

# Control engineering exam paper - Model answers

## Question 1

- The controllability matrix is

$$\mathcal{R} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

which has rank three. Hence the system is controllable. The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

which has rank three. Hence the system is observable.

[4 marks]

- Note that  $y = Cx = x_2$ , hence  $\dot{y} = \dot{x}_2 = x_3 = C_1x$ , with  $C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ .
- The feedback is given by

$$u = -k^2x_2 - kx_3.$$

The closed-loop system is described by the equation

$$\dot{x} = A_{cl}x = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -k^2 & -k \end{bmatrix} x.$$

The characteristic polynomial of  $A_{cl}$  is

$$\det(\lambda I - A_{cl}) = \lambda^3 + (k+1)\lambda^2 + k(k+1)\lambda + (k^2-1).$$

Routh test shows that all roots of the polynomial have negative real part for all  $k > 1 = k_*$ .

[6 marks]

- Since  $\dot{y} = x_3$ ,  $\ddot{y} = \dot{x}_3 = x_1 + u = C_2x + u$ , with  $C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ .
- Note that

$$u = -k^3y - k^2\dot{y} - k\ddot{y} = -k^3x_2 - k^2x_3 - k(x_1 + u).$$

This can be rewritten as

$$u = -\frac{k^3}{1+k}x_2 - \frac{k^2}{1+k}x_3 - \frac{k}{1+k}x_1,$$

which is well-defined for all  $k \neq -1$ . The closed-loop system is described by the equation

$$\dot{x} = \tilde{A}_{cl}x = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 - \frac{k}{1+k} & -\frac{k^3}{1+k} & -\frac{k^2}{1+k} \end{bmatrix} x.$$

The characteristic polynomial of  $\tilde{A}_{cl}$  is

$$\det(\lambda I - \tilde{A}_{cl}) = \lambda^3 + \frac{k^2+k+1}{k+1}\lambda^2 + k^2\lambda + \frac{k^3-1}{k+1}.$$

Routh test shows that all roots of the polynomial have negative real part for all  $k > 1 = k_*$ .

[6 marks]

## Question 2

- The closed-loop system is described by the equation  $\dot{x} = Ax$ , with

$$A = \begin{bmatrix} -\alpha_1 & \alpha_1 & 0 \\ \alpha_2 & -\alpha_2 - \alpha_3 & \alpha_3 \\ 0 & \alpha_4 & -\alpha_4 \end{bmatrix}.$$

[4 marks]

- Note that

$$\det A = 0,$$

which shows that  $A$  has a zero eigenvalue.

[2 marks]

- The characteristic polynomial of  $A$  is (recall that it has a zero eigenvalue)

$$\det(\lambda I - A) = \lambda(\lambda^2 + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\lambda + (\alpha_1\alpha_4 + \alpha_4\alpha_2 + \alpha_1\alpha_3)).$$

Selecting, for example,

$$\alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = 1, \quad \alpha_4 = 1,$$

yields

$$\det(\lambda I - A) = \lambda(\lambda^2 + 4\lambda + 3) = \lambda(\lambda + 3)(\lambda + 1).$$

[6 marks]

- The differential equations are

$$\dot{z}_{12} = 3z_2 - 2z_1 - z_3, \quad \dot{z}_{23} = z_1 - 3z_2 + 2z_3.$$

[2 marks]

These can be rewritten as

$$\dot{z}_{12} = -2z_{12} + z_{23}, \quad \dot{z}_{23} = z_{12} - 2z_{23}.$$

As a result

$$F = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

The characteristic polynomial of  $F$  is

$$\det(\lambda I - F) = (\lambda + 1)(\lambda + 3),$$

which shows that the matrix  $F$  has eigenvalues equal to  $-1$  and  $-3$ .

[4 marks]

The above implies that

$$\lim_{t \rightarrow \infty} x_1(t) - x_2(t) = \lim_{t \rightarrow \infty} x_2(t) - x_3(t) = 0,$$

which is the same as condition  $(*)$ .

[2 marks]

### Question 3

- The state equations are

$$x_1(k+1) = a_1x_1(k) + x_2(k) + a_1u(k), \quad x_2(k+1) = a_0x_1(k) + a_0u(k),$$

$$y(k) = x_1(k) + b_2u(k).$$

As a result

$$y(k+1) = x_1(k+1) + u(k+1) = a_1x_1(k) + x_2(k) + a_1u(k) + u(k+1),$$

and

$$\begin{aligned} y(k+2) &= a_1x_1(k+1) + x_2(k+1) + a_1u(k+1) + u(k+2) \\ &= a_1y(k+1) + x_2(k+1) + u(k+2) \\ &= a_1y(k+1) + a_0y(k) + u(k+2), \end{aligned}$$

which shows that the state-space and the input-output descriptions are equivalent.

[8 marks]

- The reachability matrix is

$$\mathcal{R} = \begin{bmatrix} a_1 & a_1^2 + a_0 \\ a_0 & a_0a_1 \end{bmatrix},$$

and

$$\det \mathcal{R} = -a_0^2.$$

As a result the system is reachable if  $a_0 \neq 0$ . If  $a_0 = 0$  the system is controllable (regardless of the values of  $a_1$ ).

[6 marks]

- The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ a_1 & 1 \end{bmatrix},$$

hence the system is observable for any value of the constants  $a_0$  and  $a_1$ .

[2 marks]

- Selecting  $a_0 = 0$  yields

$$y(k+2) + a_1y(k+1) = u(k+2)$$

and replacing  $k+1$  with  $k$  yields the given system.

[4 marks]

## Question 4

- The equilibria of the system satisfy the equations

$$x_1 = \alpha \sin x_2, \quad x_2 = -\alpha \sin x_1.$$

Eliminating  $x_2$  yields

$$x_1 = \alpha \sin(-\alpha \sin x_1).$$

This equation, for  $\alpha \in (0, 1]$  has the unique solution  $x_1 = 0$ , hence  $(0, 0)$  is the only equilibrium of the system. [4 marks]

- The linearization of the system around the zero equilibrium is given by

$$\delta_x^+ = A\delta_x + B\delta_u,$$

with

$$A = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

[4 marks]

- The characteristic polynomial of the matrix  $A$  is

$$\det(\lambda I - A) = \lambda^2 + \alpha^2,$$

which has roots inside the unity disk for all  $|\alpha| < 1$ . For  $\alpha = 1$ , the roots are  $\pm j$ . As a result, the linearized system is asymptotically stable for all  $|\alpha| < 1$ , and stable for  $\alpha = 1$ . [4 marks]

- Let  $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$  and note that

$$A + BK = \begin{bmatrix} K_1 & \alpha + K_2 \\ \alpha & 0 \end{bmatrix}.$$

Hence, selecting  $K_1 = 0$  and  $K_2 = -\alpha$ , yields two eigenvalues at zero. [4 marks]

- The observability matrix is

$$\mathcal{O} = \begin{bmatrix} c_1 & c_2 \\ -c_2\alpha & c_1\alpha \end{bmatrix},$$

and  $\det \mathcal{O} = \alpha(c_1^2 + c_2^2)$ . The system is therefore observable provided that  $c_1^2 + c_2^2 \neq 0$ .

Let  $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$  and note that

$$A + LC = \begin{bmatrix} L_1c_1 & \alpha + L_1c_2 \\ -\alpha + L_2c_1 & L_2c_2 \end{bmatrix}.$$

Hence, selecting  $L_1 = -\alpha \frac{c_2}{c_1^2 + c_2^2}$  and  $L_2 = \alpha \frac{c_1}{c_1^2 + c_2^2}$  yields two eigenvalues at zero.

[4 marks]