EE4-23
EE9CS2-1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

MSc and EEE/ISE PART IV: MEng and ACGI

STABILITY AND CONTROL OF NON-LINEAR SYSTEMS

Tuesday, 15 May 2:30 pm

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : D. Angeli
Second Marker(s) : E.C. Kerrigan

1. Consider the planar nonlinear system of equations:

$$\begin{aligned} \dot{x}_1 &= -e^{x_1} + x_2 \\ \dot{x}_2 &= (-x_2^3 + x_2 - x_1)(x_1 - 1). \end{aligned}$$

a) Find all equilibria of the system (hint: use nullclines). [3]

b) Identify the 7 connected regions in which nullclines partition the state space; which ones are forward invariant? Which ones are backward invariant? Identify the possible directions of $\dot{x}$ at the boundary and within each of the 7 regions. [5]

c) Linearize the system around each equilibrium and discuss the local phase-portrait. [6]

d) Sketch the global phase portrait of the system by taking into account the clues collected in items a), b) and c). [6]

2. Consider the following scalar nonlinear system affected by disturbances:

$$\dot{x} = -\text{sat}(x + d_1) + d_2,$$

with state $x \in \mathbb{R}$, and inputs $d_1, d_2 \in \mathbb{R}$. Assume a piecewise linear saturation function:

$$\text{sat}(x) = \begin{cases} x & \text{if } |x| < 1 \\ 1 & \text{if } x \geq 1 \\ -1 & \text{if } x \leq -1 \end{cases} .$$

a) Show that when $d_2 = 0$ the system is Input-to-State Stable with respect to input d_1. Give an example of a gain function relating d_1 to x. [6]

b) Set $d_1 = 0$. Show that the system is not Input-to-State Stable with respect to d_2. [6]

c) Consider the following nonlinear system:

$$\begin{aligned} \dot{x}_1 &= -\text{sat}(x_1 + x_2 + d) \\ \dot{x}_2 &= -4x_2^5 + x_1^5 \end{aligned}$$

Show that the system is Input-to-State Stable with respect to input d. (Hint: regard the system as a feedback interconnection of ISS systems). [8]

3. Consider the equations of three water tanks in series with each other:

$$
\begin{aligned}
\dot{x}_1 &= -\sqrt{x_1} + u \\
\dot{x}_2 &= \sqrt{x_1} - \sqrt{x_2} \\
\dot{x}_3 &= \sqrt{x_2} - \sqrt{x_3}
\end{aligned}
$$

with state x taking values in $[0, +\infty)^3$, and input u taking values in $\mathbb{R}$. The variable x_i denotes the amount of water in tank i, for $i \in \{1, 2, 3\}$ respectively.

a) Compute, for each constant value of u, the corresponding equilibrium. [4]

b) Discuss, for all $u > 0$, the local stability of the equilibria previously computed. [4]

c) Let r denote a constant set-point for the desired total amount of water in the three tanks. Let the output equation be defined as

$$y = x_1 + x_2 + x_3 - r$$

which represents the tracking error. What is the relative degree with respect to this output selection? Write the system of equations in normal form and highlight the zero-dynamics. [6]

d) Design, by means of feedback linearization and disregarding positivity of state and input variables, a controller to steer the output y to zero; what kind of internal stability property can be guaranteed? (Hint: check local asymptotic stability of the zero dynamics). [6]

4. Consider the following parameter-dependent nonlinear system:

$$\begin{aligned} \dot{x}_1 &= -x_1 + 2x_1x_2^4 \\ \dot{x}_2 &= -ax_2 - 2x_1^2x_2 \end{aligned}$$

with state $x \in \mathbb{R}^2$ and parameter $a \in \mathbb{R}$.

a) Compute the set of equilibria as a function of the parameter a. [6]

b) Show that for $a > 0$ the system is globally asymptotically stable (Hint: use a Lyapunov function of the type $V(x) = c_1x_1^\alpha + c_2x_2^\beta$ for some even integers α and β and some positive reals c_1, c_2). [7]

c) Discuss the local phase portraits of equilibria for $a \neq 0$. [7]

5. Consider the following nonlinear control system:

$$\begin{aligned}
\dot{x}_1 &= -x_1 + u \\
\dot{x}_2 &= (1+u)x_3 + u \\
\dot{x}_3 &= -(1+u)x_2 \\
\\
y &= x_1^3 + x_2
\end{aligned}$$

with state $x = [x_1, x_2, x_3] \in \mathbb{R}^3$, input $u \in \mathbb{R}$ and output $y \in \mathbb{R}$.

a) Decompose the system in order to show that it can be recast as the parallel interconnection of two subsystems. [4]

b) Show that each of the subsystems is passive and write explicitly the appropriate storage functions. [6]

c) Prove that the overall system is also passive (Hint: what is the storage function?). [2]

d) Design a static output feedback controller in order to globally asymptotically stabilize (GAS) the equilibrium $x = 0$. [4]

e) Assume that only controls $u \in [-1, 1]$ are allowed. Design an alternative output feedback that achieves GAS of the origin and fulfills the constraint on the input signal. [4]

6. Consider the following system of coupled differential equations:

$$\begin{aligned}\ddot{\theta} &= -\theta\dot{\theta} + (1+\sin^2(\theta))\delta \\ \dot{\delta} &= a\delta + u\end{aligned}$$

where $a \in [-1,1]$ is an unknown constant parameter, while $u \in \mathbb{R}$ is an exogenous input.

a) Choose a suitable state vector and write the system in state space representation. [4]

b) By using δ as a virtual input, design a state feedback that achieves global asymptotic stability of the θ subsystem at its zero equilibrium by using feedback linearization. [5]

c) Find a Lyapunov function for the stabilized closed-loop θ subsystem. [4]

d) Use the previous Lyapunov function to backstep the virtual feedback through the δ equation when $a = 0$. [3]

e) Let now a be unknown in the interval $[-1,1]$. Modify the previous feedback to make sure that the terms involving the uncertain parameter a are dominated in the derivative of the Lyapunov function used to prove global asymptotic stability of 0. [4]

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SOLUTIONS: STABILITY AND CONTROL OF NONLINEAR SYSTEMS 2012
MASTER IN CONTROL

1. Exercise

a) The system admits 2 equilibria. As it can be seen by plotting nullclines on the plane these occur in $(0,1)'$ and $(1,e)'$.

b) The nullclines partition the plane in 7 connected regions, (see Fig. 1.1), denoted by $R_1, R_2, R_3, R_4, R_5, R_6$ and R_7. By using cardinal point notation for identifying $\dot{x}$ directions, the regions are of type SW, SE, NE, NW, SE, SW and NW respectively. Region R_1 borders regions R_2, R_4 and R_7 respectively. Notice that the vector-field on the boundary is always pointing from the neighbouring region towards R_1. Hence R_1 is forward invariant. Similarly, R_2 borders with regions R_1 and R_3. The vector-field on the boundary always points from R_2 towards the neighbouring regions. Hence R_2 is backwards invariant.

c) Computing the Jacobian of the vector-field yields:

$$\frac{\partial \dot{x}}{\partial x} = \begin{bmatrix} -e^{x_1} & 1 \\ 1-2x_1-x_2^3+x_2 & (x_1-1)(1-3x_2^2) \end{bmatrix}.$$

Evaluating around the equilibrium in $(0,1)$ yields the linearized system:

$$\dot{\delta x} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \delta x.$$

The characteristic polynomial of the matrix is $s^2 - s - 3$ and eigenvalues are in $(1 \pm \sqrt{13})/2$. Hence they are real and have opposite sign. The first equilibrium is therefore a saddle point. Evaluating around the equilibrium in $(1,e)$ yields the linearized system:

$$\dot{\delta x} = \begin{bmatrix} -e & 1 \\ -e^3+e-1 & 0 \end{bmatrix} \delta x.$$

The characteristic polynomial is $s^2 + es + e^3 - e + 1$. Its roots are in:

$$-e/2 \pm j\sqrt{4e^3 - e^2 - 4e + 4})/2 \approx -1.36 \pm 4j.$$

Hence, the second equilibrium is a stable focus.

d) A phase portrait consistent with the information collected in items a), b) and c) is sketched in Fig. 1.2.

2. Exercise

a) Consider the candidate Lyapunov function $V(x) = x^2/2$. Taking derivatives of $V(x)$ for $d_2 = 0$ yields:

$$\frac{\partial V}{\partial x}(x) f(x, d_1) = -x\mathrm{sat}(x + d_1).$$

Hence, for any $\varepsilon > 0$ we have:

$$|x| \geq (1+\varepsilon)|d_1| \Rightarrow \frac{\partial V}{\partial x}(x) f(x, d_1) = -x\mathrm{sat}(x+d_1) \leq -x\mathrm{sat}(\varepsilon x/(1+\varepsilon)).$$

This implies Input-to-State Stability with gain $\gamma_1(r) = (1+\varepsilon)r$.

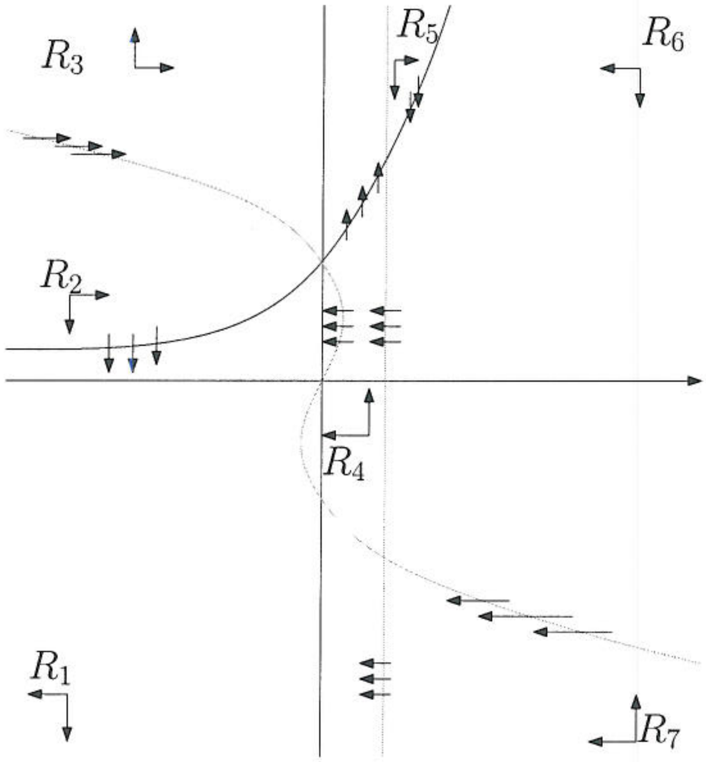

Figure 1.1 Nullclines and regions in which $\mathbb{R}^2$ is partitioned

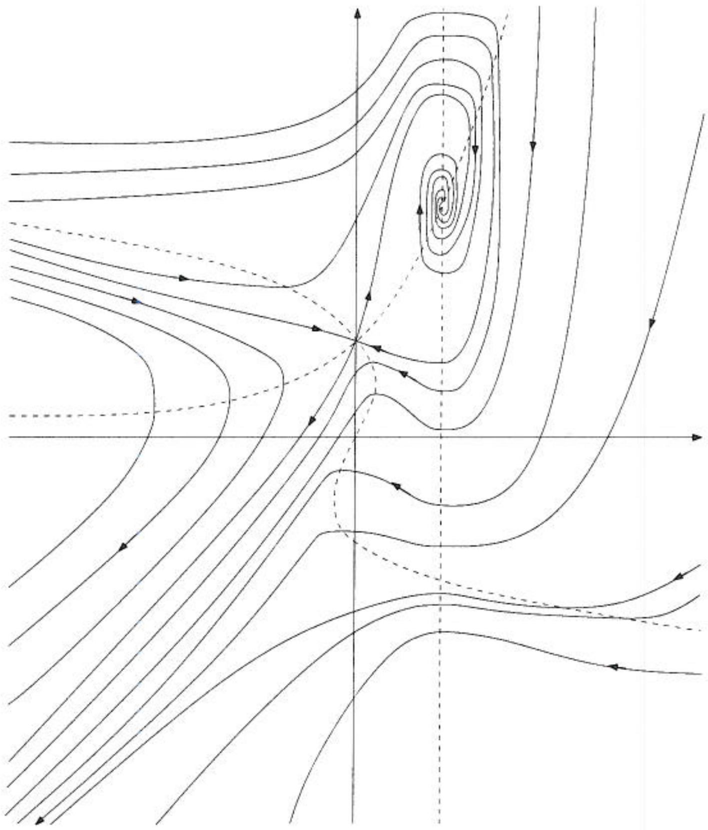

Figure 1.2 Sketch of phase-portrait for Exercise 1

b) In order to prove that ISS is violated take a constant input $d_2 = 2$. For $d_1 = 0$ we have:

$$\dot{x} = -\text{sat}(x) + 2 \geq 1.$$

Hence $x(t) \geq x(0) + t$ for all $t \geq 0$. This implies that $x(t) \to +\infty$ as $t \to +\infty$ which violates the BIBS property.

c) Consider the system:

$$\dot{x} = -4x^5 + d^5.$$

We show that this system is Input-to-State Stable and compute its gain. Again, letting $V(x) = x^2/2$ we see that:

$$\frac{\partial V}{\partial x}(x)[-4x^5 + d^5] = -4x^6 + xd^5 \leq -4|x|^6 + |x||d|^5.$$

Notice that for all sufficiently small $\varepsilon > 0$:

$$|x| \geq \sqrt[5]{\frac{1}{4-\varepsilon}}|d| \Rightarrow \frac{\partial V}{\partial x}(x)[-4x^5 + d^5] \leq -\varepsilon|x|^6$$

which implies ISS with respect to d with gain:

$$\gamma_2(r) = \sqrt[5]{\frac{1}{4-\varepsilon}}r.$$

We regard the system

$$\begin{aligned} \dot{x}_1 &= \text{sat}(x_1 + x_2 + d) \\ \dot{x}_2 &= -4x_2^5 + x_1^5 \end{aligned}$$

as the feedback interconnection of

$$\begin{aligned} \dot{x}_1 &= -\text{sat}(x_1 + d_1 + d) & d_1 &= x_2 \\ \dot{x}_2 &= -4x_2^5 + d_2^5 & d_2 &= x_1. \end{aligned}$$

Notice that the composition of gains fulfills:

$$\gamma_1(\gamma_2(r)) = (1+\varepsilon)\sqrt[5]{\frac{1}{4-\varepsilon}}r < r \qquad \forall r > 0$$

provided $\varepsilon > 0$ is chosen sufficiently small. Hence, by the small-gain theorem, the interconnected system is Input-to-State Stable with respect to the input d.

3. Exercise

a) In order to compute the equilibria of the system we need to solve the following system of equations:

$$\begin{cases} 0 = -\sqrt{x_1} + u \\ 0 = -\sqrt{x_2} + \sqrt{x_1} \\ 0 = -\sqrt{x_3} + \sqrt{x_1} \end{cases}$$

From the first equation we have $x_1 = u^2$ and substituting in the second and third equations we obtain $x_1 = x_2 = x_3 = u^2$. Hence, for each non-negative value of the input u the corresponding equilibrium u is in $x = [u^2, u^2, u^2]'$.

b) Computing the Jacobian matrix yields:

$$\frac{\partial \dot{x}}{\partial x} = \begin{bmatrix} -\frac{1}{2\sqrt{x_1}} & 0 & 0 \\ \frac{1}{2\sqrt{x_1}} & -\frac{1}{2\sqrt{x_2}} & 0 \\ 0 & \frac{1}{2\sqrt{x_2}} & -\frac{1}{2\sqrt{x_3}} \end{bmatrix}.$$

The linearized system around the equilibrium $[u^2, u^2, u^2]$ is therefore given by:

$$\dot{\delta x} = \begin{bmatrix} -\frac{1}{2u} & 0 & 0 \\ \frac{1}{2u} & -\frac{1}{2u} & 0 \\ 0 & \frac{1}{2u} & -\frac{1}{2u} \end{bmatrix} \delta x.$$

Given the lower-triangular structure of the above system, it follows that diagonal entries are eigenvalues of the A matrix. Hence, the system has 3 eigenvalues in $-1/2u < 0$ and is locally asymptotically stable for all $u > 0$. Notice that the equilibrium achieved for $u = 0$ is $[0, 0, 0]$ but the vector field is not differentiable in 0.

c) Taking derivatives of y along the systems equations yields:

$$\dot{y} = -\sqrt{x_3} + u,$$

hence the relative degree equals 1 and is globally well-defined. The I-O linearizing feedback:

$$u = \sqrt{x_3} + v$$

brings the system in the following normal form:

$$\begin{aligned} \dot{y} &= v \\ \dot{x}_2 &= \sqrt{y + r - x_2 - x_3} - \sqrt{x_2} \\ \dot{x}_3 &= \sqrt{x_2} - \sqrt{x_3} \end{aligned},$$

where the x_2 and x_3 equations define the internal dynamics, while $\xi = [x_2, x_3]'$ is the internal state. The zero dynamics are achieved for $y = 0$ and read:

$$\begin{aligned} \dot{x}_2 &= \sqrt{r - x_2 - x_3} - \sqrt{x_2} \\ \dot{x}_3 &= \sqrt{x_2} - \sqrt{x_3} \end{aligned}.$$

d) A simple linear feedback $v = -y$ renders the output equation asymptotically stable and steers the output to 0. In order to validate the design, however, stability of zero-dynamics needs to be checked. The zero-dynamics have a unique equilibrium at $[r/3, r/3]$. Computing the Jacobian of the zero-dynamics yields:

$$\frac{\partial \dot{\xi}}{\partial \xi} = \begin{bmatrix} -\frac{1}{2\sqrt{r - x_2 - x_3}} - \frac{1}{2\sqrt{x_2}} & -\frac{1}{2\sqrt{r - x_2 - x_3}} \\ \frac{1}{2\sqrt{x_2}} & -\frac{1}{2\sqrt{x_3}} \end{bmatrix}.$$

Evaluating at the equilibrium yields the following linearized zero dynamics:

$$\dot{\delta \xi} = \begin{bmatrix} -\frac{1}{\sqrt{r/3}} & -\frac{1}{2\sqrt{r/3}} \\ \frac{1}{2\sqrt{r/3}} & -\frac{1}{2\sqrt{r/3}} \end{bmatrix} \delta \xi$$

Notice that trace and determinant of the above matrix are respectively negative and positive for all positive values of r. Hence the zero dynamics are locally asymptotically stable (the nonlinear system is locally minimum-phase). By the Lyapunov direct method local asymptotic stability of the overall system is guaranteed.

4. Exercise

a) In order to compute equilibria as a function of a we need to solve the following parameter-dependent system of equations:

$$\begin{cases} 0 = -x_1 + 2x_1 x_2^4 \\ 0 = -a x_2 - 2x_1^2 x_2. \end{cases}$$

From the first equation we see that:

$$x_1 = 0 \text{ or } x_2 = \pm\sqrt[4]{\frac{1}{2}}.$$

Notice that:

$$x_1 = 0 \text{ and } a \neq 0 \Rightarrow x_2 = 0.$$

On the other hand, if $x_1 = 0$ and $a = 0$, then x_2 can be arbitrary.

$$x_2 = \pm\sqrt[4]{\frac{1}{2}} \text{ and } a < 0 \Rightarrow x_1 = \pm\sqrt{\frac{-a}{2}}.$$

Summarizing the previous result we have that the set of equilibria $X(a)$ is given by:

i) $\{0\}$ if $a > 0$;

ii) $\{x : x_1 = 0\}$ if $a = 0$;

iii) $\{(0,0), (\sqrt{-\frac{a}{2}}, \sqrt[4]{\frac{1}{2}}), (\sqrt{-\frac{a}{2}}, -\sqrt[4]{\frac{1}{2}}), (-\sqrt{-\frac{a}{2}}, \sqrt[4]{\frac{1}{2}}), (-\sqrt{-\frac{a}{2}}, -\sqrt[4]{\frac{1}{2}})\}$ if $a < 0$.

b) Consider the candidate Lyapunov function $V(x) = \frac{x_1^2}{2} + \frac{x_2^4}{4}$. Taking derivatives along solutions yields:

$$\dot{V}(x) = x_1[-x_1 + 2x_1x_2^4] + x_2^3[-ax_2 - 2x_1^2x_2] = -x_1^2 - ax_2^4.$$

Notice that $V(x)$ is positive definite and radially unbounded. Moreover for $a > 0$ the derivative $\dot{V}(x)$ is negative definite. Hence the equilibrium at the origin is globally asymptotically stable.

c) Computing the Jacobian of $f(x)$ yields:

$$\frac{\partial f}{\partial x}(x) = \begin{bmatrix} -1 + 2x_2^4 & 8x_1x_2^3 \\ -4x_1x_2 & -a - 2x_1^2 \end{bmatrix}.$$

Hence, linearization around equilibrium $(0,0)$ for $a < 0$ yields:

$$\dot{\delta x} = \begin{bmatrix} -1 & 0 \\ 0 & -a \end{bmatrix} \delta x.$$

Eigenvalues are real and have opposite sign. Hence the local phase portrait is that of a saddle point. Around the 2 equilibria in $(\pm\sqrt{-\frac{a}{2}}, \pm\sqrt[4]{\frac{1}{2}})$ we have the following linearized dynamics:

$$\dot{\delta x} = \begin{bmatrix} 0 & 8\sqrt{-\frac{a}{2}}\sqrt[4]{\frac{1}{8}} \\ -4\sqrt{-\frac{a}{2}}\sqrt[4]{\frac{1}{2}} & 0 \end{bmatrix} \delta x.$$

Similarly, around the 2 equilibria in $(\pm\sqrt{-\frac{a}{2}}, \mp\sqrt[4]{\frac{1}{2}})$ we have the following linearized dynamics:

$$\dot{\delta x} = \begin{bmatrix} 0 & -8\sqrt{-\frac{a}{2}}\sqrt[4]{\frac{1}{8}} \\ 4\sqrt{-\frac{a}{2}}\sqrt[4]{\frac{1}{2}} & 0 \end{bmatrix} \delta x.$$

Eigenvalues are therefore purely imaginary. The local phase portrait cannot be inferred by Hartman-Grossman theorem as the equilibria are not hyperbolic.

5. Exercise

a) We define a scalar subsystem with state $\chi_a = x_1$ and a bidimensional subsystem with state $\chi_b = [x_2, x_3]'$. Accordingly we may define subsystem a and b as follows

$$\dot{x}_1 = -x_1 + u_a \qquad y_a = x_1^3,$$

and:

$$\begin{array}{rcl} \dot{x}_2 & = & (1+u_b)x_3 + u_b \\ \dot{x}_3 & = & -(1+u_b)x_2 \end{array} \qquad y_b = x_2.$$

The overall system is recovered by letting $u = u + a = u_b$ and $y = y_a + y_b$ which is in fact a parallel interconnection of subsystems.

b) For subsystem a we have, letting $V_a(x_1) = x_1^4/4$:

$$y_a u_a = x_1^3 u_a = x_1^3 \dot{x}_1 + x_1^4 = x_1^4 + \frac{\partial V_a}{\partial x_1}\dot{x}_1 \geq \frac{\partial V_a}{\partial x_1}\dot{x}_1.$$

Hence the first subsystem is passive with storage function V_a. For subystem b we observe that:

$$V_b(x_2, x_3) = x_2^2/2 + x_3^2/2$$

fulfills the equation:

$$\frac{\partial V_b}{\partial x_2}\dot{x}_2 + \frac{\partial V_b}{\partial x_3}\dot{x}_3 = u_b x_2 = y_b u_b$$

Hence the b subsystem is passive (and lossless) with storage function V_b.

c) The parallel interconnection is passive with storage function $V(x) = V_a(x_1) + V_b(x_2, x_3)$.

d) A simple static and passive output feedback controller is achieved by letting $u = -y$. We show below that this achieves global asymptotic stability of the equilibrium at the origin. Take the candidate Lyapunov function $V(x)$ previously defined. This is positive definite and radially unbounded. Moreover:

$$\dot{V}(x) = \frac{\partial V_a}{\partial x_1}\dot{x}_1 + \frac{\partial V_b}{\partial x_2}\dot{x}_2 + \frac{\partial V_b}{\partial x_3}\dot{x}_3 = -x_1^4 + yu = -x_1^4 - (x_1^3 + x_2)^2.$$

This shows that $\dot{V}(x)$ is negative semi-definite. It is zero iff $x \in \{x : x_1 = 0 \text{ and } x_1^3 + x_2 = 0\} = \{x : x_1 = 0 \text{ and } x_3 = 0\}$. Assume next $x_1(t) \equiv 0$ and $x_2(t) \equiv 0$. This implies $\dot{x}_2(t) \equiv 0$ which in turn yields $(1 + u(t))x_3(t) \equiv 0$, that is $x_3(t) \equiv 0$. Hence, the largest invariant set contained in $\{x : x_1 = 0 \text{ and } x_3 = 0\}$ is the origin and by the Lasalle's invariance principle this implies global asymptotic stability of 0.

e) In order to respect the input constraint we may modify the static feedback as follow:

$$u = -\mathrm{sat}(y).$$

Notice that this choice results in:

$$\dot{V} = -x_1^4 - (x_1^3 + x_2)\mathrm{sat}(x_1^3 + x_2),$$

which is also negative semi-definite. Moreover $\dot{V}$ vanishes on the same set $\{x : x_1 = 0 \text{ and } x_2 = 0\}$ so that the same argument can be used to prove global asymptotic stability of 0.

6. Exercise

a) We may choose $x = [\theta, \dot{\theta}, \delta]' = [x_1, x_2, x_3]'$, so that the resulting equations read:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 x_2 + (1 + \sin(x_1)^2) x_3 \\ \dot{x}_3 &= a x_3 + u. \end{aligned}$$

b) The θ subsystem reads:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 x_2 + (1 + \sin(x_1)^2) x_3. \end{aligned}$$

Letting x_3 be the virtual control variable we see that the relative degree with respect to the output x_1 is 2 and the following choice:

$$x_{3v} = \frac{x_1 x_2 - x_1 - x_2}{1 + \sin^2(x_1)}$$

achieves global input-to-state linearization of the θ subsystem. Moreover the closed-loop equations read:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2. \end{aligned}$$

and are globally asymptotically stable.

c) To find a Lyapunov function for the system we may solve the Lyapunov equation:

$$A'P + PA = -I.$$

where:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}.$$

This gives $p_{11} = 3/2$, $p_{12} = 1/2$ and $p_{22} = 1$. Hence, the desired Lyapunov function is quadratic and given by $V(x_1, x_2) = [x_1, x_2] P [x_1, x_2]'$.

d) Consider next the error $\xi = x_3 - x_{3v}$. Clearly:

$$\dot{\xi} = a x_3 + u - \dot{x}_{3v}.$$

While $\dot{x}_{3v}$ could be computed explicitly, we don't do it as we will cancel it with u later on. Next we may define the candidate Lyapunov function $W(x) = V(x_1, x_2) + \frac{\xi^2}{2}$. This is positive definite and radially unbounded. Taking derivatives along the solutions of the system yields:

$$\dot{W}(x) = -x_1^2 - x_2^2 + \frac{\partial V}{\partial x_2}\xi + \xi\dot{\xi} =$$

$$= -x_1^2 - x_2^2 + \xi[\frac{\partial V}{\partial x_2} + a x_3 - \dot{x}_{3v} + u].$$

We would like to make $\dot{W}(x)$ negative definite. For $a = 0$ we can achieve this just by letting:

$$u = -\frac{\partial V}{\partial x_2} + \dot{x}_{3v} - k\xi$$

for any $k > 0$. The corresponding $\dot{W}$ is in fact:

$$\dot{W} = -x_1^2 - x_2^2 - k\xi^2.$$

e) For $a \in [-1,1]$ however, we have:

$$\begin{aligned}
\dot{W} &= -x_1^2 - x_2^2 - k\xi^2 + ax_3\xi \\
&= -x_1^2 - x_2^2 - k\xi^2 + a\xi^2 + a\xi\tfrac{x_1x_2 - x_1 - x_2}{1+\sin^2(x_1)} \\
&\leq -x_1^2 - x_2^2 - k\xi^2 + \xi^2 + |\xi||x_1|x_2| + |\xi||x_1| + |\xi||x_2|
\end{aligned}$$

While the terms $|\xi||x_1|$ and $|\xi||x_2|$ and ξ^2 can be dominated by choosing k sufficiently large. The term $|\xi||x_1||x_2|$ is not quadratic and requires higher order negative terms in order to be dominated. For this reason we modify u as follows:

$$u = -\frac{\partial V}{\partial x_2} + \dot{x}_{3v} - k\xi - k\xi(x_1^2 + x_2^2).$$

This results in:

$$\dot{W}(x) \leq -x_1^2 - x_2^2 - k\xi^2 + \xi^2 + |\xi||x_1|x_2| + |\xi||x_1| + |\xi||x_2| - k\xi^2x_1^2 - k\xi^2x_2^2.$$

Notice that:

$$-x_1^2/2 + |\xi||x_1|x_2| - k\xi^2x_2^2 \leq 0$$

for all $k > 1/2$. Similarly all quadratic terms can be dominated for k sufficiently large. Hence, this is a suitable choice of feedback.