Master - May 08

C2.2

SYSTEM IDENTIFICATION, Exam of May 2008, Solutions

Question 1. (a) We regard the complex number $x+iy$ as being equivalent to the vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. Since a_k and b_k are independent, the probability density of the random vector $\begin{bmatrix} a_k \\ b_k \end{bmatrix}$ is $f(x,y) = f_a(x) \cdot f_b(y)$ where f_a and f_b are the densities of a_k and b_k, respectively. Since a_k and b_k are normalized Gaussian, we have $f_a(x) = f_b(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, so that

$$f(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} .$$

(b) Since $e^{i\psi_k} c_k$ is a rotated version of c_k, and the density $f(x,y)$ is invariant under rotation (it only depends on the radius $r = \sqrt{x^2+y^2}$), it follows that $e^{i\psi_k} c_k$ has the same density as c_k (as computed in part (a)). Thus, $e^{i\psi_k} c_k$ is normalized Gaussian. Since a_j and b_j are independent of a_k and b_k (for $j \neq k$), any function of a_j, b_j is independent of any function of a_k, b_k. Thus, the terms of the sequence of random variables $e^{i\psi_k} c_k$ are independent of each other. Thus, by definition, this is normalized Gaussian white noise.

(c) White noise is ergodic. Hence, the averages of the white noise $(e^{i\nu k} c_k)$ converge (with probability 1) to $E(e^{i\nu k} c_k) = e^{i\nu k} E(c_k) = 0$.

(d) The complex conjugate random variable corresponds to the random vector $\begin{bmatrix} a_k \\ -b_k \end{bmatrix}$, which is again normalized Gaussian. Thus, $(\overline{c_k})$ and also $(e^{i\nu k}\,\overline{c_k})$ are normalized white noise signals, so that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} e^{i\nu k}\,\overline{c_k} = 0, \text{ with prob. } 1.$$

Adding this to the result from part (c), we obtain

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} e^{i\nu k}\, a_k = 0, \text{ with prob. } 1.$$

Taking here real and imaginary parts, we obtain the desired statements.

(e) Assume that $w_k = g_0 a_k + g_1 a_{k-1} \dots + g_n a_{k-n}$.

Then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \cos(\nu k)\, w_k = g_0 \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \cos(\nu k)\, a_k$$

$$+ g_1 \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \cos(\nu k)\, a_{k-1} \dots + g_n \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \cos(\nu k)\, a_{k-n}$$

where each limit on the right-hand side is zero. For $\sin(\nu k)$ in place of $\cos(\nu k)$ the proof is similar.

(f) We compute

$$c = \frac{1}{10^6}\sum_{k=1}^{10^6} u_k \cos(0.01 k), \quad s = \frac{1}{10^6}\sum_{k=1}^{10^6} u_k \sin(0.01 k).$$

Since $u_k = A\cos(0.01\,k)\sin\varphi + A\sin(0.01 k)\cos\varphi + w_k$, using the statement from (e) we obtain that

$$c \approx \frac{1}{2} A \sin\varphi, \quad s \approx \frac{1}{2} A \cos\varphi.$$

From here we can easily estimate A and φ.

Question 2. (a) $w^2 = \lambda + \dfrac{u^2}{\alpha^2} + \dfrac{v^2}{\beta^2}$,

hence
$$\underbrace{w_k^2}_{y_k} = \underbrace{\begin{bmatrix} 1 & u_k^2 & v_k^2 \end{bmatrix}}_{\varphi_k} \underbrace{\begin{bmatrix} \lambda \\ 1/\alpha^2 \\ 1/\beta^2 \end{bmatrix}}_{\theta} + e_k .$$

(b) $J(\theta)$ has a unique minimum at $\theta = \hat{\theta}$ if and only if $\Phi^*\Phi$ is invertible, where $\Phi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{200} \end{bmatrix}$.

Equivalently, Φ should have full column rank, i.e., 3 independent columns.

If this is the case, then $\hat{\theta} = \Phi^{\#} y$, where
$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{200} \end{bmatrix} \quad \text{and} \quad \Phi^{\#} = (\Phi^*\Phi)^{-1}\Phi^* .$$

(c) If $u_k^2 - v_k^2 = 18$ of the last two columns of Φ gives 18 times the first column, so that J has no unique minimum. We now have the model
$$w_k^2 = \lambda + \frac{u_k^2}{\alpha^2} + \frac{u_k^2 - 18}{\beta^2} + e_k$$
$$= \lambda + u_k^2\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right) - \frac{18}{\beta^2} + e_k = \begin{bmatrix} 1 & u_k^2 \end{bmatrix} \begin{bmatrix} \lambda - \frac{18}{\beta^2} \\ \frac{1}{\alpha^2} + \frac{1}{\beta^2} \end{bmatrix} + e_k .$$

From here, we can estimate the two numbers $\lambda - \frac{18}{\beta^2}$ and $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ in the standard way.

(d)
$$\widehat{\mathrm{Var}(e_k)} = \frac{1}{197}\|y - \Phi\hat{\theta}\|^2 = \frac{1}{197} y^*(I - \Phi\Phi^{\#})y .$$
(197 = 200 − 3)

(e)
$$\widehat{\mathrm{Cov}\,\hat{\theta}} = \widehat{\mathrm{Var}(e_k)}\,(\Phi^*\Phi)^{-1} .$$

(f) For N measurements $(k = 1, 2, \ldots N)$ we have

$$\Phi^*\Phi = \begin{bmatrix} N & \sum_{k=1}^{N} u_k^2 & \sum_{k=1}^{N} v_k^2 \\ \sum_{k=1}^{N} u_k^2 & \sum_{k=1}^{N} u_k^4 & \sum_{k=1}^{N} u_k^2 v_k^2 \\ \sum_{k=1}^{N} v_k^2 & \sum_{k=1}^{N} u_k^2 v_k^2 & \sum_{k=1}^{N} v_k^4 \end{bmatrix}.$$

Since u_k and v_k are independent white noise signals, they are jointly ergodic. Therefore

$$\Phi^*\Phi \approx N \begin{bmatrix} 1 & E(u_k^2) & E(v_k^2) \\ E(u_k^2) & E(u_k^4) & E(u_k^2 v_k^2) \\ E(v_k^2) & E(u_k^2 v_k^2) & E(v_k^4) \end{bmatrix}.$$

The 3×3 matrix on the right-hand side above is independent of N. Thus, $\Phi^*\Phi$ grows proportionally to N. According to our result at part (e) or, more precisely, because of

$$\mathrm{Cov}\,\hat{\theta} = \mathrm{Var}(e_k)\,(\Phi^*\Phi)^{-1},$$

$\mathrm{Cov}\,\hat{\theta}$ is inverse proportional to N. Thus, for 800 measurements (instead of 200) we expect $\mathrm{Cov}\,\hat{\theta}$ to be 4 times smaller.

Question 3. (a) Denote $\mathring{\alpha}_k = \alpha_k - E(\alpha_k)$, and similarly for $\mathring{\beta}_k, \mathring{\gamma}_k$, so that $\mathring{\gamma}_k = \mathring{\alpha}_k + \mathring{\beta}_k$. We have $C_\tau^{\gamma\gamma} = E(\mathring{\gamma}_k \cdot \mathring{\gamma}_{k-\tau}) = E(\mathring{\alpha}_k \cdot \mathring{\alpha}_{k-\tau}) + E(\mathring{\alpha}_k \mathring{\beta}_{k-\tau}) + E(\mathring{\beta}_k \cdot \mathring{\alpha}_{k-\tau}) + E(\mathring{\beta}_k \mathring{\beta}_{k-\tau})$. Since (α_k) and (β_k) are independent signals, the two middle terms are zero and we get

$$C_\tau^{\gamma\gamma} = C_\tau^{\alpha\alpha} + C_\tau^{\beta\beta}.$$

Applying the $\mathcal{Z}$ transformation, $S^{\gamma\gamma} = S^{\alpha\alpha} + S^{\beta\beta}$.

(b) Denote $A(z) = 1 + a_1 z^{-1} \dots + a_4 z^{-4}$, $B(z) = b_0 + b_1 z^{-1} \dots + b_4 z^{-4}$, then by (1) $A(z)\hat{p}(z) = B(z)\hat{u}(z) + \hat{v}(z)$. From $\hat{y} = \hat{p} + \hat{w}$ we get $A\hat{y} = A\hat{p} + A\hat{w} = B\hat{u} + \hat{v} + A\hat{w}$. According to the problem statement, A^{-1} is stable. Denoting $\hat{\delta} = \hat{w} + A^{-1}\hat{v}$, we obtain

$$A(z)\,\hat{y}(z) = B(z)\,\hat{u}(z) + A(z)\,\hat{\delta}(z). \qquad (*)$$

According to our result from part (a) we have $S^{\delta\delta} = S^{ww} + |A^{-1}|^2 S^{vv}$. By the problem statement we have $S^{ww} \geq 0.1$, hence $S^{\delta\delta} \geq 0.1$. Since δ is Gaussian, this implies that δ can be represented as

$$\hat{\delta} = \Xi\,\hat{e}, \quad \Xi, \Xi^{-1} \text{ stable}, \quad e \text{ white noise}.$$

Since Ξ is stable, its impulse response (ξ_k) tends to zero and we can approximate Ξ by truncating its impulse response:

$$\Xi(z) \approx 1 + \xi_1 z^{-1} + \xi_2 z^{-2} \dots + \xi_n z^{-n} = \Xi_n(z).$$

The coefficient ξ_0 has been taken $= 1$, which is possible by rescaling e. Now $(*)$ becomes

$$A(z)\,\hat{y}(z) = B(z)\,\hat{u}(z) + A(z)\,\Xi_n(z)\,\hat{e}(z),$$

which is the desired ARMAX model.

(c) Denoting $C(z) = A(z)\,\Xi_n(z)$, the ARMAX equation from part (b) is $A\hat{y} = B\hat{u} + C\hat{e}$. By assumption, A^{-1} is stable. Since Ξ^{-1} from part (b) is stable, we may assume that also Ξ_n^{-1} is stable. This implies that C^{-1} is stable. Divide the ARMAX equation by C: $(A/C)\hat{y} = (B/C)\hat{u} + \hat{e}$, and introduce the impulse responses of A/C and B/C:

$$\frac{A(z)}{C(z)} = 1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \dots$$

$$\frac{B(z)}{C(z)} = \beta_0 + \beta_1 z^{-1} + \beta_2 z^{-2} + \dots$$

Since C^{-1} is stable, the sequences (α_k) and (β_k) tend to zero. Hence, by truncating A/C and B/C to polynomials (in z^{-1}) of a high order m, we get good approximations of these functions, and the approximate ARX model

$$y_k + \alpha_1 y_{k-1} + \alpha_2 y_{k-2} \dots + \alpha_m y_{k-m} = \beta_0 u_k + \beta_1 u_{k-1} \dots + \beta_m u_{k-m} + e_k .$$

(d) We have

$$y_k = \underbrace{\left[-y_{k-1}\ -y_{k-2}\ \dots\ -y_{k-m}\quad u_k\ u_{k-1}\ \dots\ u_{k-m}\right]}_{\varphi_k}\underbrace{\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_m\\ \beta_0\\ \beta_1\\ \vdots\\ \beta_m\end{bmatrix}}_{\theta} + e_k .$$

Denoting

$$y = \begin{bmatrix} y_1\\ y_2\\ \vdots\\ y_{20,000}\end{bmatrix}, \quad \phi = \begin{bmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_{20,000}\end{bmatrix}, \qquad \boxed{\phi^{\#} = (\phi^*\phi)^{-1}\phi^*}$$

the optimal least squares estimate of θ is $\hat{\theta} = \phi^{\#} y$.

(e) After having estimated θ from part (d), we can estimate (e_k) using the ARX equation. Now we rewrite the ARMAX equation from the top of this page as $y_k = \tilde{\varphi}_k\tilde{\theta} + \delta_k$, where

$$\tilde{\varphi}_k = \left[-y_{k-1}\ -y_{k-2}\ \dots -y_{k-4}\quad u_k\ u_{k-1}\ \dots\ u_{k-4}\quad e_{k-1}\ e_{k-2}\ \dots\ e_{k-q}\right],$$

$$\tilde{\theta}^T = \left[\,a_1\ \ a_2\ \dots\ a_4\ \ b_0\ \ b_1\ \dots\ b_4\ \ c_1\ \ c_2\ \dots\ c_q\,\right],$$

and $\delta_k = e_k +$ new modeling error. From here we can estimate $\tilde{\theta}$ in the usual way $(q = n+4)$.

Question 4. (a) The impedance Z of the three components in parallel is given by

$$\frac{1}{Z(s)} = Cs + \frac{1}{R} + \frac{1}{L_l s} = \frac{CRL_l s^2 + L_l s + R}{RL_l s},$$

so that
$$Z(s) = \frac{RL_l s}{CRL_l s^2 + L_l s + R}.$$

The transfer function from u to y is

$$G(s) = \frac{Z(s)}{Z(s) + Ls} = \frac{RL_l s}{RL_l s + Ls(CRL_l s^2 + L_l s + R)}$$

$$= \frac{\frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{L+L_l}{LCL_l}} = \frac{b_0}{s^2 + a_1 s + a_0}.$$

Note that b_0 is known, while a_1, a_0 are unknown. G is stable, because a_1 and a_0 are >0.

(b) The sum of the two inductor voltages is u. If u is a positive constant, then the inductor currents will grow to infinity, since L times the derivative of the current through the inductor L is the voltage of this inductor. Hence, the system is unstable. (More precisely, zero is an eigenvalue of the system, and the corresponding eigenvector is unobservable.)

(c) We denote by $G^e(i\omega_k)$ the values of the transfer function determined using a sinusoidal signal u (here, $k = 1, 2, \dots 25$). We have

$$b_0 = \left[(i\omega_k)^2 + a_1(i\omega_k) + a_0\right] G^e(i\omega_k) - e_k,$$

where e_k are the equation errors (due to measurement errors and model mismatch).

Thus,
$$\underbrace{(i\omega_k)^2 G^e(i\omega_k) - b_0}_{y_k} = \underbrace{\begin{bmatrix} -i\omega_k & -1 \end{bmatrix} G^e(i\omega_k)}_{\varphi_k} \underbrace{\begin{bmatrix} a_1 \\ a_0 \end{bmatrix}}_{\theta} + e_k .$$

(d) We are searching for the optimal real θ. We put $\tilde{y}_k = \mathrm{Re}\, y_k$, $\tilde{\varphi}_k = \mathrm{Re}\, \varphi_k$ for $k = 1, 2, \dots 25$, and $\tilde{y}_k = \mathrm{Im}\, y_{k-25}$, $\tilde{\varphi}_k = \mathrm{Im}\, \varphi_{k-25}$ for $k = 26, 27, \dots 50$.

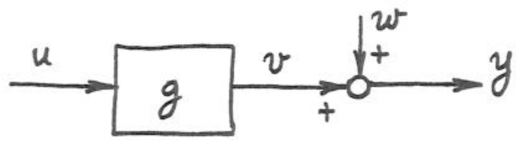
The new error terms $\tilde{e}_k$ $(k = 1, 2, \dots 50)$ are defined similarly. Then $\tilde{y}_k = \tilde{\varphi}_k \theta + \tilde{e}_k$ for $k = 1, 2, \dots 50$, and $\sum_{k=1}^{50} \tilde{e}_k^2 = \sum_{k=1}^{25} |e_k|^2$. The optimal θ (which minimizes $\sum_{k=1}^{50} \tilde{e}_k^2$) is given by $\hat{\theta} = \tilde{\Phi}^{\#} \tilde{y}$, where $\tilde{y} = [\tilde{y}_1, \dots \tilde{y}_{50}]^T$,
$\tilde{\Phi} = \begin{bmatrix} \tilde{\varphi}_1 \\ \vdots \\ \tilde{\varphi}_{50} \end{bmatrix}$, $\tilde{\Phi}^{\#} = (\tilde{\Phi}^* \tilde{\Phi})^{-1} \tilde{\Phi}^*$. From the estimated a_1 we estimate R, and then (from a_0) L_1.

(e)
$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} b_0 & 0 \end{bmatrix}, \quad D = 0.$$

(f)
$$A^d = e^{AT}, \quad B^d = \left(e^{AT} - I\right) A^{-1} B,$$
$$G^d(z) = C(zI - A^d)^{-1} B^d + D$$

(this is exact discretisation). Alternatively, we get a good approximation to G^d by Tustin's formula:
$$G^d(z) \approx G\left(\frac{2}{T} \cdot \frac{z-1}{z+1}\right),$$
valid if the absolute values of the poles of G are much smaller than $2\pi/T$, the sampling frequency in rad/sec. In the specific example, G is stable hence also G^d is stable.

Question 5.



(a) If u and y are jointly ergodic, then the expectation of any function of u and y (which may depend on current and past values) can be approximated by averaging over a long time. Thus, for example, $E(u_k) \overset{a.s.}{=} \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} u_j$, where the abreviation a.s. ("almost sure") means that the equality holds with probability 1. A similar formula holds for $E(y_k)$, obviously. Denote

$$\overset{\circ}{u}_k = u_k - E(u_k), \qquad \overset{\circ}{y}_k = y_k - E(y_k),$$

then for any $\tau \in \mathbb{Z}$, ergodicity implies

$$C_\tau^{uu} = E(\overset{\circ}{u}_k \cdot \overset{\circ}{u}_{k-\tau}) \overset{a.s.}{=} \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \overset{\circ}{u}_j \overset{\circ}{u}_{j-\tau},$$

$$C_\tau^{yu} = E(\overset{\circ}{y}_k \cdot \overset{\circ}{u}_{k-\tau}) \overset{a.s.}{=} \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \overset{\circ}{y}_j \overset{\circ}{u}_{j-\tau}.$$

In practice, we have only finitely many data, so that in all the above formulas, we have to replace $\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N}$ with $\frac{1}{N}\sum_{j=a}^{a+N}$, where N is large (and the starting time a depends on the data that we have). In our specific case, when u_k and y_k are given for $k = 1, 2, \ldots 6000$ and $\tau = 0, 1, \ldots 30$, we approximate

$$C_\tau^{uu} \approx \frac{1}{6000-\tau-1}\sum_{j=\tau+1}^{6000} (u_j - \bar{u})(u_{j-\tau} - \bar{u}),$$

where $\bar{u}$ is the average of all available u_j (so that $\bar{u} \approx E(u_k)$). A similar approximation can be used for C_τ^{yu}.

(b) $C_\tau^{yu} = E(\overset{\circ}{y}_k \cdot \overset{\circ}{u}_{k-\tau}) = E(\overset{\circ}{v}_k \overset{\circ}{u}_{k-\tau}) + E(\overset{\circ}{w}_k \overset{\circ}{u}_{k-\tau}) =$
$= C_\tau^{vu} + C_\tau^{wu}$, so that $C^{yu} = C^{vu} + C^{wu}$
$= g * C^{uu} + C^{wu}$.

(c) If u and w are independent of each other, then $C^{wu}=0$, so that (according to the result from part (b)), $C^{yu}=g * C^{uu}$. This can be written as an infinite matrix equation:

$$\begin{bmatrix} C_0^{uu} & C_{-1}^{uu} & C_{-2}^{uu} & \cdots \\ C_1^{uu} & C_0^{uu} & C_{-1}^{uu} & \cdots \\ C_2^{uu} & C_1^{uu} & C_0^{uu} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \cdot \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} C_0^{yu} \\ C_1^{yu} \\ C_2^{yu} \\ \vdots \end{bmatrix}. \qquad (**)$$

Since $g_k \to 0$ (by stability), we can approximate $g_k \approx 0$ for $k>30$. Looking only at the first 31 equations, we now get 31 equations with 31 unknowns $g_0, g_1, \cdots g_{30}$. The coefficients C_τ^{uu} and C_τ^{yu} are not known exactly, but they have been estimated in (a). Recall that $C_{-\tau}^{uu}=C_\tau^{uu}$.

(d) u is persistent of order N if the $N \times N$ truncation of the infinite matrix from $(**)$ is invertible. If this is the case, and the coefficients in the equation have been estimated sufficiently accurately, then we can solve the truncated equation for $g_0, g_1, \cdots g_{N-1}$.

The matrix from $(**)$ is $\geqslant 0$, hence any $N \times N$ truncation of it is also $\geqslant 0$. A matrix $P \geqslant 0$ is invertible if and only if $P>0$, i.e., $x^* P x > 0$ for any vector $x \neq 0$ of matching dimension. This implies that if $P>0$ and we truncate P, keeping only its first m rows and first m columns, then the truncated matrix is again >0 (hence, invertible). definition of $P>0$

(e) $\hat{u}(z)=(1-0.3z^{-1})\hat{e}(z)$, $S^{uu}(z)=|1-0.3z^{-1}|^2$, it is easy to see that $|1-0.3z^{-1}| \geqslant 0.7$ for all z with $|z|=1$, hence the claim.

(f) The difference equation of the FIR filter is $\begin{pmatrix} u = \text{input}, \\ v = \text{output} \end{pmatrix}$

$$v_k = g_0 u_k + g_1 u_{k-1} + g_2 u_{k-2} \cdots + g_{N-1} u_{k-N+1}.$$