

[B] — bookwork

[E] — new example

[T] — new theory

Q1

$$\begin{aligned}
 a) \quad P(\text{ill} | +) &= \frac{P(+ | \text{ill}) P(\text{ill})}{P(+)} & [1A] \\
 &= \frac{P(+ | \text{ill}) P(\text{ill})}{P(+ | \text{ill}) P(\text{ill}) + P(+ | \text{healthy}) P(\text{healthy})} & [1A] \\
 &= \frac{0.9 \times 10^{-4}}{0.9 \times 10^{-4} + 0.1 \times (1 - 10^{-4})} & [1A] \\
 &= \frac{9}{9 + 10^4 - 1} & [1A] \\
 &= \frac{9}{10008} \approx 9 \times 10^{-4} & [1A]
 \end{aligned}$$

$$b) \quad \text{If } y = \tan^{-1} x, \text{ then } \frac{dy}{dx} = \frac{1}{1+x^2} \quad [1A]$$

$$\begin{aligned}
 \text{So } f_y(y) &= \frac{1}{dy/dx} f_x(x) = (1+x^2) \frac{\alpha/\pi}{x^2 + x^2} & [1A] \\
 &= (1+x^2) f_x(\tan y) & [1A] \\
 &= [1 + (\tan y)^2] \frac{\alpha/\pi}{x^2 + (\tan y)^2} & [1A]
 \end{aligned}$$

We observe that if this is uniform, then [1A]

$$\alpha = 1$$

(c) i) $f_Z(z)$ is the convolution of $f_X(x)$ and $f_Y(y)$.

$$f_Z(z) = \int_0^z f_X(z-y) f_Y(y) dy \quad [2E]$$

$$= \int_0^z e^{-(z-y)} e^{-y} dy \quad [2E]$$

$$= \int_0^z e^{-z} dy \quad [1E]$$

$$= z e^{-z} \quad z > 0$$

ii) Define $y' = -y$ so that $Z = X + y'$.

We note the pdf of y' is given by

$$f_{Y'}(y') = e^{y'} \quad y' < 0$$

$f_Z(z)$ is the convolution of $f_X(x)$ and $f_{Y'}(y')$. [1E]

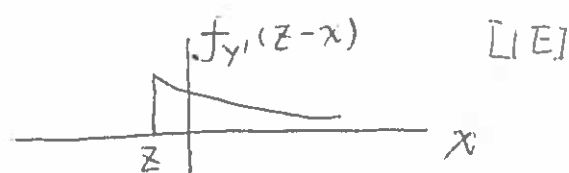
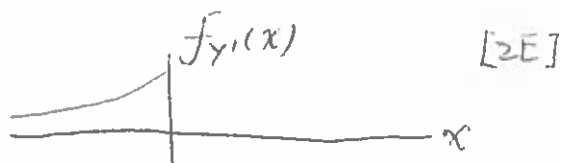
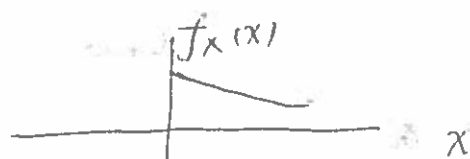
$$f_Z(z) = f_X(z) \otimes f_{Y'}(z)$$

$$= \int_0^\infty e^{-x} e^{z-x} dx, \quad z < 0$$

$$\left\{ \int_z^\infty e^{-x} e^{z-x} dx, \quad z > 0 \right.$$

$$= \begin{cases} \frac{1}{2} e^z, & z < 0 \\ \frac{1}{2} e^{-z}, & z > 0 \end{cases}$$

$$= \frac{1}{2} e^{-|z|} \quad -\infty < z < \infty$$



iii) $F_Z(z) = P(Z \leq z) = P(X + Y \leq z)$ [1E]

$$= \int_0^\infty \int_0^{z/y} f_X(x) f_Y(y) dx dy \quad [1E]$$

$$f_Z(z) = \int_0^\infty \frac{1}{y} f_X\left(\frac{z}{y}\right) f_Y(y) dy \quad [2E]$$

$$= \int_0^\infty \frac{1}{y} e^{-(\frac{z}{y} + y)} dy \quad z > 0 \quad [1E]$$

Q2

a) For n samples we have

$$f(\underline{x}, c) = c^{4n} (x_1 \dots x_n)^{3n} e^{-c(x_1 + \dots + x_n)} \quad [2E]$$

$$\frac{\partial f(\underline{x}, c)}{\partial c} = 4n c^{4n-1} (x_1 \dots x_n)^{3n} e^{-c(x_1 + \dots + x_n)} - (x_1 + \dots + x_n) c^{4n} (x_1 \dots x_n)^{3n} e^{-c(x_1 + \dots + x_n)} \quad [2E]$$

$$= \left[\frac{4n}{c} - (x_1 + \dots + x_n) \right] f(\underline{x}, c) \quad [1E]$$

$$= 0$$

$$c = \frac{4n}{x_1 + \dots + x_n} \quad [1E]$$

In this problem, $n = 5$

$$c = \frac{4 \times 5}{80} = \frac{4}{6} = \frac{2}{3} \quad [2E]$$

b) i) Note that the transfer function is

$$H(z) = \frac{1}{1 - \alpha z^{-1}} = \sum_{n=0}^{\infty} \alpha^n z^{-n} \quad [1B]$$

So

$$h(n) = \alpha^n \quad n \geq 0 \quad [1B]$$

Therefore,

$$\begin{aligned} R_y(n) &= R_x(n) \otimes h(-n) \otimes h(n) \\ &= h(-n) \otimes h(n) \end{aligned} \quad [1B]$$

Since $R_x(n) = \delta(n)$.

$$R_y(n) = \begin{cases} \sum_{k=0}^{\infty} \alpha^{-(n-k)} \alpha^k & n < 0 \\ \sum_{k=n}^{\infty} \alpha^{-(n-k)} \alpha^k & n > 0 \end{cases} \quad [1B]$$

$$= \begin{cases} \alpha^{-n} \sum_{k=0}^{\infty} \alpha^{2k} & n < 0 \\ \alpha^n \sum_{k=0}^{\infty} \alpha^{2k} & n > 0 \end{cases} \quad [1B]$$

$$= \begin{cases} \alpha^{-n} \frac{1}{1 - \alpha^2} & n < 0 \\ \alpha^n \frac{1}{1 - \alpha^2} & n > 0 \end{cases} \quad [1B]$$

$$= \alpha^{|n|} \frac{1}{1 - \alpha^2} \quad [1B]$$

ii) The Wiener-Hopf equation reads

$$\begin{pmatrix} R_y(0) & R_y(1) & \dots & R_y(n-1) \\ R_y(1) & R_y(0) & \dots & R_y(n-2) \\ & & \ddots & \\ R_y(n-1) & \dots & & R_y(0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} R_y(n) \\ R_y(n-1) \\ \vdots \\ R_y(1) \end{pmatrix}$$

that is

$$\begin{pmatrix} 1 & \alpha & \dots & \alpha^{n-1} \\ \alpha & 1 & \dots & \alpha^{n-2} \\ & & \ddots & \\ \alpha^{n-1} & \dots & & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \alpha^n \\ \alpha^{n-1} \\ \vdots \\ \alpha \end{pmatrix} \quad [2E]$$

whose solution is

$$c_n = \alpha, \quad c_i = 0 \quad i < n \quad [3E]$$

Consequently, the MMSE prediction is

$$y(n+1) = \alpha y(n) \quad [1E]$$

The mean-square error is given by

$$MSE = E[y(n+1) - \alpha y(n)]^2 \quad [1E]$$

$$= E[y^2(n+1) - 2\alpha y(n+1)y(n) + \alpha^2 y^2(n)] \quad [1E]$$

$$= R_y(0) - 2\alpha R_y(1) + \alpha^2 R_y(0) \quad [1E]$$

$$= \frac{1 - 2\alpha^2 + \alpha^2}{1 - \alpha^2} \quad [1E]$$

$$= 1$$

Q3

$$\begin{aligned} a) \ i) \quad E[X(t)] &= E[A_t \cos(\omega t + \theta)] \\ &= E[A_t] \cdot \cos(\omega t + \theta) \quad [2A] \\ &= 0 \end{aligned}$$

$$\begin{aligned} E[X^2(t)] &= E[A_t^2] \cos^2(\omega t + \theta) \\ &= \sigma^2 \cos^2(\omega t + \theta) = \text{Var}[X(t)] \quad [2A] \end{aligned}$$

Since the variance is a function of t , it is not stationary. [1A]

$$\begin{aligned} ii) \quad E[X(t)] &= E[A_t] \cdot E[\cos(\omega t + \theta)] \\ &= 0 \quad [1A] \end{aligned}$$

$$\begin{aligned} E[X(t)X(t+\tau)] &= E[A_t A_{t+\tau}] E[\cos(\omega t + \theta) \cos(\omega(t+\tau) + \theta)] \\ &= \begin{cases} 0 & \tau \neq 0 \\ \sigma^2 E[\cos^2(\omega t + \theta)] & \tau = 0 \end{cases} \\ &= \begin{cases} 0 & \tau \neq 0 \\ \frac{\sigma^2}{2} & \tau = 0 \end{cases} \quad [3A] \\ &= \frac{\sigma^2}{2} \delta(\tau) \quad \text{only a function of } \tau \end{aligned}$$

So it is wide-sense stationary [1A]

b) i) By Chebyshev's inequality [1E]

$$\begin{aligned} P\{|X(t+\tau) - X(t)| > a\} &\leq \frac{E[|X(t+\tau) - X(t)|^2]}{a^2} \quad [2E] \\ &= \frac{2[R(0) - R(\tau)]}{a^2} \quad [2E] \end{aligned}$$

$$\begin{aligned}
ii) \quad & \sum_{i,k} a_i a_k^* R(\bar{t}_i - \bar{t}_k) \\
&= \sum_{i,k} a_i a_k^* \frac{1}{2\pi} \int S(\omega) e^{j\omega(\bar{t}_i - \bar{t}_k)} d\omega \quad \text{inverse Fourier transform} \quad [2B] \\
&= \frac{1}{2\pi} \int S(\omega) \left| \sum_i a_i e^{j\omega \bar{t}_i} \right|^2 d\omega \quad \text{rearrangement} \quad [2B] \\
&\geq 0 \quad [1B]
\end{aligned}$$

iii) If we suppress ω_2 in $\phi(\omega_1, \omega_2)$, we recover the characteristic function of a Gaussian r.v.

$$\phi(\omega) = \exp\left(-\frac{\sigma^2 \omega^2}{2}\right) = E[e^{j\omega X}]$$

Then

$$\begin{aligned}
E[Y(t)] &= E[I e^{aX(t)}] \\
&= I E[e^{aX(t)}] \quad [2E] \\
&= I \exp\left(\frac{\sigma^2 a^2}{2}\right) \quad \text{definition of C.F.} \\
&= I \exp\left(\frac{\sigma^2 R(0)}{2}\right) \quad \sigma^2 = R(0)
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
R_Y(\tau) &= E[Y(t) Y(t+\tau)] \quad [2E] \\
&= I^2 E[e^{aX(t)} e^{aX(t+\tau)}] \\
&= I^2 \exp\left(\frac{\sigma^2 a^2 + 2R(\tau)a^2 + \sigma^2 a^2}{2}\right) \quad \text{definition of C.F.} \\
&= I^2 \exp\{\sigma^2 [R(0) + R(\tau)]\} \quad [1E]
\end{aligned}$$

$$\phi(\omega_1, \omega_2) = E[e^{j(X_1 \omega_1 + X_2 \omega_2)}]$$

Q4

$$a) E[S_{n+1} | S_n, \dots, S_1]$$

$$= E[X_1 + X_2 + \dots + X_n + X_{n+1} | S_n, \dots, S_1] \quad [2E]$$

$$= E[S_n + X_{n+1} | S_n, \dots, S_1]$$

$$= S_n + E[X_{n+1} | S_n, \dots, S_1] \quad [2E]$$

$$= S_n + 0$$

$$= S_n \quad [1E]$$

Therefore, $\{S_n\}$ is a martingale.

b) Denote by π the limiting distribution.

$$\pi = \pi P \quad [1E]$$

$$\pi_1 = \pi_1 p + \pi_m q \Rightarrow \pi_1 = \pi_m$$

$$\pi_2 = \pi_1 p + \pi_2 q \Rightarrow \pi_2 = \pi_1 \quad [2E]$$

$$\pi_3 = \pi_2 p + \pi_3 q \Rightarrow \pi_3 = \pi_2$$

\vdots

$$\pi_m = \pi_{m-1} p + \pi_m q \Rightarrow \pi_m = \pi_{m-1}$$

$$\text{Thus, } \pi_i = \frac{1}{m} \quad 1 \leq i \leq m \quad [2E]$$

$$c) i) P(X_n = j | X_{n+1} = i)$$

$$= P(X_n = j | X_{n+1} = i, X_{n+2}, X_{n+3}, \dots)$$

This is needed to prove the reversed chain is Markov

$$\begin{aligned}
& P(X_n = j \mid X_{n+1}, X_{n+2}, X_{n+3}, \dots) \\
&= \frac{P(X_n = j, X_{n+1} = i, X_{n+2} = k, \dots)}{P(X_{n+1} = i, X_{n+2} = k, \dots)} \quad [1T] \\
&= \frac{\xi_j \cdot P_{ji} P_{ik} \dots}{\xi_i P_{ik} \dots} \quad \text{forward chain is Markov and it is steady} \quad [2T] \\
&= \frac{\xi_j P_{ji}}{\xi_i} \quad [1T] \\
&= \frac{P(X_n = j, X_{n+1} = i)}{P(X_{n+1} = i)} = P(X_n = j \mid X_{n+1} = i) \quad [1T]
\end{aligned}$$

ii) If the chain is reversible,

$$\begin{aligned}
P_{ij} P_{jk} P_{ki} &= P_{ij}^* P_{jk}^* P_{ki}^* \quad \text{reversible} \quad [1T] \\
&= \frac{\xi_j P_{ji}}{\xi_i} \cdot \frac{\xi_k P_{ik}}{\xi_j} \cdot \frac{\xi_i P_{ij}}{\xi_k} \quad [2T] \\
&= P_{ik} P_{kj} P_{ji} \quad [2T]
\end{aligned}$$

iii) We need show $\{\xi_i\}$ is invariant under transformation P , i.e.,

$$\underline{\xi} = \underline{\xi} P \quad [1T]$$

The j th element of $\underline{\xi} P$ is

$$\sum_i \xi_i P_{ij} \stackrel{(1)}{=} \sum_i \xi_j P_{ji} \stackrel{(2)}{=} \xi_j \sum_i P_{ji} \stackrel{(3)}{=} \xi_j$$

where

(1) is due to reversibility $P_{ij} \xi_i = P_{ji} \xi_j$ [3T]

(2) is by rearrangement

(3) is because $\sum_i P_{ji} = 1$

Therefore, $\{\xi_i\}$ is the steady state distribution [1T]