Imperial College London

[E 2.9 (Maths 4) 2009]

B.ENG. AND M.ENG. EXAMINATIONS 2009

PART II Paper 4 : MATHEMATICS (ELECTRICAL ENGINEERING)

Date Thursday 4th June 2009 2.00 - 4.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer FOUR questions.

Please answer question from Section A and Section B in separate answerbooks.

A mathematical formulae sheet is provided.

Statistical data sheets are provided.

[Before starting, please make sure that the paper is complete; there should be 5 pages, with a total of 6 questions. Ask the invigilator for a replacement if your copy is faulty.]

SECTION A

1. Find the eigenvalues and normalized eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & \sqrt{12} & 0 \\ \sqrt{12} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} .$$

Using these, or otherwise, show that the matrix

$$P = \begin{pmatrix} \sqrt{4/7} & \sqrt{3/7} & 0 \\ \sqrt{3/7} & -\sqrt{4/7} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

diagonalizes A such that

$$P^T A P = \operatorname{diag}(5, -2, 3) .$$

2. Show that the quadratic form

$$Q = x_1^2 + 2\sqrt{6}\, x_1 x_2 + 2x_2^2 + 6x_3^2$$

can be written as

$$Q = \boldsymbol{x}^T A \boldsymbol{x}$$

where $\boldsymbol{x} = (x_1, x_2, x_3)^T$ and A is a real symmetric matrix, which is to be found. Hence show that Q can be re-expressed in the diagonal form

$$Q = 4y_1^2 - y_2^2 + 6y_3^2 ,$$

by finding a matrix P that satisfies

$$\boldsymbol{x} = P\boldsymbol{y} \qquad \text{where} \qquad \boldsymbol{y} = (y_1, y_2, y_3)^T .$$

Find y_1, y_2 and y_3 in terms of x_1, x_2 and x_3 from the matrix P.

PLEASE TURN OVER

SECTION B

3. Consider the discrete random variables X and Y with joint probability mass function $p(x, y) = \mathrm{P}(X = x, Y = y)$ given by the table below.

	$p(x,y)$	$y = 0$	$y = 1$
	0	0.4k	0.2k
x	1	0.2k	0.4k
	2	0.6k	0.2k

(i) Show that $k = 1/2$.

(ii) Find the marginal distribution of X and the marginal distribution of Y.

(iii) Find $\mathrm{E}(X)$ and $\mathrm{E}(Y)$.

(iv) Find $\mathrm{Var}(X)$.

(v) Find $\mathrm{cov}(X, Y)$.

(vi) Are X and Y uncorrelated? Give your reasoning.

(vii) Are X and Y independent? Give your reasoning.

4. (i) Let X be *Exponential* (1).

Find the cumulative distribution function (cdf) of the random variable X^2.

(ii) Let X_1, X_2, X_3, X_4 be independent *Exponential* (λ) distributed random variables.

Find the cdf of $\min(X_1, X_2, X_3, X_4)$.

(iii) Let X and Y be independent *Exponential* (λ) distributed random variables.

(a) Find $\mathrm{cov}(X, Y)$.

(b) Find $\mathrm{E}(X - Y)$.

(c) Find $\mathrm{Var}(X - Y)$.

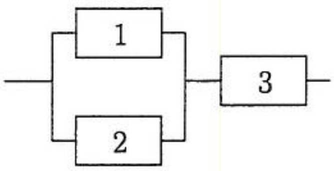
(d) Find the probability density function (pdf) of $X + Y$.

5. (i) Suppose that a system consists of k components in series, i.e. it functions as long as all components function. Suppose that the states of the components are independent and that each component functions with probability 0.999.

   (a) What is the probability that the system functions?

   (b) It is required that the system will fail with a probability of less than 0.01. What is the maximal number of components that can be used?

   (ii) Consider a system in which there are three components with component 3 in series to the parallel components 1 and 2 as in the sketch.



   Suppose that the failure times of the components are independent and follow an exponential distribution with parameters $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$, respectively.
   What is the probability that the system fails before a specified time t?

   (iii) Let T be a nonnegative random variable with hazard rate

$$h(t) = at^b$$

   for some $a > 0$, $b > -1$.

   (a) Find the cumulative distribution function F of T.

   (b) Find the probability density function f of T.

PLEASE TURN OVER

6. (i) Let $X_1, \ldots, X_n$ be a random sample from an $N(\mu, 1)$ distribution, where μ is unknown.

   (a) Find the maximum likelihood estimator $\hat{\mu}$ of μ.

   (b) What is the distribution of $\hat{\mu}$?

   Find $\mathrm{P}(\hat{\mu} > \mu + 2/\sqrt{n})$.

(ii) The random variable Y has density function

$$f(y) = \frac{1}{6}\lambda^4 y^3 e^{-\lambda y} \text{ on } (0, \infty), \text{ with } \lambda > 0.$$

   (a) Find $\mathrm{E}(Y^{-1})$.

   You may use that $\int_0^\infty y^k e^{-\lambda y} dy = \frac{k!}{\lambda^{k+1}}$ for integers $k \geq 0$.

   (b) A random sample $(y_1, \ldots, y_n)$ is obtained from the Y-distribution.

   Show that the estimator

$$T = \frac{3}{n} \sum_{i=1}^{n} y_i^{-1}$$

   is unbiased for λ and find the mean-square error of T.

   You may use that $\mathrm{Var}(Y^{-1}) = \lambda^2/18$.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b} \times \mathbf{c} = \mathbf{b}\,.\,\mathbf{c} \times \mathbf{a} = \mathbf{c}\,.\,\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b\,;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i \sinh z\,; \quad \sinh iz = i \sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a + \theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h,\, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2!\,\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right]_{a,b} + \cdots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two

estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

1. $\boxed{\text{Probabilities for events}}$

For events A, B, and C $\quad P(A \cup B) \;=\; P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) \;=\; \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of A $\quad P(A) \,/\, P(\overline{A})$

Conditional probability $\quad P(A \mid B) \;=\; \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\quad P(A \cap B \cap C) \;=\; P(A)\, P(B \mid A)\, P(C \mid A \cap B)$

Bayes' rule $\quad P(A \mid B) \;=\; \dfrac{P(A)\, P(B \mid A)}{P(A)\, P(B \mid A) \;+\; P(\overline{A})\, P(B \mid \overline{A})}$

A and B are independent if $\quad P(B \mid A) = P(B)$

A, B, and C are independent if $\quad P(A \cap B \cap C) \;=\; P(A)P(B)P(C)$, and

$$P(A \cap B) \;=\; P(A)P(B), \quad P(B \cap C) \;=\; P(B)P(C), \quad P(C \cap A) \;=\; P(C)P(A)$$

2. $\boxed{\text{Probability distribution, expectation and variance}}$

The probability distribution for a discrete random variable X is called the probability mass function (pmf) and is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) \;=\; \mu \;=\; \sum_x x p_x$

For function $g(x)$ of x, $\quad E\{g(X)\} = \sum_x g(x) p_x$, $\quad$ so $\quad E(X^2) = \sum_x x^2 p_x$

Sample mean $\quad \bar{x} \;=\; \dfrac{1}{n} \sum_k x_k$ $\quad$ estimates μ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \mathrm{var}\,(X) \;=\; \sigma^2 \;=\; E\{(X-\mu)^2\} \;=\; E(X^2) \;-\; \mu^2$

Sample variance $\quad s^2 \;=\; \dfrac{1}{n-1} \left\{ \sum_k x_k^2 \;-\; \dfrac{1}{n} \left(\sum_j x_j \right)^2 \right\}$ $\quad$ estimates σ^2

Standard deviation $\quad \mathrm{sd}\,(X) \;=\; \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness $\quad \beta_1 \;=\; E\left(\dfrac{X-\mu}{\sigma}\right)^3$ $\quad$ is estimated by $\quad \dfrac{1}{n-1} \sum \left(\dfrac{x_i - \bar{x}}{s}\right)^3$

Kurtosis $\quad \beta_2 \;=\; E\left(\dfrac{X-\mu}{\sigma}\right)^4 - 3$ $\quad$ is estimated by $\quad \dfrac{1}{n-1} \sum \left(\dfrac{x_i - \bar{x}}{s}\right)^4 - 3$

Sample median $\quad \tilde{x}$ or x_{med}. Half the sample values are smaller and half larger

If the sample values $x_1, \ldots, x_n$ are ordered as $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$,
then $\tilde{x} \;=\; x_{(\frac{n+1}{2})}$ if n is odd, and $\tilde{x} \;=\; \frac{1}{2}\left(x_{(\frac{n}{2})} \;+\; x_{(\frac{n+2}{2})}\right)$ if n is even

<u>α-quantile</u> $Q(\alpha)$ is such that $P(X \leq Q(\alpha)) = \alpha$

<u>Sample α-quantile</u> $\widehat{Q}(\alpha)$ Proportion α of the data values are smaller

<u>Lower quartile</u> $\mathrm{Q1} = \widehat{Q}(0.25)$ one quarter are smaller

<u>Upper quartile</u> $\mathrm{Q3} = \widehat{Q}(0.75)$ three quarters are smaller

<u>Sample median</u> $\tilde{x} = \widehat{Q}(0.5)$ estimates the <u>population median</u> $Q(0.5)$

3. Probability distribution for a continuous random variable

The <u>cumulative distribution function</u> (cdf) $\quad F(x) = P(X \leq x) = \displaystyle\int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$

The <u>probability density function</u> (pdf) $\quad f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$E(X) = \mu = \displaystyle\int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x\,, \quad \mathrm{var}\,(X) = \sigma^2 = E(X^2) - \mu^2,$ where $E(X^2) = \displaystyle\int_{-\infty}^{\infty} x^2\, f(x)\mathrm{d}x$

4. Discrete probability distributions

<u>Discrete Uniform</u> *Uniform* (n)

$p_x = \dfrac{1}{n} \quad (x = 1, 2, \ldots, n) \qquad \mu = (n+1)/2, \;\; \sigma^2 = (n^2-1)/12$

<u>Binomial distribution</u> *Binomial* (n, θ)

$p_x = \dbinom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n) \qquad \mu = n\theta\,, \;\; \sigma^2 = n\theta(1-\theta)$

<u>Poisson distribution</u> *Poisson* (λ)

$p_x = \dfrac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots)$ (with $\lambda > 0$) $\qquad \mu = \lambda\,, \;\; \sigma^2 = \lambda$

<u>Geometric distribution</u> *Geometric* (θ)

$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots) \qquad \mu = \dfrac{1}{\theta}\,, \;\; \sigma^2 = \dfrac{1-\theta}{\theta^2}$

5. Continuous probability distributions

<u>Uniform distribution</u> *Uniform* (α, β)

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \qquad \mu = (\alpha+\beta)/2\,, \;\; \sigma^2 = (\beta-\alpha)^2/12$$

<u>Exponential distribution</u> *Exponential* (λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \leq 0). \end{cases} \qquad \mu = 1/\lambda\,, \;\; \sigma^2 = 1/\lambda^2$$

Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\} \quad (-\infty < x < \infty), \quad E(X) = \mu, \quad \mathrm{var}(X) = \sigma^2$$

Standard normal distribution $N(0,1)$

If X is $N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$

6. Reliability

For a device in continuous operation with failure time random variable T having pdf $f(t)$ $(t > 0)$

The reliability function at time t $\quad R(t) = P(T > t)$

The failure rate or hazard function $\quad h(t) = f(t)/R(t)$

The cumulative hazard function $\quad H(t) = \int_0^t h(t_0)\, dt_0 = -\ln\{R(t)\}$

The Weibull(α, β) distribution has $\quad H(t) = \beta t^{\alpha}$

7. System reliability

For a system of k devices, which operate independently, let

$R_i = P(D_i) = P($"device i operates"$)$

The system reliability, R, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$$

A system of devices in parallel operates if any device operates

$$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1-R_1)(1-R_2)\cdots(1-R_k)$$

8. Covariance and correlation

The covariance of X and Y $\quad \mathrm{cov}(X,Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample covariance $\quad s_{xy} = \dfrac{1}{n-1} S_{xy}$ estimates $\mathrm{cov}(X,Y)$

Correlation coefficient $\quad \rho = \mathrm{corr}(X,Y) = \dfrac{\mathrm{cov}(X,Y)}{\mathrm{sd}(X) \cdot \mathrm{sd}(Y)}$

Sample correlation coefficient $\quad r = \dfrac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$ estimates ρ

9. Sums of random variables

$$E(X+Y) = E(X)+E(Y)$$

$$\text{var}(X+Y) = \text{var}(X)+\text{var}(Y)+2\,\text{cov}(X,Y)$$

$$\text{cov}(aX+bY,\ cX+dY) = (ac)\,\text{var}(X)+(bd)\,\text{var}(Y)+(ad+bc)\,\text{cov}(X,Y)$$

If X is $N(\mu_1,\sigma_1^2)$, Y is $N(\mu_2,\sigma_2^2)$, and $\text{cov}(X,Y)=c$, then $X+Y$ is $N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2+2c)$

10. Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

Bias of t	$\text{bias}(t)$	$= E(T)-\theta$
Standard error of t	$\text{se}(t)$	$= \text{sd}(T)$
Mean square error of t	MSE(t)	$= E\{(T-\theta)^2\} = \{\text{se}(t)\}^2+\{\text{bias}(t)\}^2$

If $\overline{x}$ estimates μ, then $\text{bias}(\overline{x})=0$, $\text{se}(\overline{x})=\sigma/\sqrt{n}$, $\text{MSE}(\overline{x})=\sigma^2/n$, $\widehat{\text{se}}(\overline{x})=s/\sqrt{n}$

Central limit property If n is fairly large, $\overline{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ.

For a random sample $x_1,x_2,\ldots,x_n$

$$\ell(\theta;\ x_1,x_2,\ldots,x_n) = P(X_1=x_1\,|\,\theta)\ \cdots\ P(X_n=x_n\,|\,\theta) \qquad \text{(discrete distribution)}$$

$$\ell(\theta;\ x_1,x_2,\ldots,x_n) = f(x_1\,|\,\theta)\,f(x_2\,|\,\theta)\ \cdots\ f(x_n\,|\,\theta) \qquad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\widehat{\theta}$ for which the likelihood is a maximum

12. Confidence intervals

If $x_1,x_2,\ldots,x_n$ are a random sample from $N(\mu,\sigma^2)$ and σ^2 is known, then the 95% confidence interval for μ is $(\overline{x}-1.96\frac{\sigma}{\sqrt{n}},\ \overline{x}+1.96\frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0=t_{n-1,0.05}$

The 95% confidence interval for μ is $(\overline{x}-t_0\frac{s}{\sqrt{n}},\ \overline{x}+t_0\frac{s}{\sqrt{n}})$

13. Standard normal table — Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table — Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

m	$p=$ 0.10	0.05	0.02	0.01	m	$p=$ 0.10	0.05	0.02	0.01
1	6.31	12.71	31.82	63.66	9	1.83	2.26	2.82	3.25
2	2.92	4.30	6.96	9.92	10	1.81	2.23	2.76	3.17
3	2.35	3.18	4.54	5.84	12	1.78	2.18	2.68	3.05
4	2.13	2.78	3.75	4.60	15	1.75	2.13	2.60	2.95
5	2.02	2.57	3.36	4.03	20	1.72	2.09	2.53	2.85
6	1.94	2.45	3.14	3.71	25	1.71	2.06	2.48	2.78
7	1.89	2.36	3.00	3.50	40	1.68	2.02	2.42	2.70
8	1.86	2.31	2.90	3.36	∞	1.645	1.96	2.326	2.576

15. Chi-squared table — Values $\chi^2_{k,p}$ of x for which $P(X > x) = p$, when X is χ^2_k and $p = .995, .975$, *etc*

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum_y \dfrac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of χ^2_k with significance point p,

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ

17. Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$$p_{x\bullet} = \sum_y p_{xy} \quad \text{and} \quad P(X = x \,|\, Y = y) = \frac{p_{xy}}{p_{\bullet y}}$$

Continuous distribution

Joint cdf $\quad F(x,y) = P(\{X \le x\} \cap \{Y \le y\}) = \displaystyle\int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0, y_0)\, \mathrm{d}x_0\, \mathrm{d}y_0$

Joint pdf $\quad f(x,y) = \dfrac{\mathrm{d}^2 F(x,y)}{\mathrm{d}x\, \mathrm{d}y}$

Marginal pdf of X $\quad f_X(x) = \displaystyle\int_{-\infty}^{\infty} f(x, y_0)\, \mathrm{d}y_0$

Conditional pdf of X given $Y = y$ $\quad f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. Linear regression

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \dots, (x_n, y_n)$, the least squares fit is $\quad \widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}, \quad \widehat{\beta} = \dfrac{S_{xy}}{S_{xx}}$

The residual sum of squares $\quad \mathrm{RSS} = S_{yy} - \dfrac{S_{xy}^2}{S_{xx}}$

$\widehat{\sigma^2} = \dfrac{\mathrm{RSS}}{n-2} \qquad \dfrac{n-2}{\sigma^2}\widehat{\sigma^2}$ is from χ^2_{n-2}

$$E(\widehat{\alpha}) = \alpha, \quad E(\widehat{\beta}) = \beta,$$

$$\mathrm{var}(\widehat{\alpha}) = \frac{\sum x_i^2}{n\, S_{xx}}\sigma^2, \quad \mathrm{var}(\widehat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \mathrm{cov}(\widehat{\alpha}, \widehat{\beta}) = -\frac{\overline{x}}{S_{xx}}\sigma^2$$

$$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x, \qquad E(\widehat{y}_x) = \alpha + \beta x, \qquad \mathrm{var}(\widehat{y}_x) = \left\{\frac{1}{n} + \frac{(x - \overline{x})^2}{S_{xx}}\right\}\sigma^2$$

$\dfrac{\widehat{\alpha} - \alpha}{\widehat{\mathrm{se}}(\widehat{\alpha})}, \quad \dfrac{\widehat{\beta} - \beta}{\widehat{\mathrm{se}}(\widehat{\beta})}, \quad \dfrac{\widehat{y}_x - \alpha - \beta x}{\widehat{\mathrm{se}}(\widehat{y}_x)}$ are each from t_{n-2}

	EXAMINATION QUESTIONS/SOLUTIONS 2008-09 E2.9 (Maths 4) Solutions 2009	Course (4) EE2
Question 1		Marks & seen/unseen
Parts	$A=\begin{pmatrix}2 & \sqrt{12} & 0\\ \sqrt{12} & 1 & 0\\ 0 & 0 & 3\end{pmatrix}$ Charac. eqn $\lambda_3=-3$ and $(\lambda-1)(\lambda-2)-12=\lambda^2-3\lambda-10=0$ $\therefore\ \lambda_1=5,\ \lambda_2=-2,\ \lambda_3=3.$ $(\lambda+2)(\lambda-5)=0$	2
	$\lambda_1=5;\ \begin{pmatrix}-3 & \sqrt{12}\\ \sqrt{12} & -4\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=0,\ b=a\sqrt{\tfrac{3}{4}},\ c=0$, $\underline{a}_1=(1,\sqrt{\tfrac{3}{4}},0)^T$ $\underline{a}_{1,norm}=\left(\sqrt{\tfrac{4}{7}},\sqrt{\tfrac{3}{7}},0\right)^T$	4
	$\lambda_2=-2\ \begin{pmatrix}4 & \sqrt{12}\\ \sqrt{12} & 3\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=0,\ b=-a\sqrt{\tfrac{4}{3}},\ c=0$, $a_{2,norm}=\left(\sqrt{\tfrac{3}{7}},-\sqrt{\tfrac{4}{7}},0\right)^T$	4
	$\lambda_3=3$ $\underline{a}_3=(0,0,1)^T$	2
	Now $A^T=A$ so $\underline{a}_i$ orthogonal. Define the mx $P=(\underline{a}_1\,\underline{a}_2\,a_3)$ which has the properties i) $P^TP=I\Rightarrow P^{-1}=P^T$ ii) $AP=P\Lambda$ where $\Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$ $\therefore\ P^{-1}AP=\Lambda=P^TAP$	4
	Thus P is $P=\begin{pmatrix}\sqrt{\tfrac{4}{7}} & \sqrt{\tfrac{3}{7}} & 0\\ \sqrt{\tfrac{3}{7}} & -\sqrt{\tfrac{4}{7}} & 0\\ 0 & 0 & 1\end{pmatrix}$ Can check that $P^TP=I$. $\Lambda=\begin{pmatrix}5 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 3\end{pmatrix}$.	4
	If the student wishes to hack out P^TAP long-hand, then that is O.K.	20
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	EXAMINATION QUESTIONS/SOLUTIONS 2008-09	Course (4) EE2
Question 2		Marks & seen/unseen

Parts

Write Q as $Q = \underline{x}^T A \underline{x}$ $\quad A = \begin{pmatrix} 1 & \sqrt{6} & 0 \\ \sqrt{6} & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

Evs are $\lambda_3 = 6$ & roots of $(\lambda-1)(\lambda-2)-6 = \lambda^2 - 3\lambda - 4 = 0$

$\therefore \lambda_1 = 4, \ \lambda_2 = -1.$ — 2

$\lambda_1 = 4 \quad \underline{a}_1 = (1, \sqrt{\tfrac{3}{2}}, 0) \to (\sqrt{\tfrac{2}{5}}, \sqrt{\tfrac{3}{5}}, 0)$ normalized — 3

$\lambda_2 = -1 \quad \underline{a}_2 = (1, -\sqrt{\tfrac{2}{3}}, 0) \to (\sqrt{\tfrac{3}{5}}, -\sqrt{\tfrac{2}{5}}, 0)$ " . — 3

$\lambda_3 = 6 \quad \underline{a}_3 = (0, 0, 1)$. <u>Note</u>: $\underline{a}_i^T \underline{a}_j = \delta_{ij}$. — 1

Construct $P = (\underline{a}_1 \ \underline{a}_2 \ \underline{a}_3)$, so we have.

$P^T = P^{-1}$ (orthog. property)

With $\underline{x} = P\underline{y}$ or $\underline{y} = P^T \underline{x}$ — 4

$Q = \underline{x}^T A P = \underline{y}^T (P^T A P) \underline{y}$

However, because $AP = P\Lambda \Rightarrow P^T A P = \Lambda$

$\therefore Q = \underline{y}^T \Lambda \underline{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$ — 4

$= 4y_1^2 - y_2^2 + 6y_3^2.$

Because $\underline{y} = P^T \underline{x} = (\underline{a}_1 \ \underline{a}_2 \ \underline{a}_3)^T \underline{x}$

$P = \begin{pmatrix} \sqrt{2/5} & \sqrt{3/5} & 0 \\ \sqrt{3/5} & -\sqrt{2/5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad P^T = \begin{pmatrix} \sqrt{2/5} & \sqrt{3/5} & 0 \\ \sqrt{3/5} & -\sqrt{2/5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ — 3

$\therefore y_1 = x_1\sqrt{2/5} + x_2\sqrt{3/5}$

$y_2 = x_1\sqrt{3/5} - x_2\sqrt{2/5}$

$y_3 = x_3$

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	EXAMINATION SOLUTIONS 2008-09	Course EE2(4)
Question 3		Marks & seen/unseen
Parts		sim. seen $\Downarrow$
(i)	Since $1 = \sum_{x,y} p(x,y) = 2k$, we have $k = 1/2$.	2
(ii)	$\mathrm{P}(X=0) = 0.4k + 0.2k = 0.3$ $\mathrm{P}(X=1) = 0.2k + 0.4k = 0.3$ $\mathrm{P}(X=2) = 0.6k + 0.2k = 0.4$ $\mathrm{P}(Y=0) = 0.4k + 0.2k + 0.6k = 0.6$ $\mathrm{P}(Y=1) = 0.2k + 0.4k + 0.2k = 0.4$	5
(iii)	$\mathrm{E}(X) = 0 \cdot 0.3 + 1 \cdot 0.3 + 2 \cdot 0.4 = 1.1$ $\mathrm{E}(Y) = 0 \cdot 0.6 + 1 \cdot 0.4 = 0.4$	3
(iv)	$\mathrm{E}(X^2) = 0^2 \cdot 0.3 + 1^2 \cdot 0.3 + 2^2 \cdot 0.4 = 1.9$ Hence, $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = 1.9 - 1.1^2 = 1.9 - 1.21 = 0.69$	3
(v)	$\mathrm{E}(XY) = 0 + 1 \cdot 0.4k + 2 \cdot 0.2k = 0.8k = 0.4$ $\mathrm{cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y) = 0.4 - 1.1 \cdot 0.4 = -0.04$	3
(vi)	No, they are not uncorrelated since $\mathrm{cov}(X,Y) \neq 0$.	2
(vii)	No, they are not independent since they are not uncorrelated.	2
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	EXAMINATION SOLUTIONS 2008-09	Course EE2(4)
Question 4		Marks & seen/unseen

Parts

sim. seen ⇓

(i) For $x \geq 0$:

$$F_{X^2}(x) = \mathrm{P}(X^2 \leq x) = \mathrm{P}(X \leq \sqrt{x}) = 1 - e^{-\sqrt{x}}$$

$F_{X^2}(x) = 0$ for $x < 0$. 4

(ii) For $x \geq 0$:

$$F_{\min_i X_i}(x) = \mathrm{P}(\min_i X_i \leq x) = 1 - \mathrm{P}(X_i > x, i = 1, \ldots, 4)$$
$$= 1 - \prod_{i=1}^{4} e^{-\lambda x} = 1 - e^{-4\lambda x}$$

For $x < 0$: $F_{\min_i X_i}(x) = 0$. 4

(iii)

(a) $\mathrm{cov}(X, Y) = 0$ since X and Y are independent. 2

(b) From the formula sheet: $\mathrm{E}(X) = 1/\lambda = \mathrm{E}(Y)$.
Hence, $\mathrm{E}(X - Y) = \mathrm{E}(X) - \mathrm{E}(Y) = 0$. 2

(c) From the formula sheet: $\mathrm{Var}(X) = \mathrm{Var}(Y) = 1/\lambda^2$.

$$\begin{aligned}\mathrm{Var}(X - Y) &= \mathrm{Var}(X + (-Y)) \\ &= \mathrm{Var}(X) + \mathrm{Var}(-Y) + \mathrm{cov}(X, -Y) \\ &= \mathrm{Var}(X) + \mathrm{Var}(Y) = 1/\lambda^2 + 1/\lambda^2 = 2/\lambda^2\end{aligned}$$

3

(d) For $t \geq 0$:

$$\begin{aligned}f_{X+Y}(t) &= \int_{-\infty}^{\infty} f_X(x) f_Y(t - x) dx = \int_0^t f_X(x) f_Y(t - x) dx \\ &= \int_0^t \lambda e^{-\lambda x} \lambda e^{-\lambda(t-x)} dx \\ &= \lambda^2 \int_0^t e^{-\lambda t} dx = \lambda^2 t e^{-\lambda t}\end{aligned}$$

For $t < 0$: $f_{X+Y}(t) = 0$ 5

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	EXAMINATION SOLUTIONS 2008-09	Course EE2(4)
Question 5		Marks & seen/unseen

Parts

sim. seen ⇓

(i) (a)

$$\begin{aligned} \mathrm{P}(\text{system functions}) &= \mathrm{P}(\text{all components function}) \\ &= \prod_{i=1}^{k} \mathrm{P}(\text{component } i \text{ functions}) = 0.999^k \end{aligned}$$

3

(b) Want:
$\mathrm{P}(\text{system fails}) = 1 - \mathrm{P}(\text{system functions}) = 1 - 0.999^k \leq 0.01$. 2
This is equivalent to $0.999^k \geq 0.99$.
Hence, $k \log 0.999 \geq \log 0.99$ and $k \leq \log 0.99 / \log 0.999 \approx 10.05$. 3
Since k needs to be a natural number, at most 10 components can be used. 1

(ii) Let C_i denote the event "component i fails before time t".
From the formula sheet: $\mathrm{P}(C_i) = 1 - \exp(-\lambda_i t)$. 2
Using the system structure and the independence we get

$$\begin{aligned} &\mathrm{P}(\text{system fails before time } t) = \mathrm{P}((C_1 \cap C_2) \cup C_3) \\ &= 1 - \mathrm{P}([(C_1 \cap C_2) \cup C_3]^c) = 1 - \mathrm{P}([C_1 \cap C_2]^c \cap C_3^c) \\ &= 1 - \mathrm{P}([C_1 \cap C_2]^c)\, \mathrm{P}(C_3^c) \\ &= 1 - (1 - \mathrm{P}(C_1 \cap C_2))(1 - \mathrm{P}(C_3)) \\ &= 1 - (1 - \mathrm{P}(C_1)\, \mathrm{P}(C_2))(1 - \mathrm{P}(C_3)) \\ &= 1 - [1 - (1 - e^{-t})(1 - e^{-2t})]e^{-3t} \\ &= 1 - (e^{-t} + e^{-2t} - e^{-3t})e^{-3t} \\ &= 1 - e^{-4t} - e^{-5t} + e^{-6t} \end{aligned}$$

4

(iii) (a) $H(t) = \int_0^t h(s)ds = \frac{a}{b+1} t^{b+1}$.
From the formula sheet, $H(t) = -\log(\mathrm{P}(T > t))$. Thus, for $t \geq 0$,

$$\begin{aligned} F(t) &= \mathrm{P}(T \leq t) = 1 - \mathrm{P}(T > t) \\ &= 1 - \exp(-H(t)) = 1 - \exp(-\frac{a}{b+1} t^{b+1}) \end{aligned}$$

3

(b) $f(t) = \frac{d}{dt} F(t) = a t^b \exp(-\frac{a}{b+1} t^{b+1})$ for $t \geq 0$ 2

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	EXAMINATION SOLUTIONS 2008-09	Course EE2(4)
Question 6		Marks & seen/unseen

Parts

(i) (a) The likelihood function is

$$L(\mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2}}.$$

sim. seen ⇓

and its logarithm is

3

$$\log L(\mu) = -\sum_{i=1}^{n}(x_i-\mu)^2/2 - n/2\log(2\pi).$$

Taking derivatives yields

$$\log L(\mu) = \sum_{i=1}^{n}(x_i-\mu) = \sum_{i=1}^{n} x_i - n\mu.$$

Equating this to 0 gives the candidate $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$ for the maximum likelihood estimator. 3

$\hat{\mu}$ is indeed a maximum since $\left(\frac{d}{d\mu}\right)^2 \log L(\mu) = -n < 0$. 1

(b) First note that $\mathrm{E}(\hat{\mu}) = \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}(X_i) = \mu$ and $\mathrm{Var}(\hat{\mu}) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i) = \frac{1}{n}$. Since $X_1,\dots,X_n$ are independent normal random variables (and thus jointly normal) this implies that $\hat{\mu}$ is $N(\mu, \frac{1}{n})$. 3

Thus

$$\begin{aligned}\mathrm{P}(\hat{\mu} > \mu + 2/\sqrt{n}) &= \mathrm{P}(\sqrt{n}(\hat{\mu}-\mu) > 2) = 1 - \mathrm{P}(\sqrt{n}(\hat{\mu}-\mu) \le 2)\\ &= 1 - \Phi(2) = 1 - 0.977 = 0.023\end{aligned}$$

2

(ii) (a)

$$\begin{aligned}\mathrm{E}(Y^{-1}) &= \int_0^\infty y^{-1}\frac{1}{6}\lambda^4 y^3 e^{-\lambda y}dy = \frac{1}{6}\lambda^4\int_0^\infty y^2 e^{-\lambda y}dy\\ &= \frac{1}{6}\lambda^4 2/\lambda^3 = \lambda/3\end{aligned}$$

3

(b) $\mathrm{E}(T) = \frac{3}{n}\sum_{i=1}^{n}\mathrm{E}(y_i^{-1}) = \frac{3}{n}\sum_{i=1}^{n}\frac{\lambda}{3} = \lambda$. Thus T is unbiased. 2

In general, $\mathrm{MSE}(T) = \mathrm{Var}(T) + \mathrm{bias}^2(T)$.

Since T is unbiased, $\mathrm{bias}(T) = 0$.

Thus, $\mathrm{MSE}(T) = \mathrm{Var}(T) = \frac{9}{n^2}\sum_{i=1}^{n}\mathrm{Var}(Y_i^{-1}) = \frac{9}{n}\lambda^2/18 = \frac{\lambda^2}{2n}$. 3

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