EE4-04
EE9SC6

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

MSc and EEE/ISE PART IV: MEng and ACGI

Corrected Copy

P1, P3, P5

ADVANCED DATA COMMUNICATIONS

Wednesday, 9 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer THREE questions.

All questions carry equal marks. The maximum mark for each subquestion is shown in brackets.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : M.K. Gurcan
Second Marker(s) : E. Gelenbe

Instructions to Candidates
Useful equations

$$\frac{8}{3}Q\left(\frac{d}{2\sigma}\right) = \frac{8}{3}Q\left(\frac{1}{\sqrt{0.1}}\right) = 2.09\times 10^{-3}$$

$$Q\left(\frac{d}{2\sigma}\right) = Q\left(3.162\right) = 7.827\times 10^{-4}$$

For $P_e = 10^{-7}$ the gap value $\Gamma = 9.8dB$

For $P_e = 10^{-6}$ the gap value $\Gamma = 8.8dB$

$$\text{For } P_e \leq Q\left(\sqrt{\frac{3SNR}{\frac{31}{32}M-1}}\right) < 10^{-7}$$

$$\frac{3SNR}{\frac{31}{32}M-1} = 14.2 \text{ dB}$$

$$\text{For } P_e \overset{=}{\leq} Q\left(\sqrt{\frac{3SNR}{M-1}}\right) \overset{=}{<} 10^{-7}$$

CORRECTION ②

$$SNR = 21.2 \text{ dB}$$

For a $M-QAM$ constellation the probablity of error is assumed to be

$$P_M = 1-\left(1-P_{\sqrt{M}}\right)^2.$$

With $P_M = 10^{-6}$ we have $P_{\sqrt{M}} = 5\times 10^{-7}$ where $P_{\sqrt{M}} = 2\times\left(1-\frac{1}{4}\right)Q\left[\sqrt{\frac{3\varepsilon_x}{15\times 2\times 10^{-10}}}\right]$. For

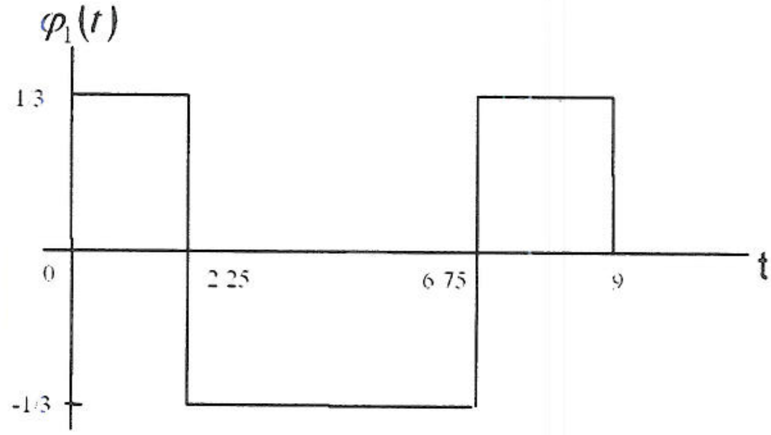
$$2\times\left(1-\frac{1}{M}\right)Q\left[\sqrt{\frac{3\varepsilon_x}{(M-1)N_0}}\right] = 5\times 10^{-7}$$

we have $\varepsilon_x = 24.70\times 10^{-9}$.

Questions

1. Answer the following subquestions

(a) Consider the two orthonormal basis functions given in Figure 1.1



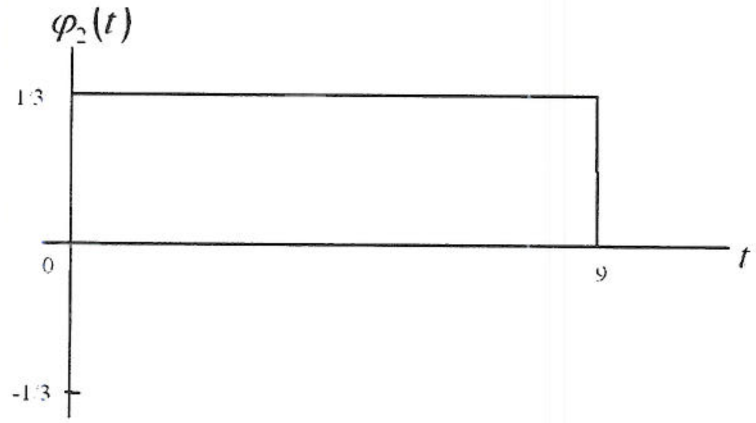


Figure 1.1 Basis functions.

i. Use the basis functions given above to find the modulated waveforms $u(t)$ and $v(t)$ given the data symbols $\overrightarrow{u} = [1, 1]$ and $\overrightarrow{v} = [2, 1]$. It is sufficient to draw $u(t)$ and $v(t)$. [4]

ii. For the same $u(t)$ and $v(t)$, a different set of two orthonormal basis functions is employed for which $\overrightarrow{u} = [\sqrt{2}, 0]$ produces $u(t)$. Draw the new basis functions and find the $\overrightarrow{v}$ which produces $v(t)$. [5]

(b) Consider the signal set shown in Figure 1.2 with an AWGN channel with the noise variance $\sigma^2 = 0.1$.

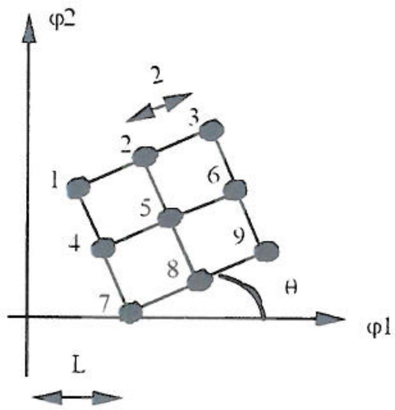


Figure 1.2 A rotated signal constellation.

Answer the following parts

i. Does the probability of error P_e depend on L and θ? [2]

ii. Find the nearest neighbour union bound on P_e for the Maximum Likelihood detector assuming $p_x(i) = \frac{1}{9}$ for $i = 1, \cdots 9$. [2]

iii. Find P_e exactly using the assumptions of the previous part. How far off was the NNUB? [2]

iv. Suppose we have a minimum energy constraint on the signal constellation. How would we change the constellation of this problem without changing the P_e? [2]

v. How does θ affect the constellation energy? [2]

(c) Either a square or cross QAM can be used on an AWGN channel with SNR = 30.2 dB and symbol rate $\frac{1}{T} = 10^6$ Hz.

i. Select a QAM constellation and specify a corresponding integer number of bits per symbol, b, for a modem with the highest data rate such that $P_e < 10^{-6}$. [2]

ii. Compute the data rate for part 1.(c).i. [2]

iii. Repeat part 1.(c).i if $P_e < 2 \times 10^{-7}$ is the new probability of error constraint. [1]

iv. Compute the data rate for part 1.(c).iii. [1]

$P_e < 10^{-7}$

CORRECTION

1

2. Answer the following subquestions.

(a) A 32 Cross (CR) QAM modulation is used for transmission on an AWGN channel with $\frac{N_0}{2} = .001$ W/Hz. The symbol rate is $\frac{1}{T}$= 400 kHz.

i. Find the data rate R. [1]

ii. What SNR is required for $P_e < 10^{-7}$? (ignore N_e). [2]

iii. In actual transmitter design the analog filter is rarely normalized and has some gain/attenuation, unlike a basis function. Thus, the average power in the constellation is calibrated to the actual power measured at the analog input to the channel. Suppose $\bar{\mathcal{E}}_x = 1$ corresponds to 0 dBm (1 mW), then what is the power of the signals entering the transmission channel for the 32CR with $P_e < 10^{-7}$? [1]

iv. Without increasing the transmit power or changing $\frac{N_0}{2} = .001$ (W/Hz), design a QAM system which achieves the same P_e at 3.2 Mbps on this same AWGN channel. [3]

(b) The QAM constellation shown in Figure 2.1 is used for transmission on an AWGN channel with symbol rate 10MHz and a carrier frequency of 100 MHz.

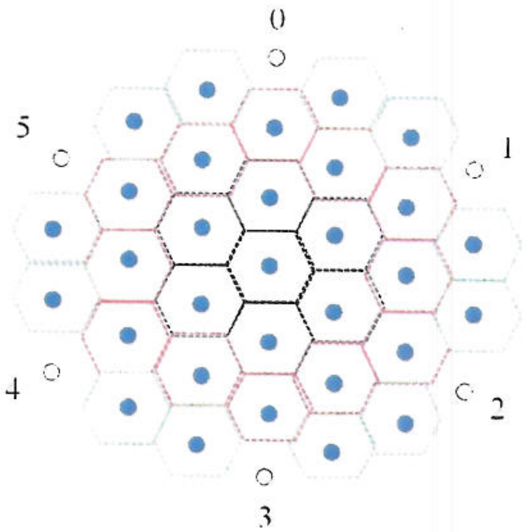


Figure 2.1 Constellation for QAM signal.

Each of the solid constellation symbols is located at the center of a perfect hexagon and the distance to any of the closest sides of the hexagon is $\frac{d}{2}$. The 6 empty points represent a possible message also, but each is used only every 6 symbol instants, so that for instance, the point labelled 0 is a potential message only on symbol instants that are integer multiples of 6. The 1 point can only be transmitted on symbol instants that are integer multiples of 6 plus one, the 2 point only on symbol instants that are integer multiples of 6 plus two, and so on. At any symbol instant, any of the points possible on that symbol are equally likely.

i. What is the number of messages that can be transmitted on any single symbol? [1]

ii. What are the values for the number of bits b per symbol and the rate $\bar{b}$ per dimension? [1]

iii. What is the data rate? [1]

iv. Draw the decision boundaries for time 0 of a ML receiver. [2]

v. What is the value of the minimum distance d_{min}? [2]

vi. What are ε_x and $\bar{\varepsilon}_x$ for this constellation in terms of d? [1]

vii. What is the average number of nearest neighbours? [1]

viii. Determine the NNUB (nearest neighbor union bound) expression that tightly upper bounds P_e for this constellation in terms of SNR. It is sufficient to express the probability of error in terms of a Q function with the appropriate argument. [1]

ix. Compare this constellation fairly to Cross QAM transmission. [1]

x. Describe an equivalent maximum likelihood (ML) receiver which uses time-invariant decision boundaries and a constant decision device with a simple preprocessor to the decision device. [2]

(c) Consider the signal given in Figure 2.2

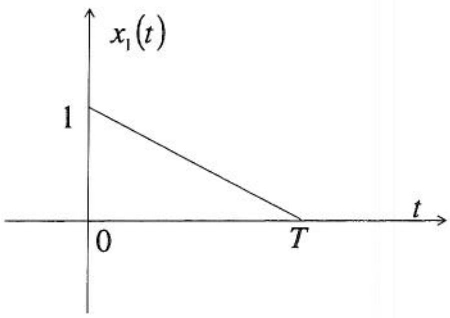


Figure 2.2 Transmission waveform.

i. Sketch the impulse response of the filter matched to the pulse shown in the figure. [2]

ii. ~~Determine and~~ sketch the output of the matched filter ~~at $t = T$.~~ [3]

and determine its value at $t=T$.

3. Answer the following subquestions.

(a) For the AWGN channel with the transfer function shown in Figure 3.1, a transmitted signal cannot exceed 1 mW (0 dBm) and the power spectral density is also limited according to -83 dBm/Hz(two sided psd).

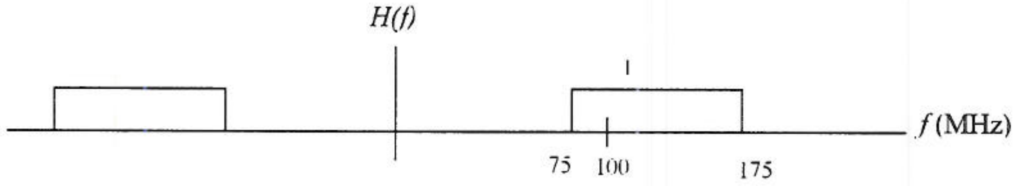


Figure 3.1 Channel response.

The two-sided noise power spectral density is $\sigma^2 = -98$ dBm/Hz. The carrier frequency is $f_c = 100$ MHz for QAM transmission. The probability of error is $P_e = 10^{-6}$.

i. Find the largest symbol rate that can be used with the 100 MHz carrier frequency. [2]

ii. What is the maximum signal power at the channel output with QAM? [2]

iii. What QAM data rate can be achieved with the symbol rate of part 3.a.ii? [2]

iv. Find the carrier frequency that allows the best QAM data rate. [2]

(b) For the channel $P(\omega) = \sqrt{T}\,(1 + 0.9 \exp(j\omega T))$ for $|\omega| < \frac{\pi}{T}$, consider using binary antipodal modulation with the average energy $\bar{\varepsilon}_x = 1$ and having the filter matched-filter-bound SNR $SNR_{MFB} = 10dB$.

i. Calculate the pulse energy. [3]

ii. Find the normalized pulse response $\varphi(t)$ and the deterministic autocorrelation function $q(t)$ and its samples $q_k = q(kT)$. Remember that $q_0 = 1$. [3]

iii. Find the discrete time Fourier Transform $Q(\exp(-j\omega T))$ of the sequence q_k and also the D transform $Q(D)$. [2]

iv. Compute the SNR achieved by the zero forcing equalizers. [2]

(c) A 4 kHz bandpass channel is to be used for the transmission of data at a rate of 9600 bits/sec. If $\frac{N_0}{2} = 10^{-10}$ W/Hz is the spectral density of the zero-mean, additive-white-Gaussian-noise in the channel, design a QAM modulation and determine the average power that achieves a bit-error probability of 10^{-6}. Use a signal pulse with a raised cosine spectrum having a roll-off factor of at least 50%. [7]

4. Answer the following sub-questions.

(a) Consider a multi-tone system with, K parallel channels and channel SNRs g_k for $k = 1, \cdots K$. A total constrained energy E_T is available when operating the system with the gap value of Γ. Explain how

i. the rate adaptive (RA) and [5]

ii. the margin adaptive (MA) [5]

water filling algorithms are used to allocate energies E_k for each channel for $k = 1, \cdots K$.

(b) Assume that a High Speed Downlink Packet Access (HSDPA) SiSO transmission system with K parallel channels has the received signal covariance matrix $\mathbf{C}$ when the matched filter receiver sequence is given by $\mathbf{Q} = [\overrightarrow{q}_1, \cdots, \overrightarrow{q}_K]$.

i. Explain the system model for the HSDPA SISO system. [8]

ii. Explain how the energies are allocated iteratively when using the HSDPA system with, K parallel channels, a constrained total energy E_T and noise variance σ^2 per dimension. [7]

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Answers 2012

1. Answer for each subquestion is as follows.

   (a) We have the solutions for the basis question

      i.

      (**2 pts**) The functions u(t) and v(t) are given by,

$$u(t) = \begin{cases} \frac{2}{3} & \text{if } t\epsilon[0, 2.25] \\ 0 & \text{if } t\epsilon[2.25, 6.75] \\ \frac{2}{3} & \text{if } t\epsilon[6.75, 9] \end{cases} \quad v(t) = \begin{cases} 1 & \text{if } t\epsilon[0, 2.25] \\ -\frac{1}{3} & \text{if } t\epsilon[2.25, 6.75] \\ 1 & \text{if } t\epsilon[6.75, 9] \end{cases}$$

      See figure 2.

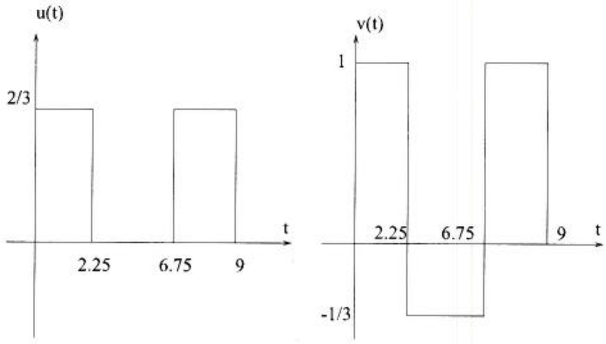


Figure 2: $u(t)$ and $v(t)$

      ii.

      (**3 pts**) The new basis functions are,

$$\phi_1(t) = \begin{cases} \frac{\sqrt{2}}{3} & \text{if } t\epsilon[0, 2.25] \\ 0 & \text{if } t\epsilon[2.25, 6.75] \\ \frac{\sqrt{2}}{3} & \text{if } t\epsilon[6.75, 9] \end{cases} \quad \phi_2(t) = \begin{cases} 0 & \text{if } t\epsilon[0, 2.25] \\ \frac{\sqrt{2}}{3} & \text{if } t\epsilon[2.25, 6.75] \\ 0 & \text{if } t\epsilon[6.75, 9] \end{cases}$$

      See figure 3.
      And obviously, we have

$$v = \left[\ \frac{3}{\sqrt{2}} - \frac{1}{\sqrt{2}} \ \right]$$

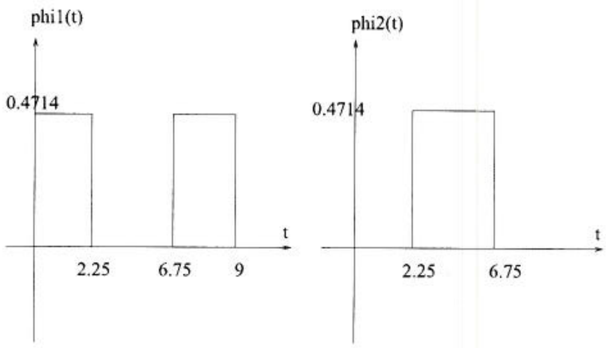


Figure 3: New Basis function

(b) For the rotated constellation we have

i.

(1 pt) P_e is invariant under rotation or translation of signal constellation. Therefore, P_e does not depend on L or θ.

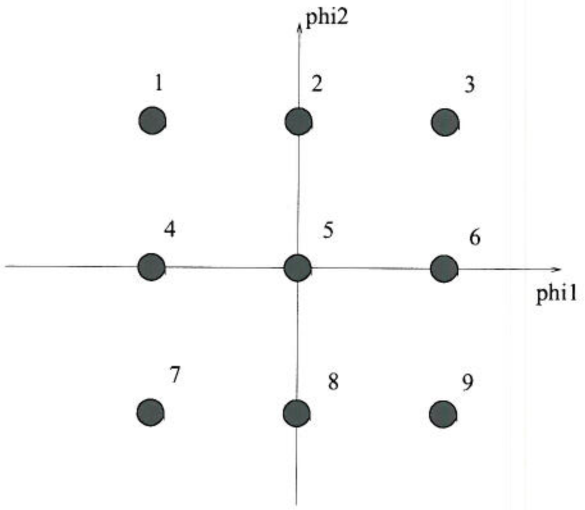
ii.

(2 pts) First, we get the average number of nearest neighbors:

$$\begin{aligned} N_e &= \sum_{i=0}^{M-1} N_i p_{\boldsymbol{x}}(i) \\ &= \frac{4+4\cdot 2+4\cdot 3}{9} \\ &= \frac{8}{3}. \\ P_e &\le \frac{8}{3} Q\left(\frac{d}{2\sigma}\right) \\ &= \frac{8}{3} Q\left(\frac{2}{2\sqrt{0.1}}\right) \\ &= 2.09\times 10^{-3}. \end{aligned}$$

iii.

(5 pts) By part (a), the constellation shown in the figure gives the same P_e:



$$\begin{aligned} P_{c|i=5} &= \left(1 - 2Q\left(\frac{d}{2\sigma}\right)\right)^2 \\ P_{c|i=2,4,6,8} &= \left(1 - 2Q\left(\frac{d}{2\sigma}\right)\right)\left(1 - Q\left(\frac{d}{2\sigma}\right)\right) \\ P_{c|i=1,3,5,9} &= \left(1 - Q\left(\frac{d}{2\sigma}\right)\right)^2. \end{aligned}$$

Hence,

$$\begin{aligned} P_c &= \frac{1}{9}\left[\left(1 - 4Q + 4Q^2\right) + 4\left(1 - 3Q + 2Q^2\right) + 4\left(1 - 2Q + Q^2\right)\right], \\ P_e &= 1 - P_c \\ &= \frac{8}{3}Q - \frac{16}{9}Q^2, \end{aligned}$$

where $Q = Q\left(\frac{d}{2\sigma}\right) = Q(3.162) = 7.827 \times 10^{-4}$. So,

$$\begin{aligned} P_e &= 2.09 \times 10^{-3} - 1.09 \times 10^{-6} \\ &= 2.09 \times 10^{-3}. \end{aligned}$$

In this case, NNUB was off by $\frac{16}{9}Q^2 = 1.09 \times 10^{-6}$, a small quantity compared to P_e.

iv.

(2 pts) To get a constellation with minimum energy, we subtract from the constellation its mean. So, a possible choice is the constellation of part c). The energy of the original constellation would change with θ, whereas the energy of the minimum-energy constellation would be independent of θ.

2. Answers for each subsection is as follows

(a) For the 32 CR we have

i.

(1 pt) The data rate is computed as $R = b/T = 5 \cdot 4 \cdot 10^5 = 2$Mbps.

ii.

(1 pt) The data rate is computed as $R = b/T = 5 \cdot 4 \cdot 10^5 = 2$Mbps.

(2 pts) Since it is suggested that N_e be neglected, therefore the P_e is,

$$P_e \leq Q\left(\sqrt{\frac{3 \cdot SNR}{\frac{31}{32}M - 1}}\right) < 10^{-7} \qquad \text{, for CR-QAM}$$

From the graph of the $Q(\cdot)$ function, we see that to obtain $P_e < 10^{-7}$, we need,

$$\frac{3 \cdot SNR}{\frac{31}{32}M - 1} = 14.2 \text{ dB}$$

Solving, the SNR required is 24.2dB.

iii.

(1 pt) $SNR = \frac{\overline{\mathcal{E}_x}}{N_0/2} = 24.2$ dB. Noise $N_0/2 = 0.001 = -30$dB. Therefore, $\overline{\mathcal{E}_x} = 24.2 - 30 = -5.8$dB. Finally, since $\overline{\mathcal{E}_x} = 1$ corresponds to 0 dBm, hence, the transmit power of our 32 CR system is -5.8 dBm.

iv.

(4 pts) We have two ways to go. We can try increasing SNR and thus increasing $\mathcal{E}_x$. This will let us transmit a larger size constellation as P_e depends on SNR as

$$P_e \approx 4Q\left(\sqrt{\frac{3SNR}{M-1}}\right)$$

An increased SNR lets us increase M and hence b while keeping P_e same. But, because $P_x = \frac{\mathcal{E}_x}{T}$ is constant, we have to reduce the symbol rate. The other way is to increase symbol rate. But, this will reduce SNR and cause a reduction in constellation size for the same P_e. The end effect on R is not immediately clear in either case. The problem can be done by trial and error, or by noting the gap approximation,

$$R = \frac{b}{T} = \frac{1}{T}\log_2\left(1 + \frac{SNR}{\Gamma}\right)$$

When power is constant, SNR and $\frac{1}{T}$ are inversely proportional. But, SNR has only a log effect on rate. This indicates that if we increase the symbol rate and thereby cause a reduction in SNR, the net effect is positive. So, increasing symbol rate is the way to go. To this end, choose $\frac{1}{T} = 800$ kHz. Then, a 16 SQ-QAM constellation will be sufficient to achieve $R = 3.2$ Mbps. Since the symbol rate has doubled, $\mathcal{E}_x$ should reduce by half to maintain the old transmit power. Therefore, the SNR will halve (decrease by 3 dB). Thus, new $SNR = 24.2 - 3 = 21.2$ dB.

Its easy to check that the 16-QAM system with an SNR of 21.2 dB achieves,

$$P_e \leq Q\left(\sqrt{\frac{3 \cdot SNR}{M-1}}\right) = Q(14.2dB) = 10^{-7}$$

as desired.

(b) The honeycomb constellation subquestions are answered as follows.

i. Counting we have $M = 32$.

ii. $b = \log_2 M = 5$, $\bar{b} = \frac{5}{2} = 2.5$ bits per symbol.

iii. $R = \frac{b}{T}$ bits/s.

iv.

(**2 pts**) The decision regions are shown in figure 1.

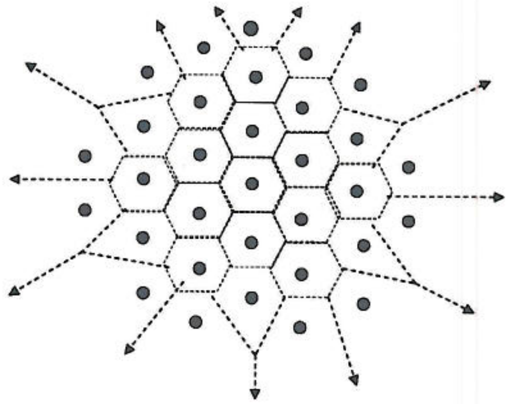


Figure 1: Decision Regions

v.

$$d_{\min} = d$$

vi.

(**3 pts**) First let us figure out the basic distances within a single hexagon. Note that we can divide the hexagon into 6 equilateral triangles. The length of the perpendicular bisector of each triangle is given to us as $\frac{d}{2}$. This bisector creates a $30° - 60° - 90°$ triangle as shown in figure 2. Therefore, the distance between the center of the hexagon to any of the vertices can be computed as

$$\frac{\frac{d}{2}}{\cos\frac{\pi}{6}} = \frac{d}{\sqrt{3}}$$

We are now ready to calculate the constellation's energy. Notice we can breakdown the constellation points based on distance to origin into these 6 cases:

i. The constellation point at origin has distance 0.

ii. The 6 closest points to the origin are at distance d.

iii. We have 6 points at distance $2d$ from the origin.

iv. We have 6 points at distance $\sqrt{3}d$ from the origin.

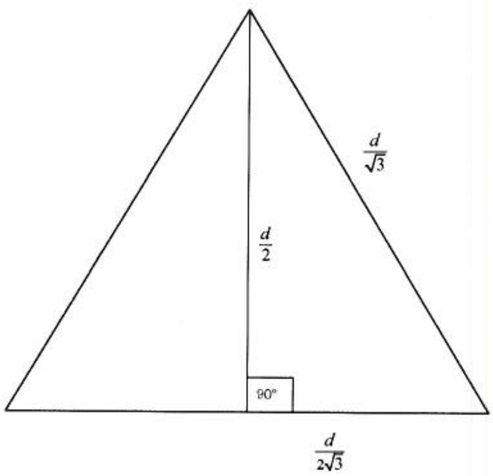


Figure 2: One of the 6 triangles created by the hexagon

v. The distance to the 12 outermost (except the "rotating" point) points could be calculated as follows. Notice that the distance to the center of an edge shared by each of these adjacent pair of points is

$$\frac{1}{\sqrt{3}}\left[\frac{d}{2}+d+2d+d\right]=\frac{9d}{2\sqrt{3}}$$

Since the distance from the center of the dividing edge of these outermost points to the actual constellation point is $\frac{d}{2}$, the distance of these points could be calculated as the hypotenuse of a right triangle and so we have:

$$distance=\sqrt{d^2\left(\frac{9}{2\sqrt{3}}\right)^2+\frac{d^2}{4}}=\sqrt{7}d$$

vi. The constellation always also has one of the "rotating" points at distance $3d$.

Hence, the constellation's energy is

$$\mathcal{E}_x=\frac{1}{32}\left[1(0)+6d^2+6(\sqrt{3}d)^2+6(2d)^2+12(\sqrt{7}d)^2+1(3d)^2\right]=\frac{141}{32}d^2=4.406d^2$$

and the energy per dimension is

$$\bar{\mathcal{E}}_x=\frac{\mathcal{E}_x}{2}=2.203d^2$$

vii.

(1 pts) Looking at the figure (and including any of the "rotating" points in the constellation) we notice that there are

i. 14 points with 6 nearest neighbors - the "inside" points.

ii. 5 points with 5 nearest neighbors - those points that border one of the 5 "rotating" constellation points not allowed to be transmitted at this instant.

iii. 12 points with 4 nearest neighbors - the 12 outer most points (except "rotating" point).

iv. 1 point with 3 nearest neighbors - the "rotating" symbol.

Adding all of these up we have,

$$N_e = \frac{1}{32}\left[6(14) + 5(5) + 4(12) + 3(1)\right] = 5$$

viii.

(**2 pts**) From NNUB we have,

$$\bar{P}_e \leq \frac{5}{2}Q\left[\frac{d}{2\sigma}\right] = \frac{5}{2}Q\left[\sqrt{\frac{\frac{2\bar{\mathcal{E}}_x}{4.406}}{4\sigma^2}}\right] = 2.5Q\left[\sqrt{.1135\,SNR}\right]$$

ix.

(**1 pts**) For a fair comparison we hold $\bar{\mathcal{E}}_x$ and $\bar{b}$ constant and compare $\bar{P}_e$. Holding $\bar{\mathcal{E}}_x$ and $\bar{b}$ constant means that SNR for both constellations is the same and that QAM-CR has $M = 2^{2\bar{b}} = 32$ constellation points.

Hence, .

$$\bar{P}_e \leq 2\left(1 - \frac{1}{\sqrt{2M}}\right)Q\left[\sqrt{\frac{3\,SNR}{\frac{31}{32}M - 1}}\right] = 1.75Q\left[\sqrt{.1\,SNR}\right]$$

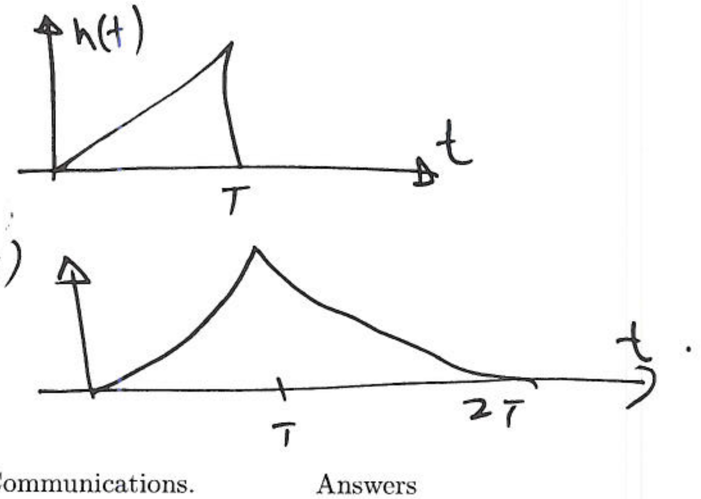
Since the argument of the Q-function dominates the probability of error the hexagonal constellation is better since the argument in its Q-function is larger.

x.

Since the argument of the Q-function dominates the probability of error the hexagonal constellation is better since the argument in its Q-function is larger.

(**1 pts**) An equivalent ML receiver that uses constant decision regions could be implemented by simply rotating the 2-dimensional constellation by 60° at every consecutive symbol instant. Or, multiply the channel output by $exp^{j\frac{\pi}{3}k}$ for $k = 0, 1 \ldots 5$.

2.c



3. The answers for question 3 are as follows

(a) The answers for the channel with the spectrum between 75 MHz to 175 MHz is as follows

i.

(2 pts) $\frac{1}{2}H_{bb}(f)$ is shown in Figure 1.

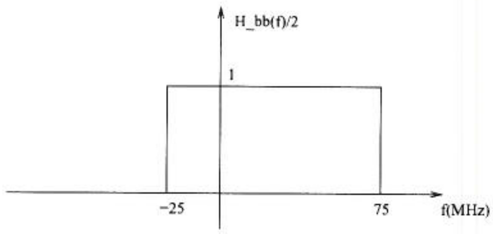


Figure 1: Baseband equivalent channel

ii.

(1 pts) The largest symbol rate that doesn't make the transmit signal spectrum to fall outside the $[75, 175]$ MHz range (when centered around 100 MHz) is clearly $\frac{1}{T} = 50$ MHz.

iii.

(2 pts) Maximum allowed signal power =

$$min(1 \text{ mW}, 10^{-\frac{83}{10}}\text{mW/Hz} \times 2 \times 50 \text{ MHz}) = 10^{-0.3} \text{ mW} = -3 \text{ dBm}.$$

iv.

(2 pts) For QAM the data rate can be approximated by the gap formula:

$$\bar{b} = \frac{1}{2}\log_2(1 + \frac{\bar{\mathcal{E}}_x/\sigma^2}{\Gamma}) = 1.1849,$$

using:

$$\bar{\mathcal{E}}_x = \frac{\mathcal{E}_x}{2} = \frac{P_x \cdot T}{2} = \frac{1}{2} \times 10^{-0.3}\text{mW} \times \frac{1}{50\text{MHz}} = 10^{-\frac{83}{10}}\text{mW/Hz}$$

$$\sigma^2 = 10^{-\frac{98}{10}}\text{mW/Hz},$$

and

$$\Gamma = 8.8 \text{ dB} = 10^{0.88}, \text{ for uncoded QAM with } \bar{P}_e = 10^{-6}.$$

Considering 4-QAM transmission then (which results in a $\bar{P}_e$ slightly less than 10^{-6} because $\bar{b} = 1 < 1.1849$):

$$R = \frac{b}{T} = \frac{2\bar{b}}{T} = 100\text{Mbps}.$$

ii

(2 pts) The center of the passband of the channel is the best carrier frequency, because it allows use of more bandwidth, which in turn means larger symbol rate. (In this case, because of the limit on the transmit signal PSD, one can also increase the transmit power by using more bandwidth.)

$$f_c = \frac{75+175}{2} \text{ MHz } = 125 \text{ MHz}.$$

iii vi.

(2 pts) The center of the passband of the channel is the best carrier frequency, because it allows use of more bandwidth, which in turn means larger symbol rate. (In this case, because of the limit on the transmit signal PSD, one can also increase the transmit power by using more bandwidth.)

$$f_c = \frac{75+175}{2} \text{ MHz } = 125 \text{ MHz}.$$

IV vii.

(3 pts) As in previous parts:

$$\frac{1}{T} = 100 \text{ MHz},$$

$$P_x = min(1 \text{ mW}, 10^{-\frac{83}{10}} \frac{\text{mW}}{\text{Hz}} \times 2 \times 100 \text{ MHz}) = 1 \text{ mW } = 0 \text{ dBm},$$

$$\bar{\mathcal{E}}_x = \frac{\mathcal{E}_x}{2} = \frac{P_x \cdot T}{2} = 1 \text{ mW} \times \frac{1}{100 \text{ MHz}} \times \frac{1}{2},$$

and therefore:

$$\bar{b} = \frac{1}{2}\log_2(1 + \frac{\bar{\mathcal{E}}_x/\sigma^2}{\Gamma}) = 1.1835.$$

With $\bar{b} = 1$ then:

$$R = \frac{2\bar{b}}{T} = 200\text{Mbps}.$$

(b) The equalizer question solution is as follows

i. The pulse energy is calculated as follows

The first step is to obtain a discrete-time ISI channel model(ie. $Q(D)$) from the continuous-time channel $P(\omega)$. To this end, consider

$$P(f) = \sqrt{T}(1 + 0.9 \cdot e^{j2\pi fT}) \cdot rect(fT)$$

Therefore, we calculate

$$\begin{aligned} p(t) &= F^{-1}\{P(f)\} \\ &= \sqrt{T}[\delta(t) + 0.9\delta(t+T)] * [\frac{1}{T}sinc(\frac{t}{T})] \\ &= \frac{1}{\sqrt{T}}[sinc(\frac{t}{T}) + 0.9 \cdot sinc(\frac{t+T}{T})] \\ p^*(-t) &= \frac{1}{\sqrt{T}}[sinc(\frac{t}{T}) + 0.9 \cdot sinc(\frac{t-T}{T})] \end{aligned}$$

since sinc(t) is an even function. Therefore,

$$\begin{aligned} p(t) * p^*(-t) &= 1.81 \cdot sinc(\frac{t}{T}) + 0.9 \cdot sinc(\frac{t+T}{T}) + 0.9 \cdot sinc(\frac{t-T}{T}) \\ \|p\|^2 &= \int_{-\infty}^{\infty} |p(t)|^2 \, dt = [p(t) * p^*(-t)]_{t=0} = 1.81 \end{aligned}$$

where we've used the fact that $sinc(t/T-n) * sinc(t/T-m) = T \cdot sinc(t/T-m-n)$ (this is easy to see in the Fourier domain).

$$\begin{aligned} q(t) &= \frac{1}{\|p\|^2} p(t) * p^*(-t) \\ &= sinc(\frac{t}{T}) + \frac{0.9}{1.81} \cdot sinc(\frac{t+T}{T}) + \frac{0.9}{1.81} \cdot sinc(\frac{t-T}{T}) \end{aligned}$$

ii.

by sampling $q(t)$ at $t = kT$, we get

$$\begin{aligned} q_k &= \delta_k + \frac{0.9}{1.81}\delta_{k+1} + \frac{0.9}{1.81}\delta_{k-1} \\ Q(D) &= \frac{0.9}{1.81}D^{-1} + 1 + \frac{0.9}{1.81}D \end{aligned}$$

iii.

$$SNR_{ZFE} = \frac{\bar{\mathcal{E}}_x}{5.26\frac{N_0}{2}} = 0.2\ dB$$

(c) The answer for the pulse shaping filter question is as follows

The bandwidth of the bandpass channel is $W = 4$ KHz. Hence, the rate of transmission should be less or equal to 4000 symbols/sec. If a 8-QAM constellation is employed, then the required symbol rate is $R = 9600/3 = 3200$. If a signal pulse with raised cosine spectrum is used for shaping, the maximum allowable roll-off factor is determined by

$$1600(1+\alpha) = 2000$$

which yields $\alpha = 0.25$. Since α is less than 50%, we consider a larger constellation. With a 16-QAM constellation we obtain

$$R = \frac{9600}{4} = 2400$$

and

$$1200(1+\alpha) = 2000$$

Or $\alpha = 2/3$, which satisfies the required conditions. The probability of error for an M-QAM constellation is given by

$$P_M = 1 - (1 - P_{\sqrt{M}})^2$$

where

$$P_{\sqrt{M}} = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left[\sqrt{\frac{3\mathcal{E}_{av}}{(M-1)N_0}}\right]$$

With $P_M = 10^{-6}$ we obtain $P_{\sqrt{M}} = 5 \times 10^{-7}$ and therefore

$$2 \times (1 - \frac{1}{4})Q\left[\sqrt{\frac{3\mathcal{E}_{av}}{15 \times 2 \times 10^{-10}}}\right] = 5 \times 10^{-7}$$

Using the last equation and the tabulation of the $Q[\cdot]$ function, we find that the average transmitted energy is

$$\mathcal{E}_{av} = 24.70 \times 10^{-9}$$

Note that if the desired spectral characteristic $X_{rc}(f)$ is split evenly between the transmitting and receiving filter, then the energy of the transmitting pulse is

$$\int_{-\infty}^{\infty} g_T^2(t)dt = \int_{-\infty}^{\infty} |G_T(f)|^2 df = \int_{-\infty}^{\infty} X_{rc}(f)df = 1$$

Hence, the energy $\mathcal{E}_{av} = P_{av}T$ depends only on the amplitude of the transmitted points and the symbol interval T. Since $T = \frac{1}{2400}$, the average transmitted power is

$$P_{av} = \frac{\mathcal{E}_{av}}{T} = 24.70 \times 10^{-9} \times 2400 = 592.8 \times 10^{-7}$$

4. The answers for the multi-tone, Levin Campello and HSDPA SISO questions are as follows.

(a) The answers for the water filling theorem for multi-tone modulation are as follows

i. The rate adaptive loading:
An N dimensional system with

$$m = 1, \cdots, N$$

The relationship between the bit value b_m and SNR_m is

$$b_m = \frac{1}{2}\log_2\left(1 + \frac{SNR_m}{\Gamma}\right)$$

Total number of bits is $b = \sum_{m=1}^{N} b_m$ where the total rate

$$R = \frac{b}{T} = \frac{\sum_{m=1}^{N} b_m}{T} = \sum_{m=1}^{N} R_m$$

RA loading objective is to maximize

$$\max_{\varepsilon_m}\left(\sum_{m=1}^{N}\frac{1}{2}\log_2\left(1 + \frac{g_m \quad \varepsilon_m}{\Gamma}\right)\right)$$

subject to

$$N\bar{\varepsilon}_x = \sum_{m=1}^{N}\varepsilon_m$$

The algorithm operates as follows:
make

$$i = 0$$

A. make

$$N^* = N - i$$

B. order channel SNR

$$g_1 = \max_m g_m$$

and

$$g_{N^*} = \min_m g_m$$

C. calculate the rate adaptive water filling constant

$$K = \frac{1}{N^*}\left[N\bar{\varepsilon}_x + \Gamma\sum_{m=1}^{N^*}\frac{1}{g_m}\right]$$

D. calculate the energy for each dimension

$$\varepsilon_m = K - \frac{\Gamma}{g_m}$$

$\forall m = 1, \cdots, N^*$. if

$$\varepsilon_{N^*} \leq 0$$

discard g_{N^*} and make $i = i + 1$ go to A.

ii. The margin adaptive loading:

MA loading is to minimize the total energy

$$\min_{\varepsilon_m} \sum_{m=1}^{N} \varepsilon_m$$

subject to ensuring that the total number of bits is given by

$$b_T = \frac{1}{2} \sum_{m=1}^{N} \log_2 \left(1 + \frac{g_m \quad \varepsilon_m}{\Gamma}\right)$$

such that between the total available energy $N\bar{\varepsilon}_x$ and the total used energy $\sum_{m=1}^{N} \varepsilon_m$ we have a margin of

$$\gamma_{\max} = \frac{N\bar{\varepsilon}_x}{\sum_{m=1}^{N} \varepsilon_m}$$

As we have $\varepsilon_m = K_{ma} - \frac{\Gamma}{g_m}$. Total number of bits can be expressed as

$$b_T = \frac{1}{2} \sum_{m=1}^{N^*} \log_2 \left(\frac{g_m \quad K_{ma}}{\Gamma}\right).$$

Express K_{ma} as follows

$$K_{ma} = \Gamma \left(\frac{2^{2b}}{\prod_{m=1}^{N^*} g_m}\right)^{\frac{1}{N^*}}$$

The MA operates as follows. set $i = N$

A. set $N^* = i$ and order g_m largest to smallest for

$$m = 1, \cdots, N^*$$

B. compute

$$K_{ma} = \Gamma \left(\frac{2^{2b}}{\prod_{m=1}^{N^*} g_m}\right)^{\frac{1}{N^*}}$$

C. If

$$\varepsilon_m = K_{ma} - \frac{\Gamma}{g_m} < 0$$

then $i = i - 1$ and discard g_m go to step A else compute solution with

$$\varepsilon_m = K_{ma} - \frac{\Gamma}{g_m}$$

$m = 1, \cdots, N^*$ and calculate bits

$$b_m = \frac{1}{2}\log_2\left(\frac{g_m \quad K_{ma}}{\Gamma}\right)$$

compute margin

$$\gamma_{\max} = \frac{N\bar{\varepsilon}_x}{\sum_{m=1}^{N^*} \varepsilon_m}$$

(b) The HSDPA SISO system

i. System model for the HSDPA SISO system: Bookwork.

Transmission symbols are used to produce a $(N^{(x)} \times 1)$-dimensional symbol vector $\vec{x}_k = [x_k(1), \cdots, x_k(\rho), \cdots, x_k(N^{(x)})]^T$ for each vector $\vec{d_k}$. The entire block of transmission can be represented as an $(N^{(x)} \times K)$ dimensional transmit symbol matrix defined as

$$\begin{aligned} \mathbf{X} &= [\vec{x}_1, \cdots, \vec{x}_k, \cdots, \vec{x}_K] \\ &= \left[\overrightarrow{y}(1), \cdots, \overrightarrow{y}(\rho), \cdots \overrightarrow{y}\left(N^{(x)}\right)\right]^T. \end{aligned}$$

The transmitted vector $\overrightarrow{y}(\rho) = [y_1(\rho), \cdots, y_k(\rho), \cdots, y_K(\rho)]^T$ contains the symbols, over the symbol period $\rho = 1, \cdots, N^{(x)}$, with the unit average energy $E(y_k(\rho)\, y_k^*(\rho)) = 1$ for $k = 1, \ldots, K$. the symbols are weighted with an amplitude matrix $\mathbf{A} = diag\left(\sqrt{E_1}, \cdots, \sqrt{E_k}, \cdots, \sqrt{E_K}\right)$ and spread with an $N \times K$ dimensional signature sequence matrix

$$\mathbf{S} = [\overrightarrow{s}_1, \cdots, \overrightarrow{s}_k, \cdots, \overrightarrow{s}_K].$$

This results in the size N transmission column vector expressed as $\overrightarrow{z}(\rho) = [z_1(\rho), \cdots, z_n(\rho), \cdots, z_N(\rho)]^T = \mathbf{SA}\overrightarrow{y}(\rho)$. Each element, $z_n(\rho)$, of the transmission vector $\overrightarrow{z}(\rho)$, for $n = 1, \cdots, N$, is then filtered using a pulse shaping function at regular intervals of chip period T_c before being modulated with an up converter to transmit the data at the desired frequency. For the duration of packet transmission, the link between the transmitter and receiver antennas is then modelled using the multipath radio channel impulse response vector

$$\overrightarrow{h} = [\ h_0 \ \ \ldots \ \ h_{L-1}\ ]^T.$$

The $((N + L - 1) \times N)$-dimensional channel convolution matrix $\mathbf{H}$ is formed as follows

$$\mathbf{H} = \begin{bmatrix} h_0 & 0 & & 0 \\ \vdots & h_0 & & \vdots \\ h_{L-1} & \vdots & \ddots & 0 \\ 0 & h_{L-1} & & h \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & h_{L-1} \end{bmatrix}$$

In the presence of more than one resolvable path ($L > 1$), the despreading signature sequences at the receiver antenna would be longer than the spreading signature sequences at the transmit antenna. The channel convolution matrix $\mathbf{H}$, which convolves with the transmission signature sequence matrix $\mathbf{S}$, produces the $(N + L - 1) \times K$ dimensional receiver matched filter signature sequence matrix as

$$\mathbf{Q} = \mathbf{HS} = [\overrightarrow{q}_1, \cdots \overrightarrow{q}_k, \cdots \overrightarrow{q}_K]$$

where $\overrightarrow{q}_k$ is the receiver matched filter receiver sequence. The vector $\vec{q}_k = \mathbf{H}\vec{s}_k$ is an $(N + L - 1)$-dimensional receiver signature sequence which is a function of an $(N \times 1)$-dimensional signature sequence $\vec{s}_k$. At the receiver, it is assumed that the receiver carrier and clocks are fully synchronized with the transmitter carrier and clocks. The received signal at the receiver antenna is down converted to the baseband. The baseband signal is passed through the receiver chip matched filter (CMF) and the filtered signal is sampled at the chip period intervals T_c.The signal vector $\vec{r}(\rho) = \begin{bmatrix} r_1(\rho) & \ldots & r_{N+L-1}(\rho) \end{bmatrix}^T$ of size $(N + L - 1)$ gives the received matched filtered signal samples at the ρ-th symbol period for $\rho = 1, \ldots, N^{(x)}$. The ISI is incorporated into the system model by producing the $(N + L - 1) \times 3N$ dimensional extended matched filter matrix

$$\mathbf{Q}_e = \left[\mathbf{Q}, \quad \left(\mathbf{J}^T_{N+L-1}\right)^N \mathbf{Q}, \quad \mathbf{J}^N_{N+L-1}\mathbf{Q}\right].$$

The $(N+L-1)\times(N+L-1)$-dimensional matrix is defined as $\mathbf{J}_{N+L-1} = \begin{bmatrix} \overrightarrow{\mathbf{0}}^T_{(N+L-2)} & 0 \\ \mathbf{I}_{(N+L-2)} & \overrightarrow{\mathbf{0}}_{(N+L-2)} \end{bmatrix}$

The ISI interference signature sequence matrices $\left(\mathbf{J^T}\right)^{\mathbf{N}} \mathbf{Q}$ and $\mathbf{J^N Q}$ are expressed

$$\left(\mathbf{J}^T\right)^N \mathbf{Q} = \left(\mathbf{J}^T\right)^N \mathbf{HS} = [\overrightarrow{q}_{1,1}, \cdots \overrightarrow{q}_{k,1}, \cdots \overrightarrow{q}_{K,1}]$$

and

$$\mathbf{J}^N\mathbf{Q} = \mathbf{J}^N\mathbf{HS} = [\overrightarrow{q}_{2,1}, \cdots \overrightarrow{q}_{k,2}, \cdots \overrightarrow{q}_{K,2}]$$

Both $\vec{q}_{k,1} = \left(\mathbf{J}^T\right)^N \vec{q}_k$ and $\vec{q}_{k,2} = \mathbf{J}^N\vec{q}_k$ are the receiver signature sequences corresponding to the previous and the next symbol periods and are used to handle the ISI. The $(N + L - 1)$ dimensional received signal vector is given in terms of the transmitter vector $\overrightarrow{y}(\rho)$ as

$$\vec{r}(\rho) = \mathbf{Q}_e\left(\mathbf{I}_3 \otimes \mathbf{A}\right) \begin{bmatrix} \overrightarrow{y}(\rho) \\ \overrightarrow{y}(\rho - 1) \\ \overrightarrow{y}(\rho + 1) \end{bmatrix} + \overrightarrow{n}(\rho)$$

where $\otimes$ is the Kronecker product and the $(N + L - 1)$ dimensional noise vector $\overrightarrow{n}(\rho)$ has the noise covariance matrix $E\left(\overrightarrow{n}(\rho)\overrightarrow{n}^H(\rho)\right) = 2\sigma^2\mathbf{I}_{N+L-1}$with the one dimensional noise variance $\sigma^2 = \frac{N_0}{2}$. The received signal vector $\vec{r}(\rho)$ is used to produce the size K column vector $\widehat{\overrightarrow{y}}(\rho) = \left[\widehat{y}_1(\rho), \cdots, \widehat{y}_k(\rho), \cdots, \widehat{y}_K(\rho)\right]^T$ as an estimate of the transmitted symbol vector $\overrightarrow{y}(\rho)$ as follows

$$\widehat{\overrightarrow{y}}(\rho) = \mathbf{W}^H \vec{r}(\rho).$$

The $(N + L - 1) \times K$ dimensional matrix $\mathbf{W} = [\overrightarrow{w}_1, \cdots, \overrightarrow{w}_k, \cdots, \overrightarrow{w}_K]$ has the MMSE linear equalizer despreading filter coefficients $\overrightarrow{w}_k$ for $k = 1, \cdots, K$. In order to ensure that $\vec{w}_k^H \vec{q}_k = 1$ and the cross-correlations $\vec{w}_k^H \vec{q}_j$ are minimized for $j \neq k$, a normalized MMSE despreading filter coefficient vector,

$$\vec{w}_k = \frac{\mathbf{C}^{-1}\vec{q}_k}{\vec{q}_k^H \mathbf{C}^{-1}\vec{q}_k}$$

ii. The iterative energy calculation uses the $(N+L-1)\times(N+L-1)$ dimensional covariance matrix $\mathbf{C} = E\left(\vec{r}(\rho)\vec{r}^H(\rho)\right)$ of the received signal vector $\vec{r}(\rho)$ in the following form

$$\mathbf{C} = \mathbf{Q}_e\left(\mathbf{I}_3 \otimes \mathbf{A}^2\right)\mathbf{Q}_e^H + 2\sigma^2 \mathbf{I}_{N+L-1}$$

for $k = 1, \cdots, K$. At the output of each receiver, the mean-square-error $\varepsilon_k = E\left(|\widehat{y}_k(\rho) - y_k(\rho)|^2\right)$ between the transmitted signal $y_k(\rho)$ and the estimated signal $\widehat{y}_k(\rho)$ is given by

$$\begin{aligned}\varepsilon_k &= 1 - E_k\, \vec{q}_k^H \mathbf{C}^{-1}\vec{q}_k \\ &= \frac{1}{1+\gamma_k} = 1 - \lambda_k\end{aligned}$$

for $k = 1, \cdots, K$.Where $\gamma_k = \dfrac{1-\varepsilon_k}{\varepsilon_k}$ is the signal-to-noise-ratio (SNR) at the output of each receiver and λ_k is the system value given as

$$\begin{aligned}\lambda_k &= 1 - \varepsilon_k = \frac{\gamma_k}{1+\gamma_k}, \\ &= E_k\, \vec{q}_k^H \mathbf{C}^{-1}\vec{q}_k.\end{aligned}$$

In the rate adaptive loading schemes based on the total MMSE minimization criterion, it is assumed that the total energy E_T at the transmitter will be distributed to each channel with a transmission energy E_k such that

$$\sum_{k=1}^{K} E_k \leq E_T.$$

Objective of the energy allocation is to minimize the total MMSE $\varepsilon_T = \sum_{k=1}^{K}\varepsilon_k$ to maximize the total rate

$$b_T = \sum_{k=1}^{K} b_{p_k}$$

where b_{p_k} is the number of bits allocated to each spreading sequence symbol for $k = 1, \cdots, K$. In the optimization the energy E_k is related to the system value λ_k by means of $E_k = \dfrac{\lambda_k}{\vec{q}_k^H \mathbf{C}^{-1}\vec{q}_k}$. The discrete bit rates b_{p_k} are related to the SNR γ_k in terms of $b_{p_k} = \log_2\left(1 + \frac{\gamma_k}{\Gamma}\right)$ where Γ is the gap value. The target SNR $\gamma^*(b_{p_k})$ is given by

$$\gamma^*(b_{p_k}) = \Gamma\left(2^{b_{p_k}} - 1\right),$$

and the target system value λ_k^* required to transmit b_{p_k} bits per symbol is given by

$$\lambda_k^* = \frac{\Gamma\left(2^{b_{p_k}} - 1\right)}{1 + \Gamma\left(2^{b_{p_k}} - 1\right)} = 1 - \varepsilon_k^*.$$