

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

MSc and EEE/ISE PART III/IV: MEng, BEng and ACGI

Corrected Copy

MATHEMATICS FOR SIGNALS AND SYSTEMS

Wednesday, 6 May 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer THREE questions.

Q2 —

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : M.M. Draief
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1. Consider the space $\mathbb{R}^{3 \times 3}$ of three-by-three matrices. We define the inner product $\langle A, B \rangle = \text{tr}(B^T A)$, and we define the corresponding norm $N(A) = \sqrt{\text{tr}(A^T A)}$. Let

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

- a) Find the range and the kernel of S . [1]
- b) Compute $N(S)$ the norm of S as defined before. [2]
- c) Compute the matrix norm of S given by $\|S\| = \sup_{\|x\| \leq 1} \|Sx\|$ where $\|x\|^2 = x^T x$. [3]
- d) A matrix A is said to be S -invariant if $AS = SA$. Let \mathcal{S} be the set of S -invariant matrices.
 - (i) Show that \mathcal{S} is a vector space. [2]
 - (ii) Determine the dimension of \mathcal{S} . [4]
 - (iii) Show that if A and B are two elements of \mathcal{S} then $AB \in \mathcal{S}$ and $AB = BA$. [3]
 - (iv) Show that if $A \in \mathcal{S}$ then A has a unique eigenvalue. [2]
 - (v) Explicitly describe all the matrices $A \in \mathcal{S}$ for which $A^T \in \mathcal{S}$. [3]

2. Consider \mathcal{C}_0 the space of real-valued ~~continuous~~ functions on the interval $[-1, 1]$. For $f, g \in \mathcal{C}_0$, we define the inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$.

a) Check that the above inner product is indeed an inner product and find the expression of the corresponding norm. [3]

b) Let

$$\mathcal{E} = \{f \in \mathcal{C}_0, f(-x) = f(x)\}$$

be the set of even functions and

$$\mathcal{O} = \{f \in \mathcal{C}_0, f(-x) = -f(x)\}$$

be the set of odd functions.

(i) Show that \mathcal{E} and \mathcal{O} are two vector spaces. [3]

(ii) Show that \mathcal{E} and \mathcal{O} are orthogonal, i.e., for any $f \in \mathcal{E}$ and $g \in \mathcal{O}$, we have $\langle f, g \rangle = 0$. [4]

(iii) For $f \in \mathcal{C}_0$, and let $g(x) = f(x) + f(-x)$ and $h(x) = f(x) - f(-x)$. Show that $g \in \mathcal{E}$ and $h \in \mathcal{O}$. [3]

(iv) Show that any $f \in \mathcal{C}_0$ can be decomposed in a **unique** way as $f(x) = g(x) + h(x)$ where $g \in \mathcal{E}$ and $h \in \mathcal{O}$. [3]

(v) Determine the orthogonal projections on \mathcal{E} and \mathcal{O} . [4]

3. Let u_1, \dots, u_n be a set of orthonormal vectors in \mathbb{R}^n , i.e., pairwise orthogonal

$$u_i^T u_j = 0, \text{ for } i \neq j \quad \text{and} \quad u_i^T u_i = 1, \text{ for all } i.$$

We denote by $\|x\|^2 = x^T x$.

- a) Let $U = [u_1, \dots, u_n]$ be a matrix in $\mathbb{R}^{n \times n}$.

- (i) Show that $U^T U = U U^T = I$, I being the identity matrix. [2]
- (ii) Prove that $x = \sum_{i=1}^n (u_i^T x) u_i$ for x in \mathbb{R}^n . [3]
- (iii) Show that u_1, \dots, u_n is an orthonormal basis of \mathbb{R}^n . [3]
- (iv) Show that $\|Ux\| = \|x\|$. [3]

- b) Let $V_1 = \text{Span}(u_1, \dots, u_k)$ and $V_2 = \text{Span}(u_{k+1}, \dots, u_n)$.

- (i) Show that $\mathbb{R}^n = V_1 \oplus V_2$, i.e. V_1 and V_2 are complementary. [3]
- (ii) Let $p(x) = \sum_{i=1}^k (u_i^T x) u_i$; show that p is a projection. [3]
- (iii) Let $s(x) = \sum_{i=1}^k (u_i^T x) u_i - \sum_{i=k+1}^n (u_i^T x) u_i$; show that s is a reflexion.

[3]

4. Let A be a matrix in $\mathbb{R}^{m \times n}$, $m \geq n$.

- a) Show that if A has a left inverse, i.e. there exists a C such that $CA = I$ (I the identity matrix in $\mathbb{R}^{n \times n}$), then A has zero-null space. [3]
- b) Assume that A is zero-null space.
 - (i) Show that $A^T A$ is a positive definite matrix. [3]
 - (ii) Find a left inverse for A . [3]
 - (iii) Let $y \in \mathbb{R}^n$. Find a condition on y so that the equation $Ax = y$ admits a solution. [2]
- c) For $y \in \mathbb{R}^n$, let $\hat{x} = (A^T A)^{-1} A^T y$. We consider the inner product $x^T y$ and the associated norm $\|\cdot\|$.
 - (i) Show that, for any vector $x \in \mathbb{R}^n$, $A(x - \hat{x})$ is orthogonal to $A\hat{x} - y$. [3]
 - (ii) Show that $\|Ax - y\| \geq \|A\hat{x} - y\|$. [3]
 - (iii) Suppose that $y \notin \text{Range}(A)$. Relate the above to the linear least-square problem. [3]

5. a) Show that if A is a positive definite matrix then if λ is an eigenvalue of A then $\lambda > 0$. [3]

- b) Let A be a symmetric matrix with $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $x_1, \dots, x_n \in \mathbb{R}^n$ as its eigenvalues and eigenvectors respectively, i.e., $Ax_i = \lambda_i x_i$, $i = 1, \dots, n$. Show that if for all $i = 1, \dots, n$, $\lambda_i > 0$ then A is positive definite. [4]

Hint: Use the fact that if $A \in \mathbb{R}^{n \times n}$ is symmetric then (x_1, \dots, x_n) is an orthonormal basis, i.e., $x_i^T x_j = 0$ if $i \neq j$ and $x_i^T x_i = 1$.

c) Let $A = \frac{1}{5} \begin{pmatrix} 3 & -6 & 26 \\ 4 & -8 & -7 \\ 0 & 4 & 4 \\ 0 & -3 & -3 \end{pmatrix}$.

Show that $A^T A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 30 \end{pmatrix}$. [2]

- d) We now want to solve the linear least-square problem with A above and $y = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

- (i) Show that is equivalent to solving the linear problem

$$\begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 30 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7/5 \\ -13/5 \\ 4 \end{pmatrix}.$$

[2]

- (ii) Using the Cholesky decomposition, show that

$$\begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 30 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{pmatrix}.$$

[5]

- (iii) Show that the solution is $\hat{x} = \frac{1}{25} \begin{pmatrix} 41 \\ 4 \\ 1 \end{pmatrix}$. [4]

Hint: Cholesky Decomposition: Let $A \in \mathbb{R}^{n \times n}$ such that

$$A = \begin{pmatrix} a_{11} & A_{21}^T \\ A_{21} & A_{22} \end{pmatrix}$$

where a_{11} is a scalar, $A_{21} \in \mathbb{R}^{(n-1) \times 1}$, and $A_{22} \in \mathbb{R}^{(n-1) \times (n-1)}$ symmetric.

- Calculate the first column of L : $l_{11} = \sqrt{a_{11}}$ and $L_{21} = \frac{1}{l_{11}} A_{21}$.
- Compute the Cholesky factor L_{22} of the matrix $A_{22} - \frac{1}{a_{11}} A_{21} A_{21}^T$.
- The Cholesky factor L of a positive definite matrix A is given by

$$L = \begin{pmatrix} l_{11} & 0 \dots 0 \\ L_{21} & L_{22} \end{pmatrix}$$

MATHEMATICS FOR SIGNAL & SYSTEMS

(1)

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(Q 1.)

a) Range (S) = Span { vector columns of S }.

$$\text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

not Kernel (S).

$$S \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_1 = x_2 = 0$$

$$\text{Kernel (S)} = \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

b) $S^T S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$N(S) = \sqrt{\lambda(S^T S)} = \sqrt{2}.$$

c) $Sx = \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix} \quad \|Sx\| = \sqrt{x_1^2 + x_2^2}$

$$x \neq 0 \quad \frac{\|Sx\|}{\|x\|} \leq 1$$

$$\text{and } S \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\text{Hence } \|S\| = 1.$$

(Q1)

(2)

d) (i) We will show that \mathcal{J} is a subspace of $\mathbb{R}^{3 \times 3}$.

* $\mathbf{I} \in \mathcal{J}$; $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ & $\mathbf{I}\mathbf{S} = \mathbf{S}\mathbf{I} = \mathbf{S}$.

* $\mathbf{A}, \mathbf{B} \in \mathcal{J}$; $\lambda \in \mathbb{R}$

$$(\lambda \mathbf{A} + \mathbf{B})\mathbf{S} = \mathbf{S}(\lambda \mathbf{A}) + \mathbf{S}\mathbf{B} = \mathbf{S}(\lambda \mathbf{A} + \mathbf{B}) \Rightarrow \lambda \mathbf{A} + \mathbf{B} \in \mathcal{J}$$

Hence \mathcal{J} is a vector space.

(ii) $\mathbf{A}\mathbf{S} = \begin{pmatrix} a_{12} & a_{13} & 0 \\ a_{22} & a_{23} & 0 \\ a_{32} & a_{33} & 0 \end{pmatrix}$; $\mathbf{S}\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

$$\mathbf{A}\mathbf{S} = \mathbf{S}\mathbf{A} \Rightarrow \mathbf{A} = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \alpha & 0 \\ \gamma & \beta & \alpha \end{pmatrix} ; \alpha, \beta, \gamma \in \mathbb{R}$$

Therefore $\dim(\mathcal{J}) = 3$.

(iii) If $\mathbf{A} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 \\ \gamma_1 & \beta_1 & \alpha_1 \end{pmatrix}$ & $\mathbf{B} = \begin{pmatrix} \alpha_2 & 0 & 0 \\ \beta_2 & \alpha_2 & 0 \\ \gamma_2 & \beta_2 & \alpha_2 \end{pmatrix}$

then ; $\mathcal{J} \ni \mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A} = \begin{pmatrix} \alpha_1 \alpha_2 & 0 & 0 \\ \beta_1 \alpha_2 + \alpha_1 \beta_2 & \alpha_1 \alpha_2 & 0 \\ \gamma_1 \alpha_2 + \beta_1 \beta_2 + \alpha_1 \gamma_2 & \beta_1 \alpha_2 + \alpha_1 \beta_2 & \alpha_1 \alpha_2 \end{pmatrix}$

(iv) from description in (ii) $\mathbf{A} \in \mathcal{J}$

has a unique eigenvalue α .

(v) $\mathbf{A} \in \mathcal{S}$; $\mathbf{A}^T = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \in \mathcal{S} \Rightarrow \beta = \gamma = 0 \Rightarrow \mathbf{A} = \alpha \mathbf{I}$.

Q2

a) 1) $\langle f, g \rangle = \langle g, f \rangle$

2) $\langle \lambda f + g, h \rangle = \lambda \langle f, h \rangle + \langle g, h \rangle$

3) $\langle f, f \rangle = \int_{-1}^1 f(x) f(x) dx \geq 0$

4) $\langle f, f \rangle = 0 \Rightarrow f = 0$

$$\|f\| = \sqrt{\int_{-1}^1 f(x)^2 dx}$$

b)

(i) \mathcal{E}, \mathcal{O} subspaces of \mathcal{C}_0 .

$$(x \mapsto 0) \in \mathcal{E} \quad ; \quad (x \mapsto 0) \in \mathcal{O}$$

$$\lambda f + g(-x) = \lambda f(-x) + g(-x) = \begin{cases} \lambda f(x) + g(x) & ; f, g \in \mathcal{E} \\ -\lambda f(x) - g(x) & ; f, g \in \mathcal{O} \end{cases}$$

(ii) $f \in \mathcal{E}, g \in \mathcal{O}$ then $f(-x)g(x) = -f(x)g(x)$

$$\begin{aligned} \langle f, g \rangle &= \int_{-1}^1 f(x) g(x) dx = \int_0^1 f(x) g(x) dx + \int_{-1}^0 f(x) g(x) dx \\ &= \int_0^1 f(x) g(x) dx - \int_0^1 f(x) g(x) dx \\ &= 0 \end{aligned}$$

(iii) $g(-x) = g(x) \quad ; \quad h(-x) = -h(x)$.

Q2

b).
(iv)

$$g(u) = \frac{f(u) + f(-u)}{2}$$

$$h(u) = \frac{f(u) - f(-u)}{2}$$

(if $f = g_1 + h_1$ then $g_1 = g$ & $h_1 = h$).

(v) orthogonal projection of f on \mathcal{E} is given
by $u \mapsto \frac{f(u) + f(-u)}{2}$.

orthogonal projection of f on \mathcal{O} is given by
 $u \mapsto \frac{f(u) - f(-u)}{2}$.

(Q3.)

Similar to problem solved during lectures.

(5)

a)

$$(i) \quad U^T U = \begin{pmatrix} u_1^T \\ \vdots \\ u_n^T \end{pmatrix} (u_1 \dots u_n) = \left(u_i^T u_j \right)_{i,j=1 \dots n} = I$$

$$U^{-1} = U^T \Rightarrow U^T U = U U^T = I.$$

(ii)

$$x = I x = U U^T x$$

$$= U \begin{pmatrix} u_1^T \\ \vdots \\ u_n^T \end{pmatrix} x$$

$$= U \begin{pmatrix} u_1^T x \\ \vdots \\ u_n^T x \end{pmatrix} = \sum_{i=1}^n u_i^T x u_i$$

(iii)

$u_1 \dots u_n$ linearly independent

since if $\lambda_1 u_1 + \dots + \lambda_n u_n = 0$ then

$$u_i^T (\lambda_1 u_1 + \dots + \lambda_n u_n) = \lambda_i \Rightarrow \lambda_i = 0 \text{ for all } i$$

& by (ii) $\text{Span} \{ u_1 \dots u_n \} = \mathbb{R}^n$

$\Rightarrow (u_1 \dots u_n)$ is a basis of orthonormal

vectors thus an orthonormal basis.

Q3

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a)
(iv)

$$\|Ux\|^2 \quad (U_2)^T Ux = x^T \widetilde{\frac{1}{11}} \\ \therefore x^T x = \|x\|^2$$

b).

(i)

$$x \in V_1 \cap V_2$$

$$x = \sum_{i=1}^h \lambda_i u_i \\ = \sum_{j=k+1}^n \lambda_j u_j$$

$$x^T x = 0$$

since u_i 's are orthogonal
 $\Rightarrow x = 0$.

ans!

~~$V_1 \oplus V_2$~~ $V_1 + V_2 = \mathbb{R}^n$ by a) (ii)

\Rightarrow

$$V_1 \oplus V_2 = \mathbb{R}^n.$$

(ii)

$$p^2(x) = p(x) \Rightarrow p \text{ projection}$$

(iii)

$$s^2(x) = x \Rightarrow s \text{ reflexion}$$

Q4

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a) $\exists C \quad CA = I$

then $Ax = 0 \Rightarrow x: CAx = C0 = 0 \Rightarrow x = 0.$

Kernel $\{A\} = \{0\}.$

b)

$$x^T (A^T A) x = (Ax)^T Ax \geq 0$$

if $x^T (A^T A) x = 0 \Rightarrow Ax = 0 \Rightarrow x = 0$

$\therefore A$ has zero null-space

(ii) $\left[(A^T A)^{-1} A^T \right] A = I$

(iii) $Ax = y$ has a solution if $y \in \text{Range}(A)$

c) (i) $\left[A(x - \hat{x}) \right]^T (A\hat{x} - y) = (x - \hat{x})^T A^T (A\hat{x} - y).$
 $= (x - \hat{x})^T (A^T A \hat{x} - A^T y)$
 $= (x - \hat{x})^T (A^T y - A^T y) = 0$

(ii) $\|Ax - y\|^2 = \| (Ax - A\hat{x}) \|^2 + \| (A\hat{x} - y) \|^2$
 $\geq \| A\hat{x} - y \|^2$

(24)

(iii)

 $y \notin \text{Range}(A)$ then \hat{x} is the least square solution(the one that minimizes $\|Ax - y\|$).for the linear problem $Ax = y$;

Q5

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a) $Ax = \lambda x \quad (x \neq 0) \quad x^T Ax = \lambda x^T x$

$$\lambda = \frac{x^T Ax}{x^T x} > 0 \quad \text{since } x^T Ax > 0 \text{ if } x \neq 0$$

b)

x_1, \dots, x_n is a basis.

for $x \in \mathbb{R}^n \quad x = \alpha_1 x_1 + \dots + \alpha_n x_n.$

$$Ax = \alpha_1 \lambda_1 x_1 + \dots + \alpha_n \lambda_n x_n.$$

$$x^T Ax = \alpha_1^2 \lambda_1 + \dots + \alpha_n^2 \lambda_n > 0 \text{ if } x \neq 0.$$

c)

~~first~~ $ATA = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 3 \end{pmatrix}$ by direct matrix multiplication

d) (i) we want to find $\hat{u} = (ATA)^{-1} ATy$

that is to say solve

$$ATA \hat{u} = ATy$$

i.e. $\begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 3 \end{pmatrix} u = ATy = \begin{pmatrix} 7/5 \\ -13/5 \\ 4 \end{pmatrix}$

(ii) Immediate application of Cholesky.

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(iii) First solve.

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 1 & 5 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 7/5 \\ -13/5 \\ 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 7/5 \\ 1/5 \\ 1/5 \end{pmatrix}$$

and then solve.

$$\begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 7/5 \\ 1/5 \\ 1/5 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 41/25 \\ 4/25 \\ 1/25 \end{pmatrix}$$