IMPERIAL COLLEGE LONDON

E4.05
CS7.22
SO7

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

MSc and EEE PART IV: MEng and ACGI

Corrected Copy

Q5, Q6

TRAFFIC THEORY & QUEUEING SYSTEMS

Thursday, 30 April 2:30 pm

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : J.A. Barria

Second Marker(s) : M.M. Draief

Special instructions for students

1. Erlang Loss formula recursive evaluation:

$$E_N(\rho) = \frac{\rho E_{N-1}(\rho)}{N + \rho E_{N-1}(\rho)}$$
$$E_0(\rho) = 1.$$

2. Engset Loss formula recursive evaluation (for a fixed M and $p = \alpha/1+\alpha$):

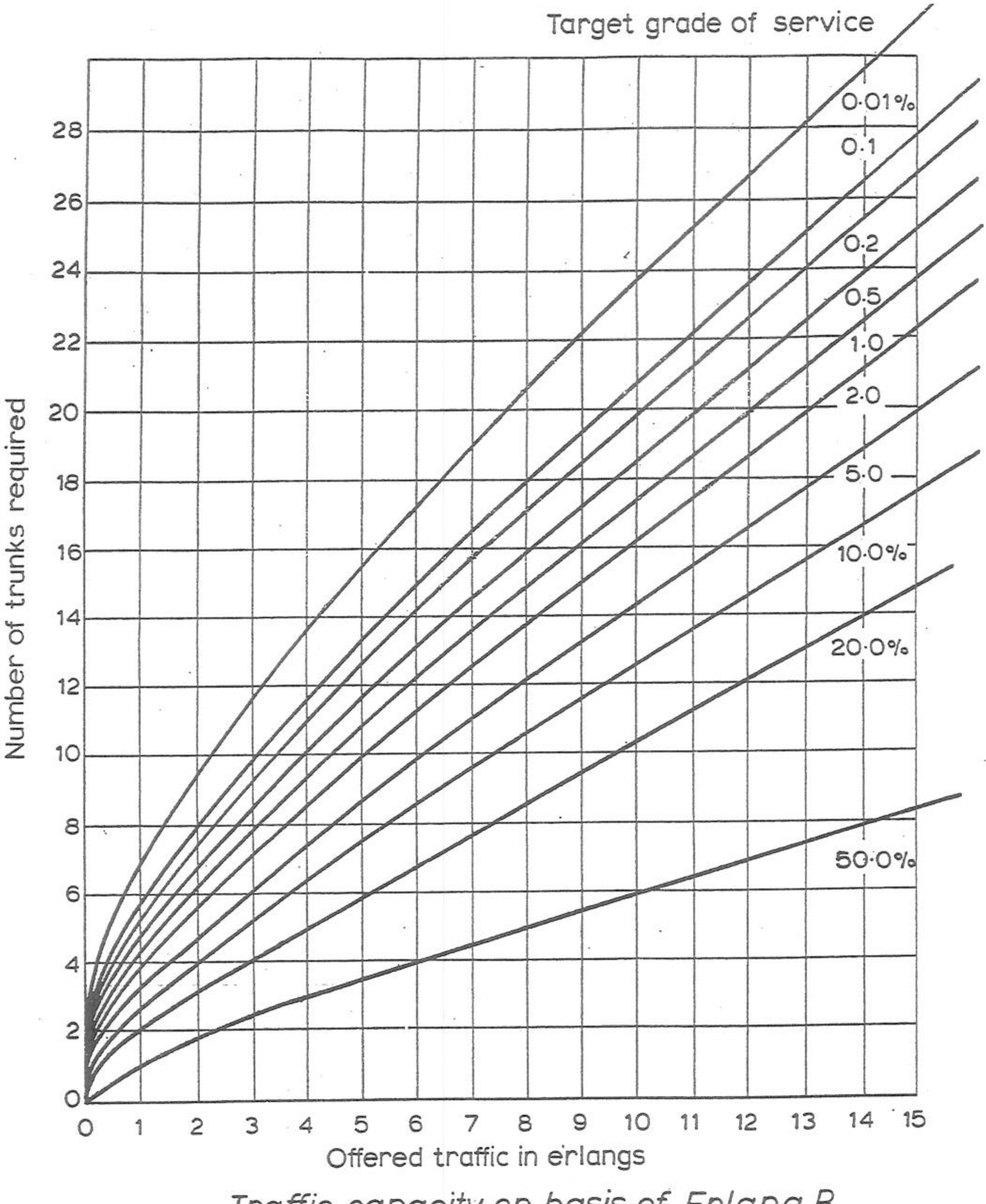
$$e_N = \frac{(M-N+1)\alpha e_{N-1}}{N + (M-N+1)\alpha e_{N-1}}$$
$$e_0 = 1.$$
$$\alpha = \lambda/\mu.$$

3. Traffic capacity on basis of Erlang B formula (next page).

Note: for large ρ, N is approximately linear: $N \approx 1.33\rho + 5$

4. Expected residual time

$$E[R] = \frac{1}{2}\sum_{k=1}^{m} \lambda_k E[S_k^2]$$



Traffic capacity on basis of Erlang B. formula.

1.

a) Design a switching exchange multi-channel link operating with a loss probability of 0.005.

Assume:
- Incoming calling rate: 1320 calls/hour.
- Average call duration: 150 s.

i) Determine the total offered traffic for the link. [3]

ii) Determine the total carried traffic for the link. [3]

iii) Estimate the size of the link. [4]

b) For the Erlang model.

i) Discuss the assumptions of the model. [3]

ii) Derive the local balance equations. [3]

iii) Derive the global balance equations. [3]

iv) Is the system reversible? [1]

2.

a) For an M/M/K system:

i) Derive the distribution of $\mathrm{P}[Q_t = i \mid Delay]$.

[4]

ii) Derive $\mathrm{E}[Q_t = i \mid Delay]$.

[4]

iii) Derive $\mathrm{Var}[Q_t = i \mid Delay]$.

[6]

b) ATM admission control mechanisms.

i) Discuss the assumptions and approximation made when using the stationary approximation to derive the equivalent capacity function.

[3]

ii) Discuss the assumptions and approximation made when using the fluid-flow approximation to derive the equivalent capacity function.

[3]

3.

a) A switching exchange can obtain the following information of one of its outgoing links:

Number of channels = N = 65,
Carried traffic = 44.8 Erlangs,
Mean call duration = 2.5 minutes.

i) Estimate the offered traffic. [4]

ii) Obtain the ~~call~~ Call blocking probability B_C. [4]

iii) Estimate the call arrival rate. [4]

b) In the context of a fluid flow approximation framework:

i) Derive an expression of the cumulative probability distribution $F_i(t+\Delta t, x)$ at time $t+\Delta t$, with the system in state $\boldsymbol{i}$:

- as a function of $F_i(t,x)$ and $F_i(t, x-\Delta x)$. [4]

ii) Define and derive Δx and $F_i(t, x-\Delta x)$. [4]

4.

a) For an M/M/K/N system:

i) State the relation between N and the buffer size B. [2]

ii) Derive $\mathrm{E}[Q_t \mid Delay]$. [7]

iii) Derive the expected waiting time $\mathrm{E}[W \mid Delay]$. [4]

Explain clearly all steps of your derivations

b)

i) Describe the characteristics of an Interrupted Poisson process (IPP). [3]

ii) Give example and describe a traffic processes that could be modelled using an IPP. [4]

5.

A Poisson stream of packets arrives to a single-channel communication link at a rate of $\lambda = 300$[packets/s].

The arrivals consist of a random mixture of two (2) types of traffic with the following packet sizes:

Traffic Type	Packet size [bits]	Probability of Arrival
Type 1	320	25 %
Type 2	160	75 %

Assume:

- Type 2 traffic is given non-pre-emptive priority.

- The transmission rate of the link is 64[Kbits/s].

i) Determine the mean message length. [2]

ii) Determine the mean square message length. [2]

iii) Calculate ρ and $\cancel{E(r)}$. E(R) [4]

iv) Determine the mean transit time for Type 1 traffic. [4]

v) Determine the mean transit time for Type 2 traffic. [4]

vi) Determine the overall mean transit time. [4]

6.

Consider the degradable system MRM model of Figure ~~5.1~~. 6.1.

Assume:

- Failure rates: $\lambda_1 = 2$; $\lambda_2 = 1$.
- Restoration Strategy 1: $\mu_1 = 6$; $\mu_2 = 1$
- Restoration Strategy 2: $\mu_1 = 1$; $\mu_2 = 6$
- Reward Structure $R = [r_1, r_2, r_3]$. And, $r_1 \geq r_2 \geq r_3 = 0$

Note :

$$Y(t) = \sum_{i=0}^{N} r_i \tau_i \quad \text{(accumulated reward up to time t)}$$

$$W(t) = \frac{Y(t)}{t}$$

i) From the MRM shown in Figure 6.1, derive the transition matrices for Restoration Strategy 1 and Restoration Strategy 2.

[4]

ii) For Restoration Strategy 1 and Restoration Strategy 2 obtain: $\lim_{t\to\infty} \mathrm{E}[W(t)]$.

[6]

iii) Using $\lim_{t\to\infty} \mathrm{E}[W(t)]$ as benchmark for comparison, which strategy would you recommend.

[4]

iv) Find the relationship between r_i for Restoration Strategies 1 and Restoration Strategy 2 to accomplish the following requirement:

$$\lim_{t\to\infty} \mathrm{E}[W(t)] = \frac{3}{10}\left[\sum_{i=1}^{3} r_i\right]$$

[6]

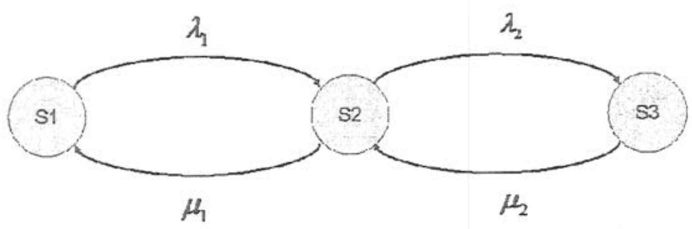


Figure 6.1

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Q1 a) i) 1320 calls/s ⟶ 22 calls/m

150 s ⟶ 2.5 m

Offered traffic $22 \cdot 2.5 = 55$ Erlangs

ii) Carried traffic $55(1-B_E) = 55(0.995) = 54.725$ Erlangs

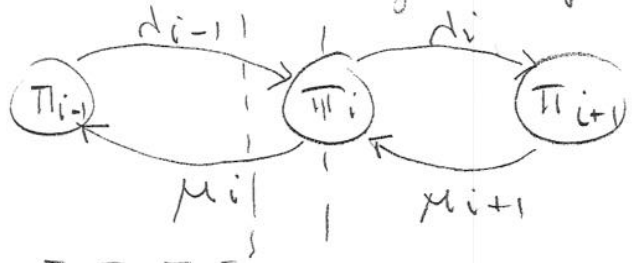
iii) $N \sim 1.33\rho + 5 = 1.33 \times 55 + 5 = 78.15$

$N = 79$

b) i) - Arrival stream Poisson (λ)

- Channel holding time are independent and exponential R.V. mean holding time $1/\mu$

- Access switch gives full availability



ii) - - - -

$$\pi_i = \left(\frac{\lambda_{i-1}}{\mu_i}\right)\pi_{i-1}$$

iii)

$$\pi_{i-1}\lambda_{i-1} + \pi_{i+1}\mu_{i+1} = \pi_i\lambda_i + \pi_i\mu_i$$

iv) yes

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Q2

a) M/M/K

i) $$P[Q_i = i \mid N_t \geq K] = \frac{P[N_t = K+i]}{\sum_{j=0}^{\infty} P[N_t = K+j]}$$

$$= \frac{\pi_K \rho^i}{\sum_{j=0}^{\infty} \pi_K \rho^i} = \frac{\rho^i}{\frac{1}{1-\rho}}$$

$$= \rho^i (1-\rho)$$

ii) ii) $$E[Q_t = i \mid N_K \geq K] = \frac{\rho}{1-\rho}$$

iii)) $$\mathrm{var}[Q_t = i \mid N_K \geq K] = \sum_{i=0}^{\infty} i^2 (1-\rho)\rho^i - \frac{\rho^2}{(1-\rho)^2}$$

$$\sum_{i=0}^{\infty} i^2 (1-\rho)\rho^i = (1-\rho)(1\rho + 4\rho^2 + 9\rho^3 + 16\rho^4 + \dots)$$

$$= \frac{(1-\rho)^3}{(1-\rho)^2}(\rho + 4\rho^2 + 9\rho^3 + 16\rho^4 + \dots)$$

$$= \rho - 3\rho^2 + 3\rho^3 - \rho^4 + 4\rho^2 - 12\rho^3 + 12\rho^4 - 4\rho^5 + 9\rho^3 - 27\rho^4 + 16\rho^4 \dots$$

$$= \frac{(\rho + \rho^2)}{(1-\rho)^2} \Rightarrow \mathrm{Var} = \frac{\rho}{(1-\rho)^2}$$

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Q4 b) i) Equivalent capacity stationary approximation

- Large number of sources multiplexed

$N \gg 1$, $p \ll 1$

$$\pi_i = \binom{N}{i} p^i (1-p)^{N-i}$$

can be approximated closely by the normal distribution

$$P_L = \frac{1}{m}\int_{J_0}^{\infty} \frac{e^{-(x-m)^2/2\sigma^2}(x-C)}{\sqrt{2\pi\sigma^2}}\,dx$$

$$\varepsilon = \frac{1}{m}\int_{J_0}^{\infty} \frac{e^{-(x-m)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dx$$

ii) The fluid flow approximation

- take into account the server buffer
- cell arrival can be represented by a fluid
- The capacity of the server is high

$$G(x) \sim A_N \rho^N e^{-\beta R x / R_p}$$

$$R = (1-\rho)\left(1-\frac{\alpha}{\beta}\right) \Big/ \left(1-\frac{C_L}{NR_p}\right)$$

$$\rho = \frac{N p R_p}{C_L}$$

$$P_L \sim e^{-\beta R x / R_p}$$

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Q3

a) $N = 65$

carried traffic $= 44.8$ Erlang

mean call duration $= 2.5$ minutes

i) carried traffic $= \rho(1 - B_c) = 44.8$

$$1.33\rho + 5 = N \Rightarrow \rho = \frac{N - 5}{1.33} = 45.11$$

$$45.11(1 - B_c) = 44.8 \Rightarrow B_c = \frac{45.11 - 44.8}{45.11}$$

$B_c = 0.00687$

iii) $\rho \sim 45.11$

ii) 45.11 Erlangs $= \lambda \cdot 2.5 = 45.11$

$$\lambda = \frac{45.11}{2.5} = 18.044 \ \frac{\text{calls}}{m}$$

5

Q3 b) i) $F_i(t+\Delta t, x)$ = probability that hopper occupancy is less than or equal to x with i sources 'on' at time $t+\Delta t$.

$$F_i(t+\Delta t, x) = [N-(i-1)]\lambda\Delta t\, F_{i-1}(t,x)$$
$$+ (i+1)\alpha\Delta t\, F_{i+1}(t,x)$$
$$+ \{1-[(N-i)\lambda + i\alpha]\Delta t\} F_i[t, x-(i-c)\alpha\Delta t]$$
$$+ o(\Delta t)$$

Explain:

$i\alpha - \alpha c = h$ = rate of fillup hopper

hopper should start at : $x - h\Delta t$

ii)

$$\Delta x = (i-c)\alpha\Delta t$$

$$F_i(t, x-\Delta x) = F_i[t, x-(i-c)\alpha\Delta t]$$

- one voice source will generate cell at a rate V cells/s during a talk spurt of average length $1/\alpha$ s.
- x is incremented by V/α cells during a talk spurt
- system capacity Vc cells/s
- Equivalent capacity $\dfrac{Vc}{\frac{V}{\alpha}} = \alpha c$
- i sources on $\Rightarrow$ iV cells/s $\Rightarrow$ αi

$\Delta x = (i-c)\alpha\Delta t$

$$\frac{\Delta x}{\Delta t} = (i-c)\alpha$$

6

Q4

a) $M/M/K/N$

i) $N = K + B$

ii) $P[Q_t \mid \text{Delay}] = P[Q_t = i \mid K \le N_t < K+B]$

$$= \frac{P[N_t = K+i]}{P[\text{Delay}]}$$

$$= \frac{\pi_K \rho^i}{\pi_K \left[\frac{1-\rho^B}{1-\rho}\right]} \qquad i = 0, \dots, B$$

$$= \rho^i \left[\frac{1-\rho}{1-\rho^B}\right] \qquad i = 0, \dots, B$$

ii) $$E[Q_t \mid \text{Delay}] = \sum_{i=1}^{B-1} i \frac{1}{1-\rho^B} [\rho^i (1-\rho)] \qquad i = 0, 1, \dots B-1$$

$B = 4$, $B-1 = 3$

$$\frac{1}{1-\rho^4}\left[(1-\rho)0 + (1-\rho)\rho \cdot 1 + (1-\rho)\rho^2 \cdot 2 + (1-\rho)\rho^3 \cdot 3\right]$$

$$\frac{1}{1-\rho^4}\left[\rho - \rho^2 + 2\rho^2 - 2\rho^3 + 3\rho^3 - \rho^4 3\right]$$

$$\frac{1}{1-\rho^4}\left[\underbrace{\rho + \rho^2 + \rho^3 + \rho^4}_{\sum_{j=1}^{4} \rho^j = \frac{\rho}{1-\rho}(1-\rho^4)} - 4\rho^4\right]$$

$$\frac{1}{1-\rho^4}\left[\frac{\rho}{1-\rho}(1-\rho^4) - 4\rho^4\right] = \frac{\rho}{1-\rho} - \frac{B\rho^B}{1-\rho^B}\Bigg|_{B=4}$$

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Q4 a) iii) Using Little's theorem.

For items accepted into buffer (i.e. not rejected) the entry rate is

$$\lambda_A = \lambda [1 - P(\text{loss})]$$

Then applying Little's theorem to the buffer

$$E[W] = \left(\frac{1}{\lambda_A}\right) E[Q_t]$$

For delayed arrivals, we shall have

$$E[W|\text{delay}] = \left(\frac{1}{\lambda_A}\right) E[Q_t|\text{delay}]$$

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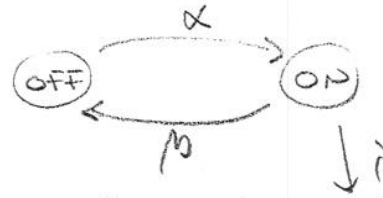
Q4 a) IPP:



i) ON : the arrival rate is Poisson (λ)

OFF: No arrival is possible

Assume ON and OFF sojourn time are exponentially distributed

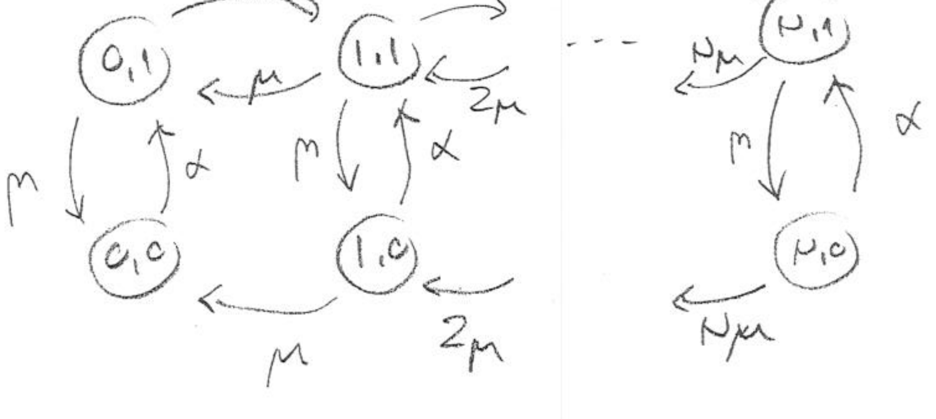
ii) Example: over-flow traffic

could be represented by a 2-state Markov process

$$y_t = \begin{cases} 0 & \text{- arrival stream is OFF} \\ 1 & \text{- arrival stream is ON} \end{cases}$$

The joint process $\{N_t, Y_t\}$ N_t being the number of busy channels on the overflow link.

state transition diagram



μ = call holding time

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Q5 M/G/1

mean message length $= \frac{5}{4}B = \frac{3}{4}B + \frac{1}{4}2B$

mean square message length $= \frac{7}{4}B^2 = \frac{3}{4}B^2 + \frac{1}{4}(2B)^2$

mean message length $= \frac{25}{8}$ ms. $= E(S)$

mean square message length $= \frac{175}{16}(\text{ms})^2 \sim 11\,(\text{ms})^2$

$$\rho = \lambda E(S) = 0.94 = 300 \frac{\times 25}{8} = 0.9375$$

$$E(R) = \frac{1}{2}\lambda E(S^2) = 1.65\ \text{ms} = \frac{1}{2}300 \times 11$$

i) Type 1

$$E(W_1) = \frac{E(R)}{1-\rho_1} = \frac{1.65}{1-0.5625} = 3.77\ \text{ms}$$

$$\rho_1 = \lambda_1 E(S_1) = 300 \times \frac{3}{4} \cdot 2.5\ \text{ms} = 0.5625$$

$2.5\ \text{ms} \times 64\ \text{kb/s} = 160$ bits

$$E(T_1) = E(W_1) + E(S_1) = 6.27\ \text{ms}$$

ii) Type 2 $\quad E(W_2) = \left[\frac{E(W_1)}{1-\rho_1-\rho_2}\right] = \frac{3.77}{1-0.9375} = 60.3\ \text{ms}$

$$E(T_2) = E(W_2) + E(S_2) = 65.3\ \text{ms}$$

iii) $E(T) = \frac{3}{4}E(T_1) + \frac{1}{4}E(T_2) = 21.0\ \text{ms}$

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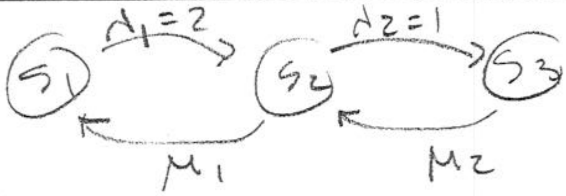
Q6 i)



$$R = [R_1 \; , \; R_2 \; , \; R_3]$$

a) $$Q = \begin{bmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 6 & -6 \end{bmatrix}$$

b) $$Q = \begin{bmatrix} -2 & 2 & 0 \\ 6 & -7 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

ii) $$Q^T \pi = 0 \; , \; \pi e = 1 \Rightarrow \begin{bmatrix} -2 & 1 & 0 \\ 2 & -2 & 6 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 = \frac{3}{10} \; , \; x_2 = \frac{6}{10} \; , \; x_3 = \frac{1}{10}$$

$$Q^T \pi = 0 \; , \; \pi e = 1 \Rightarrow \begin{bmatrix} -2 & 6 & 0 \\ 2 & -7 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 = \frac{3}{5} \; , \; x_2 = \frac{1}{5} \; , \; x_3 = \frac{1}{5}$$

$$E[W(t)]_A = \frac{3}{10} R_1 + \frac{6}{10} R_2 + \frac{1}{10} R_3$$

$$E[W(t)]_B = \frac{6}{10} R_1 + \frac{2}{10} R_2 + \frac{2}{10} R_3$$

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Q6

iii)

$R_3 = 0$

$$\frac{3}{10} R_1 + \frac{6 R_2}{10} \ (?) \ \frac{6}{10} R_1 + \frac{2}{10} R_2$$

$$4 R_2 \ (?) \ 3 R_1$$

I) if $R_1 = \frac{4}{3} R_2 \longrightarrow E[W(t)]_A = E[W(t)]_B$

II if $R_1 > \frac{4}{3} R_2 \longrightarrow$ System B

III if $R_1 < \frac{4}{3} R_2 \longrightarrow$ System A

iv)

System A

$$\frac{3}{10}\left[R_1 + R_2 + R_3\right] = \frac{3}{10}\left[R_1 + 2R_2 + \frac{1}{3} R_3\right]$$

$$R_2 + \frac{R_3}{3} = R_3$$

$$R_2 = \frac{2}{3} R_3$$

System B

$$\frac{3}{10}\left[R_1 + R_2 + R_3\right] = \frac{3}{10}\left[2R_1 + \frac{2}{3} R_2 + \frac{2}{3} R_3\right]$$

$$R_1 = \frac{1}{3} R_2 + \frac{1}{3} R_3$$