IMPERIAL COLLEGE LONDON

EE4-04
EE9SC6

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

MSc and EEE/ISE PART IV: MEng and ACGI

Corrected Copy

ADVANCED DATA COMMUNICATIONS

Thursday, 5 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer THREE questions.

All questions carry equal marks. The maximum mark for each subquestion is shown in brackets.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : M.K. Gurcan

Second Marker(s) : E. Gelenbe

Instructions to Candidates
Useful equations

$$\text{For } P_e = 10^{-7} \text{ the gap value } \Gamma = 9.8dB$$

$$\text{For } P_e = 10^{-6} \text{ the gap value } \Gamma = 8.8dB$$

$$\text{For } 4Q\left(\sqrt{x}\right) = 2 \times 10^{-7} \implies x = 10^{1.45}$$

$$\text{For } 4Q\left(\sqrt{x}\right) = 2 \times 10^{-6} \implies x = 10^{1.39}$$

$$2.4 \times Q\left(\sqrt{4.47}\right) = 1.6 \times 10^{-5}.$$

$$\int \frac{dx}{p + q\cos(ax)} = \frac{2}{a\sqrt{p^2 - q^2}} \tan^{-1} \sqrt{(p-q)(p+q)} \tan\frac{ax}{2}$$

Questions

1. Answer the following sub-questions

   (a) Consider the four waveforms shown in Figure 1.1

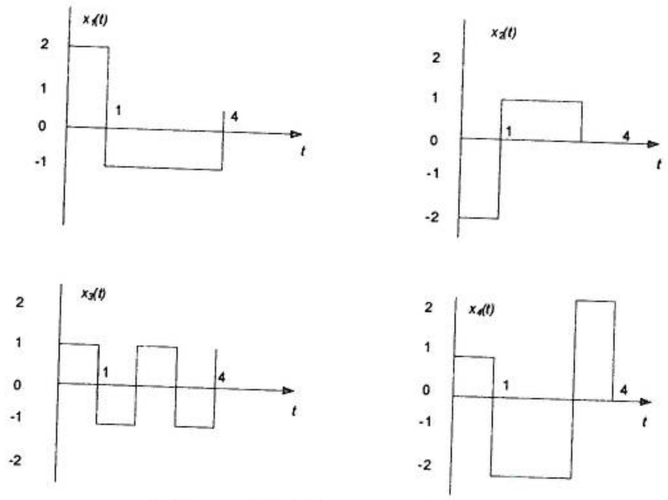

Figure 1.1 Time waveforms.

i. Determine the dimensionality, N, of the waveforms and the basis functions $\phi_1(t), \cdots, \phi_N(t)$. [3]

ii. Represent the four waveforms by vectors $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$, and $\vec{x}_4$, when using the basis functions. Determine the minimum distance between any pair of vectors. [2] [2]

(b) Consider the following signals

$$x_0(t) = \begin{cases} \frac{2}{\sqrt{T}} \cos\left(\frac{2\pi t}{T} + \frac{\pi}{6}\right) & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

$$x_1(t) = \begin{cases} \frac{2}{\sqrt{T}} \cos\left(\frac{2\pi t}{T} + \frac{5\pi}{6}\right) & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

$$x_2(t) = \begin{cases} \frac{2}{\sqrt{T}} \cos\left(\frac{2\pi t}{T} + \frac{3\pi}{2}\right) & 0 \le t \le T \\ 0 & otherwise. \end{cases}$$

i. Find two orthonormal basis functions for this signal set and show that they are orthonormal. Hint use the identity $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$. [2]

ii. Find the data signal corresponding to the signals above for the basis functions you found in part (b).i. [2]

iii. Find the following inner products.

A. $\langle x_0(t), x_0(t) \rangle$; [1]

B. $\langle x_0(t), x_1(t) \rangle$; [1]

C. $\langle x_0(t), x_2(t) \rangle$. [1]

(c) Consider the signal

$$x(t) = \begin{cases} \dfrac{At}{T}\cos(2\pi f_c t) & \text{for } 0 \leq t \leq T \\ 0 & \text{otherwise.} \end{cases}$$

i. Determine the impulse response of the matched filter for the signal. [3]

ii. Determine the output of the matched filter at $t = T$. [4]

(d) Either a square or a cross QAM constellation can be used when transmitting information over an AWGN channel having a $SNR = 30.2dB$ and symbol rate $\frac{1}{T} = 10^6$.

i. Select one of the two QAM constellations and specify a corresponding integer number of bits per symbol for a modem which will have the highest **data rate** such that the probability of error, $P_e \leq 2 \times 10^{-6}$.

ii. Compute the data rate for part (d).i. [2]

iii. Repeat part (d).i when $P_e \leq 2 \times 10^{-7}$. typo announced 11:09 [1]

iv. Compute the data rate for part (d).iii. [1]

2. Answer the following sub-questions.

   (a) Consider the signal constellation, shown in Figure 2.1, that is to be used over an AWGN channel with noise variance $\sigma^2 = 0.05$.

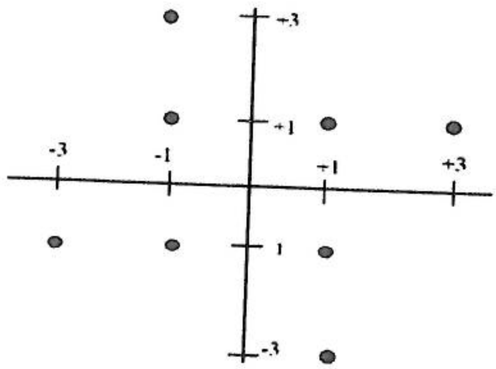

Figure 2.1 Constellation for 8 points.

Assuming that the signal points are to be used with equal probability,

   i. Find the total energy ε_x and average energy $\bar{\varepsilon}_x$ per dimension for this constellation. [2]

   ii. Find the average number of bits, $\bar{b}$ per dimension, the minimum distance d_{min} and average number of neighbours, N_e, for this constellation. [2]

   iii. Find the union bound for the probability of error P_e, when a maximum likelihood (ML) detector is employed at the receiver. [2]

   (b) When using an M-ary PAM transmission system show that

   i. the average energy $\bar{\varepsilon}_x$ per dimension and the minimum distance d are related to each other by [2]

$$d^2 = \frac{12}{M^2 - 1}\bar{\varepsilon}_x$$

   ii. the gap value Γ is given by [2]

$$\Gamma = \left(\frac{d}{2\sigma}\right)^2 \frac{1}{3}$$

where σ^2 is the sampled noise variance per dimension.

(c) Consider the octal signal-point constellations in Figure 2.2.

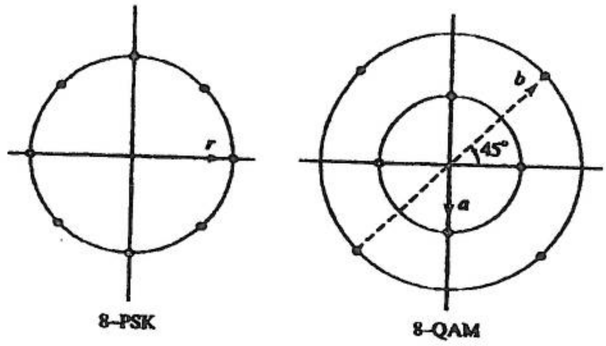

Figure 2.2. Modulation schemes with 8 constellation points.

Answer the following sub-questions.

i. The nearest neighbour signal points in the 8-QAM signal constellation are separated in distance by A units. Determine the radii a and b of the inner and outer circles. [2]

ii. The adjacent signal points in the 8-PSK are separated by a distance of A units. Determine the radius r of the circle. [2]

iii. Determine the average transmitter powers for the two signal constellations and compare the two powers. What is the relative power advantage of one constellation over the other? (Assume that an signal points are equally probable). [2] [1]

(d) A QAM system is to be used to transmit over an AWGN channel with SNR=27.5 dB at a symbol rate of $1/T = 5$ M symbol/s. The desired probability of symbol error is $P_e \leq 10^{-6}$. Answer the following parts

i. List two basis functions that you would use for modulation. [2]

ii. Estimate the highest average bit rate $\bar{b}$ per dimension, and the data rate, R, that can be achieved with the QAM system. [2]

iii. Determine which signal constellation is to be used. [2]

iv. Find how much (in dB) the average energy, $\bar{\mathcal{E}}_x$, per dimension would need to be increased to have 5 Mbps more data rate at the same probability of error? [2]

3. Answer the following sub-questions.

(a) A Hadamard matrix is defined as a matrix whose elements are ± 1 and its row vectors are pairwise orthogonal. In the case when n is a power of 2, an $n \times n$ Hadamard matrix is constructed by means of the recursion

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and

$$H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}$$

i. let c_i denote the i^{th} row of an $n \times n$ Hadamard matrix as defined above. Show that the waveforms constructed as [4]

$$s_i(t) = \sum_{k=1}^{n} c_{i,k} p(t - kT_c), \qquad i = 1, \cdots, n$$

are orthogonal, where $p(t)$ is an arbitrary pulse confined to the time interval $0 \le t \le T_c$.

ii. Show that the n matched filters (or cross correlators) for the n waveforms $\{s_i(t)\}$ can be realized by a single filter (or correlator) matched to the pulse $p(t)$ followed by n correlators using the code words $\{c_i\}$. [4]

(b) Suppose the Fourier transform of the pulse response of a strictly bandlimited channel using binary PAM is

$$P(\omega) = \begin{cases} \sqrt{T}\left(1 + 0.9\exp(j\omega T)\right) & |\omega| \le \dfrac{\pi}{T} \\ 0 & |\omega| > \dfrac{\pi}{T} \end{cases}$$

and the pulse energy $\|p\|^2$, for the channel is $\|p\|^2 = \dfrac{1}{2\pi}\displaystyle\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} |P(\omega)|^2 d\omega = 1.81$. The function characterizing the inter-symbol-interference (ISI) is

$$Q(D) = \frac{0.9D^{-1} + 1.81 + 0.9D}{1.81} = \frac{1}{1.81}(1 + 0.9D)\left(1 + 0.9D^{-1}\right).$$

Given that the matched filter signal-to-noise ratio bound, SNR_{MFB}, is 10dB, answer the following sub-questions:

i. Find the feedforward filter transfer function , $W(D)$ for zero forcing equalizer. [4]

ii. Find the minimum-mean-square-error linear equalizer transfer function $W(D)$. [4]

iii. Find the mean-square-error values σ^2_{ZFE} and σ^2_{ZFE} for the zero forcing and also mean-square-error equalizers. [3]

(c) Data symbols are transmitted over a 4 kHz voice-band telephone (bandpass) channel. Assuming that the transmitter pulse shape has a raised cosine spectrum with a 50% roll-off, determine the bit rate that can be transmitted through the channel if each of the following modulation methods are used:

i. binary PSK, [2]

ii. four-phase PSK [2]

iii. 8-point QAM. [2]

4. Answer the following sub-questions.

(a) The Levin-Campello loading algorithm will be used to improve the energy utilization for PAM/QAM signals when transmitting them over the multi-tone modulation channel with $1 + 0.5D^{-1}$. Assume that the system has $N = 8$ dimensions and operates at a bit error rate of $P_e = 10^{-6}$ when the matched filter bound signal-to-noise-ratio $SNR_{MFB} = 10dB$ and the average energy per dimension $\bar{\mathcal{E}}_x = 1$. Answer the following questions.

i. Create a table of incremental energies $e(n)$ vs. the channel number $n = 0, \cdots, 4$. [2]

ii. Use the EF algorithm to make the average number of bits per dimension $\bar{b} = 1$. [2]

iii. Use the E-Tightening algorithm to find the largest $\bar{b}$. [2]

iv. The total number of bits b obtained in part (a.iii) is to be reduced by 2 bits. Use the EF and B-Tightening algorithms to maximize the margin. What is the maximum margin? [2]

(b) A multi-tone modulation system operates over the channel $H(f) = 1 + 0.5e^{(j2\pi f)}$. The system operational parameters are: the matched filter bound SNR $SNR_{MFB} = 10dB$, the average energy per dimension $\bar{\mathcal{E}}_x = 1$ and the system dimension $N = 8$. Using the Rate-Adaptive water-filling optimization method answer the following questions.

i. Calculate the optimal distribution of energy for the sub-channels and the maximum bit rate assuming that the gap, $\Gamma = 1$ (0dB). [3]

ii. Calculate the gap for PAM/QAM which produces an argument of the Q−function equal to 9dB. (The gap for $\bar{b} \geq 1$ is the difference between the SNR derived from capacity and the argument of the Q-function for a particular probability of error). [3]

iii. Calculate the optimal distribution of energy for the sub-channels and the maximum bit rate using the gap found in part (b.ii). [3]

(c) For the system in problem (4.b), the system margin will be maximized using the Margin-Adaptive water filling method for a system dimension of $N = 8$. Answer the following questions.

i. Is transmission of uncoded QAM/PAM at $P_e = 10^{-7}$ at a data rate of 1 possible? Justify your answer. What will the margin be when operating at the data rate of 1? [3] [3]

ii. For the data rate of 1, what gap provides a margin value equal to zero? [2]

1.a

i As an orthonormal set of basis functions we consider the set

$$\psi_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad \psi_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases}$$

$$\psi_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases} \qquad \psi_4(t) = \begin{cases} 1 & 3 \le t < 4 \\ 0 & \text{o.w} \end{cases}$$

In matrix notation, the four waveforms can be represented as

$$\begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \\ \psi_4(t) \end{pmatrix}$$

Note that the rank of the transformation matrix is 4 and therefore, the dimensionality of the waveforms is 4

ii The representation vectors are

$$\begin{aligned} \mathbf{s}_1 &= \begin{bmatrix} 2 & -1 & -1 & -1 \end{bmatrix} \\ \mathbf{s}_2 &= \begin{bmatrix} -2 & 1 & 1 & 0 \end{bmatrix} \\ \mathbf{s}_3 &= \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \\ \mathbf{s}_4 &= \begin{bmatrix} 1 & -2 & -2 & 2 \end{bmatrix} \end{aligned}$$

iii The distance between the first and the second vector is

$$d_{1,2} = \sqrt{|\mathbf{s}_1 - \mathbf{s}_2|^2} = \sqrt{\left|\begin{bmatrix} 4 & -2 & -2 & -1 \end{bmatrix}\right|^2} = \sqrt{25}$$

Similarly we find that

$$\begin{aligned} d_{1,3} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} 1 & 0 & -2 & 0 \end{bmatrix}\right|^2} = \sqrt{5} \\ d_{1,4} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 1 & 1 & 1 & -3 \end{bmatrix}\right|^2} = \sqrt{12} \\ d_{2,3} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} -3 & 2 & 0 & 1 \end{bmatrix}\right|^2} = \sqrt{14} \\ d_{2,4} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} -3 & 3 & 3 & -2 \end{bmatrix}\right|^2} = \sqrt{31} \\ d_{3,4} &= \sqrt{|\mathbf{s}_3 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 0 & 1 & 3 & -3 \end{bmatrix}\right|^2} = \sqrt{19} \end{aligned}$$

Thus, the minimum distance between any pair of vectors is $d_{\min} = \sqrt{5}$.

1b

By using the following trigonometric identity

$$\cos(a+b) = \cos(a)\cos b - \sin(a)\sin(b)$$

the signals $x_0(t)$, $x_1(t)$, $x_2(t)$ can be written as

$$x_0(t) = \sqrt{2}\left[\phi_1(t)\cos\left(\tfrac{\pi}{6}\right) - \phi_2(t)\sin\left(\tfrac{\pi}{6}\right)\right]$$

$$x_1(t) = \sqrt{2}\left[\phi_1(t)\cos\left(\tfrac{5\pi}{6}\right) - \phi_2(t)\sin\left(\tfrac{5\pi}{6}\right)\right]$$

$$x_2(t) = \sqrt{2}\left[\phi_1(t)\cos\left(\tfrac{\pi}{2}\right) + \phi_2(t)\sin\left(\tfrac{\pi}{2}\right)\right]$$

where

$$\phi_1(t) = \begin{cases} \sqrt{\frac{2}{T}}\left(\cos\left(\frac{2\pi t}{T}\right)\right) & \text{for } 0<t<T \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_2(t) = \begin{cases} \sqrt{\frac{2}{T}}\left(\sin\left(\frac{2\pi t}{T}\right)\right) & \text{for } 0<t<T \\ 0 & \text{otherwise} \end{cases}$$

are orthonormal

$$\int_0^T \phi_1(t)\phi_2(t)\,dt = \int_0^T \frac{2}{T}\sin\left(\frac{2\pi t}{T}\right)\cos\left(\frac{2\pi t}{T}\right)dt = \int_0^T \sin\left(\frac{4\pi t}{T}\right)dt = 0$$

$$\int_0^T \phi_1^2(t)\,dt = \int_0^T \frac{2}{T}\cos^2\left(\frac{2\pi t}{T}\right)dt = \int_0^T \frac{1}{T}\left[1+\cos\left(\frac{4\pi t}{T}\right)\right]dt = 1$$

$$\int_0^T \phi_2^2(t)\,dt = \int_0^T \frac{2}{T}\sin^2\left(\frac{2\pi t}{T}\right)dt = \int_0^T \frac{1}{T}\left[1-\cos\left(\frac{4\pi t}{T}\right)\right]dt = 1$$

—////—

$$x_0 = \left[\sqrt{\tfrac{3}{2}}, -\tfrac{\sqrt{2}}{2}\right],$$

$$x_1 = \left[-\sqrt{\tfrac{3}{2}}, -\tfrac{\sqrt{2}}{2}\right]$$

$$x_3 = \left[0, \sqrt{2}\right]$$

—///—

$$\langle x_0(t), x_0(t)\rangle = \tfrac{3}{2} + \tfrac{1}{2} = 2$$

$$\langle x_0(t), x_1(t)\rangle = -\tfrac{3}{2} + \tfrac{1}{2} = -1$$

$$\langle x_0(t), x_1(t)\rangle = -1$$

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MODEL ANSWERs and MARKING SCHEME

First Examiner:	**Gurcan, M.K.**	**Paper Code :**	**E4.04, SC6, ISE4.9**
Second Examiner:	**Gelenbe, E.**	**Question** 1	**Page** 3 ***out of*** 16

Question labels in left margin **Marks allocations in right margin**

i) The impulse response of the matched filter is

$$s(t) = u(T-t) = \begin{cases} \frac{A}{T}(T-t)\cos(2\pi f_c(T-t)) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

ii) The output of the matched filter at $t = T$ is

$$\begin{aligned}
g(T) &= u(t) * s(t)|_{t=T} = \int_0^T u(T-\tau)s(\tau)d\tau \\
&= \frac{A^2}{T^2}\int_0^T (T-\tau)^2\cos^2(2\pi f_c(T-\tau))d\tau \\
&\overset{v=T-\tau}{=} \frac{A^2}{T^2}\int_0^T v^2\cos^2(2\pi f_c v)dv \\
&= \frac{A^2}{T^2}\left[\frac{v^3}{6} + \left(\frac{v^2}{4\times 2\pi f_c} - \frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c v) + \frac{v\cos(4\pi f_c v)}{4(2\pi f_c)^2}\right]\Bigg|_0^T \\
&= \frac{A^2}{T^2}\left[\frac{T^3}{6} + \left(\frac{T^2}{4\times 2\pi f_c} - \frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c T) + \frac{T\cos(4\pi f_c T)}{4(2\pi f_c)^2}\right]
\end{aligned}$$

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1-d

$$P_e \leqslant 4\,Q\left(\sqrt{\frac{3\,SNR}{M-1}}\right)$$

$$P_e \leqslant 4\,Q\left(\sqrt{\frac{3\,SNR}{\frac{31}{32}M-1}}\right)$$

—///—

$\frac{3\,SNR}{M-1} = 10^{1.39}$ for SQ-QAM $P_e = 2\times10^{-6} \Rightarrow Q(x)$, $x = 10^{1.39}$

$\frac{3\,SNR}{\frac{31}{32}M-1} = 10^{1.39}$ for CR-QAM $SNR = 10^{3.02}$

$M = 129 \Rightarrow b = \log_2 M = 7.0$ for SQ-QAM

$M = 139 \Rightarrow b = \log_2 M = 7.1$ for CR-QAM

SQ-QAM requires even b ⇒ b=6 SQ-QAM
CR-QAM requires odd b ⇒ b = 7 CR-QAM
we would use 128 point CR-QAM

—///—

Data rate $R = \frac{b}{T} = 7\times10^{6} = 7$ mbps.

—///—

$P_e < 2\cdot10^{-7}$ we need ⇒ $2\times10^{-7} = 4\,Q(\sqrt{x}) \Rightarrow x = 10^{1.45}$

$\frac{3\,SNR}{M-1} = 10^{1.45}$ for SQ-QAM $SNR = 10^{3.02}$

$\frac{3\,SNR}{\frac{31}{32}M-1} = 10^{1.45}$ for CR-QAM $SNR = 10^{3.02}$

$M = 112 \Rightarrow b = \log_2 112 = 6.81$ for SQ-QAM

$M = 133 \Rightarrow b = \log_2 133 = 6.86$ for CR-QAM

highest SQ-QAM is obtained when b=6
highest CR-QAM is obtained when b=5
Therefore we select 64 QAM

Data rate $R = \frac{b}{T} = 6\times10^{6} = 6$ mbps.

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Second Examiner:	**Gelenbe, E.**	**Question** 2	**Page** 5 ***out of*** 16

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2-a

$$\mathcal{E}_x = \frac{1}{2}(2+10) = 6$$

$$\bar{\mathcal{E}}_x = \frac{\mathcal{E}_x}{N} = 3$$

$$\bar{b} = \frac{\log_2 M}{N} = 1.5$$

$$d_{min} = 2$$

—///—

$$N_e = 4$$

$$\bar{N}_e = 2$$

—//—

$$P_e \leq N_e Q\left(\frac{d_{min}}{2\sigma}\right)$$

$$\leq 4\, Q(4.47)$$

$$\leq 1.6 \times 10^{-5}$$

$$P_e \leq 8 \times 10^{-6}$$

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MODEL ANSWERs and MARKING SCHEME

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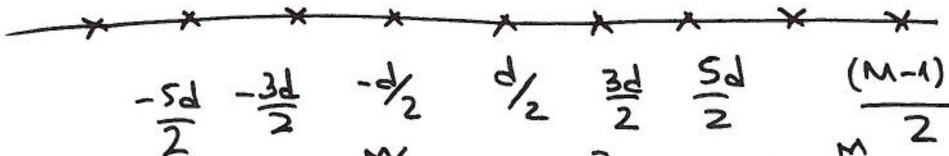
2-b



$$\mathcal{E}_x = \bar{\mathcal{E}}_x = \frac{1}{M}(2)\sum_{K=1}^{M/2}\left(\frac{2K-1}{2}\right)^2 d^2 = \frac{d^2}{2M}\sum_{K=1}^{M/2}(2K-1)^2$$

$$\sum_{K=1}^{A}(2K-1)^2 = \frac{A\,(2A+1)(2A-1)}{3}$$

simplification

$$\sum_{K=1}^{M/2}(2K-1)^2 = \frac{\frac{M}{2}\,(M+1)(M-1)}{3} = \frac{M(M^2-1)}{6}$$

Energy simplification

$$\mathcal{E}_x = \frac{d^2}{2M}\,\frac{M(M^2-1)}{6} = d^2\,\frac{M^2-1}{12}$$

$$\mathcal{E}_x = \bar{\mathcal{E}}_x = \frac{d^2}{16}\,(M^2-1)$$

$$d^2 = \frac{12\,\bar{\mathcal{E}}_x}{M^2-1}$$

$$\left(\frac{d}{2\sigma}\right)^2 = \frac{12\,\bar{\mathcal{E}}_x}{4(M^2-1)\sigma^2} = \frac{3}{M^2-1}\,SNR$$

$$M^2-1 = \frac{SNR}{\left(\frac{d}{2\sigma}\right)^2 \frac{1}{3}} \Rightarrow M^2 = 1 + \frac{SNR}{\left(\frac{d}{2\sigma}\right)^2 \frac{1}{3}}$$

$$\log_2 M^2 = \log_2\left(1+\frac{SNR}{\left(\frac{d}{2\sigma}\right)^2\frac{1}{3}}\right) = \log_2\left(1+\frac{SNR}{\Gamma}\right)$$

$$\Gamma = \left(\frac{d}{2\sigma}\right)^2 \frac{1}{3}$$

2-c

1) Consider the QAM constellation of Fig. P-7.46. Using the Pythagorean theorem we can find the radius of the inner circle as

$$a^2 + a^2 = A^2 \Longrightarrow a = \frac{1}{\sqrt{2}} A$$

The radius of the outer circle can be found using the cosine rule. Since b is the third side of a triangle with a and A the two other sides and angle between then equal to $\theta = 75^o$, we obtain

$$b^2 = a^2 + A^2 - 2aA\cos 75^o \Longrightarrow b = \frac{1+\sqrt{3}}{2} A$$

2) If we denote by r the radius of the circle, then using the cosine theorem we obtain

$$A^2 = r^2 + r^2 - 2r\cos 45^o \Longrightarrow r = \frac{A}{\sqrt{2-\sqrt{2}}}$$

3) The average transmitted power of the PSK constellation is

$$P_{\text{PSK}} = 8 \times \frac{1}{8} \times \left(\frac{A}{\sqrt{2-\sqrt{2}}}\right)^2 \Longrightarrow P_{\text{PSK}} = \frac{A^2}{2-\sqrt{2}}$$

whereas the average transmitted power of the QAM constellation

$$P_{\text{QAM}} = \frac{1}{8}\left(4\frac{A^2}{2} + 4\frac{(1+\sqrt{3})^2}{4}A^2\right) \Longrightarrow P_{\text{QAM}} = \left[\frac{2+(1+\sqrt{3})^2}{8}\right] A^2$$

The relative power advantage of the PSK constellation over the QAM constellation is

$$\text{gain} = \frac{P_{\text{PSK}}}{P_{\text{QAM}}} = \frac{8}{(2+(1+\sqrt{3})^2)(2-\sqrt{2})} = 1.5927 \text{ dB}$$

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MODEL ANSWERs and MARKING SCHEME

First Examiner:	**Gurcan, M.K.**	**Paper Code :**	**E4.04, SC6, ISE4.9**
Second Examiner:	**Gelenbe, E.**	**Question** 2	**Page** 8 ***out of*** 16

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2-d

i Since QAM transmission is used.

$$\begin{aligned}\phi1 &= -\sqrt{\frac{2}{T}}sinc(\frac{t}{T})\sin\omega_c t\\ \phi2 &= \sqrt{\frac{2}{T}}sinc(\frac{t}{T})\cos\omega_c t\end{aligned}$$

where $T = 2 \cdot 10^{-7}$.

ii For QAM and $P_e = 10^{-6}$ the gap is $\Gamma = 8.8\ dB$. Hence,

$$\bar{b} = \frac{1}{2}\log_2(1 + \frac{SNR}{\Gamma}) \simeq 3$$

iii Then, $R = \frac{b}{T} = \frac{2 \cdot 3}{2 \cdot 10^{-7}} = 30\ Mbps$.

Since $\bar{b} = 3, b = 6$, we have $M = 2^b = 2^6 = 64$. Therefore this is a 64-square QAM signal constellation.

iv In order to achieve $R = 35\ Mbps$, we should have $b = 7$, thus employing 128-cross QAM. We require the same probability of error, so:

$$\left(\frac{d_{min}}{2\sigma}\right)^2 = \frac{3SNR}{\frac{31}{32}M - 1} = 13.5\ dB \Rightarrow$$
$$SNR = 917.9 = 29.6\ dB$$

3a i

The inner product of $s_i(t)$ and $s_j(t)$ is

$$\int_{-\infty}^{\infty} s_i(t)\, s_j(t)\,dt = \int_{-\infty}^{\infty} \sum_{k=1}^{n} c_{ik} p(t-kT_c) \sum_{l=1}^{n} c_{jl} p(t-lT_c)\,dt$$

$$= \sum_{k=1}^{n}\sum_{l=1}^{n} c_{ik} c_{jl} \int_{-\infty}^{\infty} p(t-kT_c)\, p(t-lT_c)\,dt$$

$$= \sum_{k=1}^{n}\sum_{l=1}^{n} c_{ik} c_{jl} \mathcal{E}_p \delta_{kl} = \mathcal{E}_p \sum_{k=1}^{n} c_{ik} c_{jk}$$

The quantity $\sum_{k=1}^{n} c_{ik} c_{jk}$ is the inner product of the row vectors $\underline{c}_i$ and $\underline{c}_j$. Since the rows of the matrix H_n are orthogonal by construction, we obtain

$$\int_{-\infty}^{\infty} s_i(t)\, s_j(t)\,dt = \mathcal{E}_p \sum_{k=1}^{n} c_{ik}^2 \delta_{ik} = n\, \mathcal{E}_p \delta_{ij}$$

Thus the waveforms $s_i(t)$ and $s_j(t)$ are orthogonal.

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ii

First we consider the signal

$$y(t) = \sum_{k=1}^{n} c_k \delta(t-kT_c)$$

The signal $y(t)$ has duration $T = nT_c$ and its matched filter is

$$g(t) = y(T-t) = y(nT_c - t) = \sum_{k=1}^{n} c_n \delta(nT_c - kT_c - t)$$

$$= \sum_{i=1}^{n} c_{n-i+1} \delta((i-1)T - t) = \sum_{i=1}^{n} c_{n-i+1} \delta(t-(i-1)T)$$

that is a sequence of impulses starting at $t=0$ and weighted by the mirror image sequence of $\{c_i\}$. Since

$$s(t) = \sum_{k=1}^{n} c_k\, p(t-kT_c) = p(t) * \sum_{k=1}^{n} c_k\, \delta(t-kT_c)$$

the Fourier transform of the signal $s(t)$ is

$$S(f) = P(f) \sum_{k=1}^{n} c_k \exp(-j2\pi f k T_c)$$

and therefore, the Fourier transform of the signal matched to $s(t)$ is

$$H(f) = S^*(f)\exp(-j2\pi f T) = S^*(f)\exp(-j2\pi f n T_c)$$

$$= P^*(f) \sum_{k=1}^{n} c_k \exp(j2\pi f k T_c)\exp(-j2\pi f n T_c)$$

$$= P^*(f) \sum_{i=1}^{n} c_{n-i+1} \exp(-j2\pi f (i-1) T-c)$$

$$= P^*(f)\, \mathcal{F}[g(t)]$$

Thus the matched filter $H(f)$ can be considered as the cascade of a filter, with impulse response $p(-t)$, matched to the pulse $p(t)$ and a filter, with impulse response $g(t)$, matched to the signal

$$y(t) = \sum_{k=1}^{n} c_k\, \delta(t-kT_c).$$

The output of the matched filter at $t = nT_c$ is

$$\int_{-\infty}^{\infty} |s(t)|^2 = \sum_{k=1}^{n} c_{1k} \int_{-\infty}^{\infty} p^2(t - kT_c)\,dt$$

$$= T_c \sum_{k=1}^{n} c_{1k}^2$$

Where we have used the fact $p(t)$ is a rectangular pulse of unit amplitude and duration T_c.

Using the above result we obtain the filter matched to the waveform

$$s_i(t) = \sum_{k=1}^{n} c_{ik}\, p(t - kT_c)$$

can be realised as the cascade of a filter matched to $p(t)$ followed by a discrete-time filter matched to the vector $\underline{c_i} = [c_{i,1}, \cdots c_{i,n}]$. Since the pulse $p(t)$ is common to all the signal waveforms $s_i(t)$ we conclude that the n matched filters can be realised by a filter matched to $p(t)$ followed by n discrete-time filters matched to vectors $\underline{c_i}$, $i = 1 \cdots n$.

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First Examiner: **Gurcan, M.K.** **Paper Code : E4.04, SC6, ISE4.9**

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3.b

$$P(D) = 1 + 0.9D^{-1} \Rightarrow |P|^2 = 1.81$$

$$Q(D) = \frac{1.81 + 0.9D^{-1} + 0.9D}{1.81}$$

i. ZF equaliser $W(D) = \frac{1}{|P|\,Q(D)} = \frac{1.81}{\sqrt{1.81}\,(1.81 + 0.9D^{-1} + 0.9D)}$

$$W(D) = \frac{\sqrt{1.81}\,D}{0.9 + 1.81D + 0.9D^2}$$

ii linear MMSE

$$W(D) = \frac{1}{|P|\left(Q(D) + \frac{1}{SNR_{MFB}}\right)} = \frac{1}{\sqrt{1.81}\left(\frac{0.9}{1.81}D^{-1} + 1.1 + \frac{0.9}{1.81}D\right)}$$

iii

$$\sigma^2_{ZFE} = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \frac{\frac{N_0}{2}}{|P|^2\,Q(\exp(-jwT))}\,dw$$

$$= \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \frac{\frac{1.81}{1.81}\,\frac{N_0}{2}}{1.81 + 1.8\cos(wT)}\,dw = \frac{N_0}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{1.81 + 1.8\cos w}\,du$$

$$= \frac{N_0}{2}\,\frac{1}{2\pi}\,\frac{4}{\sqrt{1.81^2 - 1.8^2}}\,\frac{\pi}{2} = \frac{N_0}{2}\;5.26$$

$$\sigma^2_{MMSE} = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \frac{\frac{N_0}{2}}{1.81 + 1.8\cos(wT) + \frac{1.81}{10}}\,dw$$

$$= \frac{N_0}{2}\,\frac{1}{\sqrt{1.991^2 - 1.8^2}} = 0.181 \times 1.175$$

$$SNR = \frac{\mathcal{E}_x}{\sigma^2_{MMSE}} - 1 = \frac{1 - 0.181 \times 1.175}{0.181 \times 1.175} = 3.7$$

3.
c

The bandwidth of the bandpass channel is $W = 4$ KHz. Hence, the rate of transmission should be less or equal to 4000 symbols/sec. If a 8-QAM constellation is employed, then the required symbol rate is $R = 9600/3 = 3200$. If a signal pulse with raised cosine spectrum is used for shaping, the maximum allowable roll-off factor is determined by

$$1600(1+\alpha) = 2000$$

which yields $\alpha = 0.25$. Since α is less than 50%, we consider a larger constellation. With a 16-QAM constellation we obtain

$$R = \frac{9600}{4} = 2400$$

and

$$1200(1+\alpha) = 2000$$

Or $\alpha = 2/3$, which satisfies the required conditions. The probability of error for an M QAM constellation is given by

$$P_M = 1 - (1 - P_{\sqrt{M}})^2$$

where

$$P_{\sqrt{M}} = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left[\sqrt{\frac{3\mathcal{E}_{av}}{(M-1)N_0}}\right]$$

With $P_M = 10^{-5}$ we obtain $P_{\sqrt{M}} = 5 \times 10^{-7}$ and therefore

$$2 \times (1 - \frac{1}{4})Q\left[\sqrt{\frac{3\mathcal{E}_{av}}{15 \times 2 \times 10^{-10}}}\right] = 5 \times 10^{-7}$$

Using the last equation and the tabulation of the $Q[\cdot]$ function, we find that the average transmitted energy is

$$\mathcal{E}_{av} = 24.70 \times 10^{-9}$$

Note that if the desired spectral characteristic $X_{rc}(f)$ is split evenly between the transmitting and receiving filter, then the energy of the transmitting pulse is

$$\int_{-\infty}^{\infty} g_T^2(t)dt = \int_{-\infty}^{\infty} |G_T(f)|^2 df = \int_{\infty}^{\infty} X_{rc}(f)df = 1$$

Hence, the energy $\mathcal{E}_{av} = P_{av}T$ depends only on the amplitude of the transmitted points and the symbol interval T. Since $T = \frac{1}{2400}$, the average transmitted power is

$$P_{av} = \frac{\mathcal{E}_{av}}{T} = 24.70 \times 10^{-9} \times 2400 = 592.8 \times 10^{-7}$$

If the points of the 16-QAM constellation are evenly spaced with minimum distance between them equal to d, then there are four points with coordinates $(\pm\frac{d}{2}, \pm\frac{d}{2})$, four points with coordinates $(\pm\frac{3d}{2}, \pm\frac{3d}{2})$, four points with coordinates $(\pm\frac{3d}{2}, \pm\frac{d}{2})$, and four points with coordinates $(\pm\frac{d}{2}, \pm\frac{3d}{2})$. Thus, the average transmitted power is

$$P_{av} = \frac{1}{2 \times 16}\sum_{i=1}^{16}(A_{mc}^2 + A_{ms}^2) = \frac{1}{2}\left[4 \times \frac{d^2}{2} + 4 \times \frac{9d^2}{2} + 8 \times \frac{10d^2}{4}\right] = 20d^2$$

Since $P_{av} = 592.8 \times 10^{-7}$, we obtain

$$d = \sqrt{\frac{P_{av}}{20}} = 0.00172$$

4a

(i) We will proceed using the gap approximation: i.e. $E_n(b_n) = \frac{\Gamma}{g_n}(2^{2b_n} - 1) * k$, where k=1 if PAM and k=2 if QAM.

So, we first need to find $g_n = \frac{|H_n|^2}{\sigma_n^2}$. From the system parameters $\sigma_n^2 = .125$. So, we have the following table (the center 3 are QAM channels)

subchannel	0	1	2	3	4
g_n	18	15.6569	10	4.3431	2

Now, using the above formula, we get:

subchannel	0	1	2	3	4
$e_n(1)$	1.2463	.9690	1.5172	3.4932	11.37
$e_n(2)$	5.07	1.938	3.043	6.9860	45.5233
$e_n(3)$	19.2287	3.8760	6.0686	13.97	182.0567
$e_n(4)$	81.9150	7.7521	12.1372	27.94	728.2345

Note for the QAM channels we could have used $e_n(b_n) = \frac{\Gamma}{g_n} 2^{b}_n$

(ii) With the above table, it is obvious that the bit allocations are as follows:

subchannel	0	1	2	3	4
b_n	2	3	2	1	0
$E_n(b_n)$	6.3215	6.7830	4.5515	3.4932	0

Bits were chosen in the following order: 1,0,2,1,2,3,1,0

(iii) $N * \bar{E_x} = 8$, so we are way over budget. Working backwards, we get

subchannel	0	1	2	3	4
b_n	1	2	1	0	0
$E_n(b_n)$	1.2463	2.9070	1.5172	0	0

(iv) Again, we just work backwards

subchannel	0	1	2	3	4
b_n	1	1	0	0	0
$E_n(b_n)$	1.2463	.9690	0	0	0

The margin in this case is $10 * log_{10}(\frac{8}{1.2463+.9690}) = 5.577dB$

4. b

(i)

```
g =

    4.3431   10.0000   15.6569   18.0000   15.6569   10.0000    4.3431    2.000

e =

    1.1124    1.1041    1.0680    0.9377    0.6680

b =

    2.1970    2.0964    1.7730    1.1714    0.6120

N =

    8

b_bar =

    1.6113
```

(ii) gap $= 10^{9}/3 = 2.6478 = 4.23\text{dB}$

(iii) Check the following results

$R = 1.0622$

```
g =

    4.3431   10.0000   15.6569   18.0000   15.6569   10.0000    4.3431    2.00
e =

    1.2977    1.2757    1.1800    0.8351    0.1206

b =

    1.6478    1.5472    1.2239    0.6223    0.0629

N =

    8
b_bar =

    1.0622
```

4.c

i From Problem 4-6, a rate of 1.06 is achieved with a gap of 4.23 dB. Since $\Gamma = 9.8dB$ for $P_e = 10^{-7}$, about 4.5 dB more energy is needed. So, the P_e goal cannot be achieved. We can reach the same conclusion by doing MA, which will give us a negative margin.

ii Since we get $R = 1.0622$ for Γ =4.23dB from 4.6, an easy way to solve this one would be to try a slightly larger gap than 4.23dB. To get the exact gap value, we need to solve MA again. The same approach as in problem 4 will give us Γ = 4.79dB, and the corresponding $P_e = 1.3 \cdot 10^{-3}$.

ii Part ii gives us Γ = 4.79dB with no margin. So, with $\Gamma = 0$, margin = 4.79dB.