

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING  
EXAMINATIONS 2014

EEE/EIE PART III/IV: MEng, Beng and ACGI

Corrected Copy

**CONTROL ENGINEERING**

Wednesday, 22 January 10:00 am

Time allowed: 3:00 hours

**There are FOUR questions on this paper.**

**Answer ALL questions.**

*All questions carry equal marks*

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible      First Marker(s) :      A. Astolfi

Second Marker(s) :      D. Angeli



## CONTROL ENGINEERING

1. Consider a linear, single-input, single-output, continuous-time, system of dimension  $n = 3$ , that is  $x = [x_1, x_2, x_3]'$ , with

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [0 \quad 1 \quad 0].$$

- a) Show that the system is controllable and observable. [ 4 marks ]  
b) Compute  $\dot{y}$  and write  $\dot{y}$  as a function of the state  $x$ . [ 2 marks ]  
c) Consider the feedback law

$$u = -k^2 y - k \dot{y}.$$

Show that there exists a value  $k_* > 0$  such that the closed-loop system is asymptotically stable for all  $k > k_*$ .

(Hint: compute the characteristic polynomial of the closed-loop system and then use Routh test.) [ 6 marks ]

- d) Compute  $\ddot{y}$  and write  $\ddot{y}$  as a function of the state  $x$  and the input  $u$ . [ 2 marks ]  
e) Consider now the feedback law

$$u = -k^3 y - k^2 \dot{y} - k \ddot{y}.$$

Discuss for which values of  $k$  this feedback law is well-defined and show that there exists a value  $k_o > 0$  such that the closed-loop system is asymptotically stable for all  $k > k_o$ . [ 6 marks ]

2. Consider a linear, continuous-time, system described by the equations

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = u_3,$$

with  $x_i(t) \in \mathbb{R}$  and  $u_i(t) \in \mathbb{R}$ , for  $i = 1, 2, 3$ , and the problem of designing the feedback signals  $u_i$ , for  $i = 1, 2, 3$ , such that

$$\lim_{t \rightarrow \infty} (x_1(t) - x_2(t)) = \lim_{t \rightarrow \infty} (x_2(t) - x_3(t)) = 0, \quad (*)$$

Let

$$u_1 = \alpha_1(x_2 - x_1) \quad u_2 = \alpha_2(x_1 - x_2) + \alpha_3(x_3 - x_2) \quad u_3 = \alpha_4(x_2 - x_3),$$

with  $\alpha_i$  constant, for  $i = 1, 2, 3, 4$ .

- a) Write the equations describing the closed-loop system in the form  $\dot{x} = Ax$ , with  $x = [x_1, x_2, x_3]^T$ . Write explicitly the matrix  $A$ . [ 4 marks ]
- b) Show that the matrix  $A$  has always an eigenvalue equal to zero. [ 2 marks ]
- c) Select the parameters of the matrix  $A$  such that the matrix has one eigenvalue equal to  $-1$  and one eigenvalue equal to  $-3$ .  
(Hint: the selection is not unique. Any selection is acceptable.) [ 6 marks ]
- d) Using the parameters selected in part c) show that the condition  $(*)$  holds. This can be shown as follows.
  - i) Let  $z_{12} = x_1 - x_2$  and  $z_{23} = x_2 - x_3$ . Write differential equations for the variables  $z_{12}$  and  $z_{23}$ . [ 2 marks ]
  - ii) Write the differential equations in part d.i) in the form

$$\begin{bmatrix} \dot{z}_{12} \\ \dot{z}_{23} \end{bmatrix} = F \begin{bmatrix} z_{12} \\ z_{23} \end{bmatrix}.$$

Write explicitly the matrix  $F$  and show that it has two eigenvalues equal to  $-1$  and  $-3$ . [ 4 marks ]

- iii) Exploiting the results of part d.ii) show that condition  $(*)$  holds. [ 2 marks ]

3. Consider the discrete-time system described by the equations

$$x(k+1) = \begin{bmatrix} a_1 & 1 \\ a_0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + u(k),$$

in which  $a_0$  and  $a_1$  are constant.

- a) Show that the sequence  $y(k)$  is such that

$$y(k+2) - a_1 y(k+1) - a_0 y(k) = u(k+2),$$

[ 8 marks ]

- b) Study the reachability and controllability properties of the system as a function of  $a_0$  and  $a_1$ . [ 6 marks ]

- c) Study the observability properties of the system as a function of  $a_0$  and  $a_1$ . [ 2 marks ]

- d) Determine values of  $a_0$  and  $a_1$  such that the output sequence is generated by a relation of the form

$$y(k+1) - \alpha y(k) = u(k+1),$$

for some  $\alpha$  constant.

(Hint: exploit the results of part b) in your selection.)

[ 4 marks ]

4. Consider a nonlinear, discrete-time, system described by the equations

$$x^+ = f(x, u) = \begin{bmatrix} \alpha \sin x_2 \\ -\alpha \sin x_1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u,$$

with  $\alpha \in (0, 1]$  and constant,  $x(t) = [x_1(t), x_2(t)]' \in \mathbb{R}^2$  and  $u(t) \in \mathbb{R}$ .

Assume  $u = 0$ .

- a) Show that the point  $(0, 0)$  is the only equilibrium of the system. [ 4 marks ]
- b) Compute the linearization of the system around the equilibrium point  $(0, 0)$  and write explicitly the matrices  $A$  and  $B$  of the linearized system. [ 4 marks ]
- c) Study the stability properties of the linearized system as a function of  $\alpha$ . (Recall that  $\alpha \in (0, 1]$ .) [ 4 marks ]
- d) Consider the linearized system determined in part b). Design a linear state feedback control law which assigns all eigenvalues of the closed-loop system to zero. [ 4 marks ]
- e) Consider the linearized system determined in part b) with output  $y = Cx$ , and  $C = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$ . Study the observability properties of the linearized system. Assume  $c_1^2 + c_2^2 \neq 0$  and design an observer such that the estimation error system has both eigenvalues at 0. [ 4 marks ]