EE4-45
EE9SO22

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

MSc and EEE/ISE PART IV: MEng and ACGI

WAVELETS AND APPLICATIONS

Thursday, 26 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.L. Dragotti

Second Marker(s) : K.D. Harris

Special Information for the Invigilators: NONE

Information for Candidates:

Sub-sampling by an integer N:

$$x_{\downarrow N}[n] \longleftrightarrow \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N}) = \frac{1}{N}\sum_{k=0}^{N-1} X(W_N^k z^{1/N}),$$

where

$$W_N^k = e^{-j2\pi k/N}.$$

Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(t-n) = \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k)e^{j2\pi kt}.$$

The Questions

1. *Multirate Signal Processing:*

   (a) The deterministic autocorrelation function for a real-valued stable sequence $x[n]$ is defined as

   $$c_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]x[k+n].$$

   i. Show that the z-transform of $c_{xx}[n]$ is $C_{xx}(z) = X(z)X(z^{-1})$.

   [6]

   ii. Using the above result, express the orthogonality condition $\langle g[n], g[n-2k] \rangle = \delta_k$ in the z-domain.

   [6]

   (b) *Interpolation followed by decimation:* Given an input $x[n]$, consider upsampling by 2 followed by interpolation with a filter $H(z)$. Then to recover the original signal, apply filtering with a filter $G(z)$ followed by downsampling by 2 in order to obtain a reconstruction $\hat{x}[n]$ (see Fig. 1a). What does the product $P(z) = H(z)G(z)$ have

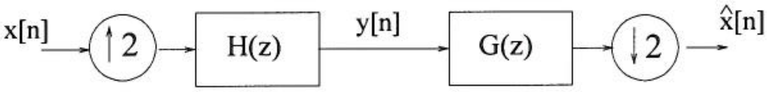


Figure 1a: Interpolation followed by decimation.

   to satisfy in order for $\hat{x}[n]$ to be equal to $x[n]$?

   [6]

   (c) *Successive interpolation:* Consider the system in Fig. 1b where the output $y^{(1)}[n]$ is an interpolated version of the input sequence $x[n]$. We would like $y^{(1)}[2n] = x[n]$,

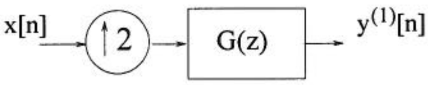


Figure 1b: Interpolation of $x[n]$.

   while $y^{(1)}[2n+1]$ is interpolated. What conditions does that impose on $G(z)$?

   [7]

2. You want to construct a wavelet transform that uses powers of three instead of powers of two. For this purpose, you will consider the centered box function $\varphi(x)$ such that $\varphi(x) = 1$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$ and $\varphi(x) = 0$ otherwise.

   (a) Start by showing that $\varphi(x)$ is a valid scaling function. That is, show that

      i. it satisfies partition of unity: $\sum_n \varphi(x-n) = 1$.

      [4]

      ii. it satisfies the Riesz basis criterion:

      $$0 < A \leq \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega + 2\pi k)|^2 \leq B < \infty.$$

      [4]

      iii. it satisfies a 3-scale equation $\varphi(x/3) = \sqrt{3} \sum_n h_0[n] \varphi(x-n)$.

      [4]

   (b) Construct two orthogonal and compactly supported wavelets $\psi_a(t)$ and $\psi_b(t)$. We want $\psi_a(t)$ to be symmetric and $\psi_b(t)$ to be anti-symmetric. Moreover, we want the shortest possible solution. [Hint: recall that $\psi_a(x/3) = \sqrt{3} \sum_n g_a[n] \varphi(x-n)$ and that $\psi_b(x/3) = \sqrt{3} \sum_n g_b[n] \varphi(x-n)$. Thus, you just need to find the coefficients $g_a[n]$ and $g_b[n]$ satisfying all the above requirements: shortest possible support, (anti-)symmetry and orthogonality].

   [8]

   (c) Give the corresponding perfect reconstruction three-channel filter bank.

   [5]

3. *Spectral factorization.* Following the footsteps of some great wavelet researchers, you

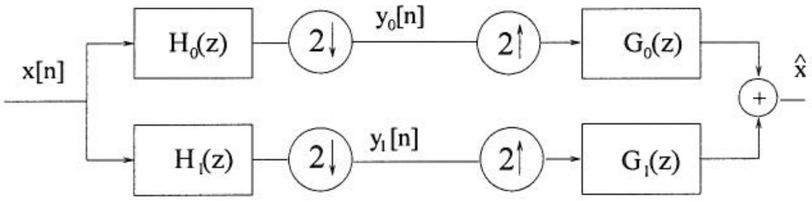


Figure 1c: Two-channel filter bank.

want to design your own family of orthogonal and biorthogonal filter banks using the spectral factorization method. You are an original person: instead of considering $z = -1$, you want to put a maximum number of zeros at $z = 1$.

(a) Start by determining the shortest symmetric polynomials of the form $P(z) = (1 - z)^L(1 - z^{-1})^L B(z)$ for $L = 1$ and $L = 2$ such that $P(z) + P(-z) = 2$.

[8]

(b) Based on the polynomial $P(z)$ of part (a), construct an orthogonal filter bank.

[8]

(c) Based on the polynomial $P(z)$ in part (a), construct all possible linear phase biorthogonal filter banks using the constraint $H_0(-1) = 1$.

[9]

4. *Meyer's Wavelet.* We are going to design a valid scaling function $\varphi(t)$ and the corresponding wavelet in the frequency domain. We are going to follow a methodology first proposed by Meyer. Construct the scaling function $\varphi(t)$ such that:

$$\hat{\varphi}(\omega) = \begin{cases} \sqrt{a(2+\frac{3\omega}{2\pi})} & \omega \leq 0 \\ \\ \sqrt{a(2-\frac{3\omega}{2\pi})} & \omega \geq 0 \end{cases}$$

where $\hat{\varphi}(\omega)$ is the Fourier transform of $\varphi(t)$ and

$$a(\omega) = \begin{cases} 0 & \omega \leq 0 \\ \\ 3\omega^2 - 2\omega^3 & 0 \leq \omega \leq 1 \\ \\ 1 & \omega \geq 1 \end{cases}$$

Show that $\varphi(t)$ satisfies the three criteria of a valid scaling function. More specifically:

(a) Show that $\varphi(t)$ satisfies partition of unity:

$$\sum_{n=-\infty}^{\infty} \varphi(t-n) = 1.$$

[6]

(b) Show that $\langle \varphi(t-n), \varphi(t-m) \rangle = \delta_{n,m}$. This is equivalent to showing that $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of the space $V_0 = \text{span}\{\varphi(t-n)\}_{n\in\mathbb{Z}}$. [Hint: operate in the frequency domain].

[7]

(c) Finally, show pictorially and by operating in the frequency domain, that the two-scale equation

$$\varphi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n]\varphi(2t-n)$$

is satisfied when $G_0(e^{j\omega}) = \sqrt{2}\sum_{k=-\infty}^{\infty} \hat{\varphi}(2\omega + 4\pi k)$, where $G(e^{j\omega})$ is the discrete-time Fourier transform of $g_0[n]$.

[6]

(d) Given $\varphi(t)$, the corresponding wavelet $\psi(t)$ satisfies the following two-scale relation:

$$\psi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_1[n]\varphi(2t-n).$$

Derive the frequency domain expression of $g_1[n]$.

[6]

SOLUTIONS Wavelets and Applications 2011 1/12

QUESTION 1

EE 4-45
EE 9 SO 22

(a)

i.
$$C_{xx}(z) = \sum_{m=-\infty}^{\infty} c_{xx}[m] z^{-m}$$

$$= \sum_m \sum_k x[k]\, x[k+m]\, z^{-m}$$

REPLACE m WITH. $\ell = k+m$.

THIS LEADS TO

$$C_{xx}(z) = \sum_\ell \sum_k x[k]\, x[\ell]\, z^{k-\ell}$$

$$= \sum_\ell x[\ell] z^{-\ell} \cdot \sum_k x[k] z^{k}$$

$$= X(z) \cdot X(z^{-1})$$

$\square$

ii

$$\langle g[n], g[n-2k] \rangle = \sum_n g[n] \cdot g[n-2k]$$

WE DENOTE

$$p[k] = \sum_n g[n]\, g[n-k]$$

BECAUSE OF (i) $P(z) = G(z) \cdot G(z^{-1})$

$\langle g[n], g[n-2k]\rangle$ IS EQUIVALENT TO SUB-SAMPLING $p[k]$ BY A FACTOR 2.

THEREFORE

THE Z-TRANFORM OF $\langle g[n], g[n-2k]\rangle$ IS

$$\frac{1}{2} P\left(z^{+1/2}\right) + \frac{1}{2} P\left(-z^{1/2}\right) = \frac{1}{2} G\left(z^{1/2}\right) G\left(z^{-1/2}\right) + \frac{1}{2} G\left(-z^{1/2}\right) G\left(-z^{-1/2}\right)$$

PUTTING EVERYTHING TOGETHER, WE OBTAIN

$$\langle g[n], g[n-2k]\rangle = \delta_k \Longleftrightarrow \frac{1}{2} G\left(z^{1/2}\right) G\left(z^{-1/2}\right) + \frac{1}{2} G\left(-z^{1/2}\right) G\left(-z^{-1/2}\right) = 1$$

WHICH LEADS TO THE FOLLOWING ORTHOGONALITY CONDITION:

$$G(z)\, G\left(z^{-1}\right) + G(-z)\, G\left(-z^{-1}\right) = 2$$

b)

$$Y(z) = H(z)X(z^2)$$

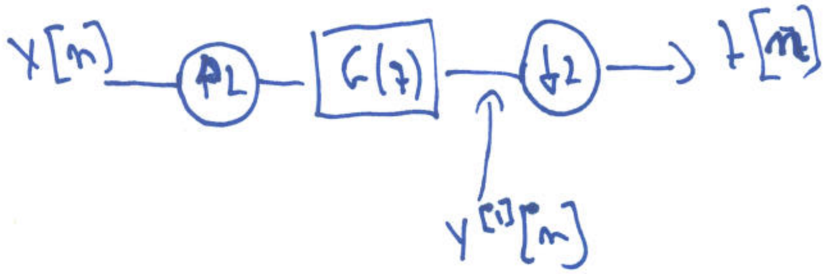
$$\hat{X}(z) = \frac{X(z)}{2}\left[G(z^{1/2})H(z^{1/2}) + G(-z^{1/2})H(-z^{1/2})\right]$$

THEREFORE

$$\hat{X}(z) = X(z) \iff G(z)H(z) + G(-z)H(-z) = 2$$

c)

CONSIDER THE FOLLOWING SYSTEM:



THE CONDITION $y^{(1)}[2n] = x[n]$

IS EQUIVALENT TO IMPOSING

THAT $z[n] = x[n]$ WHICH IMPLIES THAT

$$X(z) = Z(z) = \frac{1}{2}\left[G(z^{1/2}) + G(-z^{1/2})\right]X(z)$$

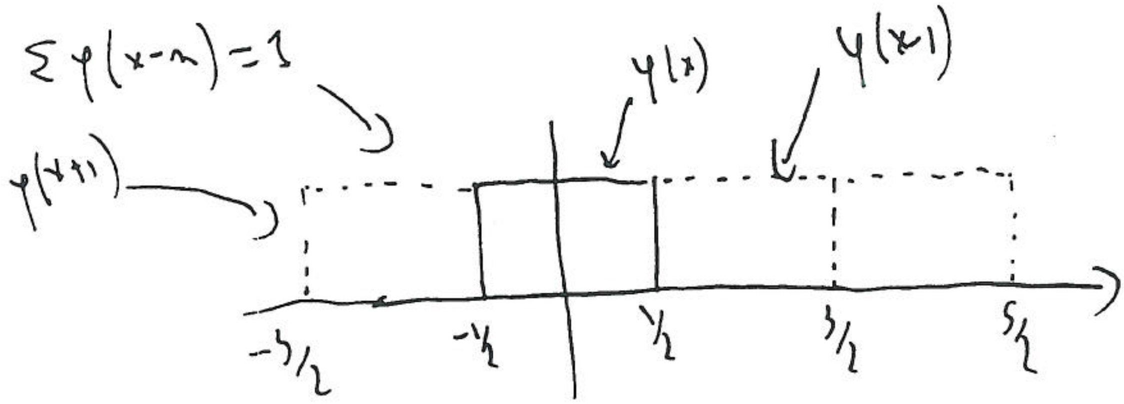
$$\Updownarrow$$

$$G(z) + G(-z) = 2$$

QUESTION 2

2) (a)

i) $\varphi(x)$ CLEARLY SATISFIES PARTITION OF UNITY.



ii) MOREOVER

$$\langle \varphi(x), \varphi(x-n) \rangle = \delta_n$$

THUS $\sum_k |\hat{\varphi}(\omega + 2k\pi)|^2 = 1 \Rightarrow$

$\varphi(x)$ IS AN ORTHONORMAL BASIS

iii) ONE CAN EASILY SEE GRAPHICALLY THAT

$$\varphi\left(\tfrac{x}{3}\right) = \sum_n \sqrt{3}\, h_0[n]\, \varphi(x-n)$$

WITH $\begin{cases} h_0[-1] = h_0[0] = h_0[1] = 1/\sqrt{3} \\ \text{AND } h_0[n] = 0 \quad n \neq -1, 0, 1. \end{cases}$

(b)

~~$\psi_a(x)$ MUST BE SYMMETRIC~~

$$\psi_a\left(\frac{x}{3}\right) = \sqrt{3} \sum_n g_a[n]\, \varphi(x-n)$$

$\psi_a(x)$ MUST BE SYMMETRIC. MOREOVER

$$(*) \quad \langle \psi_a(x), \psi_a(x-3m) \rangle = \delta_m$$

$$(**) \quad \langle \psi_a(x), \varphi(x-3n) \rangle = 0$$

IF $g_a[n] = 0$ FOR $n \neq \{-1, 1, 0\}$

CONDITION $(*)$ IS SATISFIED UP TO A CONSTANT FACTOR

FOR SYMMETRY $g_a[1] = g_a[-1]$

THUS CONDITIONS $\Rightarrow$ $\begin{cases} \langle \psi_a(x), \psi_a(x) \rangle = 1 \\ \langle \psi_a(x), \varphi(x) \rangle = 0 \end{cases}$ IMPLIES

THAT $g_a[0] = \frac{2}{\sqrt{6}}$

$g_a[-1] = g_a[1] = -1/\sqrt{6}$

$$\psi_b\left(\frac{x}{3}\right) = \sqrt{3} \sum_n g_b[n]\, \varphi(x-n)$$

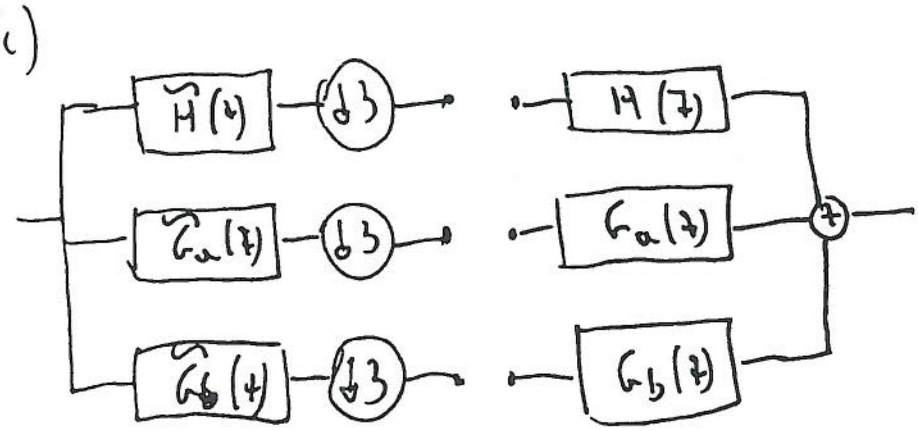
$\psi_b(x)$ MUST BE ANTY-SYMMETRIC

AND HAS TO SATISFY THE ORTHOGONALITY CONDITIONS. THIS LEADS TO

$$g_b[n] = 0 \qquad n \notin \{-1, 1\}$$

AND $\qquad g_b[1] = -g_b[-1] = \frac{1}{\sqrt{3}}$

(c)



WHERE THE ANALYSIS FILTERS ARE THE TIME REVERSED VERSIONS OF THE SYNTHESIS ONES.

QUESTION 3

4

a) $B(t) = a + b(t+t^{-1})$

$$P(t)+P(-t) = (1-t)(1-t^{-1})(a+b(t+t^{-1})) +$$
$$(1+t)(1+t^{-1})(a-b(t+t^{-1})) =$$
$$= (4a-4b) - 2b(t^2+t^{-2}) = 2 \Rightarrow$$

$$\begin{cases} b=0 \\ a=\frac{1}{2} \end{cases} \Rightarrow \quad B(t)=\frac{1}{2}$$

$L=2$

$$P(t)+P(-t) = (1-t)^2(1-t^{-1})^2(a+b(t+t^{-1})) +$$
$$(1+t)^2(1+t^{-1})^2(a-b(t+t^{-1})) =$$
$$= (t^{-2}+t^2)(2a-8b) + (12a-20b) = 2$$
$$\Downarrow$$
$$\begin{cases} 2a-8b=0 \\ 12a-20b=2 \end{cases}$$
$$\Downarrow$$
$$b=\frac{1}{8} \quad a=\frac{1}{2} \rightarrow B(t)=\frac{1}{8}(4+(t+t^{-1}))$$

DUE TO THE CONSTRUCTION THIS IS THE SHORTEST SOLUTION.

b) ORTHOGONALITY IMPLIES

$$P(z) = G_0(z)\, G_0(z^{-1}) \qquad \text{AND} \qquad H_0(z) = G_0(z^{-1})$$

THUS

$$L=1 \Rightarrow G_0(z) = \frac{1}{\sqrt{2}}\left(1-z^{-1}\right) = H_0(z^{-1})$$

$$L=2 \Rightarrow G_0(z) = \frac{1}{2\sqrt{2}}\left(1-z^{-1}\right)^2\left(2+\sqrt{3}+z^{-1}\right)\frac{1}{\sqrt{2+\sqrt{3}}} = H_0(z^{-1})$$

THE FILTERS $G_1(z)$ AND $H_1(z)$ ARE IN BOTH CASES GIVEN BY:

$$G_1(z) = -z^{-1} G_0(-z^{-1})$$

$$H_1(z) = G_1(z^{-1})$$

c) BIORTHOGONAL CASE

$$L=1 \qquad P(z) = (1-z)\left(1-z^{-1}\right)\frac{1}{2} \qquad \text{WITH } H_0(-1)=1$$

$H_0(z)$	$G_0(z)$
1	$\frac{1}{2}(1-z)(1-z^{-1})$
$\frac{1}{2}(1-z^{-1})$	$(1-z)$
$\frac{1}{4}(1-z)(1-z^{-1})$	2

$L=2$

$$P(t) = (1-t)^2 (1-t^{-1})^2 (t+4+t^{-1}) \frac{1}{8} \quad \text{with } H_0(-1)=1$$

$H_0(t)$	$G_0(t)$		
1	$(1-t)^2(1-t^{-1})^2(t+4+t^{-1}) \cdot \frac{1}{8}$		
$\frac{1}{2}(1-t)$	$(1-t)(1-t^{-1})^2$	"	$\frac{1}{4}$
$\frac{1}{4}(1-t)^2$	$(1-t^{-1})^2$	"	$\frac{1}{2}$
$\frac{1}{8}(1-t)^2(1-t^{-1})$	$(1-t^{-1})$	"	
$\frac{1}{16}(1-t^2)(1-t^{-1})^2$		"	2

10

QUESTION 4

(a) USING POISSON SUM FORMULA

WE HAVE THAT

$$\sum_{n=-\infty}^{\infty} \varphi(t-n) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(2\pi k) e^{j2\pi k t}$$

WE THUS NEED TO PROOVE THAT

$$\sum_{k=-\infty}^{\infty} \hat{\varphi}(2\pi k) e^{j2\pi k t} = 1$$

OR EQUIVALENTLY

$$\hat{\varphi}(2\pi k) = \begin{cases} 1 & \text{FOR } k=0 \\ 0 & \text{FOR } k \neq 0 \end{cases}$$

BUT BY CONSTRUCTION

$$\hat{\varphi}(0) = a(2) = 1$$

AND

$$\cancel{\hat{\varphi}(2\pi k) =}$$

$$\hat{\varphi}(2\pi k) = \begin{cases} a(2-3k) = 0 & k>0 \\ a(2+3k) = 0 & k<0 \end{cases}$$

$\square$

(b) [redacted] $\{\varphi(t-n)\}_{n\in\mathcal{Z}}$ is an orthonormal family from $L_2(\mathcal{R})$. To that end, we [redacted] show that [redacted]

$$\sum_{k\in\mathcal{Z}} |\Phi(\omega+2k\pi)|^2 = 1. \qquad (1)$$

From Figure 4.8 it is clear that for $\omega \in [-(2\pi/3) - 2n\pi, (2\pi)/3 - 2n\pi]$

$$\sum_k |\Phi(\omega+2k\pi)|^2 = |\Phi(\omega+2n\pi)|^2 = 1.$$

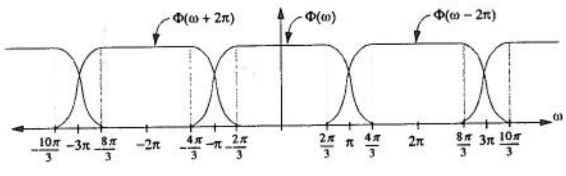


Figure 4.8 Pictorial proof that $\{\varphi(t-n)\}_{n\in\mathcal{Z}}$ form an orthonormal family in $L^2(\mathcal{R})$.

The only thing left is to show ((1)) holds in overlapping regions. Thus, take for example, $\omega \in [(2\pi)/3, (4\pi)/3]$:

$$\begin{aligned}
\Phi(\omega)^2 + \Phi(\omega-2\pi)^2 &= a\left(2 - \frac{3\omega}{2\pi}\right) + a\left(2 + \frac{3(\omega-2\pi)}{2\pi}\right) \\
&= a\left(2 - \frac{3\omega}{2\pi}\right) + a\left(-1 + \frac{3\omega}{2\pi}\right) \\
&= a\left(2 - \frac{3\omega}{2\pi}\right) + a\left(1 - \left(2 - \frac{3\omega}{2\pi}\right)\right) \\
&= 1.
\end{aligned}$$

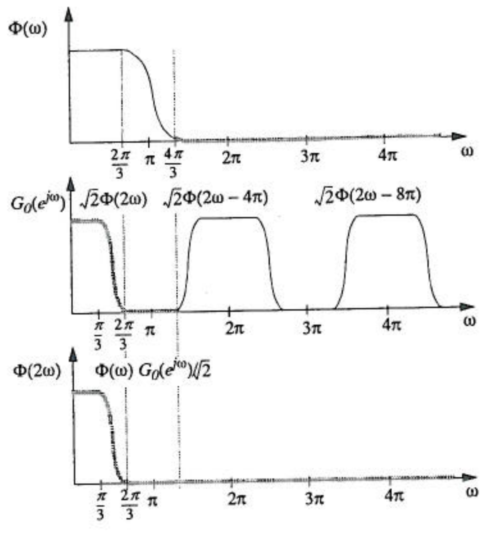
The last equation follows from the definition of [redacted]). $a(w)$.

(c) WE NEED TO ~~SHOW~~ FIND A $G\left(e^{j\omega}\right)$ SUCH THAT THE FOLLOWING IS SATISFIED:

$$\hat{\varphi}(2\omega) = \frac{G_0\left(e^{j\omega}\right)}{\sqrt{2}}\,\hat{\varphi}(\omega) \qquad (2)$$

WE CAN SEE PICTORIALLY FROM THE FIGURE BELOW THAT EQ. (2) IS SATISFIED WHEN

$$G_0\left(e^{j\omega}\right) = \sqrt{2} \sum_{n=-\infty}^{\infty} \hat{\varphi}(2\omega + 4n\pi)$$



(d) BECAUSE OF ORTHOGONALITY

$$G_1\left(e^{j\omega}\right) = -G_0^*\left(e^{j(\omega+\pi)}\right)e^{-j\omega}$$

THUS

$$\hat{\psi}(\omega) = -\frac{1}{\sqrt{2}}\, e^{-j\frac{\omega}{2}} \sum_{n=-\infty}^{\infty} \hat{\varphi}\left(\omega + (4n+1)\pi\right)\hat{\varphi}\left(\frac{\omega}{2}\right)$$