

B.ENG. AND M.ENG. EXAMINATIONS 2008

MATHEMATICS (INFORMATION SYSTEMS ENGINEERING E1.11)

Date Wednesday 4th June 2008 10.00 am - 1.00 pm

Answer ANY SEVEN questions.

Answers to questions from Section A and Section B should be written in different answer books.

A mathematical formulae sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 7 pages, with a total of 9 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. Show that

$$\int_0^{2\pi} \exp(in\theta) d\theta = 0,$$

if n is any non-zero integer.

Hence, by expressing $\cos(\theta)$ in terms of complex exponentials, evaluate

$$\int_0^{2\pi} \cos^{2n} \theta d\theta.$$

2. (i) State whether the following series converge or diverge, explaining your reasons:

(a)
$$\sum_{n=1}^{\infty} \frac{n+3}{(n+2)(n+1)},$$

(b)
$$\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{(n+2)(n+3)}.$$

- (ii) Explain what is meant by the *radius of convergence* of a power series. Calculate the radii of convergence of the following two power series:

(a)
$$\sum_{n=0}^{\infty} \frac{2n+1}{\sqrt{n^2+1}} z^n,$$

(b)
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n.$$

PLEASE TURN OVER

3. (i) Evaluate the limits

(a) $\lim_{x \rightarrow 0} \frac{\exp(2x)}{\cosh x},$

(b) $\lim_{x \rightarrow \pi/4} \frac{2 \sin^2 x - 1}{\tan x - 1},$

(c) $\lim_{n \rightarrow \infty} [n \{ (n^2 + 3)^{1/2} - (n^3 + n)^{1/3} \}].$

(ii) Write down the Maclaurin series for the two functions $\exp(x^2)$ and $\cos x$.

Hence calculate the first three non-zero terms of the Maclaurin series for the product

$$\exp(x^2) \cos x.$$

4. Evaluate the definite integrals

(i)

$$\int_0^{\pi/2} x \cos x \, dx,$$

(ii)

$$\int_0^1 x^2 \ln x \, dx,$$

(iii)

$$\int_1^\infty \frac{1}{x(x+1)(x+2)} \, dx.$$

PLEASE TURN OVER

5. Solve the ordinary differential equations

(i)

$$\frac{dy}{dx} = \frac{x + 2y}{2x + y};$$

(ii)

$$\frac{dy}{dx} + 3x^2y = \exp(-x^3), \quad \text{with } y(0) = 0;$$

(iii)

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \exp(-x), \text{ with } y(0) = 1, \text{ and } y'(0) = 1.$$

In each case, find the most general solution possible.

PLEASE TURN OVER

SECTION B

6. (i) Find $\frac{dy}{dx}$ if $xy^2 - 2x \sin y = 5$.

(ii) If $z = x^2 + y^2$ where $x = r^2 \cos \theta$ and $y = r \sin 2\theta$,
find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$, in terms of r and θ .

(iii) Find the stationary points of the function

$$f(x, y) = \frac{1}{3}x^3 + y^2 - 2xy$$

and determine their nature.

7. Find the Fourier cosine series for the function

$$f(x) = x, \quad 0 \leq x \leq \pi.$$

Use this to evaluate the sum

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

PLEASE TURN OVER

8. (i) Use the method of Laplace transforms to solve the differential equation

$$\frac{dy}{dx} + 2y = 5 \cos x$$

with $y(0) = 1$.

No credit will be given if you use a different method.

- (ii) Use Laplace transforms to find functions $y = y(x)$, $z = z(x)$ satisfying the simultaneous differential equations

$$\frac{dy}{dx} + \frac{dz}{dx} + y = 0,$$

$$\frac{dy}{dx} + 2\frac{dz}{dx} - y = e^{-x},$$

where $y(0) = 0$, $z(0) = 0$.

The Laplace transform of $f(x)$ is defined as

$$\mathcal{L}(f(x)) = F(t) = \int_0^\infty e^{-tx} f(x) dx.$$

You may assume that

$$\mathcal{L}(f'(x)) = -f(0) + t\mathcal{L}(f(x)).$$

PLEASE TURN OVER

9. (i) Consider the three planes

$$\begin{aligned} \mathbf{r} \cdot (1, 1, 1) &= 1, \\ \mathbf{r} \cdot (2, 1, a) &= -1, \\ \mathbf{r} \cdot (1, -1, 1) &= b, \end{aligned}$$

where $\mathbf{r} = (x, y, z)$. For which values of a and b do the three planes

- (a) meet in exactly one point ,
- (b) meet in a line ,
- (c) not meet at all?

- (ii) Let

$$A = \begin{pmatrix} 1 & -7 \\ 0 & 8 \end{pmatrix}.$$

Find a 2×2 matrix P such that $P^{-1}AP$ is a diagonal matrix.

Find a 2×2 matrix B such that $B^3 = A$.

END OF PAPER

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$a = a_1i + a_2j + a_3k = (a_1, a_2, a_3)$$

Scalar (dot) product:

$$a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$$

Vector (cross) product:

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix};$$

Scalar triple product:

$$[a, b, c] = a \cdot b \times c = b \cdot c \times a = c \cdot a \times b = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $a \times (b \times c) = (c \cdot a)b - (b \cdot a)c$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{(n+1)!} + \dots \quad (-1 < x \leq 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

$$\cosh iz = \cos z; \quad \cosh iz = \cos z; \quad \sinh iz = i \sinh z; \quad \sinh iz = i \sin z.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f D^n g + \binom{n}{1} Df D^{n-1} g + \dots + \binom{n}{n} D^n f g.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^{(n)}(a) + \epsilon_n(h),$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a + \theta h)/(n+1)!$, $0 < \theta < 1$.

(c) Taylor's expansion of $f(x, y)$ about (a, b) :

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2! [h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}]_{a,b} + \dots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\frac{dF}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\frac{dF}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.

Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.

If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;

If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;

If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\frac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

- (a) An important substitution: $\tan(\theta/2) = t$:
 $\sin \theta = 2t/(1+t^2)$, $\cos \theta = (1-t^2)/(1+t^2)$, $d\theta = 2 dt/(1+t^2)$.
- (b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1} \left(\frac{x}{a} \right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1} \left(\frac{x}{a} \right) = \ln \left\{ \frac{x}{a} + \left(1 + \frac{x^2}{a^2} \right)^{1/2} \right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1} \left(\frac{x}{a} \right) = \ln \left| \frac{x}{a} + \left(\frac{x^2}{a^2} - 1 \right)^{1/2} \right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a} \right) \tan^{-1} \left(\frac{x}{a} \right).$$

6. NUMERICAL METHODS

- (a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and $x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2, \dots$

(Newton Raphson method).

- (b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

- i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x) dx \approx (h/2) [y_0 + y_1]$.
- ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x) dx \approx (h/3) [y_0 + 4y_1 + y_2]$.

- (c) Richardson's extrapolation method: Let $I = \int_a^b f(x) dx$ and let I_1, I_2 be two

estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t) dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	$d^2 f/dt^2$	$s^2 F(s) - sf(0) - f'(0)$
$e^{at} f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial \alpha) f(t, \alpha)$	$(\partial/\partial \alpha) F(s, \alpha)$	$\int_0^t f(t) dt$	$F(s)/s$
$\int_0^t f(u)g(t-u) du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2, \dots)$	$n!/s^{n+1}, (s > 0)$
e^{at}	$1/(s-a), (s > a)$	$\sin \omega t$	$\omega/(s^2 + \omega^2), (s > 0)$
$\cos \omega t$	$s/(s^2 + \omega^2), (s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	$e^{-sT}/s, (s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \dots, \text{ and}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

Parseval's theorem

$$\frac{1}{L} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$