SOLUTIONS: SYSTEMS IDENTIFICATION

1. Solution

a) Consider the scheme shown in Fig. 1.1 in the text of the exam paper. As we are dealing with linear systems, the transfer function from the input process $\eta(t)$ to the output $y(t)$ can be determined setting $\xi(t) = 0, \forall t \geq 0$.

Let us denote the transfer function from the input $\eta(t)$ to the output $y(t)$ as $H_{\eta y}(z)$. Hence, by inspection of the scheme, we have:

$$H_{\eta y}(z) = \frac{1+z^{-1}}{1+\frac{1}{2}z^{-1}} - z^{-1} = \frac{1-\frac{1}{2}z^{-2}}{1+\frac{1}{2}z^{-1}} = \frac{z^2-\frac{1}{2}}{z(z+\frac{1}{2})}$$

where the last term has been equivalently rewritten in terms of positive powers of z.

Again considering the scheme shown in Fig. 1.1 in the text of the exam paper, the transfer function from the input $\xi(t)$ to the output $y(t)$ can be determined setting $\eta(t) = 0, \forall t \geq 0$. Then, by denoting with $H_{\xi y}(z)$ the transfer function from the input $\xi(t)$ to the output $y(t)$, we immediately get:

$$H_{\xi y}(z) = \frac{1}{1-\frac{1}{3}z^{-1}} = \frac{z}{z-\frac{1}{3}}$$

where, again, the last term has been equivalently rewritten in terms of positive powers of z.

[4 Marks]

b) Notice that, by assumption, $\eta(\cdot) \sim WN(0,1)$ and $\xi(\cdot) \sim WN(0,4)$ are white uncorrelated stochastic processes. Moreover, the processes $v(\cdot)$ and $w(\cdot)$ are stationary because output processes of asymptotically stable systems driven by white processes (the transfer functions determined in the answer to Question 2b) have poles lying strictly inside the unit circle). Hence

$$y(t) = v(t) + w(t) \Longrightarrow \Gamma_y(\omega) = \Gamma_v(\omega) + \Gamma_w(\omega)$$

where $\Gamma_v(\omega)$ and $\Gamma_w(\omega)$ denote the spectra of $v(\cdot)$ and $w(\cdot)$, respectively.

Let us first compute the spectrum of $\Gamma_v(\omega)$:

$$\Gamma_v(\omega) = \left|H_{\eta y}(e^{j\omega})\right|^2 \text{var}[\eta(t)] = \frac{\left|e^{2j\omega}-\frac{1}{2}\right|^2}{\left|e^{j\omega}\right|^2\left|e^{j\omega}+\frac{1}{2}\right|^2} \cdot 1 = \frac{\left|\cos(2\omega)+j\sin(2\omega)-\frac{1}{2}\right|^2}{\left|\cos(\omega)+j\sin(\omega)+\frac{1}{2}\right|^2}$$
$$= \cdots = \frac{5-4\cos(2\omega)}{5+4\cos(\omega)}$$

Now, we compute the spectrum of $\Gamma_w(\omega)$:

$$\Gamma_w(\omega) = \left|H_{\xi y}(e^{j\omega})\right|^2 \text{var}[\xi(t)] = \frac{\left|e^{j\omega}\right|^2}{\left|e^{j\omega}-\frac{1}{3}\right|^2} \cdot 4 = \frac{4}{\left|\cos(\omega)+j\sin(\omega)-\frac{1}{3}\right|^2}$$
$$= \cdots = \frac{36}{10-6\cos(\omega)}$$

Finally, the spectrum $\Gamma_y(\omega)$ is given by:

$$\Gamma_y(\omega) = \Gamma_v(\omega) + \Gamma_w(\omega) = \frac{5-4\cos(\omega)}{5+4\cos(\omega)} + \frac{36}{10-6\cos(\omega)}$$
$$= \cdots = \frac{230+114\cos(\omega)-40\cos(2\omega)+24\cos(\omega)\cos(2\omega)}{50+10\cos(\omega)-24[\cos(\omega)]^2}$$

[8 Marks]

c) As stated in the answer to Question 1b), the spectrum $\Gamma_y(\omega)$ is given by $\Gamma_y(\omega) = \Gamma_v(\omega) + \Gamma_w(\omega)$. The spectra behaviours of $\Gamma_v(\omega)$, $\Gamma_w(\omega)$, and $\Gamma_y(\omega)$ in the interval $\omega \in [-\pi, \pi]$ are shown in Figs. 1.1, 1.2, 1.3, respectively.

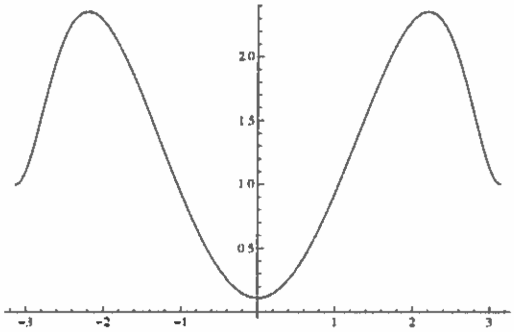

Figure 1.1 Plot of the spectrum $\Gamma_v(\omega) = \frac{5-4\cos(\omega)}{5+4\cos(\omega)}$.

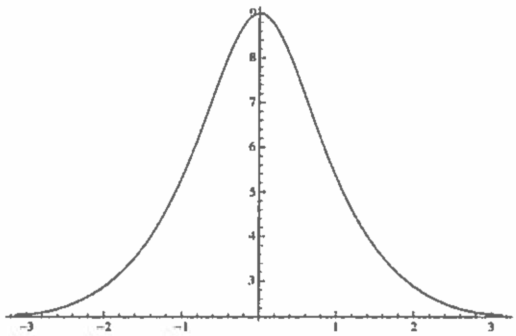

Figure 1.2 Plot of the spectrum $\Gamma_w(\omega) = \frac{36}{10-6\cos(\omega)}$.

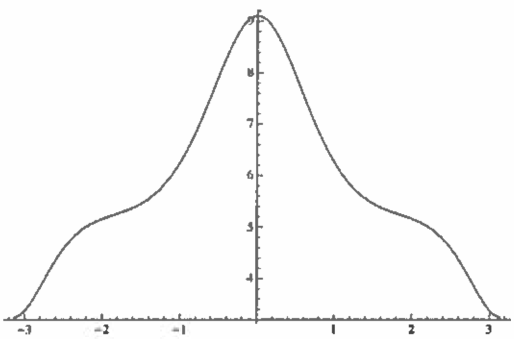

Figure 1.3 Plot of the spectrum $\Gamma_y(\omega) = \frac{230+114\cos(\omega)-40\cos(2\omega)+24\cos(\omega)\cos(2\omega)}{50+10\cos(\omega)-24[\cos(\omega)]^2}$.

To sketch the plots in Figs. 1.1, 1.2, 1.3, a few values of the spectra can be computed directly from the analytical expressions of $\Gamma_v(\omega), \Gamma_w(\omega), \Gamma_y(\omega)$ provided in the answer to Question 1b).

Alternatively, a few values can be easily computed by geometric considerations:

$$\Gamma_v: \begin{cases} \Gamma_v(0) = \frac{\left(1-\frac{1}{\sqrt{2}}\right)^2\left(1+\frac{1}{\sqrt{2}}\right)^2}{\left(1+\frac{1}{2}\right)^2} = \frac{1}{9} \\ \Gamma_v(\pi/2) = \frac{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2}\right)}{\left(1+\frac{1}{4}\right)} = \frac{9}{5} \\ \Gamma_v(\pi) = \frac{\left(1-\frac{1}{\sqrt{2}}\right)^2\left(1+\frac{1}{\sqrt{2}}\right)^2}{\frac{1}{4}} = 1 \end{cases}$$

$$\Gamma_w: \begin{cases} \Gamma_w(0) = \frac{1}{\frac{4}{9}} \cdot 4 = 9 \\ \Gamma_w(\pi/2) = \frac{1}{1+\frac{1}{9}} \cdot 4 = \frac{18}{5} \\ \Gamma_w(\pi) == \frac{1}{\left(1+\frac{1}{3}\right)^2} \cdot 4 = \frac{9}{4} \end{cases}$$

$$\Gamma_y: \begin{cases} \Gamma_y(0) = \frac{1}{9} + 9 = \frac{82}{9} \\ \Gamma_y(\pi/2) = \frac{9}{5} + \frac{18}{5} = \frac{27}{5} \\ \Gamma_y(\pi) = 1 + \frac{9}{4} = \frac{13}{4} \end{cases}$$

[8 Marks]

2. Solution

a) The state equations

$$\begin{cases} x_1(t+1) = \frac{1}{2}x_1(t) + 3x_2(t) + v_1(t) \\ x_2(t+1) = -\frac{1}{2}x_2(t) + v_2(t) \\ y(t) = x_2(t) \end{cases}$$

can be rewritten as follows (with the usual interpretation of z as a one-step forward shift operator):

$$\begin{cases} (z-\frac{1}{2})x_1(t) = 3x_2(t) + v_1(t) \\ (z+\frac{1}{2})x_2(t) = v_2(t) \\ y(t) = x_2(t) \end{cases}$$

Then, the transfer function from $v_2(t)$ to $x_2(t)$ is

$$H_{v_2x_2}(z) = \frac{1}{z+\frac{1}{2}}$$

Since the input process $v_2(\cdot)$ is stationary and the pole of the transfer function $H_{v_2x_2}(z)$ lies strictly inside the unit circle, it can be concluded that the steady-state process $x_2(\cdot)$ is stationary.

Let us now consider the other state equation. Again, with the usual interpretation of z as a one-step forward shift operator, we can write:

$$x_1(t) = \frac{1}{z-\frac{1}{2}}[3x_2(t) + v_1(t)]$$

If we introduce a new input stochastic process $v(t) = 3x_2(t) + v_1(t)$, it follows immediately that the transfer function from $v(t)$ to $x_1(t)$ is

$$H_{vx_1}(z) = \frac{1}{z-\frac{1}{2}}$$

$v(\cdot)$ is a stationary stochastic process because it is a linear combination of the stationary stochastic processes $x_2(\cdot)$ and $v_1(\cdot)$. Moreover, the pole of the transfer function $H_{vx_1}(z)$ lies strictly inside the unit circle, and hence it can be concluded that the steady-state process $x_1(\cdot)$ is stationary as well.

[5 Marks]

b) Consider the state equation for $x_2(t)$:

$$x_2(t+1) = -\frac{1}{2}x_2(t) + v_2(t)$$

and apply the expected value operator $\mathbb{E}[\cdot]$ on both sides getting

$$\mathbb{E}[x_2(t+1)] = -\frac{1}{2}\mathbb{E}[x_2(t)] + \mathbb{E}[v_2(t)]$$

Due to the stationarity of $x_2(\cdot)$ established in the answer to Question 2a), we get

$$m_2 = -\frac{1}{2}m_2 + 1 \Longrightarrow m_2 = \frac{2}{3}$$

[4 Marks]

c) Consider the state equation for $x_1(t)$:

$$x_1(t+1) = \frac{1}{2}x_1(t) + 3x_2(t) + v_1(t)$$

and apply the expected value operator $\mathbb{E}[\cdot]$ on both sides getting

$$\mathbb{E}[x_1(t+1)] = \frac{1}{2}\mathbb{E}[x_1(t)] + 3m_2 + \mathbb{E}[v_1(t)]$$

Again, due to the stationarity of $x_1(\cdot)$ established in the answer to Question 2a), we get

$$m_1 = \frac{1}{2}m_1 + 3\cdot\frac{2}{3} \Longrightarrow m_1 = 4$$

[4 Marks [

d) It is convenient to introduce the following zero-mean stochastic processes:

$$\begin{aligned}\tilde{x}_1(t) &= x_1(t) - m_1\\ \tilde{x}_2(t) &= x_2(t) - m_2\\ \tilde{v}_2(t) &= v_2(t) - 1\end{aligned}$$

Then, the original state equations can be equivalently rewritten as

$$\begin{cases}\tilde{x}_1(t+1) + m_1 = \frac{1}{2}\tilde{x}_1(t) + \frac{1}{2}m_1 + 3\tilde{x}_2(t) + 3m_2 + v_1(t)\\ \tilde{x}_2(t+1) = -\frac{1}{2}\tilde{x}_2(t) + \tilde{v}_2(t)\\ y(t) = \tilde{x}_2(t) + m_2\end{cases}$$

and hence

$$\begin{cases}\tilde{x}_1(t+1) = \frac{1}{2}\tilde{x}_1(t) + 3\tilde{x}_2(t) + v_1(t)\\ \tilde{x}_2(t+1) = -\frac{1}{2}\tilde{x}_2(t) + \tilde{v}_2(t)\\ y(t) = \tilde{x}_2(t) + \frac{2}{3}\end{cases}$$

Consider the state equation for $\tilde{x}_2(t)$

$$\tilde{x}_2(t+1) = -\frac{1}{2}\tilde{x}_2(t) + \tilde{v}_2(t)$$

and apply the variance operator $\mathrm{var}[\cdot]$ on both sides. Since the process $\tilde{x}_2(\cdot)$ is stationary and uncorrelated from the process $\tilde{v}_2(\cdot)$, we obtain

$$\mathrm{var}[x_2(t)] = \left(-\frac{1}{2}\right)^2 \mathrm{var}[x_2(t)] + 1$$

and hence

$$\lambda_{22} = \mathrm{var}[x_2(t)] = \mathbb{E}[(x_2(t) - m_2)^2] = \mathrm{var}[\tilde{x}_2(t)] = \frac{4}{3}$$

Let us compute

$$\lambda_{12} = \mathrm{cov}[x_1(t), x_2(t)] = \mathbb{E}[(x_1(t) - m_1)(x_2(t) - m_2)] = \mathbb{E}[\tilde{x}_1(t)\tilde{x}_2(t)] = \mathrm{cov}[\tilde{x}_1(t), \tilde{x}_2(t)]$$

by first multiplying the state equations for $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$:

$$\tilde{x}_1(t+1)\tilde{x}_2(t+1) = \left[\frac{1}{2}\tilde{x}_1(t) + 3\tilde{x}_2(t) + v_1(t)\right]\cdot\left[-\frac{1}{2}\tilde{x}_2(t) + \tilde{v}_2(t)\right]$$

Now, recall the stationarity of $\bar{x}_1(\cdot)$ and $\bar{x}_2(\cdot)$ and the mutual un-correlation between $\bar{x}_1(t)$ and $v_1(t)$ and $\bar{v}_2(t)$ and between $\bar{x}_2(t)$ and $v_1(t)$ and $\bar{v}_2(t)$. Then, by applying the expected value operator $\mathbb{E}[\cdot[$ on both sides we get immediately

$$\mathbb{E}[\bar{x}_1(t+1)\bar{x}_2(t+1)] = -\frac{1}{4}\mathbb{E}[\bar{x}_1(t)\bar{x}_2(t)] - \frac{3}{2}\mathbb{E}[\bar{x}_2(t)^2]$$

Due to stationarity, we have

$$\mathbb{E}[\bar{x}_1(t+1)\bar{x}_2(t+1)] = \mathbb{E}[\bar{x}_1(t)\bar{x}_2(t)]$$

and hence

$$\lambda_{12} = -\frac{1}{4}\lambda_{12} - \frac{3}{2}\lambda_{22} \Longrightarrow \lambda_{12} = -\frac{8}{5}$$

Finally, we compute λ_{11} by applying the variance operator var$[\cdot]$ on both sides of the state equation

$$\bar{x}_1(t+1) = \frac{1}{2}\bar{x}_1(t) + 3\bar{x}_2(t) + v_1(t)$$

thus getting

$$\begin{aligned}\mathrm{var}[\bar{x}_1(t+1)] &= \mathbb{E}\left\{\left[\frac{1}{2}\bar{x}_1(t) + 3\bar{x}_2(t) + v_1(t)\right]^2\right\}\\ &= \frac{1}{4}\mathrm{var}[\bar{x}_1(t)] + 9\mathrm{var}[\bar{x}_2(t)] + 1 + 3\mathbb{E}[\bar{x}_1(t)\cdot\bar{x}_2(t)]\end{aligned}$$

and hence

$$\lambda_{11} = \frac{1}{4}\lambda_{11} + 12 + 1 - \frac{24}{5} \Longrightarrow \lambda_{11} = \frac{164}{15}$$

[7 Marks]

3. Solution

a) From the difference equation (see (3.1) in the exam paper)

$$v(t) = \frac{1}{4}v(t-1) + \eta(t) + 2\eta(t-1)$$

we obtain (with the usual interpretation of z^{-1} as a one-step backward shift operator)

$$\left(1 - \frac{1}{4}z^{-1}\right)v(t) = (1+2z^{-1})\eta(t)$$

and hence the transfer function from the input process $\eta(t)$ to the output process $v(t)$ is

$$H_{\eta v}(z) = \frac{z+2}{z-\frac{1}{4}}$$

Since the input process $\eta(\cdot)$ is stationary and the pole of the transfer function $H_{\eta v}(z)$ lies strictly inside the unit circle, it can be concluded that the steady-state process $v(\cdot)$ is stationary.

To compute the expected value $\mathbb{E}(v)$, we apply the expected value operator $\mathbb{E}(\cdot)$ on both sides of the above difference equation, getting

$$\mathbb{E}[v(t)] = \frac{1}{4}\mathbb{E}[v(t-1)] + \mathbb{E}[\eta(t)] + 2\mathbb{E}[\eta(t-1)]$$

and thus, owing to the stationarity of processes $\eta(\cdot)$ and $v(\cdot)$, it follows that

$$\mathbb{E}[v(t)] = \frac{1}{4}\mathbb{E}[v(t)] \Longrightarrow \mathbb{E}[v(t)] = 0$$

As $\mathbb{E}[v(t)] = 0$, we have

$$\begin{aligned}
\text{var}[v(t)] = \mathbb{E}[v(t)^2] = \gamma_v(0) &= \mathbb{E}\left\{\left[\frac{1}{4}v(t-1) + \eta(t) + 2\eta(t-1)\right]^2\right\} \\
&= \frac{1}{16}\mathbb{E}[v(t-1)^2] + \mathbb{E}[\eta(t)^2] + 4\mathbb{E}[\eta(t-1)^2] \\
&+ \frac{1}{2}\mathbb{E}[v(t-1)\eta(t)] + \mathbb{E}[v(t-1)\eta(t-1)] + 4\mathbb{E}[\eta(t)\eta(t-1)]
\end{aligned}$$

and hence

$$\gamma_v(0) = \frac{1}{16}\gamma_v(0) + \frac{3}{2} \Longrightarrow \gamma_v(0) = \frac{8}{5}$$

Analogously, we compute $\gamma_v(1)$:

$$\begin{aligned}
\gamma_v(1) = \mathbb{E}[v(t)v(t-1)] &= \mathbb{E}\left\{\left[\frac{1}{4}v(t-1) + \eta(t) + 2\eta(t-1)\right]v(t-1)\right\} \\
&= \frac{1}{4}\gamma_v(0) + \mathbb{E}[\eta(t)v(t-1)] + 2\mathbb{E}[\eta(t-1)v(t-1)] = \frac{1}{4}\cdot\frac{8}{5} + 2\cdot\frac{1}{4} = \frac{9}{10}
\end{aligned}$$

[4 Marks]

b) First of all, notice that the model

$$v(t) = \frac{1}{4}v(t-1) + \eta(t) + 2\eta(t-1)$$

is not in spectral canonical form. The model in spectral canonical form is

$$A(z)v(t) = C(z)\xi(t)$$

where $\xi(\cdot) \sim WN(0,1)$ and

$$A(z) = 1 - \frac{1}{4}z^{-1}, \quad C(z) = 1 + \frac{1}{2}z^{-1}$$

Let's carry out one iteration of polynomial division of $C(z)$ by $A(z)$:

$$\begin{array}{llll} 1 & \frac{1}{2}z^{-1} & 1 & -\frac{1}{4}z^{-1} \\ -1 & \frac{1}{4}z^{-1} & 1 & \\ // & \frac{3}{4}z^{-1} & & \end{array}$$

Then, it follows that the transfer function of the one-step ahead predictor of $v(t+1)$ from the white noise process $\xi(t)$ is given by

$$\hat{W}(z) = \frac{C(z)}{A(z)} = 1 + z^{-1}\frac{\frac{3}{4}}{1 - \frac{1}{4}z^{-1}}$$

and thus the transfer function of the one-step ahead predictor of $v(t+1)$ from the past data $v(t)$ is

$$W_1(z) = \frac{\frac{3}{4}}{1 + \frac{1}{2}z^{-1}}$$

Hence, the difference equation implementing the one-step ahead predictor of $v(t+1)$ from the data $v(t)$ is

$$\hat{v}(t+1|t) = -\frac{1}{2}\hat{v}(t|t-1) + \frac{3}{4}v(t)$$

[5 Marks]

c) As

$$\hat{W}(z) = \frac{C(z)}{A(z)} = 1 + z^{-1}\frac{\frac{3}{4}}{1 - \frac{1}{4}z^{-1}}$$

the variance of the one-step ahead prediction error $\text{var}[\varepsilon_1(t)]$ is given by:

$$\text{var}[\varepsilon_1(t)] = \text{var}[v(t+1) - \hat{v}(t+1|t)] = 1 \cdot \text{var}[\xi(t+1)] = 1$$

] 3 Marks]

d) We start from the model in spectral canonical form determined in the Answer to Question 3b):

$$A(z)v(t) = C(z)\xi(t)$$

where $\xi(\cdot) \sim WN(0,1)$ and

$$A(z) = 1 - \frac{1}{4}z^{-1}, \quad C(z) = 1 + \frac{1}{2}z^{-1}$$

Next, we carry out another iteration of polynomial division of $C(z)$ by $A(z)$:

$$
\begin{array}{lll@{\qquad\qquad}ll}
1 & \frac{1}{2}z^{-1} & & 1 & -\frac{1}{4}z^{-1} \\[1ex]
-1 & \frac{1}{4}z^{-1} & & 1+\frac{3}{4}z^{-1} & \\[1ex]
// & \frac{3}{4}z^{-1} & & & \\[1ex]
// & -\frac{3}{4}z^{-1} & \frac{3}{16}z^{-2} & & \\[1ex]
// & // & \frac{3}{16}z^{-2} & &
\end{array}
$$

Then, it follows that the transfer function of the two-steps ahead predictor of $v(t+2)$ from the white noise process $\xi(t)$ is given by

$$\hat{W}(z) = \frac{C(z)}{A(z)} = 1 + \frac{3}{4}z^{-1} + z^{-2}\frac{\frac{3}{16}}{1-\frac{1}{4}z^{-1}}$$

and thus the transfer function of the two-steps ahead predictor of $v(t+2)$ from the past data $v(t)$ is

$$W_2(z) = \frac{\frac{3}{16}}{1+\frac{1}{2}z^{-1}}$$

Finally, the difference equation implementing the two-step ahead predictor of $v(t+2)$ from the data $v(t)$ is

$$\hat{v}(t+1|t) = -\frac{1}{2}\hat{v}(t|t-1) + \frac{3}{16}v(t)$$

[4 Marks]

e) As

$$\hat{W}(z) = \frac{C(z)}{A(z)} = 1 + \frac{3}{4}z^{-1} + z^{-2}\frac{\frac{3}{16}}{1-\frac{1}{4}z^{-1}}$$

the variance of the two-steps ahead prediction error $\mathrm{var}[\varepsilon_2(t)]$ is given by:

$$\mathrm{var}[\varepsilon_2(t)] = \mathrm{var}[v(t+2) - \hat{v}(t+2|t)] = 1\cdot\mathrm{var}[\xi(t+2)] + \left(\frac{3}{4}\right)^2\cdot\mathrm{var}[\xi(t+1)] = \frac{25}{16}$$

Let us compare $\mathrm{var}[\varepsilon_1(t)]$ with $\mathrm{var}[\varepsilon_2(t)]$: we have

$$\mathrm{var}[\varepsilon_2(t)] = \frac{25}{16} > 1 = \mathrm{var}[\varepsilon_1(t)]$$

This confirms that the variance of the prediction error $\mathrm{var}[\varepsilon_r(t)]$ increases with the number r of steps-ahead of the prediction we are computing. Indeed, as expected, we have that

$$\lim_{r\to\infty}\mathrm{var}[\varepsilon_r(t)] = \mathrm{var}[v(t)] = \frac{8}{5} > \frac{25}{16} > 1$$

[4 Marks]

4. Solution

a) To compute the numerical estimates $\hat{\gamma}_v(\tau)$ of the first three values of the correlation function $\gamma_v(\tau)$, $\tau = 0,1,2$, we use the standard empirical approximation of the expected value operator $\mathbb{E}(\cdot)$, thus getting the following formula:

$$\hat{\gamma}_v(\tau) = \frac{1}{N-\tau}\sum_{i=0}^{N-\tau-1} v(i)v(i+\tau), \quad \tau \geq 0$$

where N denotes the number of data samples available which, in this case, is $N = 8$.

Then, we have:

$$\hat{\gamma}_v(0) = \frac{1}{8}\sum_{i=0}^{7} v(i)v(i) \simeq 7.357$$

$$\hat{\gamma}_v(1) = \frac{1}{7}\sum_{i=0}^{6} v(i)v(i+1) \simeq 6.746$$

$$\hat{\gamma}_v(2) = \frac{1}{6}\sum_{i=0}^{5} v(i)v(i+2) \simeq 6.218$$

[7 Marks]

b) Moving-average models of order n with $n \geq 1$ (typically denoted by $MA(n)$) have a correlation function γ such that $\gamma(\tau) = 0, \forall\tau\colon |\tau| > n$. Therefore, by noticing that $\hat{\gamma}_v(2) \neq 0$, it is immediate to conclude that moving-average first-order models MA(1) have a correlation function whose form is not in agreement with $\hat{\gamma}_v(\tau)$. Hence, we conclude that the model describing the process $v(\cdot)$ takes on the form

$$AR(1): \quad v(t) = av(t-1) + \xi(t)$$

[3 Marks]

c) We follow a least-squares approach exploiting the one step-ahead predictor for AR(1) processes. Define the following cost function:

$$J = \frac{1}{N-1}\sum_{t=1}^{N-1}[v(t) - \hat{v}(t|t-1)]^2 = \frac{1}{N-1}\sum_{t=1}^{N-1}[v(t) - av(t-1)]^2$$

where N denotes the number of data samples available which, in this case, is $N = 8$ and $\hat{v}(t|t-1)$ is the one-step ahead optimal prediction for an AR(1) process.

Now, we compute the partial derivative of J with respect to a thus getting

$$\frac{\partial J}{\partial a} = \frac{dJ}{da} = -\frac{2}{N-1}\sum_{t=1}^{N-1}\left[v(t)v(t-1) - av(t-1)^2\right]$$

Hence

$$\frac{dJ}{da} = 0 \Longrightarrow \hat{a} = \frac{\sum_{t=1}^{7} v(t)v(t-1)}{\sum_{t=1}^{7} v(t-1)^2} \simeq 0.809$$

[5 Marks]

d) The process $v(\cdot)$ is stationary by assumption. Hence, for a sufficiently large number of data samples, the empirical estimates $\hat{\gamma}_v(\tau)$ of the correlation function approximate reasonably well the true values $\gamma_v(\tau)$.

Owing to the answer to Question 4b), the model of the process is AR(1). Hence, from the Yule-Walker equations we have

$$\gamma_v(\tau) = a\gamma_v(\tau - 1), \quad \tau \geq 1$$

and thus

$$\gamma_v(1) = a\gamma_v(0)$$

Replacing in the equation above the true values of the correlation function $\gamma_v(0), \gamma_v(1)$ with the respective empirical estimates $\hat{\gamma}_v(0), \hat{\gamma}_v(1)$, we get

$$\hat{\gamma}_v(1) = \hat{a}\hat{\gamma}_v(0) \Longrightarrow \hat{a} = \frac{\hat{\gamma}_v(1)}{\hat{\gamma}_v(0)} \simeq 0.917$$

where $\hat{a}$ denotes the approximated value of the unknown parameter a of the AR(1) model.

The estimate $\hat{a}$ is slightly different from the one computed in the answer to Question 4c) because $\hat{\gamma}_v(0)$ has been computed taking advantage of one more samples of $v(t)$. This difference shows up because the number N of data samples is small. For large values of N, the difference would become much smaller.

[5 Marks [