Final

Imperial College London

[E2.11 (Maths) ISE 2010]

B.ENG. and M.ENG. EXAMINATIONS 2010

MATHEMATICS (INFORMATION SYSTEMS ENGINEERING E2.11)

Date Thursday 3rd June 2010 2.00 - 4.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Answer FOUR questions, to include at least one from Section B.

Answers to questions from Section A and Section B should be written in different answer books.

Mathematical and statistics formulae sheets are provided.

[Before starting, please make sure that the paper is complete; there should be 7 pages, with a total of 6 questions. Ask the invigilator for a replacement if your copy is faulty.]

Section A

1. The Fourier transform (FT) of $f(t)$ is given by

$$\widehat{f}(\omega) = FT\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt\ .$$

   (i) Establish the following results:

   (a) $FT\{f'(t)\} = i\,\omega\,\widehat{f}(\omega)$;

   (b) $FT\{tf(t)\} = i\dfrac{d}{d\omega}\,\widehat{f}\,(\omega)$,

   stating any assumptions that you make about the behaviour of $f(t)$ as $t \to \pm\infty$.

   (ii) Use the results above to show that

$$FT\left\{2f'(t) + tf(t)\right\} = ie^{-\omega^2}\,\frac{d}{d\omega}\left(e^{w^2}\widehat{f}(\omega)\right)\ .$$

   (iii) Apply the result obtained in (ii) to show that if

$$f(t) = e^{-t^2/4}\ ,$$

   then

$$\widehat{f}(\omega) = Ae^{-\omega^2}\ ,$$

   where A is a constant.

   (iv) Compute the constant A by evaluating $\widehat{f}(0)$ directly, making use of the result

$$\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}\ .$$

   (v) Deduce the value of

$$\int_{0}^{\infty} e^{-t^2/4}\cos\omega t\,dt\ .$$

PLEASE TURN OVER

2. The Laplace transform (LT) of $f(t)$ is given by

$$F(s) = LT\{f(t)\} = \int_0^\infty f(t)e^{-st}dt\,.$$

(i) Prove the convolution theorem, i.e. if $F(s)$, $G(s)$ are the Laplace transforms of $f(t)$, $g(t)$, then

$$LT\left\{\int_0^t f(t-u)g(u)du\right\} = F(s)\,G(s)\,.$$

The function $y(t)$ satisfies

$$y(t) = e^{-\alpha t} - \int_0^t y(t-u)\,e^{-\beta u}\,du$$

for $t > 0$, with α, β positive constants.

(ii) By taking the Laplace transform of this equation and using the tables provided, show that

$$Y(s) = \frac{s+\beta}{(s+\alpha)(s+\beta+1)}\,,$$

where $Y(s)$ is the Laplace transform of $y(t)$.

(iii) Use the residue theorem to invert $Y(s)$ and hence find $y(t)$, distinguishing between the cases $\alpha \neq \beta+1$ and $\alpha = \beta+1$.

If $F(s)$ has a pole of order m at $s=a$, then the residue of $F(s)$ at $s=a$ is given by

$$\frac{1}{(m-1)!}\lim_{s\to a}\left\{\frac{d^{m-1}}{ds^{m-1}}\left((s-a)^m F(s)\right)\right\}\,.$$

3. (i) Sketch the region of the $x - y$ plane over which the integral

$$I = \int_0^a \left\{ \int_{\sqrt{(ax-x^2)}}^{\sqrt{(a^2-x^2)}} \frac{2xy}{x^2+y^2}\, e^{-(x^2+y^2)}\, dy \right\} dx$$

is taken.

Use polar coordinates to evaluate I.

(ii) Show that

$$I(\alpha) = \int_C (3x^2y^2 + \alpha y\, e^{xy})\, dx + (2x^3y + xe^{xy})\, dy$$

is path-independent for a particular value of α (α^*, say) to be determined.

For this value of α, find a suitable potential function and hence evaluate $I(\alpha^*)$ in the case where C starts at (0, 0) and ends at (1, 2).

Verify this answer by finding a suitable parameterization when C is the straight line segment leading from (0, 0) to (1, 2), and evaluating $I(\alpha^*)$ using this parameterization.

PLEASE TURN OVER

4. (i) Classify the singularities of the function

$$f(z) = \frac{e^{i\alpha z}}{z^2 + 2z + 2} \ , \quad (\alpha > 0) \ ,$$

in the complex z-plane, and determine the residues at the poles.

(ii) Consider

$$I = \int_C f(z)\, dz \ ,$$

where $f(z)$ is as given in (i) and C is the closed semi-circular path formed by the segment of the real axis from $-R$ to R followed by the path $z = \mathrm{R}e^{i\theta}$, $(0 \le \theta \le \pi)$.

Determine the value of I both when $R < \sqrt{2}$ and when $R > \sqrt{2}$.

(iii) Show that the integral over the curved part of the path tends to zero as $R \to \infty$.

(iv) Hence evaluate the integrals

$$\int_{-\infty}^{\infty} \frac{\cos \alpha x}{x^2 + 2x + 2}\, dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin \alpha x}{x^2 + 2x + 2}\, dx \ .$$

Deduce the inverse Fourier transform of

$$\frac{1}{\omega^2 + 2\omega + 2} \ .$$

PLEASE TURN OVER

5. The number of errors, $X_i,\ i = 1, 2, 3$, made per day in three numeric database fields $F_i,\ i = 1, 2, 3$, are well modelled by independent Poisson distributions with,

$$\mathrm{P}(X_i = x) \;=\; \frac{\mathrm{e}^{-\lambda_i}\lambda_i^x}{x!}, \quad x \;=\; 0, 1, 2, \ldots,$$

where $\lambda_1 = 3, \lambda_2 = 2$ and $\lambda_3 = 1$. A fourth field, F_4, is correctly evaluated if both F_1 and F_2 are error free.

(i) Find the probability that

(a) F_1 has more than one error;

(b) F_2 has more than one error;

(c) F_3 has more than one error;

(d) F_1, F_2 and F_3 all have more than one error;

(e) F_4 returns an error.

(ii) An independent diagnostic test is run every day. If F_3 has more than one error and F_4 returns an error then the database has a 90% chance of failing the diagnostic test, otherwise it has a 20% chance of failing the test.

(a) Find the probability that the diagnostic test fails.

(b) If the diagnostic test has failed what is the probability that F_3 has more than one error and F_4 returns an error?

(c) Over 5 consecutive days what is the probability that the diagnostic test will fail no more than once?

PLEASE TURN OVER

6. The functioning lifetime, T, of a component, in days, has probability density function

$$f(t) = \begin{cases} \dfrac{c}{t^{\alpha+1}} & t > 1; \\ \\ 0 & \text{otherwise.} \end{cases}$$

for $\alpha > 0$.

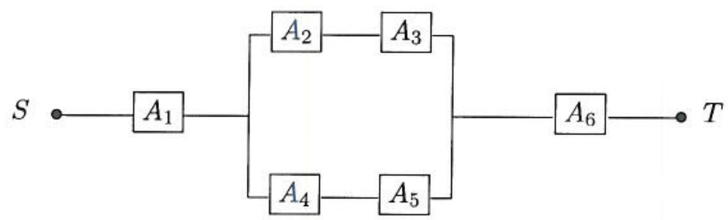
(i) (a) Determine the normalising constant c.

(b) For $t > 1$, determine the reliability function, $R(t)$, and show that the hazard function, $h(t)$, is given by

$$h(t) \;=\; \frac{\alpha}{t}, \qquad t > 1\,.$$

(c) What is the probability that such a component is functioning after 2 days?

(ii) A system is made up of six such components, operating independently with $\alpha = 0.5$, which operates if there is a path of functioning components between S and T.



(a) Determine the value for the system reliability at 2 days.

(b) To improve the system reliability, components A_1 and A_6 are replaced with two identical components of a different type. Find the minimum reliability of the replacement component to ensure the overall reliability at 2 days exceeds 0.5.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix};$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n\frac{x^{2n}}{(2n)!} + \dots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \dots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n\frac{x^{n+1}}{(n+1)} + \dots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\cos b + \cos a\sin b\,;$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i\sinh z\,; \quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \dots + \binom{n}{r} D^r f\,D^{n-r}g + \dots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \dots + h^nf^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}\right]_{a,b} + \dots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

1. **Probabilities for events**

For events A, B, and C $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of A $\quad P(A) \,/\, P(\overline{A})$

Conditional probability $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\quad P(A \cap B \cap C) = P(A)\, P(B \mid A)\, P(C \mid A \cap B)$

Bayes' rule $\quad P(A \mid B) = \dfrac{P(A)\, P(B \mid A)}{P(A)\, P(B \mid A) + P(\overline{A})\, P(B \mid \overline{A})}$

A and B are independent if $\quad P(B \mid A) = P(B)$

A, B, and C are independent if $\quad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

2. **Probability distribution, expectation and variance**

The probability distribution for a discrete random variable X is called the probability mass function (pmf) and is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) = \mu = \sum_x x p_x$

For function $g(x)$ of x, $\quad E\{g(X)\} = \sum_x g(x) p_x$, $\quad$ so $\quad E(X^2) = \sum_x x^2 p_x$

Sample mean $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ $\quad$ estimates μ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \mathrm{var}\,(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$

Sample variance $\quad s^2 = \dfrac{1}{n-1} \left\{ \sum_k x_k^2 - \dfrac{1}{n} \left(\sum_j x_j \right)^2 \right\}$ $\quad$ estimates σ^2

Standard deviation $\quad \mathrm{sd}\,(X) = \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness $\quad \beta_1 = E\left(\dfrac{X - \mu}{\sigma}\right)^3$ $\quad$ is estimated by $\quad \dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^3$

Kurtosis $\quad \beta_2 = E\left(\dfrac{X - \mu}{\sigma}\right)^4 - 3$ $\quad$ is estimated by $\quad \dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^4 - 3$

Sample median $\quad \tilde{x}$ or x_{med}. Half the sample values are smaller and half larger

If the sample values $x_1, \ldots, x_n$ are ordered as $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$,
then $\tilde{x} = x_{(\frac{n+1}{2})}$ if n is odd, and $\tilde{x} = \frac{1}{2}\left(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})}\right)$ if n is even

α-quantile $Q(\alpha)$ is such that $P(X \le Q(\alpha)) = \alpha$

Sample α-quantile $\widehat{Q}(\alpha)$ Proportion α of the data values are smaller

Lower quartile Q1 $= \widehat{Q}(0.25)$ one quarter are smaller

Upper quartile Q3 $= \widehat{Q}(0.75)$ three quarters are smaller

Sample median $\tilde{x} = \widehat{Q}(0.5)$ estimates the population median $Q(0.5)$

3. $\boxed{\text{Probability distribution for a continuous random variable}}$

The cumulative distribution function (cdf) $F(x) = P(X \le x) = \int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$

The probability density function (pdf) $f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x$, $\mathrm{var}\,(X) = \sigma^2 = E(X^2) - \mu^2$, where $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\mathrm{d}x$

4. $\boxed{\text{Discrete probability distributions}}$

Discrete Uniform *Uniform* (n)

$p_x = \dfrac{1}{n}$ $(x = 1, 2, \ldots, n)$ $\mu = (n+1)/2$, $\sigma^2 = (n^2-1)/12$

Binomial distribution *Binomial* (n, θ)

$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x}$ $(x = 0, 1, 2, \ldots, n)$ $\mu = n\theta$, $\sigma^2 = n\theta(1-\theta)$

Poisson distribution *Poisson* (λ)

$p_x = \dfrac{\lambda^x e^{-\lambda}}{x!}$ $(x = 0, 1, 2, \ldots)$ (with $\lambda > 0$) $\mu = \lambda$, $\sigma^2 = \lambda$

Geometric distribution *Geometric* (θ)

$p_x = (1-\theta)^{x-1}\theta$ $(x = 1, 2, 3, \ldots)$ $\mu = \dfrac{1}{\theta}$, $\sigma^2 = \dfrac{1-\theta}{\theta^2}$

5. $\boxed{\text{Continuous probability distributions}}$

Uniform distribution *Uniform* (α, β)

$$f(x) = \begin{cases} \dfrac{1}{\beta-\alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases}$$

$\mu = (\alpha+\beta)/2$, $\sigma^2 = (\beta-\alpha)^2/12$

Exponential distribution *Exponential* (λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \le 0). \end{cases}$$

$\mu = 1/\lambda$, $\sigma^2 = 1/\lambda^2$

Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \quad (-\infty < x < \infty), \quad E(X) = \mu, \quad \mathrm{var}(X) = \sigma^2$$

Standard normal distribution $N(0,1)$

If X is $N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$

6. Reliability

For a device in continuous operation with failure time random variable T having pdf $f(t)$ $(t > 0)$

The reliability function at time t $\quad R(t) = P(T > t)$

The failure rate or hazard function $\quad h(t) = f(t)/R(t)$

The cumulative hazard function $\quad H(t) = \int_0^t h(t_0)\,\mathrm{d}t_0 = -\ln\{R(t)\}$

The Weibull(α, β) distribution has $\quad H(t) = \beta t^\alpha$

7. System reliability

For a system of k devices, which operate independently, let

$R_i = P(D_i) = P($"device i operates"$)$

The system reliability, R, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$$

A system of devices in parallel operates if any device operates

$$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1-R_1)(1-R_2)\cdots(1-R_k)$$

8. Covariance and correlation

The covariance of X and Y $\quad \mathrm{cov}(X,Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample covariance $\quad s_{xy} = \dfrac{1}{n-1} S_{xy}$ estimates $\mathrm{cov}(X,Y)$

Correlation coefficient $\quad \rho = \mathrm{corr}(X,Y) = \dfrac{\mathrm{cov}(X,Y)}{\mathrm{sd}(X) \cdot \mathrm{sd}(Y)}$

Sample correlation coefficient $\quad r = \dfrac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$ estimates ρ

9. Sums of random variables

$$E(X+Y) = E(X)+E(Y)$$

$$\text{var}(X+Y) = \text{var}(X)+\text{var}(Y)+2\,\text{cov}(X,Y)$$

$$\text{cov}(aX+bY,\ cX+dY) = (ac)\,\text{var}(X)+(bd)\,\text{var}(Y)+(ad+bc)\,\text{cov}(X,Y)$$

If X is $N(\mu_1,\sigma_1^2)$, Y is $N(\mu_2,\sigma_2^2)$, and $\text{cov}(X,Y)=c$, then $X+Y$ is $N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2+2c)$

10. Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

Bias of t	$\text{bias}(t)$	$= E(T)-\theta$
Standard error of t	$\text{se}(t)$	$= \text{sd}(T)$
Mean square error of t	$\text{MSE}(t)$	$= E\{(T-\theta)^2\} = \{\text{se}(t)\}^2+\{\text{bias}(t)\}^2$

If $\overline{x}$ estimates μ, then $\text{bias}(\overline{x})=0$, $\text{se}(\overline{x})=\sigma/\sqrt{n}$, $\text{MSE}(\overline{x})=\sigma^2/n$, $\widehat{\text{se}}(\overline{x})=s/\sqrt{n}$

Central limit property If n is fairly large, $\overline{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ.

For a random sample $x_1,x_2,\ldots,x_n$

$$\ell(\theta;\ x_1,x_2,\ldots,x_n) = P(X_1=x_1\,|\,\theta)\ \cdots\ P(X_n=x_n\,|\,\theta) \qquad \text{(discrete distribution)}$$

$$\ell(\theta;\ x_1,x_2,\ldots,x_n) = f(x_1\,|\,\theta)\,f(x_2\,|\,\theta)\ \cdots\ f(x_n\,|\,\theta) \qquad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\widehat{\theta}$ for which the likelihood is a maximum

12. Confidence intervals

If $x_1,x_2,\ldots,x_n$ are a random sample from $N(\mu,\sigma^2)$ and σ^2 is known, then
the 95% confidence interval for μ is $(\overline{x}-1.96\frac{\sigma}{\sqrt{n}},\ \overline{x}+1.96\frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0=t_{n-1,0.05}$

The 95% confidence interval for μ is $(\overline{x}-t_0\frac{s}{\sqrt{n}},\ \overline{x}+t_0\frac{s}{\sqrt{n}})$

13. **Standard normal table** Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. **Student t table** Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

m	p= 0.10	0.05	0.02	0.01	m	p= 0.10	0.05	0.02	0.01
1	6.31	12.71	31.82	63.66	9	1.83	2.26	2.82	3.25
2	2.92	4.30	6.96	9.92	10	1.81	2.23	2.76	3.17
3	2.35	3.18	4.54	5.84	12	1.78	2.18	2.68	3.05
4	2.13	2.78	3.75	4.60	15	1.75	2.13	2.60	2.95
5	2.02	2.57	3.36	4.03	20	1.72	2.09	2.53	2.85
6	1.94	2.45	3.14	3.71	25	1.71	2.06	2.48	2.78
7	1.89	2.36	3.00	3.50	40	1.68	2.02	2.42	2.70
8	1.86	2.31	2.90	3.36	∞	1.645	1.96	2.326	2.576

15. **Chi-squared table** Values $\chi^2_{k,p}$ of x for which $P(X > x) = p$, when X is χ^2_k and p = .995, .975, *etc*

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum_y \frac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of χ^2_k with significance point p,

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ

17. Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$$p_{x\bullet} = \sum_y p_{xy} \quad \text{and} \quad P(X = x \,|\, Y = y) = \frac{p_{xy}}{p_{\bullet y}}$$

Continuous distribution

Joint cdf $F(x,y) = P(\{X \le x\} \cap \{Y \le y\}) = \int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0, y_0)\,\mathrm{d}x_0\,\mathrm{d}y_0$

Joint pdf $f(x,y) = \frac{\mathrm{d}^2 F(x,y)}{\mathrm{d}x\,\mathrm{d}y}$

Marginal pdf of X $f_X(x) = \int_{-\infty}^{\infty} f(x, y_0)\,\mathrm{d}y_0$

Conditional pdf of X given $Y = y$ $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. Linear regression

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is $\widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}$, $\widehat{\beta} = \frac{S_{xy}}{S_{xx}}$

The residual sum of squares $\text{RSS} = S_{yy} - \frac{S_{xy}^2}{S_{xx}}$

$\widehat{\sigma^2} = \frac{\text{RSS}}{n-2}$ $\quad \frac{n-2}{\sigma^2}\,\widehat{\sigma^2}$ is from χ^2_{n-2}

$E(\widehat{\alpha}) = \alpha$, $E(\widehat{\beta}) = \beta$,

$$\operatorname{var}(\widehat{\alpha}) = \frac{\sum x_i^2}{n\,S_{xx}}\sigma^2, \quad \operatorname{var}(\widehat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \operatorname{cov}(\widehat{\alpha}, \widehat{\beta}) = -\frac{\overline{x}}{S_{xx}}\sigma^2$$

$$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x, \quad E(\widehat{y}_x) = \alpha + \beta x, \quad \operatorname{var}(\widehat{y}_x) = \left\{\frac{1}{n} + \frac{(x - \overline{x})^2}{S_{xx}}\right\}\sigma^2$$

$\frac{\widehat{\alpha} - \alpha}{\widehat{\text{se}}\,(\widehat{\alpha})}$, $\frac{\widehat{\beta} - \beta}{\widehat{\text{se}}\,(\widehat{\beta})}$, $\frac{\widehat{y}_x - \alpha - \beta x}{\widehat{\text{se}}\,(\widehat{y}_x)}$ are each from t_{n-2}

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course
	E2.11 ise2 maths -	ISE2
Question Solution 1		Marks & seen/unseen

Parts

(i)(a) $FT\{f'(t)\} = \int_{-\infty}^{\infty} f'(t) e^{-i\omega t} dt$

$$\underset{\text{(by parts)}}{=} \underbrace{\left[f(t)e^{-i\omega t}\right]_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} (-i\omega) f(t) e^{-i\omega t} dt$$

assuming $|f(t)| \to 0$ as $t \to \pm\infty$ — 2

$$= \underline{i\omega \hat{f}(\omega)}$$

[transform exists provided $|tf(t)| \to 0$ as $t \to \pm\infty$ sufficiently quickly] — 2 (for assumptions)

(i)(b) $FT\{tf(t)\} = \int_{-\infty}^{\infty} f(t). t e^{-i\omega t} dt$

$$= \frac{d}{d\omega} \int_{-\infty}^{\infty} f(t) \frac{1}{-i} e^{-i\omega t} dt = \underline{i \frac{d}{d\omega} \hat{f}(\omega)}$$ — 2 } 6

(ii) from above, $FT\{2f' + tf\} = 2i\omega \hat{f}(\omega) + i\hat{f}'(\omega)$ — 2

and $ie^{-\omega^2} \frac{d}{d\omega}(e^{\omega^2}\hat{f}) = ie^{-\omega^2}(2\omega e^{\omega^2}\hat{f} + e^{\omega^2}\hat{f}') = 2i\omega\hat{f} + i\hat{f}'$ — 2 } 4

$\therefore$ LHS = RHS.

(iii) Given, $f(t) = e^{-t^2/4}$ then $2f' + tf = 2(-\frac{2t}{4})e^{-t^2/4} + te^{-t^2/4} = \underline{0}$ — 2

$\therefore$ from (ii): $ie^{-\omega^2} \frac{d}{d\omega}(e^{\omega^2}\hat{f}) = 0 \Rightarrow e^{\omega^2}\hat{f} = \text{const} = A, \text{ say}.$ — 2 } 4

Hence $\underline{\hat{f} = Ae^{-\omega^2}}$

(iv) Directly: $\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-t^2/4} e^{-i\omega t} dt$ and so $\hat{f}(0) = \int_{-\infty}^{\infty} e^{-t^2/4} dt$

Subst $t = 2u$: $\hat{f}(0) = \int_{-\infty}^{\infty} e^{-u^2} 2\, du = \underline{2\sqrt{\pi}}$ using given integral } 2

Hence $\underline{A = \hat{f}(0) = 2\sqrt{\pi}}$

(v) We have that $FT(e^{-t^2/4}) = 2\sqrt{\pi} e^{-\omega^2}$,

i.e. $\int_{-\infty}^{\infty} e^{-t^2/4} \underbrace{e^{-i\omega t}}_{\cos\omega t - i\sin\omega t} dt = 2\sqrt{\pi} e^{-\omega^2}$ — 1

$\therefore$ since $\int_{-\infty}^{\infty} \underbrace{e^{-t^2/4}\sin\omega t}_{\text{ODD}} dt = 0$ we have $\int_{-\infty}^{\infty} \underbrace{e^{-t^2/4}\cos\omega t}_{\text{even}} dt = 2\sqrt{\pi} e^{-\omega^2}$ — 2

and hence $\int_0^{\infty} e^{-t^2/4} \cos\omega t\, dt = \underline{\sqrt{\pi} e^{-\omega^2}}$ — 1 } 4

Total 20

Setter's initials	Checker's initials	Page number
AGW	XW	1 of 4

All unseen except for part (i), but similar done.

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: ISE2

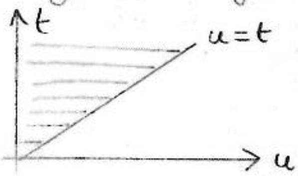
Question: Solution 2

Marks & seen/unseen

Parts

(i) $LHS = \int_{t=0}^{t=\infty} \left\{ \int_{u=0}^{u=t} f(t-u) g(u)\, du \right\} e^{-st}\, dt$ — 1

Region of integration:



Reverse order of integration & cover with vertical strips: t goes from u to ∞ and then u goes 0 to ∞. — 2

So, $LHS = \int_{u=0}^{u=\infty} g(u) \int_{t=u}^{t=\infty} f(t-u) e^{-st}\, dt\, du$ — 5

In inner integral let $y = t-u$, $dy = dt$

$LHS = \int_0^\infty g(u) e^{-su} \left(\int_{y=0}^{y=\infty} f(y) e^{-sy}\, dy \right) du$ — 2

$= \underline{G(s)F(s)}$, as req'd

(ii) Given $y(t) = e^{-\alpha t} - \int_0^t y(t-u) e^{-\beta u}\, du$

Take LT: $Y(s) = LT(e^{-\alpha t}) - Y(s)\, LT(e^{-\beta t})$ — 2

Using tables: $Y(s) = \frac{1}{s+\alpha} - \frac{Y(s)}{s+\beta}$ i.e. $Y(s)\left(1 + \frac{1}{s+\beta}\right) = \frac{1}{s+\alpha}$ — 2 — 5

Hence $Y(s) = \underline{\frac{s+\beta}{(s+\alpha)(s+\beta+1)}}$, as req'd — 1

(iii) $Y(s)$ has poles at $s = -\alpha$, $s = -\beta - 1$. Provided $\underline{\alpha \neq \beta+1}$ these are simple. — 1

$\operatorname{Res}_{s=-\alpha} Y(s)e^{st} = \lim_{s \to -\alpha} \frac{\cancel{(s+\alpha)}(s+\beta)e^{st}}{\cancel{(s+\alpha)}(s+\beta+1)} = \frac{(\beta-\alpha)\, e^{-\alpha t}}{\beta - \alpha + 1}$ // — 2 — 6

$\operatorname{Res}_{s=-\beta-1} Y(s)e^{st} = \lim_{s \to -\beta-1} \frac{\cancel{(s+\beta+1)}(s+\beta)e^{st}}{(s+\alpha)\cancel{(s+\beta+1)}} = \frac{-e^{-(\beta+1)t}}{\alpha - \beta - 1}$ // — 2

$\therefore y(t) = \text{sum of residues} = \underline{\frac{1}{\beta-\alpha+1}\left\{ (\beta-\alpha)e^{-\alpha t} + e^{-(\beta+1)t} \right\}}$ — 1

$(\alpha \neq \beta+1)$

$\underline{\text{If } \alpha = \beta+1}$ then $Y(s) = \frac{s+\beta}{(s+\alpha)^2}$ and has a double pole at $s = -\alpha$ — 1

$\operatorname{Res}_{s=-\alpha} Y(s)e^{st} = \lim_{s \to -\alpha} \left\{ \frac{d}{ds}\left(\cancel{(s+\alpha)^2} \frac{(s+\beta)}{\cancel{(s+\alpha)^2}} e^{st} \right) \right\} = \lim_{s \to -\alpha} \left(e^{st} + t\underbrace{(s+\beta)}_{(\alpha-1)} e^{st} \right)$ — 2 — 4

$= e^{-\alpha t}(1-t)$ //. Hence $\underline{y(t) = (1-t)e^{-\alpha t}}$ $(\alpha = \beta+1)$ — 1

Setter's initials: AGW

Checker's initials: XW

All unseen (except (i)) but similar done.

Total 20

EXAMINATION QUESTIONS/SOLUTIONS 2009-2010

Course: ISE2

Question: Solution 3

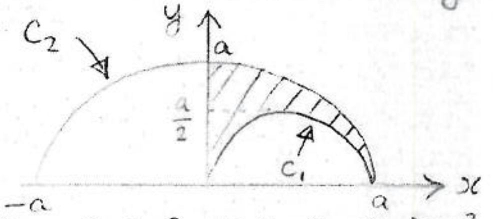
Marks & seen/unseen

Parts

(i) Inner limits: $y=\sqrt{(ax-x^2)} \Rightarrow x^2+y^2-ax=0$ ← Curve C_1

$\Rightarrow (x-\frac{a}{2})^2+y^2=(\frac{a}{2})^2$ with $y>0$, arc of circle centre $(\frac{a}{2},0)$ radius $a/2$ — 1

$y=\sqrt{(a^2-x^2)} \Rightarrow x^2+y^2=a^2$ with $y>0$, arc of circle centre $(0,0)$ radius a (curve C_2) — 1



Outer limit has $x: 0 \to a$ so integral is taken over shaded region. — 2

In polars, C_2 is $r=a$, C_1 is $r^2-ar\cos\theta=0$ i.e. $r=a\cos\theta$ — 2

$dx\,dy \to r\,dr\,d\theta$; $\theta: 0 \to \pi/2$ — 1

Thus $$I=\int_{\theta=0}^{\theta=\pi/2}\sin\theta\cos\theta\int_{r=a\cos\theta}^{r=a} 2re^{-r^2}\,dr\,d\theta$$

$$=\int_0^{\pi/2}\sin\theta\cos\theta\left[-e^{-r^2}\right]_{a\cos\theta}^{a}d\theta$$ — 2

$$=\int_0^{\pi/2}\sin\theta\cos\theta\left(e^{-a^2\cos^2\theta}-e^{-a^2}\right)d\theta$$

$$=\left[\frac{1}{2a^2}e^{-a^2\cos^2\theta}\right]_0^{\pi/2}-e^{-a^2}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}$$ — 2

$$=\underline{\frac{1}{2a^2}(1-e^{-a^2})-\frac{1}{2}e^{-a^2}}$$ — 1

} 12

(ii) Let $P=3x^2y^2+\alpha ye^{xy}$, $Q=2x^3y+xe^{xy}$

Then $\partial P/\partial y=6x^2y+\alpha e^{xy}+\alpha xye^{xy}$; $\partial Q/\partial x=6x^2y+xye^{xy}+e^{xy}$ — 2

$P_y=Q_x$ provided $\underline{\alpha=1=\alpha^*}$ and then I is $\underline{\text{path-independent}}$. — 1

Potential F found from $\partial F/\partial x=P$, $\partial F/\partial y=Q$

From 1st equation $F=x^3y^2+e^{xy}+G(y) \Rightarrow \frac{\partial F}{\partial y}=2x^3y+xe^{xy}+G'(y)$

Setting this equal to Q, we find $G'(y)=0$ & hence $G=\text{const}$.

$\therefore$ Potential $\underline{F(x,y)=x^3y^2+e^{xy}}+\text{const}$ — 2

$\Rightarrow I(\alpha^*)=F(1,2)-F(0,0)=4+e^2-1=\underline{3+e^2}$ — 1

Final part: Take $x=t$, $y=2t$, $dx=dt$, $dy=2\,dt$, $0\le t\le 1$.

Then $$I(\alpha^*)=\int_0^1\left(12t^4+2te^{2t^2}+(4t^4+te^{2t^2})2\right)dt$$

$$=\int_0^1 20t^4+4te^{2t^2}\,dt=\left[\frac{20t^5}{5}+e^{2t^2}\right]_0^1$$

$=4+e^2-1=\underline{3+e^2}$, in agreement with result found using potential. — 2

} 8

Total 20

Setter's initials: Agw

Checker's initials: XW

Unseen, but similar done.

	EXAMINATION QUESTIONS/SOLUTIONS 2009-2010	Course ISE2
Question Solution 4		Marks & seen/unseen

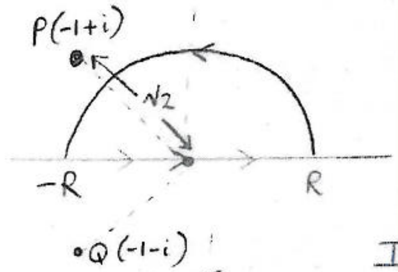
Parts

(i) $f(z)$ has pole singularities when $z^2+2z+2=0$, i.e when $z=(-2\pm\sqrt{(4-8)})/2 = \underset{\sim\sim}{-1\pm i}$ Distinct roots, so poles are simple } 4

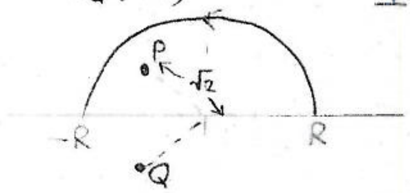
$$\operatorname*{Res}_{z=-1+i} f(z) = \lim_{z\to -1+i} \frac{\cancel{(z-(-1+i))}\, e^{i\alpha z}}{\cancel{(z-(-1+i))}(z-(-1-i))} = e^{-\alpha(1+i)}/2i$$ } 2

$$\operatorname*{Res}_{z=-1-i} f(z) = \lim_{z\to -1-i} \frac{\cancel{(z-(-1-i))}\, e^{i\alpha z}}{(z-(-1+i))\cancel{(z-(-1-i))}} = \frac{e^{\alpha(1-i)}}{-2i}$$ } 2

(ii)



If $R<\sqrt{2}$ then both poles lie outside C. $f(z)$ is $\therefore$ analytic in & on C and so $\int_C f(z)\,dz = 0$ by Cauchy's theorem } 2



If $R>\sqrt{2}$ then pole at $-1+i$ lies inside C.

$$\therefore \int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1+i} f(z)$$

using (i) $= 2\pi i\, e^{-\alpha(1+i)}/2i$

$= \pi e^{-\alpha(1+i)}$ } 2

(iii) Let curved part be Γ.

On Γ, $z = Re^{i\theta}$, $dz = Rie^{i\theta}$

$$\left|\int_\Gamma f(z)\,dz\right| \leq \int_0^\pi |f(Re^{i\theta})|R\underbrace{|ie^{i\theta}|}_{=1}\,d\theta \leqslant \int_0^\pi \frac{e^{-\alpha R\sin\theta}}{|2-R^2|} R\,d\theta \longrightarrow 0 \text{ as } R\to\infty$$

$\therefore \int_\Gamma f(z)\,dz \to 0$ as $R\to\infty$. } 3

(iv) We have
$$\underbrace{\int_C f(z)\,dz}_{=\pi e^{-\alpha(1+i)} \text{ by (ii)}} = \underbrace{\lim_{R\to\infty}\int_\Gamma f(z)\,dz}_{=0 \text{ by (iii)}} + \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$$ } 3

$$\therefore \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x^2+2x+2}\,dx = \pi e^{-\alpha(1+i)} = \pi e^{-\alpha}(\cos\alpha - i\sin\alpha)$$

Taking re & imag parts:

$$\int_{-\infty}^{\infty} \frac{\cos\alpha x}{x^2+2x+2}\,dx = \underset{\sim\sim}{\pi e^{-\alpha}\cos\alpha} \quad \& \quad \int_{-\infty}^{\infty} \frac{\sin\alpha x}{x^2+2x+2}\,dx = \underset{\sim\sim}{-\pi e^{-\alpha}\sin\alpha}$$

(v)
$$(FT)^{-1}\left(\frac{1}{\omega^2+2\omega+2}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega^2+2\omega+2}\,d\omega = \frac{1}{2\pi}\pi e^{-t(1+i)} \text{ (from above)} = \frac{1}{2}e^{-(1+i)t}$$ } 2

Setter's initials	Checker's initials	Page number
AGW	XW	4 of 4

All unseen but similar done.

Total 20

Question 5.

Marks & seen/unseen

Parts

(i) (a)

$$\begin{aligned} P(X_1 > 1) &= 1 - (P(X_1 = 0) + P(X_1 = 1)) \\ &= 1 - (e^{-3} + 3e^{-3}) = 1 - 4e^{-3} = \boxed{0.8009} \end{aligned}$$

$\boxed{3}$

(b)

$$\begin{aligned} P(X_2 > 1) &= 1 - (P(X_2 = 0) + P(X_2 = 1)) \\ &= 1 - (e^{-2} + 2e^{-2}) = 1 - 3e^{-2} = \boxed{0.5940} \end{aligned}$$

$\boxed{2}$

(c)

$$\begin{aligned} P(X_3 > 1) &= 1 - (P(X_3 = 0) + P(X_3 = 1)) \\ &= 1 - (e^{-1} + e^{-1}) = 1 - 2e^{-1} = \boxed{0.2642} \end{aligned}$$

$\boxed{2}$

(d)

$$\begin{aligned} P((X_1 > 1) \cap (X_2 > 1) \cap (X_3 > 1)) &= P(X_1 > 1)P(X_2 > 1)P(X_3 > 1) \\ &= (1 - 4e^{-3})(1 - 3e^{-2})(1 - 2e^{-1}) \\ &= \boxed{0.1260} \end{aligned}$$

$\boxed{2}$

(e)

$$\begin{aligned} P(X_4 > 0) &= 1 - P(X_4 = 0) = 1 - P(X_1 = 0) \cap (X_2 = 0)) \\ &= 1 - e^{-3}e^{-2} = 1 - e^{-5} = \boxed{0.9933} \end{aligned}$$

$\boxed{2}$

(ii) Let T = event test fails and E = event that $X_3 > 1$ and $X_4 > 0$.

(a)

$$\begin{aligned} P(T) &= P(T \mid E)P(E) + P(T \mid \overline{E})P(\overline{E}) \\ P(E) &= P((X_3 > 1) \cap (X_4 > 0)) = (1 - 2e^{-1})(1 - e^{-5}) = \boxed{0.2625} \\ P(T) &= 0.9 \times P(E) + 0.2 \times (1 - P(E)) = \boxed{0.3838} \end{aligned}$$

$\boxed{4}$

(b)

$$P(E \mid T) = \frac{P(T \mid E)P(E)}{P(T)} = \frac{0.9(1 - 2e^{-1})(1 - e^{-5})}{P(T)} = \boxed{0.3837}$$

$\boxed{2}$

(c) Let N= number of times that the diagnostic test fails, then $N \sim Binomial(5, p)$ where $p = P(T)$.

$$\begin{aligned} P(N \leq 1) &= P(N = 0) + P(N = 1) \\ &= (1 - p)^5 + 5p(1 - p)^4 = \boxed{0.3656} \end{aligned}$$

$\boxed{3}$

Question 6.

Marks & seen/unseen

Parts

(i) (a)

$$\int_1^\infty \frac{c}{t^{\alpha+1}}\,dt = 1 \Rightarrow c\left[-\frac{t^{-\alpha}}{\alpha}\right]_1^\infty = 1 \Rightarrow \boxed{c=\alpha}$$

[3]

(b)

$$R(t) = \mathrm{P}(T>t) = \int_t^\infty \frac{\alpha}{x^{\alpha+1}}\,dx = \left[-x^{-\alpha}\right]_t^\infty = \boxed{\tfrac{1}{t^\alpha},\ t>1}$$

[3]

$$h(t) = \frac{f(t)}{R(t)} = \frac{\alpha}{t^{\alpha+1}}\,t^\alpha = \boxed{\tfrac{\alpha}{t},\ t>1}$$

[2]

(c)

$$\boxed{R(2) = \tfrac{1}{2^\alpha}}$$

[1]

(ii) (a) Let $R(t)$ be the system reliability at time t, A_i be the event that component A_i is functioning at time t.

$$\begin{aligned}
R(t) &= \mathrm{P}(A_1 \cap ((A_2\cap A_3)\cup(A_4\cap A_5))\cap A_6)\\
&= \mathrm{P}(A_1)\mathrm{P}(A_6)\mathrm{P}((A_2\cap A_3)\cup(A_4\cap A_5))\\
&= \mathrm{P}(A_1)\mathrm{P}(A_6)(\mathrm{P}(A_2)\mathrm{P}(A_3)+\mathrm{P}(A_4)\mathrm{P}(A_5)-\mathrm{P}(A_2\cap A_3\cap A_4\cap A_5))\\
&= \left(\frac{1}{t^\alpha}\right)^2\left[\left(\frac{1}{t^\alpha}\right)^2+\left(\frac{1}{t^\alpha}\right)^2-\left(\frac{1}{t^\alpha}\right)^4\right]\\
&= \frac{1}{t^{4\alpha}}\left(2-\frac{1}{t^{2\alpha}}\right)
\end{aligned}$$

$$R(2) = \frac{1}{4}\left(2-\frac{1}{2}\right) = \frac{3}{8} = \boxed{0.375}$$

[7]

(b) Let p be the reliability at 2 days of the new component, let $R_c(t)$ be the reliability of the other components at time t, then the reliability at 2 days for the new system is

$$\begin{aligned}
R(2) &= p^2\left(2R_c(2)^2 - R_c(2)^4\right)\\
R(2) > 0.5 &\Rightarrow p^2\left(2R_c(2)^2 - R_c(2)^4\right) > 0.5\\
&\Rightarrow p^2\left(2\frac{1}{2}-\frac{1}{4}\right) > 0.5 \Rightarrow p^2\left(\frac{3}{4}\right) > 0.5\\
&\Rightarrow p^2 > \frac{2}{3} \Rightarrow \boxed{p > 0.8165}
\end{aligned}$$

[4]