## THE ANSWERS

*Notations:*

(a) B - Bookwork

(b) E - New example

(c) A - New application

1. a) User channels are orthogonal. We can simply apply

$$\mathbf{p}_1 = \begin{bmatrix} 3 & 1 & 4 & -1 \end{bmatrix}^T / \sqrt{27},$$

and

$$\mathbf{p}_2 = \begin{bmatrix} 1 & 2 & -1 & 1 \end{bmatrix}^T / \sqrt{7},$$

to null out the multi-user interference.

[ 3 - E ]

This is a (normalized) transmit matched filter w.r.t. the co-scheduled user channel. Given that the user channel are orthogonal the matched filter nulls out the interference and acts as a ZFBF. Complexity is very low as there is no need for any pseudo-inverse to null out interference (as in conventional ZFBF).

[ 3 - E ]

b) i) The diversity gain is the slope at high SNR of the error curve vs. the SNR on a log-log scale, i.e. $-\lim_{\rho\rightarrow\infty} \frac{\log(P_e(\rho))}{\log(\rho)}$ with $\rho$ being the SNR. For (a), the diversity gain is 1 as the error rate decreases by $10^{-1}$ when the SNR is increased from 50dB to 60dB.

[ 1 - E ]

For (b), the diversity gain is 2 as the error rate decreases by $10^{-2}$ when the SNR is increased from 30dB to 40dB.

[ 1 - E ]

For (c), the diversity gain is 3 as the error rate decreases by $10^{-3}$ when the SNR is increased from roughly 26dB to 36dB.

[ 1 - E ]

For (d), the diversity gain is 4 as the error rate decreases by $10^{-4}$ when the SNR is increased from 10dB to 20dB.

[ 1 - E ]

ii) The simplest approach is to perform

for (a), receive matched filter with $n_r = 1$, $n_t = 1$

[ 1 - E ]

for (b), receive matched filter with $n_r = 2$, $n_t = 1$

[ 1 - E ]

for (c), receive matched filter with $n_r = 3$, $n_t = 1$

[ 1 - E ]

for (d), receive matched filter with $n_r = 4$, $n_t = 1$

[ 1 - E ]

Alternative strategies are possible, for instance selection combining at the receiver for all 4 cases. We could also perform transmit diversity based on space-time coding and use O-STBC for (b),(c),(d) to achieve diversity order of 2 with $n_t = 2$ and $n_r = 1$, 3 with $n_t = 3$ and $n_r = 1$, 4 with $n_t = 4$ and $n_r = 1$, respectively.

c) i) The multiplexing gain is the pre-log factor of the ergodic capacity at high SNR, i.e. $g_s = \lim_{\rho\to\infty} \frac{C_{CDIT}}{\log_2(\rho)}$. Hence by increasing the SNR by 3dB (e.g. from 17dB to 20dB), the ergodic capacity increases by $g_s$ bits/s/Hz.

(a) $g_s = 3$.

[ 1 - E ]

(b) $g_s = 2$.

[ 1 - E ]

(c) $g_s = 2$.

[ 1 - E ]

(d) $g_s = 1$.

[ 1 - E ]

(e) $g_s = 1$.

[ 1 - E ]

ii) There are several possible configurations that satisfy to $n_r + n_t = 8$, namely $5\times 3$, $3\times 5$, $6\times 2$, $2\times 6$, $7\times 1$ and $1\times 7$, $4\times 4$. The matching between curves and antenna configurations is easily identified by using the following two arguments: 1) The multiplexing gain with CDIT at high SNR is given by $\min\{n_t, n_r\}$. 2) With CDIT only, the input covariance matrix in i.i.d. channel is $\mathbf{Q} = 1/n_t \mathbf{I}_{n_t}$. This implies that $6\times 2$ and $7\times 1$ outperform $2\times 6$ and $1\times 7$, respectively.

(a) $n_r \times n_t = 5\times 3$ or $3\times 5$

[ 1 - E ]

(b) $n_r \times n_t = 6\times 2$

[ 1 - E ]

(c) $n_r \times n_t = 2\times 6$

[ 1 - E ]

(d) $n_r \times n_t = 7\times 1$

[ 1 - E ]

(e) $n_r \times n_t = 1\times 7$

[ 1 - E ]

d) i) Assume that the flat fading channel remains constant over the two successive symbol periods, and is denoted by $\mathbf{h} = [h_1\ h_2]$.

Two symbols $c_1$ and $c_2$ are transmitted simultaneously from antennas 1 and 2 during the first symbol period, followed by symbols $-c_2^*$ and

$c_1^*$, transmitted from antennas 1 and 2 during the next symbol period:

$$y_1 = \sqrt{E_s} h_1 \frac{c_1}{\sqrt{2}} + \sqrt{E_s} h_2 \frac{c_2}{\sqrt{2}} + n_1, \qquad \text{(first symbol period)}$$

$$y_2 = -\sqrt{E_s} h_1 \frac{c_2^*}{\sqrt{2}} + \sqrt{E_s} h_2 \frac{c_1^*}{\sqrt{2}} + n_2. \qquad \text{(second symbol period)}$$

The two symbols are spread over two antennas and over two symbol periods.

[ 2 - B ]

ii) Equivalently

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = \sqrt{E_s} \underbrace{\begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix}}_{\mathbf{H}_{eff}} \underbrace{\begin{bmatrix} c_1/\sqrt{2} \\ c_2/\sqrt{2} \end{bmatrix}}_{\mathbf{c}} + \begin{bmatrix} n_1 \\ n_2^* \end{bmatrix}.$$

Applying the matched filter $\mathbf{H}_{eff}^H$ to the received vector y effectively decouples the transmitted symbols as shown below

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{H}_{eff}^H \begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = \sqrt{E_s} \left[ |h_1|^2 + |h_2|^2 \right] \mathbf{I}_2 \begin{bmatrix} c_1/\sqrt{2} \\ c_2/\sqrt{2} \end{bmatrix} + \mathbf{H}_{eff}^H \begin{bmatrix} n_1 \\ n_2^* \end{bmatrix}$$

For both symbol $c_i$, $i = 1,2$, we can apply a SISO ML decoder that will find the symbol in the constellation that minimizes $\left| z_i - \sqrt{E_s/2} \left[ |h_1|^2 + |h_2|^2 \right] c_i \right|$.

[ 3 - B ]

iii) If the channel is not constant over two consecutive symbol durations, by applying the same procedure, we can see that the decoupling does not hold anymore and symbols are still subject to inter-stream interference. In order to benefit from full diversity, a MIMO ML decoder is necessary, which leads to higher complexity. [ 3 - A ]

e) i) In such regime, the interference is so strong that each user performs SIC by decoding the interfering message first and subtracting it from the received signal before decoding its own message. Each transmitter $i$ can communicate with its receiver $i$ at a rate $R_i = \log_2(1 + \tilde{\eta}_{i,i})$ for $i = 1,2$, as in the absence of any interference, therefore leading to a multiplexing gain per user of 1. $\tilde{\eta}_{i,i}$ is the instantaneous SNR between transmitter $i$ and receiver $i$.

[ 4 - B ]

ii) When user 1 decodes user 2's signal in the very strong interference regime, it treats its own signal as noise. Hence, for user 1 to be able to cancel correctly user 2's signal, the interfering channel between transmitter 2 and user 1 has to be strong enough to support $R_2$. Similarly, the interfering channel between transmitter 1 and user 2 has to be strong enough to support $R_1$. This leads to the conditions

$$\log_2(1 + \tilde{\eta}_{2,2}) \leq \log_2(1 + \frac{\tilde{\eta}_{1,2}}{1 + \tilde{\eta}_{1,1}}),$$

$$\log_2(1 + \tilde{\eta}_{1,1}) \leq \log_2(1 + \frac{\tilde{\eta}_{2,1}}{1 + \tilde{\eta}_{2,2}}),$$

where $\tilde{\eta}_{1,2}$ and $\tilde{\eta}_{2,1}$ are the instantaneous INR between transmitter 2 and user 1 and transmitter 1 and user 2, respectively. [ 4 - B ]

2. a) We transmit $c' = c'_1 + c'_2$ (with power of $c'_q$ denoted as $s_q$). User 1 cancels user 2's signal $c'_2$ so as to be left with its own Gaussian noise. User 2 decodes its signal by treating user 1's signal $c'_1$ as Gaussian noise. This is called superposition coding with SIC. The achievable rates of such strategy (with sum-power constraint $s_1 + s_2 = E_s$)

$$R_1 = \log_2\left(1 + \frac{\Lambda_1^{-1} s_1}{\sigma_{n,1}^2} |h_1|^2\right)$$

$$R_2 = \log_2\left(1 + \frac{\Lambda_2^{-1} |h_2|^2 s_2}{\sigma_{n,2}^2 + \Lambda_2^{-1} |h_2|^2 s_1}\right).$$

For user 1 to be able to correctly cancel user 2's signal, user 1's channel has to be good enough to support $R_2$, i.e.

$$R_2 \leq \log_2\left(1 + \frac{\Lambda_1^{-1} |h_1|^2 s_2}{\sigma_{n,1}^2 + \Lambda_1^{-1} |h_1|^2 s_1}\right).$$

The channel gains normalized w.r.t. their respective noise power should be ordered

$$\frac{\Lambda_2^{-1} |h_2|^2}{\sigma_{n,2}^2} \leq \frac{\Lambda_1^{-1} |h_1|^2}{\sigma_{n,1}^2}.$$

If the ordering condition is satisfied, the above strategy achieves the boundary of the capacity region of the two-user SISO BC for any power allocation $s_1$ and $s_2$ satisfying $s_1 + s_2 = E_s$.

[ 6 - B ]

b) The codebook only contains 3 precoders. Looking at the precoder, no stream can be transmitted on the fourth antenna. The fourth antenna is therefore useless. This implies that the system effectively looks like a MISO with 3 transmit antenna where transmit antenna selection is performed. With $n_r$ antennas, receive antenna selection achieves a diversity gain of $n_r$ (see lecture notes). Similarly, transmit antenna selection achieves a diversity equal to 3 in this scenario. Hence the proposed strategy will achieve a diversity gain of 3. Hence the statement is wrong.

[ 6 - A ]

c) The statement is correct in point-to-point channel where the imperfect CSIT leads to a SNR loss but not multiplexing loss. The rate achieved with a precoder $\mathbf{w}$ is given by $R = \log_2(1 + \rho\,|\mathbf{h}\mathbf{w}|^2)$. The optimal precoder is $\mathbf{w} = \mathbf{h}^H / \|\mathbf{h}\|$. At high SNR, $R \approx \log_2(\rho) + \log_2(|\mathbf{h}\mathbf{w}|^2)$, hence leading to multiplexing gain of 1 irrespectively of $\mathbf{w}$. In MU-MISO on the other hand, the imperfect CSIT leads to multi-user interference that scales with the transmit power and becomes dominant at high SNR (compared to noise) and induces a sum-rate ceiling which translate into a multiplexing gain loss.

[ 6 - A ]

d) A PF scheduler aims at finding the scheduled user set K to maximize the weighted sum-rate $\sum_{k \in K} \gamma_k \frac{R_k(t)}{\bar{R}_k(t)}$ where $\gamma_k$ is the QoS of user $k$ (which is the same for all users in the question), $R_k(t)$ is user $k$ rate at time $t$ and $\bar{R}_k(t)$ is the long-term average rate of user $k$ at time $t$. $\bar{R}_k(t)$ is computed using a low-pass filter and is updated at every time instant based on the past estimates $\bar{R}_k(t-1)$ and instantaneous rate $R_k(t-1)$, as

$$\bar{R}_k(t) = \begin{cases} (1-1/t_c)\bar{R}_k(t-1) + 1/t_c R_k(t-1), & q \in \mathbf{K}^\star \\ (1-1/t_c)\bar{R}_q(k-1), & q \notin \mathbf{K}^\star \end{cases}$$

where $t_c$ is the scheduling time scale and $\mathbf{K}^\star$ refers to the scheduled user set at time $k$.

In the limit of very small scheduling time scale $t_c$, assuming all users have the same QoS, the scheduler divides the available resources equally among users without accounting for the relative strength of the user channels. Hence the scheduler is fair but cannot exploit any multi-user diversity.

[ 6 - A ]

e) By the distance-product criterion, the diversity gain is given by $n_r L_{min} = n_r \min l_{\mathbf{C},\mathbf{E}}$ where $\min l_{\mathbf{C},\mathbf{E}}$ refers to the minimum effective length over all possible non-zero error matrices. Given the three codewords, the possible non-zero error matrices are given by

$$\mathbf{a}-\mathbf{b} = \begin{bmatrix} 0 & a-d & b-a & c-b \end{bmatrix},$$
$$\mathbf{a}-\mathbf{c} = \begin{bmatrix} a-b & 0 & b-a & c-b \end{bmatrix},$$
$$\mathbf{b}-\mathbf{c} = \begin{bmatrix} a-b & d-a & 0 & 0 \end{bmatrix}.$$

Hence, $l_{\mathbf{a},\mathbf{b}} = 3$, $l_{\mathbf{a},\mathbf{c}} = 3$ and $l_{\mathbf{b},\mathbf{c}} = 2$, leading to $L_{min} = 2$ and a total diversity gain of 2.

[ 6 - A ]

3. a) The received signal of terminal 1 in cell 1 writes as

$$y_1 = \mathbf{h}_{1,1}\mathbf{p}_{1,1}c_{1,1} + \mathbf{h}_{1,1}\mathbf{p}_{1,2}c_{1,2} + \mathbf{h}_{1,2}\mathbf{p}_2 c_2 + n_1$$

where $y_1$ is the $[1 \times 1]$ received signal at user 1 in cell 1, $\mathbf{h}_{1,i}$ is the channel between transmitter $i$ and user 1, $\mathbf{P}_1 = [\mathbf{p}_{1,1}, \mathbf{p}_{1,2}]$ is the $[2 \times 2]$ precoder at transmitter 1 made of two sub-precoder $\mathbf{p}_{1,1}$ and $\mathbf{p}_{1,2}$ of size $[2 \times 1]$ (precoding data for terminal 1 and 2 respectively), and $\mathbf{c}_1 = [c_{1,1}, c_{1,2}^T]^T$ is the $[2 \times 1]$ transmit symbol vector at transmitter 1 whose entries are unit-average energy independent symbols. $\mathbf{P}_2 = \mathbf{p}_2$ is the vector precoder at transmitter 2 and $\mathbf{c}_2 = c_2$ is the transmit symbol at transmitter 2. The transmit power at each transmitter writes as $\mathrm{Tr}\left\{\mathcal{E}\left\{\mathbf{P}_i\mathbf{c}_i(\mathbf{P}_i\mathbf{c}_i)^H\right\}\right\} = \mathrm{Tr}\left\{\mathbf{P}_i\mathbf{P}_i^H\right\} = P$. Hence the power allocation to each stream is naturally accounted for in the precoder. The first term is the intended signal, the second term refers to the intra-cell interference and the third term refers to the inter-cell interference.

[ 4 - A ]

b) We can simply write the received signal as follows

$$y_1 = \mathbf{h}_{1,1}\mathbf{p}_{1,1}c_{1,1} + \mathbf{n}_1'$$

with $\mathbf{n}_1' = \mathbf{h}_{1,1}\mathbf{p}_{1,2}c_{1,2} + \mathbf{h}_{1,2}\mathbf{p}_2 c_2 + n_1$. The term $\mathbf{n}_1'$ now expresses the combined noise plus interference (intra-cell and inter-cell interference) seen by stream 1. The power of $\mathbf{n}_1'$ is given by $\sigma_n^2 + |\mathbf{h}_{1,1}\mathbf{p}_{1,2}|^2 + |\mathbf{h}_{1,2}\mathbf{p}_2|^2$.

[ 2 - A ]

The rate achievable at receiver 1 is given by

$$R_1 = \log_2\left(1 + \frac{|\mathbf{h}_{1,1}\mathbf{p}_{1,1}|^2}{\sigma_n^2 + |\mathbf{h}_{1,1}\mathbf{p}_{1,2}|^2 + |\mathbf{h}_{1,2}\mathbf{p}_2|^2}\right).$$

[ 2 - A ]

c) i) To increase the rate, terminal 1 aims at recommending the precoder $\mathbf{p}_2$ that minimizes $|\mathbf{h}_{1,2}\mathbf{p}_2|^2$ where $\mathbf{h}_{1,2} = \begin{bmatrix} 1 & e^{\pi/3} \end{bmatrix}$ and $\mathbf{p}_2$ can take any of the four codewords in the codebook. The precoder that terminal 1 will recommend is the third one, i.e. $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & j \end{bmatrix}^T$, for which $|\mathbf{h}_{1,2}\mathbf{p}_2|^2 = 0.1340P$. [ 3 - A ]

ii) In i.i.d. channel, the precoders of the codebook should be as far as each other so that the best codebook is the one that maximizes the minimum of $1 - |\mathbf{w}_k^H\mathbf{w}_l|^2$ among all distinct ($k \neq l$) pairs of precoders $\mathbf{w}_k^H$, $\mathbf{w}_l$ in the codebook. For codebook 1, we compute

$$abs(\mathcal{W}_1^H\mathcal{W}_1) = \begin{bmatrix} 1.0000 & 0 & 0.7071 & 0.7071 \\ 0 & 1.0000 & 0.7071 & 0.7071 \\ 0.7071 & 0.7071 & 1.0000 & 0 \\ 0.7071 & 0.7071 & 0 & 1.0000 \end{bmatrix}.$$

For codebook 2, we compute

$$abs(\mathcal{W}_2^H\mathcal{W}_2) = \begin{bmatrix} 1.0000 & 0.5774 & 0.5774 & 0.5774 \\ 0.5774 & 1.0000 & 0.5774 & 0.5774 \\ 0.5774 & 0.5774 & 1.0000 & 0.5774 \\ 0.5774 & 0.5774 & 0.5774 & 1.0000 \end{bmatrix}.$$

Codebook 2 is better as it leads to a larger minimum value of $1 - |\mathbf{w}_k^H\mathbf{w}_l|^2$. [ 4 - A ]

d) We ignore the inter-cell interference and focus exclusively on cell 1. The received signal at user 1 and user 2 write as

$$y_1 = \mathbf{h}_{1,1}\mathbf{p}_{1,1}c_{1,1} + \mathbf{h}_{1,1}\mathbf{p}_{1,2}c_2 + \mathbf{n}_1,$$

$$y_2 = \mathbf{h}_{2,1}\mathbf{p}_{1,1}c_{1,1} + \mathbf{h}_{2,1}\mathbf{p}_{1,2}c_2 + \mathbf{n}_2.$$

i) The transmit precoder follows from the zero forcing beamforming, i.e. the multi-user interference is completely zero-forced. Hence we choose $\mathbf{p}_{1,2}$ such that $\mathbf{h}_{1,1}\mathbf{p}_{1,2} = 0$ and $\mathbf{p}_{1,1}$ such that $\mathbf{h}_{2,1}\mathbf{p}_{1,1} = 0$. Writing $\mathbf{h}_{1,1} = [h_{1,1,a}\, h_{1,1,b}]$ and $\mathbf{h}_{2,1} = [h_{2,1,a}\, h_{2,1,b}]$, we take

$$\mathbf{p}_{1,2} = [-h_{1,1,b}\, h_{1,1,a}]^T / \sqrt{|h_{1,1,a}|^2 + |h_{1,1,b}|^2}\sqrt{s_2}.$$

Similarly,

$$\mathbf{p}_{1,1} = [-h_{2,1,b}\, h_{2,1,a}]^T / \sqrt{|h_{2,1,a}|^2 + |h_{2,1,b}|^2}\sqrt{s_1}$$

where $s_1$ and $s_2$ are powers allocated to stream 1 and 2, respectively, with $s_1 + s_2 = P$.

[ 5 - A ]

ii) Rate of user 1 writes as

$$R_1 = \log_2\left(1 + \frac{|\mathbf{h}_{1,1}\mathbf{p}_{1,1}|^2}{\sigma_n^2}\right).$$

where $|\mathbf{h}_{1,2}\mathbf{p}_2|^2 = 0$ because inter-cell interference has been mitigated and $|\mathbf{h}_{1,1}\mathbf{p}_{1,2}|^2 = 0$ because of the ZF precoder as given in previous question. Similarly,

$$R_2 = \log_2\left(1 + \frac{|\mathbf{h}_{2,1}\mathbf{p}_{1,2}|^2}{\sigma_n^2}\right).$$

At high SNR, with $s_1 = \alpha P$ and $s_2 = (1-\alpha)P$, $R_1 + R_2 \approx \log_2(P/\sigma_n^2) + \log_2(P/\sigma_n^2) + C$ Hence a multiplexing gain of 2 at high SNR.

[ 5 - A ]

iii) From above,

$$R_1 + R_2 = \log_2\left(1 + \frac{|\mathbf{h}_{1,1}\mathbf{w}_{1,1}|^2 s_1}{\sigma_n^2}\right) + \log_2\left(1 + \frac{|\mathbf{h}_{2,1}\mathbf{w}_{1,2}|^2 s_2}{\sigma_n^2}\right)$$

where $\mathbf{p}_{1,1} = \mathbf{w}_{1,1}\sqrt{s_1}$ and $\mathbf{p}_{1,2} = \mathbf{w}_{1,2}\sqrt{s_2}$, subject to $s_1 + s_2 = P$. The optimal power allocation is given by the water filling solution $s_1^\star = \left(\mu - \frac{\sigma_n^2}{|\mathbf{h}_{1,1}\mathbf{w}_{1,1}|^2}\right)^+$ and $s_2^\star = \left(\mu - \frac{\sigma_n^2}{|\mathbf{h}_{2,1}\mathbf{w}_{1,2}|^2}\right)^+$ where $\mu$ is chosen such that the power constraint is satisfied.

[ 5-B ]