1. a) Figure 1.1 illustrates an RLC circuit. The capacitor has capacitance C, the inductor has inductance L and the resistor resistance R. The input is the applied voltage $v_i(t)$ and the output is the voltage across the capacitor $v_o(t)$. Assume that the capacitor is initially uncharged.

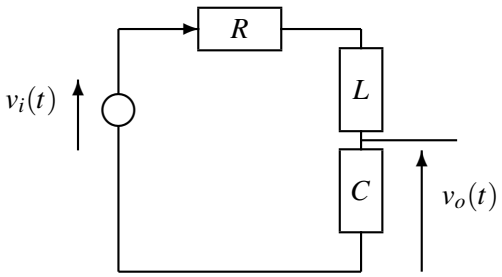


Figure 1.1

i) Determine the transfer function $G(s)$ relating v_o to v_i. [5]

ii) By writing $G(s)$ above in the standard form, determine the DC gain, the undamped natural frequency ω_n and the damping ratio ζ. [5]

iii) Let $v_i(t)$ be a unit step applied at $t = 0$. Use the final value theorem, which should be stated, to find the steady–state value of $v_o(t)$. Comment on your answer. [5]

iv) Set $R = 1\ k\Omega$. Derive the value of L and C so that, when $v_i(t)$ is a unit step applied at $t = 0$, the settling time is 8 μs and the maximum overshoot is no more than 5%. [5]

b) In Figure 1.2 below, $G(s) = 0.5(s-1)/(s+1)$ and $K(s)$ is a compensator.

i) Draw the Nyquist diagram accurately for $G(s)$. [5]

ii) Let $K(s) = 1$. Use the Nyquist criterion, which should be stated, to show that the closed-loop is stable. Find the gain margin. [5]

iii) Let $K(s) = 10$. Use the Nyquist criterion to determine how many unstable poles the closed-loop has. [5]

iv) Let $K(s) = k/(s+2)$. Use the Routh-Hurwitz stability criterion to determine the range of values of k for closed-loop stability. [5]

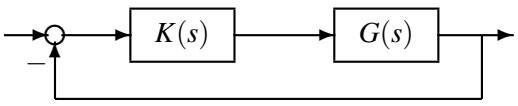


Figure 1.2

2. Consider the feedback control system in Figure 2.1 below. Here,

$$G(s) = \frac{(s-1)}{(s+1)(s+2)}$$

and $K(s)$ is the transfer function of a compensator.

a) i) Sketch the Nyquist diagram of $G(s)$, clearly indicating the low and high frequency portions. Use the Routh array to find the real–axis intercepts. [4]

ii) What is the value of the gain margin? [4]

iii) Comment on the phase margin. [4]

b) Let $K(s) = K$ be a nondynamic compensator. State the Nyquist stability criterion and use the Nyquist diagram to determine the number of unstable closed-loop poles for all values of K. [8]

c) Explain what is meant by a Proportional-plus-Derivative (PD) compensator. Your answer should include an expression for such a compensator, and a description of its frequency response. [5]

d) Without doing any actual design, describe how a PD compensator would affect the stability margins and the steady–state tracking properties of the loop. [5]

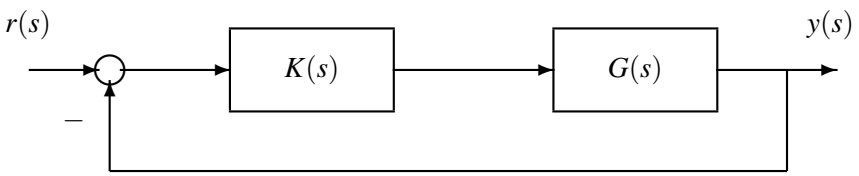


Figure 2.1

3. Let $G(s) = 1/s^2$ and consider the feedback loop shown in Figure 3.1 below.

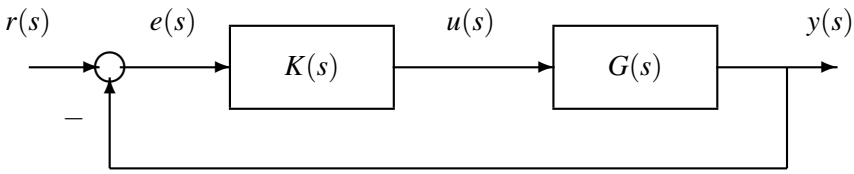


Figure 3.1

A PD compensator is required such that the following specifications in response to a step reference signal are satisfied

- The settling time is at most 2 seconds.
- The maximum overshoot is 5%.

a) Let $K(s) = K$ be a constant compensator with $K > 0$. Find the closed-loop characteristic equation in terms of K. Comment on the closed-loop stability as K varies from 0 to ∞. [5]

b) Find the location of the closed-loop poles that achieves the design specifications above. [5]

c) Let $K(s)$ be a PD compensator.

i) Give a general form of $K(s)$ and find the closed-loop characteristic equation. [5]

ii) Design a PD compensator $K(s)$ that achieves the design specifications above. [10]

d) Figure 3.2 illustrates an implementation of the PD compensator $K(s)$. Here, $C_i = 1\ \mu$F. Find the values of R_i and R_f. [5]

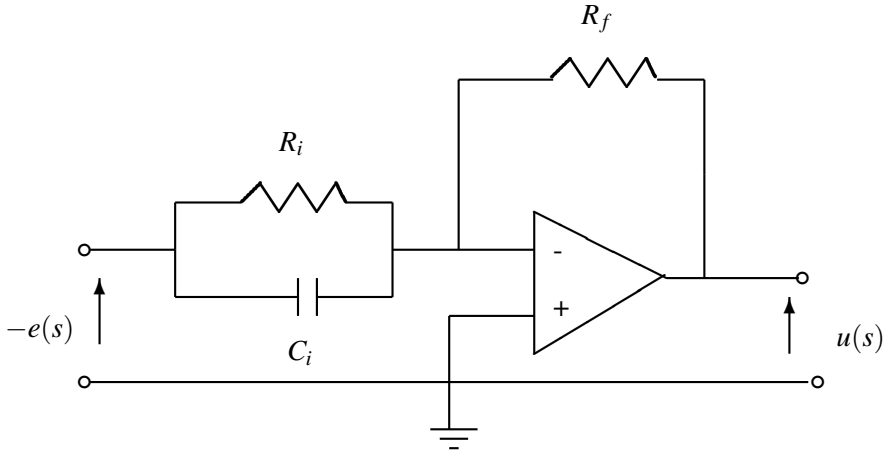


Figure 3.2

SOLUTIONS: Feedback Systems

1. a) i) Using the potential divider rule and the impedances we have

$$G(s) := \frac{v_o(s)}{v_i(s)} = \frac{(LC)^{-1}}{s^2 + sRL^{-1} + (LC)^{-1}}.$$

ii) Writing $G(s)$ in the standard form

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

we get that the DC response is

$$G(0) = 1,$$

and

$$\begin{aligned} \omega_n &= 1/\sqrt{LC}, \\ \zeta &= \frac{1}{2}R\sqrt{C/L}. \end{aligned}$$

iii) Using the final value theorem and the fact that $v_i(s) = 1/s$,

$$\begin{aligned} \lim_{t\to\infty} v_o(t) &= \lim_{s\to 0} s v_o(s) = \lim_{s\to 0} sG(s)v_i(s) = \lim_{s\to 0} sG(s)\frac{1}{s} = G(0) \\ &= 1. \end{aligned}$$

In the steady-state, the capacitor acts as an open circuit and the output voltage is the same as the input voltage.

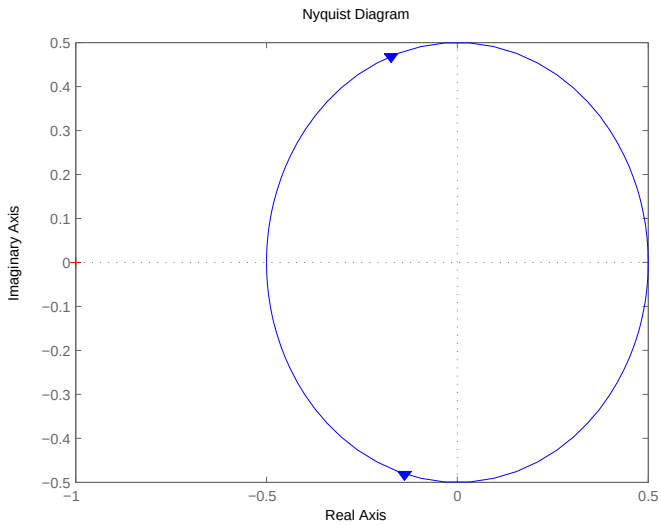
iv) The settling time is given by

$$T_s = \frac{4}{\zeta\omega_n}.$$

For a maximum overshoot of 5%, we need ζ to be at least $1/\sqrt{2}$. Putting in the numbers we get

$$\begin{aligned} L &= 1\,mH, \\ C &= 2\,nF. \end{aligned}$$

b) i) The Nyquist diagram is shown below.



ii) Let $K(s) = 1$. The Nyquist criterion states that $N = Z - P$, where N is the number of clockwise encirclements by the Nyquist diagram of the point $-K^{-1} = -1$, P is the number of unstable open–loop poles and Z is the number of unstable closed–loop poles. Since $G(s)$ is stable, $P = 0$. From the diagram, $N = 0$ and so $Z = 0$ and the closed-loop is stable. Since the real-axis intercept is at -0.5, the gain margin is 2.

iii) When $K = 10$, $N = 1$ and so $Z = 1$. Therefore there is one unstable closed-loop pole.

iv) The characteristic equation for the closed-loop is

$$1 + K(s)G(s) = 1 + \frac{k(s-1)}{2(s+1)(s+2)} = 0 \Rightarrow 2s^2 + (6+k)s + 4 - k = 0$$

The Routh array is:

$$\begin{array}{c|cc} s^2 & 2 & 4-k \\ s^1 & 6+k & \\ 1 & 4-k & \end{array}$$

For stability we need the first column to be positive, so

$$-6 < k < 4.$$

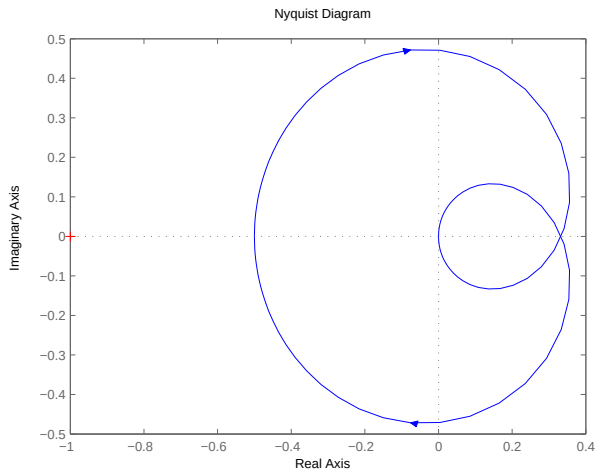
2. a) i) The characteristic equation for the closed-loop is

$$1+KG(s)=1+\frac{K(s-1)}{(s+1)(s+2)}=0 \Rightarrow s^2+(3+K)s+(2-K)=0$$

The Routh array is:

$$\begin{array}{c|cc} s^2 & 1 & 2-K \\ s & 3+K & \\ 1 & 2-K & \end{array}$$

Therefore the real-axis intercepts are 0, $-1/2$ and $1/3$. The Nyquist diagram is shown below.



ii) The gain margin is 2.

iii) Since the Nyquist diagram lies within the unit circle, the phase margin is infinite.

b) We have $N=Z-P$, where N is the number of clockwise encirclements by the Nyquist diagram of the point $-K^{-1}$, P is the number of unstable open–loop poles and Z is the number of unstable closed–loop poles. Here, $P=0$.

- When $-\infty<K<-3$, $N=2$ so $Z=2$.
- When $-3<K<0$, $N=0$ so $Z=0$.
- When $0<K<2$, $N=0$ so $Z=0$.
- When $2<K<\infty$, $N=1$ so $Z=1$.

c) A PD compensator has the form $K(s)=K_P+K_Ds$, with $K_P>0,\ K_D>0$. A PD compensator is a special form of phase-lead compensation and has gain close to K_P at low frequencies and large gain at high frequencies. The phase is close to 0° at low frequencies and tends to 90° at high frequencies.

d) Thus PD compensation can increase the phase at high frequencies and hence improve the phase margin. However, it also has large gain at high frequency which may deteriorate the gain margin. It tends to degrade steady–state tracking since it has small gain at low frequencies.

3. a) The closed-loop characteristic equation is

$$1 + KG(s) = 0$$

or

$$s^2 + K = 0.$$

It follows that the closed-loop poles are given as

$$s = \pm j\sqrt{K}$$

and so the closed-loop is marginally stable for all K.

b) For a maximum overshoot of 5%, the real and imaginary parts of the pole are equal. For a settling time of 2 seconds, the real part must be equal to -2. Thus the closed–loop poles must be placed at

$$s_1,\ \bar{s}_1 = -2 \pm j2.$$

c) i) A PD compensator has the form

$$K(s) = k(s+z).$$

The characteristic equation is

$$1 + \frac{k(s+z)}{s^2} = 0$$

or

$$s^2 + ks + kz = 0.$$

ii) Since the closed-loop poles must be equal to $s_1,\ \bar{s}_1 = -2 \pm j2$, it follows that $k = 4$ and $z = 2$.

d) The transfer function from $-e(s)$ to $u(s)$ in the figure is given by

$$R_f C_i (s + \frac{1}{R_i C_i}).$$

Thus

$$R_i = 0.5\ M\Omega$$

and

$$R_f = 4\ M\Omega.$$