IMPERIAL COLLEGE LONDON EE2-5

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

EEE/ISE PART II: MEng, BEng and ACGI

SIGNALS AND LINEAR SYSTEMS

Time allowed: 2:00 hours

There are THIRTEEN questions on this paper.

This paper is accompanied by four tables of formulae.

ANSWER ALL QUESTIONS.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First marker: P.Y.K. Cheung
Second marker: A. Manikas

Special instructions for invigilators: None

Information for candidates: None

1. The unit step sequence u[n] is defined as

$$u[n]=\begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}.$$

A discrete-time signal x[n] is shown in Figure 1.1.

a) Sketch the functions u[-n] and u[1-n].

[4]

b) Sketch and label each of the following signals:

i) $x[n]u[1-n]$

ii) $x[n]\left(u[n+2]-u[n]\right)$

iii) $x[n]\,\delta[n-1]$

[6]

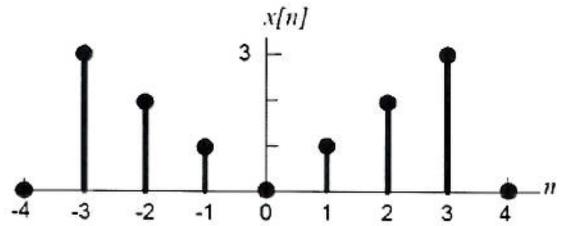


Figure 1.1

2. Sketch the even and odd components of the signal $x(t)=4e^{-0.5t}u(t)$ as shown in Figure 2.1.

[5]

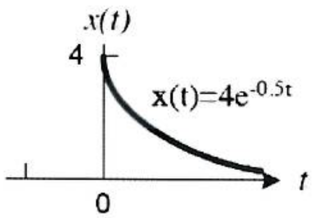


Figure 2.1

3. Compute the output *y(t)* for a continuous-time LTI system whose impulse response *h(t)* and the input *x(t)* are given by

$$h(t)=e^{-\alpha t}u(t) \qquad x(t)=e^{\alpha t}u(-t) \qquad \alpha>0.$$

[6]

4. Consider the capacitor shown in Figure 4.1. Let input $x(t) = i(t)$ and output $y(t)=v_c(t)$.

a) Find the input-output relationship in the form of a differential-integral equation.

[2]

b) Determine with justifications whether the system is:
   i) memoryless;
   ii) causal;
   iii) linear;
   iv) time-invariant.

[4]

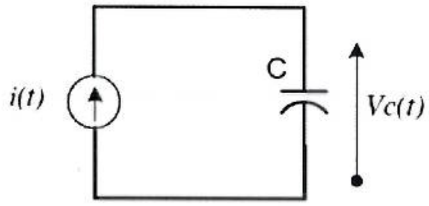


Figure 4.1

5. Consider a continuous-time system whose input $x(t)$ and output $y(t)$ are related by the following differential equation:

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

where a is a constant.

a) Given the initial condition $y(0)=y_0$, find the zero-input response $y_{zi}(t)$ of the system.

[3]

b) Derive the impulse response $h(t)$. If the input $x(t) = Ke^{-bt}u(t)$, derive the zero-state response of the system. You may use the formula tables provided.

[4]

c) Hence or otherwise, derive the general expression for $y(t)$.

[3]

6. The output $y(t)$ of a continuous-time LTI system is found to be $2e^{-3t}u(t)$ when the input $x(t)$ is $u(t)$.

a) Derive the transfer function H(s) of the system. Hence or otherwise, find the impulse response $h(t)$ of the system.

[5]

b) Find the output $y(t)$ when the input $x(t)$ is $e^{-t}u(t)$.

[5]

7. Find the inverse Laplace transform of

$$X(s) = \frac{5s+13}{s(s^2+4s+13)}.$$

[6]

8. The frequency response of a system is given by:

$$H(\omega) = \frac{10^4(1 + j\omega)}{(10 + j\omega)(100 + j\omega)}.$$

Sketch the Bode amplitude plot for this system.

[8]

9. Plot on the s-plane the pole and zero locations for a system with the transfer function:

$$H(s) = \frac{2s + 4}{s^2 + 4s + 20}.$$

[4]

10. It is known that $g(t) = x(t)\cos t$ and that the Fourier transform of $g(t)$ is:

$$G(\omega) = \begin{cases} 1, & -2 \le \omega \le +2 \\ 0, & otherwise \end{cases}.$$

a) Derive an expression for $G(\omega)$ in terms of the Fourier transform $X(\omega)$ of $x(t)$. Sketch $G(\omega)$ and $X(\omega)$.

[6]

b) Hence determine $x(t)$.

[3]

11. A continuous-time signal

$$f(t) = 10\cos 2000\pi t + \sqrt{2}\sin 3000\pi t + 2\cos(5000\pi t + \frac{\pi}{4})$$

is sampled at a rate of 4000 samples/second.

a) Derive an expression for the discrete-time signal f $[k]$ in terms of k, where k is the sample number.

[5]

b) Explain with justification why this sampling rate causes aliasing.

[3]

c) Determine the maximum sampling period T that can be used to sample this signal without aliasing.

[2]

12. A discrete-time shift-invariant system is described by the difference equation

$$y[n] = 0.5y[n-1] + bx[n].$$

a) Derive the frequency response $H(\omega)$ of the system.

[4]

b) Find the value of b so that $|H(\omega)|$ is equal to 1 at $\omega = 0$.

[3]

c) Find the frequency ω at which the output power is half that of its peak value.

[3]

13. Find the z-transform of each of the following sequences:

a) $x[n] = 2^n u[n] + 3\left(\frac{1}{2}\right)^n u[n]$

[3]

b) $x[n] = \cos[n\omega_0]u[n]$

[3]

[THE END]

Table of formulae for E2.5 Signals and Linear Systems

(For use during examination only.)

Convolution Table

No.	$x_1(t)$	$x_2(t)$	$x_1(t) * x_2(t) = x_2(t) * x_1(t)$
1	$x(t)$	$\delta(t-T)$	$x(t-T)$
2	$e^{\lambda t}u(t)$	$u(t)$	$\dfrac{1-e^{\lambda t}}{-\lambda}u(t)$
3	$u(t)$	$u(t)$	$tu(t)$
4	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\dfrac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}u(t) \quad \lambda_1 \neq \lambda_2$
5	$e^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$te^{\lambda t}u(t)$
6	$te^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$\dfrac{1}{2}t^2e^{\lambda t}u(t)$
7	$t^N u(t)$	$e^{\lambda t}u(t)$	$\dfrac{N!e^{\lambda t}}{\lambda^{N+1}}u(t) - \displaystyle\sum_{k=0}^{N}\dfrac{N!t^{N-k}}{\lambda^{k+1}(N-k)!}u(t)$
8	$t^M u(t)$	$t^N u(t)$	$\dfrac{M!N!}{(M+N+1)!}t^{M+N+1}u(t)$
9	$te^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\dfrac{e^{\lambda_2 t}-e^{\lambda_1 t}+(\lambda_1-\lambda_2)te^{\lambda_1 t}}{(\lambda_1-\lambda_2)^2}u(t)$
10	$t^M e^{\lambda t}u(t)$	$t^N e^{\lambda t}u(t)$	$\dfrac{M!N!}{(N+M+1)!}t^{M+N+1}e^{\lambda t}u(t)$
11	$t^M e^{\lambda_1 t}u(t)$	$t^N e^{\lambda_2 t}u(t)$	$\displaystyle\sum_{k=0}^{M}\dfrac{(-1)^k M!(N+k)!t^{M-k}e^{\lambda_1 t}}{k!(M-k)!(\lambda_1-\lambda_2)^{N+k+1}}u(t)$
	$\lambda_1 \neq \lambda_2$		$+\displaystyle\sum_{k=0}^{N}\dfrac{(-1)^k N!(M+k)!t^{N-k}e^{\lambda_2 t}}{k!(N-k)!(\lambda_2-\lambda_1)^{M+k+1}}u(t)$
12	$e^{-\alpha t}\cos(\beta t+\theta)u(t)$	$e^{\lambda t}u(t)$	$\dfrac{\cos(\theta-\phi)e^{\lambda t}-e^{-\alpha t}\cos(\beta t+\theta-\phi)}{\sqrt{(\alpha+\lambda)^2+\beta^2}}u(t)$
			$\phi = \tan^{-1}[-\beta/(\alpha+\lambda)]$
13	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(-t)$	$\dfrac{e^{\lambda_1 t}u(t)+e^{\lambda_2 t}u(-t)}{\lambda_2-\lambda_1} \quad \mathrm{Re}\,\lambda_2 > \mathrm{Re}\,\lambda_1$
14	$e^{\lambda_1 t}u(-t)$	$e^{\lambda_2 t}u(-t)$	$\dfrac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_2-\lambda_1}u(-t)$

Laplace Transform Table

No.	$x(t)$	$X(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5	$e^{\lambda t} u(t)$	$\frac{1}{s-\lambda}$
6	$te^{\lambda t} u(t)$	$\frac{1}{(s-\lambda)^2}$
7	$t^n e^{\lambda t} u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
8a	$\cos bt\, u(t)$	$\frac{s}{s^2+b^2}$
8b	$\sin bt\, u(t)$	$\frac{b}{s^2+b^2}$
9a	$e^{-at} \cos bt\, u(t)$	$\frac{s+a}{(s+a)^2+b^2}$
9b	$e^{-at} \sin bt\, u(t)$	$\frac{b}{(s+a)^2+b^2}$
10a	$re^{-at} \cos(bt+\theta)\, u(t)$	$\frac{(r\cos\theta)s+(ar\cos\theta - br\sin\theta)}{s^2+2as+(a^2+b^2)}$
10b	$re^{-at} \cos(bt+\theta)\, u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb} + \frac{0.5re^{-j\theta}}{s+a+jb}$
10c	$re^{-at} \cos(bt+\theta)\, u(t)$ $r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$ $\theta = \tan^{-1}\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right)$ $b = \sqrt{c-a^2}$	$\frac{As+B}{s^2+2as+c}$
10d	$e^{-at}\left[A\cos bt + \frac{B-Aa}{b}\sin bt\right]u(t)$ $b = \sqrt{c-a^2}$	$\frac{As+B}{s^2+2as+c}$

Fourier Transform Table

No.	$x(t)$	$X(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$	
13	$\cos\omega_0 t\, u(t)$	$\dfrac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\dfrac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0 t\, u(t)$	$\dfrac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\dfrac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\, u(t)$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\dfrac{t}{\tau}\right)$	$\tau\operatorname{sinc}\left(\dfrac{\omega\tau}{2}\right)$	
18	$\dfrac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\dfrac{\omega}{2W}\right)$	
19	$\Delta\left(\dfrac{t}{\tau}\right)$	$\dfrac{\tau}{2}\operatorname{sinc}^2\left(\dfrac{\omega\tau}{4}\right)$	
20	$\dfrac{W}{2\pi}\operatorname{sinc}^2\left(\dfrac{Wt}{2}\right)$	$\Delta\left(\dfrac{\omega}{2W}\right)$	
21	$\displaystyle\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\displaystyle\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\dfrac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	

z-transform Table

No.	$x[n]$	$X[z]$
1	$\delta[n-n]$	z^{-k}
2	$u[n]$	$\frac{z}{z-1}$
3	$nu[n]$	$\frac{z}{(z-1)^2}$
4	$n^2u[n]$	$\frac{z(z+1)}{(z-1)^3}$
5	$n^3u[n]$	$\frac{z(z^2+4z+1)}{(z-1)^4}$
6	$\gamma^n u[n]$	$\frac{z}{z-\gamma}$
7	$\gamma^{n-1}u[n-1]$	$\frac{1}{z-\gamma}$
8	$n\gamma^n u[n]$	$\frac{\gamma z}{(z-\gamma)^2}$
10	$\frac{n(n-1)(n-2)\cdots(n-m+1)}{\gamma^m m!}\gamma^n u[n]$	$\frac{z}{(z-\gamma)^{m+1}}$
11a	$\lvert\gamma\rvert^n \cos\beta n\, u[n]$	$\frac{z(z-\lvert\gamma\rvert\cos\beta)}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
11b	$\lvert\gamma\rvert^n \sin\beta n\, u[n]$	$\frac{z\lvert\gamma\rvert\sin\beta}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12a	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$	$\frac{rz[z\cos\theta-\lvert\gamma\rvert\cos(\beta-\theta)]}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12b	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$ $\quad \gamma=\lvert\gamma\rvert e^{j\beta}$	$\frac{(0.5re^{j\theta})z}{z-\gamma}+\frac{(0.5re^{-j\theta})z}{z-\gamma^*}$
12c	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$	$\frac{z(Az+B)}{z^2+2az+\lvert\gamma\rvert^2}$

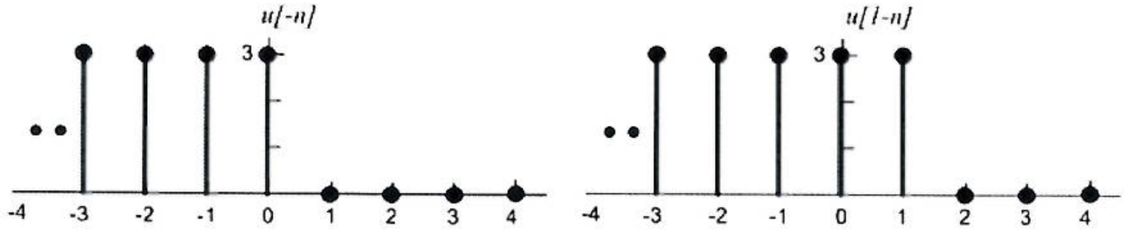
$$r=\sqrt{\frac{A^2\lvert\gamma\rvert^2+B^2-2AaB}{\lvert\gamma\rvert^2-a^2}} \qquad \beta=\cos^{-1}\frac{-a}{\lvert\gamma\rvert} \qquad \theta=\tan^{-1}\frac{Aa-B}{A\sqrt{\lvert\gamma\rvert^2-a^2}}$$

E2.5 Signals and Linear Systems
Solutions 2011

All questions are UNSEEN and covers the whole syllabus of this course.

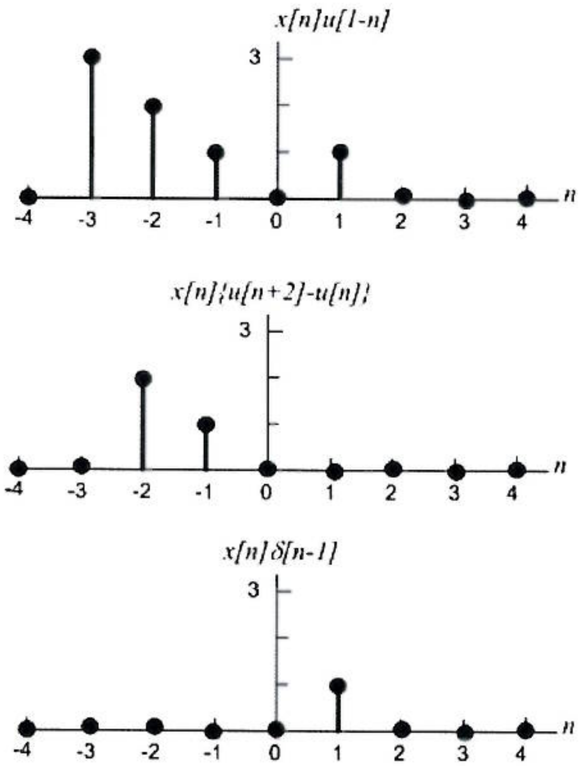
Answer to Question 1 (Topic: Signal modelling)

a)



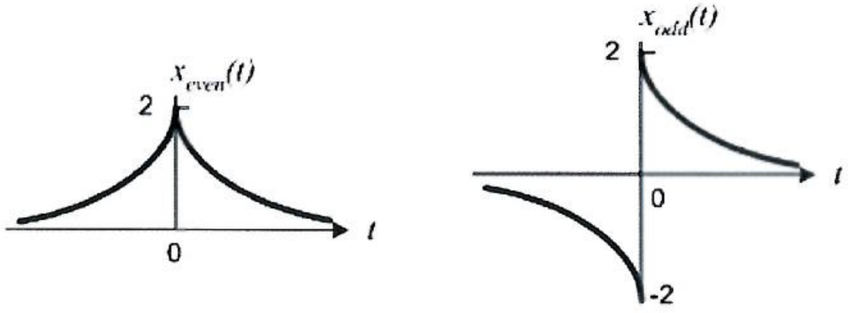
[4]

b)



[6]

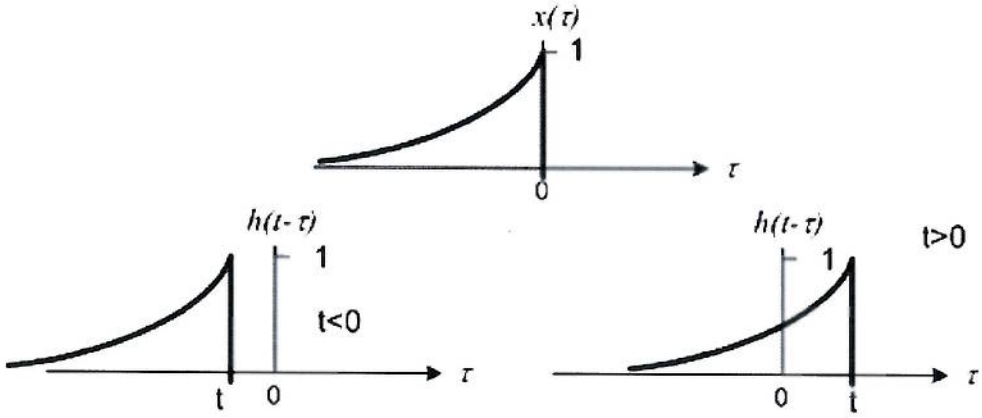
Answer to Question 2 (Topic: Signal modelling & classification)



[5]

Answer to Question 3 (Topic: Convolution)

Here are the plots for x(τ) and h(t-τ):

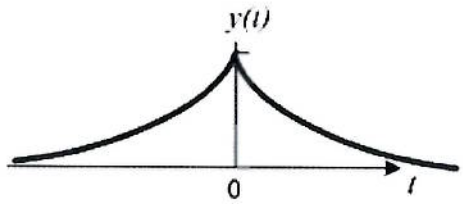


We can see from this diagram that for t < 0, x(τ) and h(t- τ) overlap from τ = -∞ to τ = t. For t > 0, they overlap from τ = -∞ to τ = 0. Hence:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$= \begin{cases} \displaystyle\int_{-\infty}^{t} e^{\alpha\tau}e^{-\alpha(t-\tau)}\,d\tau = e^{-\alpha t}\int_{-\infty}^{t} e^{2\alpha\tau}\,d\tau = \frac{1}{2\alpha}e^{\alpha t} & for \quad t<0 \\ \displaystyle\int_{-\infty}^{0} e^{\alpha\tau}e^{-\alpha(t-\tau)}\,d\tau = e^{-\alpha t}\int_{-\infty}^{0} e^{2\alpha\tau}\,d\tau = \frac{1}{2\alpha}e^{-\alpha t} & for \quad t>0 \end{cases}$$

Therefore the shape of y(t) is:



$$y(t) = \frac{1}{2a}e^{-\alpha|t|} \qquad \alpha > 0$$

Alternatively an easier solution is to use Table 1, Pair 13. This yield an equivalent solution:

$$y(t) = \frac{e^{-\alpha t}u(t) + e^{\alpha t}u(-t)}{2\alpha}$$

[6]

Answer to Question 4 (Topic: System classification & time-domain analysis)

a)

This is a simple circuit where the capacitor C is charged by a constant current x(t). The output voltage y(t) across the capacitor and the input current x(t) are related by:

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$

[2]

b)

i) From the above equation, the output depends on both the past and the present input, there the system is not memoryless.

ii) Since the output y(t) does not depend on the future values of the input, the system is causal.

iii) Since, the integration operation obeys the principle of superposition, therefore the system is linear.

iv) Let $x_1(t) = x(t - t_0)$, then

$$y_1(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau - t_0)d\tau = \frac{1}{C}\int_{-\infty}^{t-t_0} x(\lambda)d\lambda = y(t - t_0)$$

Therefore the system is time-invariant.

[4]

Answer to Question 5 (Topic: Time-domain analysis)

a)

$$\frac{dy(t)}{dx} + ay(t) = x(t)$$

Characteristic equation: $\lambda + a = 0$
Therefore, characteristic root is $\lambda = -a$
Characteristic mode is e^{-at}

Hence $y_{zi}(t) = c_1 e^{-at}$.

Since $y(0) = y_0$, *therefore* $c_1 = y_0$.

Therefore the zero-input response is:

$$y_{zi}(t) = y_0 e^{-at}$$

[3]

b)

$h(t) = [P(D)y_n(t)]u(t)$, P(D) = 1 in this case.

From a), $y_n(t) = ke^{-at}$, where k is a constant.

Now, we know that $y_n(0)=1$. Therefore k = 1.

Therefore the impulse response is:
$h(t) = e^{-at}u(t)$.

Given that $x(t) = Ke^{-bt}u(t)$,

We also know that the zero-state response is:

$$y_{zs}(t) = h(t) * x(t)$$

Use the convolution pair:

$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}u(t)$	$\lambda_1 \neq \lambda_2$

$$y_{zs}(t) = e^{-at}u(t) * Ke^{-bt}u(t)$$
$$= \frac{K}{a-b}(e^{-bt} - e^{-at})u(t)$$

[4]

c)

$$y(t) = y_o e^{-at} + \frac{K}{a-b}(e^{-bt} - e^{-at})u(t)$$

[3]

Answer to Question 6 (Topic: Transfer function and Laplace Transform)

a)

The step response of the system is: $2e^{-3t}u(t)$, i.e.

$$y(t) = 2e^{-3t}u(t), \quad x(t) = u(t).$$

Take the Laplace Transform of these:

$$X(s) = \frac{1}{s}, \quad Y(s) = \frac{2}{s+3}.$$

Hence, the system transfer function is:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s}{s+3} = 2 - \frac{6}{s+3}.$$

Taking the inverse Laplace transform of H(s):

$$h(t) = 2\delta(t) - 6e^{-3t}u(t).$$

Alternatively, one could calculate *h(t)* by differentiating the step response:

$$h(t) = \frac{d}{dt}(2e^{-3t}u(t)) = 2\delta - 6e^{-3t}u(t).$$

[5]

b)

$$x(t) = e^{-t}u(t) \leftrightarrow \frac{1}{s+1}.$$

Thus,

$$Y(s) = X(s)H(s) = \frac{2s}{(s+1)(s+3)}.$$

Use partial fraction expansions:

$$Y(s) = \frac{2s}{(s+1)(s+3)} = -\frac{1}{s+1} + \frac{3}{s+3}.$$

Take inverse Laplace transform of Y(s), we get:

$$y(t) = (-e^{-t} + 3e^{-3t})u(t).$$

[5]

Answer to Question 7 (Topic: Inverse Laplace Transform)

Using partial fraction expansion gives:

$$X(s) = \frac{5s+13}{s(s^2+4s+13)} = \frac{5s+13}{s(s+2+j3)(s+2-j3)}$$
$$= \frac{c_1}{s} + \frac{c_2}{s-(-2+j3)} + \frac{c_3}{s-(-2-3j)}.$$

$$c_1 = \left.\frac{5s+13}{(s^2+4s+13)}\right|_{s=0} = 1$$

$$c_2 = \left.\frac{5s+13}{s(s+2+j3)}\right|_{s=-2+j3} = -\frac{1}{2}(1+j)$$

$$c_3 = \left.\frac{5s+13}{s(s+2-j3)}\right|_{s=-2-j3} = -\frac{1}{2}(1-j).$$

Therefore,

$$X(s) = \frac{1}{s} + \frac{-\frac{1}{2}(1+j)}{s-(-2+j3)} + \frac{-\frac{1}{2}(1-j)}{s-(-2-3j)}.$$

Use the Laplace Transform table:

$$x(t) = u(t) - \frac{1}{2}(1+j)e^{(-2+j3)t}u(t) - \frac{1}{2}(1-j)e^{(-2-j3)t}u(t).$$

Therefore,

$$x(t) = [1 - e^{-2t}(\cos 3t - \sin 3t)]u(t).$$

An alternative and equivalent solution is:

$$X(s) = \frac{5s+13}{s(s^2+4s+13)} = \frac{P}{s} + \frac{Qs+R}{(s^2+4s+13)}$$

Equating coefficients of the numerator yields: P = 1, Q = -1, R = 1. Therefore:

$$X(s) == \frac{1}{s} + \frac{1-s}{(s^2+4s+13)}$$

Using transform table pairs 2 and 10c gives:

$$x(t) = [1 + \sqrt{2}\, e^{-2t}\cos(3t + \pi/4)]\, u(t)$$

[6]

Answer to Question 8 (Topic: Frequency Response and Bode plot)

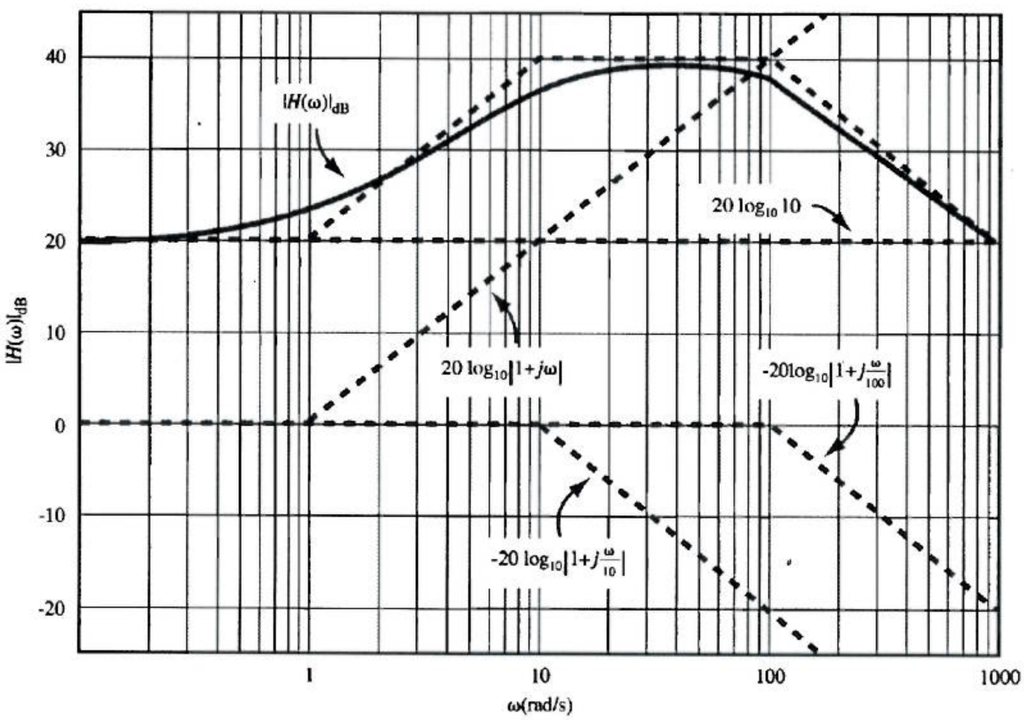
$$H(\omega) = \frac{10(1 + j\omega)}{(1 + j\omega/10)(1 + j\omega/100)}.$$

Therefore,

$$|H(\omega)|_{dB} = 20\log_{10} 10 + 20\log_{10}|1 + j\omega| - 20\log_{10}|1 + j\omega/10| - 20\log_{10}|1 + j\omega/100|$$

Therefore the corner frequencies are: ω = 1, 10 and 100, and the gain at each corner frequencies are:

$$H(1)|_{dB} = 20 + 20\log_{10}\sqrt{2} - 20\log_{10}\sqrt{1.01} - 20\log_{10}\sqrt{1.0001} = 23dB$$
$$H(10)|_{dB} = 20 + 20\log_{10}\sqrt{101} - 20\log_{10}\sqrt{2} - 20\log_{10}\sqrt{1.01} = 37dB$$
$$H(100)|_{dB} = 20 + 20\log_{10}\sqrt{10001} - 20\log_{10}\sqrt{101} - 20\log_{10}\sqrt{2} = 37dB$$

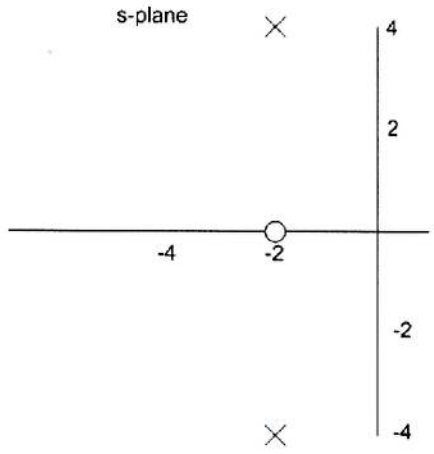


[8]

Answer to Question 9 (Topic: Poles & Zeroes)

$$H(s) = \frac{2s+4}{s^2+4s+20} = \frac{2(s+2)}{(s-(-2+4j)(s-(-2-4j)}$$

The zero is therefore at s = -2, the conjugate poles are at s = $-2 \pm 4j$.



[4]

Answer to Question 10 (Topic: Fourier Transform)

a)

From the Fourier Transform table, we know that

$x(t)$	$X(\omega)$
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$.

Therefore

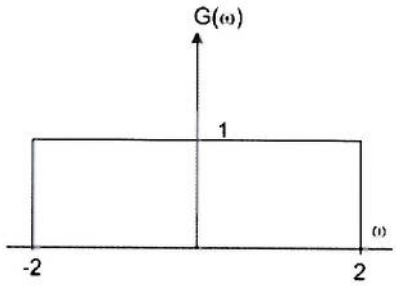
$$w(t) = \cos(t) \quad \leftrightarrow \quad W(\omega) = \pi[\delta(\omega - 1) + \delta(\omega + 1)]$$

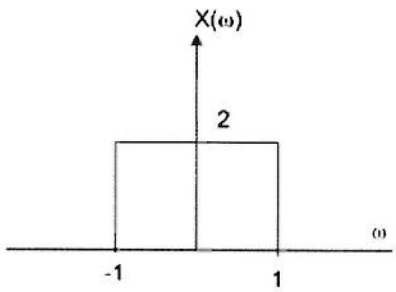
Further more we know:

$$g(t) = x(t)\cos(t) \quad \leftrightarrow \quad G(\omega) = \frac{1}{2\pi}\{X(\omega) * W(\omega)\}$$

Therefore,

$$G(\omega) = \frac{1}{2}X((\omega - 1)) + \frac{1}{2}X((\omega + 1)).$$





[6]

b)

Inverse Fourier transform of X(jω) gives a sinc-function:

$$x(t) = \frac{2\sin t}{\pi t}.$$

[3]

Answer to Question 11 (Topic: Sampling)

a) The sampling period T = 1/4000. Hence we can replace t with $kT = k/4000$. Hence

$$
\begin{aligned}
f[k] &= 10\cos\frac{\pi}{2}k + \sqrt{2}\sin\frac{3\pi}{4}k + 2\cos\left(\frac{5\pi}{4}k + \frac{\pi}{4}\right) \\
&= 10\cos\frac{\pi}{2}k + \sqrt{2}\sin\frac{3\pi}{4}k + 2\cos\left(\frac{-3\pi}{4}k + \frac{\pi}{4}\right) \\
&= 10\cos\frac{\pi}{2}k + \sqrt{2}\sin\frac{3\pi}{4}k + 2\cos\left(\frac{3\pi}{4}k - \frac{\pi}{4}\right)
\end{aligned}
$$

This can be simplified further using trigonometric identity, but is not compulsory:

$$
\begin{aligned}
f[k] &= 10\cos\frac{\pi}{2}k + \sqrt{2}\sin\frac{3\pi}{4}k + 2\cos\left(\frac{3\pi}{4}k - \frac{\pi}{4}\right) \\
&= 10\cos\frac{\pi}{2}k + \sqrt{2}\sin\frac{3\pi}{4}k + 2\cos\frac{3\pi}{4}k\cos\frac{\pi}{4} + 2\sin\frac{3\pi}{4}k\sin\frac{\pi}{4} \\
&= 10\cos\frac{\pi}{2}k + \sqrt{2}\sin\frac{3\pi}{4}k + \sqrt{2}\cos\frac{3\pi}{4}k + \sqrt{2}\sin\frac{3\pi}{4}k \\
&= 10\cos\frac{\pi}{2}k + 2\sqrt{2}\sin\frac{3\pi}{4}k + \sqrt{2}\cos\frac{3\pi}{4}k \\
&= 10\cos\frac{\pi}{2}k + \sqrt{10}\cos\left(\frac{3\pi}{4}k - 1.107\right)
\end{aligned}
$$

[5]

b)

Since the frequency $5\pi/4$ has been reduced to $3\pi/4$, there is aliasing.

[3]

c)

The highest frequency in the signal is $w = 5000\pi$ or 2500Hz. Therefore to avoid aliasing, the signal must be sampled at a minimum of 5000Hz, or T = 0.2ms.

[2]

Answer to Question 12 (Topic: DFT)

a)

$$y[n] = 0.5y[n-1] + bx[n]$$

Therefore the transfer function is found by taking the z-transform of both sides:

$$Y(z) = 0.5z^{-1}Y(z) + bX(z)$$
$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{1-0.5z^{-1}}$$

To find the frequency response, substitute $z = e^{j\omega}$,

$$H(z)\big|_{z=e^{j\omega}} = \frac{b}{1-0.5e^{-j\omega}}.$$

[4]

b)

$$|H(\omega)|^2 = \frac{b^2}{(1-0.5e^{-j\omega})(1-0.5e^{+j\omega})} = \frac{b^2}{1.25-\cos\omega}$$

When $\omega = 0$, $|H(\omega)|^2 = 1$.

Therefore

$$\frac{b^2}{1.25-1} = 1 \Rightarrow b = \pm 0.5.$$

[3]

c) To find the half-power point,

$$|H(\omega)|^2 = \frac{0.25}{1.25-\cos\omega} = 0.5$$
$$\Rightarrow \cos\omega = 0.75$$
$$\Rightarrow \omega = 0.23\pi.$$

[3]

Answer to Question 13 (Topic: z-transform)

a)

Use the z-transform pair:

$$\gamma^n u[n] \qquad\qquad \frac{z}{z-\gamma}$$

we have:

$$X[z] = \frac{1}{1-2z^{-1}} + \frac{3}{1-0.5z^{-1}}$$
$$= \frac{4-\frac{13}{2}z^{-1}}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})}$$

[3]

b)

$$x[n] = \cos[n\omega_0]u[n] = \frac{1}{2}[e^{jn\omega_0} + e^{-jn\omega_0}]u[n].$$

Therefore

$$X[z] = \frac{1}{2}\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1-e^{-j\omega_0}z^{-1}}$$
$$= \frac{1-\cos\omega_0 z^{-1}}{1-2\cos\omega_0 z^{-1} + z^{-2}}.$$

[3]