1. The two questions 1.a and 1.b below are independent.

We say that two subspaces V and W of $\mathbb{R}^n$ are complementary, denoted by $V \oplus W = \mathbb{R}^n$, if (i) $V \cap W = \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$, and (ii) any vector $x \in \mathbb{R}^n$ can be written as $x = v + w$ where $v \in V$ and $w \in W$.

a) Let P be the matrix defined as

$$P = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

i) Describe a basis of Ker(P) the null-space (kernel) of P and Ran(P) the range of P. Justify your answer. [3]

ii) Show that $\mathbb{R}^4 = \text{Ker}(P) \oplus \text{Ran}(P)$. [2]

iii) Show that for $x \in \text{Ker}(P)$ and $y \in \text{Ran}(P)$ then $x^T y = 0$. [2]

iv) Conclude that P is an orthogonal projection. [3]

SOLUTION

1.a.i)

Ran(P)

We show an alternative method to the previous examples. The range of P is $\mathbf{y}$ such that $P\mathbf{x} = \mathbf{y}$ for some vector $\mathbf{x}$. That is,

$$\frac{1}{2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Using Gaussian elimination:

$$\left.\begin{array}{cccc|c} 1 & 0 & -1 & 0 & a \\ 0 & 1 & 0 & -1 & b \\ -1 & 0 & 1 & 0 & c \\ 0 & -1 & 0 & 1 & d \end{array}\right. \rightsquigarrow \left.\begin{array}{cccc|c} 1 & 0 & -1 & 0 & a \\ 0 & 1 & 0 & -1 & b \\ 0 & 0 & 0 & 0 & c+a \\ 0 & 0 & 0 & 0 & d+b \end{array}\right. = [A|\mathbf{w}]$$

Solutions exist iff rank A = rank $A|\mathbf{w}$, implying $a + c = 0, b + d = 0$. Hence, solutions have the form

$$\mathbf{x} = \begin{bmatrix} a \\ b \\ -a \\ -b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad a, b \in \mathbb{R}.$$

. Furthermore, since they are linearly independent, $\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}\right\}$ forms a basis of Ran (P).

Ker(P)

We find the solution space in x of $P\mathbf{x} = \mathbf{0}$ using the Gaussian eliminations from the above, i.e. using matrix A:

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x = z, y = t$. Hence $\mathbf{x} = \begin{bmatrix} x \\ y \\ x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ $x, y \in \mathbb{R}$. And since they are linearly independent, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ forms a basis of Ker (P).

1.a.ii) By the usual techniques, the four vectors above can be shown to be linearly independent. Therefore, they form a basis for $\mathbb{R}^4$, and so the desired result follows.

1.a.iii) $\mathbf{x} \in \mathrm{Ker}(P) \Rightarrow \mathbf{x} = \begin{bmatrix} \gamma \\ \delta \\ \gamma \\ \delta \end{bmatrix}$ for some $\gamma, \delta \in \mathbb{R}$, and $\mathbf{y} \in \mathrm{Ran}(P) \Rightarrow \mathbf{y} = \begin{bmatrix} \alpha \\ \beta \\ -\alpha \\ -\beta \end{bmatrix}$ for some $\alpha, \beta \in \mathbb{R}$. Hence $\mathbf{x}^T\mathbf{y} = \alpha\gamma + \beta\delta - \alpha\gamma - \beta\delta = 0$.

1.a.iv) We show P is a projection: $\mathbf{y} \in \mathrm{Ran}(P)$ has form $\begin{bmatrix} \alpha \\ \beta \\ -\alpha \\ -\beta \end{bmatrix}$ and

$$P\mathbf{y} = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ -\alpha \\ -\beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ -\alpha \\ -\beta \end{bmatrix} = \mathbf{y}$$

Hence $P^2 = P$ and P is projection. By part 1.a.iii), $\mathrm{Ran}(P) \perp \mathrm{Ker}(P)$, and so P is an orthogonal projection.

b) Define the matrix A_m as follows

$$A_m = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & m & 0 & 0 \\ 1 & 0 & -m & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

where $m \in \mathbb{R}$ is a parameter.

i) Derive bases for $\mathrm{Ker}(A_m)$ and $\mathrm{Ran}(A_m)$. [3]

ii) For $m \neq 0$, show that $\mathrm{Ran}(A_m) \oplus \mathrm{Ker}(A_m) = \mathbb{R}^4$. [2]

iii) We now fix $m = 0$. Compute A_0^3. [2]

iv) Do we have $\mathrm{Ran}(A_0^3) \oplus \mathrm{Ker}(A_0^3) = \mathbb{R}^4$?

Justify your answer. [3]

SOLUTION

1.b.i) Similar to the above approach, we derive that $\mathrm{Ran}(A) = \mathrm{span}\left\{\begin{bmatrix}-1\\m\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\}$

and $\mathrm{Ker}(A) = \mathrm{span}\left\{\begin{bmatrix}m\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right\}$ where bases are given in both cases.

1.b.ii) We show that the four vectors are linearly independent when $m \neq 0$, and so by the same justification as above, the result follows.

$$\begin{bmatrix}1 & 0 & 0 & 1\\ m & 0 & 1 & 0\\ -1 & m & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 0 & 1 & -m\\ -1 & m & 0 & 2\\ 0 & 0 & 1 & 0\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & 0 & 0 & 1\\ 0 & m & 0 & 2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -m\end{bmatrix}$$

hence, $m \neq 0 \Rightarrow$ the rows are linearly independent, implying the vectors are linearly independent.

1.b.iii) $A_0^3 = \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$.

1.b.iv) $\mathrm{Ran}(A_0^3) = \mathbf{0}$ and $\mathrm{Ker}(A_0^3) = \mathbb{R}^4$, therefore $\mathrm{Ran}(A_0^3) \oplus \mathrm{Ker}(A_0^3) = \mathbb{R}^4$.

2. Let $A = (a_{ij})_{i,j=1,\dots,n} \in \mathbb{R}^{n\times n}$ be a symmetric matrix, i.e, $A^T = A$ such that for all $x \in \mathbb{R}^n$ with $x \neq \mathbf{0}$ we have

$$x^T A x > 0.$$

Matrices satisfyng the above properties are known as *positive-definite matrices*

a) Let $e_i \in \mathbb{R}^n$ with all its entries equal to 0 except the i-th entry which is equal to 1. Show that, for $i = 1,\dots,n$, we have $a_{ii} = e_i^T A e_i > 0$. [1]

b) Let C be the Schur complement of a_{11} in A, i.e.

$$C = A_{22} - \frac{1}{a_{11}} A_{21} A_{12},$$

where

$$A = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with a_{11} is a scalar, $A_{21} \in \mathbb{R}^{n-1}$, and $A_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$ and $A_{12} \in \mathbb{R}^{1\times(n-1)}$.

i) Justify the fact that $C = A_{22} - \frac{1}{a_{11}} A_{21} A_{21}^T$. [1]

ii) Let $v \in \mathbb{R}^{n-1}$ and define $x \in \mathbb{R}^n$ such that

$$x = \begin{pmatrix} -(1/a_{11}) A_{21}^T v \\ v \end{pmatrix}.$$

Show that $x^T A x = v^T C v$ and that C is a positive-definite matrix. [3]

c) In what follows we will show that there exists a lower-triangular matrix $L \in \mathbb{R}^{n\times n}$ such that $A = LL^T$. This factorisation is known as the *Cholesky decomposition.*

i) Let L be given by

$$L = \begin{pmatrix} l_{11} & \mathbf{0}^T \\ L_{21} & L_{22} \end{pmatrix}$$

with l_{11} is a scalar, $L_{21} \in \mathbb{R}^{n-1}$, and $L_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\mathbf{0} \in \mathbb{R}^{n-1}$. Write the block structure of the matrix LL^T. [2]

ii) Let $A = LL^T$. Show that $l_{11} = \sqrt{a_{11}}$, $L_{21} = (1/l_{11}) A_{21}$, and $L_{22} L_{22}^T = A_{22} - L_{21} L_{21}^T$. [2]

iii) Describe a recursive procedure to construct the lower-triangular matrix L such that $A = LL^T$. [4]

iv) Describe how one would use the above procedure to solve the linear equation $Ax = y$ for $A \in \mathbb{R}^{n\times n}$ positive definite. [3]

d) Define the following matrix A

$$A = \begin{pmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{pmatrix}$$

i) Apply the Cholesky decomposition to the matrix A above. [2]

ii) Use it to solve the equation $Ax = y$ where $y = \begin{pmatrix} 30 \\ 15 \\ -16 \end{pmatrix}$. [2]

SOLUTION [For a matrix $A = [a_{ij}]$, the notation $[A]_{ij}$ means the element a_{ij}]

1. Ae_i picks out the i'th colum of A and $e_i^T(Ae_i)$ picks out the the i'th row of Ae_i, that being a single element. Hence, the i'th diagonal element is picked.

More formally, Ae_i is a $n \times 1$ matrix (column vector) with $[Ae_i]_{k1} = \sum_{t=1}^{n}[A]_{kt}[e_i]_{t1}$. Now $[e_i]_{i1} = 1$ and $[e_i]_{t1} = 0$ for $t \neq i$. Thus $[Ae_i]_{k1} = \sum_{t=1}^{n}[A]_{kt}[e_i]_{t1} = [A]_{ki}[e_i]_{i1} = a_{ki}$. Now $e_i^T Ae_i = \sum_{t=1}^{n}[e_i^T]_{1t}[Ae_i]_{t1} = \sum_{t=1}^{n}[e_i^T]_{1t}a_{ti}$, and since $[e_i^T]_{1i} = 1$ and $[e_i^T]_{1t} = 0$ for $t \neq i$, we have $e_i^T Ae_i = a_{ii}$.

2. $A_{12} = A_{21}^T$ because A is symmetric.

3. Observe $x^T = \begin{bmatrix}(-(1/a_{11})A_{21}^T v)^T & v^T\end{bmatrix} = \begin{bmatrix}-(1/a_{11})v^T A_{21} & v^T\end{bmatrix}$. Hence,

$$\begin{aligned}
x^T Ax &= \begin{bmatrix}-(1/a_{11})v^T A_{21} & v^T\end{bmatrix}\begin{bmatrix}a_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}-(1/a_{11})A_{21}^T v\\ v\end{bmatrix}\\
&= \begin{bmatrix}0 & -\frac{1}{a_{11}}v^T A_{21}A_{12} + v^T A_{22}\end{bmatrix}\begin{bmatrix}-(1/a_{11})A_{21}^T v\\ v\end{bmatrix}\\
&= -\frac{1}{a_{11}}v^T A_{21}A_{12}v + v^T A_{22}v\\
&= v^T A_{22}v - \frac{1}{a_{11}}v^T A_{21}A_{21}^T v
\end{aligned}$$

$$v^T Cv = v^T\left(A_{22} - \frac{1}{a_{11}}A_{21}A_{21}^T\right)v = v^T A_{22}v - \frac{1}{a_{11}}v^T A_{21}A_{21}^T v$$

thus $x^T Ax = v^T Cv$. Furthermore, given A, any $v \in \mathbb{R}^{n-1}$ defines an $x \in \mathbb{R}^n$ such that $v^T Cv = x^T Ax > 0$. Thus, $v^T Cv > 0$ for any $v \in \mathbb{R}^{n-1}$, implying C is positive-definite.

4. (a)

$$LL^T = \begin{bmatrix}l_{11} & 0^T\\ L_{21} & L_{22}\end{bmatrix}\begin{bmatrix}l_{11} & L_{21}^T\\ 0 & L_{22}^T\end{bmatrix} = \begin{bmatrix}l_{11}^2 & l_{11}L_{21}^T\\ l_{11}L_{21} & L_{21}L_{21}^T + L_{22}L_{22}^T\end{bmatrix} \tag{2.1}$$

(b)Equating elements from A and the RHS of (2.1), we see that: $a_{11} = l_{11}^2 \Rightarrow l_{11} = \sqrt{a_{11}}$; $A_{21} = l_{11}L_{21} \Rightarrow L_{21} = (1/l_{11})A_{21}$; $A_{22} = L_{21}L_{21}^T + L_{22}L_{22}^T \Rightarrow L_{22}L_{22}^T = A_{22} - L_{21}L_{21}^T$

(c)

<u>CholeskyLD(A)</u> // A is a positive-definite matrix. Return is a the lower triangular matrix L

BEGIN

1. $l_{11} \leftarrow \sqrt{a_{11}}$
2. If A is a 1×1 matrix, return l_{11}
3. $L_{21} \leftarrow (1/l_{11})A_{21}$
4. $C \leftarrow A_{22} - \frac{1}{a_{11}}A_{21}A_{21}^T$
5. $L_{22} \leftarrow$ CholeskyLD(C)
6. return $\begin{bmatrix}l_{11} & 0\\ L_{21} & L_{22}\end{bmatrix}$

END

Note that in line 4., $C \leftarrow A_{22} - \frac{1}{a_{11}}A_{21}A_{21}^T = A_{22} - L_{21}L_{21}^T = L_{22}L_{22}^T$ by part (b) above. Furthermore, we know from part 3. of this exercise that C is positive-definite, and so is valid input to the function.

(d) $Ax = y$. Compute $L =$ CholeskyLD(A), trasnpose (a copy of) that to give L^T. Then we know that $A = LL^T$. Thus, the equation is $L(L^T x) = y$. Let $z = L^T x$, solve $Lz = y$ for

z, using forward substitution (which is an easy computation). Now solve $L^T x = z$ using backward substitution.

(e) $A = \begin{bmatrix} 25 & 15 & -1 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$, $L = \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix}$, as follows:

<u>CholeskyLD(A)</u>

1. $l_{11} \leftarrow 5$
2. -
3. $L_{21} \leftarrow \begin{bmatrix} 3 \\ -1 \end{bmatrix}$
4. $C \leftarrow \begin{bmatrix} 9 & 3 \\ 3 & 10 \end{bmatrix}$
5. $L_{22} \leftarrow \text{CholeskyLD}(C) = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$
6. return $\begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix}$

$C = \begin{bmatrix} 9 & 3 \\ 3 & 10 \end{bmatrix}$

<u>CholeskyLD(C)</u>

1. $l_{11} \leftarrow 3$
2. -
3. $L_{21} \leftarrow [1]$
4. $C \leftarrow [9]$
5. $L_{22} \leftarrow 3$
6. return $\begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$

$$Ax = \begin{bmatrix} 25 & 15 & -1 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix} \left(\begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 30 \\ 15 \\ -16 \end{bmatrix}$$

Letting $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and solving $Lz = y$, we get $z_1 = 6, z_2 = -1, z_3 - 3$

Then

$$\begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -3 \end{bmatrix}$$

We get $x_3 = -1, x_2 = 0, x_1 = 1$. Thus, $x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

3. Let m and n be two positive integers with $m \leq n$. We consider $A \in \mathbb{R}^{(n+1)\times(m+1)}$ the matrix defined by

$$A = \begin{pmatrix} 1 & x_0 & \dots & x_0^m \\ 1 & x_1 & \dots & x_1^m \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^m \end{pmatrix},$$

where $x_0, \dots, x_n$ are n distinct real numbers.

Let $\mathbf{0}$ be the vector with all its entries equal to 0 (we will use the same notation for both the zero vector of $\mathbb{R}^{m+1}$ and the one of $\mathbb{R}^{n+1}$). In what followed we define the vector

$$v = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_m \end{pmatrix} \in \mathbb{R}^{m+1}.$$

a) i) Show that if $Av = \mathbf{0}$ then $v = \mathbf{0}$. [1]

Hint: Use the fact if the polynomial $P(x) = v_0 + v_1 x + \dots + v_m x^m$ has n distinct zeros then $P(x) = 0$.

ii) Using the previous question, show that if $A^T Av = \mathbf{0}$ then $v = \mathbf{0}$. [2]

iii) Fix $y \in \mathbb{R}^{n+1}$. Justify the fact that the linear equation $A^T Ax = A^T y$ admits a unique solution w. [2]

b) In the remainder of this problem, we will denote the solution in 2. a) iii) by w, i.e.

$$A^T Aw = A^T y.$$

For $v \in \mathbb{R}^{m+1}$ and $y \in \mathbb{R}^{n+1}$, define $g(v) = (y - Av)^T (y - Av)$.

i) Show that $g(w) = y^T y - y^T Aw$, with w defined in 2. a) iii). [2]

ii) Prove that $g(v) - g(w) = (w - v)^T A^T A(w - v)$. [2]

Hint: Use the fact that $||A(w-v)||^2 = ||(Aw-y) - (Av-y))||^2$.

iii) Show that for all $v \in \mathbb{R}^{m+1}$, we have $g(v) \geq g(w)$ and that $g(v) = g(w)$ if and only if $v = w$. [3]

c) Let P be a polynomial such that $P(x) = \sum_{k=0}^{m} v_k x^k$. We define the quantity

$$\Phi_m(P) = \sum_{i=0}^{n} (y_i - P(x_i))^2 .$$

Let $y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^{n+1}$.

i) Show that $\Phi_m(P) = g(v)$. [2]

ii) Using question 3.b.iii), show that there exisits a polynomial P_w such that $\Phi_m(P) \geq \Phi_m(P_w)$. [2]

d) Let $n = m = 3$, $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $y_0 = 1$, $y_1 = 2$, , $y_2 = 1$, $y_3 = 0$.

i) Solve $A^T Av = A^T y$. [2]

ii) Derive the expression of the polynomial in $\mathbb{R}_3[X]$ that minimizes Φ_3 and give the minimum value of Φ_3 on $\mathbb{R}_3[X]$. Justify your answer. [2]

SOLUTION

3.a.i) If $Av = 0$ then $P(x_0) = P(x_1) = \dots P(x_n) = 0$, implying $P(x) = 0$, i.e., that $v_0 = v_1 = \dots v_m = 0$.

3.a.ii) $v^T A^T Av = ||Av||^2 = 0 \Rightarrow Av = 0 \Rightarrow v = 0$.

3.a.iii) Suppose $A^T Ax = A^T y$ and $A^T Ax' = A^T y$. Then $A^T A(x - x') = 0 \Rightarrow x = x'$.

3.b.i)

$$\begin{aligned} g(w) &= (y - Aw)^T (y - Aw) \\ &= y^T y - 2y^T Aw + (Aw)^T Aw \\ &= y^T y - 2y^T Aw + w^T A^T Aw \\ &= y^T y - 2y^T Aw + w^T A^T y \\ &= y^T y - 2y^T Aw + y^T Aw \\ &= y^T y - y^T Aw \end{aligned}$$

3.b.ii) Note that

$$\begin{aligned} (w-v)^T A^T A(w-v) &= ||A(w-v)||^2 \\ &= ||(Aw-y) - (Av-y))||^2 \\ &= ||y - Av||^2 + ||y - Aw||^2 - 2(Av-y)^T (Aw-y) \\ &= ||y - Av||^2 + ||y - Aw||^2 - 2v^T A^T (Aw-y) + 2y^T (Aw-y) \\ &= ||y - Av||^2 + ||y - Aw||^2 - 0 - 2g(w) \quad \text{by 2.a.i and the definition of } w. \\ &= g(v) + g(w) - 2g(w) = g(v) - g(w) \end{aligned}$$

3.b.iii) Follows from previous question we have

$$g(v) - g(w) = ||A(v-w)||^2$$

and we conclude using question 3.a.ii).

3.c.i) Note that

$$P(x_i) = \sum_{k=0}^{m} v_k x_i^k = (Av)_i$$

Hence

$$\Phi_m(P) = ||y - Av||^2 = (y - Av)^T (y - Av) = g(v)$$

3.c.ii) Direct consquence of 3b.iii)

3.d.i) Here

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{pmatrix},$$

and

$$A^TA = \begin{pmatrix} 4 & 2 & 6 & 8 \\ 2 & 6 & 8 & 18 \\ 6 & 8 & 18 & 32 \\ 8 & 18 & 32 & 66 \end{pmatrix},$$

and

$$A^Ty = \begin{pmatrix} 4 \\ 0 \\ 2 \\ 0 \end{pmatrix},$$

Hence

$$w = \begin{pmatrix} 2 \\ -1/3 \\ -1 \\ 1/3 \end{pmatrix},$$

3.d.ii) Simple calculations yield $\Phi_m(P_w) = 0$.