

Question 1

Linear optimal Control

E4.22

C1.2

Solutions - 2008

En 4.57

(a) Bookwork.

* For step $k \leq N$

Dynamic programming iteration is:

$$J_k(x_k) = \min_{u_k} x_k' Q x_k + u_k' R u_k + J_{k+1}(A x_k + B u_k).$$

where the cost-to-go function is assumed to be quadratic, i.e.

$$J_{k+1}(x_{k+1}) = x_{k+1}' P_{k+1} x_{k+1} \quad \text{Assume } P_{k+1} \geq 0 \quad (\text{see below})$$

$$\begin{aligned} \Rightarrow J_k(x_k) &= \min_{u_k} x_k' Q x_k + u_k' R u_k + (A x_k + B u_k)' P_{k+1} (A x_k + B u_k) \\ &= x_k' Q x_k + \min_{u_k} u_k' R u_k + u_k' B' P_{k+1} A x_k + x_k' A' P_{k+1} B u_k + x_k' A' P_{k+1} A x_k \end{aligned} \quad [3]$$

Differentiating with respect to u_k and setting to zero: [2]

$$\begin{aligned} (R + B' P_{k+1} B) u_k + B' P_{k+1} A x_k &= 0 \\ \text{Assume } P_{k+1} \geq 0 \text{ and } R > 0 &\Rightarrow R + B' P_{k+1} B > 0 \quad [2] \\ \Leftrightarrow u_k &= - (R + B' P_{k+1} B)^{-1} B' P_{k+1} A x_k \\ & \quad (\text{this guarantees that } u_k \text{ is a minimum \& unique}) \quad [2] \\ & \quad \& \text{ that if } P_{k+1} \geq 0 \Rightarrow P_k \geq 0 \end{aligned}$$

Substitute the above into expression for $J_k(x_k)$, we get [3]

$$J_k(x_k) = x_k' \left[A' P_{k+1} A + Q - A' P_{k+1} B (B' P_{k+1} B + R)^{-1} B' P_{k+1} A \right] x_k$$

$$\Rightarrow P_k = A' (P_{k+1} - P_{k+1} B (B' P_{k+1} B + R)^{-1} B' P_{k+1}) A + Q \quad [2]$$

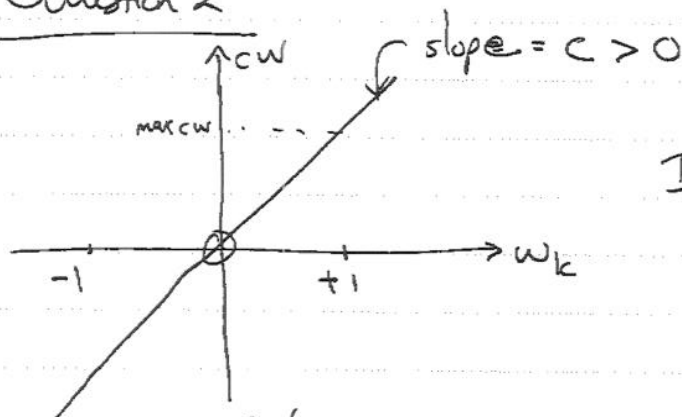
Step $k=N$ The same holds for the boundary condition with $P_N = Q$
 $\Rightarrow P_{N-1} = A' (P_N - P_N B (B' P_N B + R)^{-1} B' P_N) A + Q$ [2]

(1b) Replace Q with $\frac{1}{2}(Q+Q')$ and R with $\frac{1}{2}(R+R')$ [2]

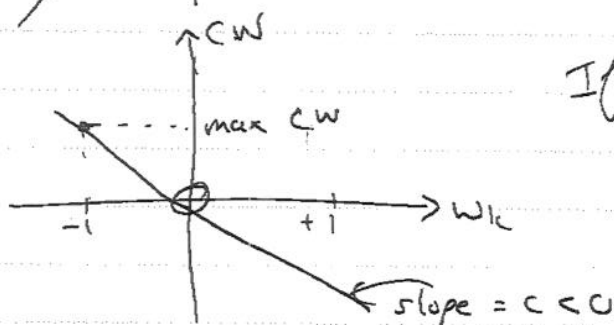
This is because $x'Mx = \frac{1}{2}x'(M+M')x$ for any matrix M , where $M+M'$ is symmetric. [2]

Question 2

a)



$$\text{If } c \geq 0 \Rightarrow \max_{-1 \leq w_k \leq 1} CW_k = c \quad [1]$$



$$\text{If } c < 0 \Rightarrow \max_{-1 \leq w_k \leq 1} CW_k = -c \quad [1]$$

$$\text{If } c \text{ is any scalar} \\ \Rightarrow \max_{-1 \leq w_k \leq 1} CW_k = |c| \quad [2]$$

~~(Partly done in class)~~

b) ~~If $c=0$~~ Partly done in class, but new problem

$$\Rightarrow x_{k+1} = (a+L)x_k + w_k$$

$$x_1 = (a+L)x_0 + w_0$$

$$x_2 = (a+L)x_1 + w_1 = (a+L)^2 x_0 + (a+L)w_0 + w_1$$

\vdots

$$x_k = \cancel{(a+L)^k x_0} + \sum_{i=0}^{k-1} (a+L)^i w_{k-i-1}$$

$$\Rightarrow \text{if } x_0 = 0 \Rightarrow x_k = \sum_{i=0}^{k-1} (a+L)^i w_{k-i-1} \quad [2]$$

$$\begin{aligned} & \text{(from part a)} \\ \Rightarrow \max_{-1 \leq w_0, \dots, w_{k-1} \leq 1} x_k &= \sum_{i=0}^{k-1} \cancel{(a+L)^i} |a+L|^i \quad [2] \end{aligned}$$

$$\text{also, } \max_{-1 \leq w_0, \dots, w_{k-1} \leq 1} -x_k = \sum_{i=0}^{k-1} |-(a+L)^i| = \sum_{i=0}^{k-1} |a+L|^i$$

$$\Rightarrow \max_{w_0, \dots, w_{k-1}} x_k = \max_{w_0, \dots, w_{k-1}} -x_k \Rightarrow \max_{w_0, \dots, w_{k-1}} x_k = \max_{w_0, \dots, w_{k-1}} |x_k| \quad [2]$$

Since $S_{k+1} - S_k = |a+L|^k \geq 0$, the sequence is non-decreasing. $[2]$

2b) continued...

The sequence $S_k = \sum_{i=0}^{k-1} |a+L|^i$ is convergent if and only if $|a+L| < 1$. Since $\{S_k\}$ is an increasing sequence, it is bounded iff $|a+L| < 1$. [1]

2c)

2c) Since $\{S_k\}$ is non-decreasing, we have that

$$\begin{aligned} \max_{k=0,1,\dots} S_k &= \lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} \sum_{i=0}^{k-1} |a+L|^i \\ &= \sum_{i=0}^{\infty} |a+L|^i = \frac{1}{1-|a+L|} \Leftrightarrow |a+L| < 1 \end{aligned} \quad [2]$$

We have two cases: ~~two cases~~ [1]

I) $0 \leq a+L < 1 \Rightarrow |a+L| = a+L$

$$\Rightarrow \arg \min_{-a \leq L < 1-a} \frac{1}{1-a-L} = L^*$$

$\frac{1}{1-a-L}$ is minimised if the denominator is maximised, i.e.

we need to make L as small as possible $\Rightarrow L^* = -a$
 $(-L \text{ as large})$ [2]

II) $-1 < a+L \leq 0 \Rightarrow |a+L| = -a-L$

$$\Rightarrow L^* = \arg \min_{-1-a \leq L \leq -a} \frac{1}{1+a+L}$$

$\frac{1}{1+a+L}$ is minimised if $1+a+L$ is maximised, i.e. we need to make L as large as possible $\Rightarrow L^* = -a$ [2]

$$\Rightarrow \underline{L^* = -a} \quad \text{QED.} \quad [1]$$

Question 3

a) Bookwork.

$$\begin{aligned} (\bar{Q}^{1/2}, \bar{A}) & \text{ detectable} \\ (\bar{A}, \bar{B}) & \text{ stabilisable} \\ \bar{Q} & \geq 0 \\ \bar{R} & > 0 \end{aligned}$$

[4]

b) New problem.

With the given choice of $Q(t)$ and $R(t)$ we have

$$u^*(\cdot) = \arg \min_u \int_0^\infty e^{2\alpha t} x(t)' \bar{Q} x(t) + e^{2\alpha t} u(t)' \bar{R} u(t) dt \quad [2]$$

Comparing with the cost in part (a) and referring to the hint, these two problems are equivalent if we make the ~~sub~~ change of variables: [2]

$$z(t) = e^{\alpha t} x(t) \text{ and } v(t) = e^{\alpha t} u(t). \quad [2]$$

$$\begin{aligned} \Rightarrow \dot{z} &= \alpha e^{\alpha t} x + e^{\alpha t} \dot{x} = \alpha z + e^{\alpha t} (Ax + Bu) \\ &= \alpha z + A e^{\alpha t} x + B e^{\alpha t} u \\ &= \alpha z + A z + B v \\ &= (A + \alpha I) z + B v \end{aligned} \quad [4]$$

$$\Rightarrow \bar{A} = (A + \alpha I) \text{ and } \bar{B} = B \quad [2]$$

$$\begin{aligned} \Rightarrow \bar{u}^*(t) &= e^{-\alpha t} u^*(t) = -e^{-\alpha t} \bar{R}^{-1} \bar{B}' \bar{P} z(t) \\ &= -\bar{R}^{-1} \bar{B}' \bar{P} x(t) \quad (\text{because } x(t) = e^{-\alpha t} z(t)) \end{aligned} \quad [2]$$

where \bar{P} is obtained from ARE in part a) with $\bar{A} = A + \alpha I$
 $\bar{B} = B.$

$$\Rightarrow \bar{P} (A + \alpha I)' \bar{P} + \bar{P} (A + \alpha I) + \bar{Q} - \bar{P} \bar{B} \bar{R}^{-1} \bar{B}' \bar{P} = 0 \quad [2]$$

QED

Question #4

a) Bookwork.

Any control policy which is optimal over the interval $[i, N]$ is necessarily optimal over the interval $[i+1, N]$ [4]

b) New problem.

For step $k \leq N$

The Dynamic Programming iteration is:

$$J_k(x_k) = \min_{u_k} c'x_k + d(u_k) + E_{\substack{w_k \\ A_k}} [J_{k+1}(A_k x_k + \beta(u_k) + w_k)] \quad [2]$$

Assume that the ~~set~~ $J_{k+1}(\cdot)$ is linear + constant, i.e.

$$J_{k+1}(x_{k+1}) = m_{k+1}' x_{k+1} + n_{k+1} \quad [1]$$

$$\Rightarrow J_k(x_k) = \min_{u_k} c'x_k + d(u_k) + E_{\substack{w_k \\ A_k}} [m_{k+1}' A_k x_k + m_{k+1}' \beta(u_k) + m_{k+1}' w_k + n_{k+1}]$$

$$= c'x_k + m_{k+1}' E[A_k] x_k + m_{k+1}' E[\beta(u_k)] + m_{k+1}' E[w_k] + n_{k+1}$$

$$+ \min_{u_k} \{d(u_k) + m_{k+1}' \beta(u_k)\} \quad [3]$$

$$\Rightarrow m_k' = c' + m_{k+1}' E[A_k] + m_{k+1}' E[\beta(u_k)]$$

$$\text{and } n_k = m_{k+1}' E[w_k] + n_{k+1} + \min_{u_k} \{d(u_k) + m_{k+1}' \beta(u_k)\} \quad [2]$$

$$\Rightarrow J_k(x_k) = m_k' x_k + n_k$$

~~For $u_k = u^*$~~
 The same has to hold for $k=N$ where
 $m_N = c'$ and $n_N = 0$
 ~~$\Rightarrow J_N(x_N) = m_N' x_N + n_N$~~

4b) continued...
For $k=N$

Check that the same holds with $m_N = C$ and $n_N = 0$

$$\begin{aligned}\Rightarrow J_{N-1}(x_{N-1}) &= m'_{N-1} x_{N-1} + n_{N-1} \\ &= (C' + C' E[A_{N-1}]) x_{N-1} \\ &\quad + C E[W_{N-1}] + \min_{u_{N-1}} \{d(u_{N-1}) + C' \beta(u_{N-1})\}.\end{aligned}$$

QED \Rightarrow [2]

4c) "Certainty ~~to~~ equivalence" holds if the optimal control law for the non-deterministic problem is the same as the optimal control law for the deterministic problem, where the ~~the~~ uncertainty is replaced with the expected value of the uncertainty. [3]

In the above problem, we see that the minimisation with respect to u_k is

$$u_k^* = \arg \min_{u_k} d(u_k) + (C' + m'_{k+2} E[A_k]) \beta(u_k),$$

~~station~~ as in the deterministic case ~~above~~
which is the same, if $E[A_k]$ is replaced with A_k .

\Rightarrow Certainty equivalence does hold. [3].

Question 5

a) $R > 0 \Rightarrow \min_u \ell(x, u)$ is obtained from. [1]
 $\frac{\partial \ell}{\partial u} = 2Ru + 2S'z = 0 \Leftrightarrow u = -R^{-1}S'z$ [2]

$$\begin{aligned} \Rightarrow L(z) &= \cancel{2z'SR^{-1}S'z} \\ &= z'Qz + z'SR^{-1}RR^{-1}S'z \\ &\quad - 2z'SR^{-1}S'z \\ &= z'(Q - SR^{-1}S')z \end{aligned} \quad [2]$$

$$L(z) \geq 0 \Leftrightarrow Q - SR^{-1}S' \geq 0. \quad [1].$$

b) New problem.

$$\begin{aligned} z_{k+1} = \begin{pmatrix} x_{k+1} \\ u_k \end{pmatrix} &= \begin{pmatrix} Ax_k + Bu_k \\ u_k \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_k \\ u_{k-1} \end{pmatrix} + \begin{pmatrix} B \\ I \end{pmatrix} u_k \\ &= \bar{A} z_k + \bar{B} u_k, \text{ where } \bar{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \bar{B} = \begin{pmatrix} B \\ I \end{pmatrix} \end{aligned} \quad [4]$$

$$\Rightarrow y_k = Cx_k = \begin{pmatrix} C & 0 \end{pmatrix} z_k.$$

The stage cost can be written as

$$\begin{aligned} z_k' \begin{pmatrix} C' \\ 0 \end{pmatrix} M \begin{pmatrix} C & 0 \end{pmatrix} z_k + u_k' V u_k + u_k' W u_k + u_{k-1}' W u_{k-1} \\ \quad - 2 u_k' W u_{k-1} \\ = z_k' \begin{pmatrix} C' M C & 0 \\ 0 & W \end{pmatrix} z_k + u_k' (V + W) u_k + 2 u_k' (0 - W) \begin{pmatrix} x_k \\ u_{k-1} \end{pmatrix} \end{aligned}$$

$$\Rightarrow \underline{Q} = \begin{pmatrix} C' M C & 0 \\ 0 & W \end{pmatrix}, \underline{R} = V + W, \underline{S} = \begin{pmatrix} 0 & -W \end{pmatrix} \quad [6]$$

c) $V \geq 0, W \geq 0$ and either $V > 0$ or $W > 0 \Rightarrow R > 0$

If, in addition $M \geq 0 \Rightarrow$ the stage cost is positive definite
 $\Rightarrow Q - S'R^{-1}S \geq 0 \quad [4].$

Question 6

New computed example.

(a) 1) $R = \frac{1}{c} > 0$

[1]

2) $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$

[1]

3) $A = \begin{pmatrix} 0 & 1 \\ \gamma & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow$ reachability matrix = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow (A, B)$ reachable.

[2]

4) $\begin{pmatrix} Q^{1/2} \\ Q^{1/2} A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ is full rank $\Rightarrow (Q^{1/2}, A)$ observable.

[2]

(b) $P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} \Rightarrow PA = \begin{pmatrix} p_2 \gamma & p_1 \\ p_3 \gamma & p_2 \end{pmatrix}$

$A'P = P^T A = \begin{pmatrix} p_2 \gamma & p_3 \gamma \\ p_1 & p_2 \end{pmatrix}$

$P \cdot B R^{-1} B^T P = \begin{pmatrix} c p_2^2 & c p_2 p_3 \\ c p_2 p_3 & c p_3^2 \end{pmatrix}$

\Rightarrow the individual terms of the ARE are:

$0 = 2 p_2 \gamma - c p_2^2 + 1$

①

$0 = p_1 + p_3 \gamma - c p_2 p_3$

②

$0 = 2 p_2 - c p_3^2$

③

[6]

① $\Leftrightarrow p_2 = \frac{\gamma \pm \sqrt{\gamma^2 + c}}{c}$

③ $\Leftrightarrow p_3 = \frac{\gamma \pm \sqrt{\gamma^2 + c}}{c} \pm \frac{1}{c} \sqrt{2 p_2}$

$\Rightarrow p_2 = \frac{\gamma + \sqrt{\gamma^2 + c}}{c}$ because otherwise $p_2 < 0 \Leftrightarrow p_3$ imaginary

(P is real.)

[2]

6b) continued.

$$\Rightarrow p_3 = \pm \frac{\sqrt{2}}{c} \sqrt{\gamma + \gamma^2 + c} \quad [2]$$

$$(2) \Leftrightarrow p_1 = p_3(\sqrt{\gamma^2 + c})$$

But for $p > 0$, we need $p_1 > 0 \Rightarrow p_3 > 0$ [2]

$$\Rightarrow p_3 = \frac{\sqrt{2}}{c} \sqrt{\gamma + \gamma^2 + c}$$

$$u^*(t) = -(c p_2 \quad c p_3) x(t) \quad [2]$$

$$\Rightarrow u^*(t) = -\left(\gamma + \sqrt{\gamma^2 + c} \quad \sqrt{2(\gamma + \sqrt{\gamma^2 + c})}\right) x(t)$$

Q.E.D.

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