

Please write on this side only, legibly and neatly, between the margins

$$(i) (a) \frac{1+i}{7-i} = \frac{(1+i)(7+i)}{7^2 + 1^2}$$

$$= \frac{3}{25} + \frac{4}{25} i$$

$$(b) (1+3i)^3 = 1 + 9i - 27 - 27i$$

$$= -26 - 18i$$

$$(c) (1-i)^{17} = \left[\sqrt{2} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right) \right]^{17}$$

$$= 2^{17/2} \left(\cos \left(-\frac{17\pi}{4} \right) + i \sin \left(-\frac{17\pi}{4} \right) \right)$$

$$= 2^8 \sqrt{2} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right)$$

$$= 2^8 (1-i)$$

(ii) (a) The equation means that z is equidistant from ± 1 ; so it represents the y -axis.

$$(b) \operatorname{Re}(z^3) = \operatorname{Re}(z) \Leftrightarrow x^3 - 3xy^2 = x$$

$$\Leftrightarrow x(x^2 - 3y^2 - 1) = 0. \text{ So the}$$

Setter : WILSON

Setter's signature :

J. Wilson
MLW

Checker : LIEBECK

Checker's signature :

Please write on this side only, legibly and neatly, between the margins

figure is the y -axis together with the hyperbola $x^2 - 3y^2 = 1$.

$$(iii) (a) \sinh z = 0 \Leftrightarrow e^z - e^{-z} = 0$$

$$\Leftrightarrow e^{2z} = 1 \Leftrightarrow 2z = \ln 1.$$

So the solutions are $z = i\pi n$
(where n is an integer).

$$(b) \sinh z + \cosh z = 0 \Leftrightarrow$$

$$e^z - e^{-z} + e^z + e^{-z} = 0$$

$$\Leftrightarrow e^z = 0.$$

So there are no solutions.

Setter : WILSON

Setter's signature :

J. Wilson

Checker : LIEBECK

Checker's signature :

ML

Please write on this side only, legibly and neatly, between the margins

$$(1) (a) \frac{(x+2)^{\frac{1}{2}} - 2}{x-2} = \frac{[(x+2)^{\frac{1}{2}} - 2][(x+2)^{\frac{1}{2}} + 2]}{(x-2)((x+2)^{\frac{1}{2}} + 2)}$$

$$= \frac{x-2}{(x-2)((x+2)^{\frac{1}{2}} + 2)} = \frac{1}{(x+2)^{\frac{1}{2}} + 2}$$

$$\text{As } x \rightarrow 2, \text{ this } \rightarrow \frac{1}{2+2} = \frac{1}{4}$$

[alter: use l'Hôpital's rule]

$$(b) |x \sin(\tan x)| \leq |x|, \text{ so the limit as } x \rightarrow 0 \text{ is } 0.$$

$$(c) x^{-9} \left\{ (x+3)^{10} - (x+1)^{10} \right\} =$$

$$x^{-9} \left\{ x^{10} + 30x^9 + \dots - (x^{10} + 10x^9 + \dots) \right\}$$

$$= x^{-9} \left\{ 20x^9 + \text{lower terms} \right\}$$

$$= 20 + (\text{terms in powers of } \frac{1}{x})$$

So the limit as $x \rightarrow \infty$ is 20.

Setter : WILSON

Setter's signature :

J. Wilson

Checker : LIEBECK

Checker's signature :

Mur

Please write on this side only, legibly and neatly, between the margins

(ii) (a) let $y = \ln \{x + (1+x^2)^{\frac{1}{2}}\};$

then $\frac{dy}{dx} = \frac{1 + x(1+x^2)^{-\frac{1}{2}}}{x + (1+x^2)^{\frac{1}{2}}}$

$= \frac{(1+x^2)^{-\frac{1}{2}} [(1+x^2)^{\frac{1}{2}} + x]}{x + (1+x^2)^{\frac{1}{2}}}$

$= (1+x^2)^{-\frac{1}{2}}$

(b) let $y = (\sin x)^x$, then

$\ln y = x \ln (\sin x)$, so

$\frac{1}{y} \frac{dy}{dx} = \ln (\sin x) + x \frac{\cos x}{\sin x}$

Hence $\frac{dy}{dx} = (\sin x)^x \{ \ln (\sin x) + x \cot x \}$.

Setter : WILSON

Setter's signature :

J. Wilson

Checker : LIEBECK

Checker's signature :

MWR

Please write on this side only, legibly and neatly, between the margins

$$(i) (a) \frac{|u_{n+1}|}{|u_n|} = \frac{2^{n+1} n^7}{(n+1)^7 2^n} =$$

$$\frac{2}{(1 + \frac{1}{n})^7} \rightarrow 2 > 1 \text{ as } n \rightarrow \infty. \text{ By the}$$

ratio test, the series diverges.

$$(b) u_n = \frac{1 + \frac{1}{n}}{10 + \frac{1}{n}} \rightarrow \frac{1}{10}. \text{ Since}$$

$u_n \not\rightarrow 0$, the series diverges

$$(c) \frac{|u_{n+1}|}{|u_n|} = \frac{e^{n+1} n!}{(n+1)! e^n} = \frac{e}{n+1},$$

which $\rightarrow 0 < 1$ as $n \rightarrow \infty$. By the ratio test, the series converges.

$$(ii)(a) \frac{|u_{n+1}|}{|u_n|} = \frac{(n+1)^3 |x|^{n+1}}{n^3 |x|^n} = \left(1 + \frac{1}{n}\right)^3 |x|,$$

which $\rightarrow |x|$ as $n \rightarrow \infty$. So by

the ratio test $R = 1$.

Setter : WILSON

Checker : LIEBECK

Setter's signature :

Checker's signature :

Wilson
MLR

Please write on this side only, legibly and neatly, between the margins

$$(b) \frac{|u_{n+1}|}{|u_n|} = \frac{(n+1)! (n+2)! |x|^{n+1} (2n+1)!}{(2n+3)! n! (n+1)! |x|^n}$$

$$= \frac{(n+2)(n+1)}{(2n+3)(2n+2)} |x| \rightarrow \frac{1}{4} |x| \text{ as } n \rightarrow \infty.$$

$$\text{So } R = 4.$$

$$(iii) \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

$$\therefore \ln \frac{1+x}{1-x} = 2 \left\{ x + \frac{x^3}{3} + \dots \right\},$$

$$\text{and } \ln \sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

Setter : WILSON

Checker : LIEBECK

Setter's signature :

Checker's signature :

J. Wilson
MWR

Please write on this side only, legibly and neatly, between the margins

(i) Set $t = \tan \frac{x}{2}$; then

$$\int \frac{dx}{\sin x} = \int \frac{2dt/(1+t^2)}{2t/(1+t^2)} = \int \frac{dt}{t}$$

$$= \ln |t| + c = \ln \left| \tan \frac{x}{2} \right| + c$$

(ii) Set $u = x^2$, $du = 2x dx$. Then

$$\int \frac{x dx}{(1-x^2)^{3/2}} = \int (1-u)^{-3/2} \cdot \frac{1}{2} du$$

$$= (1-u)^{-1/2} + c = \frac{1}{\sqrt{1-x^2}} + c$$

(iii) Set $x = \sin u$, $dx = \cos u du$. Then

$$\int \frac{x^2 dx}{(1-x^2)^{3/2}} = \int \frac{\sin^2 u \cos u du}{\cos^3 u}$$

$$= \int \tan^2 u du = \int (\sec^2 u - 1) du$$

$$= \tan u - u + c = \frac{x}{\sqrt{1-x^2}} - \sin^{-1} x + c$$

Setter : WILSON

Setter's signature :

Checker : LIEBECH

Checker's signature :

f. Wilson
ML

Please write on this side only, legibly and neatly, between the margins

$$(iv) \quad \text{let } \frac{2x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}.$$

$$\begin{aligned} \text{Then } 2x &= A(x^2+1) + (Bx+C)(x+1) \\ &= (A+B)x^2 + (B+C)x + A+C, \end{aligned}$$

$$\text{So } A+B=0, \quad B+C=2, \quad A+C=0,$$

$$\text{whence } A=-1, \quad B=C=1.$$

$$\therefore \int \frac{2x dx}{(x+1)(x^2+1)} = \int \left(\frac{-1}{x+1} + \frac{x+1}{x^2+1} \right) dx$$

$$\begin{aligned} &= -\ln|x+1| + \frac{1}{2} \ln|x^2+1| + \tan^{-1} x \\ &\quad + C. \end{aligned}$$

Setter : WILSON

Setter's signature :

J. Wilson

Checker : LIEBECK

Checker's signature :

MLR

Please write on this side only, legibly and neatly, between the margins

(i) Separable : $\frac{dy}{1+y^2} = (1+x^2)dx$

$\therefore \tan^{-1}y = x + \frac{x^3}{3} + c, \text{ i.e.}$

$y = \tan\left(x + \frac{x^3}{3} + c\right).$

(ii) Linear with IF $e^{\int \frac{dx}{x}} = x.$

$xy' + y = x \sin x, \text{ i.e.}$

$(xy)' = x \sin x.$

$\therefore xy = \int x \sin x dx = -x \cos x + \int \cos x dx$
 $= -x \cos x + \sin x + c. \text{ So}$

$y = -\cos x + x^{-1}(\sin x + c).$

(iii) The auxiliary equation is $r^2 + 2r - 3 = 0,$

i.e. $(r+3)(r-1) = 0, \text{ so } r = 1 \text{ or } -3.$

Thus the complementary function is

Setter : WILSON

Checker : LIEBECHE

Setter's signature :

Checker's signature :

f. Wilson
MVR

Please write on this side only, legibly and neatly, between the margins

$c_1 e^x + c_2 e^{-3x}$. We seek a particular
integral $y = Ax e^x$, so $y' = A(x+1)e^x$,
 $y'' = A(x+2)e^x$, and $y'' + 3y' + y =$
 $A e^x (x+2 + 3(x+1) - 3x) = 4A e^x$.

To solve the given ODE we take $A = \frac{1}{4}$.

Thus the general solution is

$$y = c_1 e^x + c_2 e^{-3x} + \frac{1}{4} x e^x.$$

(iv)

Thus $y(0) = 0 \Leftrightarrow c_1 + c_2 = 0$.

And $y' = c_1 e^x - 3c_2 e^{-3x} + \frac{1}{4}(x+1)e^x$,

so $y'(0) = 0 \Leftrightarrow c_1 - 3c_2 + \frac{1}{4} = 0$.

To satisfy both conditions we need

$c_2 = -c_1 = \frac{1}{16}$, giving

$$y = -\frac{1}{16} e^x + \frac{1}{16} e^{-3x} + \frac{1}{4} x e^x.$$

Setter : WILSON

Setter's signature :

f. Wilson

Checker : LIEBECK

Checker's signature :

MUR

Please write on this side only, legibly and neatly, between the margins

(i) Char. eqn. is $\begin{vmatrix} -10-\lambda & 9 \\ -18 & 17-\lambda \end{vmatrix} = 0$, i.e. $\lambda^2 - 7\lambda - 8 = 0$

roots $\lambda = -1, 8$.

$\lambda = -1$ Eigenvectors ^{are} \underline{a} are solutions of $\begin{pmatrix} -9 & 9 \\ -18 & 18 \end{pmatrix} x = 0$ i.e. $\underline{a} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ($a \neq 0$)

$\lambda = 8$ Eigenvectors are solutions of $\begin{pmatrix} -18 & 9 \\ -18 & 9 \end{pmatrix} x = 0$ i.e. $\underline{b} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ($b \neq 0$)

(ii) Let $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Then $P^{-1}AP = \begin{pmatrix} -1 & 0 \\ 0 & 8 \end{pmatrix}$ diagonal.

(iii) Write $D = \begin{pmatrix} -1 & 0 \\ 0 & 8 \end{pmatrix}$, so $A = PDP^{-1}$.

If $C = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$ then $C^3 = D$, so

$$\begin{aligned} (PCP^{-1})^3 &= PCP^{-1} \cdot PCP^{-1} \cdot PCP^{-1} = PC^3P^{-1} \\ &= PDP^{-1} = A. \end{aligned}$$

So take

$$\begin{aligned} B = PCP^{-1} &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 3 \\ -6 & 5 \end{pmatrix}, \end{aligned}$$

Setter : LIGBECK

Checker : WILSON

Setter's signature : M. L. Beck

Checker's signature : J. W.

Please write on this side only, legibly and neatly, between the margins

7

$$i) f_x = x^2 + y^2 - 2 + 2x(x+y) = 3x^2 + 2xy + y^2 - 2$$

$$f_y = x^2 + y^2 - 2 + 2y(x+y) = x^2 + 2xy + 3y^2 - 2$$

At stat. pts, $f_x = f_y = 0$, so subscripts, $x^2 - y^2 = 0$, $x = \pm y$

If $x = y$, $6x^2 = 2 \rightarrow x = \pm \frac{1}{\sqrt{3}}$.


If $x = -y$, $2x^2 = 2 \rightarrow x = \pm 1$.

So stat. pts $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, $(1, -1)$, $(-1, 1)$.

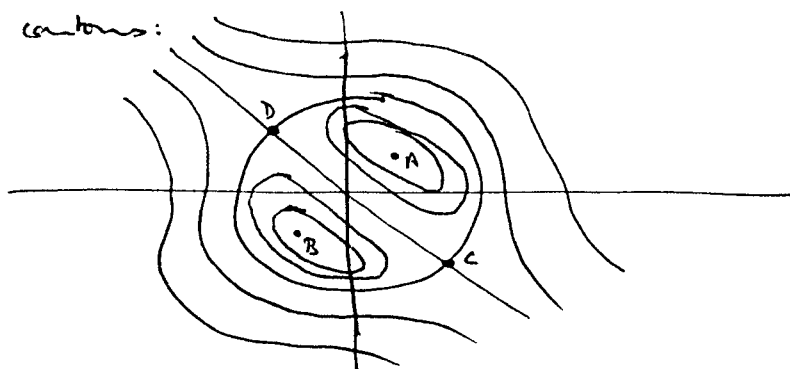
Nature: $f_{xx} = 6x + 2y$, $f_{yy} = 2x + 6y$, $f_{xy} = 2x + 2y$.

At stat. pts

	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx}f_{yy} - f_{xy}^2$		
A $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$	$\frac{8}{\sqrt{3}}$	$\frac{8}{\sqrt{3}}$	$\frac{4}{\sqrt{3}}$	> 0	MIN	2
B $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$	$-\frac{8}{\sqrt{3}}$	$-\frac{8}{\sqrt{3}}$	$-\frac{4}{\sqrt{3}}$	> 0	MAX	2
C $(1, -1)$	4	-4	0	< 0	SADDLE	3
D $(-1, 1)$	-4	4	0	< 0	SADDLE	

(ii) Contour $f = 0$ is (line $y = -x$) \cup (circle $x^2 + y^2 = 2$): 

(iii) Further contours:



Setter : LIEBECIC

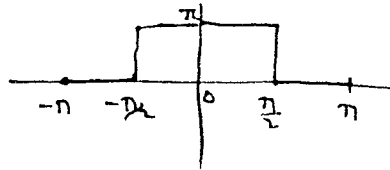
Setter's signature : MLN

Checker : WILSON

Checker's signature : J.V.

Please write on this side only, legibly and neatly, between the margins

(a) Cosine series is for even function :



Series is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \cdot \frac{\pi^2}{2} = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \pi \cos nx dx = 2 \left[\frac{\sin nx}{n} \right]_0^{\pi/2} = \frac{2}{n} \sin \frac{n\pi}{2}$$

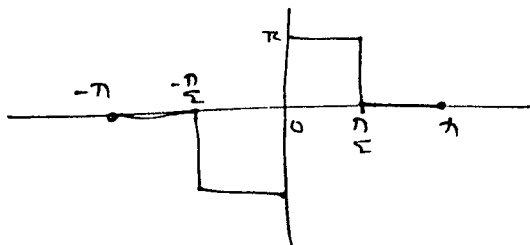
$$= \frac{2}{n} \sin \frac{n\pi}{2}$$

$$= \begin{cases} 0, & n \text{ even} \\ \frac{2}{2m+1} \cdot (-1)^m, & n = 2m+1 \text{ odd.} \end{cases}$$

So cosine series is

$$\frac{\pi}{2} + 2 \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \cos (2m+1)x.$$

(b) Sine series is for odd function :



Setter : LIEBEEK

Checker : WILSON

Setter's signature : MLW

Checker's signature : J. W.

Please write on this side only, legibly and neatly, between the margins

Series $\sum_{n=1}^{\infty} b_n \sin nx$ where

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \pi \sin nx \, dx = \frac{2}{\pi} \left[-\frac{\cos nx}{n} \right]_0^{\frac{\pi}{2}} \\ &= \frac{2}{n} \left(1 - \cos \frac{n\pi}{2} \right) \\ &= \begin{cases} \frac{2}{n}, & n \text{ odd} \\ \frac{2}{n} (1 - (-1)^m), & n = 2m \text{ even.} \end{cases} \end{aligned}$$

So sine series is

$$2 \sum_{n=1}^{\infty} \frac{\sin(2m+1)x}{2m+1} + 2 \sum_{m=0}^{\infty} \frac{\sin(4m+2)x}{2m+1}$$

Put $x = 0$ in the cosine series, we have

$$\pi = \frac{\pi}{2} + 2 \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}$$

hence

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} = \frac{\pi}{4}.$$

Setter : LIEBECK

Checker : WILSON

Setter's signature :

Checker's signature :

MLW
J.W.

6

4

Please write on this side only, legibly and neatly, between the margins

Sketch:

Note

$x \rightarrow t$

$t \rightarrow s$

in accordance
with Honor Style.

$$L(H_0 - H_1) = \int_0^1 e^{-tx} dx = \frac{1}{t} (1 - e^{-t})$$

Taking Laplace transforms,

$$\begin{aligned} -y'(0) - ty(0) + t^2 L(y) + 2(-y(0) + tL(y)) + L(y) \\ = \frac{1}{t} (1 - e^{-t}) \end{aligned}$$

$$\Rightarrow (t^2 + 2t + 1) L(y) = \frac{1}{t} (1 - e^{-t})$$

$$\Rightarrow L(y) = \frac{1}{t(t+1)^2} - \frac{e^{-t}}{t(t+1)^2}$$

Partial fractions: $\frac{1}{t(t+1)^2} = \frac{1}{t} - \frac{1}{t+1} - \frac{1}{(t+1)^2}$

So

$$y = L^{-1} \left(\frac{1}{t} - \frac{1}{t+1} - \frac{1}{(t+1)^2} - \frac{e^{-t}}{t} + \frac{e^{-t}}{t+1} + \frac{e^{-t}}{(t+1)^2} \right)$$

$$= 1 - e^{-x} - xe^{-x} - H_1(x) + H_1(x)e^{-(x-1)} + H_1(x)(x-1)e^{-(x-1)}$$

$$= 1 - e^{-x} - xe^{-x} + H_1(x) (-1 + e^{-x+1} + (x-1)e^{-x+1})$$

$$= 1 - e^{-x} - xe^{-x} - H_1(x) + H_1(x) \cdot xe^{1-x}$$

Setter: LIEBEK

Checker: WILSON

Setter's signature: MLW

Checker's signature: J.W.