IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

EEE/ISE PART I: MEng, BEng and ACGI

INTRODUCTION TO SIGNALS AND COMMUNICATIONS

Wednesday, 13 June 10:00 am

Time allowed: 2:00 hours

There are THREE questions on this paper.

Answer ALL questions.
Q1 carries 40% of the marks. Questions 2 and 3 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : K.K. Leung
Second Marker(s) : M.K. Gurcan

Special Instructions for Invigilator: **None**

Information for Students:

Some Fourier Transforms

$$\cos \omega_o t \quad <=> \quad \pi[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)]$$

Some useful trigonometric identities

$$\cos x \cos y = \tfrac{1}{2}\cos(x - y) + \tfrac{1}{2}\cos(x + y)$$
$$\sin(x - y) = \sin x \cos y - \sin y \cos x$$
$$a \cos x + b \sin x = c \cos(x + \theta)$$

where $c = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(-b / a)$

Complex exponential

$$e^{jx} = \cos x + j \sin x$$

1. This is a general question. (40%)

   a. Consider a time function $f(t) = a$ for $-b \le t \le b$ and 0 otherwise, where b is positive.

      i. Derive the Fourier transform $F(\omega)$ of $f(t)$. [4]

      ii. Sketch the frequency spectrum of $f(t)$. [3]

      iii. Consider a special case of $f(t)$ where $a = \frac{1}{2b}$ and b is reduced to zero (i.e., $b \to 0$). What is the Fourier transform of this special case of $f(t)$? [2]

      iv. Now consider $\hat{f}(t)$ as a time shifted version of $f(t)$ by t_o amount of time. That is, $\hat{f}(t) = a$ for $-b + t_o \le t \le b + t_o$ and 0 otherwise. Let $\hat{F}(\omega)$ denote the Fourier transform of $\hat{f}(t)$. How are $\hat{F}(\omega)$ and $F(\omega)$ related to each other? [3]

   b. Consider two orthogonal signals $x(t)$ and $y(t)$ over $-\infty < t < \infty$, each of which is real and has finite energy.

      i. Give a mathematical condition under which $x(t)$ and $y(t)$ are orthogonal to each other. [3]

      ii. Let signal $z(t) = x(t) + y(t)$. Let E_x, E_y and E_z denote the energy for $x(t)$, $y(t)$ and $z(t)$, respectively. Show that $E_z = E_x + E_y$. [3]

      iii. Consider two other signals $z_1(t)$ and $z_2(t)$ where $z_1(t) = c_1 x(t) + d_1 y(t)$, $z_2(t) = c_2 x(t) + d_2 y(t)$, and c_1, d_1, c_2 and d_2 are constants. If $z_1(t)$ and $z_2(t)$ have also been determined to be orthogonal to each other for $-\infty < t < \infty$, what can be said about the relationships among c_1, d_1, c_2 and d_2 and why? [4]

   c. Consider two forms of amplitude modulation (AM) signal, namely, the full AM and double-sideband with suppressed carrier (DSB-SC). Let their waveforms be denoted by $\phi_{AM}(t)$ and $\phi_{DSB}(t)$, respectively. Let ω_c be the carrier angular frequency in radians/second, A be the amplitude of the carrier, and $m(t) = B\cos(\omega_m t)$ be the modulating signal where B and ω_m are the amplitude and the angular frequency of the modulating signal, respectively. Assume that $\omega_c > \omega_m$.

      i. Give the expressions for $\phi_{AM}(t)$ and $\phi_{DSB}(t)$. [2]

      ii. Sketch the frequency spectrum for the waveform $\phi_{AM}(t)$. [2]

      iii. What is the bandwidth for both forms of AM signals? [2]

      iv. What is the condition under which an envelope-detection receiver can be used to properly recover the modulating signal $m(t)$ from the full AM signal $\phi_{AM}(t)$ and why? [2]

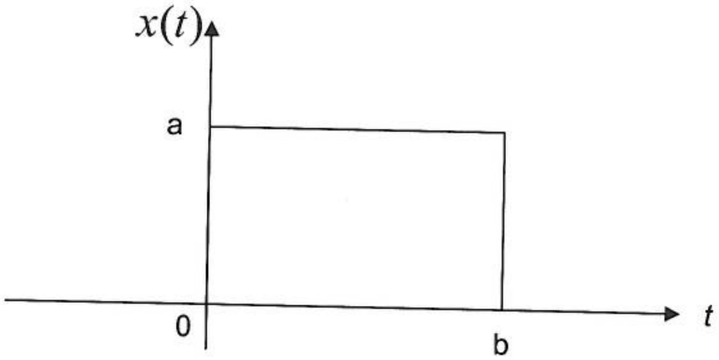
1. This is a general question. (Continued)

   d. Consider an angle modulated signal that has the form
   $u(t) = 100\cos[2\pi f_c t + 4\sin(2000\pi t)]$ where $f_c = 10\,\text{MHz}$.

      i. Determine the average transmitted power. [2]
      ii. Is this a frequency modulation (FM) or phase modulation (PM) signal? Explain. [2]
      iii. Determine the frequency deviation Δf. [3]
      iv. Using Carson's rule, find the bandwidth of the modulated signal. [3]

2. Signals. (30%)

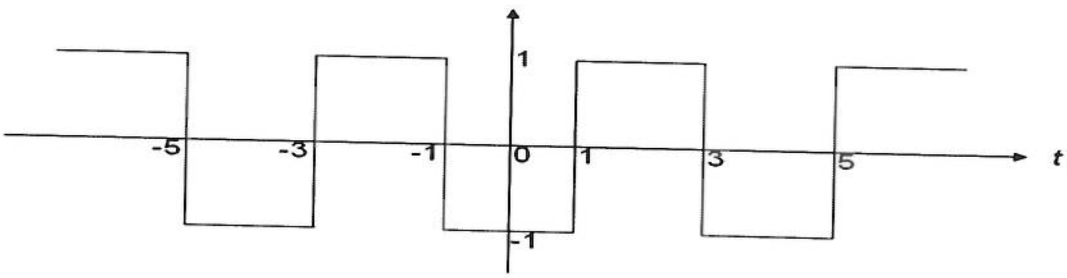
   a. Consider the following signal $x(t)$ where a and b are positive constants.



Let $y(t)$ be the self-convolution of $x(t)$. That is, $y(t) = x(t) * x(t)$.

i. Express $y(t)$ as an integral of $x(t)$. [2]
ii. Carry out the convolution integration to obtain and sketch $y(t)$. [7]
iii. If $x(t)$ is the impulse response of a linear time-invariant system, what does $y(t)$ represent physically? [3]
iv. Let $X(\omega)$ and $Y(\omega)$ denote the Fourier transforms of $x(t)$ and $y(t)$, respectively. How are $X(\omega)$ and $Y(\omega)$ related to each other? Prove their relationship. [6]

   b. Consider the following periodic signal $x(t)$.



i. What is the fundamental frequency ω_0 of $x(t)$ in radians/second? [2]
ii. What is the dc component of $x(t)$? [2]
iii. Determine the Fourier series coefficients a_n 's and b_n 's for $x(t)$ where

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t).$$ [8]

3. Communications techniques. (30%)

   a. Communication systems often are designed to carry many independent signals (e.g., different voice or TV channels), each of which occupies a separate part of an available frequency band. These individual signals are expected to be recovered at the base band at the receiver(s). Consider one such communication system that simultaneously transmits two voice signals, $m_1(t)$ and $m_2(t)$, each of which has a bandwidth of 5 KHz. A frequency band from 75 to 95 KHz is available for transmission by the system. Further, we assume that two sinusoidal signals of 10 and 80 KHz are also available for the system.

      i. Describe a method of amplitude modulation (AM) by using the sinusoidal signals of 10 and 80 KHz to transmit the voice signals $m_1(t)$ and $m_2(t)$ over the 75-95 KHz band so that the voice signals can be recovered by the same sinusoidal frequencies at the receiver. Sketch the frequency spectrum of the transmitted signal in the 75-95 KHz band. [6]
      ii. Give an expression for the transmitted signal in the 75-95 KHz band. [5]
      iii. Assume that ideal filters are available for the receiver design. Draw a block diagram of the receiver and show mathematically how the voice signals $m_1(t)$ and $m_2(t)$ are recovered by the sinusoidal frequencies of 10 and 80 KHz at the receiver. [7]
      iv. If one sinusoidal signal at a particular frequency can be included as part of the transmitted signal to simplify the receiver design, what is the preferred frequency of the tone and why can that help? [4]

   b. Consider a digital communication system where the modulating signal $g(t)$ has a bandwidth of B Hz, and is sampled at a frequency of f_S Hz to obtain the sampled signal $\tilde{g}(t)$. Let the Fourier transforms of $g(t)$ and $\tilde{g}(t)$ be denoted by $G(\omega)$ and $\tilde{G}(\omega)$, respectively. Further, let the sampling be represented by applying a train of periodic impulses $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$ to $g(t)$ where $T_s = \frac{1}{f_S}$. As the Fourier series of a periodic signal, we can express $s(t)$ as

$$s(t) = \frac{1}{T_S}[1 + 2\cos(\omega_s t) + 2\cos(2\omega_s t) + 2\cos(3\omega_s t) + \ldots] \qquad \text{where } \omega_S = 2\pi f_S = \frac{2\pi}{T_s}.$$

      i. Express $\tilde{g}(t)$ in terms of $g(t)$ and the cosine terms of $s(t)$. [2]
      ii. From the frequency-domain perspective, what is the physical interpretation of each term in $\tilde{g}(t)$? [2]
      iii. Based on part ii above, draw the frequency spectrum for $\tilde{g}(t)$. [2]
      iv. From the result in part iii, determine the relationship between B and f_S such that the modulating signal $g(t)$ can be fully recovered from $\tilde{g}(t)$. [2]

1. a. i. $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$

$= \int_{-b}^{b} a e^{-j\omega t} dt$

$= \frac{a}{-j\omega} e^{-j\omega t} \Big|_{-b}^{b}$

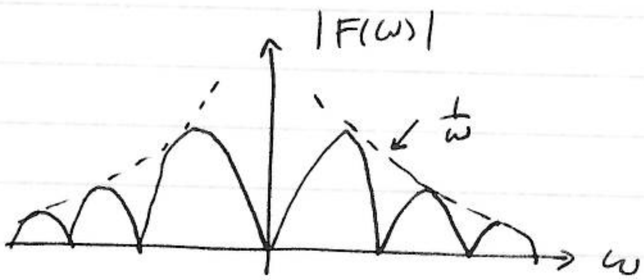
$\Rightarrow F(\omega) = \frac{a}{-j\omega} [e^{-j\omega b} - e^{j\omega b}]$

$= \frac{a}{-j\omega} [\cos \omega b - j \sin \omega b - \cos \omega b - j \sin \omega b]$

$= \frac{a}{-j\omega} [-2j \sin \omega b]$

$\Rightarrow F(\omega) = \frac{2a \sin \omega b}{\omega}$

ii.



iii. when $b \to 0$, $f(t) = \delta(t)$.

Therefore, The Fourier Transform $F(\omega) = 1$.

iv. $\hat{F}(\omega) = \int_{-b+t_0}^{b+t_0} a e^{-j\omega t} dt = \int_{-b}^{b} a e^{-j\omega(t'+t_0)} d(t'+t_0)$

$\Rightarrow \hat{F}(\omega) = e^{-j\omega t_0} \int_{-b}^{b} a e^{-j\omega t'} dt' = e^{-j\omega t_0} F(\omega)$.

1.b. i.

$$\int_{-\infty}^{\infty} x(t)\, y(t)\, dt = 0$$

ii.

$$E_z = \int_{-\infty}^{\infty} [z(t)]^2 dt$$

$$= \int_{-\infty}^{\infty} [x(t) + y(t)]^2 dt$$

$$= \int_{-\infty}^{\infty} [x(t)]^2 dt + \int_{-\infty}^{\infty} [y(t)]^2 dt + \int_{-\infty}^{\infty} 2\, x(t)\, y(t)\, dt$$

$$\Rightarrow \quad E_Z = E_X + E_Y + 0 \quad \because \int_{-\infty}^{\infty} x(t)\, y(t)\, dt = 0$$

iii. We have only two possible cases

A: $c_1 \neq 0$, $d_1 = 0$, $c_2 = 0$ and $d_2 \neq 0$; or

B: $c_1 = 0$, $d_1 \neq 0$, $c_2 \neq 0$ and $d_2 = 0$.

In essence, these cases indicate that both c_1 and c_2 cannot be non-zero simultaneous. The same comment applies to d_1 and d_2.

This is so because if c_1 and c_2 are non-zero simultaneous, then

$$\int_{-\infty}^{\infty} z_1(t)\, z_2(t) \neq 0$$

which contradicts the orthogonality of $z_1(t)$ & $z_2(t)$.

The same argument applies to d_1 & d_2.

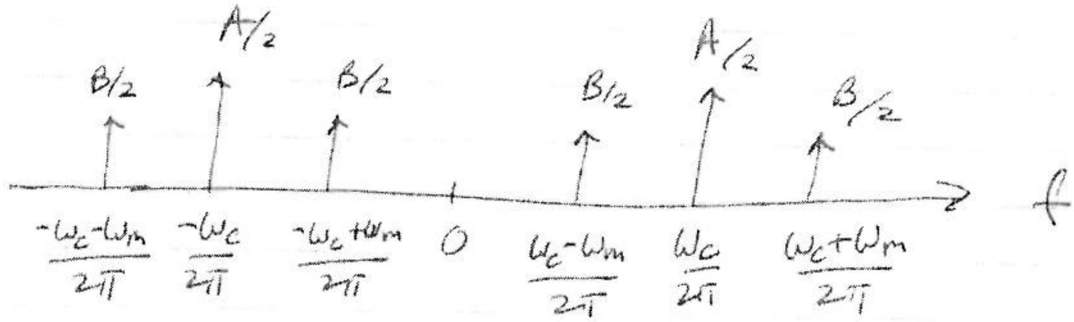
1\. c. i. $\phi_{AM}(t) = [A + m(t)]\cos(\omega_c t)$

$$= [A + B\cos(\omega_m t)]\cos(\omega_c t)$$

$$\phi_{DSB}(t) = m(t)\cos(\omega_c t)$$

$$= B\cos(\omega_m t)\cos(\omega_c t)$$

ii.



iii. The bandwidth of both waveforms is

$$\frac{2\omega_m}{2\pi}\ \text{Hz}, \quad \text{i.e.}\ \frac{\omega_m}{\pi}\ \text{Hz}$$

iv. $A > B$ s.t. $A + B\cos(\omega_m t) > 0$ for $\forall t$.

1.d. i. Since an angle modulated signal is a sinusoidal signal with constant amplitude, the transmitted power is

$$P = \frac{A^2}{2} = \frac{100^2}{2} = 5,000$$

ii. The angle modulated signal can be interpreted both as a PM and an FM signal. It is a PM signal,

$$u(t) = 100 \cos\left[2\pi f_c t + k_p \sin(2000\pi t)\right]$$

where $k_p = 4$.

It is an FM signal,

$$u(t) = 100 \cos\left[2\pi f_c t + k_f \int_{-\infty}^{t} \cos(2000\pi t')\,dt'\right]$$

where $k_f = 8000\pi$

iii. The instantaneous frequency is

$$f_i = f_c + \frac{1}{2\pi}\frac{d}{dt}\theta(t)$$

$$= f_c + \frac{k_f}{2\pi}\cos(2000\pi t)$$

$$= f_c + \frac{8000\pi}{2\pi}\cos(2000\pi t)$$

$$\Rightarrow \quad \Delta f = 4000 \text{ Hz}$$

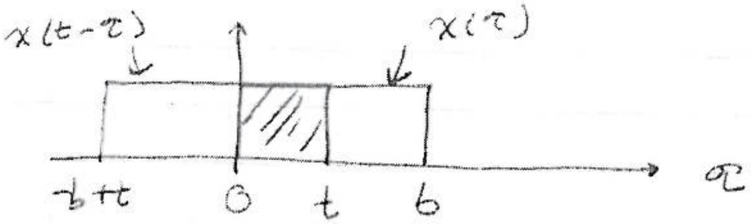
iv. Baseband signal bandwidth $B = 1$ KHz

$$\Rightarrow \quad B_{FM} = 2(\Delta f + B) = 10 \text{ KHz}.$$

2. a. i. $$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, x(t-\tau)\, d\tau$$

ii. When $t<0$, $y(t)=0$ $\because$ $x(\tau)$ & $x(t-\tau)$ do not overlap

For $0 < t \leq b$,

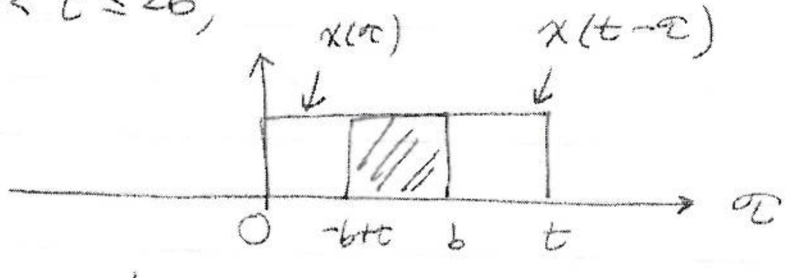


$$y(t) = \int_0^t x(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_0^t a^2\, d\tau$$

$\Rightarrow$ $y(t) = a^2 t$

For $b < t \leq 2b$,



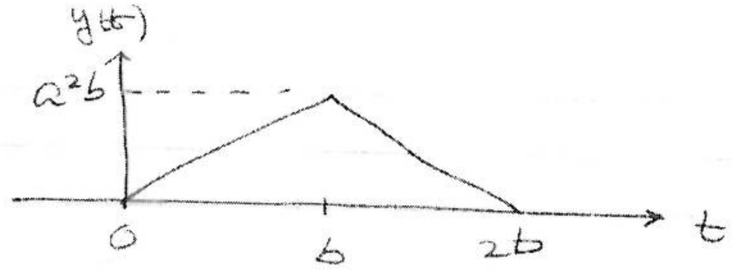
$$y(t) = \int_{-b+t}^{b} x(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-b+t}^{b} a^2\, d\tau$$

$$= a^2 \left[b + b - t\right]$$

$\Rightarrow$ $y(t) = a^2\left[2b - t\right]$

graphically,



iii. If $x(t)$ is the impulse response of a LTIS, $y(t)$ is the output of the system when the input is $x(t)$.

iv. $$Y(\omega) = X(\omega)\cdot X(\omega)$$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{t=-\infty}^{\infty}\int_{\tau=-\infty}^{\infty} x(\tau)\, x(t-\tau)\, d\tau\, e^{-j\omega t}\, dt$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\int_{t=-\infty}^{\infty} x(t-\tau)\, e^{-j\omega(t-\tau)}\cdot e^{-j\omega\tau}\, dt\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau} \int_{t=-\infty}^{\infty} x(t-\tau)\, e^{-j\omega(t-\tau)}\, dt\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, X(\omega)\, d\tau$$

$$\Rightarrow Y(\omega) = X(\omega)\int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$\Rightarrow Y(\omega) = X(\omega)\cdot X(\omega)$$

2.b. i. The period $T = 4$ sec

So, $\omega_0 = 2\pi f = \frac{2\pi}{4} = \frac{\pi}{2}$ radians/sec

ii. The dc component of $x(t) = 0$

$\because$ Equal area above and below the x-axis.

iii. Since the function is even, $b_n = 0$ for all n

For $n \geq 1$,

$$a_n = -\int_0^1 \cos\left(\frac{n\pi}{2} t\right) dt + \int_1^2 \cos\left(\frac{n\pi}{2}\right) dt$$

$$\Rightarrow a_n = \frac{-4}{n\pi} \cdot \sin\left(\frac{n\pi}{2}\right)$$

$\because x(t)$ is even

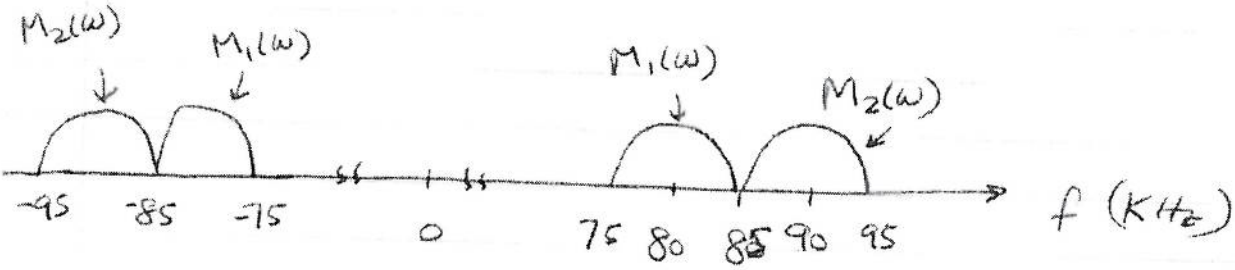
3.a i. The modulation method:

A. Modulate $m_2(t)$ by multiplying it with $\cos(20{,}000\pi t)$ i.e., sinusoidal at 10KHz

B. Add the baseband $m_1(t)$ to the result in Step A.

C. Modulate the resultant signal in Step B by multiplying it with $\cos(160{,}000\pi t)$ i.e., sinusoidal at 80KHz

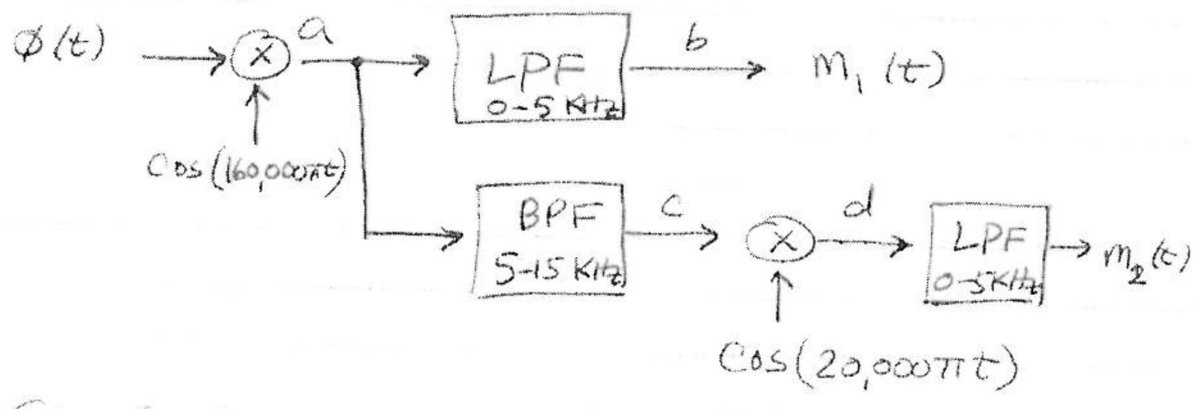
The resultant spectrum is



ii. The transmitted signal is

$$\phi(t) = m_1(t)\cos(160{,}000\pi t) + m_2(t)\cos(180{,}000\pi t)$$

iii.



Signal at point a:

$$S_a(t) = \phi(t)\cos(160{,}000\pi t)$$

$$= \left[m_1(t)\cos(\omega_{80K}t) + m_2(t)\cos(\omega_{90K}t)\right] \cdot \cos(\omega_{80K}t)$$

$$S_a(t) = \frac{1}{2}\left[m_1(t)\right]\left[\cos(2\omega_{80K}t) + 1\right]$$
$$+ m_2(t)\cos(\omega_{90K}t)\cdot\cos(\omega_{80K}t)$$
$$= \frac{1}{2}\left[m_1(t) + m_1(t)\cos(2\omega_{80K}t)\right]$$
$$+ \frac{1}{2}m_2(t)\left[\cos((\omega_{90K} - \omega_{80K})t) + \cos((\omega_{90K} + \omega_{80K})t)\right]$$

At point b,

$S_b(t)$ is the output of the LPF with input $S_a(t)$ (0-5KHz)

So, $S_b(t) = \frac{1}{2}m_1(t)$

At point c, the output of the BPF (5-15KHz) is

$$S_c(t) = \frac{1}{2}m_2(t)\cos(\omega_{10K}t)$$

Signal at point d:

$$S_d(t) = S_c(t).\cos(\omega_{10K}t)$$
$$= \frac{1}{2}m_2(t)\cos^2(\omega_{10K}t)$$
$$= \frac{1}{4}m_2(t)\left[\cos 2\omega_{10K}t + 1\right]$$
$$= \frac{1}{4}m_2(t)\left[1 + \underbrace{\cos(2\omega_{10K}t)}_{\text{filter out by LPF}}\right]$$

Thus, the final output is $\frac{1}{4}m_2(t)$

3.a. iv. The preferred frequency to be included in the transmitted signal is 80 KHz. This is so because the receiver needs both signals of 10 KHz & 80 KHz. However, once 80 KHz signal is received, taking every 8 cycles (2^3) as one cycle can lead to a sinusoidal of 10 KHz.

3. b. i.

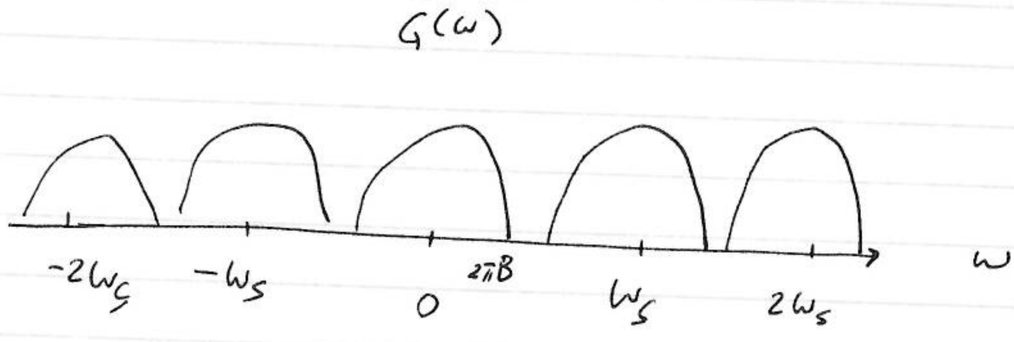
$$\tilde{g}(t) = g(t) \cdot s(t)$$

$$= \frac{1}{T_s} \left[g(t) + 2g(t)\cos(\omega_s t) \right.$$
$$+ 2g(t)\cos(2\omega_s t)$$
$$\left. + 2g(t)\cos(3\omega_s t) + \cdots \right]$$

ii. Each term in $\tilde{g}(t)$ is basically $g(t)\cos(n\omega_c t)$

From the frequency-domain perspective, $g(t)\cos(n\omega_c t)$ corresponds to shifting $g(t)$ to a frequency $n\omega_c$, similar to AM operation.

iii.



iv.

$$\omega_s \geq 2(2\pi B)$$

$$\Rightarrow \quad 2\pi f_s \geq 2(2\pi B)$$

$$\Rightarrow \quad f_s \geq 2B$$