

# Modelling and control of multibody mechanical systems

## Model answers

### Question 1

- a) i) The position vector of the centre of mass is given by

$$\mathbf{r} = x'\mathbf{i}' + y'\mathbf{j}',$$

and therefore the velocity vector by differentiation is

$$\dot{\mathbf{r}} = (\dot{x}' - y'\dot{\psi})\mathbf{i}' + (\dot{y}' + x'\dot{\psi})\mathbf{j}'.$$

[ 2 marks ]

- ii) The equations of the rolling constraint are

$$\dot{x}' - y'\dot{\psi} + a\dot{\theta} = 0, \quad (1)$$

$$\dot{y}' + x'\dot{\psi} = 0. \quad (2)$$

[ 4 marks ]

- iii) By substituting Equations 1 and 2 in the velocity vector expression

$$\dot{\mathbf{r}} = -a\dot{\theta}\mathbf{i}'.$$

Then by differentiation the acceleration vector is

$$\ddot{\mathbf{r}} = -a\ddot{\theta}\mathbf{i}' - a\dot{\theta}\dot{\psi}\mathbf{j}'.$$

[ 4 marks ]

- b) i)  $\mathbf{F} = m\ddot{\mathbf{r}}$ , or

$$F_{long}\mathbf{i}' + F_{lat}\mathbf{j}' = -ma\ddot{\theta}\mathbf{i}' - ma\dot{\theta}\dot{\psi}\mathbf{j}',$$

therefore

$$F_{long} = -ma\ddot{\theta},$$

and

$$F_{lat} = -ma\dot{\theta}\dot{\psi}.$$

[ 4 marks ]

- ii) The angular momentum vector of the body about  $C$  is

$$\mathbf{H} = I_{yy}\dot{\theta}\mathbf{j}' + I_{zz}\dot{\psi}\mathbf{k}',$$

therefore

$$\frac{d\mathbf{H}}{dt} = -I_{yy}\dot{\theta}\dot{\psi}\mathbf{i}' + I_{yy}\ddot{\theta}\mathbf{j}' + I_{zz}\ddot{\psi}\mathbf{k}'.$$

The total moment vector acting on the body is

$$\mathbf{N} = a\mathbf{k}' \times F_{long}\mathbf{i}' + a\mathbf{k}' \times F_{lat}\mathbf{j}',$$

or

$$\mathbf{N} = -ma^2\ddot{\theta}\mathbf{j}' + ma^2\ddot{\psi}\mathbf{i}'.$$

Therefore

$$N = \frac{dH}{dt},$$

which gives the equations of motion

$$(I_{yy} + ma^2)\dot{\theta}\dot{\psi} = 0,$$

$$(I_{yy} + ma^2)\ddot{\theta} = 0,$$

$$I_{zz}\ddot{\psi} = 0.$$

This implies that either one of  $\theta$  or  $\psi$  will be constant.

[ 6 marks ]

## Question 2

- a) i) We consider the moment of inertia about the axis of symmetry ( $z$  axis):

$$\begin{aligned} I_{zz} &= \int r^2 \cos^2 \phi \, dm = \rho \int_V r^2 \cos^2 \phi \, dV = \rho \int \int \int r^2 \cos^2 \phi \, r \cos \phi \, d\theta \, r \, d\phi \, dr = \\ &= \rho \int_0^a \int_0^{\frac{\pi}{2}} \int_0^{2\pi} r^4 \cos^3 \phi \, d\theta \, d\phi \, dr = 2\pi\rho \int_0^a \int_0^{\frac{\pi}{2}} r^4 \cos^3 \phi \, d\phi \, dr = \\ &= 2\pi\rho \int_0^a r^4 \left( \frac{1}{3} \cos^2 \phi \sin \phi + \frac{2}{3} \sin \phi \right) \Big|_0^{\pi/2} dr = \frac{4}{3}\pi\rho \int_0^a r^4 \, dr = \frac{4}{15}\pi\rho a^5. \end{aligned}$$

Note now that

$$m = \rho V = \rho \times \frac{1}{2} \times \frac{4}{3}\pi a^3,$$

and therefore

$$I_{zz} = \frac{2}{5}ma^2.$$

[ 6 marks ]

- ii) We consider the moment of inertia about the  $x$  axis which is perpendicular to  $z$ :

$$\begin{aligned} I_{xx} &= \int r^2 \cos^2 \phi \, dm = \rho \int_V r^2 \cos^2 \phi \, dV = \rho \int \int \int r^2 \cos^2 \phi \, r \cos \phi \, d\theta \, r \, d\phi \, dr = \\ &= \rho \int_0^a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} r^4 \cos^3 \phi \, d\theta \, d\phi \, dr = \pi\rho \int_0^a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^4 \cos^3 \phi \, d\phi \, dr = \\ &= \pi\rho \int_0^a r^4 \left( \frac{1}{3} \cos^2 \phi \sin \phi + \frac{2}{3} \sin \phi \right) \Big|_{-\pi/2}^{\pi/2} dr = \frac{4}{3}\pi\rho \int_0^a r^4 \, dr = \frac{4}{15}\pi\rho a^5. \end{aligned}$$

Note now that

$$m = \rho V = \rho \times \frac{1}{2} \times \frac{4}{3}\pi a^3,$$

and therefore

$$I_{xx} = \frac{2}{5}ma^2.$$

[ 6 marks ]

- b) The principal moments of inertia are all the same due to symmetry and have a value of  $\frac{2}{5}ma^2$ . For example,

$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}ma^2,$$

where  $I_{yy}$  is the moment of inertia about the  $y$  axis which is perpendicular to both  $x$  and  $z$  axes. Any set of three mutually orthogonal axes passing through the centre of the flat side of the hemisphere are principal axes. [ 4 marks ]

- c) We consider the moment of inertia about the parallel  $s$  axis at perpendicular distance  $a$  from the axis of symmetry (passing through the centre of mass). By using the parallel axis theorem:

$$I_s = I_{zz} + ma^2 = \frac{2}{5}ma^2 + ma^2 = \frac{7}{5}ma^2.$$

[ 4 marks ]

### Question 3

a)  $\mathbf{r}_M = x\mathbf{i}$  and  $\mathbf{r}_m = x\mathbf{i} + l\mathbf{e}_r$ . [ 1 mark ]

b) By differentiating the position vector  $\dot{\mathbf{r}}_M = \dot{x}\mathbf{i}$  and

$$\dot{\mathbf{r}}_m = \dot{x}\mathbf{i} + l\dot{\theta}\mathbf{e}_\theta = \dot{x}\cos\theta\mathbf{e}_r + (-\dot{x}\sin\theta + l\dot{\theta})\mathbf{e}_\theta.$$

[ 2 marks ]

c)

$$T = \frac{1}{2}M\dot{\mathbf{r}}_M \cdot \dot{\mathbf{r}}_M + \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}m\dot{\mathbf{r}}_m \cdot \dot{\mathbf{r}}_m = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}m(\dot{x}^2 - 2\dot{x}\dot{\theta}l\sin\theta + l^2\dot{\theta}^2).$$

[ 2 marks ]

d) The horizontal level at the wheel centre is taken as the zero potential energy level, therefore

$$V = -m\mathbf{r}_m \cdot \mathbf{g} = mgl\sin\theta.$$

[ 2 marks ]

e)

$$L = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}m(\dot{x}^2 - 2\dot{x}\dot{\theta}l\sin\theta + l^2\dot{\theta}^2) - mgl\sin\theta.$$

[ 2 marks ]

f)  $x + a\phi = 0$  or  $\dot{x} + a\dot{\phi} = 0$

[ 1 mark ]

g) The Lagrangian equation with respect to the generalised coordinate  $x$  is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \lambda = 0,$$

yielding

$$\frac{d}{dt}(M\dot{x} + m\dot{x} - m\dot{\theta}l\sin\theta) + \lambda = 0,$$

hence

$$(M + m)\ddot{x} - ml\sin\theta\ddot{\theta} - ml\dot{\theta}^2\cos\theta + \lambda = 0,$$

and finally it gives

$$-\lambda = (M + m)\ddot{x} - ml\sin\theta\ddot{\theta} - ml\dot{\theta}^2\cos\theta.$$

The Lagrangian equation with respect to the generalised coordinate  $\theta$  is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0,$$

yielding

$$\frac{d}{dt}(-m\dot{x}l\sin\theta + ml^2\dot{\theta}) + m\dot{x}\dot{\theta}l\cos\theta + mgl\cos\theta = 0,$$

hence

$$-\sin\theta\ddot{x} + l\ddot{\theta} + g\cos\theta = 0.$$

The Lagrangian equation with respect to the generalised coordinate  $\phi$  is

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} + a\lambda = T_d,$$

yielding

$$\frac{d}{dt} (I\dot{\phi}) + a\lambda = T_d,$$

and by substituting  $\lambda$  from the equation above

$$I\ddot{\phi} - a \left( (M + m)\ddot{x} - ml \sin \theta \ddot{\theta} - ml \dot{\theta}^2 \cos \theta \right) = T_d.$$

Finally substituting  $\ddot{x} = -a\ddot{\phi}$  from the constraint equation and  $l\ddot{\theta} = \sin \theta \ddot{x} - g \cos \theta$  from the above Lagrangian equation yields

$$(I + Ma^2 + ma^2 \cos^2 \theta) \ddot{\phi} + mal \cos \theta \dot{\theta}^2 - mga \sin \theta \cos \theta = T_d.$$

[ 10 marks ]

## Question 4

- a) Three single-axis-rotation transformation matrices are needed.

$$D_\psi = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the rotation matrix by angle  $\psi$  about a  $z$  axis.

$$C_\theta = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix},$$

which is the rotation matrix by angle  $\theta$  about a  $y$  axis.

$$D_\phi = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the rotation matrix by angle  $\phi$  about a  $z$  axis.

- i) The complete transformation from Earth-fixed coordinates to body-fixed coordinates is  $A = D_\phi C_\theta D_\psi$  and it amounts to

$$\begin{bmatrix} \cos \phi \cos \theta \cos \psi - \sin \phi \sin \psi & \cos \phi \cos \theta \sin \psi + \sin \phi \cos \psi & -\cos \phi \sin \theta \\ -\sin \phi \cos \theta \cos \psi - \cos \phi \sin \psi & -\sin \phi \cos \theta \sin \psi + \cos \phi \cos \psi & \sin \phi \sin \theta \\ \sin \theta \cos \psi & \sin \theta \sin \psi & \cos \theta \end{bmatrix}$$

[ 8 marks ]

- ii) The complete transformation from body-fixed coordinates to Earth-fixed coordinates is  $A^{-1} = A^T$ .

[ 4 marks ]

- b)

$$\begin{aligned} \Omega &= \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + D_\psi^T \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + D_\psi^T C_\theta^T \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} \\ &= \begin{bmatrix} \dot{\phi} \cos \psi \sin \theta - \dot{\theta} \sin \psi \\ \dot{\phi} \sin \psi \sin \theta + \dot{\theta} \cos \psi \\ \dot{\psi} + \dot{\phi} \cos \theta \end{bmatrix} \end{aligned}$$

[ 8 marks ]