EE II(3)

	EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2007-08	Course
	EE2 - MATHS PAPER 3 - SOLUTIONS 2008	1 Core
Question 8		Marks & seen/unseen
Parts		

a) Take the principal branch $-\pi < \arg z \le \pi$. which is well-defined everywhere except at $z=0$ — (2)

b) Yes, since $\frac{dw}{dz} = 1/z$, mapping is conformal except at $z=0$. — (4)

c) If $z = re^{i\theta}$

then $w = u + iv = \log r + i\theta. \Rightarrow \begin{matrix} u=\log r \\ v=\theta \end{matrix} \Rightarrow$ — (1)

If $\theta = 0$, the positive real axis is mapped into the u-axis. — (2)

d) A straight line $\theta = \alpha$, $-\pi < \theta \le \pi$ in the z-plane is mapped into the straight line $v = \alpha$ in the w-plane. — (2)

e) A ~~center~~ circle with radius $r = a > 0$ & center at the origin is mapped onto the line segment $\log a + i\theta$ $-\pi < \theta \le \pi$ in the w-plane. — (4)

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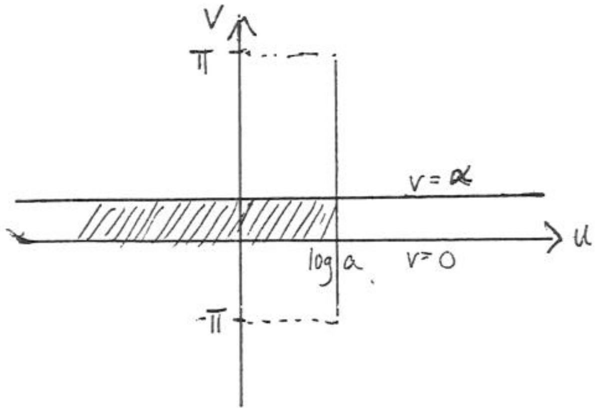
Parts f)		5
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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course EE2 Pap 3 (2)
Question 2		Marks & seen/unseen
Parts	$$f(z) = \frac{z}{(z-1)^2(z-i)^2}$$	
i)	Res at the double pole at $z=1$ is $$\lim_{z\to 1}\frac{d}{dz}\left\{(z-1)^2 f(z)\right\} = \lim_{z\to 1}\frac{d}{dz}\left[\frac{z}{(z-i)^2}\right]$$ $$= \lim_{z\to 1}\left\{\frac{(z-i)^2 - 2z(z-i)}{(z-i)^4}\right\} = -\lim_{z\to 1}\frac{(z+i)}{(z-i)^3}$$ $$= -\frac{(1+i)}{(1-i)^3} = -(1+i)(1-3i-3+i)^{-1} = \frac{1}{2}$$	7
ii)	Res at the double pole at $z=i$ is $$\lim_{z\to i}\frac{d}{dz}\left\{(z-i)^2 f(z)\right\} = \lim_{z\to i}\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$ $$= \lim_{z\to i}\frac{(z-1)^2 - 2z(z-1)}{(z-1)^4} = -\lim_{z\to i}\frac{z+1}{(z-1)^3}$$ $$= -\frac{i+1}{(i-1)^3} = \frac{i+1}{(1-i)^3} = -1/2 \text{ from (i)}$$	7
iii)	$$\oint_C f(z)\,dz = 2\pi i\left\{\tfrac{1}{2} - \tfrac{1}{2}\right\} \text{ by R.T.}$$ $$= 0$$ as both poles at $z=1$ & $z=i$ lie within C.	6
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EXAMINATION QUESTIONS/SOLUTIONS 2007-08

Course 3

EE2 pap 3

Question 3

Marks & seen/unseen

Parts

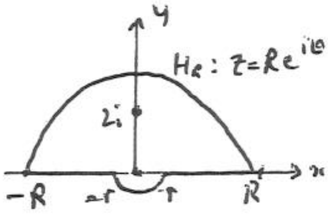
i) $F(z) = \dfrac{e^{iz}}{z(z^2+4)}$ Pole at $z=0$

Poles at $z = \pm 2i$ — 3

ii) Res at $z=0$ is $\lim_{z\to 0}\left\{\dfrac{z e^{iz}}{z(z^2+4)}\right\} = 1/4$ — 2

" " $z = 2i$ is $\lim_{z\to 2i}\left\{\dfrac{(z-2i)e^{iz}}{z(z^2+4)}\right\} = -\dfrac{e^{-2}}{8}$ — 3

iii) $\oint_c \dfrac{e^{iz}}{z(z^2+4)} \overset{R.T.}{=} 2\pi i\left\{\dfrac{1}{4} - \dfrac{e^{-2}}{8}\right\}$ — 2



Now split up the contour into

$$\oint_c = \left(\int_{-R}^{-r} + \int_r^R\right)\frac{e^{ix}dx}{x(x^2+4)}$$

indent: $z = re^{i\theta}$

$\pi \le \theta \le 2\pi$

$$+ \int_{H_R}\frac{e^{iz}dz}{z(z^2+4)} + \int_{\smile}\frac{e^{iz}dz}{z(z^2+4)}$$ — 4

By Jordan's Lemma $\lim_{R\to 0}\int_{H_R} f(z)e^{imz}dz = 0$

because a) only singularities are poles

b) $m = 1 > 0$ — 2

c) $|f(z)| \to 0$ fast enough as $R \to 0$.

Thus we take the 2 limits $R\to\infty$, $r\to 0$

$$\therefore 2\pi i\left\{\frac{1}{4} - \frac{e^{-2}}{8}\right\} = \int_{-\infty}^{\infty}\frac{e^{ix}dx}{x(x^2+4)} + \lim_{r\to 0}\int_{\smile}\frac{e^{iz}dz}{z(z^2+4)}$$

and $\displaystyle\lim_{r\to 0}\int_{\pi}^{2\pi}\frac{e^{ire^{i\theta}}\, iz\,d\theta}{z(r^2e^{2i\theta}+4)} = \frac{i}{4}\int_{\pi}^{2\pi}d\theta = \frac{\pi i}{4}$ — 2

$$\therefore \int_{-\infty}^{\infty}\frac{e^{ix}dx}{x(x^2+4)} = \pi i\left\{\frac{1}{4} - \frac{e^{-2}}{4}\right\}$$

Now $\int_{-\infty}^{\infty} f(x)\cos x\,dx = 0$ as $f(x)$ is odd, thus

$$\int_{-\infty}^{\infty}\frac{\sin x\,dx}{x(x^2+4)} = \frac{\pi}{4}\left\{1 - e^{-2}\right\}$$ — 2

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Question 4

Marks & seen/unseen

Parts

$$\int_{-\infty}^{\infty} \frac{e^{iqt}}{t}\,dt = \int_{t=-\infty}^{t=+\infty} \frac{e^{i\theta}}{\theta}\,d\theta \qquad \theta = qt$$

$q>0 \quad t \in [-\infty,\infty] \rightarrow \theta \in [-\infty,\infty]$

$q<0 \qquad\qquad\qquad \rightarrow \theta \in [\infty,-\infty]$

Thus $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iqt}\,dt}{t} = \mathrm{sgn}[q]\int_{-\infty}^{\infty} \frac{e^{i\theta}}{\theta}\,d\theta = i\pi\,\mathrm{sgn}[q]$ 6

F.T. of $\dfrac{\sin\frac{1}{2}t}{\frac{1}{2}t}$ is $\displaystyle 2\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{t}\sin(\tfrac{1}{2}t)\,dt$ 2

$$= \frac{1}{i}\int_{-\infty}^{\infty} e^{-i\omega t}\,\frac{\left(e^{\frac{1}{2}it} - e^{-\frac{1}{2}it}\right)}{t}\,dt$$

$$= \frac{1}{i}\left\{ \underbrace{\int_{-\infty}^{\infty} \frac{e^{i(\frac{1}{2}-\omega)t}}{t}\,dt}_{1} - \underbrace{\int_{-\infty}^{\infty} \frac{e^{-i(\frac{1}{2}+\omega)t}}{t}\,dt}_{2} \right\}$$ 4

Now

$$1 = \begin{cases} i\pi & \omega < \frac{1}{2} \\ -i\pi & \omega > \frac{1}{2} \end{cases} \qquad 2 = \begin{cases} i\pi & \omega < -\frac{1}{2} \\ -i\pi & \omega > -\frac{1}{2} \end{cases}$$ 4

$$\therefore \text{F.T} = \begin{cases} 0 & \omega < -\frac{1}{2} & \text{Cancellation} \\ 0 & \omega > \frac{1}{2} & \text{Cancellation} \\ 2\pi & -\frac{1}{2} < \omega < \frac{1}{2} & \text{Addition.} \end{cases}$$ 4

	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course (5) EE2 Pap3
Question 5		Marks & seen/unseen
Parts		
i)	$\int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt$	
	$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \bar{f}(\omega)\, e^{+i\omega t} d\omega\right)\left(\int_{-\infty}^{\infty} \bar{g}(\omega')\, e^{i\omega' t} d\omega'\right)^* dt$	Seen.
	$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{f}(\omega)\, \bar{g}^*(\omega')\, e^{i(\omega-\omega')t}\, dt\, d\omega'\, d\omega$	
	Given that $\int_{-\infty}^{\infty} e^{i(\omega-\omega')t} dt = 2\pi\,\delta(\omega-\omega')$ we have	4
	$\int_{-\infty}^{\infty} f(t) g^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{f}(\omega)\, \bar{g}^*(\omega')\, \delta(\omega-\omega')\, d\omega' d\omega$	
	$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(\omega)\, \bar{g}^*(\omega)\, d\omega \quad \square$	4
ii)	With $f(t) = e^{-\lvert t\rvert}$ $\qquad \lvert t\rvert = \begin{cases} t & t>0 \\ -t & t<0 \end{cases}$	
	$\bar{f}(\omega) = \int_{-\infty}^{0} e^{+(1-i\omega)t} dt + \int_{0}^{\infty} e^{-(1+i\omega)t} dt$	Unseen.
	$= \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \frac{2}{1+\omega^2}.$	4
	$\bar{g}(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-i\omega t}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) dt$	
	$= \pi \left\{ \delta(\omega-\omega_0) + \delta(\omega+\omega_0) \right\}$	4
	$\therefore \int_{-\infty}^{\infty} e^{-\lvert t\rvert} \cos\omega_0 t\, dt = \frac{\pi}{2\pi}\int_{-\infty}^{\infty} \frac{2[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]}{1+\omega^2} d\omega$	
	$= \frac{2}{1+\omega_0^2}$	4
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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course 6 EE2 Pap3
Question 6		Marks & seen/unseen
Parts	$\mathcal{L}(\dot{x}) = s\bar{x}(s) - x(0)$ Table $\mathcal{L}(\ddot{x}) = s^2\bar{x}(s) - sx(0) - \dot{x}(0)$	
	$\mathcal{L}(\dddot{x}) = \int_0^\infty e^{-st}\dddot{x}\,dt = \int_0^\infty e^{-st} d(\ddot{x})$	
	$= [\ddot{x}\, e^{-st}]_0^\infty + s\,\mathcal{L}(\ddot{x})$	
	$= s^3\bar{x}(s) - s^2 x(0) - s\dot{x}(0) - \ddot{x}(0)$	5
	Now $x(0) = \dot{x}(0) = \ddot{x}(0) = 0$; thus LT the ODE	
	$(s^3 + 3s^2 + 3s + 1)\bar{x}(s) = \bar{f}(s)$	
	$\therefore \bar{x}(s) = \dfrac{\bar{f}(s)}{(s+1)^3}$	5
	Now we know i) $\mathcal{L}(t^n) = n!/s^{n+1}$ (table)	2
	ii) Shift Thm $\mathcal{L}(e^{at}f(t)) = \bar{f}(s-a)$ (table)	2
	$\therefore \mathcal{L}\left[\frac{1}{2}e^{-t}t^2\right] = \dfrac{1}{(s+1)^3}$ choosing $n=2$ in (i) and $a=-1$ in (ii)	2
	If $\bar{x}(s) = \bar{f}(s)\bar{g}(s)$ where $f(t)$ given, $g(t) = \frac{1}{2}e^{-t}t^2$	
	The Lap. conv. thm says	
	$x(t) = \int_0^t f(t-u)\,g(u)\,du = f * g$	
	Thus $x(t) = \frac{1}{2}\int_0^t f(t-u)\,e^{-u}u^2\,du.$	4
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Course 7 EE2 Pap3

Question 7

Marks & seen/unseen

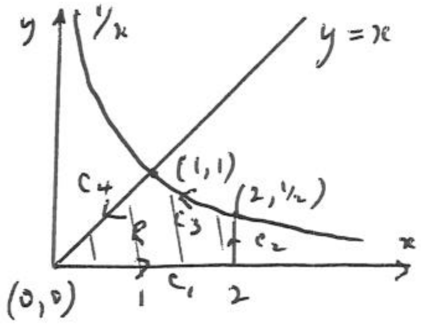
Parts

Choose $\underline{v} = \underline{i}P + \underline{j}Q \Rightarrow \text{curl}\,\underline{v} = \underline{k}(Q_x - P_y)$

$\therefore \left.\begin{array}{l}\underline{v}\cdot d\underline{r} = P\,dx + Q\,dy \\ \underline{k}\cdot\text{curl}\,\underline{v} = Q_x - P_y\end{array}\right\}$ $\therefore$ G.T. $\Rightarrow$ 2D, Stokes' Thm. — 6

When $\underline{v} = \frac{1}{2}(y^2\underline{\hat{i}} + x^2\underline{\hat{j}})$: $\text{curl}\,\underline{v} = \underline{k}(x-y)$

$\therefore \iint_R (x-y)\,dx\,dy = \frac{1}{2}\oint_C (y^2dx + x^2dy)$ — 2



C_1 : $y=0 \quad 0\le x \le 2$

$\int_{C_1} = 0$ — 2 (pic)

C_2 : $x=2 \quad \int_{C_2} = 2\int_0^{1/2} dy = 1$

C_3 : $y = 1/x \quad dy = -dx/x^2$

$\int_{C_3} = \frac{1}{2}\int_2^1 \left(\frac{dx}{x^2} - dx\right) = -\frac{1}{2}\left[x + \frac{1}{x}\right]_2^1 = \frac{1}{2}\left[x+\frac{1}{x}\right]_1^2$ — 4×2 (for each $\int_c$)

$= \frac{1}{2}(2+\frac{1}{2}) - \frac{1}{2}(1+1) = 1/4$

C_4 : $y = x$: $dy = dx \quad \int_{C_4} = \frac{1}{2}\int_1^0 x^2dx + x^2dx$

$\therefore \int_{C_4} = -\int_0^1 x^2dx = -1/3$ — 2

Total $= 0 + 1 + 1/4 - 1/3 = 5/4 - 1/3 = 11/12$.

Via the double integral is also acceptable:

$\iint_R (x-y)\,dx\,dy = \int_0^1 dx\left\{\int_0^x (x-y)\,dy\right\} + \int_1^2\left\{\int_0^{1/x}(x-y)\,dy\right\}dx$

$= \int_0^1\left\{x^2 - \frac{1}{2}x^2\right\}dx + \int_1^2\left(1 - \frac{1}{2}\frac{1}{x^2}\right)dx$

$= \frac{1}{6} + 1 + \frac{1}{2}\left(\frac{1}{2} - 1\right) = 1/6 + 3/4 = \frac{22}{24} = 11/12$

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2nd Year.

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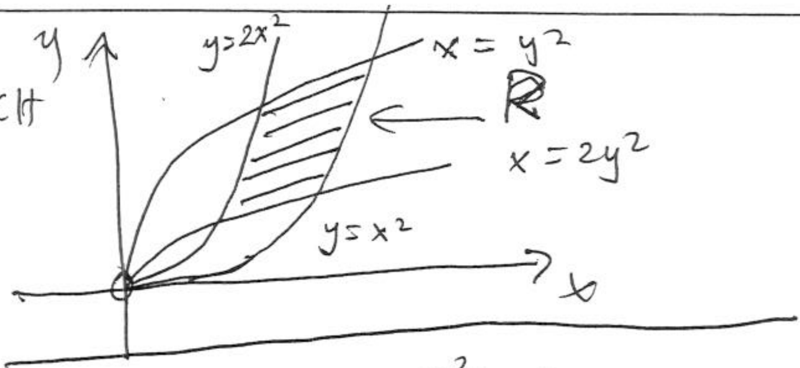
Solution C3

Marks & seen/unseen

Parts

(i) SKETCH



When $y=x^2$ $u = \frac{x^2}{y} = 1$

when $y=2x^2$ $u = \frac{x^2}{y} = \frac{1}{2}$

when $x=y^2$ $v = \frac{y^2}{x} = 1$

When $x=2y^2$ $v = \frac{y^2}{x} = \frac{1}{2}$ ~~3~~ 4

$x^3 = \left(\frac{x^2}{y}\right)^2\left(\frac{y^2}{x}\right) = u^2 v \Rightarrow x = (u^2 v)^{1/3}$ ~~3~~ 4

$y^3 = \left(\frac{y^2}{x}\right)^2\left(\frac{x^2}{y}\right) = v^2 u \Rightarrow y = (uv^2)^{1/3}$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3}u^{-1/3}v^{1/3} & \frac{1}{3}u^{2/3}v^{-2/3} \\ \frac{1}{3}u^{-2/3}v^{2/3} & \frac{2}{3}u^{1/3}v^{-1/3} \end{vmatrix}$$

$$= \frac{4}{9} - \frac{1}{9} = \frac{1}{3}$$

Or else use $J = \frac{1}{J'}$

where $J' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 3$

~~3~~ 4

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Solution ~~Question~~ C3		Marks & seen/unseen

Parts

~~Then~~ Further $xy = uv$

So Int $= \int_{\frac12}^{1}\int_{\frac12}^{1} v\, e^{uv} J\, du\, dv$

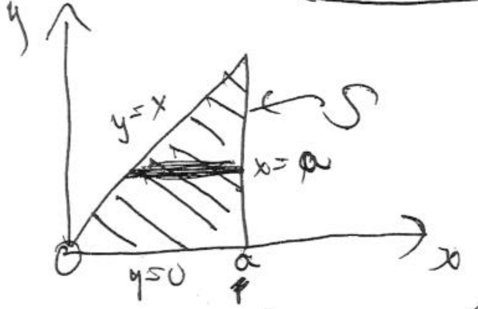
$= \frac{1}{3}\int_{\frac12}^{1} v \left[\frac{1}{v} e^{uv}\right]_{u=\frac12}^{1} dv$ 4

$= \frac{1}{3}\int_{\frac12}^{1} \left(e^{v} - e^{+\frac12 v}\right) dv$ 3

$= \frac{1}{3}\left[e^{v} - 2e^{\frac12 v}\right]_{\frac12}^{1} = \left(e - 2e^{\frac12} - e^{\frac12} + 2e^{\frac14}\right)/3$

$= \left(e - 3e^{\frac12} + 2e^{\frac14}\right)/3.$

(ii)



omit

$I_{Oz} = \int_{y=0}^{y=a}\left[\int_{x=y}^{x=a} (x^2+y^2)\, dx\right] dy$

$= \int_{y=0}^{a} \left[\frac{x^3}{3} + xy^2\right]_{x=y}^{x=a} dy$

$= \int_{0}^{a} \left[\frac{a^3}{3} + ay^2 - \frac{y^3}{3} - y^3\right] dy$ 5

$= a^4\left(\frac13 + \frac13 - \frac{1}{12} - \frac{1}{4}\right) = \frac{1}{3} a^4$ 20

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Question C4

Solution

Marks & seen/unseen

Parts

$$\operatorname{grad}\varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$$

$$\operatorname{curl}\operatorname{grad}\varphi = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_x\varphi & \partial_y\varphi & \partial_z\varphi \end{vmatrix}$$

$$= \underline{i}\,(\partial_y\partial_z\varphi - \partial_z\partial_y\varphi) - \underline{j}\,(\partial_x\partial_z\varphi - \partial_z\partial_x\varphi) + \underline{k}\,(\partial_x\partial_y\varphi - \partial_y\partial_x\varphi) = 0.$$

4

$$\operatorname{curl}\underline{E} = \underline{i}\,(\partial_y E_3 - \partial_z E_2) - \underline{j}\,(\partial_x E_3 - \partial_z E_1) + \underline{k}\,(\partial_x E_2 - \partial_y E_1)$$

$$\operatorname{div}\operatorname{curl}\underline{E} = \partial_x(\partial_y E_3 - \partial_z E_2) - \partial_y(\partial_x E_3 - \partial_z E_1) + \partial_z(\partial_x E_2 - \partial_y E_1)$$

cancellation of terms in pairs gives result.

4

From $\underline{E} = \underline{A} + \operatorname{grad}\varphi$ it follows that $\operatorname{curl}\underline{E} = \operatorname{curl}A + \operatorname{curl}\operatorname{grad}\varphi$. Last term is zero so result ~~fol~~ follows.

3

$$\operatorname{curl}\underline{E} = \underline{i}(0) - \underline{j}(0) + \underline{k}\,(a e^{ax}\sin y + e^{ax}\sin y)$$

$$\operatorname{curl}\underline{A} = \underline{i}(1-1) - \underline{j}(0) + \underline{k}\,(+2 e^{ax}\sin y)$$

These are equal if $a = 1$.

4

Parts

If $a=1$ $\quad \text{grad}\,\varphi = \underset{\sim}{E} - \underset{\sim}{A}$ so

$$\frac{\partial \varphi}{\partial x} = -e^{x}\cos y - x$$

$$\frac{\partial \varphi}{\partial y} = e^{x}\sin y - z$$

$$\frac{\partial \varphi}{\partial z} = z^2 - y$$

Thus $\varphi(x,y,z) = -e^{x}\cos y - \frac{x^2}{2} + f(y,z)$

$$\varphi(x,y,z) = -e^{x}\cos y - zy + g(x,z)$$

$$\varphi(x,y,z) = \frac{z^3}{3} - yz + h(x,y)$$

By comparison (or otherwise)

$$\varphi(x,y,z) = -e^{x}\cos y + \frac{z^3}{3} - yz - \frac{x^2}{2} + C.$$

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Parts

$$\frac{\partial g}{\partial x} = -e^{x}(y\cos y + x\sin y + \sin y)$$
$$\frac{\partial f}{\partial y} = e^{x}(-x\sin y - \sin y - y\cos y)$$

} Equal. Hence conservative. **3**

$$\frac{\partial \varphi}{\partial x} = f = e^{x}(x\cos y - y\sin y) \quad \text{—}(1)$$
$$\frac{\partial \varphi}{\partial y} = g = -e^{x}(y\cos y + x\sin y) \quad \text{—}(2)$$

Integrate (1) w.r.t. x to get

$$\varphi(x,y) = e^{x}(x\cos y - y\sin y) - \int e^{x}\cos y\,dx$$
$$= e^{x}(x\cos y - y\sin y - \cos y) + h(y) \quad \text{—}(3)$$

Substitute (3) into (2) to get

$$e^{x}(-x\sin y - \sin y - y\cos y + \sin y) + \frac{dh}{dy}$$
$$= -e^{x}(y\cos y + x\sin y)$$

$$\Rightarrow \frac{dh}{dy} = 0 \Rightarrow h(y) = C$$ **7**

Thus $\varphi(x,y) = e^{x}(x\cos y - y\sin y - \cos y) + C$

Line integral is <u>independent of path</u> for a <u>conservative</u> field.

Hence $\text{Int} = \varphi(B) - \varphi(A)$

$= \varphi(1,\frac{\pi}{[illegible]}) - \varphi(0,0) = $ [illegible]

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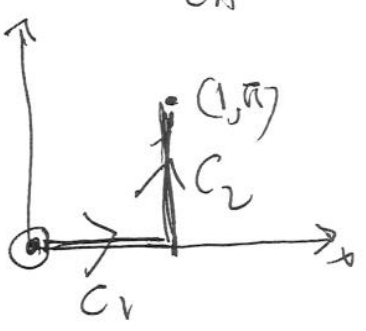
Parts

$\text{Int} = e\left(1(-1) - 1.0 - (-1)\right) - 1\left(0.1 - 0.0 - 1\right)$ **3**

$= 1$

Second integral $=$ First integral $+ \int_{CA}^{B} \left(x\,dx + \sin\frac{y}{2}\,dy\right)$

C is path shown



$\therefore$ Second int $= 1 + \int_{C_1} + \int_{C_2}$

On C_1 $y = 0$; $dy = 0$ x ranges from 0 to 1

On C_2 $x = 1$, $dx = 0$ y ranges from 0 to π

Thus Second int $= 1 + \int_0^1 x\,dx + \int_0^{\pi} \sin\frac{y}{2}\,dy$

$= 1 + \frac{1}{2} + \left[-2\cos\frac{y}{2}\right]_0^{\pi}$

$= 1 + \frac{1}{2} + 2 = 3\frac{1}{2}$. **7**

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Question 11

Marks & seen/unseen

Parts

(i) $1 = \int_0^\pi k\sin(x)dx = -k\cos(x)|_{x=0}^{\pi} = 2k.$
Hence, $k = \frac{1}{2}$. 3

(ii) $\mathrm{P}(1 \le X \le 4) = \int_1^\pi \frac{\sin(x)}{2}dx = -\frac{1}{2}\cos(x)|_{x=1}^{\pi} = \frac{1}{2}(\cos(1)+1) \approx 0.770$ 4

(iii) $\mathrm{P}(X > 2|X > 1) = \frac{\mathrm{P}(X>2)}{\mathrm{P}(X>1)} = \frac{\int_2^\pi \frac{\sin(x)}{2}dx}{\frac{1}{2}(\cos(1)+1)} = \frac{\frac{1}{2}(\cos(2)+1)}{\frac{1}{2}(\cos(1)+1)} \approx 0.380$ 4

(iv) $\mathrm{E}(X) = \int_0^\pi x\sin(x)/2dx = -x\cos(x)/2|_{x=0}^{\pi} - \int_0^\pi -\cos(x)/2dx$
$= \frac{\pi}{2} + \sin(x)/2|_{x=0}^{\pi} = \frac{\pi}{2} \approx 1.57$ 4

$\mathrm{E}(X^2) = \int_0^\pi x^2\sin(x)/2dx = \frac{1}{2}[(2-x^2)\cos(x) + 2x\sin(x)]_{x=0}^{\pi}$
$= \frac{1}{2}[(2-\pi^2)(-1) - 2] = \pi^2/2 - 2 \approx 2.93$ 3

$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \pi^2/2 - 2 - \pi^2/4 = \frac{1}{4}\pi^2 - 2 \approx 0.467$ 2

seen similar

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EXAMINATION SOLUTIONS 2007-08

Course EE2(3) 12

Question 12

Marks & seen/unseen

Parts

(i) A time series $\{e_t\}$ is called white noise if
$\mathrm{E}(e_t) = 0$ for all t,
$\mathrm{cov}(e_t, e_s) = 0$ for all $t \neq s$
$\mathrm{Var}(e_t)$ does not depend on t. 3

Seen similar

(ii)
$$\gamma(t,t) = \mathrm{Var}(y_t) = 0.3^2\,\mathrm{Var}(e_t) + 0.5^2\,\mathrm{Var}(e_{t-1}) + 0.2^2\,\mathrm{Var}(e_{t-2})$$
$$= 0.3^2 + 0.5^2 + 0.2^2 = 0.38$$
$$\gamma(t,t+1) = \mathrm{cov}(y_t, y_{t+1}) = 0.3 \cdot 0.5\,\mathrm{cov}(e_t, e_t) + 0.5 \cdot 0.2\,\mathrm{cov}(e_{t-1}, e_{t-1})$$
$$= 0.3 \cdot 0.5 + 0.5 \cdot 0.2 = 0.25$$
$$\gamma(t,t+2) = 0.3 \cdot 0.2 = 0.06$$
$$\gamma(t,t+k) = 0 \quad \text{for } k = 3, 4, \ldots$$
8

(iii) The covariance $\gamma(t, t+s)$ is independent of t by (ii).

$\mu_t = \mathrm{E}(y_t) = 0.3\,\mathrm{E}(e_t) + 0.5\,\mathrm{E}(e_{t-1}) + 0.2\,\mathrm{E}(e_{t-2}) = 0$

Since both μ_t and $\gamma(t, t+s)$ does not depend on t, the time series is stationary. 3

(iv)
$\gamma_1 = 0.25/0.38 \approx 0.658$
$\gamma_2 = 0.06/0.38 \approx 0.157$
$\gamma_k = 0$ for $k = 3, 4, \ldots$. 3

(v) The spectrum is given by

$$f(\omega) = \gamma_0 + 2\sum_{k=1}^{\infty} \gamma_k \cos(k\omega) = 0.38 + 0.5\cos(\omega) + 0.12\cos(2\omega)$$

3

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Question		Marks & seen/unseen
1		

Parts

i) $A\underline{e}_i = \lambda_i \underline{e}_i$ — ① Transpose & c.c.

$\underline{e}_i^{*T} A^{T*} = \lambda_i^* \underline{e}_i^{*T}$ RH multiply by $\underline{e}_i$

$\therefore \underline{e}_i^{*T} A^{T*} \underline{e}_i = \lambda_i^* \underline{e}_i^{*T} \underline{e}_i$ — ② 4

Now take ① & LH multiply by $\underline{e}_i^{*T}$

$\underline{e}_i^{*T} A \underline{e}_i = \lambda_i \underline{e}_i^{*T} \underline{e}_i$ — ③

Compare ② & ③ using $A^{*T} = A$: $\Rightarrow \lambda_i = \lambda_i^*$ 4

Hence λ_i are real.

ii) Take ① & LH multiply by $\underline{e}_j^T$

$\underline{e}_j^T A \underline{e}_i = \lambda_i \underline{e}_j^T \underline{e}_i$ — ④ 4

& consider $A\underline{e}_j = \lambda_j \underline{e}_j \Rightarrow \underline{e}_j^T A = \lambda_j \underline{e}_j^T$

RH multiply this by $\underline{e}_i$

$\underline{e}_j^T A \underline{e}_i = \lambda_j \underline{e}_j^T \underline{e}_i$ — ⑤

Subtract ⑤ from ④ : $\underline{e}_j^T \underline{e}_i (\lambda_j - \lambda_i) = 0$ 4

We know $\lambda_i \neq \lambda_j \Rightarrow \underline{e}_j^T \underline{e}_i = 0$ — ⑥

iii) $P = \{\underline{e}_1\ \underline{e}_2 \dots \underline{e}_n\}$ a row of col-vecs

$P^T = \begin{pmatrix} \underline{e}_1^T \\ \underline{e}_2^T \\ \vdots \\ \underline{e}_n^T \end{pmatrix}$ Col of row-vecs. $\therefore P^T P = \{\underline{e}_i^T \underline{e}_j\} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I$ from ⑥ 4

Question 2 | Marks & seen/unseen

Parts

$$A = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 1 & \sqrt{2} \\ 0 & \sqrt{2} & 1 \end{pmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & \sqrt{2} & 0 \\ \sqrt{2} & 1-\lambda & \sqrt{2} \\ 0 & \sqrt{2} & 1-\lambda \end{vmatrix} = 0$$

$\therefore (1-\lambda)\left[(1-\lambda)^2-2\right] = \sqrt{2}\left[\sqrt{2}(1-\lambda)\right]$

$\therefore \lambda = 1$ and $(1-\lambda)^2 = 4 \quad \therefore \lambda^2 - 2\lambda - 3 = 0 \quad \lambda = -1,\ \lambda = 3$ — 3

$\lambda_1 = 3 \quad \lambda_2 = 1 \quad \lambda_3 = -1$

$\lambda_1 = 3 \quad \begin{pmatrix} -2 & \sqrt{2} & 0 \\ \sqrt{2} & -2 & \sqrt{2} \\ 0 & \sqrt{2} & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad \begin{matrix} \sqrt{2}a = b \\ c + a = \sqrt{2}b \\ b = \sqrt{2}c \end{matrix} \quad \underline{e}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$ — 2

$\lambda_2 = 1 \quad \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad \begin{matrix} b = 0 \\ c = -a \\ b = 0 \end{matrix} \quad \underline{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ — 2

$\lambda_3 = -1 \quad \begin{pmatrix} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & \sqrt{2} \\ 0 & \sqrt{2} & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad \begin{matrix} b = -\sqrt{2}a \\ a + c = -\sqrt{2}b \\ b = -\sqrt{2}c \end{matrix} \quad \underline{e}_3 = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$ — 2

$Q = \underline{x}^T A \underline{x}$ with A as above; write $\underline{x} = P\underline{y}$ — 2

$\underline{x}^T = \underline{y}^T P^T \Rightarrow Q = \underline{y}^T (P^T A P)\, \underline{y}$

$P^T A P = \Lambda$ if $P = \{\underline{e}_1\ \underline{e}_2\ \underline{e}_3\}$ — reason $AP = P\Lambda$ $\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix}$; $P^{-1}AP = \Lambda \to P^T A P = \Lambda$ — 3

$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \quad \therefore Q = \underline{y}^T \Lambda \underline{y} = 3y_1^2 + y_2^2 - y_3^2$ — 2

$\underline{y} = P^{-1}\underline{x} = P^T\underline{x} \quad y_1 = \frac{1}{2}x_1 + \frac{1}{\sqrt{2}}x_2 + \frac{1}{2}x_3$

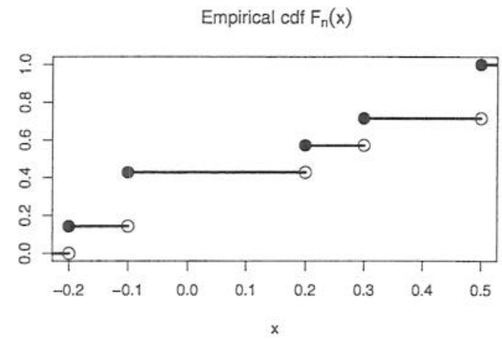
$y_2 = \frac{1}{\sqrt{2}}(x_1 - x_3)$ — 4

$y_3 = \frac{1}{2}x_1 - \frac{1}{\sqrt{2}}x_2 + \frac{1}{2}x_3$

Question 3 — Marks & seen/unseen

Parts

(i)



3

seen similar

(ii) $\bar{x} = \frac{1}{7}(-0.2 + 0.3 + \ldots + 0.5) = 1.1/7 \approx 0.157$ 2

Median: 0.2 1

(iii) Let $x_1, \ldots, x_7$ denote the observed values.

$$s^2 = \frac{1}{7-1}\left(\sum_{k=1}^{7} x_k^2 - \frac{1}{7}\left(\sum_{j=1}^{7} x_j\right)^2\right) = \frac{1}{6}\left(0.69 - \frac{1}{7}(1.1)^2\right)$$
$$\approx \frac{1}{6}(0.69 - 0.1729) = \frac{0.5171}{6} \approx 0.0862$$

3

so $s = \sqrt{s^2} \approx 0.2936$. 1

(iv) From the Student t table we find $t_0 = t_{7-1,0.1} = 1.94$. 2

The 90% confidence interval for μ is thus

$$(\bar{x} - t_0\frac{s}{\sqrt{7}}, \bar{x} + t_0\frac{s}{\sqrt{7}}) = (0.157 - 1.94\frac{0.2936}{\sqrt{7}}, 0.157 + 1.94\frac{0.2936}{\sqrt{7}})$$
$$\approx (-0.058, 0.372)$$

2

(v) The test statistic is $t = \frac{\bar{x}-0}{s/\sqrt{7}} = \frac{1.1/7}{0.2936/\sqrt{7}} \approx 1.42$ 2

Since we have one-sided hypotheses one should use a one-sided test. Taking into account that the formula sheet only gives two-sided values, the critical value is given by $t_0 = t_{6,0.1} = 1.94$. 2

Since $t < 1.94$ the hypothesis H_0 is not rejected. 2

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Course 4
EE2(4)

Question 4

Marks & seen/unseen

Parts

(i) We know that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = 1$. Hence,

$$\begin{aligned}
1 &= k\int_0^1\int_0^{1-y} xy^2 dxdy = k\int_0^1 \frac{1}{2}y^2(1-y)^2dy \\
&= k\frac{1}{2}\int_0^1 (y^4-2y^3+y^2)dy = k\frac{1}{2}\left[\frac{1}{5}y^5 - \frac{1}{2}y^4 + \frac{1}{3}y^2\right]_{y=0}^1 \\
&= k\frac{1}{2}\left(\frac{1}{5}-\frac{2}{4}+\frac{1}{5}\right) = k\frac{1}{60}
\end{aligned}$$

seen similar

Hence, k=60. 4

(ii) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$. Hence, $f_X(x) = 0$ for $x < 0$ or $x > 1$.
For $0 \leq x \leq 1$,

$$f_X(x) = \int_0^{1-x} kxy^2dy = kx[y^3/3]_{y=0}^{1-x} = 20x(1-x)^3.$$

3

(iii) $\mathrm{E}[X] = \int_{-\infty}^{\infty} xf_X(x) = 20\int_0^1 x^2(1-x)^3dx$
$= 20\int_0^1(-x^5+3x^4-3x^3+x^2)dx$
$= 20[-x^6/6+3x^5/5-3x^4/4+x^3/3]_{x=0}^1$
$= 20(-1/6+3/5-3/4+1/3) = 1/3$ 4

(iv)

$$\begin{aligned}
\mathrm{E}(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy = 60\int_0^1\int_0^{1-y} x^2y^3dxdy \\
&= 60\int_0^1 y^3(1-y)^3/3dy = 20\int_0^1(-y^6+3y^5-3y^4+y^3)dy \\
&= 20(-1/7+3/6-3/5+1/4) = 1/7(\approx 0.143)
\end{aligned}$$

Hence, 3

$$\mathrm{cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y) = \frac{1}{7} - \frac{1}{3}\frac{1}{2} = -\frac{1}{42} \approx -0.024$$

2

(v) No, since $\mathrm{cov}(X,Y) \neq 0$. 2

(vi) No, since they are correlated. 2

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EXAMINATION SOLUTIONS 2007-08

Course EE2(4) 5

Question 5

Marks & seen/unseen

Parts

(i) For the ith student let
Y_i denote the result of the spinner,
let $Z_i = 1$ if the student has cheated and $Z_i = 0$ otherwise and
let X_i denote the number the student reports.

Unseen

One can assume that Y_i and Z_i are independent. Hence,

$$\begin{aligned}\mathrm{P}(X_i = 1) &= \mathrm{P}(Y_i = 1, Z_i = 1) + \mathrm{P}(Y_i = 0, Z_i = 0) = \theta p + (1-\theta)(1-p)\\ &= \theta(2p-1) + (1-p).\end{aligned}$$

3

(ii) $\mathrm{P}(X_i = 0) = 1 - \mathrm{P}(X_i = 1) = 1 - (\theta(2p-1) + (1-p)) = \theta(1-2p) + p$

1

(iii) The likelihood function is given by

$$\begin{aligned}L(\theta) &= \mathrm{P}(X_i = 1)^{\sum_{i=1}^n X_i}\,\mathrm{P}(X_i = 0)^{n-\sum_{i=1}^n X_i}\\ &= (\theta(2p-1) + (1-p))^{\sum_{i=1}^n X_i}(\theta(1-2p) + p)^{n-\sum_{i=1}^n X_i}\end{aligned}$$

4

Differentiating the loglikelihood leads to the equation

$$\sum_{i=1}^n X_i \frac{2p-1}{\hat\theta(2p-1) + (1-p)} + (n - \sum_{i=1}^n X_i)\frac{1-2p}{\hat\theta(1-2p) + p} = 0$$

4

for the maximum likelihood estimator $\hat\theta$. Since $p \neq 1/2$ this is equivalent to

$$-\sum_{i=1}^n X_i(\hat\theta(1-2p) + p) + (n - \sum_{i=1}^n X_i)(\hat\theta(2p-1) + (1-p)) = 0$$

Hence,

$$\begin{aligned}\hat\theta &= \frac{p\sum_{i=1}^n X_i - (n - \sum_{i=1}^n X_i)(1-p)}{-\sum_{i=1}^n X_i(1-2p) + (n - \sum_{i=1}^n X_i)(2p-1)}\\ &= \frac{\sum_{i=1}^n X_i - n(1-p)}{n(2p-1)} = \frac{\frac{1}{n}\sum_{i=1}^n X_i - (1-p)}{2p-1}\end{aligned}$$

4

(iv) $\mathrm{E}_\theta(\hat\theta) = \frac{1}{2p-1}(\frac{1}{n}\sum_{i=1}^n \mathrm{E}(X_i) + p - 1) = \frac{1}{2p-1}(\theta(2p-1) + 1 - p + p - 1)$
$= \theta$

Hence, $\hat\theta$ is an unbiased estimator for θ.

4

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Question 6		Marks & seen/unseen

Parts

(i) For $0 < x < 1$:

$F_{X^5}(x) = \mathrm{P}(X^5 \le x) = \mathrm{P}(X \le x^{1/5}) = F_X(x^{1/5}) = x^{1/5}$ — 4

Hence, $f_{X^5}(x) = \frac{d}{dx} x^{1/5} = x^{-4/5}/5$. — 3

For $x \le 0$ or $x \ge 1$ we have $f_{X^5}(x) = 0$. — 1

(ii) Since X and Y are independent, $\mathrm{cov}(X, Y) = 0$.

Hence, from the formula sheet: $X + Y$ is $N(0+5, 1+3)$, i.e. $N(5, 4)$. — 4

(iii) For $t > 0$,

$$
\begin{aligned}
f_{X+Y}(t) &= (f_X * f_Y)(t) \\
&= \int_0^{\min(t,1)} \lambda e^{-\lambda(t-y)} dy \\
&= e^{-\lambda t} \int_0^{\min(t,1)} \lambda e^{\lambda y} dy = e^{-\lambda t}(e^{\lambda \min(t,1)} - 1) \\
&= \begin{cases} e^{-\lambda t}(e^{\lambda} - 1) & \text{if } 1 \le t \\ 1 - e^{-\lambda t} & \text{if } 0 < t < 1 \end{cases}
\end{aligned}
$$

$f_{X+Y}(t) = 0$ for $t \le 0$. — 8

seen similar

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