ANALYSIS OF CIRCUITS

**** Solutions 2014 ****

Information for Candidates:

The following notation is used in this paper:

1. The voltage waveform at node X in a circuit is denoted by $x(t)$, the phasor voltage by X and the root-mean-square (or RMS) phasor voltage by $\widetilde{X} = \frac{X}{\sqrt{2}}$. The complex conjugate of X is X^*.
2. Component and source values in a circuit are normally given in Ohms, Farads, Henrys, Volts or Amps with the unit symbol omitted. Where an imaginary number is specified, it represents the complex impedance or phasor value.
3. Times are given in seconds unless otherwise stated.
4. Unless otherwise indicated, frequency response graphs should use a linear axis for phase and logarithmic axes for frequency and magnitude.

1. a) Using nodal analysis, calculate the voltages at nodes X and Y of Figure 1.1. [5]

We can immediately label the voltages on the bottom left and top left nodes as 6 *and* $6+9=15$ *respectively. We now write down KCL equation at node X to obtain*

$$\begin{aligned}\frac{X-15}{2}+X-6+\frac{X-Y}{3} &= 0\\ \Rightarrow 3X-45+6X-36+2X-2Y &= 0\\ \Rightarrow 11X-2Y &= 81\end{aligned}$$

KCL at Y gives

$$\begin{aligned}\frac{Y-X}{3}+\frac{Y}{2}+4 &= 0\\ \Rightarrow 2Y-2X+3Y+24 &= 0\\ \Rightarrow -2X+5Y &= -24\end{aligned}$$

Combining these gives $55X-4X=405-48 \quad\Rightarrow\quad X=\frac{357}{51}=7$

form which $5Y=-24+14=-10 \quad\Rightarrow\quad Y=\frac{-10}{5}=-2$

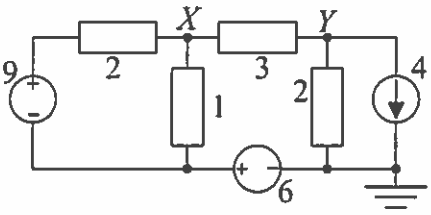

Figure 1.1

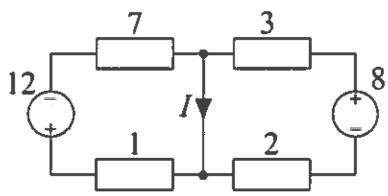

Figure 1.2

b) Use the principle of superposition to find the current I in Figure 1.2. [5]

If we short circuit the 12 V *voltage source, the* $7\,\Omega$ *and* $1\,\Omega$ *are shorted out by the central link that carries I and so we have a current of* $I_A=\frac{8}{3+2}=+1.6\,\text{A}$.

If we short circuit the 8 V *voltage source, the* $3\,\Omega$ *and* $2\,\Omega$ *are shorted out and so we have a current of* $I_B=\frac{-12}{7+1}=-1.5\,\text{A}$.

By superposition, the total current is therefore $1.6-1.5=-0.1$ A.

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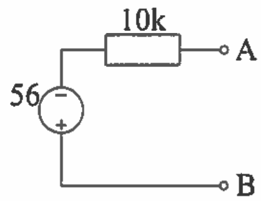
c) Draw the Thévenin equivalent circuit of the network in Figure 1.3 and find the value of its components. [5]

We can find the open circuit voltage by ignoring the 3k *resistor (since there is no current flowing through it). The* 8mA *will therefore flow upwards through the* 7k *resistor resulting in an open-circuit voltage of* $V_{AB} = -8 \times 7 = -56\,\text{V}$.

To find the Thévenin resistance, we treat the current source as an open circuit. The 4k *resistor now plays no part and the Thévenin resistance is therefore* $7+3 = 10\text{k}$.

So the complete Thévenin equivalent is:



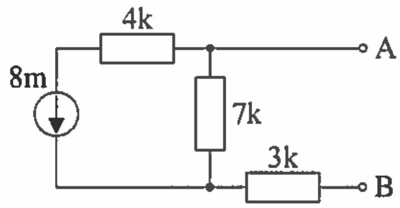


Figure 1.3

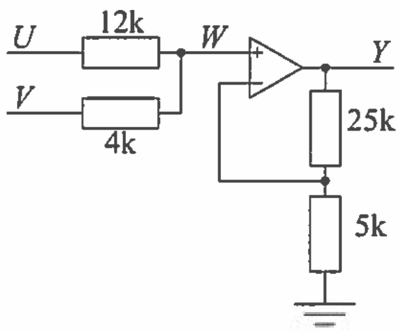


Figure 1.4

d) Assuming the opamp in the circuit of Figure 1.4 is ideal, give an expression for Y in terms of U and V. [5]

This is an non-inverting op-amp circuit and so we can write down $Y = W\left(1+\frac{25}{5}\right) = 6W$.

Applying KCL at node W gives $\frac{W-U}{12} + \frac{W-V}{4} = 0$ *from which* $W - U + 3W - 3V = 0$ *and hence* $W = \frac{U+3V}{4}$.

Putting these together gives $Y = 6W = 6 \times \frac{U+3V}{4} = 1.5U + 4.5V$.

e) The graph of Figure 1.5 plots the output voltage, Y, against the input voltage, X, for the circuit shown in Figure 1.6. The graph consists of two straight lines that intersect at the point (10, 10) and that pass through the origin and the point (20, 12) respectively. Assuming that the forward voltage drop of the diode is 0.7 V, determine the values of the resistor, R, and the voltage source, V. [5]

The diode turns on when $Y = V + 0.7$. For this to be when $Y = 10$V we must have $V = 9.3$.

To determine R, we can apply KCL at node Y when $X = 20$ and $Y = 12$:

$$\frac{12-20}{20} + \frac{12-10}{R} = 0$$

from which $-8R + 40 = 0$ which gives $R = 5$k.

Alternatively, we can say that, since the slope of the second part of the characteristic is $\frac{2}{10} = 0.2$, this must be the gain of the potential divider formed by 20k *and R. To get a gain of $\frac{1}{5}$ you need $20\text{k} = 4R$ from which $R = 5$k as before.*

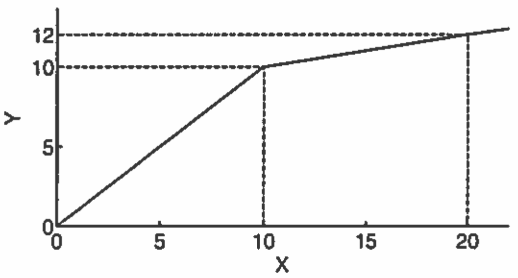


Figure 1.5

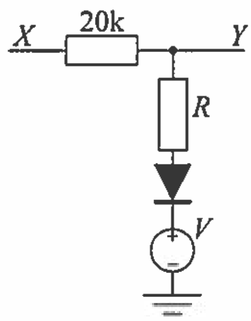


Figure 1.6

f) Determine the gain, $\frac{Y}{X}$, for the block diagram shown in Figure 1.7. The rectangular blocks are drawn with inputs at the left and outputs at the right and have gains of F and G respectively. The open circles represent adder/subtractors whose inputs have the signs indicated on the diagram and whose outputs are W and Y respectively. [5]

For node W, we can write the following equation: $W = X - GFW$ from which we get $W = \frac{1}{1+FG}X$.

For node Y, we can write $Y = FW - X = \left(\frac{F}{1+FG} - 1\right)X = \frac{F-FG-1}{1+FG}X$ so the gain is $\frac{Y}{X} = \frac{F-FG-1}{1+FG}$.

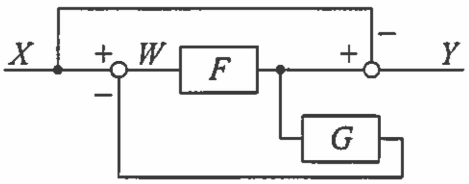


Figure 1.7

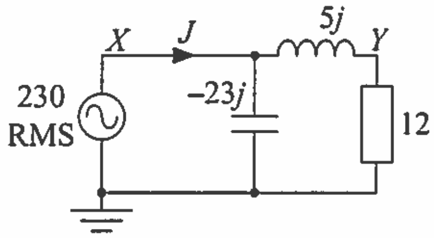


Figure 1.8

g) In the circuit of Figure 1.8, the RMS phasor $\widetilde{X} = 230$ and the component values shown indicate complex impedances. Determine the value of the RMS current $\widetilde{J}$ and of the complex power, $\widetilde{V} \times \widetilde{I}^*$, absorbed by each of the four components.

[5]

The RMS current through he capacitor is $\frac{\widetilde{X}}{-23j} = 10j$.

The RMS current through the inductor is $\frac{\widetilde{X}}{12+5j} = \frac{2760-1150j}{169} = 16.3 - 6.8j$.

The total RMS current is therefore $\widetilde{J} = 16.3 - 6.8j + 10j = 16.3 + 3.2j$.

Power absorbed by source is $-\widetilde{X} \times \widetilde{J}^* = -230(16.3 - 3.2j) = -3.76 + 0.73j$kVA.

Power absorbed by capacitor is $\widetilde{X} \times (10j)^* = -2.3j$kVA.

Power absorbed by resistor is $|\widetilde{I}|^2 R = |16.3 - 6.8j|^2 \times 12 = 313 \times 12 = 3.76$kW.

Power absorbed by inductor is $|\widetilde{I}|^2 Z_L = |16.3 - 6.8j|^2 \times 5j = 313 \times 5j = 1.57j$kVA.

As expected the powers sum to zero.

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h) Figure 1.10 shows a transmission line of length 100 m that is terminated in a resistive load, R, with reflection coefficient $\rho = +0.6$. The line has a propagation velocity of $u = 2 \times 10^8$ m/s. At time $t = 0$, a forward-travelling (i.e. left-to-right) pulse arrives at X with amplitude 4 V and duration 1.5 μs as shown in Figure 1.9.

Draw a dimensioned sketch of the waveform at Y, a point 60 m from the end of the line, for $0 \le t \le 3\,\mu$s. Assume that no reflections occur at point X. [5]

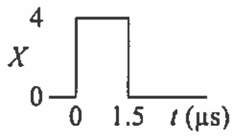


Figure 1.9

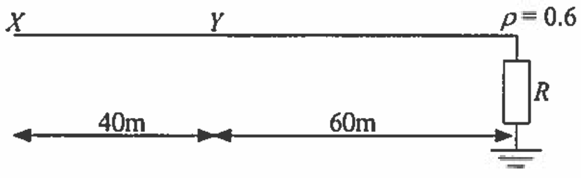
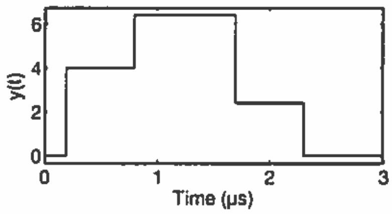


Figure 1.10

The velocity, u, is 200 m per μs. The forward wave takes 0.2 μs to reach Y and a further 0.6 μs to reflect from the end and return to Y. Therefore the waveform at Y is the sum of two overlapping waves: (i) a pulse of amplitude 4 V beginning at $t = 0.2\,\mu$s (ending at $t = 1.7\,\mu$s) and a pulse of $4 \times \rho = 2.4$ V beginning at $t = 0.8\,\mu$s (ending at $t = 2.3\,\mu$s). Where the pulses overlap, their combined voltage is $4 + 2.4 = 6.4$ V.



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2. a) Show that the transfer function of the circuit of Figure 2.1 can be written in the form

$$\frac{Y}{X}(j\omega) = \frac{1}{\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0} + 1}$$

and express the values of ω_0 and ζ in terms of the component values L, C and R. [5]

Viewing the circuit as a potential divider, the transfer function is

$$\begin{aligned}\frac{Y}{X}(j\omega) &= \frac{\frac{1}{j\omega C}}{j\omega L + R + \frac{1}{j\omega C}}\\ &= \frac{1}{(j\omega)^2 LC + j\omega RC + 1}\\ &= \frac{1}{\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0} + 1}\end{aligned}$$

where, by identifying coefficients, $\frac{1}{\omega_0^2} = LC$ *and* $\frac{2\zeta}{\omega_0} = RC$ *from which* $\omega_0 = \sqrt{\frac{1}{LC}}$ *and* $\zeta = \frac{\omega_0 RC}{2} = \frac{R}{2}\sqrt{\frac{C}{L}}$.

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b) Give expressions for the low and high frequency asymptotes of $\frac{Y}{X}(j\omega)$ and the frequency at which they have the same magnitude. [3]

LF asymptote: $\frac{Y}{X}(j\omega) = 1$. *HF asymptote:* $\frac{Y}{X}(j\omega) = \left(\frac{j\omega}{\omega_0}\right)^{-2}$. *The asymptotes have the same magnitude at* $\omega = \omega_0$.

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c) Determine the magnitude and phase of $\frac{Y}{X}(j\omega)$ at $\omega = \omega_0$. [2]

At $\omega = \omega_0$, $\left(\frac{j\omega}{\omega_0}\right)^2 = -1$ *so* $\frac{Y}{X}(j\omega) = \frac{1}{-1+2\zeta j+1} = \frac{-j}{2\zeta}$. *This has a magnitude of* $\left|\frac{Y}{X}(j\omega_0)\right| = \frac{1}{2\zeta}$ *and a phase* $\angle\frac{Y}{X}(j\omega_0) = -\frac{\pi}{2}$.

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d) Show that $\left|\frac{Y}{X}(j\omega)\right|^{-2}$ may be written as a polynomial in x with real coefficients where $x = \left(\frac{\omega}{\omega_0}\right)^2$. By differentiating this polynomial, or otherwise, show that the maximum value of $\left|\frac{Y}{X}(j\omega)\right|$ occurs at $\omega = \omega_0\sqrt{1-2\zeta^2}$. [6]

$$
\begin{aligned}
\left|\frac{Y}{X}(j\omega)\right|^{-2} &= \left|\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0} + 1\right|^2 \\
&= \left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2 \\
&= (1-x)^2 + 4\zeta^2 x \\
&= x^2 + (4\zeta^2-2)x + 1
\end{aligned}
$$

Setting the derivative of this polynomial to zero to find its minimum gives

$$x_p = -\frac{4\zeta^2-2}{2} = 1-2\zeta^2$$

Hence

$$
\begin{aligned}
\left(\frac{\omega_p}{\omega_0}\right)^2 &= 1-2\zeta^2 \\
\Rightarrow \quad \omega_p &= \omega_0\sqrt{1-2\zeta^2}
\end{aligned}
$$

This is the minimum of $\left|\frac{Y}{X}(j\omega)\right|^{-2}$ *and so must be the maximum of* $\left|\frac{Y}{X}(j\omega)\right|$.

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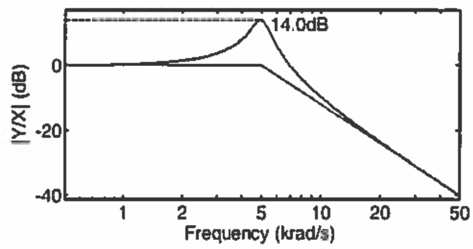
e) Determine values of C and R so that $\omega_0 = 5000\,\text{rad/s}$ and $\zeta = 0.1$ given that $L = 100\,\text{mH}$. [2]

i) Sketch a dimensioned graph of $\left|\frac{Y}{X}(j\omega)\right|$ in decibels using a logarithmic frequency axis. Your graph should include both the high and low frequency asymptotes in addition to a sketch of the true magnitude response. [3]



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ii) If $x(t) = 3\cos\omega_0 t$, determine the average power dissipation of the circuit and the peak value of the energy, $\frac{1}{2}Cy^2(t)$, stored in the capacitor. [3]

At resonance, the total impedance is R and so the average power dissipation is $\frac{1}{2}3^2 \times \frac{1}{R} = 45\,\text{mW}$.

Using phasors, $Y = \frac{Y}{X}(j\omega) \times X = \frac{1}{2\zeta j}X = \frac{1}{0.2j} \times 3 = -15j$. *Hence the peak capacitor voltage is 15 and the peak energy stored is* $\frac{1}{2}C \times 15^2 = 45\,\mu\text{J}$.

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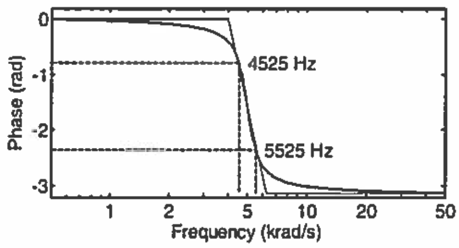
iii) Determine the values of ω for which $\angle\frac{Y}{X}(j\omega) = -45°$ and $-135°$.

Hence sketch a dimensioned graph of $\angle\frac{Y}{X}(j\omega)$ using a straight-line approximation with three segments. Your graph should use a logarithmic frequency axis and a linear phase axis. [6]

$\frac{Y}{X}(j\omega) = \frac{1}{\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0} + 1}$ *so that* $\angle\frac{Y}{X}(j\omega) = -\arctan\left(\frac{2\zeta\frac{\omega}{\omega_0}}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right)$. *We want this to equal* ± 1 *so we need* $1 - \left(\frac{\omega}{\omega_0}\right)^2 = \pm 2\zeta\frac{\omega}{\omega_0}$ *or, equivalently,* $\left(\frac{\omega}{\omega_0}\right)^2 \pm 2\zeta\frac{\omega}{\omega_0} - 1 = 0$. *The roots of this equation are* $\frac{\omega}{\omega_0} = \mp\zeta \pm \sqrt{\zeta^2+1}$.

For $\zeta = 0.1$, *this gives* $\frac{\omega}{\omega_0} = \mp 0.1 \pm \sqrt{1.01} = \pm 0.905, \pm 1.105$. *Taking the positive frequencies,* $\angle\frac{Y}{X}(j\omega) = -45°$ *at* $\omega = 4525$ *and* $\angle\frac{Y}{X}(j\omega) = -135°$ *at* $\omega = 5525$.



The blue curve shows the true phase. The central segment passes through 0 and π *at* ω_a *and* ω_b *respectively where* $\omega_a = 5000 \times \left(\frac{4525}{5000}\right)^2 = 4095$ *and* $\omega_b = 5000 \times \left(\frac{5525}{5000}\right)^2 = 6105$. *The squared frequency ratios arise because we wish to double the phase shift relative to that at* ω_0. *Alternatively, for those with a good memory, the formula given in the lecture notes gives slightly different values of* $\omega_a = 10^{-\zeta}\omega_0 = 3972$ *and* $\omega_b = 10^{+\zeta}\omega_0 = 6295$; *this is the straight line approximation plotted above.*

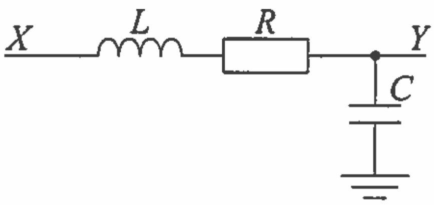


Figure 2.1

3. In the circuit of Fig. 3.1, the input, X, is driven by a voltage source as shown.

a) Derive an expression for the transfer function, $\frac{Y}{X}(j\omega)$ and determine the corner frequencies in its magnitude response. [4]

The circuit is a potential divider, and the impedance of $3R||C$ *is* $\left(\frac{1}{3R}+j\omega C\right)^{-1} = \frac{3R}{1+3j\omega RC}$ *so the transfer function is*

$$\frac{Y}{X}(j\omega) = \frac{R}{R+\frac{3R}{1+3j\omega RC}} = \frac{1+3j\omega RC}{4+3j\omega RC}.$$

The numerator corner frequency is $\omega_n = \frac{1}{3RC}$ *and the denominator corner frequency is* $\omega_d = \frac{4}{3RC}$.

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b) Determine the Thévenin equivalent voltage and resistance of the remainder of the circuit at the terminals of the capacitor. [4]

To determine the Thévenin equivalent voltage, we assume no current flows through the capacitor and determine the voltage across it as $V_{th} = 0.75X$.

To determine the Thévenin equivalent resistance, we connect the two grounds together, short-circuit the voltage source and measure the resistance of the resultant network, which consists of R *and* $3R$ *in parallel. The resistance is therefore* $R_{th} = 0.75R$.

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c) Derive the time constant of the circuit, τ, in two ways: (i) from the Thévenin resistance found in part b) and (ii) from the denominator corner frequency found in part a). [2]

The time constant is (i) $R_{th}C = 0.75RC$ *or alternatively (ii)* $\frac{1}{\omega_d} = 0.75RC$.

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d) If the input voltage, $x(t)$, is given by

$$x(t) = \begin{cases} -2 & \text{for } t < 0 \\ +3 & \text{for } t \geq 0 \end{cases},$$

determine an expression for the output waveform, $y(t)$. Sketch its waveform over approximately the range $-\tau \leq t \leq 4\tau$. [7]

From part a), the DC gain of the circuit is $\frac{Y}{X}(0) = 0.25$. *For* $t < 0$, $y(t) = -2 \times 0.25 = -0.5$.

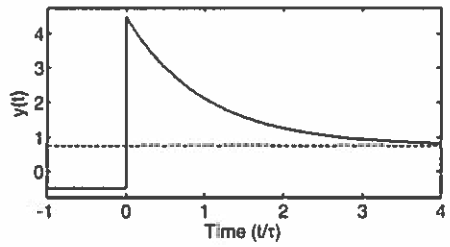
For $t \geq 0$, *the steady state solution is* $y_{SS}(t) = +3 \times 0.25 = 0.75$.

We can calculate $y(0+)$ in two ways:
(i) by ensuring the capacitor voltage, $y(t)-x(t)$, does not change instantly and
(ii) by noting that $y(0+) = y(0-) + \frac{Y}{X}(\infty) \times (x(0+) - x(0-))$.

Using method (i), at $t = 0-$ the capacitor voltage is $y(0-) - x(0-) = -0.5 - (-2) = 1.5$. At $t = 0+$, we therefore still have $1.5 = y(0+) - x(0+) = y(0+) - 3$. From this we get $y(0+) = 1.5 + 3 = 4.5$.

Alternatively, using method (ii) we have $y(0+) = y(0-) + \frac{Y}{X}(\infty) \times (x(0+) - x(0-)) = -0.5 + 1 \times (3 - (-2)) = 4.5$.

For $t \geq 0$, the output is therefore given by $y(t) = y_{SS}(t) + (y(0+) - y_{SS}(0+))e^{-\frac{t}{\tau}} = 0.75 + 3.75e^{-\frac{t}{\tau}}$ where $\tau = 0.75RC$ from part c). The dashed line in the plot below shows the asymptote $y(+\infty) = 0.75$.



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e) Assuming that the opamp in Fig. 3.2 is ideal, determine the transfer function, $\frac{V}{U}(j\omega)$. [4]

From the standard gain of a non-inverting amplifier, the gain is $\frac{V}{U} = 1 + \frac{Z}{R} = \frac{R+Z}{R}$ where Z is the impedance of the $3R||C$ combination. Note that this is just the reciprocal of the gain $\frac{Y}{X}$. Making use of the previous result, we therefore have $\frac{V}{U}(j\omega) = \frac{4+3j\omega RC}{1+3j\omega RC}$. Equivalently, we can regard the circuit as a potential divider with V as the input and U as the output (since the inverting input of the opamp is constrained by negative feedback to equal U).

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f) By considering the voltage across the capacitor, explain why an input voltage discontinuity of Δu will result in an output voltage discontinuity of the same amplitude. [2]

If $u(t)$ suddenly changes by Δu, then negative feedback will ensure that the inverting input of the opamp changes by the same amount and, since the voltage across the capacitor cannot change instantly, V, must jump by the same amount.

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g) If $R = 20\text{k}\Omega$, $C = 20\,\text{nF}$ and the input voltage, $u(t)$, is given by

$$u(t) = \begin{cases} \sin 1000t & \text{for } t < 0 \\ 2\cos 2000t & \text{for } t \geq 0 \end{cases},$$

determine expressions for the output $v(t)$ for both positive and negative t. [7]

From part e), the transfer function is $\frac{V}{U}(j\omega) = \frac{4+3j\omega RC}{1+3j\omega RC}$ *with* $RC = 4 \times 10^{-4}$.

At $\omega_1 = 1000$, $\omega_1 RC = 0.4$ and so $\frac{V}{U}(j\omega_1) = \frac{4+1.2j}{1+1.2j} = 2.23 - 1.475j$. *Using phasors, for* $t < 0$, $U_1 = -j$ *and so* $V_1 = -j \times (2.23 - 1.475j) = -1.475 - 2.23j$. *Hence, for* $t < 0$, *we have* $v(t) = -1.475\cos 1000t + 2.23\sin 1000t$.

At $\omega_2 = 2000$, $\omega_2 RC = 0.8$ and so $\frac{V}{U}(j\omega_2) = \frac{4+2.4j}{1+2.4j} = 1.44 - 1.065j$. *Using phasors, for* $t \geq 0$, $U_2 = 2$ *and so* $V_2 = 2 \times (1.44 - 1.065j) = 2.89 - 2.13j$. *Hence* $v_{2,SS}(t) = 2.89\cos 2000t + 2.13\sin 2000t$.

To determine the transient amplitude, we note that $v(0-) = -1.475$, $v_{SS}(0+) = 2.89$ and that $\Delta v = 1 \times \Delta u = 2$. Thus $v(0+) = v(0-) + \Delta v = -1.475 + 2 = 0.525$ and the transient amplitude is $0.525 - 2.89 = -2.365$.

Thus, for $t \geq 0$, we have $v_{2,SS}(t) = 2.89\cos 2000t + 2.13\sin 2000t - 2.365e^{-\frac{t}{\tau}}$ *where* $\tau = \frac{1}{3RC} = \frac{1}{1.2}\,\text{ms} = 0.833\,\text{ms}$.

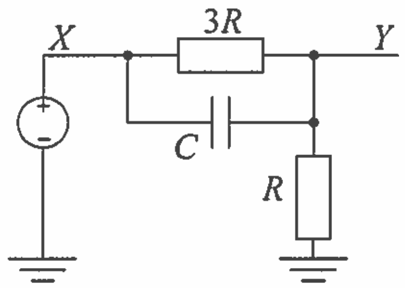

Figure 3.1

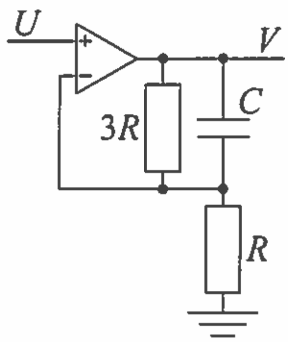

Figure 3.2