EE4-54
EE9-CS4-1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2013

MSc and EEE/EIE PART IV: MEng and ACGI

PREDICTIVE CONTROL

Tuesday, 14 May 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer FOUR questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	E.C. Kerrigan
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PREDICTIVE CONTROL

1. We are interested in solving the following optimal control problem :

$$\left(u_0^*(\hat{x}),\ldots,u_{N-1}^*(\hat{x})\right) := \arg\min_{u_0,\ldots,u_{N-1}} \sum_{k=0}^{N-1} \|Qx_k + Ru_k\|_2^2$$

subject to the constraints

$$\begin{aligned} x_0 &= \hat{x}, \\ x_{k+1} &= Ax_k + Bu_k, \quad k = 0,1,\ldots,N-1 \end{aligned}$$

where the state $x_k \in \mathbb{R}^n$, input $u_k \in \mathbb{R}^m$ and weights $Q \in \mathbb{R}^{p\times n}$ and $R \in \mathbb{R}^{p\times m}$. Assume that an estimate of the current state $\hat{x}$ is given.

a) Show how the above optimization problem can be converted into an unconstrained least squares problem of the form:

$$\min_\theta \|M\theta - b\|_2^2.$$

Be careful to state the size of all matrices and vectors that you define. [8]

b) Show that the solution to the optimal control problem is unique if R is full column rank and $Q^T R = 0$. [6]

c) Suppose that R is full column rank and $Q^T R = 0$. Give an expression for the receding horizon control law

$$\kappa_N(\hat{x}) := u_0^*(\hat{x})$$

in terms of the matrices and vectors of the least squares problem. Make sure you denote the size of any identity and zero matrices in your expression. [6]

2. We are interested in solving the following optimal control problem :

$$\min_{x_0,x_1,\dots,x_N,u_{-1},u_0,\dots,u_{N-1}} \sum_{k=0}^{N-1} \left(\|Qx_{k+1}\|_2^2 + \|R(u_k - u_{k-1})\|_1\right)$$

subject to the constraints

$$\begin{aligned} x_0 &= \hat{x}, \quad u_{-1} = \hat{u}, \\ x_{k+1} &= Ax_k + Bu_k, \quad k = 0,1,\dots,N-1, \\ \delta_\ell &\le u_k - u_{k-1} \le \delta_h, \quad k = 0,1,\dots,N-1, \end{aligned}$$

where the states $x_k \in \mathbb{R}^n$, inputs $u_k \in \mathbb{R}^m$ and weights $Q \in \mathbb{R}^{p\times n}$ and $R \in \mathbb{R}^{m\times m}$. N is the horizon length. The previous value of the input $\hat{u}$ is known and an estimate of the current state $\hat{x}$ is given.

We define the following vectors:

$$\bar{x} := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \quad \bar{u} := \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}.$$

a) Give expressions for $\overline{Q}$, $\overline{R}$ and $\overline{T}$ such that one can write the cost function as

$$\sum_{k=0}^{N-1} \left(\|Qx_{k+1}\|_2^2 + \|R(u_k - u_{k-1})\|_1\right) = \|\overline{Q}\bar{x}\|_2^2 + \|\overline{R}\bar{u} + \overline{T}\hat{u}\|_1.$$

Make sure to denote the size of any identity matrices. [8]

b) Show that the solution to the above optimal control problem can be obtained by formulating it as a QP with inequality and equality constraints of the form:

$$\min_{\theta} \theta^T H\theta + c^T\theta$$

subject to

$$\begin{aligned} D\theta &\le f \\ E\theta &= g \end{aligned}$$

where $\theta := \begin{bmatrix}\bar{x}^T & \bar{u}^T & s^T\end{bmatrix}^T$ and s is a vector of appropriate length. Make sure to denote the size of any identity matrices or column vectors of ones $\mathbf{1} := [1\ 1\ \cdots\ 1]^T$.
[12]

3. Suppose we have the following QP:

$$\min_{\theta} \frac{1}{2}\theta^T H\theta + c^T\theta$$

subject to

$$\begin{bmatrix} c_H(\theta) \\ c_S(\theta) \end{bmatrix} \leq 0,$$

were

$$c_H(\theta) := D_H\theta - f_H \leq 0$$

represents hard constraints and

$$c_S(\theta) := D_S\theta - f_S \leq 0$$

represents q soft constraints. Suppose that there exists a value of θ that satisfies the hard constraints, but that it is not possible to satisfy the hard and soft constraints simultaneously.

a) Show how you would set up an LP to compute a feasible value of θ that:

   i) minimises the sum of the violations of the soft constraints; [6]

   ii) minimises the worst case violation of the soft constraints. [6]

b) Convert the soft constraints into an exact penalty function, which is added to the original cost function, and show that the new optimization problem can be converted into a single QP where:

   i) the exact penalty function penalises the sum of the violations of the soft constraints; [4]

   ii) the exact penalty function penalises the worst case violation of the soft constraints. [4]

In all your answers, make sure to denote the size of any identity matrices or column vectors of ones $\mathbf{1} := [1\ 1\ \cdots\ 1]^T$.

4. Suppose we have a system

$$x_{k+1} = Ax_k + Bu_k,$$

where the states $x_k \in \mathbb{R}^n$ and inputs $u_k \in \mathbb{R}^m$, and a reference $r \in \mathbb{R}^q$ that we would like some linear combinations of the states and inputs

$$z_k := C_z x_k + D_z u_k$$

of our system to track, i.e. we want to design a control policy $u_k = \kappa(x_k, r)$ such that

$$\lim_{k \to \infty} z_k = r.$$

Suppose also now that we have constraints on the state and input of the form

$$-c \leq C_c x_k + D_c u_k \leq c,$$

where $c \in \mathbb{R}^s$.

a) Show how you would set up a QP to compute a target equilibrium state-input pair $(x_e^*(r), u_e^*(r))$ that satisfies the inequality constraints and ensures no error between $z_e^*(r) := C_z x_e^*(r) + D_z u_e^*(r)$ and r. Define your QP to ensure that the computed state-input pair is unique. [6]

b) Show how you would set up a constrained finite horizon optimal control problem with a quadratic cost function and horizon length N such that its solution is unique and can be used to define $\kappa(\cdot,\cdot)$ as above, provided the control problem is feasible at each time step. Clearly state all assumptions and why you make them. [4]

c) Show how you would convert the problem in part b) into a QP with inequality constraints only. [10]

5. Suppose we have a continuous-time dynamical system:

$$\dot{x}(t) = Fx(t) + Gu(t),$$

where the states $x(t) \in \mathbb{R}^n$ and inputs $u(t) \in \mathbb{R}^m$, and we choose to implement a predictive controller with a sample period of h with a zero-order hold at the input, i.e.

$$u(t) = u(kh), \quad \forall t \in [kh, kh+h),$$

where $k = 0, 1, \ldots$ denotes the sample instant. The unit of time is seconds.

a) Give the definition of the exponential of a square matrix M. [2]

b) Show how you would use the matrix exponential to compute an expression for A and B in the equivalent sampled-data model

$$x(kh+h) = Ax(kh) + Bu(kh).$$

You cannot assume that F is invertible. [4]

c) Suppose the continuous-time system is a double integrator, i.e.

$$F := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad G := \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Use your method in part b) to compute an expression for A and B as a function of h. [4]

d) Suppose $h = 1$ s and $x(0)$ is known and non-zero. Use your result in part c) to convert the following set of constraints

$$\begin{aligned} -1 \le u(t) \le 1, &\quad \forall t \in [0,2), \\ [1\ 0]x(t) \le 1, &\quad \forall t \in \{0.5, 1.0, 1.5, 2\}, \end{aligned}$$

into an equivalent finite set of constraints of the form

$$D \begin{bmatrix} u(0) \\ u(1) \end{bmatrix} \le f,$$

where D is a matrix with 8 rows and 2 columns, and f is a column vector with 8 rows. Note that some of the constraints on the state are in-between the sample instants. [10]

Question 1   Variation of material in lectures (i.e. new problem)

(a)
$$\bar{x} := \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix} = \underbrace{\begin{pmatrix} I \\ A \\ \vdots \\ A^{N-1} \end{pmatrix}}_{\Phi} \hat{x} + \underbrace{\begin{bmatrix} 0 & & & \\ B & \cdots & & \\ AB & & & \\ A^{N-2}B & A^{N-3}B & \cdots & B \end{bmatrix}}_{\Gamma} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}}_{\bar{u}}$$

$$\sum_{k=0}^{N-1} \|Qx_k + Ru_k\|_2^2 = \left\| \underbrace{\begin{bmatrix} Q & & \\ & Q & \\ & & \ddots \\ & & & Q \end{bmatrix}}_{\bar{Q}} \bar{x} + \underbrace{\begin{bmatrix} R & & \\ & R & \\ & & \ddots \\ & & & R \end{bmatrix}}_{\bar{R}} \bar{u} \right\|_2^2$$

$$= \| \bar{Q}(\Phi \hat{x} + \Gamma \bar{u}) + \bar{R}\bar{u} \|_2^2$$

$$= \| \underbrace{(\bar{Q}\Gamma + \bar{R})}_{M} \underbrace{\bar{u}}_{\theta} - \underbrace{(-\bar{Q}\Phi\hat{x})}_{b} \|_2^2$$

Φ has Nn rows, n columns

Γ has Nn rows, Nm columns

$\bar{Q}$ has Np rows & Nn columns

$M, \bar{R}$ has Np rows & Nm columns.

b has Np rows, 1 column

θ has Nm rows, 1 column.

(b) The solution is given by
$$M^T M \theta^* = M^T b$$

$$\forall v \neq 0: v^T M^T M v = v^T(\bar{Q}\Gamma + \bar{R})^T(\bar{Q}\Gamma + \bar{R})v = v^T(\bar{R}^T\bar{R} + \Gamma^T\bar{Q}^T\bar{Q}\Gamma + 2\bar{Q}^T\bar{R})v$$

$$\bar{Q}^T\bar{R} = \begin{bmatrix} Q^T R & & \\ & Q^T R & \\ & & \ddots \end{bmatrix} = 0$$

$$\Rightarrow v^T M^T M v = v^T(\bar{R}^T\bar{R} + \underbrace{\Gamma^T\bar{Q}^T\bar{Q}\Gamma}_{\geq 0})v \quad \forall v \neq 0$$

if $\bar{R}$ is full col. rank $\Rightarrow \bar{R}^T\bar{R} > 0 \Rightarrow M^T M > 0$

$\Rightarrow$ solution is unique

1 (c) if $M^T M > 0 \Rightarrow \theta^* = (M^T M)^{-1} M^T b$.

$$\Leftrightarrow \begin{bmatrix} u_0^*(\hat{x}) \\ \vdots \\ u_{N-1}^*(\hat{x}) \end{bmatrix} = (M^T M)^{-1} M^T b$$

$$\Rightarrow u_0^*(\hat{x}) = \begin{bmatrix} I_m & 0 \end{bmatrix} (M^T M)^{-1} M^T b.$$

m rows, (N-1) columns.

$\longrightarrow$

Question 2 New problem

(a) $$\sum_{k=0}^{N-1} \|Q x_{k+1}\|_2^2 = \left\| \begin{bmatrix} Q & & \\ & \ddots & \\ & & Q \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \right\|_2^2 \Rightarrow \bar{Q} = I_N \otimes Q.$$

$$\sum_{k=0}^{N-1} \|R(u_k - u_{k-1})\|_1 = \left\| \begin{pmatrix} R & & \\ & \ddots & \\ & & R \end{pmatrix} \begin{pmatrix} u_0 - u_{-1} \\ u_1 - u_0 \\ \vdots \\ u_{N-1} - u_{N-2} \end{pmatrix} \right\|_1$$

$$\begin{pmatrix} u_0 - u_{-1} \\ u_1 - u_0 \\ \vdots \\ u_{N-1} - u_{N-2} \end{pmatrix} = \begin{pmatrix} I_m & & & \\ -I_m & I_m & & \\ & -I_m & \ddots & \\ & & -I_m & I_m \end{pmatrix} \begin{pmatrix} u_0 \\ \vdots \\ u_{N-1} \end{pmatrix} + \begin{pmatrix} -I_m \\ 0 \\ \vdots \\ 0 \end{pmatrix} \hat{u}$$

$$= \underbrace{\left[I_{Nm} - \begin{pmatrix} 0 & 0 \\ I_{(N-1)m} & 0 \end{pmatrix} \right]}_{\mathbb{I}} \bar{u} + \underbrace{\begin{bmatrix} -I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{V} \hat{u}$$

$$\Rightarrow \bar{R} = (I_N \otimes R)\mathbb{I}$$

$$\bar{T} = (I_N \otimes R)V \longrightarrow$$

(b) ~~$\min \|\bar{Q}x\|_2^2 + \|\bar{R}u + \bar{T}\hat{u}\|_1$~~

First, convert inequality constraints into a suitable form:

$$\delta l \leq u_k - u_{k-1} \leq \delta h, \quad k = 0, 1, \dots, N-1 \qquad \mathbb{1}_N = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\Leftrightarrow \mathbb{1}_N \otimes \delta l \leq \mathbb{I}\bar{u} + V\hat{u} \leq \mathbb{1}_N \otimes \delta h.$$

$$\Leftrightarrow \begin{cases} \mathbb{I}\bar{u} + V\hat{u} \leq \mathbb{1}_N \otimes \delta h \\ -\mathbb{I}\bar{u} - V\hat{u} \leq -\mathbb{1}_N \otimes \delta l \end{cases}$$

Next, convert equality constraints into a suitable form:

$$\underbrace{\begin{pmatrix} I_n & & \\ -A & I_n & \\ & \ddots & \\ & -A & I_n \end{pmatrix}}_{\bar{A}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} - \underbrace{\begin{pmatrix} B & & \\ & B & \\ & & \ddots \\ & & & B \end{pmatrix}}_{\bar{B}} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = \underbrace{\begin{pmatrix} A \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\bar{\Phi}} \hat{x}$$

2b cont)

$$\Longleftrightarrow \quad \bar{A}\,\bar{x} - \bar{B}\,\bar{u} = \Phi\,\hat{x}$$

Problem now becomes

$$\min_{\bar{x},\bar{u}} \|\bar{Q}\bar{x}\|_2^2 + \|\bar{R}\bar{u} + \bar{T}\hat{u}\|_1$$

$$\text{s.t.} \quad \begin{bmatrix} \bar{T} \\ -\bar{T} \end{bmatrix} \bar{u} \leq \begin{bmatrix} \mathbb{1}_N \otimes \delta h - V\hat{u} \\ -\mathbb{1}_N \otimes \delta l + V\hat{u} \end{bmatrix} \quad (*)$$

$$\begin{bmatrix} \bar{A} & -\bar{B} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = \Phi\,\hat{x}$$

Introducing slack variable s for 1-norm term:

(no need for $s \geq 0$)

$$\min_{\bar{x},\bar{u},s} \|\bar{Q}\bar{x}\|_2^2 + \mathbb{1}_{Nm}^T s$$

$$\text{s.t. } (*) \text{ and } \quad -s \leq \bar{R}\bar{u} + \bar{T}\hat{u} \leq s$$

$$\Longleftrightarrow (*) \text{ and } \begin{cases} \bar{R}\bar{u} - s \leq -\bar{T}\hat{u} \\ -\bar{R}\bar{u} - s \leq \bar{T}\hat{u} \end{cases}$$

$$= \min_{\bar{x},\bar{u},s} \bar{x}^T \bar{Q}^T \bar{Q} x + \mathbb{1}_{Nm}^T s$$

$$\text{s.t.} \quad \underbrace{\begin{bmatrix} 0 & \mathbb{I} & 0 \\ 0 & -\mathbb{I} & 0 \\ 0 & \bar{R} & -I_{Nm} \\ 0 & -\bar{R} & -I_{Nm} \end{bmatrix}}_{D} \begin{pmatrix} \bar{x} \\ \bar{u} \\ s \end{pmatrix} \leq \underbrace{\begin{bmatrix} \mathbb{1}_N \otimes \delta h - V\hat{u} \\ -\mathbb{1}_N \otimes \delta l + V\hat{u} \\ -\bar{T}\hat{u} \\ \bar{T}\hat{u} \end{bmatrix}}_{f}$$

$$\underbrace{\begin{bmatrix} \bar{A} & -\bar{B} & 0 \end{bmatrix}}_{E} \begin{pmatrix} \bar{x} \\ \bar{u} \\ s \end{pmatrix} = \underbrace{\Phi\,\hat{x}}_{g}$$

$$\bar{x}^T \bar{Q}^T \bar{Q} x + \mathbb{1}^T s = \begin{pmatrix} \bar{x} \\ \bar{u} \\ s \end{pmatrix}^T \underbrace{\begin{pmatrix} \bar{Q}^T\bar{Q} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{H} \begin{pmatrix} \bar{x} \\ \bar{u} \\ s \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ \mathbb{1}_{Nm} \end{pmatrix}}_{C}^T \begin{pmatrix} \bar{x} \\ \bar{u} \\ s \end{pmatrix}$$

Question 3 Bookwork ~~+ new~~ + new problems.

(a) (i) $\min_{\theta, s} \mathbb{1}_q^T s$ s.t. $D_H \theta \leq f_H$

$D_S \theta - f_s \leq s$

$s \geq 0$

$$= \min_{\theta, s} \begin{pmatrix} 0 \\ \mathbb{1}_q \end{pmatrix}^T \begin{pmatrix} \theta \\ s \end{pmatrix}$$

$$\text{s.t.} \begin{pmatrix} D_H & 0 \\ D_s & -I_q \\ 0 & -I_q \end{pmatrix} \begin{pmatrix} \theta \\ s \end{pmatrix} \leq \begin{pmatrix} f_H \\ f_s \\ 0 \end{pmatrix}$$

$\longrightarrow$

(ii) $\min_{\theta, t} t$ s.t. $D_H \theta \leq f_H$

$D_s \theta - f_s \leq \mathbb{1}_q t$

$t \geq 0$

$$= \min_{(\theta, t)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} \theta \\ t \end{pmatrix}$$

$$\text{s.t.} \begin{pmatrix} D_H & 0 \\ D_s & -\mathbb{1}_q \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \theta \\ t \end{pmatrix} \leq \begin{pmatrix} f_H \\ f_s \\ 0 \end{pmatrix}$$

$\longrightarrow$

3(b)(i) $$\min_{\theta,s} \frac{1}{2}\theta^T H\theta + c^T\theta + \rho\,(\mathbb{1}_q^T s) \quad ,\rho>0$$
$$\text{s.t. } D_H\theta \le f_H$$
$$D_s\theta - f_s \le s$$
$$s \ge 0$$

$$= \min_{\theta,s} \frac{1}{2}\begin{pmatrix}\theta\\ s\end{pmatrix}^T \begin{pmatrix}H & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\theta\\ s\end{pmatrix} + \begin{pmatrix}c\\ \rho\mathbb{1}_q\end{pmatrix}^T\begin{pmatrix}\theta\\ s\end{pmatrix}$$

$$\text{s.t. } \begin{pmatrix} D_H & 0\\ D_s & -I_q\\ 0 & -I_q\end{pmatrix}\begin{pmatrix}\theta\\ s\end{pmatrix} \le \begin{pmatrix}f_H\\ f_s\\ 0\end{pmatrix}$$

(ii) $$\min_{\theta,t} \frac{1}{2}\theta^T H\theta + c^T\theta + \rho t \quad ,\rho>0$$
$$\text{s.t. } D_H\theta \le f_H$$
$$D_s\theta - f_s \le \mathbb{1}_q t$$
$$t \ge 0$$

$$= \min_{\theta,t} \frac{1}{2}\begin{pmatrix}\theta\\ t\end{pmatrix}^T \begin{pmatrix}H & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\theta\\ t\end{pmatrix} + \begin{pmatrix}c\\ \rho\end{pmatrix}^T\begin{pmatrix}\theta\\ t\end{pmatrix}$$

$$\begin{pmatrix} D_H & 0\\ D_s & -\mathbb{1}_q\\ 0 & -1\end{pmatrix}\begin{pmatrix}\theta\\ t\end{pmatrix} \le \begin{pmatrix}f_H\\ f_s\\ 0\end{pmatrix}$$

Question 4 New problem, mostly based on bookwork.

(a)

$$\left(x_e^*(r), u_e^*(r)\right) := \arg\min_{x_e, u_e} \|Q_e x_e\|_2^2 + \|R_e u_e\|_2^2$$

$$\text{s.t.} \quad C_z x_e + D_z u_e = r$$

$$-c \le C_c x_e + D_c u_e \le c$$

$$= \arg\min_{(x_e, u_e)} x_e^T Q_e^T Q_e x_e + u_e^T R_e^T R_e u_e$$

$$\text{s.t.} \quad C_z x_e + D_z u_e = r$$

$$C_c x_e + D_c u_e \le c$$

$$-C_c x_e - D_c u_e \le c$$

$$= \arg\min_{(x_e, u_e)} \begin{pmatrix} x_e \\ u_e \end{pmatrix}^T \begin{pmatrix} Q_e^T Q_e & 0 \\ 0 & R_e^T R_e \end{pmatrix} \begin{pmatrix} x_e \\ u_e \end{pmatrix}$$

$$\text{s.t.} \quad \begin{pmatrix} C_z & D_z \end{pmatrix} \begin{pmatrix} x_e \\ u_e \end{pmatrix} = r$$

$$\begin{pmatrix} C_c & D_c \\ -C_c & -D_c \end{pmatrix} \begin{pmatrix} x_e \\ u_e \end{pmatrix} \le \begin{pmatrix} c \\ c \end{pmatrix}$$

In order to ensure solution is unique, choose Q_e and R_e to be full column rank

$\therefore \Rightarrow Q_e^T Q_e > 0$ and $R_e^T R_e > 0$. $\longrightarrow$

(b)

$$\left(u_0^*(\hat{x}, r), u_1^*(\hat{x}, r), \ldots, u_{N-1}^*(\hat{x}, r)\right) := \arg\min_{u_0, \ldots, u_{N-1}} \sum_{k=0}^{N-1} \left(\|Q(x_k - x_e^*(r))\|_2^2 + \|R(u_k - u_e^*(r))\|_2^2 \right)$$

$$\text{s.t.} \quad x_0 = \hat{x}$$

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, \ldots, N-1$$

$$-c \le C_c x_k + D_c u_k \le c, \quad k = 0, 1, \ldots, N-1$$

$$\Rightarrow K(x, r) := u_0^*(x, r) \longrightarrow$$

4 (c) $$\underbrace{\bar{x} := \begin{pmatrix} x_0 \\ x_2 \\ \vdots \\ x_{N-1} \end{pmatrix}}_{} := \underbrace{\begin{pmatrix} I \\ A \\ \vdots \\ A^{N-1} \end{pmatrix}}_{\Phi} \hat{x} + \underbrace{\begin{bmatrix} 0 & & \ddots & \\ B & & & \\ AB & B & \ddots & \\ \vdots & & & \\ A^{N-2}B & & B & 0 \end{bmatrix}}_{\Gamma} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}}_{\bar{u}}$$

$$\Rightarrow \bar{x} = \Phi \hat{x} + \Gamma u .$$

$$\text{cost} = \Big\| \underbrace{\begin{pmatrix} Q & & \\ & \ddots & \\ & & Q \end{pmatrix}}_{\bar{Q}} \left[\bar{x} - \mathbb{1}_N \otimes x_e^*(r) \right] \Big\|_2^2 \qquad \left(\bar{Q} = I_N \otimes Q, \; \bar{R} = I_N \otimes R \right)$$

$$+ \Big\| \underbrace{\begin{pmatrix} R & & \\ & \ddots & \\ & & R \end{pmatrix}}_{\bar{R}} \left[\bar{u} - \mathbb{1}_N \otimes u_e^*(r) \right] \Big\|_2^2$$

$$= \left\| \begin{matrix} \bar{Q} \left[\Phi \hat{x} + \Gamma \bar{u} - \mathbb{1}_N \otimes x_e^*(r) \right] \\ \bar{R} \left[\bar{u} - \mathbb{1}_N \otimes u_e^*(r) \right] \end{matrix} \right\|_2^2$$

$$= \Big\| \underbrace{\begin{bmatrix} \bar{Q}\Gamma \\ \bar{R} \end{bmatrix}}_{M} \bar{u} - \underbrace{\begin{bmatrix} \bar{Q} \left(\mathbb{1}_N \otimes x_e^*(r) \right) - \bar{Q}\Gamma \hat{x} \\ \bar{R} \left(\mathbb{1}_N \otimes u_e^*(r) \right) \end{bmatrix}}_{b} \Big\|_2^2$$

constraints :
$$-c \le \begin{pmatrix} C_c & & \\ & \ddots & \\ & & C_c \end{pmatrix} \bar{x}_0 + \begin{pmatrix} D_c & & \\ & \ddots & \\ & & D_c \end{pmatrix} \bar{u} \le c$$

$$\Longleftrightarrow \quad \begin{matrix} (I_N \otimes C_c)\bar{x} + (I_N \otimes D_c)\bar{u} \le c \\ -(I_N \otimes C_c)\bar{x} - (I_N \otimes D_c)\bar{u} \le +c \end{matrix}$$

$$\Longleftrightarrow \quad \begin{pmatrix} I_N \otimes C_c & I_N \otimes D_c \\ -I_N \otimes C_c & -I_N \otimes D_c \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix} \le \begin{pmatrix} c \\ c \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} I_N \otimes C_c & I_N \otimes D_c \\ -I_N \otimes C_c & -I_N \otimes D_c \end{pmatrix} \begin{pmatrix} \Gamma \\ I \end{pmatrix} \bar{u} \le \begin{pmatrix} c - (I_N \otimes C_c)\bar{Q}\hat{x} \\ c + (I_N \otimes C_c)\bar{Q}\hat{x} \end{pmatrix}$$

$$\Longleftrightarrow \underbrace{\begin{pmatrix} (I_N \otimes C_c)\Gamma & I_N \times D_c \\ -(I_N \otimes C_c)\Gamma & -I_N \otimes D_c \end{pmatrix}}_{E} \bar{u} \le \underbrace{\begin{pmatrix} c - (I_N \otimes C_c)\bar{Q}\hat{x} \\ c + (I_N \otimes C_c)\bar{Q}\hat{x} \end{pmatrix}}_{f}$$

Q.E.D.

4c) cont) $\underset{u_0,\dots,u_{N-1}}{\arg\min} \quad \|M\bar{u} - b\|_2^2$

$$\text{s.t. } E\bar{u} \le f$$

$$= \underset{\bar{u}}{\arg\min} \quad \bar{u}^T M^T M \bar{u} - 2\bar{u}^T M^T b + b^T b$$

$$\text{s.t. } E\bar{u} \le f$$

$$= \underset{\bar{u}}{\arg\min} \ \frac{1}{2} \bar{u}^T H \bar{u} + g^T \bar{u}$$

$$\text{s.t. } E\bar{u} \le f$$

where $H = 2M^T M$

$g = -2b^T M$

$\longrightarrow$ P.

Question 5. ~~New problem~~.

(a) Bookwork $e^{M} := I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{M^k}{k!}$

(b) Bookwork.
Define augmented state $z = \begin{pmatrix} x \\ u \end{pmatrix}$

$$\Rightarrow \dot{z}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{u}(t) \end{pmatrix} \quad \forall t \in [kh, kh+h)$$

note $\dot{u}(t) = 0 \ \forall t \in [kh, (k+1)h) \Rightarrow \dot{z}(t) = \begin{pmatrix} Fx(t) + Gu(t) \\ 0 \end{pmatrix} = \begin{pmatrix} F & G \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}$

$$\Rightarrow z(kh+h) = e^{\begin{pmatrix} F & G \\ 0 & 0 \end{pmatrix} h} z(kh)$$

$$\Rightarrow x(kh+h) = \begin{bmatrix} I_n & 0 \end{bmatrix} e^{\left[\begin{pmatrix} F & G \\ 0 & 0 \end{pmatrix} h\right]} \begin{bmatrix} x(kh) \\ u(kh) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} I_n & 0 \end{bmatrix} e^{\begin{pmatrix} F & G \\ 0 & 0 \end{pmatrix} h}$$

New problem
(c) $\begin{pmatrix} F & G \\ 0 & 0 \end{pmatrix} h = \begin{pmatrix} 0 & h & 0 \\ 0 & 0 & h \\ 0 & 0 & 0 \end{pmatrix} = M$

$$M^2 = \begin{pmatrix} 0 & 0 & h^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow M^k = 0 \ \forall k \geq 3$$

$$\Rightarrow e^{M} = I + M + \frac{M^2}{2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & h & 0 \\ 0 & 0 & h \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & h^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & h & h^2/2 \\ 0 & 1 & h \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} A & B \end{bmatrix} = \begin{pmatrix} 1 & h & h^2/2 \\ 0 & 1 & h \end{pmatrix} \Rightarrow A = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} h^2/2 \\ h \end{pmatrix}$$

New problem
(d) Input constraints : $-1 \leq u(t) \leq 1, \ \forall t \in [0, 2)$

$$+ \text{ZOH} \iff \begin{matrix} -1 \leq u(0) \leq 1 \\ -1 \leq u(1) \leq 1 \end{matrix} \iff$$

$$\iff \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & +1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u(0) \\ u(1) \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

5d) cont) $x\left(kh+\frac{h}{2}\right) = \begin{pmatrix}1 & 0.5\\ 0 & 1\end{pmatrix} x(kh) + \begin{pmatrix}0.125\\ 0.5\end{pmatrix} u(kh)$

$h=1$

ie. $\Rightarrow x\left(k+\frac{1}{2}\right) = \begin{pmatrix}1 & 1/2\\ 0 & 1\end{pmatrix} x(k) + \begin{pmatrix}1/8\\ 1/2\end{pmatrix} u(k)$, $k=0,1$

$\Rightarrow [1\ 0]\, x\left(k+\frac{1}{2}\right) = (1\ \ 1/2)\, x(k) + (1/8)\, u(k)$, $k=0,1$

$\Rightarrow$ ~~State constraints~~

$x(k+1) = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} x(k) + \begin{pmatrix}1/2\\ 1\end{pmatrix} u(k)$, $k=0,1$

$\Rightarrow [1\ 0]\, x(k+1) = (1\ \ 1)\, x(k) + \frac{1}{2} u(k)$, $k=0,1$

$\Rightarrow$ State constraints:

$$[1\ 0]\, x(0.5) \le 1$$
$$[1\ 0]\, x(1) \le 1$$
$$[1\ 0]\, x(1.5) \le 1$$
$$[1\ 0]\, x(2) \le 1$$

$\Leftrightarrow$

$$\left.\begin{aligned} (1\ \ 1/2)\, x(0) + \tfrac{1}{8} u(0) &\le 1\\ (1\ \ 1)\, x(0) + \tfrac{1}{2} u(0) &\le 1\\ (1\ \ \tfrac{1}{2})\, x(1) + \tfrac{1}{8} u(1) &\le 1\\ (1\ \ 1)\, x(1) + \tfrac{1}{2} u(1) &\le 1 \end{aligned}\right\}$$

$$x(1) = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} x(0) + \begin{pmatrix}1/2\\ 1\end{pmatrix} u(1)$$

$$(1\ \ 1/2)\, x(1) = (1\ \ 1.5)\, x(0) + u(1)$$

$$(1\ \ 1)\, x(1) = (1\ \ 2)\, x(0) + 1.5\, u(1)$$

$\Leftrightarrow$ $\cancel{(1\ \ 0.5)\, x}$

$$\tfrac{1}{8} u(0) \le 1 - (1\ \ 0.5)\, x(0)$$
$$\tfrac{1}{2} u(0) \le 1 - (1\ \ 1)\, x(0)$$
$$\tfrac{9}{8} u(0) \le 1 - (1\ \ 1.5)\, x(0)$$
$$2\, u(1) \le 1 - (1\ \ 2)\, x(0)$$

all constraints:

$\Leftrightarrow$

$$\begin{pmatrix} 1 & 0\\ -1 & 0\\ 0 & 1\\ 0 & -1\\ 1/8 & 0\\ 1/2 & 0\\ 8 & 9/8\\ & 2 \end{pmatrix} \begin{pmatrix} u(0)\\ u(1) \end{pmatrix} \le \begin{pmatrix} 1\\ 1\\ 1\\ 1\\ 1 - (1\ \ 0.5)\, x(0)\\ 1 - (1\ \ 1)\, x(0)\\ 1 - (1\ \ 1.5)\, x(0)\\ 1 - (1\ \ 2)\, x(0) \end{pmatrix}$$

QED.