IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2010

EEE PART II: MEng, BEng and ACGI

Corrected Copy

DEVICES AND FIELDS

Wednesday, 9 June 2:00 pm

Time allowed: 2:00 hours

There are SIX questions on this paper.

Question ONE and Question FOUR are compuslory. Answer Question One, Question Four, plus one additional question from Section A and one additional question from Section B.

Questions One and Four are each weighted at 20%. Remaining questions are each weighted at 30%.

Use a separate answer book for each section.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : K. Fobelets, R.R.A. Syms

Second Marker(s) : W.T. Pike, Z. Durrani

Special instructions for invigilators

This exam consists of **2 sections**. Section A: **Devices** and section B: **Electromagnetic Fields**. Each section has to be solved in their respective answer books. Check that 2 different answer books are available for the students.

Questions 1 and 4 are obligatory.

Special instructions for students

Use different answers books for each section:

Devices: answer book **A**

Electromagnetic Fields: answer book **B**

Questions **1** and **4** are obligatory.

Constants and Formulae for section A: Devices

permittivity of free space:	$\varepsilon_o = 8.85 \times 10^{-12}$ F/m
permeability of free space:	$\mu_o = 4\pi \times 10^{-7}$ H/m
intrinsic carrier concentration in Si:	$n_i = 1.45 \times 10^{10}$ cm^{-3} at $T = 300$K
dielectric constant of Si:	$\varepsilon_{Si} = 11$
dielectric constant of SiO_2:	$\varepsilon_{ox} = 4$
thermal voltage:	$kT/e = 0.026$V at $T = 300$K
charge of an electron:	$e = 1.6 \times 10^{-19}$ C

$$\left.\begin{aligned} J_n(x) &= e\mu_n n(x)E(x) + eD_n \frac{dn(x)}{dx} \\ J_p(x) &= e\mu_p p(x)E(x) - eD_p \frac{dp(x)}{dx} \end{aligned}\right\} \quad \text{Drift-diffusion current equations}$$

$$I_{DS} = \frac{\mu C_{ox} W}{L}\left((V_{GS} - V_T)V_{DS} - \frac{V_{DS}^2}{2}\right) \quad \text{MOSFET current}$$

$$\left.\begin{aligned} J_n &= \frac{eD_n n_p}{L_n}\left(e^{\frac{eV}{kT}} - 1\right) \\ J_p &= \frac{eD_p p_n}{L_p}\left(e^{\frac{eV}{kT}} - 1\right) \end{aligned}\right\} \quad \text{Diode diffusion currents}$$

$$V_0 = \frac{kT}{e}\ln\left(\frac{N_A N_D}{n_i^2}\right) \quad \text{Built-in voltage}$$

$$c = c_0 \exp\left(\frac{eV}{kT}\right) \text{ with } \begin{cases} c = p_n \text{ or } n_p \\ c_0 \text{ bulk minority carrier concentration} \end{cases} \quad \text{Minority carrier injection under bias } V$$

$$\delta c = \Delta c \exp\left(\frac{-x}{L}\right) \text{ with } \begin{cases} \delta c = \delta p_n \text{ or } \delta n_p \\ \Delta c \text{ the excess carrier concentration} \\ \text{at the edge of the depletion region} \end{cases} \quad \text{Excess carrier concentration as a function of distance}$$

$$L = \sqrt{D\tau} \quad \text{Diffusion length}$$

$$D = \frac{kT}{e}\mu \quad \text{Einstein relation}$$

$$W_{depl} = \left[\frac{2\varepsilon V_0}{e}\frac{N_A + N_D}{N_A N_D}\right]^{1/2} \quad \text{Depletion width in pn diode}$$

$$C_{diff} = \frac{e}{kT}I\tau \quad \text{Diffusion capacitance}$$

Constants and Formulae for section B: Electromagnetic Fields

Vector calculus (Cartesian co-ordinates)

$\nabla = \underline{i}\ \partial/\partial x + \underline{j}\ \partial/\partial y + \underline{k}\ \partial/\partial z$

$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$

$\mathrm{grad}(\phi) = \nabla\phi = \underline{i}\ \partial\phi/\partial x + \underline{j}\ \partial\phi/\partial y + \underline{k}\ \partial\phi/\partial z$

$\mathrm{div}(\underline{\mathbf{F}}) = \nabla \cdot \underline{\mathbf{F}} = \partial \mathbf{F}_x/\partial x + \partial \mathbf{F}_y/\partial y + \partial \mathbf{F}_z/\partial z$

$\mathrm{curl}(\underline{\mathbf{F}}) = \nabla \times \underline{\mathbf{F}} = \underline{i}\ \{\partial \mathbf{F}_z/\partial y - \partial \mathbf{F}_y/\partial z\} + \underline{j}\ \{\partial \mathbf{F}_x/\partial z - \partial \mathbf{F}_z/\partial x\} + \underline{k}\ \{\partial \mathbf{F}_y/\partial x - \partial \mathbf{F}_x/\partial y\}$

Where ϕ is a scalar field and $\underline{\mathbf{F}}$ is a vector field

Maxwell's equations – integral form

$\iint_A \underline{\mathbf{D}} \cdot d\underline{a} = \iiint_V \rho\ dv$

$\iint_A \underline{\mathbf{B}} \cdot d\underline{a} = 0$

$\int_L \underline{\mathbf{E}} \cdot d\underline{L} = -\iint_A \partial\underline{\mathbf{B}}/\partial t \cdot d\underline{a}$

$\int_L \underline{\mathbf{H}} \cdot d\underline{L} = \iint_A [\underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t] \cdot d\underline{a}$

Where $\underline{\mathbf{D}}$, $\underline{\mathbf{B}}$, $\underline{\mathbf{E}}$, $\underline{\mathbf{H}}$, $\underline{\mathbf{J}}$ are time-varying vector fields

Maxwell's equations – differential form

$\mathrm{div}(\underline{\mathbf{D}}) = \rho$

$\mathrm{div}(\underline{\mathbf{B}}) = 0$

$\mathrm{curl}(\underline{\mathbf{E}}) = -\partial\underline{\mathbf{B}}/\partial t$

$\mathrm{curl}(\underline{\mathbf{H}}) = \underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t$

Material equations

$\underline{\mathbf{J}} = \sigma\ \underline{\mathbf{E}}$

$\underline{\mathbf{D}} = \varepsilon\ \underline{\mathbf{E}}$

$\underline{\mathbf{B}} = \mu\ \underline{\mathbf{H}}$

Theorems

$\iint_A \underline{\mathbf{F}} \cdot d\underline{a} = \iiint_V \mathrm{div}(\underline{\mathbf{F}})\ dv$ – Gauss' theorem

$\int_L \underline{\mathbf{F}} \cdot d\underline{L} = \iint_A \mathrm{curl}(\underline{\mathbf{F}}) \cdot d\underline{a}$ – Stokes' theorem

$\mathrm{curl}\ \{\mathrm{curl}(\underline{\mathbf{F}})\} = \mathrm{grad}\ \{\mathrm{div}(\underline{\mathbf{F}})\} - \nabla^2\underline{\mathbf{F}}$

Electromagnetic waves (pure dielectric media)

Time dependent vector wave equation $\nabla^2\underline{\mathbf{E}} = \mu_0\varepsilon\ \partial^2\underline{\mathbf{E}}/\partial t^2$

Time independent scalar wave equation $\nabla^2\underline{E} = -\omega^2\mu_0\varepsilon_0\varepsilon_r \underline{E}$

For z-going, x-polarized waves $d^2E_x/dz^2 + \omega^2\mu_0\varepsilon_0\varepsilon_r E_x = 0$

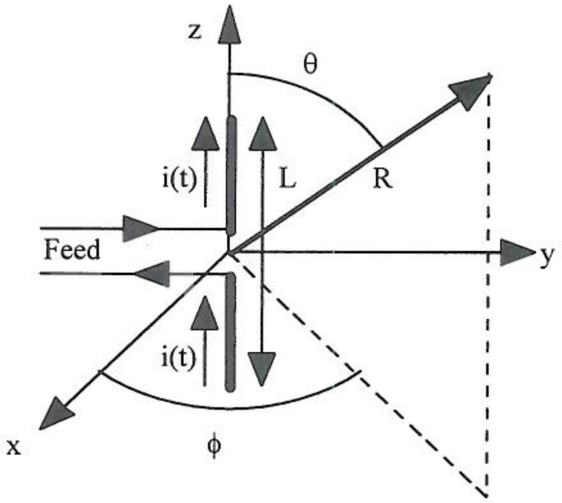
Where $\underline{E}$ is a time-independent vector field

Antenna formulae

Far-field pattern of half-wave dipole

$E_\theta = j\, 60I_0 \{\cos[(\pi/2)\cos(\theta)] / \sin(\theta)\} \exp(-jkR)/R;\ H_\phi = E_\theta/Z_0$

Here I_0 is peak current, R is range and $k = 2\pi/\lambda$



Power density $\underline{S} = 1/2\ \mathrm{Re}\ (\underline{E} \times \underline{H}^*) = S(R, \theta)$

Normalised radiation pattern $F(\theta, \phi) = S(R, \theta, \phi) / S_{max}$

Directivity $D = 1/ \{1/4\pi \iint_{4\pi} F(\theta, \phi) \sin(\theta)\, d\theta\, d\phi\}$

Gain $G = \eta D$ where η is antenna efficiency

Effective area $A_e = \lambda^2 D/4\pi$

Friis transmission formula $P_r = P_t\,(\eta_t\eta_r A_t A_r/R^2\lambda^2)$

The Questions

SECTION A: SEMICONDUCTOR DEVICES

1. Obligatory question.

a) What is the name of the minority carrier in n-doped Si? [2]

b) What is the type of current that is flowing when a carrier gradient exists? [2]

c) Briefly describe the process of recombination of carriers. [2]

e) What carrier related effect causes the switching delay in a pn diode when switching from ON to OFF? [2]

d) Sketch the variation of the minority carrier concentration in the p-region of a forward biased pn diode when the p-region is longer than the minority carrier diffusion length. Label your axes with all important parameters. [6]

f) In fig. 1.1 the different carrier fluxes that flow in a BJT in forward active mode are given. Describe the three physics related processes labelled 1, 2 and 3, indicated on the figure, that contribute to the base current i_B. [6]

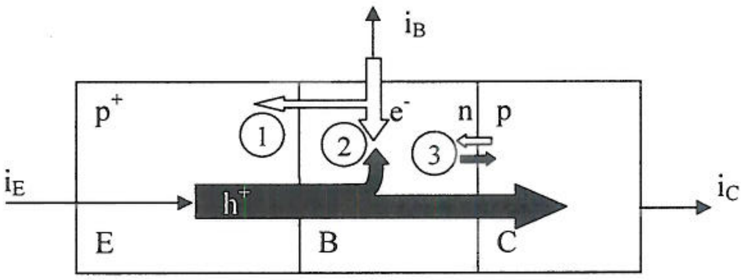


Figure 1.1: A pnp bipolar transistor in forward active mode. The black arrows indicate the hole flux, the white arrows indicate the electron flux.

SECTION A: SEMICONDUCTOR DEVICES

2. A pn diode is designed with circular cross section of $A = 100\ \mu m^2$ on a p-type substrate with a doping concentration of $N_A = 10^{16}\ cm^{-3}$. The doping density of the n layer is $N_D = 10^{19}\ cm^{-3}$. The n-layer is short with a length of $X_n = 200$ nm and the p-layer is long with a length of $X_p = 500\ \mu m$. The diffusion length of the minority carriers is: $L_n = 500\ 10^{-7}$ cm and $L_p = 400\ 10^{-7}$ cm. The device is contacted by ideal Ohmic contacts and operates under a forward bias voltage of $V = 0.26$ V at room temperature.

a) Draw the excess minority carrier concentrations $n_p(x)$ and $p_n(x)$ as a function of distance x in both layers of the diode. Label both axes with all boundary parameters and ensure relative magnitudes are correct. [6]

b) Give the value of the minority carrier concentration in both layers at:

i) the Ohmic contacts (n_{p0} & p_{n0}), [2]

ii) the edge of the depletion region (n'_p & p'_n). [2]

c) For this diode.

i) in which region (n or p) does recombination occur? [2]

ii) which type of carriers (electrons or holes) contributes the largest part of the total current? [2]

iii) which region (n or p) has the largest concentration of stored minority carrier charge? [2]

d) Calculate the stored minority carrier charge (Q_n and Q_p) in both regions. [8]

e) Based upon the answers above, make the appropriate approximations to calculate a fairly accurate value of the on current through the diode when you know that the recombination time of the minority carriers has a value of $\tau = \tau_p = \tau_n = 10^{-6}$ s. [6]

SECTION A: SEMICONDUCTOR DEVICES

3. A common emitter biased p$^+$np bipolar transistor is given in figure 3.1.

$I_E = I_n^{EB} + I_p^{EB}$

I_n^{EB} : electron current

I_p^{EB} : hole current across

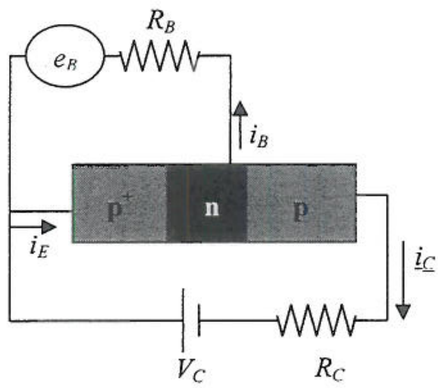


Figure 3.1: A pnp bipolar transistor in forward active mode.

a) Give the type of minority carriers (n or p) in the base. [2]

b) Assume that the emitter injection efficiency (ratio of "useful" emitter current to total emitter current) $\gamma = 1$ and the reverse base-collector bias current $I_{CB0} = 0$. Given these approximations, write the expression for:

   i) base current, I_B, [2]

   ii) collector current, I_C, [2]

   iii) current gain, β, [2]

   as a function of minority carrier charge in the base, Q_b and the appropriate time constants, τ. Define all parameters you use.

c) Give the expression of the saturation charge, Q_{sat}, in function of the parameters given in b) and fig.3.1. [8]

d) Now assume $\gamma < 1$ and the reverse base collector bias current $I_{CB0} = 0$.

   i) Give the new expression of the base current, I_B, [5]

   ii) Give the new expression of the collector current, I_C, [3]

   iii) Give the new expression of the current gain, β. [3]

e) How does the reduction of the emitter injection efficiency γ influence the current gain β of the bipolar junction transistor? [3]

The Questions

SECTION B: ELECTROMAGNETIC FIELDS

4. Obligatory question.

The relationship between ω and k for electromagnetic waves in the ionosphere has the form:

$$\omega = \sqrt{(\omega_P^2 + c^2k^2)}$$

Here c is the velocity of light and ω_P is the plasma frequency.

a) What is the ionosphere? Explaining your reasoning, sketch the dispersion diagram, and describe the nature of wave propagation in different frequency ranges.

[8]

b) In 1901, Guglielmo Marconi transmitted radio waves over the Atlantic. Explain how he was able to do this, given that the earth's curvature is such that there is no line-on-sight path. Explain the limitations of Marconi's approach, and the way these are overcome in modern long-distance free-space communications systems.

[8]

c) In 1924, Edward Appleton measured the height of the ionosphere. Explain how he was able to do this.

[4]

SECTION B: ELECTROMAGNETIC FIELDS

5.

a) A transmission line with characteristic impedance Z_1 is connected to a line with impedance Z_2. Show that the reflection coefficient for voltage waves is: $R_V = (Z_2 - Z_1)/(Z_2 + Z_1)$. Find the reflection coefficient when a 50 Ω line is terminated with a 0.5 μH inductor, at 10 MHz.

[10]

b) A section of line of impedance Z_0 and length L is terminated with a load Z_L. Find the input impedance. Assuming the load is a short circuit, and that the length L is less than a quarter of a wavelength, what kind of impedance does the terminated line represent?

[10]

c) A section of line of length L and impedance Z_3 is to be used to connect a first line of impedance $Z_1 = 50\ \Omega$ to a second line of length $Z_2 = 100\ \Omega$. Find the values of Z_3 and L that can yield zero reflection at 100 MHz. Each line is filled with a plastic whose relative dielectric constant is $\varepsilon_r = 4$. What other applications use this principle to reduce reflection?

[10]

SECTION B: ELECTROMAGNETIC FIELDS

6.

a) Give three reasons why up-shifting of the frequency is used in a radio communications system. Explain how up-shifting is achieved in amplitude-modulated radio.

[6]

b) Explain why antenna arrays are used, instead of simple dipoles. Demonstrate the performance advantage offered by a two-element array.

[12]

c) The Friis transmission formula $P_r = P_t\,(\eta_t \eta_r A_t A_r / R^2 \lambda^2)$ describes the received power in a radio communication system. Explain how it is derived, and derive the corresponding formula for the received power in a radar system. What factors govern the range of a radar system?

[12]

E2.3A DEVICES + FIELDS

The Answers 2010

SECTION A: SEMICONDUCTOR DEVICES

1. Obligatory question.

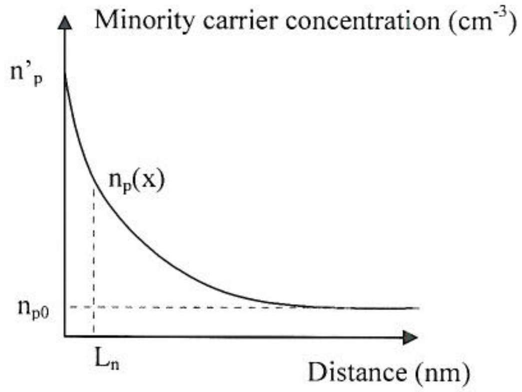
a) hole (p-type). [2]

b) Diffusion. [2]

c) The recombination process is the opposite of generation of carriers. In recombination, a hole and electron neutralise eachother and each recombination event causes of loss of 1 carrier of each type. [2]

e) Storage of minority carrier charge [2]

d)



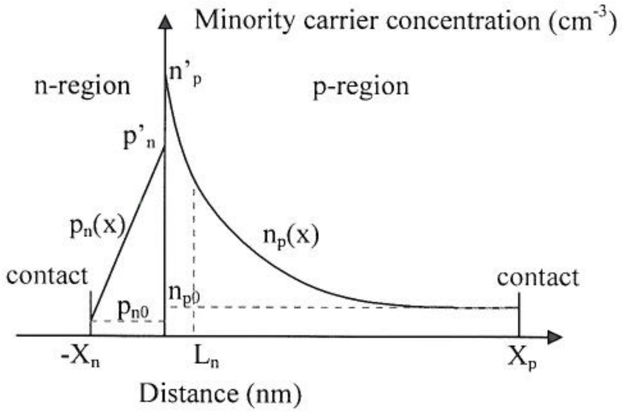
[6]

f) Base current consists of different terms that need to re-supply the majority carriers in the base due to
1: loss of electrons across the base-emitter junction
2: recombination of electrons and holes (holes are flowing through the base)
3: a small amount of electrons is gained from the reverse biased based-collector junction. [6]

SECTION A: SEMICONDUCTOR DEVICES

2.

a)



[6]

b) Give the value of the minority carrier concentration in both layers at:

i)

$$n_{p0} = \frac{n_i^2}{N_A} = \frac{(1.45\times10^{10})^2}{10^{16}} = 21025\ cm^{-3}$$

$$p_{n0} = \frac{n_i^2}{N_D} = \frac{(1.45\times10^{10})^2}{10^{19}} = 21.025\ cm^{-3}$$

[2]

ii)

$$n'_p = n_{p0}\exp\left(\frac{V}{V_T}\right) = n_{p0}\exp\left(\frac{0.26}{0.026}\right) = 21025\times22026 = 463\times10^6\ cm^{-3}$$

$$p'_n = p_{n0}\exp\left(\frac{V}{V_T}\right) = p_{n0}\exp\left(\frac{0.26}{0.026}\right) = 21.025\times22026 = 463\times10^3\ cm^{-3}$$

[2]

c) Understanding the diode.

i) In the long region, thus p. [2]

ii) The electrons because density largest. [2]

iii) The p-region because most electrons injected in that region. [2]

d) In the n^+ layer, area of triangle:

$$Q_p = \frac{e\times X_n\times(p'_n - p_{n0})\times A}{2}$$

$$Q_p = \frac{1.6\times10^{-19}\times200\times10^{-7}\times(463\times10^3-21)\times100(10^{-4}\,cm)^2}{2} = 7.41\times10^{-25}C$$

Note: p_{no} could have been easily neglected.

In the p layer, area under the carrier density is calculated via integral

$$Q_n = e \times A \times \int_0^{contact} n_p(x)dx = e \times A \times \int_0^{contact} \Delta n_p \exp\left(\frac{-x}{L_n}\right)dx$$

$$Q_n = e \times A \times L_n \times \Delta n_p \exp\left(\frac{-x}{L_n}\right)\Bigg|_0^{contact} = e \times A \times L_n \times \Delta n_p \times (0-1)$$

$$Q_n = -e \times A \times L_n \times (n'_p - n_{p0})$$ [8]

$$Q_n = -1.6 \times 10^{-19} \times 100(10^{-4})^2 \times 500 \times 10^{-7} \times (463 \times 10^6 - 21025)$$

$$Q_n = 3.7 \times 10^{-21} C$$

Note: n_{po} could have been easily neglected.

e) neglecting hole current because ND>>NA allows us to calculate the on current from the electron current only. In the p-region electron current is determined by the recombination processes and thus can be written as:

$$I_{ON} \approx I_n = \frac{Q_n}{\tau_n}$$

$$I_{ON} \approx \frac{3.7 \times 10^{-21}}{10^{-6}} = 3.7 \times 10^{-15} A$$ [6]

SECTION A: SEMICONDUCTOR DEVICES

3.

a) p or holes. [2]

b) When $\gamma = 1$ and $I_{CB0} = 0$ only recombination plays a role:

i) $I_B = \dfrac{Q_p}{\tau_p}$, Q_p is the charge in the base, τ_p is the lifetime the holes. [2]

ii) $I_C = \dfrac{Q_p}{\tau_t}$, τ_t is the transit time of the holes through the base [2]

iii) $\beta = \dfrac{I_C}{I_B} = \dfrac{\tau_p}{\tau_t}$ [2]

c) The maximum collector current that can flow is restricted by the external circuit. At that point the transistor has reached saturation. The collector current can then be expressed by both the external circuit as well as by the charge storage in the base.

$$I_C \text{ @ } saturation = I_{sat} \approx \frac{V_C}{R_C}$$

$$I_{sat} \approx \frac{Q_{sat}}{\tau_t}$$ [8]

$$\Rightarrow Q_{sat} = \frac{V_C}{R_C}\tau_t$$

d) Now assume $\gamma < 1$ and the reverse base collector bias current $I_{CB0} = 0$.

i) $\gamma < 1$ means that the loss of electrons out of the base via the emitter current (e- flux from base to emitter) $I_n^{EB} \neq 0$. Thus now both recombination as well as loss via the emitter base junction will lead to base current.

$$I_B = I_n^{EB} + \frac{Q_p}{\tau_p}$$ [5]

ii) $I_C = I_p^{EB} - \dfrac{Q_p}{\tau_p}$ [3]

iii) $\beta = \dfrac{I_C}{I_B} = \dfrac{I_p^{EB} - \dfrac{Q_p}{\tau_p}}{I_n^{EB} + \dfrac{Q_p}{\tau_p}}$ [3]

e) Current gain β decreases. [3]

SECTION B: ELECTROMAGNETIC FIELDS

4.

a) The ionosphere is a set of concentric quasi-static spherical shells surrounding the earth, which contain charged particles formed by cosmic ray bombardment of the upper atmosphere and consequently that can act somewhat like a metallic reflector. [1]

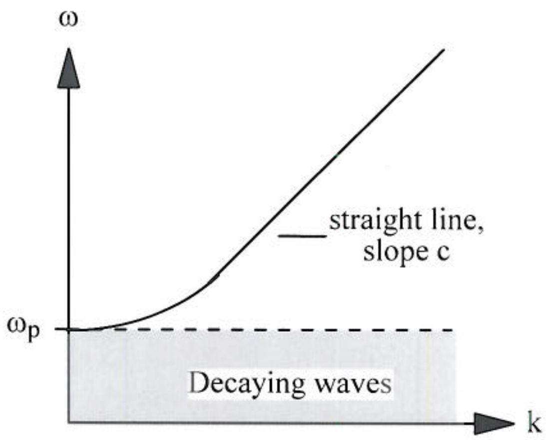
The ω-k relationship for electromagnetic waves in the ionosphere is:

$\omega = \sqrt{(\omega_P^2 + c^2k^2)}$

For small ω, this can be approximated as $\omega = \omega_P$

For large ω, this can be approximated as $\omega = ck$

Consequently the dispersion diagram must be as shown below.



[3]

Rearranging, we get:

$c^2k^2 = \omega^2 - \omega_P^2$

For $\omega > \omega_P$ we then get:

$k = 1/c \sqrt{(\omega^2 - \omega_P^2)}$

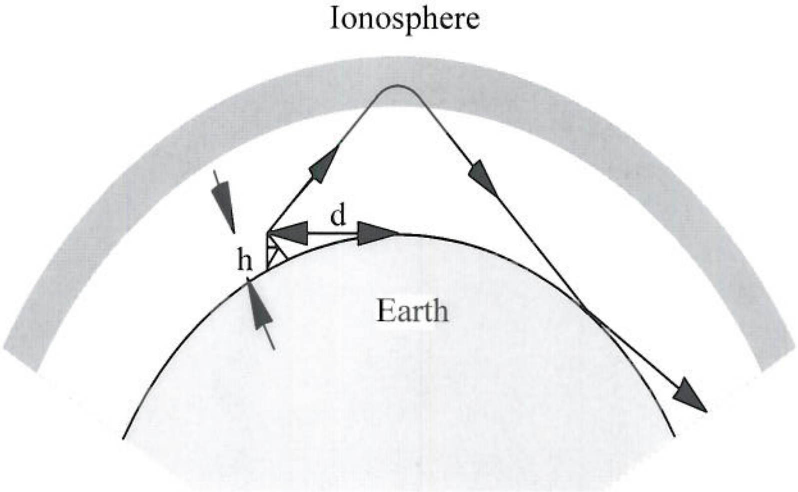
In this case, electromagnetic waves propagate as $E = E_0 \exp(-jkz)$, i.e. as travelling waves. [2]

For $\omega < \omega_P$ however we get:

$k = \pm j/c \sqrt{(\omega_P^2 - \omega^2)} = \pm jk'$

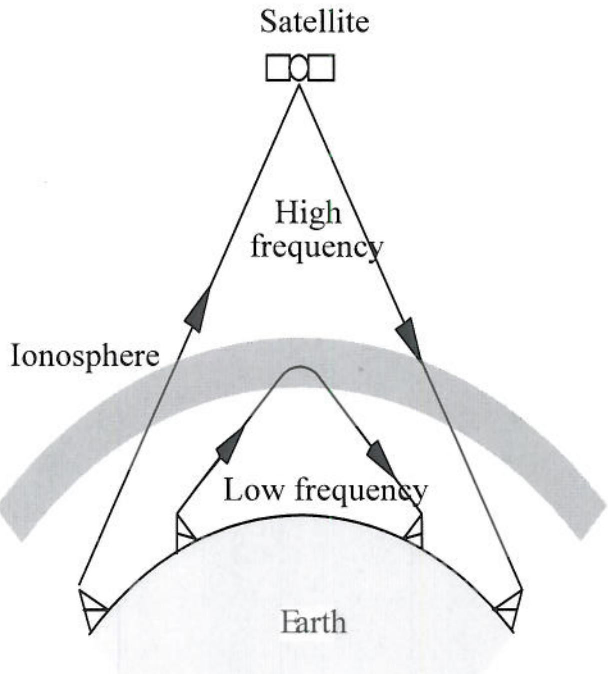
In this case, electromagnetic waves propagate as $E = E_0 \exp(-jkz) = E_0 \exp(\pm k'z)$. Travelling waves cannot exist, only exponentally decaying amplitudes, and incident waves must be reflected. [2]

b) Marconi was able to propagate low frequency waves beyond the limit set by the curvature of the earth by making use of the reflective properties of the ionosphere and the earth's surface. The latter (particularly the oceans) can operate similarly, since these also contain charged particles. Marconi must have used $\omega < \omega_P$.



[4]

Marconi's method is limited by the relative ineffectiveness of the ionosphere and the earth's surface as reflectors. Both are lossy, and the ionosphere is easily disrupted by the weather, particularly electrical storms. Modern communications systems operate by transmitting through the ionosphere at a frequency $\omega > \omega_P$ (microwaves), where there is transparency. The satellite regenerates the signal and rebroadcasts it down to a receiver.

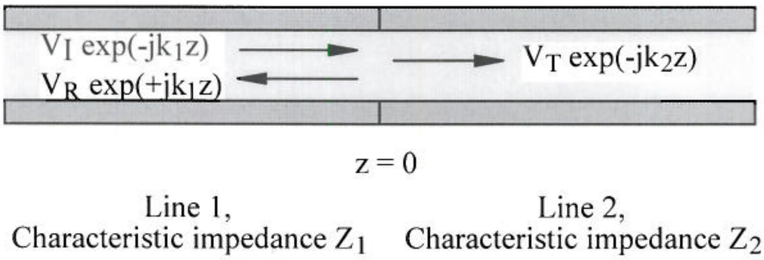


[4]

d) Appleton used a pulsed transmitter operating at $\omega < \omega_P$, transmitted the signal directly upwards and measured the time t taken for a reflected pulse to return. The height of the ionosphere was then c x t/2.

5.

a) Consider discontinuity between two lines as below. Assume there are incident and reflected waves on LHS, and transmitted wave on RHS. Assume boundary is at z = 0.



Match voltages and currents on boundary to get:

$V_I + V_R = V_T$

$V_I/Z_1 - V_R/Z_1 = V_T/Z_2$

Substituting upper equation into lower equation gives:

$V_I/Z_1 - V_R/Z_1 = (V_I + V_R)/Z_2$

Re-arranging gives

$V_I(1/Z_1 - 1/Z_2) = V_R(1/Z_1 + 1/Z_2)$

$V_I(Z_2 - Z_1)/Z_1Z_2 = V_R(Z_2 + Z_1)/Z_1Z_2$

Hence

$R = V_R/V_I = (Z_2 - Z_1)/(Z_2 + Z_1)$ [6]

If 0.5 μH inductor is attached to a line of 50 Ω characteristic impedance, the impedances at 10 MHz frequency are:

$Z_1 = 50\ \Omega$

$Z_2 = j\omega L = j\ 2\pi \times 10^7 \times 0.5 \times 10^{-6} = j\ 31.4\ \Omega$

The reflection coefficient is therefore:

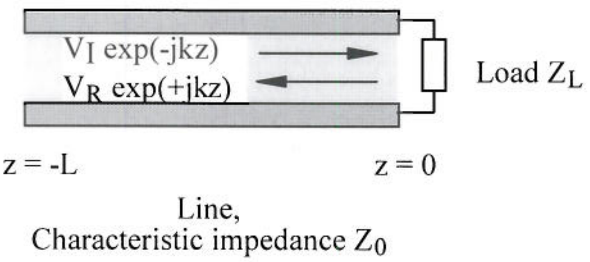
$R = (-50 + j\ 31.4) / (50 + j\ 31.4)$. Hence:

$R = (-50 + j\ 31.4) \times (50 - j\ 31.4) / (50^2 + 31.4^2)$

$R = \{(31.4^2 - 50^2) + j(2 \times 50 \times 31.4)\} / 3486.96 = \{-1513.04 + j\ 3141.6\} / 3486.96$

$R = -0.4339 + j\ 0.9009$ [4]

b) Consider a line of length L and impedance Z_0 terminated by a load as shown below.



At z = - L, the voltage and current are:

$V(-L) = V_I \exp(+jkL) + V_R \exp(-jkL)$

$I(-L) = (V_I/Z_0) \exp(+jkL) - (V_R/Z_0) \exp(-jkL)$

The input impedance is therefore $Z_{in} = V(-L)/I(-L)$ or:

$Z_{in} = \{V_I \exp(+jkL) + V_R \exp(-jkL)\} / \{(V_I/Z_0) \exp(+jkL) - (V_R/Z_0) \exp(-jkL)\}$

Since $V_R/V_I = R_V$, this result can be written as:

$$Z_{in} = \frac{Z_0 \{\exp(+jkL) + R_V \exp(-jkL)\}}{\{\exp(+jkL) - R_V \exp(-jkL)\}}$$

Since $R_V = (Z_L - Z_0)/(Z_L + Z_0)$, this result may be expanded as:

$$Z_{in} = \frac{Z_0 \{(Z_L + Z_0) \exp(+jkL) + (Z_L - Z_0) \exp(-jkL)\}}{\{(Z_L + Z_0) \exp(+jkL) - (Z_L - Z_0) \exp(-jkL)\}}$$

Finally, writing exp(+jkL) as cos(kL) + j sin(kL) we get:

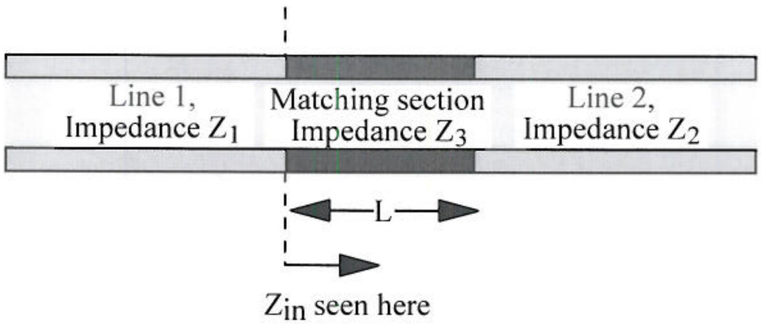
$$Z_{in} = \frac{Z_0 \{Z_L \cos(kL) + jZ_0 \sin(kL)\}}{\{Z_0 \cos(kL) + jZ_L \sin(kL)\}}$$ [7]

If $Z_L = 0$, the input impedance reduces to

$Z_{in} = jZ_0 \tan(kL)$

Now the propagation constant is $k = 2\pi/\lambda$, where λ is the wavelength in the medium. If $L < \lambda/4$, $kL < \pi/2$ and the tan function is positive. Consequently the terminated line represents an inductor. [3]

c) Consider the three-section arrangement shown below:



The input impedance seen at the point shown is:

$$Z_{in} = \frac{Z_3\{Z_2 \cos(k_3L) + jZ_3 \sin(k_3L)\}}{\{Z_3 \cos(k_3L) + jZ_2 \sin(k_3L)\}}$$

If the matching section is to suppress reflection at the junction to line 1, we need $Z_{in} = Z_1$. Generally Z_{in} is complex. However, when $k_3L = \pi/2$, we find that:

$Z_{in} = Z_3^2/Z_2$

Consequently, matching can be achieved if $Z_3^2/Z_2 = Z_1$, which requires $Z_3 = (Z_1Z_2)^{1/2}$. [4]

The impedance of the matching section should therefore be the geometric mean of the two lines. For $Z_1 = 50\ \Omega$ and $Z_2 = 100\ \Omega$ we therefore need $Z_3 = 70.7\ \Omega$. [1]

The propagation constant is $k_3 = 2\pi\varepsilon_r^{1/2}/\lambda$, where ε_r is the relative dielectric constant inside the line and λ is the wavelength. If $k_3L = \pi/2$, then $L = \lambda/4\varepsilon_r^{1/2}$, i.e. we need a section measuring a quarter of a wavelength in the dielectric medium being used.

If the frequency is f = 100 MHz, the wavelength is $\lambda = 3 \times 10^8 / (100 \times 10^6) = 3$ m.

If the relative dielectric constant is $\varepsilon_r = 4$, the length needed is $L = 3 / (4 \times 2) = 0.375$ m. [2]

The same approach is used in anti-reflection coatings at optical frequencies. The impedance of a general medium is $Z = Z_0/n$, where $Z_0 = 120\pi$ is the impedance of free space and n is the refractive index. To match between (say) air and the glass in a pair of sunglasses, we need to coat the glass with a thin layer of a further material whose thickness is one quarter of a wavelength and whose refractive index is $n^{1/2}$. [3]

6.

a) Upshifting is used in radio communications systems (which necessarily involve electromagnetic waves) so that:

The wavelength is reduced, so the wave can be launched from a small antenna

The frequency variation can be reduced, so the antenna can be narrow-band

Different carriers can be used, so the system can be frequency-multiplexed [3]

Upshifting is achieved in an AM radio system as follows. Ignoring phase, the signal can be written as $A_S \cos(\omega_S t)$ and the carrier as $A_C \cos(\omega_C t)$. The signal is first rescaled by a factor m and a DC component is added. Multiplication by the carrier then gives:

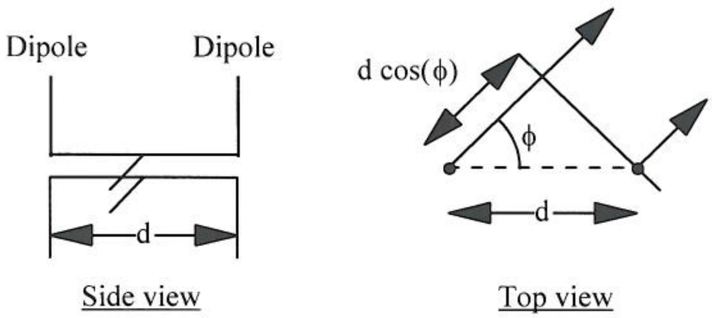
$A_C \cos(\omega_C t)\{1 + mA_S \cos(\omega_S t)\}$

$= A_C [\cos(\omega_C t) + (mA_S/2) \cos\{(\omega_C + \omega_S)t\} + (mA_S/2) \cos\{(\omega_C - \omega_S)t\}]$

The central term is then upshifted as required. [3]

b) Antenna arrays are used to improve directivity, and hence increase the power launched in or received from a given direction. [2]

An example is the broadside array, an array of otherwise undirective antennae (e.g. dipoles) arranged to have increased sensitivity in a broadside direction. The figure below shows a two-element array.



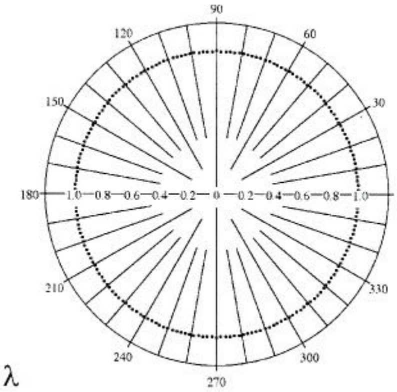
If the individual antenna in the figure below create far-field electric field patterns $E(\phi)$, the pair together generate a field $E'(\phi) = E(\phi) \{1 + \exp[-jkd \cos(\phi)]\}$

This can be written as $E'(\phi) = 2E(\phi) \exp\{-jkd \cos(\phi)/2\} \cos\{kd \cos(\phi)/2)\}$

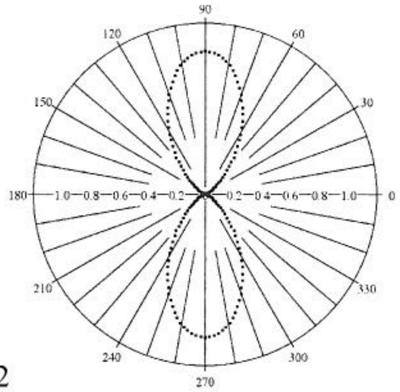
Hence, the overall radiation pattern is $F'(\phi) = \cos^2\{kd \cos(\phi)/2\}$ [4]

If $kd << \pi$, i.e. if $d << l$ the radiation pattern is approximately constant.

If $kd/2 = \pi/2$, i.e. if $d = \lambda/2$, the radiation pattern peaks when $\phi = \pi/2$, i.e. broadside on. Consequently there is much higher gain in this direction. Radiation patterns for these two cases in the antenna plane are then:



d << λ



$d = \lambda/2$ [6]

c) At a distance R from a loss-less, isotropic transmitter, the power density must be:

$S_{iso} = P_T/4\pi R^2$

A real antenna is neither loss-less nor isotropic, so the real power density is:

$S_{real} = \eta_T D_T\, S_{iso}$, where η_T and D_T is the transmitter's efficiency and directivity, respectively

Now the effective area is $A_T = \lambda^2 D_T/4\pi$ (given). Hence $D_T = 4\pi A_T/\lambda^2$ and:

$S_{real} = \eta_T\,(4\pi A_T/\lambda^2)\,S_{iso} = \eta_T\,(4\pi A_T/\lambda^2)\,(P_T/4\pi R^2) = P_T\,(\eta_T A_T/R^2\lambda^2)$

The received power is then:

$P_R = \eta_R A_R S_{real}$, where η_R and A_R are the receiver's efficiency and effective area.
Hence: $P_R = P_T\,(\eta_R A_R \eta_T A_T/R^2\lambda^2)$ – the Friis formula [4]

For a radar system, the real power density at a distance R from the transmitter is still:

$S_{real} = P_T\,(\eta_T A_T/R^2\lambda^2)$

The power intercepted by the target is:

$P_{int} = \sigma\, S_{real} = P_T\,(\sigma\eta_T A_T/R^2\lambda^2)$ where σ is the target's radar cross-section

This power is scattered isotropically. Since the transmit antenna is also used for reception the power density back at the receiver is:

$S_R = P_{int}/4\pi R^2 = P_T\,(\sigma\eta_T A_T/4\pi R^4\lambda^2)$

And the received power is:

$P_R = \eta_T A_T\, S_R = P_T\,(\sigma\eta_T^2 A_T^2/4\pi R^4\lambda^2)$ [4]

Typically, a receiver will require a threshold level of received power for successful detection, say P_{R0}. In this case we require:

$P_T (\sigma\eta_T{}^2 A_T{}^2/4\pi R^4\lambda^2) > P_{R0}$.

Inverting this, we have:

$R^4 < (P_T/P_{R0}) (\sigma\eta_T{}^2 A_T{}^2/4\pi\lambda^2)$

$R < \{(P_T/P_{R0}) (\sigma\eta_T{}^2 A_T{}^2/4\pi\lambda^2)\}^{1/4}$

Hence the range of the radar system is proportional to $P_T{}^{1/4}$. High power, a large effective area and a short wavelength are therefore required for long range. [4]