IMPERIAL COLLEGE LONDON

E4.04
SC6
ISE4.9

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

MSc and EEE/ISE PART IV: MEng and ACGI

Corrected Copy

ADVANCED DATA COMMUNICATIONS

Thursday, 21 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer THREE questions.

All questions carry equal marks. The maximum mark for each subquestion is shown in brackets.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : M.K. Gurcan

Second Marker(s) : E. Gelenbe

Instructions to Candidates
Useful equations

$$\sum_{k=1}^{A}(2k-1)^2 = \frac{A(2A-1)(2A+1)}{3}$$

$$\mathrm{sinc}^2(t) \overset{FT}{\Longleftrightarrow} \Lambda(f) = \begin{cases} f+1 & -1 \le f < 0 \\ -f+1 & 0 \le f < 1 \\ 0 & otherwise \end{cases}$$

$$\cos(2\pi f_0 t) \overset{FT}{\Longleftrightarrow} \frac{1}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right)$$

$$X_{RC}(f) = \begin{cases} T & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \frac{T}{2}\left\{1+\cos\left(\frac{\pi T}{\alpha}\left(|f| - \frac{1-\alpha}{2T}\right)\right)\right\} & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0 & \text{otherwise} \end{cases}$$

$$\overset{FT}{\Longleftrightarrow} x(t) = \mathrm{sinc}\left(\frac{t}{T}\right)\frac{\cos\left(\frac{\pi\alpha t}{T}\right)}{1-4\frac{\alpha^2 t^2}{T^2}}$$

$$\lim_{x\to 1}\frac{\cos\left(\frac{\pi}{2}x\right)}{1-x} = \lim_{x\to 1}\frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right) = \frac{\pi}{2}$$

Questions

1. Answer the following sub-questions

   (a) Consider a data transmission system using two time waveforms

$$x_0(t) = \sqrt{2}\,\mathrm{sinc}^2(t)\cos(8\pi t)$$

$$x_1(t) = \sqrt{2}\,\mathrm{sinc}^2(t)\cos(4\pi t)$$

   to transmit binary data. Answer the following questions.

   i. Find the cross correlation between these waveforms. [4]

   ii. Determine the dimensionality of the system and identify the basis functions. [3]

   iii. Find the signal space vectors $\overline{x_0}$ and $\overline{x_1}$ to represent the two waveforms. [3]

   iv. Find the signal space distance between the two signals. [2]

   (b) Sketch the impulse response of the filter matched to the transmission time waveform shown in Figure 1.a. [3]

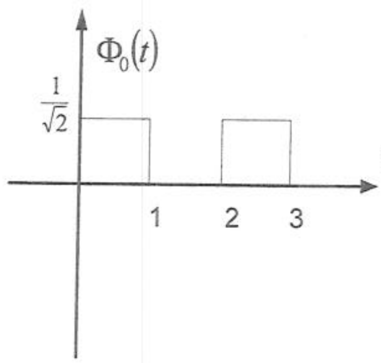

Figure 1.a

   Determine and sketch the output of the matched filter when having the waveform shown in Figure 1.b at the input of the matched filter. [7]

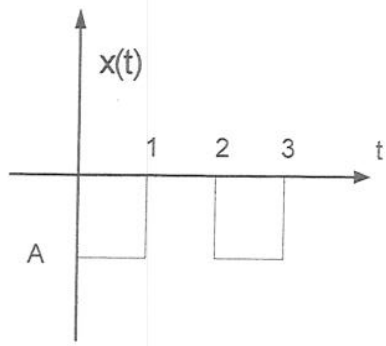

Figure 1.b

   (c) A transmission signal lasting 10 seconds is sampled at a rate 64 MHz using 16 bits/sample. The sampled signal is transmitted over four parallel channels which use $M = 16$, 64, 4 and 4 quadrature amplitude modulation (QAM) constellations. Assuming that the four channel system operates at the symbol rate $\frac{1}{T} = 300$ k symbols/sec, find how long it takes to transmit the sampled signal over these four channels. [3]

2. Answer the following sub-questions.

(a) An eight level quadrature amplitude modulation (QAM) system is shown in Figure 2.

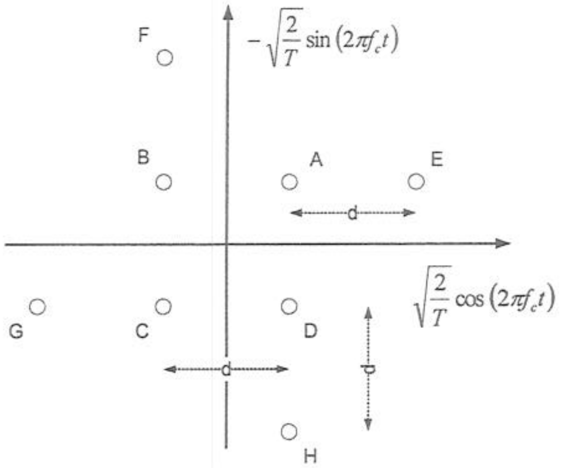

Figure 2

Answer the following sub-questions.

i. Determine the average transmitted energy for the constellation given in Figure 2 assuming that the signal points are equally probable. [3]

ii. Assign three data bits to each constellation point such that the nearest adjacent (with the minimum distance d) points differ in only one bit position. [4]

iii. When using the transmitted signal sequence [3]

$$(1,1,0,1,0,0,1,1,1,0,1,1,1,1,0,0,0,0,1,0,1,0,1,1,0,0,1,1,0,0,0,0,1,0,1,0)$$

list the constellation points to be transmitted in sequence.

iv. For the transmitted signal, plot the in-phase and quadrature time waveforms. [3]

(b) When using an M-ary square QAM transmission system, show that the average energy $\bar{\varepsilon}_x$ per dimension and the minimum distance d are related to each other by [6]

$$d^2 = \frac{12}{M-1}\bar{\varepsilon}_x.$$

(c) Either square or cross QAM will be used on an additive-white-Gaussian noise (AWGN) channel with signal-to-noise SNR=28.2 dB and symbol rate $\frac{1}{T} = 10^8$ symbols per second. Answer the following sub-questions.

i. Select a QAM constellation and specify a corresponding integer number of bits per symbol, b, for a modem with the highest data rate such that $10\log_{10}\left(\frac{d}{2\sigma}\right)^2 = 13.7$ dB, where d is the minimum distance between constellation points and σ^2 is the noise variance. [3]

ii. Compute the data rate for part 2.c.i. [1]

iii. Repeat part 2.c.i if $10\log_{10}\left(\frac{d}{2\sigma}\right)^2 = 16.4$ dB. [1]

iv. Compute the data rate for part 2.c.iii. [1]

3. Answer the following sub-questions.

(a) A multi-tone modulation system has $N = 8$ dimensions and operates over a total of $\overline{N} + 1$ channels where $\overline{N} = \dfrac{N}{2}$. The transmission system has the total transmission energy $8\bar{\varepsilon}_x = \sum_{n=1}^{8} \varepsilon_n$ where $\bar{\varepsilon}_x$ is the average energy per dimension and ε_n is the energy for each dimension $n = 1, \cdots, 8$. For each dimension $n = 1, \cdots, 8$ the channel SNR $g_n = \dfrac{|H_n|^2}{\sigma^2}$ is expressed in terms of the channel gain $|H_n|^2$ and the noise variance σ^2. Using the Lagrange multiplier method and also the following set of linear equations

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -1 \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_8 \\ K \end{bmatrix} = \begin{bmatrix} -1/g_1 \\ -1/g_2 \\ \vdots \\ -1/g_8 \\ 8\bar{\varepsilon}_x \end{bmatrix}$$

describe how the water filling theorem is used to calculate the energy distribution ε_n for $n = 1, \cdots, 8$ and also the water filling constant K when using

i. the Rate Adaptive (RA) water filling method; [6]

ii. the Margin Adaptive (MA) water filling method. [5]

(b) The number of bits per symbol for the Pulse Amplitude Modulation (PAM) and Quadrature Amplitude Modulation (QAM) systems are $b_p = 1, 2, 3 \cdots$ and $b_q = 2, 4, 6 \cdots$ respectively for integer values of p and q. The bit granularities $\beta_p = b_{p+1} - b_p$ and $\beta_q = b_{q+1} - b_q$ have values $\beta_p = 1$ and $\beta_q = 2$ for the PAM and QAM systems respectively. Consider that Γ is the gap value and σ^2 is the noise variance. Also consider that the incremental energy is $e(b_p) = \varepsilon(b_p + \beta_p) - \varepsilon(b_p)$ where $\varepsilon(b_p)$ is the energy required to transmit b_p bits over the PAM channel. Show that the incremental energies $e(b_p)$ and $e(b_q)$ are given by

i. $e(b_p) = \dfrac{\Gamma}{g} 2^{2b_p} \left(2^{2\beta_p} - 1\right)$ for the PAM system and [4]

ii. $e(b_q) = 2\dfrac{\Gamma}{g} 2^{b_q} \left(2^{\beta_q} - 1\right)$ for the QAM system. [3]

(c) A multi-tone modulation system has $N = 8$ dimensions and $\overline{N} = 8$ channels where the channel gain is equal $g_n = 1$ for each channel $n = 1, \cdots, 8$. Each channel is loaded with an identical symbol b_p bits per symbol using the PAM constellation points where $b_p = p$ for $p = 1, 2, 3, \cdots$. The total available transmission energy is $E_T = 8\bar{\varepsilon}_x$ and the gap value for the transmission system is $\Gamma = 1$. Assuming that the incremental energy $e(b_i)$ is defined in 3.b.i., answer the following questions.

i. Show that the number of bits b_p to be carried over each channel $n = 1, \cdots, 8$ is chosen to satisfy the relationship $\sum_{i=1}^{p} e(b_i) \le \dfrac{E_T}{8} < \sum_{i=1}^{p+1} e(b_i)$. [4]

ii. Assume that the residual energy $E_T - 8\sum_{i=1}^{p} e(b_i)$ will be used to transmit data at an increased rate b_{p+1} bits per symbol over m sub-channels. The remaining $(8 - m)$ will be used to transmit data at a rate b_p. Show that the number, m, of sub-channels will satisfy the relationship $m\ e(b_{p+1}) \le E_T - 8\sum_{i=1}^{p} e(b_i) < (m+1)\ e(b_{p+1})$. [3]

4. Answer the following sub-questions.

(a) Design an $M = 16$ QAM system for transmitting data at a rate 3600 bits per second and a carrier frequency of 1800 Hz over an ideal voice band telephone line which has a band-pass frequency response characteristic spanning the frequency range 600-3000 Hz. For spectral shaping, use a raised cosine pulse shaping filter.

i. Sketch a block diagram of the system and explain their functional operations; [3]

ii. Calculate the roll-off factor of the raised cosine filter. [3]

(b) Show that a pulse having the raised cosine spectrum is given by [9]

$$X_{RC}(f) = \begin{cases} T & 0 \leq |f| \leq \dfrac{1-\alpha}{2T} \\ \dfrac{T}{2}\left\{1 + \cos\left(\dfrac{\pi T}{\alpha}\left(|f| - \dfrac{1-\alpha}{2T}\right)\right)\right\} & \dfrac{1-\alpha}{2T} \leq |f| \leq \dfrac{1+\alpha}{2T} \\ 0 & \text{otherwise} \end{cases}$$

satisfies the Nyquist criterion given by equation

$$x(nT) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

for any value of roll-off factor α, where n is an integer.

(c) For a minimum-mean-square-error (MMSE) linear equalizer answer the following sub-questions

i. Show that the equalizer coefficients $W(D)$ are given by [4]

$$W(D) = \frac{1}{|p|\left(Q(D) + \frac{1}{SNR_{MFB}}\right)}$$

where $|p|$ is the transmission channel pulse energy, $Q(D)$ is the normalized channel pulse response of the transmission system. The term SNR_{MFB} is the matched filter bound SNR.

ii. Show that the noise variance at the output of the MMSE linear equalizer is given by [6]

$$\sigma^2_{mmse} = \frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \frac{\frac{N_0}{2}}{|p|^2\left(Q(\exp(-j\omega T)) + \frac{1}{SNR_{MFB}}\right)} d\omega$$

where T is the transmission symbol period.

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MODEL ANSWER and MARKING SCHEME

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Second Examiner Prof. E. GELENBE Question 1 Page 1 out of 15

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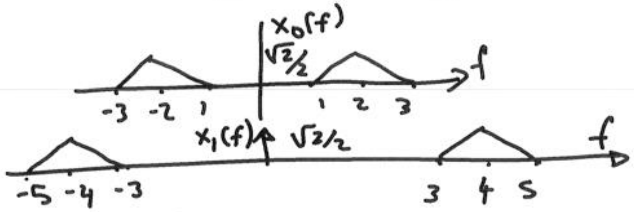
1.a.i $x_0(t) = \sqrt{2}\,\mathrm{sinc}^2(t)\cos(4\pi t)$

$$\overset{FT}{\Longleftrightarrow} X_0(f) = \frac{\sqrt{2}}{2}\Lambda(f-2) + \frac{\sqrt{2}}{2}\Lambda(f+2)$$

$$\Lambda(f) = \begin{cases} f+1 & -1 \leqslant f < 0 \\ -f+1 & 0 \leqslant f < 1 \\ 0 & \text{otherwise} \end{cases}$$

$x_1(t) = \sqrt{2}\,\mathrm{sinc}^2(t)\cos(8\pi t)$

$$\overset{FT}{\Longleftrightarrow} X_1(f) = \frac{\sqrt{2}}{2}\Lambda(f-4) + \frac{\sqrt{2}}{2}\Lambda(f+4)$$



from Parseval's theorem

$$\int_{-\infty}^{\infty} x_0(t)\,x_1^*(t)\,dt = \int_{-\infty}^{\infty} X_0(f)\,X_1(f)\,df$$

since $X_1(f)$ real and $X_1^*(f) = X_1(f)$

$$\int_{-\infty}^{\infty} X_0(f)\,X_1(f)\,df = \int_{-\infty}^{\infty}\left\{\frac{\sqrt{2}}{2}(\Lambda(f-2)+\Lambda(f+2)) \cdot \frac{\sqrt{2}}{2}(\Lambda(f-4)+\Lambda(f+4))\right\}df = 0$$

i.) As can be seen from the figure, the two signals are orthogonal.

ii) As two waveforms are orthogonal the system is two dimensional

$$\phi_0(t) = \frac{x_0(t)}{\sqrt{\mathcal{E}_0}}, \quad \phi_1(t) = \frac{x_1(t)}{\sqrt{\mathcal{E}_1}}$$

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MODEL ANSWER and MARKING SCHEME

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$$\mathcal{E}_0 = \int_{-\infty}^{\infty} x_0^2(t)\,dt = \int_{-\infty}^{\infty} X_0(f)\,df = \int_{-3}^{-2}(f+3)^2 df + \int_{-2}^{-1}(-1-f)^2 df + \int_{1}^{2}(f-1)^2 df + \int_{2}^{3}(3-f)^2 df$$

$$= \frac{1}{2}\left\{\left|\frac{f^3}{3} + 3f^2 + 9f\right|_{-3}^{-2} + \left|\frac{f^3}{3} + f^2 + f\right|_{-2}^{-1} + \left|\frac{f^3}{3} - f^2 + f\right|_{1}^{2} + \left|\frac{f^3}{3} - 3f^2 + 9f\right|_{2}^{3}\right\}$$

$$= \frac{1}{2}\Big[(-\tfrac{8}{3}+12-18+9+27-27) + (-\tfrac{1}{3}+1-1+\tfrac{8}{3}-4+2) + (\tfrac{8}{3}-4+2-\tfrac{1}{3}+1-1) + (9-27+27-\tfrac{8}{3}+12-18)\Big] = 2/3$$

$$\mathcal{E}_1 = \int_{-\infty}^{\infty} x_1^2(t)\,dt = \int_{-\infty}^{\infty} X_1^2(f)\,df$$

$$= \left(\frac{\sqrt{2}}{2}\right)^2\left\{\int_{-5}^{-3}(\Lambda(f-4))^2 df + \int_{3}^{5}(\Lambda(f+4))^2 df\right.$$

$$= \frac{1}{2}\left\{\int_{-5}^{-4}(f+5)^2 df + \int_{-4}^{-3}(-f-3)^2 df + \int_{3}^{4}(f-3)^2 df + \int_{4}^{5}(5-f)^2 df\right.$$

$$= \frac{1}{2}\left\{\int_{-5}^{-4}(f^2+10f+25)\,df + \int_{-4}^{-3}(f^2+6f+9)\,df + \int_{3}^{4}(f^2-6f+9)\,df + \int_{4}^{5}(f^2-10f+25)\,df\right.$$

$$= \frac{1}{2}\left\{\left|\frac{f^3}{3}+5f^2+25f\right|_{-5}^{-4} + \left(\frac{f^3}{3}+3f+9f\right|_{-4}^{-3} + \left|\frac{f^3}{3}-3f+9f\right|_{3}^{4} + \left|\frac{f^3}{3}-5f^2+25f\right|_{4}^{5}\right\}$$

$$= \frac{1}{2}\Big\{(-\tfrac{64}{3}+80-100+\tfrac{125}{3}-125+125) + (-\tfrac{27}{3}+3\times 9-3\times 9+\tfrac{64}{3}-3\times 16+36) + (\tfrac{64}{3}-3\times 16+36-\tfrac{27}{3}+3\times 9-9\times 3) + (\tfrac{125}{3}-125+125-\tfrac{64}{3}+80-100)$$

$$\mathcal{E}_1 = \frac{1}{2}\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right) = \frac{4}{2\times 3} = \frac{2}{3}$$

$$\mathcal{E}_1 = \frac{2}{3}$$

$$\phi_0(t) = \frac{\sqrt{2}}{\sqrt{\frac{2}{3}}}\operatorname{sinc}^2(t)\cos(4\pi t)$$

$$= \sqrt{3}\operatorname{sinc}^2(t)\cos(4\pi t)$$

$$\phi_1(t) = \sqrt{3}\operatorname{sinc}^2(t)\cos(8\pi t)$$

iii signal space vectors

$$x_{0,1} = \int_{-\infty}^{\infty} x_0(t)\phi_0(t)\,dt = \int_{-\infty}^{\infty}\frac{x_0(t)\,x_0(t)}{\sqrt{\mathcal{E}_0}}\,dt = \frac{\mathcal{E}_0}{\sqrt{\mathcal{E}_0}} = \sqrt{\mathcal{E}_0} = \sqrt{\frac{2}{3}}$$

$$x_{0,2} = \int_{-\infty}^{\infty} x_0(t)\,\phi_1(t)\,dt = 0$$

$$\bar{x}_0 = \left[\sqrt{\frac{2}{3}}\,,\,0\right]^T$$

$$x_{1,1} = \int_{-\infty}^{\infty} x_1(t)\,\phi_0(t)\,dt = 0$$

$$x_{1,2} = \int_{-\infty}^{\infty} x_1(t)\,\phi_1(t)\,dt = \int_{-\infty}^{\infty}\frac{x_1(t)\,x_1(t)}{\sqrt{\mathcal{E}_1}}\,dt = \frac{\mathcal{E}_1}{\sqrt{\mathcal{E}_1}} = \sqrt{\mathcal{E}_1} = \sqrt{\frac{2}{3}}$$

$$\bar{x}_1 = \left[0\,,\sqrt{\frac{2}{3}}\right]^T$$

iv)

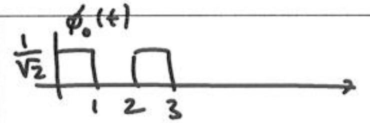
$$d_{0,1} = \sqrt{(x_{0,1}-x_{1,1})^2 + (x_{0,2}-x_{1,2})^2}$$

$$= \sqrt{\left(\sqrt{\frac{2}{3}}-0\right)^2 + \left(0-\sqrt{\frac{2}{3}}\right)^2}$$

$$= \sqrt{\frac{2}{3}+\frac{2}{3}} = \frac{2}{\sqrt{3}}$$

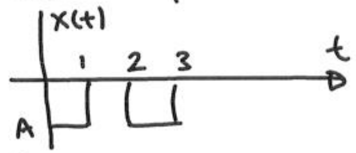
$$\boxed{d_{0,1} = \frac{2}{\sqrt{3}}}$$

1.6



The impulse response of the filter matched to the above time waveform is same as itself.

Response of the matched filter to the signal



Between $0 \le t < 1$

$$Y(t) = -A\int_0^t d\tau = -At$$

Between $1 \le t < 2$

$$Y(t) = A\int_1^{t-1} d\tau = A\,\Big|_1^{t-1} = At - A - A = A(t-2)$$

Between $2 \le t < 3$

$$Y(t) = -A\int_0^t d\tau - A\int_2^{t-2} d\tau = -A(t+t-2-2) = -2A(t-2)$$

Between $3 \le t < 4$

$$Y(t) = A\int_1^{t-1} d\tau + A\int_3^{t-3} d\tau = A(t-1-1+t-3-3) = 2A(t-4)$$

Between $4 \le t < 5$

$$Y(t) = -A\int_2^{t-2} d\tau = -A\tau\Big|_2^{t-2} = -At + 4A = -A(t-4)$$

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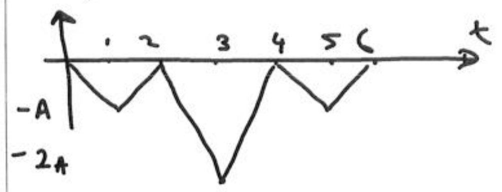
1.b cont Between $5 \le t < 6$

$$y(t) = A\int_3^{t-3} d\tau = A\tau \Big|_3^{t-3} = A(t-3-3) = A(t-6)$$

The response of the matched filter to the signal is



1.c Sampling rate

$64 \times 16 \times 10^6$ bps $= 1024 \times 10^6$ bps

Number of bits /symbol over the four channels

$b_1 = 4,\ b_2 = 6,\ b_3 = 2,\ b_4 = 2$

Data rate

$R = (4+6+2+2) \times 300 \times 10^3 = 42 \times 10^5$ bps

Transmission time

$$t = \frac{1024 \times 10^6}{42 \times 10^5} = 243.81 \text{ seconds}$$

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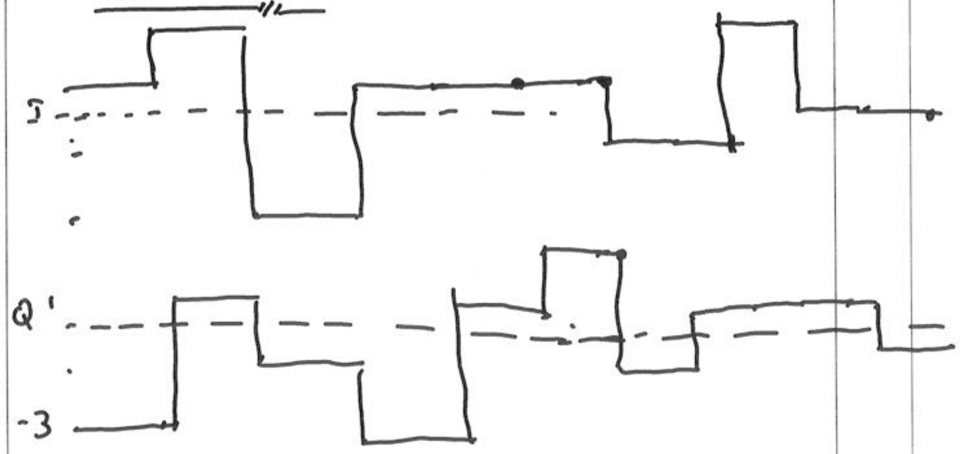
2.a i $$\mathcal{E}_x = \frac{d^2}{4} + \frac{d^2\,10}{4} = d^2\,\frac{11}{4}$$

ii

iii 110 100 111 110 000 101 011 011 001 100 001

H E G H A F C B E B D

IV



2b $\mathcal{E}_x = 2\bar{\mathcal{E}}_x = \frac{1}{M}\sum_{j,i=1}^{\sqrt{M}}(x_i^2+x_j^2) = \frac{1}{M}\left[\sqrt{M}\sum_{i=1}^{\sqrt{M}} x_i^2 + \sqrt{M}\sum_{j=1}^{\sqrt{M}} x_j^2\right.$

$= \frac{2}{\sqrt{M}}\sum_{i=1}^{\sqrt{M}} x_i^2 = \frac{2}{\sqrt{M}}2\sum_{k=1}^{\sqrt{M}/2}\frac{(2k-1)^2}{4}d^2 = \frac{d^2}{\sqrt{M}}\sum_{i=1}^{\sqrt{M}/2} x_i^2$

using the identity

$\sum_{k=1}^{A}(2k-1)^2 = \frac{A(2A-1)(2A+1)}{3} =$

$\mathcal{E}_x = \frac{d^2}{\sqrt{M}}\frac{\frac{\sqrt{M}}{2}(\sqrt{M}+1)(\sqrt{M}-1)}{3} = \frac{1}{6}d^2(M-1)$

$\bar{\mathcal{E}}_x = \frac{\mathcal{E}_x}{2} = \frac{d^2}{12}(M-1)$

2.c i $SNR = 28.3\ dB \Rightarrow SNR = 676$

$10\log_{10}\left(\frac{d}{2\sigma}\right)^2 = 13.7 \Rightarrow \left(\frac{d}{2\sigma}\right)^2 = 23.44$

for M ary QAM we have

$\frac{3\,SNR}{M-1} = \left(\frac{d}{2\sigma}\right)^2 \Rightarrow M = \frac{3\,SNR}{\left(\frac{d}{2\sigma}\right)^2}+1 = \frac{3\times 676}{23.44}+1$

$M = 87.52$ nearest M ary QAM is $M=64$ QAM

ii Data rate 6×10^8 bits/sec

iii $10\log\left(\frac{d}{2\sigma}\right)^2 = 16.4 \Rightarrow \left(\frac{d}{2\sigma}\right)^2 = 43.65$

M-ary cross QAM

$M = \frac{32}{31}\left(\frac{3\times SNR}{\left(\frac{d}{2\sigma}\right)^2}+1\right) = \frac{32}{31}\,\frac{3\times676}{43.65}+1 = 35.21$

nearest $M=32$ cross QAM

iv data rate $R = 5\times10^8$ bits/sec

3.a $b = \frac{1}{2}\sum_{m=1}^{N}\log_2\left(1+\frac{g_m\mathcal{E}_m}{\Gamma}\right)$

$\sum_{m=1}^{N}\mathcal{E}_m = N\bar{\mathcal{E}}_x$

$\frac{1}{2\ln(2)}\sum_{m=1}^{N}\ln\left(1+\frac{g_m\mathcal{E}_m}{\Gamma}\right)+\lambda\left(\sum_{m=1}^{N}\mathcal{E}_m - N\bar{\mathcal{E}}_x\right)$

$\frac{1}{2\ln(2)}\frac{\frac{g_m}{\Gamma}}{1+\frac{g_m\mathcal{E}_m}{\Gamma}} = \lambda \Rightarrow \mathcal{E}_m+\frac{\Gamma}{g_m} = \mathcal{E}_m+\frac{\sigma_m^2}{|H_m|^2} = \text{constant} = k$

Water filling equations

$\mathcal{E}_1+\frac{\Gamma}{g_1} = k$

$\mathcal{E}_2+\frac{\Gamma}{g_2} = k$

$\vdots$

$\mathcal{E}_n+\frac{\Gamma}{g_N} = K$

$\mathcal{E}_1+\mathcal{E}_2+\cdots\mathcal{E}_n = N\bar{\mathcal{E}}_x$

Water filling solution, constant K calculation

$NK = \sum_{m=1}^{N}\mathcal{E}_m+\Gamma\sum_{m=1}^{N}\frac{1}{g_m} = N\bar{\mathcal{E}}_x+\Gamma\sum_{m=1}^{N}\frac{1}{g_m}$

$k = \frac{1}{N}\left[N\bar{\mathcal{E}}_x+\Gamma\sum_{m=1}^{N}\frac{1}{g_m}\right]$ for $\forall m = 1,\ldots,N$.

3.a.i Rate adaptive loading

$$\max_{\mathcal{E}_m} \; b = \sum_{m=1}^{N} \log_2 \left(1 + \frac{g_m \mathcal{E}_m}{\Gamma}\right)$$

subject to $N\bar{\mathcal{E}}_x = \sum_{m=1}^{N} \mathcal{E}_m$

RA water filling

1. make $i = 0$
2. Make $N^* = N - i$
3. Order channel SNRs $g_1 = \max_m g_m$, $g_{N^*} = \min_m g_m$
4. calculate constant $K = \frac{1}{N^*}\left[N\bar{\mathcal{E}}_x + \Gamma \sum_{m=1}^{N^*} \frac{1}{g_m}\right]$
5. Calculate energy for each dimension $\mathcal{E}_m = K - \frac{\Gamma}{g_m}$

6 if $\mathcal{E}_{N^*} \leq 0$ discard K and make $i = i+1$ go to 2
Else
calculate all energies $\mathcal{E}_m = K - \frac{\Gamma}{g_m}$, $\forall m = 1, \dots, N^* = N - i$

7. Calculate

$$b_m = \frac{1}{2}\log_2\left(1 + \frac{g_m \mathcal{E}_m}{\Gamma}\right) = \frac{1}{2}\log_2\left(1 + \frac{g_m K}{\Gamma} - 1\right)$$

$$b_m = \frac{1}{2}\log_2\left(\frac{g_m K}{\Gamma}\right)$$

ii MA Loading

$$\min_{\mathcal{E}_m} N\bar{\mathcal{E}}_x = \sum_{m=1}^{N} \mathcal{E}_m, \quad b = \sum_{m=1}^{N} \frac{1}{2}\log_2\left(1 + \frac{g_m \mathcal{E}_m}{\Gamma}\right)$$

$\gamma_{max} = \dfrac{N\bar{\mathcal{E}}_x}{\sum_{m=1}^{N} \mathcal{E}_m}$, we need total b, we need to calculate K_{ma}.

Total number of bits $b = \frac{1}{2}\sum_{m=1}^{N^*} \log_2\left(1 + \frac{g_m \mathcal{E}_m}{\Gamma}\right)$

$$b = \frac{1}{2}\sum_{m=1}^{N^*} \log_2\left(\frac{g_m K_{ma}}{\Gamma}\right) = \frac{1}{2}\log_2\left(\prod_{m=1}^{N^*} \frac{g_m K_{ma}}{\Gamma}\right)$$

$$2^{2b} = \left(\frac{K_{ma}}{\Gamma}\right)^{N^*} \prod_{m=1}^{N^*} g_m \Rightarrow K_{ma} = \Gamma\left(\frac{2^{2b}}{\prod_{m=1}^{N^*} g_m}\right)$$

Iterative MA water filling

1-) set $i = N$
2-) set $N^* = i$ and order g_m largest to smallest $m = 1, \dots, N^*$
3-) Compute

$$K_{ma} = \Gamma\left(\frac{2^{2b}}{\prod_{m=1}^{N^*} g_m}\right)^{1/N^*}$$

4-) If $\mathcal{E}_m = K_{ma} - \frac{\Gamma}{g_m} < 0$ then $i = i - 1$ and discard g_m go to step 2
Else compute solution with

$$\mathcal{E}_m = K_{ma} - \frac{\Gamma}{g_m}, \quad m = 1, \dots, N^*$$

calculate bits

$$b_m = \frac{1}{2}\log_2\left(\frac{g_m K_{ma}}{\Gamma}\right)$$

5. compute Margin

$$\gamma_{max} = \frac{N\bar{\mathcal{E}}_x}{\sum_{m=1}^{N^*} \mathcal{E}_m}$$

3.b $\mathcal{E}(b_p) = \frac{\Gamma}{9}\left(2^{2b_p}-1\right)$

$$e(b_p) = \mathcal{E}(b_p+\beta_p) - \mathcal{E}(b_p) = \frac{\Gamma}{9}\left(2^{2b_p+2\beta_p} - 2^{2b_p}\right)$$
$$= \frac{\Gamma}{9}\, 2^{2b_p}\left(2^{2b_p}-1\right)$$

///

For QAM

$$\mathcal{E}(b_q) = 2\,\frac{\Gamma}{9}\left(2^{b_p}-1\right)$$
$$e(b_q) = \mathcal{E}(b_q+\beta_q) - \mathcal{E}(b_q) = \frac{2\Gamma}{9}\left(2^{b_q+\beta_q} - 2^{2b_q}\right)$$
$$= \frac{2\Gamma}{9}\, 2^{b_q}\left(2^{\beta_q}-1\right)$$

///

3.c.i Energy allocated to each channel is $E/8$

Energy required to transmit b_p over a single channel is $\mathcal{E}(b_p) = \sum_{i=1}^{p} e(b_i)$, we transmit b_p when

$$\mathcal{E}(b_p) \leq \frac{E_T}{8} < \mathcal{E}(b_{p+1})$$

Hence

$$\sum_{i=1}^{p} e(b_i) \leq \frac{E_T}{8} < \sum_{i=1}^{p+1} e(b_i)$$

ii) Residual energy

$$E_T - 8\sum_{i=1}^{p} e(b_i)$$

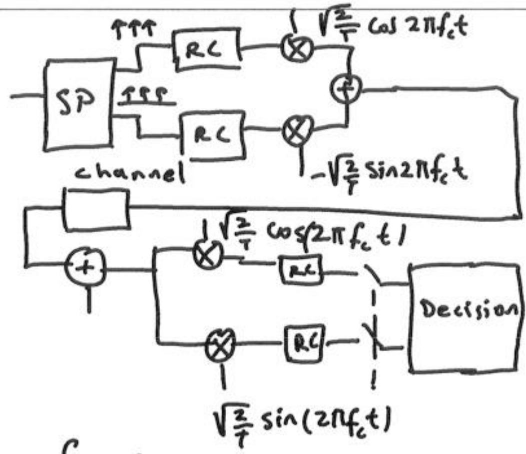
as there may be a total of m incremental energies being less than the residual energy when satisfying the relationship

$$m = \left\lfloor \frac{E_T - 8\sum_{i=1}^{p} e(b_i)}{e(b_{p+1})} \right\rfloor$$

which can be written in the form

$$m\, e(b_{p+1}) \leq E_T - 8\sum_{i}^{p} e(b_p) < (m+1)\, e(b_{p+1})$$

4.a



$f_c = 3600$ Hz

///

4a.ii

$M = 16 \qquad b = 4$

$R = \frac{b}{T} = 3600$

$\frac{1}{T} = \frac{3600}{4} = 900$

$\frac{1+\alpha}{T} = \frac{3000-600}{2} = 1200$

$1+\alpha = \frac{1200}{900} \Rightarrow \alpha = \frac{1200}{900} - 1 = \frac{1}{3}$

///

4.b The pulse $x(t)$ having the raised cosine spectrum

$$x(t) = \mathrm{sinc}\left(\frac{t}{T}\right) \frac{\cos\left(\frac{\pi\alpha t}{T}\right)}{1 - 4\frac{\alpha^2 t^2}{T^2}}$$

The function $\mathrm{sinc}\left(\frac{t}{T}\right)$ is 1 when $t=0$ and 0 when $t = nT$.

4.b on the other hand

$$g(t) = \frac{\cos\left(\frac{\pi\alpha t}{T}\right)}{1-4\frac{\alpha^2 t^2}{T^2}} = \begin{cases} 1 & t=0 \\ \text{bounded} & t\neq 0 \end{cases}$$

The function $g(t)$ needs to be checked only for those values of t such that

$$\frac{4\alpha^2 t^2}{T^2} = 1 \quad \text{or} \quad \alpha t = \frac{T}{2}$$

check

$$\lim_{\alpha T \to \frac{T}{2}} \frac{\cos\left(\frac{\pi\alpha t}{T}\right)}{1-\frac{4\alpha^2 t^2}{T^2}} = \lim_{x\to 1} \frac{\cos\left(\frac{\pi}{2}x\right)}{1-x}$$

and using L'Hospital rule

$$\lim_{x\to 1} \frac{\cos\left(\frac{\pi}{2}x\right)}{1-x} = \lim_{x\to 1} \frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right) = \frac{\pi}{2}$$

Hence

$$x(nT) = \begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases}$$

means that the pulse $x(t)$ satisfies the Nyquist criterion

4c.i The error signal is

$$E(D) = X(D) - W(D)\,Y(D)$$

where $X(D)$ is the transmitted signal, $Y(D)$ is the received signal. The error signal is orthogonal to the received signal such that

$$E\left(E(D)\,Y^*(D^{-*})\right) = 0$$

$$E\left[X(D)\,Y^*(D^{-*}) - W(D)\,Y(D)\,Y^*(D^{-*})\right] = 0$$

Cross correlation

$$R_{xy}(D) = E\left[X(D)\,Y^*(D^{-*})\right] = |P|\,Q(D)\,\bar{\mathcal{E}}_x$$

Autocorrelation

$$R_{yy}(D) = E\left(Y(D)\,Y^*(D^{-*})\right) = |P|^2\,Q^2(D)\,\bar{\mathcal{E}}_x + \frac{N_0}{2}Q(D)$$

$$R_{xy}(D) - W(D)\,R_{yy}(D) = 0$$

$$W(D) = \frac{R_{xy}(D)}{R_{yy}(D)} = \frac{|P|\,Q(D)\,\bar{\mathcal{E}}_x}{|P|^2\,Q(D)\,\bar{\mathcal{E}}_x\left(Q(D)+\frac{N_0}{2}\frac{1}{|P|^2\bar{\mathcal{E}}_x}\right)}$$

Hence

$$W(D) = \frac{1}{|P|\left(Q(D)+\frac{1}{SNR_{MFB}}\right)}$$

—— /// ——

4c.ii

$$R_{ee}(D) = E\left(E(D)\,E^*(D^{-*})\right)$$

$$= E\left(\left(X(D)-W(D)\,Y(D)\right)\left(X^*(D^{-*})-W^*(D^{-*})\,Y^*(D^{-*})\right)\right)$$

$$= \bar{\mathcal{E}}_x - W^*(D^{-*})\,R_{xy}(D) - W(D)\,R^*_{xy}(D^{-*})$$

$$+ W(D)\,R_{yy}(D)\,W^*(D^{-*})$$

$$R_{ee}(D) = \bar{\mathcal{E}}_x - \frac{Q(D)\left(|p|^2 Q(D)\,\bar{\mathcal{E}}_x + \frac{N_0}{2}\right)}{|p|^2\left(Q(D) + \frac{1}{SNR_{MFB}}\right)}$$

$$= \bar{\mathcal{E}}_x - \frac{Q(D)\,\bar{\mathcal{E}}_x}{Q(D) + \frac{1}{SNR_{MFB}}} = \frac{\frac{N_0}{2}}{|p|^2\left(Q(D) + \frac{1}{SNR_{MFB}}\right)}$$