

B.ENG. and M.ENG. EXAMINATIONS 2011

PART I : MATHEMATICS 1 (ELECTRICAL ENGINEERING)

Date Wednesday 8th June 2011 10.00 - 12.00

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer Question 1 and THREE of the remaining five

Question 1 carries twice the marks of each of the other questions.

CALCULATORS MAY NOT BE USED.

A mathematical formulae sheet is provided.

[Before starting, please make sure that the paper is complete; there should be 5 pages, with a total of SIX questions. Ask the invigilator for a replacement if your copy is faulty.]

1. (i) Express in polar form :

$$i, \quad 1 - i, \quad (2 + 2i)^2, \quad (2 + i)(2 - i), \quad 1/2 - \sqrt{3/4}i.$$

- (ii) Determine real and imaginary parts in the form

$x + iy$ where $x, y \in \mathbb{R}$ and $z \in \mathbb{C}$:

$$(1 + i)^3, \exp(z), \exp(iz), \sinh(iz), \cos(iz).$$

Note: $\sinh(x) = \frac{1}{2}(e^x - e^{-x}).$

- (iii) Find the limits :

$$\lim_{x \rightarrow \infty} \frac{(x+1)^3 - x(x+1)^2}{x^2}; \quad [\text{do not use } L'H\acute{o}pital's \text{ rule}]$$

$$\lim_{x \rightarrow \infty} y^{1/x} \quad \text{for } y > 0;$$

$$\lim_{x \rightarrow 1} \frac{x^{1/3} - x^{1/2}}{x - 1}; \quad [\text{do not use } L'H\acute{o}pital's \text{ rule}]$$

$$\lim_{x \rightarrow 1} \frac{x^{1/3} - x^{1/2}}{x - 1}. \quad [\text{use } L'H\acute{o}pital's \text{ rule}]$$

- (iv) Differentiate the following functions with respect to x :

$$f(x) = (1 + \cos(x))^4;$$

$$f(x) = \ln(\ln(x));$$

$$f(x) = x \ln(x/a);$$

$$f(x) = x^{\exp(x)}.$$

PLEASE TURN OVER

(v) Determine the following definite integrals :

$$\int_0^{2\pi} \frac{dx}{\sin^2 x + \cos^2 x} ;$$

$$\int_1^2 \frac{dx}{\sqrt{x-1}} ;$$

$$\int_1^2 \frac{x + x^5}{2x^2 + x^6} dx .$$

(vi) Determine the indefinite integral

$$\int \frac{dx}{\cos^2 x - \sin^2 x} .$$

(vii) Find the Taylor expansion of $\tan(x)$ about $x = \pi/4$ to first order (up to and including the term linear in x) and state the remainder term $R_2(x)$.

(viii) Determine the radius of convergence of the following two series :

$$\sum_{n=1}^{\infty} \frac{x^n}{n} ;$$

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{n!} .$$

(ix) Find the general solutions of the following ODEs :

$$y''(x) = 0 ;$$

$$y'(x) = \frac{y^2(x)}{x^2} ;$$

$$x y'(x) = \frac{y(x)}{x} + \frac{1}{x} .$$

(x) Find the general solutions of the following second order ODEs :

$$y''(x) + 2y'(x) + y(x) = 0 ;$$

$$3y''(x) - 2y'(x) + 2y(x) = 0 .$$

2. Find $\frac{dy}{dx}$ as a function of x in each of the following cases :

(i) $y = e^{x+x^2}$;

(ii) $y = \frac{x \sin x}{x+1}$;

(iii) $y = \sin(\ln x)$;

(iv) $y = x^{\sin x}$;

(v) $e^x + e^y = e^{x+y}$;

(vi) $e^x = \ln(x+y)$.

(vii) Show by induction that

$$\frac{d^n}{dx^n} \left(\frac{1}{1-x} \right) = \frac{n!}{(1-x)^{n+1}} \quad \forall n \geq 1 .$$

3. Evaluate the following limits :

(i) $\lim_{x \rightarrow 1} \frac{(x-2)(x+2)}{(x-3)(x+1)}$;

(ii) $\lim_{x \rightarrow \infty} \frac{\sin(\sinh x)}{x}$;

(iii) $\lim_{x \rightarrow 0} x^x$;

(iv) $\lim_{x \rightarrow -2} \frac{\sqrt{-2x} - 2}{x+2}$;

(v) $\lim_{x \rightarrow \infty} \left(\frac{x}{2} \right)^{1/2} \left[(2x+1)^{1/2} - (2x-3)^{1/2} \right]$.

Note: You can assume $\lim_{x \rightarrow 0} x \ln x = 0$.

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[E1.10 (Maths 1) 2011]

4. (i) State whether the improper integral $\int_0^1 \frac{\ln x}{x} dx$ is finite and calculate its value if so.

(ii) Integrate $\int 5^{6x+7} dx$.

(iii) Integrate $\int \frac{dx}{\sqrt{3+2x-x^2}}$.

5. (i) Find all complex number solutions of the equation

$$z^4 - z^2 + 1 = 0.$$

- (ii) Sketch the subset of the complex plane described by the equation

$$|z - i - 1| = \operatorname{Im}(z + 2i),$$

where, for $z = x + iy$ a complex number, $\operatorname{Im} z = y$ is the imaginary part.

6. (i) Find the solution $y(x)$ of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 2e^x,$$

that satisfies $y(0) = 0$, $\frac{dy}{dx}(0) = 1$.

- (ii) Find the general solution for the following ODE using the $x = e^t$ substitution (or any other method).

$$x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = 0.$$

END OF PAPER

M A T H E M A T I C S D E P A R T M E N T

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{(n+1)} + \dots \quad (-1 < x \leq 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

$$\cos iz = \cosh z; \quad \cosh iz = \cos z; \quad \sin iz = i \sinh z; \quad \sinh iz = i \sin z.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f D^n g + \binom{n}{1} Df D^{n-1} g + \dots + \binom{n}{r} D^r f D^{n-r} g + \dots + D^n f g.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \dots + h^n f^{(n)}(a)/n! + \epsilon_n(h),$$

$$\text{where } \epsilon_n(h) = h^{n+1} f^{(n+1)}(a + \theta h)/(n+1)!, \quad 0 < \theta < 1.$$

(c) Taylor's expansion of $f(x, y)$ about (a, b) :

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2! [h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}]_{a,b} + \dots$$

(d) Partial differentiation of $f(x, y)$:

$$\text{i. If } y = y(x), \text{ then } f = F(x), \text{ and } \frac{dF}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}.$$

$$\text{ii. If } x = x(t), y = y(t), \text{ then } f = F(t), \text{ and } \frac{dF}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.

Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.

If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;

If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;

If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\frac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:

$$\sin \theta = 2t/(1+t^2), \quad \cos \theta = (1-t^2)/(1+t^2), \quad d\theta = 2dt/(1+t^2).$$

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1} \left(\frac{x}{a} \right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1} \left(\frac{x}{a} \right) = \ln \left\{ \frac{x}{a} + \left(1 + \frac{x^2}{a^2} \right)^{1/2} \right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1} \left(\frac{x}{a} \right) = \ln \left| \frac{x}{a} + \left(\frac{x^2}{a^2} - 1 \right)^{1/2} \right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a} \right) \tan^{-1} \left(\frac{x}{a} \right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
 $x_{n+1} = x_n - [f(x_n)/f'(x_n)], \quad n = 0, 1, 2 \dots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x) dx \approx (h/2) [y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x) dx \approx (h/3) [y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x) dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t) dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t, \alpha)$	$(\partial/\partial\alpha)F(s, \alpha)$	$\int_0^t f(t) dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2, \dots)$	$n!/s^{n+1}, (s > 0)$
e^{at}	$1/(s-a), (s > a)$	$\sin \omega t$	$\omega/(s^2 + \omega^2), (s > 0)$
$\cos \omega t$	$s/(s^2 + \omega^2), (s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	$e^{-sT}/s, (s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \dots, \text{ and}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

Parseval's theorem

$$\frac{1}{L} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course EE1(1) (1)
Question Q1		Marks & seen/unseen
Parts	<p>i)</p> $i^0 = \underline{e^{i^0 \frac{\pi}{2}}}$ $1-i^0 = \sqrt{2} e^{-i^0 \frac{\pi}{4}} = \underline{\sqrt{2} e^{i^0 \frac{7\pi}{4}}}$ $(2+2i^0)^2 = 8i^0 = \underline{8 e^{i^0 \frac{\pi}{2}}}$ $(2+i^0)(2-i^0) = \underline{5}$ $\frac{1}{2} - \sqrt{\frac{3}{4}} i^0 = e^{-i^0 \frac{\pi}{3}} = \underline{e^{i^0 \frac{5\pi}{3}}}$ <p>ii)</p> $(1+i^0)^3 = \underline{-2 + i^0 2}$ $z = x + i^0 y$ $e^z = \underline{e^x \cos y + i^0 e^x \sin y}$ $e^{iz} = \underline{e^{-y} \cos x + i^0 e^{-y} \sin x}$ $\sinh(iz) = \frac{1}{2} \{ e^{-y} \cos x + i^0 e^{-y} \sin x - e^y \cos x + i^0 e^y \sin x \}$ $= \underline{\frac{1}{2} \{ (e^{-y} - e^y) \cos x + i^0 (e^{-y} + e^y) \sin x \}}$ <p>or $= -\sinh y \cos x + i^0 \cosh y \sin x$</p> $\cos iz = \frac{1}{2} (e^{-z} + e^z) = \underline{\frac{1}{2} (e^x + e^{-x}) \cos y + i^0 \frac{1}{2} (e^x - e^{-x}) \sin y}$ <p>or $= \cosh x \cos y + i^0 \sinh x \sin y$</p>	<p>4 seen similar</p> <p>4 seen similar</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course EFV(1) (1)
Question Q1		Marks & seen/unseen
Parts iii)	$\lim_{x \rightarrow \infty} \frac{(x+1)^3 - x(x+1)^2}{x^2} = \lim_{x \rightarrow \infty} \frac{3x^2 - 2x^2 + \dots}{x^2} = 1$ $\lim_{x \rightarrow \infty} y^{\frac{1}{x}} = y^0 = 1$ $\lim_{x \rightarrow 1} \frac{x^{1/3} - x^{1/2}}{x-1} = -\frac{1}{6}$ <p>using</p> $\frac{x^{1/3} - x^{1/2}}{x-1} = -x^{1/3} \frac{1 - x^{1/6}}{1-x}$ $= -x^{1/3} \frac{1-y}{1-y^6} = -x^{1/3} \frac{1-y}{(1+y+y^2+y^3+y^4+y^5)(1-y)}$ $= -x^{1/3} \frac{1}{1+y+y^2+y^3+y^4+y^5}$ <p>where $y = x^{1/6}$ and $\lim_{x \rightarrow 1} y = 1$</p> <p>Using L'Hopital gives</p> $\lim_{x \rightarrow 1} \frac{\frac{1}{3}x^{-2/3} - \frac{1}{2}x^{-1/2}}{1} = -\frac{1}{6}$	4 seen similar
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	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course EE1(P)
Question		Marks & seen/unseen
Parts	<p>iv) $\frac{d}{dx} (1 + \cos(x))^4 = \underline{\underline{-4 \sin(x) (1 + \cos(x))^3}}$</p> <p>$\frac{d}{dx} \ln(\ln(x)) = \underline{\underline{\frac{1}{x} \frac{1}{\ln(x)}}}$</p> <p>$\frac{d}{dx} \ln\left(\frac{x}{a}\right) x = \underline{\underline{\ln\left(\frac{x}{a}\right) + 1}}$</p> <p>$\frac{d}{dx} x^{\exp(x)} = \frac{d}{dx} \exp(\ln(x) \exp(x))$ $= \left(\frac{1}{x} \exp(x) + \ln(x) \exp(x)\right) x^{\exp(x)}$ $= \underline{\underline{x^{\exp(x)} e^x \left(\frac{1}{x} + \ln x\right)}}$</p> <p>v) $\int_0^{2\pi} \frac{dx}{\sin^2 x \cos^2 x} = \underline{\underline{2\pi}}$</p> <p>$\int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{\epsilon \rightarrow 0} \ln \left[2\sqrt{x-1} \right]_{1+\epsilon}^2 = \underline{\underline{2}}$</p> <p>$\int_1^2 \frac{x+x^5}{2x^2+x^6} dx = \frac{1}{8} \int_1^2 \frac{8x^3+8x^7}{2x^4+x^8} dx = \frac{1}{8} \left[\ln(2x^4+x^8) \right]_1^2$ $= \frac{1}{8} \ln\left(\frac{288}{3}\right) = \underline{\underline{\frac{1}{8} \ln(96)}}$</p> <p>or $\int_1^2 \frac{x+x^5}{2x^2+x^6} dx = \int_1^2 \frac{1+x^4}{2x+x^5} dx = \frac{1}{2} \int_1^2 \frac{2+5x^4-3x^4}{2x+x^5}$</p>	<p>4 Seen Similar</p> <p>4 Seen Similar</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course EE1(1) ①
Question		Marks & seen/unseen
Parts	$= \frac{1}{2} [\ln(2x+x^5)]_1^2 - \frac{3}{2} \int_1^2 \frac{x^3}{2+x^4}$ $= \frac{1}{2} \ln(12) - \frac{3}{8} [\ln(2+x^4)]_1^2$ $= \frac{1}{2} \ln(12) - \frac{3}{8} \ln(6) = \frac{1}{8} \ln(12^4/6^3)$ $= \frac{1}{8} \ln 96$ <p>vi)</p> $\int \frac{dx}{\cos^2 x - \sin^2 x} = \int \frac{dt}{1+t^2} \frac{1+t^2}{1-t^2}$ $= \frac{1}{2} \int \frac{1}{1+t} + \frac{1}{1-t} dt = \frac{1}{2} \ln \left(\frac{t+1}{t-1} \right) + C$ $= \frac{1}{2} \ln \left(\frac{\sin x + \cos(x)}{\sin x - \cos(x)} \right) + C$ <p>using $t = \tan x \quad \frac{dt}{dx} = \frac{1}{\cos^2 x} = 1+t^2$</p> $\sin^2 x = \frac{t^2}{1+t^2} \quad \cos^2 x = \frac{1}{1+t^2}$ <p>Ans can also be written:</p> $\frac{1}{2} \ln(\sec 2x + \tan 2x) + C$	<p>4 seen similar</p>
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	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course <u>EE1(1)</u> <u>(1)</u>
Question		Marks & seen/unseen
Parts i x)	$y'' = 0 \Rightarrow \underline{y(x) = Ax + B}$ $y'(x) = \frac{y^2}{x^2} \quad \text{separation of variables:}$ $-\frac{1}{y} = -\frac{1}{x} + A \Rightarrow \underline{y = \frac{x}{1-Ax}}$ <p>or via homogeneous ODE:</p> $\ln x = \int \frac{dv}{v^2 - v} \quad \text{with } v = \frac{y}{x}$ $= \int -\frac{1}{v} + \frac{1}{1-v} dv = \ln\left(\frac{1-v}{v}\right) + C$ $\Rightarrow x = \tilde{A} \frac{x-y}{y} \Rightarrow \underline{y = \frac{\tilde{A}x}{\tilde{A}+x}} = \frac{x}{1 + \frac{1}{\tilde{A}}x}$ <p style="text-align: center;">$\tilde{A} = -A$ above</p> $y'x = \frac{y}{x} + x^{-1}$ $y' - \frac{y}{x^2} = x^{-2}$ <p>Integrating factor $-R \frac{1}{x^2} = R' \Rightarrow R = e^{1/x}$</p> $\Rightarrow Ry = \int dx x^{-2} e^{1/x} = -e^{1/x} + C$ $\Rightarrow \underline{y = -1 + C e^{-1/x}}$	4 seen similar
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		Page number <u>56</u>

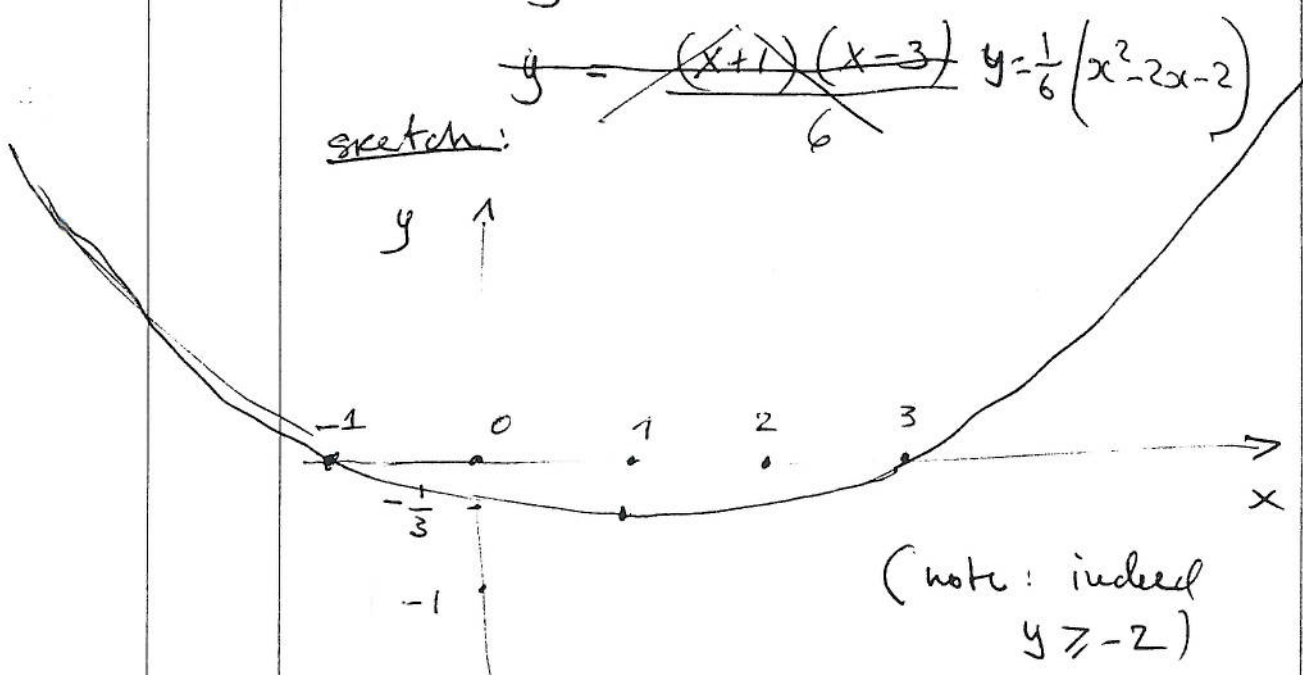
	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course DEI(1) <u>2</u>
Question		Marks & seen/unseen
Parts	<p>(i) $\frac{dy}{dx} = (1+2x)e^{x+x^2}$ Chain Rule (2)</p> <p>ii) $\frac{dy}{dx} = \frac{\sin x}{x+1} + \frac{x \cos x}{x+1} - \frac{x \sin x}{(x+1)^2}$ Product Rule (3)</p> $= \frac{\sin x + (x+1) \cos x}{(x+1)^2}$ <p>iii) $\frac{dy}{dx} = \frac{\cos(\ln x)}{x}$ (2)</p> <p>iv) $y = x^{\sin x} = e^{(\ln x) \sin x}$</p> $\frac{dy}{dx} = e^{(\ln x) \sin x} \left(\frac{\sin x}{x} + \cos x \ln x \right)$ $= x^{\sin x} \left[\frac{\sin x}{x} + (\ln x) \cos x \right]$ (3) <p>v) $e^x + e^y = e^{x+y} \Rightarrow e^y = \frac{e^x}{e^x - 1} = \frac{1}{1 - e^{-x}}$</p> $\therefore \frac{dy}{dx} e^y = \frac{-e^{-x}}{(1 - e^{-x})^2}$ $\Rightarrow \frac{dy}{dx} \left(\frac{1}{1 - e^{-x}} \right) = \frac{-e^{-x}}{(1 - e^{-x})^2} \Rightarrow \frac{dy}{dx} = \frac{-e^{-x}}{1 - e^{-x}} = \frac{1}{1 - e^x}$ (4) <p>vi) $e^x = \ln(x+y) \Rightarrow x+y = e^{e^x}$</p> $\therefore 1 + \frac{dy}{dx} = e^x e^{e^x} \Rightarrow \frac{dy}{dx} = e^x e^{e^x} - 1$ (3) <p>vii) True for $n=1$ Assume $\frac{d^n}{dx^n} \left(\frac{1}{1-x} \right) = \frac{n!}{(1-x)^{n+1}}$</p> $\text{Then } \frac{d^{n+1}}{dx^{n+1}} \frac{1}{1-x} = \frac{d}{dx} \left(\frac{n!}{(1-x)^{n+1}} \right) = \frac{(n+1)!}{(1-x)^{n+2}}$ (3)	
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	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course EE1(1) (3)
Question		Marks & seen/unseen
Parts	<p>(i) $\lim_{x \rightarrow 1} \frac{(x-2)(x+2)}{(x-3)(x+1)} = \frac{-1 \cdot 3}{-2 \cdot 2} = 3/4$</p> <p>(ii) $L = \lim_{x \rightarrow \infty} \frac{\sin(\sinh x)}{x}$ $L < \lim_{x \rightarrow \infty} \frac{1}{x} = 0$</p> <p>(iii) Let $y = x^x$ then $\ln y = x \ln x$ as $x \rightarrow 0 \Rightarrow \ln y \rightarrow 1 \therefore \lim_{x \rightarrow 0} x^x = 1$</p> <p>(iv) $\lim_{x \rightarrow -2} \frac{\sqrt{-2x} - 2}{x+2} = \lim_{x \rightarrow -2} \frac{(\sqrt{-2x} - 2)(\sqrt{-2x} + 2)}{(x+2)(\sqrt{-2x} + 2)}$ $= \lim_{x \rightarrow -2} \frac{-2x - 4}{(x+2)(\sqrt{-2x} + 2)}$ $= \lim_{x \rightarrow -2} \frac{-2}{\sqrt{-2x} + 2} = -1/2.$</p> <p>(v) $\lim_{x \rightarrow \infty} \left(\frac{x}{2}\right)^{1/2} \left[(2x+1)^{1/2} - (2x-3)^{1/2} \right]$ $= \lim_{x \rightarrow \infty} \left(\frac{x}{2}\right)^{1/2} (2x)^{1/2} \left[\left(1 + \frac{1}{2x}\right)^{1/2} - \left(1 - \frac{3}{2x}\right)^{1/2} \right]$ $= \lim_{x \rightarrow \infty} x \left[\left(1 + \frac{1}{4x} + \dots\right) - \left(1 - \frac{3}{4x} + \dots\right) \right]$ $= \lim_{x \rightarrow \infty} x \left[\frac{1}{x} + \dots \right] = 1$</p> <p>or $\lim_{x \rightarrow \infty} \left(\frac{x}{2}\right)^{1/2} = \left[(2x+1)^{1/2} - (2x-3)^{1/2} \right] \left[\frac{(2x+1)^{1/2} + (2x-3)^{1/2}}{(2x+1)^{1/2} + (2x-3)^{1/2}} \right]$ etc</p>	<p>(2)</p> <p>(3)</p> <p>(4)</p> <p>(5)</p> <p>(5)</p>
Setter's initials	JRC	Page number
Checker's initials	MHL	

Core 3
 REIC
 (4)

INTEGRAL CORE EXAM SOLUTION 2010-2011		Course
Core 3		Marks & seen/unseen
Part		
A	Consider $\int_c^1 \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2 \Big _c^1 = -\frac{1}{2}(\ln c)^2$. Taking the limit as $c \rightarrow 0^+$ yields $-\infty$. Therefore the integral diverges.	4 3
B	Let $u = 6x + 7$, so $du = 6dx$. Then the integral becomes $\frac{1}{6} \int 5^u du = \frac{1}{6} * \frac{5^u}{\ln 5} + c = \frac{5^{6x+7}}{6 \ln 5} + c$	2 4
C	$I(x) = \int \frac{dx}{\sqrt{3+2x-x^2}} = \int \frac{dx}{\sqrt{4-(x-1)^2}}$. Now let $u = x - 1$, giving $du = dx$ and hence $I(x) = \int du/\sqrt{4-u^2}$. Let $u = 2 \sin \theta \Rightarrow du = 2 \cos \theta d\theta$. $I(x) = \int \frac{2 \cos \theta d\theta}{\sqrt{4-4 \sin^2 \theta}}$ $= \int \frac{2 \cos \theta d\theta}{2 \cos \theta} = \theta + c$ $= \sin^{-1} \left(\frac{u}{2} \right) + c = \sin^{-1} \left(\frac{x-1}{2} \right) + c$	1 1 2 1 1 1
Setter's initials: DB		Page number
Checker's initials AC		

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course EE1(1) (5)
Question C4		Marks & seen/unseen
Parts	<p>(a) I first solve</p> $z^2 - z + 1 = 0$ $z_{1,2} = \frac{1 \pm i\sqrt{3}}{2} = e^{\frac{\pi}{3}i}, e^{-\frac{\pi}{3}i}$ <p>next I solve for x:</p> $x^2 = z_1 \quad x_{1,2} = \begin{cases} e^{\frac{\pi}{6}i} = \frac{\sqrt{3}}{2} + \frac{1}{2}i \\ e^{\frac{5\pi}{6}i} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i \end{cases}$ $x^2 = z_2 \quad x_{3,4} = \begin{cases} e^{-\frac{\pi}{6}i} = \frac{\sqrt{3}}{2} - \frac{1}{2}i \\ -e^{-\frac{\pi}{6}i} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i \end{cases}$ <p>(In summary $x = \pm \frac{\sqrt{3}}{2} \pm \frac{1}{2}i$)</p>	<p>4 marks</p> <p>3 marks</p> <p>3 marks</p>
	Setter's initials AC Checker's initials RB	Page number 2/3

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course Core 4
Question C4		Marks & seen/unseen
Parts	<p>(b) $z-i-1 = \text{Im}(z+2i)$</p> <p>$z = x+iy$:</p> <p>$x-1+i(y-1) = y+2$ (note: $y \geq -2$)</p> <p>square both sides:</p> <p>$(x-1)^2 + (y-1)^2 = (y+2)^2$</p> <p>$x^2 - 2x + 1 + y^2 - 2y + 1 = y^2 + 4y + 4$</p> <p>$6y = x^2 - 2x - 2$</p> <p>$y = \frac{(x+1)(x-3)}{6}$ $y = \frac{1}{6}(x^2 - 2x - 2)$</p> <p>sketch:</p>  <p>(note: indeed $y \geq -2$)</p>	<p>unseen (I hope)</p> <p>10 marks</p>
	<p>Setter's initials AC</p> <p>Checker's initials JNC</p>	Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course EEL(1) (6)
Question G10	Second order ODE (solution)	Marks & seen/unseen
Parts (i)	<p>Finding the complementary function of the ODE by solving the auxiliary equation:</p> $\lambda^2 + \lambda - 2 = 0 \Rightarrow$ $\lambda = -2, +1$ <p>CF: $y(x) = A_1 e^{-2x} + A_2 e^x$</p> <p>This is a degenerate case, therefore for the particular integral (PI) we try</p> $y = C x e^x$ $\frac{dy}{dx} = C x e^x + C e^x$ $\frac{d^2 y}{dx^2} = C x e^x + 2C e^x$ $C x e^x + 2C e^x + C x e^x + C e^x - 2C x e^x = 2e^x$ $3C = 2 \Rightarrow C = \frac{2}{3}$ <p>General solution: $y(x) = A_1 e^{-2x} + A_2 e^x + \frac{2}{3} x e^x$</p> $y(0) = A_1 + A_2 = 0$ $y'(0) = -2A_1 + A_2 + \frac{2}{3} = 1 \Rightarrow -3A_1 = \frac{1}{3} \Rightarrow A_1 = -\frac{1}{9}$ $A_2 = \frac{1}{9}$ $y(x) = -\frac{1}{9} e^{-2x} + \frac{1}{9} e^x + \frac{2}{3} x e^x$	<p>2</p> <p>2</p> <p>2</p> <p>2</p> <p>1</p> <p>1</p>
	Setter's initials V.S. Checker's initials	Page number 2

	EXAMINATION QUESTIONS/SOLUTIONS 2010-2011	Course EE1(1) (6)
Question C10	Second order ODE (solution)	Marks & seen/unseen
Parts (ii)	$x = e^t \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} e^{-t}$ $\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dt} e^{-t} \right) e^{-t} = e^{-2t} \frac{d^2y}{dt^2} - \frac{dy}{dt} e^{-2t}$ <p>Substitution gives:</p> $e^{2t} \left(e^{-2t} \frac{d^2y}{dt^2} - \frac{dy}{dt} e^{-2t} \right) + 3 e^{-t} \frac{dy}{dt} + y = 0$ $\frac{d^2y}{dt^2} + 2 \frac{dy}{dt} + y = 0$ <p>Now an equation with constant coefficients, we solve the auxiliary equation:</p> $\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda = -1$ <p>Repeated root! therefore CF is</p> $y(t) = (a + bt) e^{-t}$ $t = \log x \Rightarrow y(x) = \frac{(a + b \log x)}{x}$ <p>Note: If students used $y = Ax^b$, substitution and guessed the $\log x$ solution should get full Mark. If they only find $\frac{a}{x}$ solution 5 mark only.</p>	<p>2</p> <p>2</p> <p>2</p> <p>1</p> <p>2</p> <p>1</p>
	Setter's initials V.S.	Checker's initials All
		Page number 3