Final

Imperial College
London

[E1.11 (Maths) ISE 2010]

B.ENG. and M.ENG. EXAMINATIONS 2010

MATHEMATICS (INFORMATION SYSTEMS ENGINEERING E1.11)

Date Wednesday 2nd June 2010 10.00 am - 1.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Answer ANY SEVEN questions.

Answers to questions from Section A and Section B should be written in different answer books.

CALCULATORS MAY **NOT** BE USED.

A mathematical formulae sheet is provided.

[*Before starting, please make sure that the paper is complete; there should be 7 pages, with a total of 9 questions. Ask the invigilator for a replacement if your copy is faulty.*]

SECTION A

1. (i) If $z_1 = 1 + 7i$, and $z_2 = 4 + 3i$, find $|z_1|$ and $|z_2|$.

Find the real and imaginary parts of $\dfrac{z_1}{z_2}$.

What are the modulus and argument of this number?

Verify directly in this case that $|z_1/z_2| \;=\; |z_1|/|z_2|$.

(ii) If x and y are real, find the real and imaginary parts of

$$\cos(x + iy).$$

Hence show that

$$|\cos(x + iy)|^2 \;=\; \cos^2(x) \;+\; \sinh^2(y) \;.$$

PLEASE TURN OVER

2. (i) Evaluate the partial sum

$$S_N = \sum_{n=1}^{N} \left(\frac{1}{n(n+1)} \right).$$

Evaluate the limit, if it exists,

$$\lim_{N \to \infty} S_N.$$

State whether the infinite series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)} \right)$$

is convergent or not.

Hence, using the comparison test, state whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent or divergent.

(ii) Explain what is meant by the *radius of convergence* of a power series.

Calculate the radius of convergence of the following power series:

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} z^{2n+1} .$$

Investigate its convergence at both of the endpoints of the interval of convergence.

(iii) Calculate the Maclaurin series for the function $f(x) = \ln(1+x)$. Split this function into its odd and even parts f_{odd} and f_{even}, and write down both f_{odd} and its Maclaurin series.

3. (i) Evaluate the following limits, or else show that they do not exist:

(a)
$$\lim_{n\to 1} \frac{n^3-1}{n^4-4n+3} ,$$

(b)
$$\lim_{n\to 1} \frac{n^3-1}{n^4-4n^2+3} ,$$

(c)
$$\lim_{x\to \pi/2} \frac{\sin x - \operatorname{cosec} x}{\cos^2(x)} ,$$

(d)
$$\lim_{n\to\infty} \frac{n^2+1}{(n-2)(n-3)} .$$

(ii) Using L'Hôpital's rule, evaluate

$$\lim_{x\to 0} \frac{\sin(x^2)}{1-\cos(x)} .$$

4. Evaluate the definite integrals

(i)
$$\int_0^\infty x^3 \exp(-x)\, \mathrm{d}x ;$$

(ii)
$$\int_e^{e^2} \frac{1}{x\ln x} \mathrm{d}x ;$$

(iii)
$$\int_3^4 \frac{(x+1)}{(x^2-3x+2)}\, \mathrm{d}x .$$

5. Solve the ordinary differential equations

(i)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x+3y}{x+y} \;;$$

(ii)

$$\frac{\mathrm{d}y}{\mathrm{d}x} - xy = \exp\left(\frac{x^2}{2}\right), \quad \text{with} \;\; y(0) = 1 \;;$$

(iii)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \exp(x), \;\; \text{with} \;\; y(0) = 0, \;\; \text{and} \;\; y'(0) = 0\,.$$

PLEASE TURN OVER

SECTION B

6. (i) Let $u = u(x, y)$ where $x = s^2 + t^2$ and $y = 2st$.

   Prove that

$$s\frac{\partial u}{\partial s} + t\frac{\partial u}{\partial t} = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) .$$

   (ii) Find the stationary points of the function $f(x, y) = x^3 - 3xy^2 + 12y$, and determine their nature.

7. Define the function $f(x)$ on the interval $-\pi < x \leq \pi$ by

$$f(x) = \begin{cases} x\,, & \text{if } 0 \leq x \leq \pi \\ 0\,, & \text{if } -\pi < x < 0\,. \end{cases}$$

Show that the Fourier series of $f(x)$ is

$$\frac{\pi}{4} - \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{\cos\left[\,(2m+1)\,x\,\right]}{(2m+1)^2} + \sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{\sin(nx)}{n} .$$

Hence evaluate the infinite sums

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} \quad \text{and} \quad \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} .$$

PLEASE TURN OVER

8. (i) Use Laplace transforms to find a function $y = y(t)$ satisfying the differential equation

$$\frac{d^2y}{dt^2} - y + t^2 = 0$$

with $y(0) = 2, \ y'(0) = 0.$

(No credit will be given if you use another method.)

(ii) Use Laplace transforms to find functions x, y of t satisfying the simultaneous differential equations

$$\frac{dx}{dt} + \frac{dy}{dt} - y = 2\,,$$

$$\frac{dx}{dt} - \frac{dy}{dt} + x = t^2 + 2t\,,$$

with $x(0) = 2, \ \ y(0) = 0\,.$

9. (i) Consider the three planes given by the equations

$$\begin{aligned} \mathbf{v}.(1,1,1) &= -1, \\ \mathbf{v}.(1,-1,a) &= -3, \\ \mathbf{v}.(2,0,1) &= b, \end{aligned}$$

where $\mathbf{v} = (x, y, z)$. For which values of a and b do these three planes

(a) meet in exactly one point ?

(b) meet in a line ?

(c) not meet at all ?

(ii) Let

$$A = \begin{pmatrix} -1 & -10 \\ 5 & 14 \end{pmatrix}.$$

(a) Find an invertible 2×2 matrix P such that $P^{-1}AP$ is diagonal.

(b) Find a 2×2 matrix B such that $B^2 = A$.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b} \times \mathbf{c} = \mathbf{b}\,.\,\mathbf{c} \times \mathbf{a} = \mathbf{c}\,.\,\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b\,;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i \sinh z\,; \quad \sinh iz = i \sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a + \theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \qquad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

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SOLUTIONS -

ISE 1

E1.11

1

1/14

A1. (i) If $z_1 = 1+7i$, and $z_2 = 4+3i$, then

$$|z_1| = \sqrt{1^2+7^2} = \sqrt{50} = 5\sqrt{2}$$

2

and

$$|z_2| = \sqrt{4^2+3^2} = 5.$$

We have

$$\frac{z_1}{z_2} = \frac{1+7i}{4+3i} = \frac{(1+7i)(4-3i)}{4^2+3^2}.$$

Expanding the numerator and denominator, we find

$$\frac{z_1}{z_2} = \frac{(4+21)+(28-3)i}{25} = 1+i.$$

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SIMILAR

Thus $\Re(\frac{z_1}{z_2}) = \Im(\frac{z_1}{z_2}) = 1$. The modulus

$$|\frac{z_1}{z_2}| = \sqrt{2},$$

2

its argument is

$$\arg(\frac{z_1}{z_2}) = \arg(1+i) = \tan^{-1}(1) = \pi/4.$$

We see easily that $|z_1/z_2| = |z_1|/|z_2|$; that is

$$\sqrt{2} = \frac{5\sqrt{2}}{\sqrt{2}}.$$

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(ii) If x and y are real,

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$$\cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy) =$$

$$= \cos(x)\cosh(y) - i\sin(x)\sinh(y).$$

4

Hence

$$|\cos(x+iy)|^2 = \cos^2(x)\cosh^2(y) + \sin^2(x)\sinh^2(y) =$$

$$= \cos^2(x)(1+\sinh^2(y)) + (1-\cos^2(x))\sinh^2(y) = \cos^2(x)+\sinh^2(y),$$

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as required.

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ISE 1 (2)
E1.11

A2. (i) The partial sum

$$S_N = \sum_{n=1}^{N} \left(\frac{1}{n(n+1)} \right)$$
$$= \sum_{n=1}^{N} \left[\frac{1}{n} - \frac{1}{n+1} \right]$$
$$= 1 - \frac{1}{N+1}.$$

2

The limit exists, and is

$$\lim_{n\to\infty} S_N = 1.$$

2

Hence the infinite series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)} \right)$$

is convergent; the limit is 1.

Using the comparison test, noting that for $n > 0$,

2

$$\frac{1}{n^2} > \frac{1}{n(n+1)} > \frac{1}{(n+1)^2},$$

we see that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent.

(ii) If a power series $\sum_{n=0}^{\infty} a_n z^n$ converges for some z_0, then by the comparison test it converges for $|z| < |z_0|$. The series thus converges in some interval $|z| < R$, and diverges for $|z| > R$. R is called the *radius of convergence* of the power series. 2

To calculate the radius of convergence of the power series:

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} z^{2n+1},$$

consider the ratio of successive terms, applying the ratio test.

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$$\lim_{n\to\infty} \frac{z^{2n+1}}{2n+1} \frac{2n-1}{z^{2n-1}}$$
$$= \lim_{n\to\infty} \frac{2n-1}{2n+1} z^2 = z^2.$$

The series converges if this limit $z^2 < 1$.

For $|z| = 1$, the series diverges, by comparison with $\sum_{n=0}^{\infty} \frac{1}{n}$.

(iii) The Maclaurin series for the function $f(x) = \ln(1+x)$ is found by successively differentiating, and evaluating these derivatives at $x = 0$:

$$f(x) = \ln(1+x), \qquad f(0) = 0,$$

[illegible] [illegible]

ISE1 2
E1.11

$$f'(x) = \frac{1}{1+x}, \qquad f'(0) = 1,$$
$$f''(x) = -\frac{1}{(1+x)^2}, \qquad f''(0) = -1,$$
$$\dots$$
$$f^{(n)}(x) = (-1)^{(n+1)}(n-1)!\frac{1}{(1+x)^n}, \qquad f^{(n)}(0) = (-1)^{(n+1)}(n-1)!\dots$$

2

The series is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

Here this gives

$$f(x) = \sum_{n=1}^{\infty}(-1)^{(n+1)}\frac{x^n}{n}.$$

2

Note the zeroth term vanishes.

Splitting $f(x)$ into its odd and even parts we get

$$f_{odd} = \frac{1}{2}(\ln(1+x) - \ln(1-x)) = \ln(\sqrt{\frac{1+x}{1-x}})$$

2

and

$$f_{even} = \frac{1}{2}(\ln(1+x) + \ln(1-x)) = \ln(\sqrt{1-x^2}).$$

The Maclaurin series for f_{odd} consists of the odd power terms in the series for f:

$$f_{odd}(x) = \sum_{n=0}^{\infty} \frac{1}{2n+1} z^{2n+1}.$$

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ISE 1 (3)
E1.11

A3. (a) (i) To evaluate $\lim_{n\to 1} \frac{n^3-1}{n^4-4n+3}$, divide top and bottom by $n-1$; we get: 3

$$\lim_{n\to 1} \frac{n^2+n+1}{n^3+n^2+n-3}$$

Here the numerator has the limit 3, but the denominator has the limit zero - the limit of the quotient does not exist.

(ii) Similarly

$$\lim_{n\to 1} \frac{n^3-1}{n^4-4n^2+3} =$$
$$\lim_{n\to 1} \frac{n^2+n+1}{n^3+n^2-3n-3} =$$
$$\frac{\lim_{n\to 1} n^2+n+1}{\lim_{n\to 1} n^3+n^2-3n-3} =$$
$$\frac{3}{-4} = -\frac{3}{4}.$$

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(iii) Multiply top and bottom by $\sin(x)$:

$$\lim_{x\to\pi/2} \frac{\sin x - \operatorname{cosec} x}{\cos^2(x)} =$$
$$\lim_{x\to\pi/2} \frac{\sin^2 x - 1}{\cos^2(x)\sin x} =$$
$$\lim_{x\to\pi/2} \frac{-1}{\sin x} = -1.$$

4

(iv) Divide top and bottom by n^2:

$$\lim_{n\to\infty} \frac{n^2+1}{(n-2)(n-3)} =$$
$$\lim_{n\to\infty} \frac{1+1/n^2}{(1-2/n)(1-3/n)} = 1.$$

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(b) The numerator and denominator both vanish as $x \to 0$. Hence, using L'Hôpital's rule, we find:

$$\lim_{x\to 0} \frac{\sin(x^2)}{1-\cos(x)} =$$
$$\lim_{x\to 0} \frac{2x\cos(x^2)}{\sin(x)} =$$
$$\lim_{x\to 0} 2\frac{x}{\sin(x)} \lim_{x\to 0} \cos(x^2) = 2.$$

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ISE 1
E1.11 4

A4. (i) To evaluate

$$\int_0^\infty x^3 \exp(-x) \mathrm{d}x,$$

integrate by parts repeatedly:

$$\int_0^\infty x^3 \exp(-x) \mathrm{d}x$$

$$= \int_0^\infty 3x^2 \exp(-x) \mathrm{d}x$$

$$= \int_0^\infty 6x \exp(-x) \mathrm{d}x$$

6

$$\int_0^\infty 6 \exp(-x) \mathrm{d}x = 6[-\exp(-x)]_0^\infty = 6.$$

(ii) To evaluate $\int_e^{e^2} \frac{1}{x \ln x} \mathrm{d}x$, substitute $u = \ln(x)$, $\mathrm{d}x/x = \mathrm{d}u$:

$$\int_{x=e}^{e^2} \frac{1}{x \ln x} \mathrm{d}x =$$

$$\int_{u=1}^{2} \frac{1}{u} \mathrm{d}u =$$

7

$$[\ln(u)]_1^2 = \ln(2).$$

(iii) Put the integrand into partial fractions:

$$\int_3^4 \frac{(x+1)}{(x^2 - 3x + 2)} \mathrm{d}x =$$

$$\int_3^4 \frac{a}{x-1} + \frac{b}{x-2}$$

We find $a(x-2) + b(x-1) = x + 1$, so $a = -2$, $b = 3$.

$$\int_3^4 \frac{-2}{x-1} + \frac{3}{x-2} =$$

7

$$[-2\ln(|x-1|) + 3\ln(|x-2|)]_3^4 = -2\ln(3/2) + 3\ln(2/1)$$

$$= 5\ln(2) - 2\ln(3).$$

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ISE I
E1.11 5

A5. (i) To solve the homogeneous ode

$$\frac{dy}{dx} = -\frac{2x+3y}{x+y},$$

set $y = xu(x)$:

$$u + x\frac{du}{dx} = -\frac{2+3u}{1+u}$$

so

$$x\frac{du}{dx} = -\left(\frac{2+3u}{1+u} + \frac{u+u^2}{1+u}\right)$$
$$= -\frac{2+2u+u^2}{1+u}$$

This is separable

$$\int^x \frac{dx'}{x'} = -\int^{u(x)} \frac{(1+u')du'}{2+2u'+u'^2}$$
$$= -\frac{1}{2}\ln(2+2u+u^2) + k.$$

Thus, integrating and exponentiating,

$$x\sqrt{2+2\frac{y}{x}+\frac{y^2}{x^2}} = K,$$

that is

$$(2x^2+2xy+y^2) = K^2.$$

5

(ii)

$$\frac{dy}{dx} - xy = \exp(\frac{x^2}{2}),$$
with $y(0) = 1$.

Multiply by the integrating factor $\exp(-x^2/2)$:

$$\frac{d}{dx}(y\exp(-x^2/2)) = 1,$$

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so that

$$y\exp(-x^2/2) = x - x_0,$$

or

$$y = (x-x_0)\exp(x^2/2).$$

Set $x=0$, $y=1$, giving $1 = -x_0$, so that

$$y = (x+1)\exp(x^2/2).$$

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(iii)

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \exp(x), \qquad \text{with} \quad y(0) = 0, \text{and} \quad y'(0) = 0.$$

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ISE1 (5)
E1.11

The auxiliary equation is $m^2 + 3m + 2 = 0$; its roots are $m = -1$, $m = -2$. Hence the CF is

$$y_{CF} = A\exp(-x) + B\exp(-2x).$$

The PI must be a multiple of the exponential on the rhs - try

$$y_{PI} = \alpha\exp(x);$$

5

we see that this is a solution if $\alpha = 1/6$. Hence the general solution is

$$y = A\exp(-x) + B\exp(-2x) + \frac{1}{6}\exp(x).$$

This satisfies $y(0) = A + B + 1/6$, $y'(0) = -A - 2B + 1/6$, so $A = -1/2$, $B = 1/3$, giving the required solution:

3

$$y = -\frac{1}{2}\exp(-x) + \frac{1}{3}\exp(-2x) + \frac{1}{6}\exp(x).$$

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Course: ISE 1

Question: 6

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Parts

(a) $\frac{\partial u}{\partial s} = u_x \frac{\partial x}{\partial s} + u_y \frac{\partial y}{\partial s}$

$= 2s\, u_x + 2t\, u_y$ — 2

$\frac{\partial u}{\partial t} = 2t u_x + 2s u_y$ — 2

So

$s u_s + t u_t = s(2s u_x + 2t u_y) + t(2t u_x + 2s u_y)$

$= (2s^2 + 2t^2) u_x + 4st\, u_y$

$= 2x u_x + 2y u_y.$ — 4

(b) $f(x,y) = x^3 - 3xy^2 + 12y$

So $f_x = 3x^2 - 3y^2, \quad f_y = -6xy + 12.$ — 2

At stationary pts,

$x^2 - y^2 = 0 \quad (1)$

$xy = 2 \quad (2)$

By (1), $y = \pm x$. If $y = x$, (2) $\Rightarrow x^2 = 2$

so $y = x = \pm\sqrt{2}$.

If $y = -x$, (2) $\Rightarrow x^2 = -2$ ✗

So stationary pts are $(\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$. — 6

Now $f_{xx} = 6x$, $f_{xy} = -6y$, $f_{yy} = -6x$.

At $(\sqrt{2}, \sqrt{2})$, $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 < 0 \therefore$ MIN — 2

At $(-\sqrt{2}, -\sqrt{2})$, $f_{xx} < 0$ & $f_{xx}f_{yy} - f_{xy}^2 < 0 \therefore$ MAX — 2

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Course: ISE 1

Question: 7

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Parts

Fourier coeffs:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2}$$ 1

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left(\left[x\cdot\frac{1}{n}\sin nx\right]_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin nx\,dx\right)$$

$$= -\frac{1}{n\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi}$$

$$= \frac{1}{n^2\pi}(\cos n\pi - 1) = \begin{cases} 0, & n \text{ even} \\ \frac{-2}{n^2\pi}, & n \text{ odd.}\end{cases}$$ 5

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left(\left[x\cdot -\frac{1}{n}\cos nx\right]_0^{\pi} + \int_0^{\pi}\frac{1}{n}\cos nx\,dx\right)$$

$$= \frac{1}{\pi}\left(-\frac{\pi}{n}\cos n\pi + \left[\frac{1}{n^2}\sin nx\right]_0^{\pi}\right)$$

$$= -\frac{1}{n}\cdot(-1)^n + 0$$

$$= \frac{(-1)^{n+1}}{n}.$$ 5

So Fourier series is

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$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{\pi}{4} - \frac{2}{\pi} \sum_{n \text{ odd}} \frac{\cos nx}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}$$

$$= \frac{\pi}{4} - \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{\cos (2m+1)x}{(2m+1)^2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}$$ 2

Put $x = 0$: As $f(x)$ is continuous at $x=0$, Fourier series equals $f(0)$, so

$$0 = \frac{\pi}{4} - \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}$$

Hence $\sum \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}$. 3

Put $x = \frac{\pi}{2}$: as $\cos (2m+1)\frac{\pi}{2} = 0$

and $\sin \frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ even} \\ (-1)^k & \text{if } n = 2k+1 \end{cases}$

Fourier series gives

$$f(\tfrac{\pi}{2}) = \frac{\pi}{2} = \frac{\pi}{4} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

So $\sum \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$ 4

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Question 8

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Parts

(a) $y'' - y + t^2 = 0, \quad y(0) = 2, y'(0) = 0.$

Take Laplace transforms:

$$s^2 L(y) - 2s - L(y) + L(t^2) = 0$$ 3

As $L(t^2) = \frac{2}{s^3}$ this gives

$$(s^2-1)\,L(y) = 2s - \frac{2}{s^3} = \frac{2(s^4-1)}{s^3}$$

Hence

$$L(y) = \frac{2(s^2+1)}{s^3} = \frac{2}{s} + \frac{2}{s^3}$$ 3

Thus $\underline{y = 2 + t^2}$ 2

(b)
$$x' + y' - y = 2$$
$$x' - y' + x = t^2 + 2t$$

Take Laplace transforms:

(1) $sL(x) - 2 + sL(y) - L(y) = \frac{2}{s}$

(2) $sL(x) - 2 - sL(y) + L(x) = \frac{2}{s^3} + \frac{2}{s^2}$

So

(1) $sL(x) + (s-1)L(y) = 2 + \frac{2}{s} = \frac{2(s+1)}{s}$ 4

(2) $(s+1)L(x) - sL(y) = \frac{2}{s^3} + \frac{2}{s^2} + 2$

$$= \frac{2(s^3+s+1)}{s^3}$$

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Parts

So (1) $\times (s+1)$ — (2) $\times s$ gives

$$(2s^2-1)\,L(y) = \frac{2(s+1)^2}{s} - \frac{2(s^3+s+1)}{s^2}$$

$$= \frac{2\left(s^3+2s^2+s-s^3-s-1\right)}{s^2}$$

$$= \frac{2(2s^2-1)}{s^2} \qquad 4$$

So $\quad L(y) = \dfrac{2}{s^2}$, hence $\underline{y = 2t}$ $\qquad 2$

From (1),

$$sL(x) = 2 + \frac{2}{s} - (s-1)\,L(y)$$

$$= 2 + \frac{2}{s} - \frac{(2s-2)}{s^2}$$

$$= 2 + \frac{2}{s^2}$$

So $\quad L(x) = \dfrac{2}{s} + \dfrac{2}{s^3}$, so $\underline{x = 2 + t^2}$ $\qquad 2$

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Parts

(a) Solving the three eqns simultaneously gives system with augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & a & -3 \\ 2 & 0 & 1 & b \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & -2 & a-1 & -2 \\ 0 & -2 & -1 & b+2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & -2 & a-1 & -2 \\ 0 & 0 & -a & b+4 \end{pmatrix}$$ 4

So last eqn is $-az = b+4$.

Therefore

(i) system has 1 soln (ie. planes meet in 1 pt) if $\underline{a \neq 0}$ 2

(ii) system has a line of solns (ie. planes meet in a line) if $\underline{a = 0,\ b = -4}$ 2

(iii) system has no solns (ie. planes don't meet) if $\underline{a = 0,\ b \neq -4}$ 2

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Parts

(b) Evalues of A :

$$\begin{vmatrix} -1-\lambda & -10 \\ 5 & 14-\lambda \end{vmatrix} = \lambda^2 - 13\lambda + 36$$
$$= (\lambda-9)(\lambda-4).$$

So evalues are 4, 9. — 3

$\underline{\lambda=4}$ Evectors are solns of

$$\left(\begin{array}{cc|c} -5 & -10 & 0 \\ 5 & 10 & 0 \end{array}\right) \to \text{ an evector } \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$\underline{\lambda=9}$ $\begin{pmatrix} -10 & -10 \\ 5 & 5 \end{pmatrix} \to$ an evector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

So $\underline{P = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}}$ will work — 3

(other answers correct of course).

~~Take~~ So

$$P^{-1}AP = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}.$$

Take

$$B = P \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} P^{-1}$$
$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}.$$ — 4

Then $B^2 = (PDP^{-1})^2$ $\quad \left(D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\right)$
$= PD^2P^{-1} = A.$

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