EE4-54
EE9CS4-1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

MSc and EEE/ISE PART IV: MEng and ACGI

PREDICTIVE CONTROL

Monday, 23 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	E.C. Kerrigan
	Second Marker(s) :	S. Evangelou

PREDICTIVE CONTROL

1. a) What is meant with the term 'receding horizon principle'? [4]

   b) When and why would one want to implement an optimal control sequence in a receding horizon fashion? [4]

   c) When and why would one maybe *not* want to implement an optimal control sequence in a receding horizon fashion? [4]

   d) What are some of the potential problems that could arise in practice when implementing a receding horizon control law, and why do they occur? [4]

   e) What are some of the potential ways of solving some of the problems you mentioned in part d)? [4]

2. Consider the following finite-horizon discrete-time optimal control problem:

$$\min_{u_0,u_1,\ldots,u_{N-1}} \sum_{k=0}^{N-1} \left(\|Qx_{k+1}\|_2^2 + \|Ru_k\|_1\right)$$

where the system dynamics are given by

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0,1,\ldots,N-1,$$

where the states $x_k \in \mathbb{R}^n$, inputs $u_k \in \mathbb{R}^m$ and weighting matrices $Q \in \mathbb{R}^{p\times n}$ and $R \in \mathbb{R}^{q\times m}$.

a) What is interesting and/or challenging about solving the above problem? In other words, why is it potentially 'difficult' to solve the above problem if you have not taken this course, and what new techniques, which you learnt about in this course, allow us to solve the above problem? [6]

b) Under what practical situations would one maybe be interested in solving the above problem? [4]

c) Formulate the above problem as an equivalent linear or quadratic program in standard form if an estimate of the current state x_0 is given.

Pay particular attention to also defining the sizes of the various matrices and vectors that define the optimisation problem. [10]

3. a) Consider the following optimisation problem:

$$\theta^* := \arg\min_{\theta} f(\theta)$$

subject to the constraints

$$c(\theta) \leq 0,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $c : \mathbb{R}^n \to \mathbb{R}^m$.

With reference to the above, discuss what is meant with an 'exact penalty function' and why one might be interested in defining and working with an exact penalty function. [8]

b) Give an example of a penalty function that cannot be made exact and explain why it cannot be made exact. [2]

c) We are interested in solving the following optimal control problem :

$$\min_{(u_0,\ldots,u_{N-1})} \|Px_N\|_2^2 + \sum_{k=0}^{N-1} \left(\|Qx_k\|_2^2 + \|Ru_k\|_2^2\right),$$

where the system dynamics are given by

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \quad k = 0, 1, \ldots, N-1, \\ y_k &= Cx_k, \quad k = 0, 1, \ldots, N, \end{aligned}$$

the states $x_k \in \mathbb{R}^n$, inputs $u_k \in \mathbb{R}^m$, outputs $y_k \in \mathbb{R}^p$ and the weights $P \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$, subject to hard input constraints

$$u_\ell \leq u_k \leq u_h, \quad k = 0, 1, \ldots, N-1,$$

and soft output constraints

$$y_\ell \leq y_k \leq y_h, \quad k = 1, 2, \ldots, N.$$

Show that the above problem can be solved by formulating a quadratic programming problem in standard form, given an estimate of the current state x_0, but where you have an exact penalty function on the soft output constraints only.

Pay particular attention to also defining the sizes of the various matrices and vectors that define the optimisation problem. [10]

4. a) Show that if X is a given matrix and v is a given vector, then $X^T X v = 0$ if and only if $Xv = 0$. [2]

b) Consider now a generalised version of the least squares problem, namely the equality constrained least squares (ECLS) problem, where the minimisation

$$\theta^* := \arg\min_{\theta} \frac{1}{2}\|M\theta - b\|_2^2$$

is subject to the linear equality constraints

$$C\theta = d$$

in which θ, b, d, M and C are vectors and matrices with compatible dimensions.

Show that a solution to the ECLS problem exists for any d and is unique if and only if C is full row rank and

$$\begin{bmatrix} M \\ C \end{bmatrix}$$

is full column rank. [8]

Hint: You might want to use the Lagrangian function

$$L(\theta, \lambda) := \frac{1}{2}\|M\theta - b\|_2^2 + \lambda^T (C\theta - d),$$

where the vector λ is the Lagrange multiplier. It can be shown that θ^* is a solution to the above ECLS problem if and only if a multiplier λ^* exists such that the pair (θ^*, λ^*) is a stationary point of the Lagrangian function. However, be careful not to start by assuming that the solution to the ECLS problem is unique if and only if the stationary point of the Lagrangian is unique.

c) We are interested in solving the following optimal control problem :

$$\min_{\theta} \frac{1}{2} \sum_{k=0}^{N-1} \left(\|Qx_{k+1}\|_2^2 + \|Ru_k\|_2^2 \right)$$

subject to the constraints

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots, N-1$$

where the states $x_k \in \mathbb{R}^n$, inputs $u_k \in \mathbb{R}^m$, weights $Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times m}$ and the decision variable

$$\theta := \begin{bmatrix} u_0^T & x_1^T & u_1^T & x_2^T & u_2^T & \cdots & x_{N-1}^T & u_{N-1}^T & x_N^T \end{bmatrix}^T.$$

Using the results from above, show that the solution to the above optimal control problem exists and is unique for any given initial state x_0 if and only if

$$\begin{bmatrix} R & 0 \\ 0 & Q \\ B & -I \end{bmatrix}$$

is full column rank. [10]

5. A constant, unmeasured disturbance d is acting on a double integrator

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

where x_k is the state, u_k is the input and y_k is the measured output.

a) Can one design a stable observer to correctly estimate the disturbance d if it is an input disturbance? Justify your answer. [6]

b) Can one design a stable observer to correctly estimate the disturbance d if it is an output disturbance? Justify your answer. [2]

c) Can one design a stabilising controller such that the output tracks any constant reference signal r if the disturbance d is any constant input disturbance? Justify your answer. [6]

d) Does a steady-state exist such that the output is equal to any constant reference signal in the range $-1 \leq r \leq 1$ if there is a constant input disturbance in the range $-0.5 \leq d \leq 0.5$ and the input is subject to hard constraints in the range $-1 \leq u_k \leq 1$? Justify your answer. [6]

6. We are interested in solving the following optimal control problem :

$$V^*(x) := \min_{u_0,\ldots,u_{N-1}} \|Px_N\|_2^2 + \sum_{k=0}^{N-1} \|Qx_k + Ru_k\|_2^2$$

subject to the constraints

$$\begin{aligned} x_0 &= x, \\ x_{k+1} &= Ax_k + Bu_k, \quad k = 0,1,\ldots,N-1 \\ y_k &= Cx_k, \quad k = 0,1,\ldots,N \\ u_\ell &\le u_k \le u_h, \quad k = 0,1,\ldots,N-1 \\ y_\ell &\le y_k \le y_h, \quad k = 1,2,\ldots,N \end{aligned}$$

where the states $x_k \in \mathbb{R}^n$, inputs $u_k \in \mathbb{R}^m$, outputs $y_k \in \mathbb{R}^p$ and weights $P \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{q\times n}$ and $R \in \mathbb{R}^{q\times m}$ are such that $Q^T R = 0$.

The bounds on the constraints satisfy $u_\ell < 0 < u_h$ and $y_\ell < 0 < y_h$.

The solution to the above problem is implemented in a receding horizon fashion to form the closed-loop system.

a) Give a sufficient condition on Q that would ensure the value function $V^*(\cdot)$ is positive definite. Justify your answer. [2]

b) With reference to proving closed-loop stability, why do we care whether the value function is positive definite? [2]

c) Give a sufficient condition on P that would ensure that the origin of the closed-loop system is locally asymptotically stable. Justify your answer. [6]

d) Give the definition of an 'invariant set' for an autonomous discrete-time system. [4]

e) How could one modify the above optimal control problem to guarantee that the set of initial states for which the problem is feasible is an invariant set for the closed-loop system? [4]

f) What problem might occur in practice if the modification in part e) is not implemented? [2]

Question 1 - Bookwork / prescribed reading

(a) An optimal control problem defined over a finite horizon is solved online at each sampling instant, using the current estimate of the state and only the first input is applied to the plant.
At the next sampling instant, the process is repeated keeping the horizon length the same as before, hence ~~to~~ the horizon recedes / moves with the current time.

(b) 1) When the infinite horizon problem cannot be solved analytically or numerically, ~~to~~ but the finite horizon problem can,
2) When there is plant-model mismatch and/or disturbances etc, and we therefore want to implement a feedback policy.
3) When it is not possible to compute an explicit feedback ~~law~~ policy ~~rate~~ but it is possible to compute an open-loop input sequence, from which one can define an implicit feedback law.

Any two of the above would be OK, or other reasonable explanation.

(c) 1) When this is not appropriate to approximate an infinite-horizon problem, e.g. when we have finite horizon problems such as approaching and landing an aeroplane, when a decreasing horizon policy is perhaps more appropriate.

2) When it is too costly to solve ~~for an open-loop policy at~~ an optimisation problem at each time step - we do it only once (perhaps off-line) and then implement the whole of the ~~op~~ solution. This might be ~~useful~~ the case in motion planning, where reference shaping is done off-line.

(d) 1) Loss of stability & violation of constraints, because the horizon is too short!

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1(d) 2) Because an optimisation problem needs to be solved at each time instant, the solution might not be unique or optimal, because there is not enough computation time or the problem has not been set up to ensure uniqueness of the solution

e) 1) Add a terminal weight, which is a control Lyapunov function and add a terminal constraint which is invariant and constraint-admissable under a suitably defined control law.

2) Define the control problem to ensure uniqueness, e.g. use a positive definite weight on the inputs, or use a faster computer or optimisation algorithm which exploits the structure in the control problem.

Recycled

Gearth

Question 2. Bookwork and new problem

(a) 1) It contains a 1-norm term in the stage cost. This is a non-differentiable function, hence we cannot ~~set the d~~ compute and set the derivative of the cost function to zero in order to compute a stationary, hence optimal, point.

2) It contains a mix of quadratic and 1-norm terms, which means it cannot be solved as an unconstrained Least Squares problem, or as a linear program, but would require a quadratic program to solve. We would need to add slack variables to handle the 1-norm term.

(b) The 1-norm term ~~penalise~~ often occurs in situations where one wants to minimise the fuel used, e.g. in satellite attitude control.

The ~~2-norm~~ quadratic term arises when one wants to minimise the energy in a particular variable, e.g. the ~~power~~ energy lost in coils/wiring, or kinetic energy.

By varying Q and R one can trade off one term versus the other, fuel ~~vs. ene~~ used versus energy dissipated/lost/minimised.

(c) Cost function is equivalent to

$$\left\| \begin{matrix} Qx_1 \\ Qx_2 \\ \vdots \\ Qx_N \end{matrix} \right\|_2^2 + \left\| \begin{matrix} Ru_0 \\ \vdots \\ Ru_{N-1} \end{matrix} \right\|_1 = \left\| \bar{Q}\bar{x} \right\|_2^2 + \left\| \bar{R}\bar{u} \right\|_1 ,$$

where $\bar{x} := \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$, $\bar{u} := \begin{pmatrix} u_0 \\ \vdots \\ u_{N-1} \end{pmatrix}$, $\bar{Q} := I_N \otimes Q \in \mathbb{R}^{Np \times Nn}$

$\bar{R} := I_N \otimes R \in \mathbb{R}^{Nq \times Nm}$

2c). There are many ways to proceed from here on, the following is just an example.

The ~~optimisation~~ control problem is equivalent to the optimisation problem

$$\min_{\bar{x},\bar{u},s} \ \bar{x}\,\bar{Q}'\bar{Q}\,\bar{x} + 1_{qN}^T s$$

subject to the constraints

$$\bar{x} = \bar{A}\,x_0 + \bar{B}\,\bar{u} + \bar{P}\,\bar{x}$$

$$-s \leq \bar{R}\,\bar{u} \leq s$$

To deal with the 1-norm. Note $s \geq 0$ is implicit. $s \in \mathbb{R}^{qN}$

where $\bar{A} := \begin{bmatrix} A \\ 0_{(N-1)n\times n} \end{bmatrix}$, $\bar{B} := I_N \otimes B \in \mathbb{R}^{nN\times mN}$

$$\bar{P} := \left[\begin{array}{c|c} 0_{n\times(N-1)n} & \\ I_{(N-1)} \otimes A & 0_{Nn\times n} \end{array}\right] \in \mathbb{R}^{nN\times nN}$$

This is a quadratic program in standard form if we write it as

$$\min_{\bar{x},\bar{u},s} \begin{pmatrix} \bar{x} \\ \bar{u} \\ s \end{pmatrix}' \overbrace{\begin{pmatrix} \bar{Q}'\bar{Q} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{N(n+m+q)\text{ columns and rows}} \begin{pmatrix} \bar{x} \\ \bar{u} \\ s \end{pmatrix} + \overbrace{\begin{pmatrix} 0 \\ 0 \\ 1_{qN} \end{pmatrix}}^{N(n+m+q)\text{ components}}{}^{T} \begin{pmatrix} \bar{x} \\ \bar{u} \\ s \end{pmatrix}$$

subject to

$$N_n \text{ rows}\left\{ \overbrace{\begin{pmatrix} I_{Nn}-\bar{P} & -\bar{B} & 0 \end{pmatrix}}^{N(n+m+q)\text{ columns}} \begin{pmatrix} \bar{x} \\ \bar{u} \\ s \end{pmatrix} = \underbrace{\bar{A}}_{N_n \text{ rows, } n \text{ columns}} x_0 \right.$$

$$2qN \text{ rows}\left\{ \underbrace{\begin{pmatrix} 0 & \bar{R} & -I_{qN} \\ 0 & -\bar{R} & -I_{qN} \end{pmatrix}}_{N(n+m+q)\text{ columns}} \begin{pmatrix} \bar{x} \\ \bar{u} \\ s \end{pmatrix} \leq \overbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}^{2qN\text{ components.}} \right.$$

QED

Gearth

Question 3 Bookwork and new problem

a) An exact penalty function is a term that is added to the cost function to turn the constrained optimisation problem into an unconstrained problem. The penalty function is positive if there is constraint violation and it is zero if none of the constraints are satisfied, e.g.

$$\tilde{\theta}^* = \arg\min_{\theta} f(\theta) + \rho g(\theta)$$ scalar $\rho > 0$

where $g : \mathbb{R}^n \to \mathbb{R}$ penalises the amount by which the constraints are violated,

e.g. the function

$$c_i(\theta)^+ := \begin{cases} c_i(\theta) & \text{if } c_i(\theta) \geq 0 \\ 0 & \text{if } c_i(\theta) \leq 0 \end{cases}$$

represents the amount of constraint violation in the ith constraint. A suitable choice for $g(\theta)$ would then be $g(\theta) = \|c(\theta)^+\|_1$ or $g(\theta) = \|c(\theta)^+\|_\infty$.

A penalty function is said to be exact if one can choose the scalar $\rho > 0$ sufficiently large such that

$$\tilde{\theta}^* = \tilde{\theta}$$

i.e. the solution to the above "unconstrained" optimisation problem is equal to the solution of the constrained solution.

(b) $g(\theta) = \|c(\theta)^+\|_2^2$ cannot be made exact because it is differentiable. An essential property for exactness is that the penalty function be non-differentiable. Quadratic penalty functions therefore cannot be made exact.

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3 (c) We can reformulate the control problem as

$$\min_{u_0,\dots,u_{N-1},t} \|Px_N\|_2^2 + \sum_{k=0}^{N-1}\left(\|Qx_k\|_2^2 + \|Ru_k\|_2^2\right) + \rho t$$

scalar

$$\begin{aligned} \text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k, && k = 0,1,\dots,N-1 \\ & u_l \le u_k \le u_h, && k = 0,1,\dots,N-1 \\ & y_l - 1_p t \le Cx_k \le y_h + 1_p t, && k = 1,2,\dots,N \\ & t \ge 0 \end{aligned}$$

This is equivalent to using an ∞-norm exact penalty function.

There are many ways to convert the above problem into a QP. The following is just an example.

Cost function becomes

$$\left\| \begin{matrix} Qx_0 \\ Qx_1 \\ \vdots \\ Qx_{N-1} \\ Px_N \\ Ru_0 \\ \vdots \\ Ru_{N-1} \end{matrix} \right\|_2^2 = \Bigg\| \underbrace{\begin{pmatrix} Q & & & & & & \\ & Q & & & & & \\ & & \ddots & & & & \\ & & & Q & & & \\ & & & & P & & \\ & & & & & R & \\ & & & & & & \ddots \\ & & & & & & & R \end{pmatrix}}_{M} \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix} \Bigg\|_2^2 + \rho t$$

$$M := \begin{pmatrix} I_N \otimes Q & & 0 \\ 0 & P & 0 \\ 0 & 0 & I_N \otimes R \end{pmatrix}, \quad \bar{x} := \begin{pmatrix} x_0 \\ \vdots \\ x_N \end{pmatrix}, \quad \bar{u} := \begin{pmatrix} u_0 \\ \vdots \\ u_{N-1} \end{pmatrix}$$

$$\Rightarrow \text{Cost function} = \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix}' M'M \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix} + t = \begin{pmatrix} \bar{x} \\ \bar{u} \\ t \end{pmatrix}' \underbrace{\begin{pmatrix} M'M & 0 \\ 0 & \rho \end{pmatrix}}_{H} \begin{pmatrix} \bar{x} \\ \bar{u} \\ t \end{pmatrix}$$

is in standard form.

H: $(N+1)n + Nm + 1$ rows and $(N+1)n + Nm + 1$ columns

Gearth

3c) Constraints become — Current state estimate

$$\bar{x} = \bar{A}\hat{x}_0 + \bar{B}\bar{u} + \bar{P}\bar{x}$$

$$1_N \otimes u_l \le \bar{u} \le 1_N \otimes u_h$$

$$1_N \otimes y_l - 1_{pN} t \le \bar{C}\bar{x} \le 1_N \otimes y_h + 1_{pN} t$$

$$-t \le 0$$

where $\bar{A} := \begin{pmatrix} I_n \\ 0 \end{pmatrix} \in \mathbb{R}^{(N+1)n \times n}$

$$\bar{B} := \begin{pmatrix} 0 \\ I_N \otimes B \end{pmatrix} \in \mathbb{R}^{(N+1)n \times Nm}$$

$$\bar{P} := \left[\begin{array}{c|c} 0 & \\ I_{N-1} \otimes A & 0 \end{array}\right] \in \mathbb{R}^{(N+1)n \times (N+1)n}.$$

$$\bar{C} := \begin{bmatrix} 0 & I_N \otimes C \end{bmatrix} \in \mathbb{R}^{Np \times (N+1)n}.$$

$\Rightarrow$ Constraints become.

$$\begin{pmatrix} I_{(N+1)n} - \bar{P} & -\bar{B} & 0 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{u} \\ t \end{pmatrix} = \bar{A}\hat{x}_0$$

$$\underbrace{\begin{pmatrix} 0 & +I_{Nm} & 0 \\ 0 & -I_{Nm} & 0 \\ \bar{C} & 0 & -1_{pN} \\ -\bar{C} & 0 & -1_{pN} \\ 0 & 0 & -1 \end{pmatrix}}_{\substack{2Nm + 2pN + 1 \text{ rows and} \\ (N+1)n + Nm + 1 \text{ columns}}} \begin{pmatrix} \bar{x} \\ \bar{u} \\ t \end{pmatrix} \le \begin{pmatrix} 1_N \otimes u_h \\ -1_N \otimes u_l \\ 1_N \otimes y_h \\ -1_N \otimes y_l \\ 0 \end{pmatrix}$$

QED

Gearth

Question 4 New problem

(a) I) $Xv = 0 \Rightarrow X^TXv = 0$

II) $X^TXv = 0 \Leftrightarrow v^TX^TX = 0 \Rightarrow v^TX^TXv = 0 \Leftrightarrow \|Xv\|_2^2 = 0$

$\Leftrightarrow Xv = 0$

$\therefore X^TXv = 0 \Rightarrow Xv = 0$

QED

(b) $L(\theta, \lambda) = \frac{1}{2}\left(\theta^TM^TM\theta - 2\theta^TM^Tb + b^Tb\right) + \lambda^T(C\theta - d)$

Stationary points:

$\nabla_\theta L = M^TM\theta - M^Tb + C^T\lambda = 0$

$\nabla_\lambda L = C\theta - d = 0$

$$\Leftrightarrow \begin{pmatrix} M^TM & C^T \\ C & 0 \end{pmatrix}\begin{pmatrix} \theta \\ \lambda \end{pmatrix} = \begin{pmatrix} M^Tb \\ d \end{pmatrix} \qquad (*)$$

Clearly, $C\theta = d$ for any $d \Leftrightarrow C$ full row rank.

Suppose two solutions exist to $(*)$, namely $\begin{pmatrix} \theta_1 \\ \lambda_1 \end{pmatrix} \neq \begin{pmatrix} \theta_2 \\ \lambda_2 \end{pmatrix}$

Then

$$\begin{pmatrix} M^TM & C^T \\ C & 0 \end{pmatrix}\begin{pmatrix} \theta_1 - \theta_2 \\ \lambda_1 - \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Defining $x = \theta_1 - \theta_2$ and $y = \lambda_1 - \lambda_2$

$\Leftrightarrow M^TMx + C^Ty = 0$ and $Cx = 0$

$\Rightarrow x^TM^TMx + x^TC^Ty = 0$

$\Leftrightarrow x^TM^TMx = 0$, because $x^TC^T = 0$.

$\Leftrightarrow \|Mx\|_2^2 = 0$

$\Leftrightarrow Mx = 0$.

Summarising, $\begin{pmatrix} M \\ C \end{pmatrix} x = 0$ has to be satisfied for any two stationary points.

$\begin{pmatrix} M \\ C \end{pmatrix}$ is full column rank $\Leftrightarrow x = 0$ is the only solution.

$\Rightarrow \theta_1 = \theta_2$ is the only solution $\Leftrightarrow \begin{pmatrix} M \\ C \end{pmatrix}$ full column rank.

QED

G'earth

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4 (c) The cost function becomes

$$\frac{1}{2}\left\|\begin{matrix} R u_0 \\ Q x_1 \\ R u_1 \\ Q x_2 \\ \vdots \\ Q x_{N-1} \\ R u_{N-1} \\ Q x_N \end{matrix}\right\|_2^2 = \cancel{\frac{1}{2}\left\| \underbrace{I_N \otimes \begin{pmatrix} R & 0 \\ 0 & Q\end{pmatrix}}_{M} \Theta \right\|_2^2 , \; b = 0}$$

$$= \frac{1}{2}\left\| \underbrace{I_N \otimes \begin{pmatrix} R & 0 \\ 0 & Q \end{pmatrix}}_{M} \Theta \right\|_2^2 , \quad b = 0$$

The constraints become

$$\underbrace{\begin{pmatrix} B & -I & 0 & 0 & & & & \\ 0 & A & B & -I & & & & \\ 0 & 0 & 0 & A & & & & \\ & & & & \ddots & & & \\ & & & & & -I & 0 & 0 \\ & & & \cdots & 0 & A & B & -I \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} u_0 \\ x_1 \\ u_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ u_{N-1} \\ x_N \end{pmatrix}}_{\Theta} = \underbrace{\begin{pmatrix} -A x_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{d}$$

C is full row rank because of the ~~Eb~~ position of the I on each block row. (However, clearly a Θ always exists for any x_0, because we can always set the input sequence to zero). By rearranging columns of C this becomes obvious:

ar de
asi
only

$$\begin{pmatrix} B & & & & -I & & & \\ & B & & & A & -I & & \\ & & \ddots & & & A & -I & \\ & & & B & & & \ddots & \\ & & & & B & \cdots & 0 & A & -I \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \\ x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} -A x_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

full row rank
(all rows non-zero)

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4c) If we were to know stack M and C, we get $\begin{pmatrix} M \\ C \end{pmatrix} =$

$$\left(\begin{array}{c|c|c|c}
\begin{matrix} R & O \\ O & Q \end{matrix} & & & \\
& \begin{matrix} R & O \\ O & Q \end{matrix} & & \\
& & \ddots & \\
& & \begin{matrix} R & O \\ O & Q \end{matrix} & \\
& & & \begin{matrix} R & O \\ O & Q \end{matrix} \\
B \quad -I & & & \\
A & B \quad -I & & \\
& A & \ddots & \\
& & \ddots & B \quad -I \\
& & A & B \quad -I
\end{array}\right)$$

Clearly, this matrix has full column rank if ~~and~~

$$\begin{pmatrix} R & O \\ O & Q \\ B & -I \end{pmatrix} \text{ is full column rank}$$

because each block column is linearly independent of each other, because the ~~matr sub-matrix~~ block columns in ~~the~~

$$\left(\begin{array}{cc|cc}
R & O & & \\
O & Q & & \\
& & R & O \\
& & O & Q \\
B & -I & O & O \\
O & O & B & -I
\end{array}\right)$$

note no A here.

are "linearly independent" ~~of~~ of each other..

For the only if part, just look at last block column in $\begin{pmatrix} M \\ C \end{pmatrix}$.
This block ~~column is full colum rank~~ $\begin{pmatrix} M \\ C \end{pmatrix}$ is full column rank only if the last block column is full column rank.

QED

Gieanth

Question 5 New problem

(a) We can form the augmented system

$$x_{k+1} = Ax_k + Bu_k + B_d d_k$$
$$y_k = Cx_k + C_d d_k$$
$$d_{k+1} = d_k.$$

$$\Longleftrightarrow \begin{pmatrix} x_{k+1} \\ d_{k+1} \end{pmatrix} = \overbrace{\begin{pmatrix} A & B_d \\ 0 & I \end{pmatrix}}^{\tilde{A}} \begin{pmatrix} x_k \\ d_k \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} d_k.$$

$$y_k = \underbrace{\begin{pmatrix} C & C_d \end{pmatrix}}_{\tilde{C}} \begin{pmatrix} x_k \\ d_k \end{pmatrix}$$

We can design a stable observer $\Longleftrightarrow (\tilde{A}, \tilde{C})$ is detectable.

$B_d = B$, $C_d = 0$ for an input disturbance

$\Rightarrow (\tilde{A}, \tilde{C})$ detectable $\Longleftrightarrow \begin{pmatrix} C & C_d \\ \lambda I - A & B_d \\ 0 & \lambda I - I \end{pmatrix}$ full column rank $\forall$ e/values of $\tilde{A}$ on or outside unit circle

$\Rightarrow$ e/values of $\tilde{A}$ are e/values of A & e/values of I (all 1)

But A has all e/values at $1 \Rightarrow \lambda I - I = 0$ for all e/values

$\Rightarrow (\tilde{C}, \tilde{A})$ detectable $\Longleftrightarrow \begin{pmatrix} \overbrace{1 \quad 0}^{C} & C_d \\ \begin{matrix} 0 & -1 \\ 0 & 0 \end{matrix} & B_d \end{pmatrix}$ full column rank

$I - A$

$B_d = B$, $C_d = 0 \Rightarrow \text{rank} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0.5 \\ 0 & 0 & 1 \end{pmatrix} = 3$

$\Rightarrow$ Yes, ~~we~~ the system is detectable

$\Rightarrow$ Can design a stable observer

QED

Gieanth

Date: No.

5 b) $B_d = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $C_d = 1$ if output disturbance

$$\mathrm{rank}\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2 < 3$$

C_d

B_d

⇒ Cannot design a stable observer, because disturbance is not detectable.

QED

(c) At equilibrium

$$x_e = A x_e + B u_e + B_d d$$
$$y_e = C x_e + C_d d = r$$

$$\Leftrightarrow \begin{pmatrix} I-A & -B \\ C & 0 \end{pmatrix} \begin{pmatrix} x_e \\ u_e \end{pmatrix} = \begin{pmatrix} B_d d \\ r - C_d d \end{pmatrix}$$

If $B_d = B$, $C_d = 0$ (This disturbance is already detectable).

$$\Rightarrow \begin{pmatrix} I-A & -B \\ C & 0 \end{pmatrix} \begin{pmatrix} x_e \\ u_e \end{pmatrix} = \begin{pmatrix} \begin{matrix} 0.5d \\ 1d \end{matrix} \\ r \end{pmatrix}$$

This has a solution for any r and d iff the matrix on the LHS is full ~~column~~ row rank.

$$\mathrm{rank}\begin{pmatrix} 0 & -1 & -0.5 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} = 3 \,\&\,,$$

Reachability matrix

$$W_r = \begin{pmatrix} B & AB \end{pmatrix} = \underbrace{\begin{pmatrix} 0.5 & 1.5 \\ 1 & 1 \end{pmatrix}}^{\text{full row rank}}$$

⇒ System is reachable

⇒ Yes, we can design a stabilizing controller, such that $y = r$ in steady state.

QED

Gearth

5(d) Want to guarantee a solution to the following constraints exist ~~if~~ $\forall$ $r \in [-1, 1]$ and $d \in [-0.5, 0.5]$

$$\begin{pmatrix} 0 & -1 & -0.5 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ u \end{pmatrix} = \begin{pmatrix} 0.5d \\ 1d \\ r \end{pmatrix}$$

$$-1 \leq u \leq 1$$

Solving the linear equations we get

$$x_1 = r$$
$$u = -d$$
$$x_2 = 0$$

Hence, if $d \in [-0.5, 0.5] \Rightarrow u \in [-0.5, 0.5] \subseteq [-1, 1]$

So yes, ~~in principle we can~~ a constraint-admissible steady-state exists.

QED

Gearth

Question 6 Bookwork and new problem

(a) The stage cost is $\|Qx + Ru\|_2^2 = x'Q'Qx \underbrace{+2u'R'Qx}_{=0} + u'R'Ru$

$\Rightarrow$ The stage cost is positive definite if $Q'Q > 0$, which is the case if Q is full column rank.
$\Rightarrow$ Cost function is positive definite.

(b) We want to use the value function as a Lyapunov function for the closed-loop system. One of the conditions it has to fulfill is that it is positive define, inorder to be a Lyap. function.

(c) Let K be any stablising state feedback gain, i.e. $\rho(A + BK) < 1$.

$\Rightarrow$ Let $\forall x: \|P(A+BK)x\|_2^2 - \|Px\|_2^2 < -\|Qx + RKx\|_2^2$

$\Leftrightarrow \forall x: x^T (A+BK)^T P^T P (A+BK) x - x^T P^T P x < -x^T Q^T Q x - u^T K^T R^T R K x$

$\Leftrightarrow (A+BK)^T P^T P (A+BK) - P < -Q^T Q - K^T R^T R K$.

This means that the terminal cost is a control Lyapunov function in a neighbourhood of the origin, which is sufficient to guarantee closed-loop stability.

(d) A set S is invariant for the system $x_{k+1} = f(x_k)$ iff $f(x_k) \in S$ for all $x_k \in S$.

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6(e) Would add ~~the~~ a constraint of the form

$$Mx_N \leq b$$

where M and b are computed such that it is constraint-admissible under a stabilising gain K, i.e

$$ul \leq Kx \leq ub \quad \forall x : Mx \leq b.$$
$$yl \leq Cx \leq yl \quad \forall x : Mx \leq b$$

where $\rho(A + BK) < 1$ and it is invariant for the closed-loop system $x_{k+1} = (A+BK)x_k$, i.e.

$$M(A + BK)x \leq b \; \forall x : Mx \leq b.$$

(f) The optimisation problem might become infeasible at some sample instant, hence constraints have to be violated.

QED

Gearth