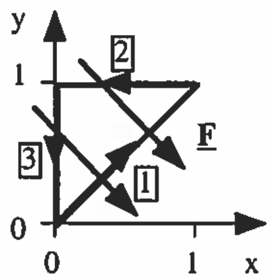
[1]

${}_0\int^2 {}_{y/2}\int^1 x^2y\, dx\, dy = {}_0\int^2 {}_{y/2}[x^3y/3]^1\, dy = {}_0\int^2 (y/3 - y^4/24)\, dy$

${}_0\int^2 (y/3 - y^4/24)\, dy = {}_0[y^2/6 - y^5/120]^2 = 4/6 - 32/120 = (80 - 32)/120 = 48/120 = 2/5$

[2]

f) Assuming that $\underline{\mathbf{F}} = 3\,\underline{i} - 3\,\underline{j}$, $\underline{\mathbf{F}}$ has the orientation shown and the line integral can be divided into three parts.



[1]

Section 1: $d\underline{L} = dx\,\underline{i} + dy\,\underline{j}$. Since $F \cdot dL = 0$, $\int_L \underline{\mathbf{F}} \cdot d\underline{L} = 0$ over this section of path

Section 2: $d\underline{L} = dx\,\underline{i}$. $\int_L \underline{\mathbf{F}} \cdot d\underline{L} = {}_1\int^0 3\, dx = -3$ over this section of path

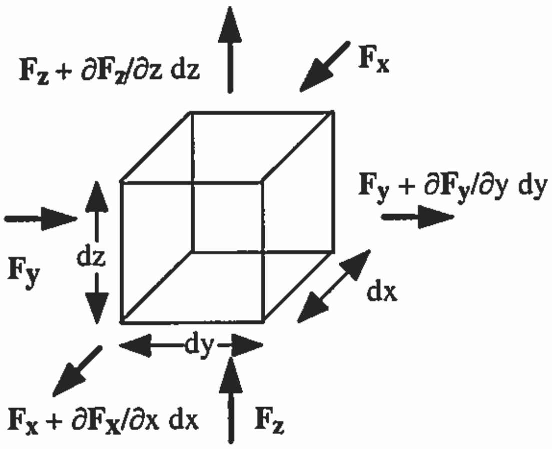
Section 3: $d\underline{L} = dy\,\underline{j}$. $\int_L \underline{\mathbf{F}} \cdot d\underline{L} = {}_1\int^0 -3\, dy = +3$ over this section of path

[3]

Hence $\int_L \underline{\mathbf{F}} \cdot d\underline{L} = 0 - 3 + 3 = 0$

[1]

g) Consider a cuboid with sides of length dx, dy and dz, with a vector field $\underline{\mathbf{F}}$ passing through it. Since the field is spatially varying, the field components at a point (x + dx, y + dy, z + dz) can be found in terms of the components at (x, y, z) as a Taylor series expansion as shown below.



[3]

For the two faces in the y-z plane, the surface normals are in the ± $\underline{i}$ direction.

The field into the rear face is $\mathbf{F_x}$,

The field out of the front face is $\mathbf{F_x} + \partial\mathbf{F_x}/\partial x\ dx$

Hence $\int\int_S \underline{\mathbf{F}} \cdot \mathbf{d\underline{a}} = \{\mathbf{F_x} + \partial\mathbf{F_x}/\partial x\ dx\}\ dy\ dz - \mathbf{F_x}\ dy\ dz = \partial\mathbf{F_x}/\partial x\ dx\ dy\ dz$

[2]

For the two faces in the x-z plane, the surface normals are in the ± $\underline{j}$ direction. Hence:

$\int\int_S \underline{\mathbf{F}} \cdot \mathbf{d\underline{a}} = \{\mathbf{F_y} + \partial\mathbf{F_y}/\partial y\ dy\}\ dx\ dz - \mathbf{F_y}\ dx\ dz = \partial\mathbf{F_y}/\partial y\ dx\ dy\ dz$

For the two faces in the x-y plane, the surface normals are in the ± $\underline{k}$ direction. Hence:

$\int\int_S \underline{\mathbf{F}} \cdot \mathbf{d\underline{a}} = \{\mathbf{F_z} + \partial\mathbf{F_z}/\partial z\ dz\}\ dx\ dy - \mathbf{F_z}\ dy\ dy = \partial\mathbf{F_z}/\partial z\ dx\ dy\ dz$

[2]

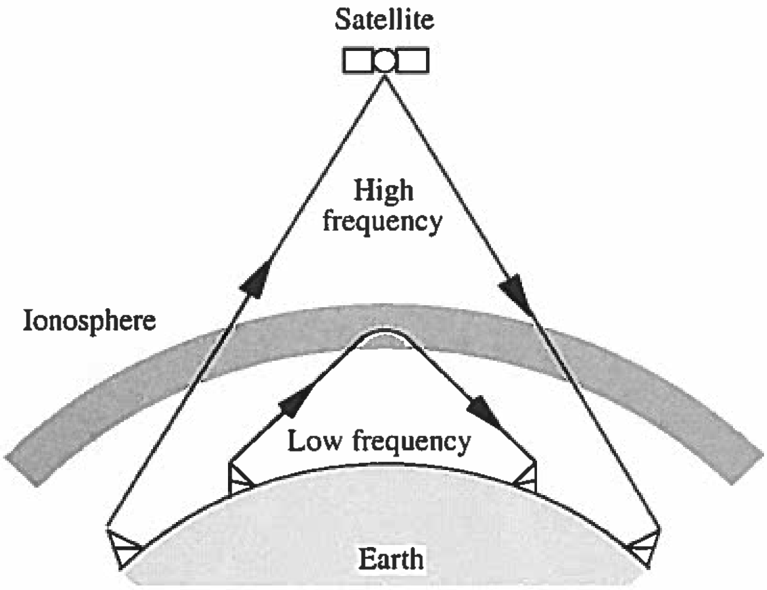
For the whole cuboid, $\int\int_S \underline{\mathbf{F}} \cdot \mathbf{d\underline{a}} = \{\partial\mathbf{F_x}/\partial x + \partial\mathbf{F_y}/\partial y + \partial\mathbf{F_z}/\partial z\}\ dx\ dy\ dz$

Hence $\int\int_S \underline{\mathbf{F}} \cdot \mathbf{d\underline{a}} = \int\int\int_V \mathrm{Div}(\underline{\mathbf{F}})\ dv$ – This is Gauss' Theorem.

[3]

2. a) The ionosphere is a set of concentric spherical layers in the upper atmosphere containing ions created by bombardment of rarefied gas with energetic particles. At low frequencies, the ionosphere reflects radio waves. It was first located in 1924 by Edward Appleton, who measured the return time of reflected radio waves. The ionosphere allows over-the horizon radio communication by multiple reflection between the ionosphere and the oceans (which also contain ions), a feature exploited by Marconi for trans-Atlantic communication. However, transmission is unreliable because the weather disrupts the ionosphere. Modern communications use higher frequencies, to which the atmosphere is transparent. The signal is transmitted to a geo-stationary satellite, regenerated and retransmitted.

[3]

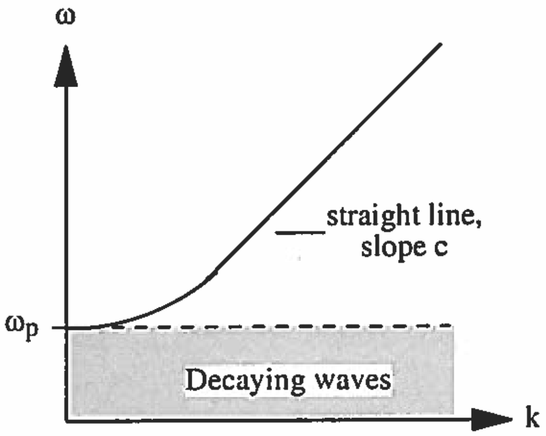


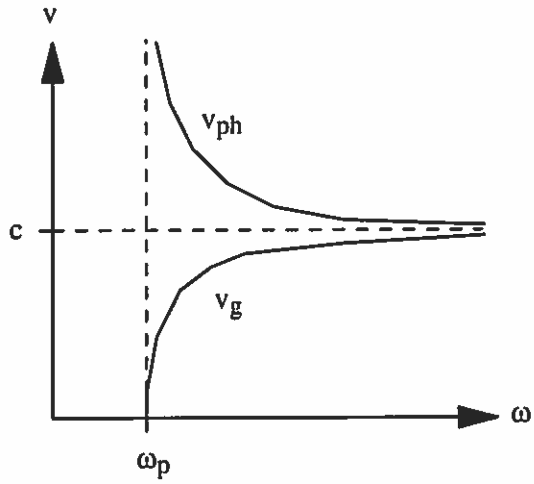
[3]

b) A dispersion diagram is a plot of angular frequency ω against propagation constant k for a material or a transmission channel. It may be used to extract the phase velocity $v_{ph} = \omega/k$ or the group velocity $v_g = d\omega/dk$.

An example might be the dispersion characteristic of the ionosphere, for which $\omega = \sqrt{(\omega_p^2 + c^2k^2)}$, which tends to the following limits: when k is small, $\omega \approx \omega_p$, and when k is large $\omega \approx ck$. Frequency bands over which there is no propagating solution (in this case, below ω_p) support decaying waves.

[3]

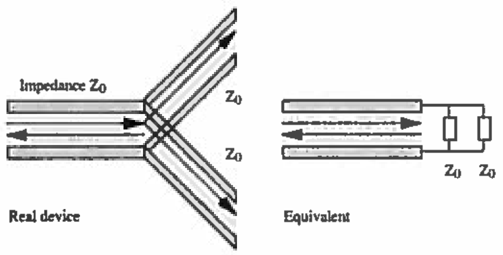




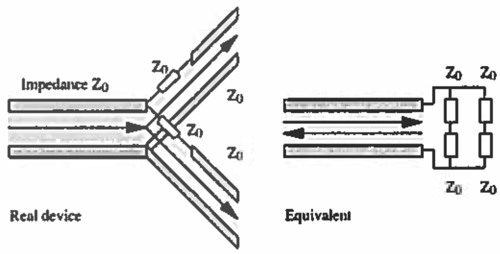
[3]

c) Simple RF splitters based on Y-connected lines do not generally have an input impedance equal to the characteristic impedance of the line. For example, consider the simple splitter shown in a) below, which has two lines of impedance Z_0 connected in parallel to a line of impedance Z_0. In this case the input impedance is $Z_0/2$. The line is therefore mismatched, and a reflection must occur. One way to reduce the reflection is to insert series resistors equal to Z_0 as shown in b). The two output lines now each present impedance $2Z_0$. Since these are in parallel, the combined impedance is now Z_0 and the line is matched at all frequencies. However, a price has been paid to achieve matching: the device consumes power because of the inserted resistors.

[3]



a)



b)

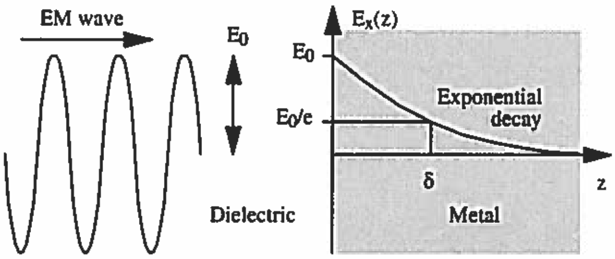
[3]

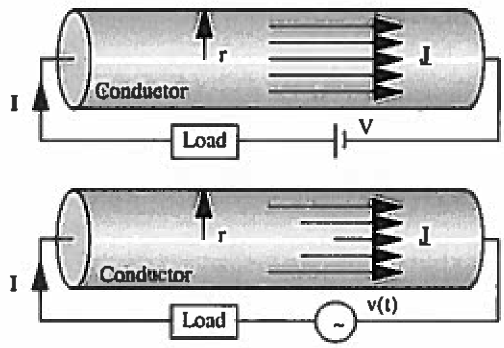
d) The skin effect is the decay of an electromagnetic field below the surface of a conductor. Generally, the field will decay to 1/e of its original amplitude when $z = (1/\pi f \mu_0 \sigma)^{1/2}$. This distance is known as the skin depth δ and reduces as the frequency rises. There are two key consequences:

i) An electromagnetic wave must decay rapidly as it travels into a metal surface, and hence cannot penetrate the metal to any significant extent.

ii) Since $\underline{J} = \sigma\underline{E}$ current densities must decay similarly, so current must be confined near the surface of a metal at RF frequency. This effect is responsible for an increase in the per-unit length resistance of a cylindrical wire from $R_{pul} = 1/\sigma\pi r^2$ at DC to $R_{pul} = 1/\sigma 2\pi r\delta$ at high frequency.

[3]

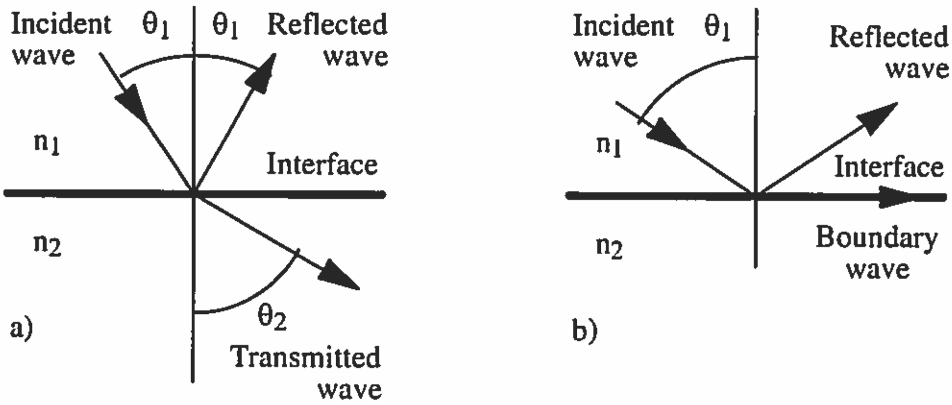




[3]

f) Total internal reflection (TIR) occurs when an optical wave strikes an interface between two dielectric media at an angle greater than the critical angle. Figure a) below shows the geometry. Assuming that the wave is incident at an angle θ_1 from a medium with refractive index n_1 onto a medium with index n_2, Snell's law implies that the transmitted wave angle θ_2 is given by $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$. Hence, $\theta_2 = \sin^{-1}\{(n_1/n_2) \sin(\theta_1)\}$.When the argument of the inverse sin reaches unity, there is no longer a transmitted wave but a boundary wave as shown in b) below. This requires $n_1 > n_2$, and occurs when $\sin(\theta_1) = n_2/n_1$, i.e. at a critical angle $\theta_{1c} = \sin^{-1}(n_2/n_1)$.

[3]



[3]

3. a) The three main factors limiting free-space communication are:

i) Absorption due to electronic transitions in the molecules of the earth's atmosphere (at visible/UV wavelengths) and molecular vibrational transitions (at infrared wavelengths). Absorption losses are concentrated near spectral bands known as absorption lines, many of which affect microwave links.

[3]

ii) Rayleigh scattering, due to inhomogeneities (e.g. water droplets and soot particles) and small-scale fluctuations in the molecular arrangement of the atmosphere. Scattering losses rise rapidly at short wavelengths (rising as $1/\lambda^4$) and hence strongly affect optical links.

[3]

iii) Diffraction, due to the spreading of a beam emitted from a source of finite extent. Diffraction effects increase rapidly as the dimensions of the beam approach that of the wavelength, and hence strongly affect radio links, which have relatively large wavelengths.

[3]

b) The spherical wave equation is $d^2E/dr^2 + (2/r)\, dE/dr + \omega^2\mu_0\varepsilon_0\, E = 0$.

Assuming the general solution $E(r) = F(r) \exp(-jk_0r)$ we then get:

$dE/dr = (dF/dr - jk_0F) \exp(-jk_0r)$

$d^2E/dr^2 = (d^2F/dr^2 - 2jk_0dF/dr - k_0^2F) \exp(-jk_0r)$

Substituting into the wave equation, we get:

$(d^2F/dr^2 - 2jk_0dF/dr - k_0^2F) \exp(-jk_0r) + (2/r)\,(dF/dr - jk_0F) \exp(-jk_0r) + \omega^2\mu_0\varepsilon_0F \exp(-jk_0r) = 0$

Cancelling exponential terms we then get:

$d^2F/dr^2 - 2jk_0dF/dr - k_0^2F + (2/r)\,(dF/dr - jk_0F) + \omega^2\mu_0\varepsilon_0F = 0$

[3]

Assuming that F is real, we can equate real and imaginary parts separately to get:

$d^2F/dr^2 + (2/r)\, dF/dr - k_0^2F + \omega^2\mu_0\varepsilon_0F = 0 \qquad (1)$

$-2jk_0dF/dr - 2jk_0F/r = 0 \qquad (2)$

From 2, we can then obtain $dF/dr = -F/r$. For the three trial solutions we then get:

i) $E(r) = E_0 \exp(-jk_0r)$ so $F = E_0$ and $dF/dr = 0$ clearly $dF/dr \neq -F/r$.

ii) $E(r) = (E_0/r) \exp(-jk_0r)$ so $F = E_0/r$ and $dF/dr = -E_0/r^2$ in this case $dF/dr = -F/r$.

iii) $E(r) = (E_0/r^2) \exp(-jk_0r)$ so $F = E_0/r^2$ and $dF/dr = -2E_0/r^3$ clearly $dF/dr \neq -F/r$.

For solution ii) we can then obtain $d^2F/dr^2 = 2E_0/r^3 = -(2/r)\ dF/dr$. Hence, from (1), this solution will be valid provided $-k_0^2F + \omega^2\mu_0\varepsilon_0F = 0$, which merely requires $k_0 = \omega\sqrt{(\mu_0\varepsilon_0)}$.

[3]

c) The Poynting vector $\underline{\mathbf{S}} = \underline{\mathbf{E}} \times \underline{\mathbf{H}}$ indicates the instantaneous power density. However, for fields oscillating at angular frequency ω, the Poynting vector contains components at 2ω, which typically, vary too fast to be measured directly. Instead, most detectors respond to time-averaged power. The irradiance is therefore defined as $\underline{S} = (1/T)\ {}_0\int^T \underline{\mathbf{E}} \times \underline{\mathbf{H}}\ dt$. Substituting $\underline{\mathbf{E}} = \mathrm{Re}\{\ \underline{E}\ \exp(j\omega t)\}$ and $\underline{\mathbf{H}} = \mathrm{Re}\{\underline{H}\ \exp(j\omega t)\}$ we can then obtain after some simple mathematics $\underline{S} = 1/2\ \mathrm{Re}\{\underline{E} \times \underline{H}^*\}$.

[3]

For a spherical wave, the magnetic field is related to the electric field as $\underline{H} = \underline{E}/Z_0$, where Z_0 is the impedance of free space. Assuming that $E_\theta = (E_0/r)\exp(-jk_0r)$, we obtain $H_\phi = (E_0/Z_0r)\exp(-jk_0r)$. The irradiance is then $S_r = 1/2\ E_0^2/Z_0r^2$, and consequently falls off as $1/r^2$.

[2]

d) The wavelength of an EM wave of frequency f is $\lambda = c/f$, where c is the velocity of light. When f = 100 MHz, $\lambda = 3 \times 10^8 / 1 \times 10^8 = 3$ m. The length of a half-wave dipole is then $\lambda/2 = 1.5$ m.

[2]

From the formula sheet, the effective area is related to the directivity by $A_R = \lambda^2D_R/4\pi$
Assuming that $\lambda = 1.5$ m and $D_R = 100$, we obtain $A_R = 1.5^2 \times 100/4\pi = 17.9\ m^2$.

[2]

Assuming a transmitter power P_T and an isotropic transmitting antenna, the power density at a radius r is $S = P_T/4\pi r^2$. The power intercepted by a lossless receiving antenna of effective area A_R is then $P_R = SA_R = P_TA_R/4\pi r^2$. Re-arranging, the transmitter power can be written as $P_T = P_R\ (4\pi r^2/A_R)$.
Assuming that $P_R = 1$ mW when r = 1 km, $P_T = 10^{-3} \times (4\pi \times 1000^2/17.9)$ W = 702 W.

[2]

Assuming that the transmitting antenna now has directivity D_T, the received power will increase to P_RD_T. Assuming that $D_T = D_R = 100$, the new value will be100 mW.

[2]

If the minimum detectable power is 1 μW, the received power can fall by a factor of 100 mW/10 μW = 10^4 before there any problem arises problem. Because the power density falls off as $1/r^2$, the link length can therefore increase by a factor of $\sqrt{(10^4)} = 100$, to 100 km.

[2]