IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

EEE/ISE PART I: MEng, BEng and ACGI

INTRODUCTION TO SIGNALS AND COMMUNICATIONS

Wednesday, 15 June 10:00 am

Time allowed: 2:00 hours

There are THREE questions on this paper.

Answer ALL questions.
Q1 carries 40% of the marks. Questions 2 and 3 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	K.K. Leung
	Second Marker(s) :	M.K. Gurcan

Special Instructions for Invigilator: **None**

Information for Students:

Some Fourier Transforms

$$\cos \omega_o t \quad \Longleftrightarrow \quad \pi[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)]$$

Some useful trigonometric identities

$$\cos x \cos y = \tfrac{1}{2}\cos(x-y) + \tfrac{1}{2}\cos(x+y)$$
$$\sin(x-y) = \sin x \cos y - \sin y \cos x$$
$$a\cos x + b\sin x = c\cos(x+\theta)$$

where $c = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(-b/a)$

Complex exponential

$$e^{jx} = \cos x + j\sin x$$

1. This is a general question. (40%)

a. Consider a time function $f(t) = a e^{-bt} u(t)$ where a and b are positive constants, and $u(t) = 0$ for $t < 0$ and 1 for $t \geq 0$.

i. Derive the Fourier transform $F(\omega)$ of $f(t)$. [4]

ii. Sketch the frequency spectrum of $f(t)$. [3]

iii. If $f(t)$ is the unit impulse response of a linear time-invariant system, what can be said about the function of the system? [2]

b. The trigonometric Fourier series for a periodic signal $g(t)$ with period T_0 is given by

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t).$$

i. Express ω_0 in terms of T_0. [1]

ii. Provide a physical interpretation of the coefficients a_n's and b_n's. [2]

iii. If $g(t)$ has zero dc component, what can be said about any of the coefficients, a_0, a_n's and b_n's, and why? [2]

iv. If $g(t)$ is an even function of t, what can be said about any of the coefficients, a_0, a_n's and b_n's, and why? [2]

c. Consider two forms of amplitude modulation (AM) signal, namely, the full AM and double-sideband with suppressed carrier (DSB-SC). Let their waveforms be denoted by $\phi_{AM}(t)$ and $\phi_{DSB}(t)$, respectively. Let $m(t)$ be the modulating signal, ω_c be the carrier angular frequency in radians/second, and A be the amplitude of the carrier.

i. Give two expressions for $\phi_{AM}(t)$ and $\phi_{DSB}(t)$, respectively. [4]

ii. What do both forms of AM do on the modulating signal $m(t)$ from the frequency-domain perspective and why? [2]

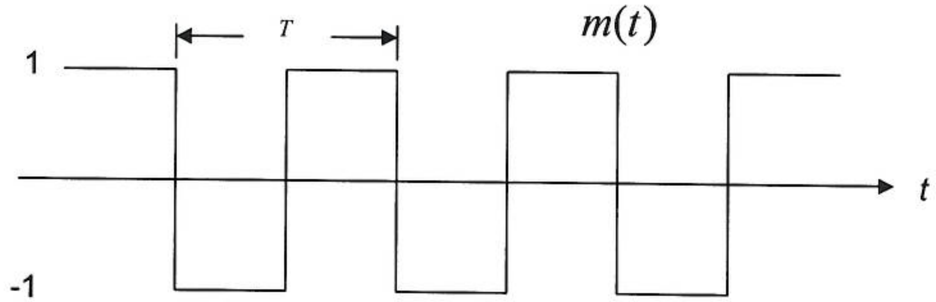
iii. If $m(t)$ has a bandwidth of B Hz, what is the bandwidth for both forms of AM signals. [2]

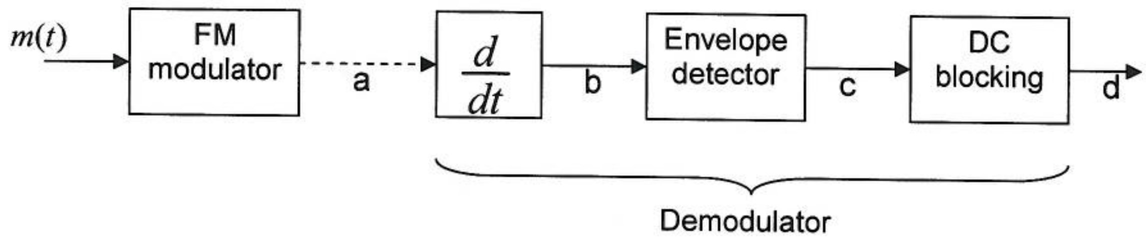
iv. How can the full AM and DSB-SC signals be demodulated at the receiver? [2]

v. What are the relative advantages and disadvantages of both forms of AM signals? [2]

1. This is a general question. (Continued)

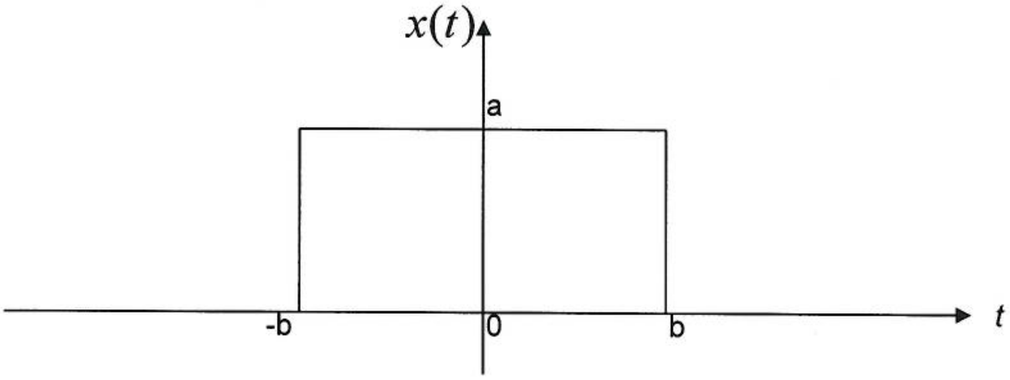
d. The following diagrams show a periodic square wave $m(t)$ and a block diagram for the frequency modulation (FM) and demodulation process, respectively. The periodic square wave $m(t)$ frequency-modulates a carrier of frequency $f_c = 10$ kHz with $\Delta f = 1$ kHz. The carrier amplitude is A. The resulting FM signal is demodulated, as shown in the diagram. Sketch the waveforms at points a, b, c and d. [12]





2. Signals. (30%)

a. Consider the following signal $x(t)$ where a and b are positive constants.



Let $y(t)$ be the self-convolution of $x(t)$. That is, $y(t) = x(t) * x(t)$.

i. Express $y(t)$ as an integral of $x(t)$. [3]

ii. Sketch $y(t)$. [6]

iii. Let $X(\omega)$ and $Y(\omega)$ denote the Fourier transforms of $x(t)$ and $y(t)$, respectively. Prove that $Y(\omega) = X(\omega)X(\omega)$. [5]

iv. Let $\hat{x}(t) = x(t - t_o)$ and $\hat{X}(\omega)$ denote the Fourier transform of $\hat{x}(t)$. Derive an expression for $\hat{X}(\omega)$ in terms of $X(\omega)$ and t_o. [4]

b. The exponential Fourier series for any periodic signal $f(t)$ with period T is given by

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_o t} \quad \text{where } D_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_o t}dt \text{ and } \omega_o = \frac{2\pi}{T}.$$

Now, consider the following signal that represents a train of unit impulses with a period of T:

$$f(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT).$$

i. Find the D_n's for $n = -\infty$ to ∞ in the exponential Fourier series for $f(t)$. [5]

ii. Sketch the frequency spectrum of $f(t)$. [4]

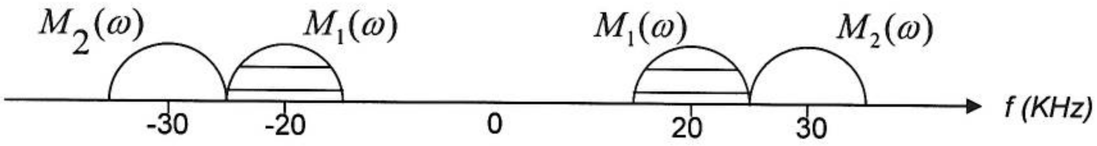
iii. Express the power of $f(t)$ in terms of the D_n's. [3]

3. Communications techniques. (30%)

a. Consider a communication system that is designed to simultaneously transmit two signals, $m_1(t)$ and $m_2(t)$, each of which has a bandwidth of 5 KHz. The frequency spectrum of the transmitted signal from the system,

$$\phi(t) = m_1(t)\cos(40{,}000\pi t) + m_2(t)\cos(60{,}000\pi t),$$

is given below where the centre of the spectrum of $m_1(t)$ and $m_2(t)$ is located at 20 and 30 KHz, respectively.



i. Use two oscillators, which can generate sinusoidal signals with required phases at frequencies 20 and 10 KHz, respectively, and ideal filters to design a receiver to recover the signals, $m_1(t)$ and $m_2(t)$, in the baseband. Draw a schematic block diagram of the receiver and specify the frequency range of each filter used. [8]

ii. Provide a mathematical justification for the receiver design. That is, explain by use of mathematics why the receiver can recover $m_1(t)$ and $m_2(t)$. [7]

iii. If a frequency tone can be included as part of the transmitted signal to simplify the receiver design, what is the preferred frequency of the tone and why can the frequency tone help? [4]

b. Consider a digital communication system where the modulating signal $m(t)$ has a bandwidth of B Hz, and is sampled at a frequency of f_S Hz.

i. Give a mathematical interpretation of the sampling operations of $m(t)$. [3]

ii. Use the results in part b of Question 2. Determine the minimum value of f_S so that the signal $m(t)$ can be faithfully represented by the samples. Draw frequency spectrum diagram(s) to explain your results. [8]

Introduction to Signals & Communications EE1-6 (1)

Solutions · 2011

1-a. i) $$F(\omega) = \int_{-\infty}^{\infty} a e^{-bt} u(t) e^{-j\omega t} dt$$

$$\Rightarrow F(\omega) = \int_0^{\infty} a e^{-bt} e^{-j\omega t} dt$$

$$= a \int_0^{\infty} e^{-(b+j\omega)t} dt$$

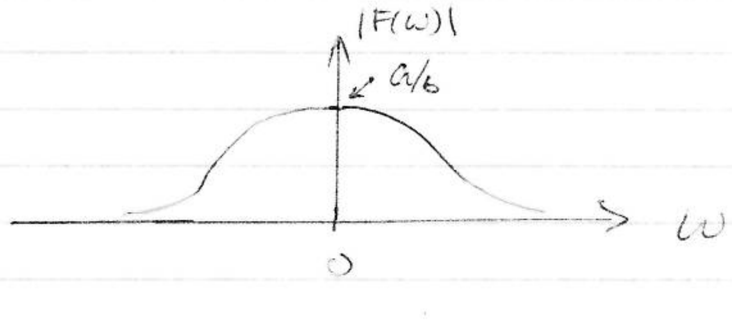
$$= \frac{a}{b+j\omega} \int_0^{\infty} d e^{-(b+j\omega)t}$$

$$\Rightarrow F(\omega) = \frac{-a}{b+j\omega} e^{-(b+j\omega)t} \Big|_0^{\infty}$$

$$\Rightarrow F(\omega) = \frac{a}{b+j\omega}$$

$$\Rightarrow F(\omega) = \frac{a/b}{1+j(\omega/b)}$$

ii)



iii) The system is a low-pass filter.

1 b. i) $\omega_0 = \frac{2\pi}{T_0}$

ii) The magnitude of a_n reflects how $g(t)$ is similar to $\cos(n\omega_0 t)$ - a sinusoidal waveform.

Similarly, b_n shows how $g(t)$ is similar to $\sin(n\omega_0 t)$ for all n.

iii) If $g(t)$ has zero dc component,

$a_0 = 0$

iv) If $g(t)$ is an even function of t,

Then $b_n = 0$ for $\forall\, n$ because $\cos(n\omega_0 t)$ is even and $\sin(n\omega_0 t)$ is an odd function of t for all n.

v) If any of a_0, a_n's or b_n's are complex, $g(t)$ is complex.

1c. i) $$\phi_{AM}(t) = \left[A + m(t) \right] \cos(\omega_c t)$$

$$\phi_{DSB}(t) = m(t) \cos(\omega_c t)$$

ii) Both forms of AM multiply the modulating signal $m(t)$ with a sinusoidal carrier. In frequency domain, that corresponds to shifting the spectrum of the modulating signal to the frequency band centering at the carrier frequency, ω_c.

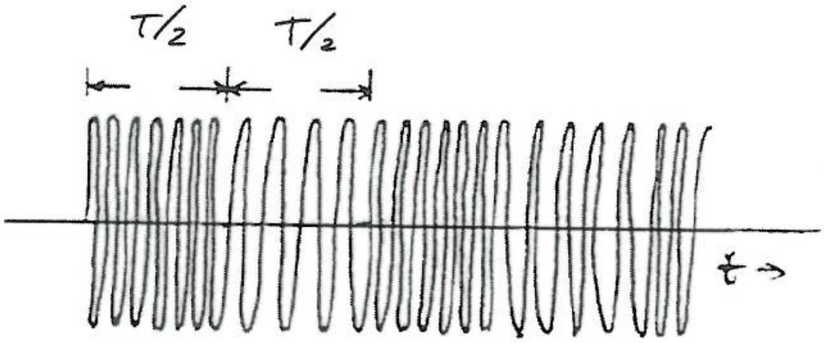
iii) The bandwidth of both forms of AM is $2B$ Hz.

iv) For the full AM, the signal can be demodulated by envelop detector, while the DSB-SC signal can be demodulated by coherent detection, i.e. multiply with the carrier frequency ω_c with the same phase as that at the transmitting side.

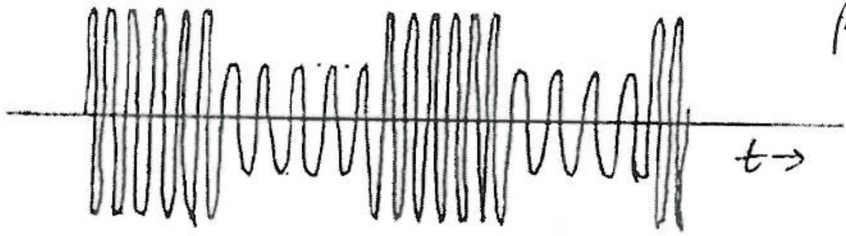
v)

	Full AM	DSB-SC
Advantages	Simple detector	power efficiency
Disadvantages	Lower power efficiency	complex phase lock loop, coherent detector

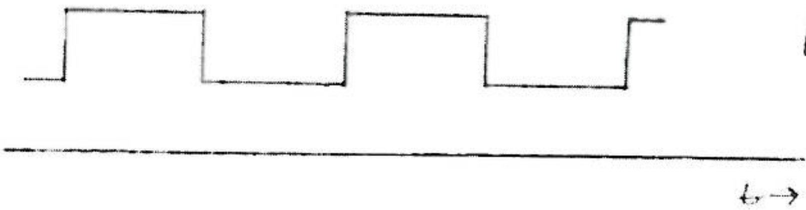
1. d.



Point a



Point b



Point c

Point d

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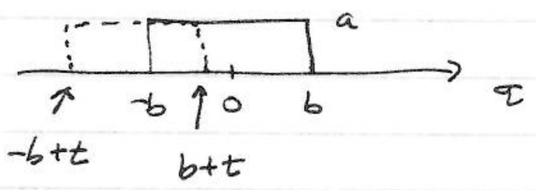
2a i) $$y(t) = \int_{-\infty}^{\infty} x(\tau)\, x(t-\tau)\, d\tau$$

ii) $y(t) = 0$ for $t < -2b$ and $t > 2b$

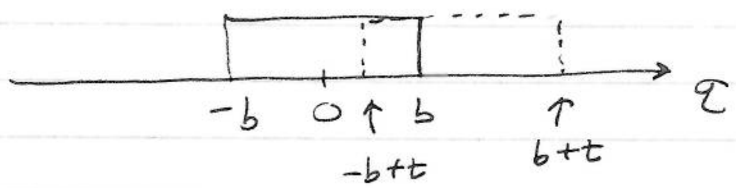
For $-2b \leq t \leq 0$



$$y(t) = \int_{-b}^{b+t} a^2\, dt = a^2\left[b+t+b\right]$$

$$\Rightarrow \quad y(t) = a^2\left[t+2b\right]$$

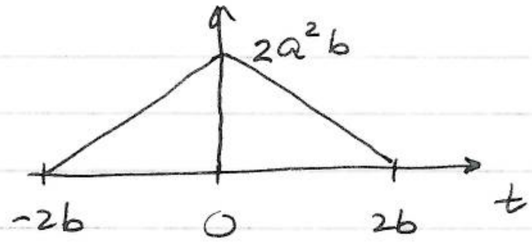
For $2b \geq t > 0$,



$$y(t) = \int_{-b+t}^{b} a^2\, dt = a^2\left[b-(-b+t)\right]$$

$$\Rightarrow \quad y(t) = a^2\left[-t+2b\right]$$

2a ii)



iii) $\mathcal{F}[y(t)] = \mathcal{F}[x(t) * x(t)]$

$$\Rightarrow \quad Y(\omega) = \int_{t=\infty}^{\infty} x(t) * x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{t=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} x(\tau)\, x(t-\tau)\, e^{-j\omega t}\, d\tau\, dt$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau) \int_{t=-\infty}^{\infty} x(t-\tau)\, e^{-j\omega(t-\tau)}\, dt\, e^{-j\omega\tau}\, d$$

$$\Rightarrow \quad Y(\omega) = \int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau} \int_{t=-\infty}^{\infty} x(t-\tau)\, e^{-j\omega(t-\tau)}\, d(t-\tau)$$

$$\cdot\, d\tau$$

$$Y(\omega) = \int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, X(\omega)\, d\tau$$

$$\Rightarrow \quad Y(\omega) = X(\omega)\, X(\omega)$$ Q.E.D.

2a.iv) $\hat{X}(\omega) = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt$

$= \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega(t-t_0)} \cdot e^{-j\omega t_0} dt$

$\Rightarrow \quad \hat{X}(\omega) = e^{-j\omega t_0} X(\omega).$

2b. i) By definition,

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t) e^{-jn\omega_0 t} dt$$

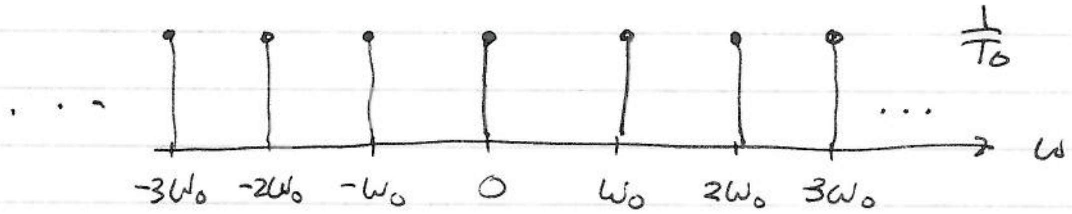
$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \sum_{m=\infty}^{\infty} \delta(t - mT_0) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jn\omega_0 t} dt$$

$$\Rightarrow D_n = \frac{1}{T_0} e^{-jn\omega_0 0} = \frac{1}{T_0} \quad \forall n.$$

ii) Since $f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$

$$\Rightarrow f(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

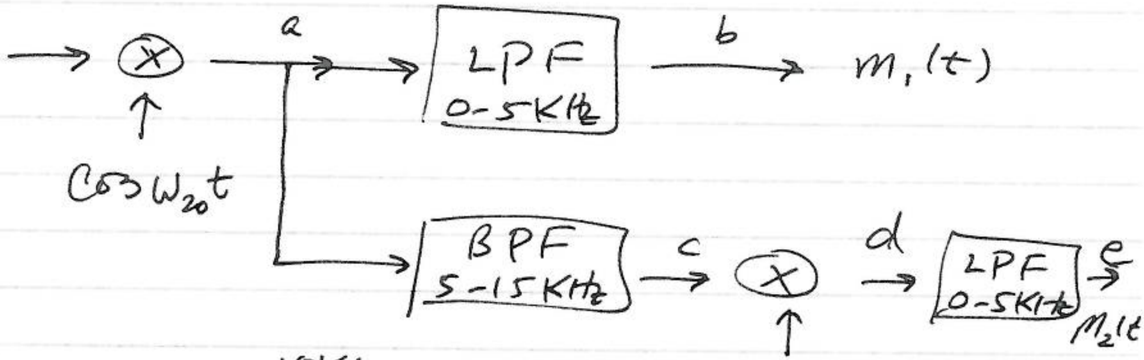


iii) By Parseval's theorem,

$$P = \sum_{n=-\infty}^{\infty} |D_n|^2 = \sum_{n=-\infty}^{\infty} \left(\frac{1}{T_0}\right)^2$$

(10)

3 a i)



where $\omega_{10} = 10K/2\pi$

$\omega_{20} = 20K/2\pi$ $\quad W_{30} = 30K/2\pi$

ii) The transmitted signal

$$\phi(t) = m_1(t)\cos W_{30}t + m_2(t)\cos W_{20}t$$

Signal at point a :

$$S_a(t) = \phi(t)\cos W_{20}t$$

$$= \left[m_1(t)\cos W_{20}t + m_2(t)\cos W_{30}t\right]\cos W_{20}t$$

$$= \frac{1}{2}\left[m_1(t)\right]\left[\cos(2W_{20}t)\right] + m_2(t)\cos W_{30}t\cos W_{20}$$

$$= \frac{1}{2}\left[m_1(t) + m_1(t)\cos(2W_{20}t)\right]$$

$$+ \frac{1}{2}m_2(t)\left[\cos(W_{30}-W_{20})t + \cos(W_{30}+W_{20})t\right]$$

Observe all the terms in $S_a(t)$, $S_b(t)$ is the output of the LPF :

$$S_b(t) = \frac{1}{2}m_1(t)$$

Signal at point c after the BPF 5-15KHz

$$S_c(t) = \frac{1}{2}m_2(t)\cos(W_{30}-W_{20})t$$

$$= \frac{1}{2}m_2(t)\cos(W_{10}t)$$

3 a. ii) Signal at point d:

$$S_d(t) = S_c(t) \cdot \cos(\omega_{10} t)$$

$$= \frac{1}{2} m_2(t) \left[\frac{1 + \cos(2\omega_{10} t)}{2} \right]$$

$$= \frac{1}{4} \left[m_2(t) + \underline{m_2(t) \cos(2\omega_{10} t)} \right]$$

beyond the range of the LPF

$$\Rightarrow \quad S_e(t) = \frac{1}{4} m_2(t)$$

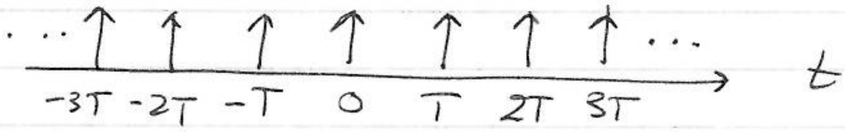
iii) The preferred frequency carrier is 20 KHz.

Then, a frequency carrier of 10KHz can also be obtained at the receiver.

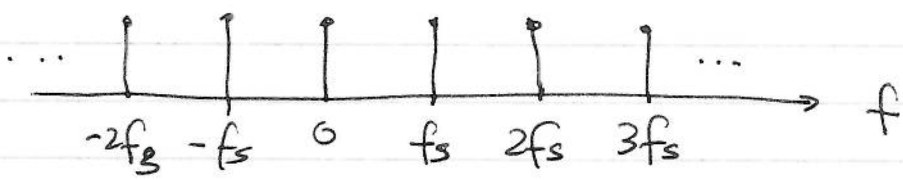
The carriers of 20 & 10 KHz in the exact phase can enable the coherent detection (i.e., multiply the signal with the corresponding sinusoidal carrier).

3b. i) The sampling operations correspond to multiplying $m(t)$ with a train of impulses with a period of $1/f_s$.

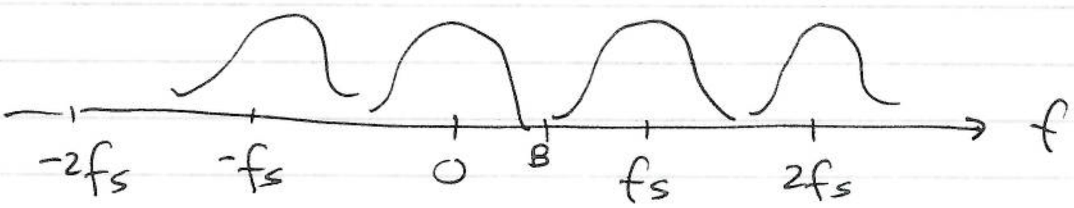
ii) Train of impulses



The corresponding frequency spectrum (from part b of Question 2)



multiplication in time domain corresponds to convolution in frequency domain. Since the spectrum of the train of sampling impulses is also a train of impulses in frequency domain, we have the spectrum of the sampled $m(t)$



To faithfully recover $m(t)$, we need

$$f_s > 2B.$$

Otherwise, the replica of $M(\omega)$ will overlap with each other and there change $M(\omega)$

That filtering is not enough for recovery of $m(t)$