EE4-27
EE9CS5-2

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

MSc and EEE/ISE PART IV: MEng and ACGI

SYSTEMS IDENTIFICATION

Wednesday, 11 May 10:00 am

Time allowed: 3:00 hours

Corrected Copy Q1

There are FIVE questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : R.B. Vinter

Second Marker(s) : S. Evangelou

Information for candidates:

The Multivarite Normal Density:

The probability density $N(m, Q)$ of an n-vector, normal random variable with mean m and covariance matrix Q $(Q > 0)$ is

$$N(m, Q)(x) = \frac{1}{(2\pi)^{\frac{n}{2}} (\det\, Q)^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x-m)^T Q^{-1}(x-m)\right\} .$$

In the case that $n = 1$, m is a scalar and $Q = \sigma^2$ $(\sigma^2 > 0)$,

$$N(m, \sigma^2)(x) \;=\; \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$

and, if X is a scalar random variable with probability density $N(m, \sigma^2)$,

$$\text{Prob}\{m - 2\sigma \;\leq\; X \;\leq m + 2\sigma\} \approx 0.95 .$$

The Cramer-Rao Lower Bound:

Take a family of probability densities $\{p(\mathbf{y}; \theta)\}$ parameterised by the k-vector θ. Let $\hat{\theta}(\mathbf{y})$ be any unbiased estimate of θ given $\mathbf{y}$. Then the covariance of $\hat{\theta}(\mathbf{y})$ satisfies

$$cov\{\hat{\theta}(\mathbf{y})\} \;\geq\; M^{-1}(\theta)$$

where $M(\theta)$ is the $k \times k$ Fisher Information Matrix, with components $\{m_{ij}\}$ defined by:

$$m_{ij} = -E_\theta\{\frac{\partial^2}{\partial\theta_i \partial\theta_j} log_e f\,(Y, \theta)\} .$$

Spectral Density Relations for Linear Systems :

Consider the stationary n-vector discrete time process y_t satisfying

$$y_t \;=\; G(z)e_t\;,$$

in which $G(z)$ is the $n \times k$ matrix transfer function of a stable linear system and e_t is a k-vector white noise sequence with covariance Σ. Then y_t has spectral density

$$\Phi_y(\omega) \;=\; G(e^{j\omega})\Sigma G^T(e^{-j\omega}) .$$

1. A zero mean stationary scalar process x_t satisfies the equation

$$x_t + a x_{t-1} = e_t \,,$$

in which e_t is white noise with variance σ_1^2. a is a given parameter, $|a| < 1$. Noisy observations y_t are taken of x_t

$$y_t = x_t + v_t$$

in which v_t is a white noise process with variance σ_2^2, independent of e_t.

(i): Develop a difference equation for y_t involving the vector noise process

$$\mathbf{w}_t = \begin{bmatrix} e_t \\ v_t \end{bmatrix}$$

of the form

$$A(z) y_t = \mathbf{b}^T(z) \mathbf{w}_t \,. \qquad [2]$$

(ii): Compute the covariance function of y_t:

$$R_y(l), \quad \text{for } l = \ldots, -1, 0, +1, \ldots \qquad [10]$$

— page 1

(iii): Using (i) and the spectral density formulae on page ~~2~~ or otherwise, show that the spectral density of y_t is

$$\Phi_y(\omega) = \frac{(1 + a^2 + 2a\cos\omega)\sigma_2^2 + \sigma_1^2}{(1 + a^2 + 2a\cos\omega)} \,.$$

[6]

(iv): Show that y_t can be realized as an ARMA process

$$y_t + a y_{t-1} = w_t + c w_{t-1}$$

involving a *scalar* white noise w_t, in which c is a root of the function

$$\gamma(z) := \cancel{\sigma_2^2} + \sigma_2^2(1 + a z^{-1})(1 + a z) \,. \qquad [2]$$

σ_1^2

2. The output y_t of a stochastic linear discrete time system is related to the applied control u_t by the equation

$$y_t \;=\; u_{t-1} + e_t + 2e_{t-1} \;,$$

in which e_t is white noise with unit variance. Notice the one-step time delay in the control term.

The control system is operated in steady state wih the feedback control

$$u_t \;=\; -ky_t$$

for some constant gain parameter k, constrained to satisfy $|k| < 1$.

(i): Derive the output variance for the system under closed loop operation

$$\sigma_y^2(k) \;:=\; E[y_t^2] \;,$$

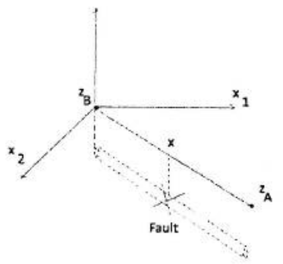
for a general value of k. [14]

(ii): Determine the value k^* of k which minimizes the output variance $\sigma_y(k)$ and also the minimum value of the variance, $\sigma_y(k^*)$. [4]

(iii): Explain why it is not practical to choose the feedback gain

$$k \;=\; 2 \;,$$

even though this value of the gain appears to give a lower output variance. [2]



3. A fault has occured in a straight water-pipe lying at constant depth under a flat field. A detector is used to measure the position $\mathbf{x}$, in 2D space, of a point in the field vertically above the fault. (See Figure.)

   N independent measurements, $\mathbf{y}_1, \ldots, \mathbf{y}_N$, of the position are made. The measurements are 2-vectors, modelled as

   $$\mathbf{y}_t \; = \; \mathbf{x} + \mathbf{e}_t \quad \text{for } t = 1, \ldots, N \, .$$

   Here the $\mathbf{e}_t$'s are independent, Gaussian, zero-mean 2-vector random variables, each with covariance matrix $\sigma^2 I_{2\times 2}$.

   Assume that the position $\mathbf{x}$ is modelled as

   $$\mathbf{x} \; = \; \mathbf{z}_B + \frac{\theta}{||\mathbf{z}_A - \mathbf{z}_B||}(\mathbf{z}_A - \mathbf{z}_B)$$

   where the 2-vectors $\mathbf{z}_A$ and $\mathbf{z}_B$ are two known locations in 2D space under which the pipe passes, and θ is an unknown parameter, giving the distance of the fault along the pipe from the point under $\mathbf{z}_B$.

   (i): Show that the linear least squares estimate $\hat{\mathbf{x}}$ is

   $$\hat{\mathbf{x}} \; = \; \mathbf{z}_B + \left(\frac{\frac{1}{N}\sum_{t=1}^{N}(\mathbf{y}_t - \mathbf{z}_B)^T(\mathbf{z}_A - \mathbf{z}_B)}{||\mathbf{z}_A - \mathbf{z}_B||^2} \right)(\mathbf{z}_A - \mathbf{z}_B) \; .$$

   [14]

   (ii): Show that the estimate $\hat{\mathbf{x}}$ of $\mathbf{x}$ is unbiased. [2]

   (iii): Now suppose that the variance $\sigma^2 = 100m^2$. How many measurements need to be taken, in order that the fault is determined within $1m$ with 0.95 probability, i.e such that

   $$P\{||\mathbf{x} - \hat{\mathbf{x}}|| \leq 1\} \; \geq \; 0.95 \; ?$$

   [4]

4. Consider a stationary, ergodic process y_t satifying

$$y_t + ay_{t-1} = n_t \tag{1}$$

in which n_t is a 'coloured noise' process satisfying

$$n_t + dn_{t-1} = e_t \ . \tag{2}$$

Here, e_t is white noise with unit variance. In these equations, a and d are unknown parameters, satisfying $|a| < 1$, $|d| < 1$.

(i): Show that the covariance function of y_t satisfies

$$\frac{R_y(1)}{R_y(0)} = -\frac{(a+d)}{(1+ad)} \ .$$

[2]

Hint: This formula can be simply derived from (1) and (2), without the need to calculate $R_y(0)$ and $R_y(1)$ separately.

(ii): An estimate $\hat{a}$ of the parameter a in (1) is obtained by the linear least squares method, without reference to the fact that the noise is coloured.

Show that the asymptotic bias in the estimate as $N \to \infty$ is

$$\hat{a} - a = \frac{d(1-a^2)}{1+ad} \ .$$

[10]

(iii): Describe in detail an algorithm for obtaining consistent estimates of both a and d. [8]

5. Measurements $y_1, \ldots, y_N$ are taken of a stationary, ergodic process described by the equation

$$y_t + a y_{t-1} \;=\; e_t \;,$$

in which a is a constant ($|a| < 1$) and e_t is Gaussian white noise with variance σ^2. a and σ^2 are to be regarded as unknown modelling parameters.

Derive the log likelihood function for a and σ^2 given the measurents $y_1, \ldots, y_N$:

$$LLF(a, \sigma^2) \;=\; p(y_1, \ldots, y_N \,|\, a, \sigma^2) \;.$$

(You may assume as starting value $y_0 = 0$.) [4]

Show that the Maximum Likelihood estimates $\hat{a}$ and $\hat{\sigma}^2$ of a and σ^2 given $(y_1, \ldots, y_N)$ are

$$\hat{a} \;=\; -\hat{R}_y(1) \,/\, \hat{R}_y(0)$$

and

$$\hat{\sigma}^2 \;=\; \hat{R}_y(0) - \hat{R}_y^2(1) \,/\, \hat{R}_y(0) \;,$$

where $\hat{R}_y(0)$ and $\hat{R}_y(1)$ are the sample covariances

$$\hat{R}_y(0) = \frac{1}{N}\sum_{t=1}^{N} y_t^2, \quad \hat{R}_y(1) = \frac{1}{N}\sum_{t=1}^{N} y_t y_{t-1} \;.$$

[10]

Show that the estimates $\hat{a}$ and $\hat{\sigma}^2$ are consistent, i.e., with probability one,

$$\hat{a} \to a \quad \text{and} \quad \hat{\sigma}^2 \to \sigma^2 \;.$$

[6]

[END]

Identification Exam 2011. Model Answers

1. (i) Eliminating x_t yields: $y_t - v_t + a(y_{t-1} - v_{t-1}) = e_t$

giving $y_t + a y_{t-1} = e_t + v_t + a v_{t-1}$ — (A)

(ii) $E\{(y_t + a y_{t-1})^2\} = E\{(e_t + v_t + a v_{t-1})^2\}$. So

$(1+a^2) R_y(0) + 2a R_y(1) = \sigma_1^2 + (1+a^2)\sigma_2^2$ — (1)

$E\{(y_t + a y_{t-1}) y_{t-1}\} = E\{(e_t + v_t + a v_{t-1}) y_{t-1}\}$

gives $R_y(1) + a R_y(0) = 0 + 0 + a R_{vy}(0)$

Also $E\{(y_t + a y_{t-1}) v_{t-1}\} = E\{(e_t + v_t + a v_{t-1}) v_{t-1}\}$

gives $a R_{vy}(0) = a \sigma_2^2$. Hence

$a R_y(0) + R_y(1) = a\sigma_2^2$ — (2)

(1) and (2) can be solved for $R_y(0)$ and $R_y(1)$:

$R_y(0) = \dfrac{\sigma_1^2 + (1-a^2)\sigma_2^2}{(1-a^2)} = (1-a^2)^{-1}\sigma_1^2 + \sigma_2^2$, $R_y(1) = -\dfrac{a}{(1-a^2)}\sigma_1^2$

From (A), $E\{(y_t + a y_{t-1}) y_{t-l}\} = R_y(l) + a R_y(l-1)$, $l \geq 2$

Hence, $R_y(l) = \dfrac{(-a)^l}{(1-a^2)}\sigma_1^2$ for $l = \pm 2, \pm 3, \ldots$

(iii) y_t can be described in terms of the vector noise process $\begin{pmatrix} e_t \\ v_t \end{pmatrix}$:

$$(1 + a z^{-1}) y_t = [1 \;\vdots\; 1 + a z^{-1}] \begin{bmatrix} e_t \\ v_t \end{bmatrix}$$

and so $y_t = G(z) \begin{bmatrix} e_t \\ v_t \end{bmatrix}$, where $G(z) = (1+az^{-1})^{-1}[1 \;\vdots\; 1+az^{-1}]$

Then
$$\Phi_y(\omega) = \frac{1}{(1+az^{-1})(1+az)} [1 \;\vdots\; 1+az^{-1}] \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1+az \end{bmatrix}\Bigg|_{z=e^{j\omega}}$$

$$= \frac{\sigma_1^2 + \sigma_2^2(1+az^{-1})(1+az)}{(1+az^{-1})(1+az)}\Bigg|_{z=e^{j\omega}} \quad — (3)$$

$$= \frac{(1 + a^2 + 2a\cos\omega)\sigma_2^2 + \sigma_1^2}{1 + a^2 + 2a\cos\omega}$$

(iv) (3) factorizes as constant $\times \dfrac{(1+cz^{-1})(1+cz)}{(1+az^{-1})(1+az)}$

where c is a root of $\gamma(z) = \sigma_1^2 + \sigma_2^2(1+az^{-1})(1+az)$.

$(1+az^{-1}) y_t = (1 + c z^{-1}) w_t$ (w_t white) provides a realisation by standard theory

2. Inserting $u_t = -k y_t$ into the system equations gives

$$y_t + k y_{t-1} = e_t + 2 e_{t-1}$$

Squaring both sides and taking expectations gives

$$(1+k^2) R_y(0) + 2k R_y(1) = 5 \qquad \text{— (1)}$$

$E\{\ldots \times y_{t-1}\}$ gives

$$R_y(1) + k R_y(0) = 0 + 2 R_{ye}(0)$$

But $E\{\ldots \times e_t\}$ gives

$$R_{ye}(0) + 0 = 1 + 0$$

So $\quad R_y(1) + k R_y(0) = 2 \qquad \text{— (2)}$

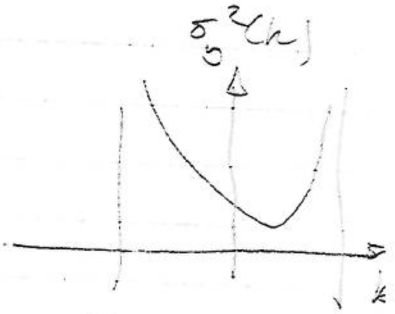
From (1) and (2)

$$(1+k^2) R_y(0) + 2k\left(2 - k R_y(0)\right) = 5$$

so $(1-k^2) R_y(0) = 5 - 4k$

and

$$\sigma_y^2(k) = E[y_t^2] = \frac{5-4k}{(1-k^2)}$$



$$\frac{\partial}{\partial k}(\;) = 0 \Rightarrow \cancel{\frac{-4}{(1-k^2)}} + \frac{(5-4k) \times 2k}{(1-k^2)^2} = 0$$

or $\quad (5-4k) \times 2k = 4(1-k^2)$

or $\quad 4k^2 - 10k + 4 = 0 \quad$ or $\quad k^2 - 2\frac{5}{4} + 1 = 0$

The roots are $\quad \frac{5}{4} \pm \sqrt{\frac{25}{16} - 1} = \frac{5}{4} \pm \frac{3}{4} = 2$ or $\frac{1}{2}$.

The values of $\sigma_k^2(k)$ are

$$\sigma_k^2(k=2) = 1 \quad \text{and} \quad \sigma_k^2(k=\tfrac{1}{2}) = \frac{5-2}{3/4} = 4$$

We cannot choose $k=2$, because this value violates the "stability" constraint on k. It is clear from the graph of $\sigma_y^2(k)$ however that $k = \frac{1}{2}$ is minimizing in the range $-1 < k < +1$

If you attempted to implement $u_t = 2y_t$, the actual closed loop system would be

$$\cancel{(1+2z^{-1})} y_t = \cancel{(1+2z^{-1})} e_t \Rightarrow y_t = e_t$$

which is stable. But even very small modelling errors for the transfer function $G(z) = (1+2z^{-1})$ would mean that the unstable pole $z = -2$ would not be cancelled, and the true closed loop system would be unstable.

3. (i) The measurements are related to x according to:

$$\begin{array}{l}\text{2-vectors}\end{array}\begin{bmatrix} y_1 - z_B \\ \vdots \\ y_N - z_B \end{bmatrix} = \|z_A - z_B\|^{-1} \begin{bmatrix} z_A - z_B \\ \vdots \\ z_A - z_B \end{bmatrix} \theta + \begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix} e$$

or $y' = X\theta + e$ where $y' = \text{col}\{y_t - z_B\}$, $X = \|z_A - z_B\|\ \text{col}\{z_A - z_B\}$

The LLS estimate $\hat{\theta}$ of θ, given y is

$$\hat{\theta} = (X^T X)^{-1} X^T y',$$

in which

$$(X^T X) = \left[(z_A - z_B)^T \dots (z_A - z_B)^T\right] \begin{bmatrix} z_A - z_B \\ \vdots \\ z_A - z_B \end{bmatrix} \times \frac{1}{\|z_A - z_B\|^2} = N$$

and

$$X^T y' = \left[(z_A - z_B)^T \dots (z_A - z_B)^T\right] \begin{bmatrix} y_1 - z_B \\ \vdots \\ y_N - z_B \end{bmatrix} \times \frac{1}{\|z_A - z_B\|} = \|z_A - z_B\|^{-1} \sum_{t=1}^{N} (y_t - z_B)^T (z_A -$$

So

$$\hat{\theta} = \|z_A - z_B\|^2 \times \frac{1}{N} \sum_{t=1}^{N} (y_t - z_B)^T (z_A - z_B)$$

The estimated position is then

$$\hat{x} = z_B + \left(\frac{\frac{1}{N} \sum_{t=1}^{N} (y_t - z_B)^T (z_A - z_B)}{\|z_A - z_B\|^2} \right) (z_A - z_B)$$

(ii) $$\hat{\theta} = (X^T X)^{-1} X^T y' = (X^T X)^{-1} X^T (X\theta + e) = \theta + (X^T X)^{-1} X^T e$$

So $E\{\hat{\theta}\} = \theta + 0$.

Then $$E\{\hat{x}\} = z_B + \frac{E[\hat{\theta}]}{\|z_A - z_B\|} (z_A - z_B)$$

$$= z_B + \frac{\theta}{\|z_A - z_B\|} (z_A - z_B) = x \quad (\text{unbiased})$$

(iii) The error variance for $\hat{\theta}$ is $E(\hat{\theta} - \theta)^2 = (X^T X)^{-1} \sigma^2 = \frac{\sigma^2}{N}$

But $\sigma^2 = 100$. So $\hat{\theta} - \theta \sim N(0, 100/N)$

Since θ measures unit length along pipe, we require $\hat{\theta} - \theta = \pm 1$ w.p. 0.95. Satisfied if $\hat{\theta} = \frac{1}{2}$ or

$$\sqrt{\frac{100}{N}} = \frac{1}{2} \implies \underline{N = 400}$$

4.(i) From $y_t + a y_{t-1} = n_t$ and $n_t + d n_{t-1} = e_t$
we deduce
$$(1 + a z^{-1})(1 + d z^{-1}) y_t = e_t \quad \text{or}$$
$$y_t + (a+d) y_{t-1} + a d y_{t-2} = e_t .$$
$E\{ \ldots \times y_{t-1} \} \Rightarrow$
$$E\{ y_t y_{t-1} + (a+d) y_{t-1} y_{t-1} + a d y_{t-1} y_{t-2} \} = E\{ e_t y_{t-1} \}$$
or $(1 + ad) R_y(1) + (a+d) R_y(0) = 0$
Hence
$$R_y(1) / R_y(0) = - \frac{a+d}{1+ad}$$

(ii) The LLSE of a based on $y_t + a y_{t-1} =$ 'noise' however is
$$\hat{a} = - \Sigma_{t=1}^{N} y_t y_{t-1} \Big/ \Sigma_{t=1}^{N} y_t^2 = - \frac{\hat{R}_y(1)}{\hat{R}_y(0)}$$
By ergodicity, as $N \to \infty$,
$$\hat{a} \to - \frac{R_y(1)}{R_y(0)} = \frac{a+d}{1+ad} = a + \frac{d(1-a^2)}{(1+ad)}$$
So the asymptotic bias is $\frac{d(1-a^2)}{1+ad}$

(iii) The Generalized Least Squares algorithm can give consistent estimates of a and d. This generates a sequence of "improved" estimates $\hat{a}_0, \hat{d}_1, \hat{a}_1, \hat{d}_2, \hat{a}_2, \ldots$, given a starting value d_0.

Step 1 Filter data: $y_t^{(0)} = (1 + d_0 z^{-1}) y_t$.
Obtain LLS estimate $\hat{a}_0$ from
$$(1 + \hat{a}_0 z^{-1}) y_t^{(0)} = \text{'error'}$$
Step 2 Calculate residuals $e_t^{(0)} = (1 + \hat{a}_0 z^{-1}) y_t$
and obtain LLS estimate $\hat{d}_1$ from
$$(1 + \hat{d}_1 z^{-1}) e_t^{(0)} = \text{"error"}$$
Go back to step 1, replacing $\hat{d}_0$ by $\hat{d}_1$ and so on.
Continue until estimates converge and $e_t^{(i)}$ satisfies a whiteness test. Algorithm is not guaranteed to converge

5. If y_t satisfies $y_t + a y_{t-1} = e_t$, $t = 0, 1, \ldots, N$ then

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ a & 1 & & \\ 0 & \ddots & a & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix} . \text{ Write } A = \begin{bmatrix} 1 & & 0 \\ a & \ddots & \\ 0 & a & 1 \end{bmatrix}, \; y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$$

So the log likelihood function of a and σ^2 given $\{y_t\}$ is

$$LLF(a, \sigma^2) = \log p(y_1, \ldots, y_N | a, \sigma^2)$$
$$= \log\left(\frac{1}{(2\pi)^{N/2}}\right) - \log(\sigma^N) - \frac{1}{2\sigma^2} y^T \left((A^{-1})(A^{-1})^T\right)^{-1} y =$$

$$= \underline{\text{constant} - \log(\sigma^N) - \frac{1}{2\sigma^2} y^T A^T A y}$$

But $Ay = [y_1 + a y_0, y_2 + a y_1, \ldots, y_N + a y_{N-1}]^T$ (where $y_0 = 0$)

So $y^T A^T A y = \sum_{i=1}^{N} |y_t + a y_{t-1}|^2 = N\left(\hat{R}_y(0) + 2a\hat{R}_y(1) + a^2 \hat{R}_y(0)\right)$

The maximum likelihood estimate $\hat{a}$ of a is given by setting

$\partial/\partial a \; LLF(a, \sigma^2) = 0$

This gives $\underline{\hat{a} = -\hat{R}_y(1) / \hat{R}_y(0)}$

Setting $\partial/\partial\sigma \; LLF(a, \sigma^2) = 0$ yields

$-N \hat{\sigma}^{-N} \hat{\sigma}^{N-1} + 1/\hat{\sigma}^3 \|\hat{A} y\|^2 = 0$ where $\hat{A} = \begin{pmatrix} 1 & & 0 \\ \hat{a} & \ddots & \\ 0 & \hat{a} & 1 \end{pmatrix}$

or $\hat{\sigma}^2 = \frac{1}{N}\|\hat{A} y\|^2 = \frac{1}{N} \sum |y_t + \hat{a} y_{t-1}|^2$

$$= \hat{R}_y(0) + 2\hat{a}\hat{R}_y(1) + \hat{a}^2 \hat{R}_y(0)$$
$$= \underline{\hat{R}_y(0) - \frac{\hat{R}_y^2(1)}{\hat{R}_y(0)}}$$

But, $E[(y_t + a y_{t-1})^2] = E[e_t^2] \Rightarrow \sigma^2 = (1 + a^2) R_y(0) + 2a R_y(1)$

and $E[(y_t + a y_{t-1}) y_{t-1}] = E[e_t y_{t-1}] = 0 \Rightarrow R_y(1) + a R_y(0) = 0$

Hence $a = -\frac{R_y(1)}{R_y(0)}$ and $\sigma^2 = \hat{R}_y(0) - \left(\frac{R_y(1)}{R_y(0)}\right)^2 \cdot R_y(0)$.

By ergodicity $\hat{R}_y(0) \to R_y(0)$, $\hat{R}_y(1) \to R_y(1)$ as $N \to \infty$

So $\hat{a} \to a$, $\hat{\sigma}^2 \to \sigma^2$ as $N \to \infty$ (consistency).