IMPERIAL COLLEGE LONDON

EE3-09

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2013

EEE/EIE PART III/IV: MEng, Beng and ACGI

CONTROL ENGINEERING

Wednesday, 16 January 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : A. Astolfi
Second Marker(s) : D. Angeli

CONTROL ENGINEERING

1. The mathematical model of a simple mobile robot is described by the equations

$$\dot{x} = \cos\theta\, v, \qquad \dot{y} = \sin\theta\, v, \qquad \dot{\theta} = \omega,$$

in which x and y denote the position of the robot with respect to a fixed reference frame, θ denotes its orientation with respect to the x-axis of the reference frame, v denotes its forward velocity, and ω denotes its angular velocity.

The robot is therefore a system with state (x, y, θ) and input (v, ω).

a) Assume $v(t) = v_0$, with v_0 constant, and $\omega(t) = 0$. Compute the solution of the differential equations describing the robot with initial condition $x(0) = 0$, $y(0) = 0$, and $\theta(0) = 0$. Argue that the solution describes a motion of the robot along a rectilinear path. [6 marks]

b) Compute the linearization of the equations of the mobile robot along the motion determined in part a).
(Hint: the linearized system is time-invariant!)
[4 marks]

c) Compute the reachability matrix of the linearized system in part b). [2 marks]

d) Show that the reachability matrix determined in part c) has rank equal to three for all $v_0 \neq 0$. Hence conclude that the linearized system is controllable.
[4 marks]

e) Show that the reachability matrix determined in part c) has rank equal to two for $v_0 = 0$. Hence conclude that the linearized system is not controllable and compute the unreachable mode. [4 marks]

2. Consider the so-called Collatz iteration, which can be described by the discrete-time system

$$x_{k+1} = \begin{cases} 3x_k + 1 & \text{if } x_k \text{ is odd,} \\ \dfrac{x_k}{2} & \text{if } x_k \text{ is even,} \end{cases}$$

with state x which is assumed to be an integer.

a) Show that if x_0 is an integer then x_k is an integer for all $k \geq 0$. [2 marks]

b) Show that the system does not have any equilibrium (not even if x is a real number).

[4 marks]

c) Show that selecting $x_0 = 1$ yields a period sequence x_k. [2 marks]

d) What happens if $x_0 = 3$ or $x_0 = 7$?
(Hint: compute no more than 10 elements of the sequence x_k.) [2 marks]

e) Consider the modified Collatz systems, with $x_k \in I\!R$, (note that 0 is even)

$$C_1 : x_{k+1} = \begin{cases} 3x_k + 1 & \text{if } k \text{ is odd,} \\ \dfrac{x_k}{2} & \text{if } k \text{ is even,} \end{cases} \qquad C_2 : x_{k+1} = \begin{cases} 3x_k + 1 & \text{if } k \text{ is even,} \\ \dfrac{x_k}{2} & \text{if } k \text{ is odd.} \end{cases}$$

i) Show that the system C_1 can be described by the equation

$$x_{k+2} = \frac{3}{2}x_k + 1.$$

Determine the equilibria of the system and study their stability properties. [8 marks]

ii) Show that the system C_2 can be described by the equation

$$x_{k+2} = \frac{3}{2}x_k + \frac{1}{2}.$$

Determine the equilibria of the system and study their stability properties. [2 marks]

(As a side comment, Collatz conjecture states that every sequence generated by the Collatz iteration *converges* to the periodic sequence 1, 4, 2, 1, ... but this is only a conjecture and "Mathematics is not yet ready for such problems"!)

3. Consider a linear, continuous-time, system described by the equations

$$\dot{x} = Ax + Bu \qquad y = Cx$$

with

$$A = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

a) Show that the system is not controllable and compute the uncontrollable modes. Show that the system is observable. [4 marks]

b) Design a state feedback control law $u = Kx$ such that the matrix $A + BK$ has eigenvalues equal to -2 and -3. Explain why this problem is solvable despite the fact that the system is not controllable. [4 marks]

c) Design an observer such that the matrix $A + LC$ has eigenvalues equal to -2 and -1. [4 marks]

d) Using the separation principle, and the results in parts b) and c) write the equations of a dynamic, output feedback, control law which stabilizes the closed-loop system. Determine the eigenvalues of the resulting closed-loop system. [4 marks]

e) Suppose the response of the closed-loop system in part d) is *too slow*, hence it is necessary to modify the design to achieve a faster response. Suppose, in addition that the designer can either redesign the state feedback or the observer (he/she cannot redesign both).
Discuss which design has to be modified, and determine a new design achieving the fastest possible response. [4 marks]

4. Consider a nonlinear, continuous-time, system described by the equations

$$\dot{x}_1 = -x_1 + x_1 x_2 \qquad \dot{x}_2 = -x_2 + x_1 x_2.$$

a) Compute the equilibrium points of the system. [4 marks]

b) Compute the linearizations of the system around the equilibrium points determined in part a). [4 marks]

c) Study the stability properties of the linearized systems determined in part b), hence establish (if possible) stability properties for the equilibrium points computed in part a). [4 marks]

d) Consider the change of coordinates

$$x_1 = \rho \cos\theta, \qquad x_2 = \rho \sin\theta,$$

with $\rho \geq 0$ and $\theta \in (-\pi, \pi]$.

i) Write a differential equation for the variable ρ^2. [4 marks]

ii) Show, exploiting the facts that $|\sin\theta\cos\theta| \leq 1/2$, and $|\sin\theta + \cos\theta| \leq 3/2$, that

$$\frac{d}{dt}\rho^2 \leq -2\rho^2 + \frac{3}{2}\rho^3.$$

[2 marks]

iii) Using the inequality in part d.ii) show that all trajectories of the system starting from initial conditions $(x_1(0), x_2(0)$ such that

$$x_1^2(0) + x_2^2(0) \leq 1$$

converge to zero. [2 marks]

Control engineering exam paper - Model answers

Question 1

- The controllability matrix is

$$\mathcal{R} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

which has rank three. Hence the system is controllable.

[2 marks]

- As indicated in the question, to evaluate the transmission zeros we build the matrix

$$\Sigma(s) = \begin{bmatrix} s+1 & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ -1 & 0 & s & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

The determinant of $\Sigma(s)$ is $s+1$, which shows that the system has $n-2=1$ transmission zero equal to $s=-1$: the transmission zero has negative real part.

[6 marks]

- Since $y = Cx = x_2$, then $\dot{y} = CAx + CBu = x_3$. Note that $CB = 0$.

[2 marks]

- The feedback is given by

$$u = -k^2 x_2 - k x_3.$$

[2 marks]

- The closed-loop system is described by the equation

$$\dot{x} = A_{cl}x = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -k^2 & -k \end{bmatrix} x.$$

The characteristic polynomial of A_{cl} is

$$\det(\lambda I - A_{cl}) = \lambda^3 + (k+1)\lambda^2 + k(k+1)\lambda + (k^2-1).$$

The Routh test shows that the roots of the polynomial have all negative real part for all $k > 1 = k_\star$.

[8 marks]

Question 2

- The controls are described by the equations

$$u_1 = k_1(x_2 - x_1), \qquad u_2 = k_2(x_1 - x_2) + k_3(x_3 - x_2), \qquad u_3 = k_4(x_2 - x_3).$$

[2 marks]

- The equations are

$$\dot{x}_1 = k_1(x_2 - x_1), \qquad \dot{x}_2 = k_2(x_1 - x_2) + k_3(x_3 - x_2), \qquad \dot{x}_3 = k_4(x_2 - x_3).$$

Hence

$$A = \begin{bmatrix} -k_1 & k_1 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_4 & -k_4 \end{bmatrix}.$$

[4 marks]

- Note that

$$\det A = 0,$$

which shows that A has a zero eigenvalue.

[2 marks]

- The characteristic polynomial of A is (recall that it has a zero eigenvalue)

$$\det(\lambda I - A) = \lambda(\lambda^2 + (k_1 + k_2 + k_3 + k_4)\lambda + (k_1 k_4 + k_4 k_2 + k_1 k_3)).$$

Selecting, for example,

$$k_1 = 1, \qquad k_2 = 1, \qquad k_3 = 1, \qquad k_4 = 1,$$

yields

$$\det(\lambda I - A) = \lambda(\lambda^2 + 4\lambda + 3) = \lambda(\lambda + 3)(\lambda + 1).$$

[4 marks]

- The differential equations are

$$\dot{z}_{12} = 3x_2 - 2x_1 - x_3, \qquad \dot{z}_{23} = x_1 - 3x_2 + 2x_3.$$

[2 marks]

These can be rewritten as

$$\dot{z}_{12} = -2z_{12} + z_{23}, \qquad \dot{z}_{23} = z_{12} - 2z_{23}.$$

As a result,

$$F = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

The characteristic polynomial of F is

$$\det(\lambda I - F) = (\lambda + 3)(\lambda + 1),$$

which shows that the matrix F has eigenvalues equal to -3 and -1.

[4 marks]

The above implies that

$$\lim_{t\to\infty} x_1(t) - x_2(t) = \lim_{t\to\infty} x_2(t) - x_3(t) = 0,$$

which is the same as condition $(\star)$.

[2 marks]

Question 3

- The state equations are

$$x_1(k+1) = a_1x_1(k) + x_2(k) + a_1b_2u(k), \qquad x_2(k+1) = a_0x_1(k) + a_0b_2u(k),$$

$$y(k) = x_1(k) + b_2u(k).$$

As a result

$$y(k+1) = x_1(k+1) + b_2u(k+1) = a_1x_1(k) + x_2(k) + a_1b_2u(k) + b_2u(k+1),$$

and

$$\begin{aligned} y(k+2) &= a_1x_1(k+1) + x_2(k+1) + a_1b_2u(k+1) + b_2u(k+2) \\ &= a_1^2x_1(k) + a_1x_2(k) + a_1^2b_2u(k) + a_0x_1 + a_0b_2u(k) + a_1b_2u(k+1) + b_2u(k+2). \end{aligned}$$

The same expression is obtained replacing the expression of $y(k)$ and $y(k+1)$, as a functions of $x_1(k)$, $x_2(k)$, $u(k)$, $u(k+1)$ and $u(k+2)$, in the equation

$$y(k+2) = a_1y(k+1) + a_0y(k) + b_2u(k+2),$$

which proves that the state-space description is equivalent to the input-output description.

[8 marks]

- The reachability matrix is

$$\mathcal{R} = \begin{bmatrix} a_1b_2 & a_1^2b_2 + a_0b_2 \\ a_0b_2 & a_0a_1b_2 \end{bmatrix},$$

and

$$\det\mathcal{R} = -a_0^2b_2^2.$$

As a result the system is reachable if a_0 and b_2 are both non-zero.
If $b_2 = 0$, the system is non-reachable. It is controllable if $a_1 = a_0 = 0$, and uncontrollable otherwise.
If $a_0 = 0$ and $b_2 \neq 0$ the system is controllable.

[6 marks]

- The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ a_1 & 1 \end{bmatrix},$$

hence the system is observable for any value of the constants a_0 and a_1.

[2 marks]

- Selecting $a_0 = 0$ yields

$$y(k+2) + a_1y(k+1) = b_2u(k+2).$$

Hence, selecting $a_1 = \alpha$ and $b_2 = 1 - \alpha$, and replacing $k+1$ with k yields the first order smoother.

[4 marks]

Question 4

- The equilibria of the system satisfy the equations

$$x_1 = k \sin x_2, \qquad x_2 = \sin x_1.$$

Eliminating x_2 yields

$$x_1 = k \sin(\sin x_1).$$

This equation, for $k \in [-1, 1]$ has the unique solution $x_1 = 0$, hence $(0, 0)$ is the only equilibrium of the system.

[4 marks]

The linearization of the system around the zero equilibrium is given by

$$\delta_x^+ = A\delta_x + B\delta_u,$$

with

$$A = \begin{bmatrix} 0 & k \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

[4 marks]

The characteristic polynomial of the matrix A is

$$\det(\lambda I - A) = \lambda^2 - k,$$

which has roots inside the unity disk for all $|k| < 1$. For $k = 1$, the roots are ± 1, whereas for $k = -1$ the roots are $\pm j$. As a result, the linearized system is asymptotically stable for all $|k| < 1$, and stable for $|k| = 1$.

[4 marks]

Let $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ and note that

$$A + BK = \begin{bmatrix} K_1 & k + K_2 \\ 1 & 0 \end{bmatrix}.$$

Hence, selecting $K_1 = 0$ and $K_2 = -k$, yields two eigenvalues at zero.

[4 marks]

- For $k = 1$, and u constant, the equilibria are solutions of the equations

$$x_1 = \sin x_2 + u, \qquad x_2 = \sin x_1.$$

Eliminating x_2 yields

$$x_1 = \sin(\sin x_1) + u,$$

which is the same as

$$x_1 - \sin(\sin x_1) = u.$$

Note that

$$\sin(\sin x_1) = \alpha(x_1)x_1,$$

with $|\alpha(x_1)| < 1$ for all $x_1 \neq 0$. Hence

$$x_1 - \sin(\sin x_1) = (1 - \alpha(x_1))x_1,$$

which shows that

$$\lim_{x_1 \to \pm\infty} (x_1 - \sin(\sin x_1)) = \pm\infty.$$

As a result, the equation $x_1 - \sin(\sin x_1) = u$ has at least one solution for any u.

[4 marks]

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b) The equations are

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Hence

$$A = \begin{bmatrix} -k_1 & k_1 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_4 & -k_4 \end{bmatrix}.$$

c) Note that

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which shows that A has a zero eigenvalue.

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