Master

E3.09
ISE3.9

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

EEE/ISE PART III/IV: MEng, BEng and ACGI

CONTROL ENGINEERING

Monday, 11 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

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CONTROL ENGINEERING

1. Two types of algae evolve, in competition, in an aqueous solution. The equations describing the evolution of the two populations of algae are

$$\dot{x}_1 = x_1\left(-x_1 + \frac{u}{1+x_2}\right), \qquad \dot{x}_2 = x_2(-x_2+u),$$

where x_1 denotes the concentration of the first type, x_2 the concentration of the second type, and u the concentration of nutrient.

a) Assume $u > 0$ and constant. Determine all equilibrium points of the system. [4 marks]

b) Write the linearized models of the system around each of the equilibrium points determined in part a). [8 marks]

c) Using the linearized models determined in part b) determine (if possible) the stability properties of the equilibrium points computed in part a). [4 marks]

d) Show that all linearized models determined in part b) are not controllable. [4 marks]

2. A cart of mass $M = 1$ has two inverted pendulums attached to it of lengths l_1 and l_2 and both of mass m. Let θ_1 and θ_2 be the angles of the pendulums with respect to a vertical axis directed upward and let f be the force on the cart.

For small values of θ_1 and θ_2 the dynamic behaviour of the pendulums is described by the differential equations

$$m(f - mg\theta_1 - mg\theta_2 + l_1\ddot{\theta}_1) = mg\theta_1, \qquad m(f - mg\theta_1 - mg\theta_2 + l_2\ddot{\theta}_2) = mg\theta_2,$$

where g denotes the gravitational acceleration.

a) Let $x_1 = \theta_1$, $x_2 = \theta_2$, $x_3 = \dot{\theta}_1$, $x_4 = \dot{\theta}_2$, $u = f$, $y = x_1 - x_2$ and $x = [x_1\ x_2\ x_3\ x_4]'$. Write a state space representation of the considered system, i.e. determine matrices A, B and C such that

$$\dot{x} = Ax + Bu \qquad y = Cx.$$

[4 marks]

b) Study the controllability property of the system as a function of the physical parameters l_1 and l_2. [6 marks]

c) Study the observability property of the system as a function of the physical parameters l_1 and l_2. [6 marks]

d) Assume $l_1 = l_2$ and write a second order differential equation describing the behaviour of $\xi = \theta_1 - \theta_2$. Use this differential equation to assess the stabilizability property of the system. [4 marks]

3. Consider a herd of cattle composed of cows and calves. Let $x_1(t)$ be the number of calves in year t and $x_2(t)$ the number of cows in year t. The dynamical behaviour of the herd is described by the equation

$$x(t+1) = Ax(t) = \begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ \frac{1-k}{2} & \frac{4}{5} \end{bmatrix} x(t),$$

where $x(t) = [x_1(t), x_2(t)]'$ and $k \in [0,1]$ denotes the portion of calves slaugthered each year.

a) Compute the equilibrium points of the system as a function of $k \in [0,1]$.
[4 marks]

b) Determine for which values of k the system is stable, asymptotically stable, unstable.
[4 marks]

c) Show that for any initial condition $x(0)$ such that $x_1(0) \geq 0$ and $x_2(0) \geq 0$, the free response $x(t)$ of the system is such that $x_1(t) \geq 0$ and $x_2(t) \geq 0$, for all $t \geq 0$.
[4 marks]

d) Assume $k = 1/2$.

i) Show that the free-response of the system converges to the line

$$5x_1 - 4x_2 = 0.$$

(Hint: write a difference equation for the variable $z(t) = 5x_1(t) - 4x_2(t)$ and show that $z(t)$ tends to zero as t tends to ∞.)
[4 marks]

ii) Suppose that for each slaughtered calf C_1 GBP are earned and that each cow costs C_2 GBP a year. The *revenue* of the herd in the year t is therefore

$$y(t) = C_1 k x_1(t) - C_2 x_2(t).$$

Determine a condition on C_1 and C_2 so that the asymptotic revenue is non-negative for each $x_1(0) \geq 0$ and $x_2(0) \geq 0$.
[4 marks]

4. The chemical reaction describing the production of water, namely

$$2H_2 + O_2 \leftrightarrow 2H_2O,$$

can be described by the nonlinear continuous-time system

$$\begin{aligned}\dot{H} &= -2k_1H^2O + 2k_2W,\\ \dot{O} &= -k_1H^2O + k_2W,\\ \dot{W} &= -2k_2W + 2k_1H^2O,\end{aligned}$$

where $H \geq 0$, $O \geq 0$ and $W \geq 0$ denote the concentrations of hydrogen, oxygen, and water, respectively, and $k_1 > 0$ and $k_2 > 0$ are positive constants which quantify the speed of the reaction.

To study the dynamical properties of the system consider the variables

$$x_1 = W, \qquad x_2 = W + 2O, \qquad x_3 = W + H.$$

a) Show that the variables (x_1, x_2, x_3) define a new set of coordinates for the system and determine (H, O, W) as a function of (x_1, x_2, x_3).
(Hint: show that there is a one-to-one relation between the variables (H, O, W) and the variables (x_1, x_2, x_3).) [4 marks]

b) Write differential equations for x_2 and x_3. Integrate the resulting differential equations and comment on the results. [4 marks]

c) Write a differential equation for x_1 and show that $\dot{x}_1$ can be written as a cubic polynomial in x_1 with coefficients that depend upon $x_2(0)$, $x_3(0)$, k_1 and k_2. In particular, show that

$$\dot{x}_1 = A - Bx_1 + Cx_1^2 - Dx_1^3, \qquad (\star)$$

where A, B, C and D are functions of $x_2(0)$, $x_3(0)$, k_1 and k_2 and take non-negative values. [4 marks]

d) Suppose that for all $x_2(0) > 0$ and $x_3(0) > 0$ the system $(\star)$ has only one equilibrium $x_1^\star$.

i) Sketch $\dot{x}_1$ as a function of x_1 and argue that the equilibrium $x_1 = x_1^\star$ is a globally asymptotically stable equilibrium for the x_1-system. [4 marks]

ii) Argue that the overal system with state (x_1, x_2, x_3) has infinitely many equilibria. Using the results of part d.i) determine the stability properties of these equilibria. [4 marks]

5. Consider a linear, time-varying, continuous-time system described by the equation

$$\dot{x} = A(t)x.$$

A common *belief* is the following.

(C) If the matrix $A(t)$ has constant eigenvalues with negative real part then the linear, time-varying, system is asymptotically stable.

To disprove the claim (C) consider the matrix

$$A(t) = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix}.$$

Let $t_0 = 0$.

a) Show that the matrix $A(t)$ has constant eigenvalues with negative real part. [2 marks]

b) Determine the state transition matrix of the system, i.e. the matrix $\Phi(t,0)$ such that

$$\Phi(0,0) = I, \qquad \frac{d\Phi(t,0)}{dt} = A(t)\Phi(t,0).$$

(Hint: integrate the differential equations describing the system.) [8 marks]

c) Show that for almost any selection of the initial conditions $x(0)$

$$\lim_{t\to\infty} \|x(t)\| = \infty.$$

Determine the set of initial conditions such that

$$\lim_{t\to\infty} \|x(t)\| = 0.$$

[4 marks]

d) Using the results in part c) conclude that the considered linear, time-varying system, is not stable. [2 marks]

e) Show that the linear, time-varying, system

$$\dot{x} = B(t)x,$$

with

$$B(t) = \begin{bmatrix} -1 & b(t) \\ 0 & -1 \end{bmatrix}$$

and $|b(t)| \le \bar{b}$, for some $\bar{b}$ positive, is asymptotically stable. [4 marks]

6. Consider a linear, single-input, single-output, system described by the equations

$$\sigma x = Ax + Bu, \qquad y = Cx,$$

where $x(t) \in I\!R^n$ is the state, $u(t) \in I\!R$ is the input, and $y(t) \in I\!R$ is the output.

Consider the problem of studying the reachability and observability properties of the system using the PBH tests.

a) Show, using the PBH reachability test, that the system is reachable if and only if there is no left eigenvector of A which is orthogonal to B.
(Hint: recall that a left eigenvector of A is a row vector w such that $wA = \lambda w$, for some λ which is an eigenvalues of A.) [4 marks]

b) Show, using the PBH observability test, that the system is observable if and only if there is no right eigenvector of A which is orthogonal to C.
(Hint: recall that a right eigenvector of A is a column vector v such that $Av = \lambda v$, for some λ which is an eigenvalues of A.) [4 marks]

c) Consider the class of linear systems described by the equations

$$\begin{aligned} \sigma x_1 &= \lambda_1 x_1 + B_1 u, \\ \sigma x_2 &= \lambda_2 x_2 + B_2 u, \\ &\vdots \\ \sigma x_n &= \lambda_n x_n + B_n u, \\ y &= C_1 x_1 + C_2 x_2 + \cdots + C_n x_n, \end{aligned}$$

with $\lambda_i \neq \lambda_j$ for $i \neq j$.

i) Using the results in part a) determine conditions on the coefficients B_i such that the system is reachable. [4 marks]

ii) Using the results in part b) determine conditions on the coefficients C_i such that the system is observable. [2 marks]

d) Let

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}.$$

Show, using the results in part a), that the system is not reachable (regardeless of the values of the coefficients B_1, B_2 and B_3). [6 marks]

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Control engineering exam paper - Model answers 2009

Question 1

a) The equilibria of the system are obtained solving the equations

$$0 = x_1\left(-x_1 + \frac{u}{1+x_2}\right), \qquad 0 = x_2(-x_2 + u),$$

with $u > 0$ and constant. The first equation yields $x_1 = 0$ or $x_1 = \frac{u}{1+x_2}$. The second equation yields $x_2 = 0$ or $x_2 = u$. There are, therefore, four equilibrium points:

$$P_1 = (0,0) \qquad P_2 = (0,u) \qquad P_3 = (u,0) \qquad P_4 = (\frac{u}{1+u}, u).$$

b) The linearized models are described by equations of the form $\dot{\delta}_x = A_i\delta_x + B_i\delta_u$, where the matrices A_i's and B_i's are the Jacobian matrices of the generating function of the system, with respect to x and u, respectively, evaluated at the point P_i. Therefore

$$A_1 = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \frac{u}{1+u} & 0 \\ 0 & -u \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ u \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -u & -u^2 \\ 0 & u \end{bmatrix}, \qquad B_3 = \begin{bmatrix} u \\ 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -\frac{u}{1+u} & -\frac{u^2}{(1+u)^3} \\ 0 & -u \end{bmatrix}, \qquad B_4 = \begin{bmatrix} \frac{u}{(1+u)^2} \\ u \end{bmatrix}.$$

c) Recall that $u > 0$. Note that

- $\lambda(A_1) = \{u\}$, hence P_1 is unstable;
- $\lambda(A_2) = \{-u, \frac{u}{1+u}\}$, hence P_2 is unstable;
- $\lambda(A_3) = \{-u, u\}$, hence P_3 is unstable;
- $\lambda(A_4) = \{-u, -\frac{u}{1+u}\}$, hence P_4 is (locally) asymptotically stable.

d) The controllability matrices of the four linearized models are

$$\mathcal{C}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathcal{C}_2 = \begin{bmatrix} 0 & 0 \\ u & -u^2 \end{bmatrix},$$

$$\mathcal{C}_3 = \begin{bmatrix} u & -u^2 \\ 0 & 0 \end{bmatrix}, \qquad \mathcal{C}_4 = \begin{bmatrix} \frac{u}{(1+u)^2} & -\frac{u^2}{(1+u)^2} \\ u & -u^2 \end{bmatrix}.$$

Note that

$$\det\mathcal{C}_1 = \det\mathcal{C}_2 = \det\mathcal{C}_3 = \det\mathcal{C}_4 = 0,$$

hence all linearized models are not controllable.

Question 2

a) With the given selection of state variables we have

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & 0 & 0 \\ a_3 & a_4 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ -1/l_1 \\ -1/l_2 \end{bmatrix} u,$$

where

$$a_1 = \frac{(m+1)g}{l_1} \qquad a_2 = \frac{mg}{l_1} \qquad a_3 = \frac{mg}{l_2} \qquad a_4 = \frac{(m+1)g}{l_2}.$$

b) The reachability matrix is

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{l_1} & 0 & -g\frac{m+1}{l_1^2} - g\frac{m}{l_1 l_2} \\ 0 & -\frac{1}{l_2} & 0 & -g\frac{m+1}{l_2^2} - g\frac{m}{l_1 l_2} \\ -\frac{1}{l_1} & 0 & -g\frac{m+1}{l_1^2} - g\frac{m}{l_1 l_2} & 0 \\ -\frac{1}{l_2} & 0 & -g\frac{m+1}{l_2^2} - g\frac{m}{l_1 l_2} & 0 \end{bmatrix},$$

and its determinant is

$$\det\mathcal{C} = -g^2 \frac{(l_1 - l_2)^2}{l_1^4 l_2^4}.$$

As a result, the system is reachable (controllable) if and only if $l_1 \neq l_2$.

c) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ g\frac{m+1}{l_1} - g\frac{m}{l_2} & -g\frac{m+1}{l_2} + g\frac{m}{l_1} & 0 & 0 \\ 0 & 0 & g\frac{m+1}{l_1} - g\frac{m}{l_2} & -g\frac{m+1}{l_2} + g\frac{m}{l_1} \end{bmatrix},$$

and its determinant is

$$\det\mathcal{O} = -g^2(2m+1)^2 \frac{(l_1 - l_2)^2}{l_1^2 l_2^2}.$$

As a result, the system is observable if and only if $l_1 \neq l_2$.

d) If $l_1 = l_2 = l$ then, subtracting the two equations describing the system yields

$$l(\ddot{\theta}_1 - \ddot{\theta}_2) = g(\theta_1 - \theta_2),$$

hence

$$l\ddot{\xi} = g\xi.$$

Note that this subsystem is not affected by the input u, and it has one positive and one negative eigenvalue, hence it is unstable. As a result, for $l_1 = l_2$ the system is not stabilizable.

Question 3

a) The equilibrium points are the (constant) solutions of the equation

$$x(t) = Ax(t)$$

hence the solutions of

$$(I - A)\bar{x} = \begin{bmatrix} \frac{1}{2} & -\frac{2}{5} \\ \frac{1-k}{2} & \frac{1}{5} \end{bmatrix} \bar{x} = 0.$$

Note that

$$\det(I - A) = \frac{2k-1}{10},$$

hence for all $k \neq \frac{1}{2}$ the system has a unique equilibrium, whereas for $k = 1/2$ the system has infinitely many equilibria given by

$$\bar{x} = \alpha \begin{bmatrix} 4 \\ 5 \end{bmatrix},$$

for any $\alpha \in I\!R$.

b) The characteristic polynomial of the matrix A is

$$p(z) = z^2 - \frac{13}{10}z + \frac{k+1}{5},$$

and its roots are

$$z_{1,2} = \frac{13}{20} \pm \frac{\sqrt{89 - 80k}}{20}.$$

Note that the roots are real and positive for all $k \in [0, 1]$, and that the root with the "$-$" sign in front of the square root is always smaller than 1. The root with the "+" sign in front of the square root is larger than 1 for $k \in [0, 1/2)$, it is equal to 1 for $k = 1/2$, and it is smaller than 1 for $k \in (1/2, 1]$. In summary, the system is unstable for $k \in [0, 1/2)$, stable for $k = 1/2$, asymptotically stable for $k \in (1/2, 1]$.

c) Recall that $x(t) = A^t x(0)$, and note that since A has all non-negative entries for $k \in [0, 1]$, A^t has non-negative entries for all $t \geq 0$. Therefore if $x(0)$ has non-negative entry then $x(t)$ is the linear combination of the entries of $x(0)$ with non-negative coefficients, hence it has non-negative components.

d) i) Note that

$$z(t+1) = 5x_1(t+1) - 4x_2(t+1) = \frac{2}{5}z(t).$$

As a result, for any initial condition,

$$z(t) = \left(\frac{2}{5}\right)^t z(0),$$

which implies that $z(t)$ tends to zero as t goes to infinity, which proves the claim.

ii) Since all trajectories converge to the line $5x_1 - 4x_2 = 0$, the asymptotic revenue is

$$\lim_{t\to\infty} y(t) = (C_1 k - \frac{5}{4}C_2) \lim_{t\to\infty} x_1(t).$$

Hence the asymptotic revenue is non-negative provided

$$C_1 k - \frac{5}{4}C_2 \geq 0.$$

Question 4

a) The relation between the variables (x_1, x_2, x_3) and (H, O, W) can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = T \begin{bmatrix} H \\ O \\ W \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} H \\ O \\ W \end{bmatrix}.$$

Note that the matrix T is invertible, hence there is a one-to-one relation between the two sets of variables. Finally

$$\begin{bmatrix} H \\ O \\ W \end{bmatrix} = T^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 - x_1 \\ \dfrac{x_2 - x_1}{2} \\ x_1 \end{bmatrix}.$$

b) Note that

$$\dot{x}_2 = \dot{W} + 2\dot{O} = 0 \qquad \dot{x}_3 = \dot{W} + \dot{H} = 0.$$

Hence

$$x_2(t) = x_2(0) \qquad x_3(t) = x_3(0),$$

which means that $x_2(t)$ and $x_3(t)$ are constant, i.e. $W(t) + 2O(t)$ and $W(t) + H(t)$ remain constant.

c) Note that

$$\dot{x}_1 = k_1 x_2 x_3^2 - (2k_2 + k_1 x_3^2 + 2k_1 x_2 x_3)x_1 + k_1(2x_3 + x_2)x_1^2 - k_1 x_1^3$$

and since $x_2(t) = x_2(0)$ and $x_3(t) = x_3(0)$

$\dot{x}_1 = k_1 x_2(0)x_3^2(0) - (2k_2 + k_1 x_3^2(0) + 2k_1 x_2(0)x_3(0))x_1 + k_1(2x_3(0) + x_2(0))x_1^2 - k_1 x_1^3.$

As a result (note that $x_2(0)$ and $x_3(0)$ are non-negative)

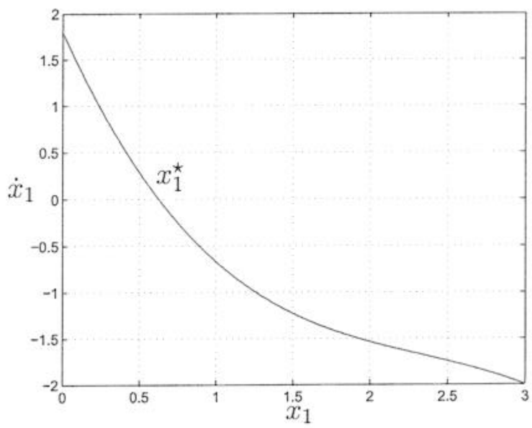
$$A = k_1 x_2(0)x_3^2(0) \geq 0, \qquad B = 2k_2 + k_1 x_3^2(0) + 2k_1 x_2(0)x_3(0) > 0,$$

$$C = 2x_3(0) + x_2(0) \geq 0, \qquad D = k_1 > 0.$$

d) i) Note that $\dot{x}_1$ is a cubic function of x_1 and that

$$\dot{x}_1|_{x_1=0} = A > 0 \qquad \lim_{x_1 \to \infty} \dot{x}_1(x_1) = -\infty.$$

As a result, $\dot{x}_1$ as a function of x_1 has the shape in the figure below.



Note that, since $\dot{x}_1 > 0$, for $x_1 < x_1^\star$, and $\dot{x}_1 < 0$, for $x_1 > x_1^\star$, the equilibrium $x_1^\star$ is globally asymptotically stable.

ii) In the (x_1, x_2, x_3) coordinates the system is described by the equations

$$\dot{x}_1 = A - Bx_1 + Cx_1^2 - Dx_1^3 \qquad \dot{x}_2 = 0 \qquad \dot{x}_3 = 0.$$

Hence, for any x_{2e} and x_{3e} there is a unique $x_{1e} = x_{1e}(x_{2e}, x_{3e})$ such that the point (x_{1e}, x_{2e}, x_{3e}) is an equilibrium. This means that the system has infinitely many equilibria, parameterized by x_{2e} and x_{3e}. The principle of stability in the first approximation cannot be used to assess stability of these equilibria. However, because of the structure of the $\dot{x}_2$ and $\dot{x}_3$ equation, and of what established in part d.i), these equilibria are stable, non-asymptotically.

Question 5

a) Since A is upper diagonal, its eigenvalues are the elements of the diagonal. As a result, the eigenvalues of A are both equal to -1, hence they are constant and with negative real part.

b) The system can be re-written as

$$\dot{x}_1 = -x_1 + e^{2t}x_2, \qquad \dot{x}_2 = -x_2,$$

hence (recall that $t_0 = 0$)

$$x_2(t) = e^{-t}x_2(0),$$

yielding

$$\dot{x}_1 = -x_1 + e^{t}x_2(0).$$

Using Lagrange formula for integrating this equation yields

$$x_1(t) = \left(x_1(0) - \frac{1}{2}x_2(0)\right)e^{-t} + \frac{1}{2}x_2(0)e^{t}.$$

Combining the expressions of $x_1(t)$ and $x_2(t)$ in matrix form yields

$$x(t) = \begin{bmatrix} e^{-t} & -\frac{1}{2}e^{-t} + \frac{1}{2}e^{t} \\ 0 & e^{-t} \end{bmatrix} x(0) = \Phi(t,0)x(0).$$

Note that $\Phi(0,0) = I$ and that

$$\frac{d\Phi(t,0)}{dt} = A(t)\Phi(t,0),$$

as requested.

c) By inspection, it is clear that, if $x_2(0) \neq 0$ then

$$\lim_{t\to\infty} \|x(t)\| = \infty.$$

Hence for almost all initial conditions the solutions are unbounded, whereas the solutions are bounded only if $x_2(0) = 0$.

d) The system is stable, if and only if, $\Phi(t,0)$ is bounded, hence the system is not stable.

e) Repeating the arguments in part a) we obtain

$$x_2(t) = e^{-t}x_2(0)$$

and

$$\begin{aligned} x_1(t) &= e^{-t}x_1(0) + \int_0^t e^{-(t-\tau)}e^{-\tau}b(\tau)d\tau \ x_2(0) \\ &= e^{-t}x_1(0) + e^{-t}\int_0^t b(\tau)d\tau \ x_2(0). \end{aligned}$$

Note now that since $b(t) \leq \bar{b}$ then

$$\left|\int_0^t b(\tau)d\tau\right| \leq \bar{b}t,$$

hence $x_1(t)$ is bounded and converges to zero. Therefore, the state transition matrix for this system is bounded and converges to zero, as $t \to \infty$, which implies that the system is asymptotically stable.

Question 6

a) The PBH reachability test states that a system is reachable if and only if

$$\text{rank}\,[sI - A \quad B] = n,$$

for all $s \in \lambda(A)$. Suppose now that there is a left eigenvector w of A which is orthogonal to B, i.e.

$$wA = \lambda w \qquad wB = 0.$$

This can be rewritten as

$$w\,[\lambda I - A \quad B] = 0,$$

which implies that the reachability pencil loses rank for $s = \lambda$. Hence, the system is reachable if and only if the reachability pencil has rank equal to n for all $s \in \lambda(A)$, which is equivalent to the fact that there is no left eigenvector of A which is orthogonal to B.
Note that we have used the fact that a matrix M is full rank if and only if $wM \neq 0$ for all vectors $w \neq 0$.

b) The PBH observability test states that a system is observable if and only if

$$\text{rank} \begin{bmatrix} sI - A \\ C \end{bmatrix} = n,$$

for all $s \in \lambda(A)$. Suppose now that there is a right eigenvector v of A which is orthogonal to C, i.e.

$$Av = \lambda v \qquad Cv = 0.$$

This can be rewritten as

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} v = 0,$$

which implies that the observability pencil loses rank for $s = \lambda$. Hence, the system is observable if and only if the observability pencil has rank equal to n for all $s \in \lambda(A)$, which is equivalent to the fact that there is no right eigenvector of A which is orthogonal to C.

c) For the considered system we have

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & & & \lambda_n \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_n \end{bmatrix} \qquad C = \begin{bmatrix} C_1 & C_2 & C_3 & \dots & C_n \end{bmatrix}.$$

i) The left eigenvectors of A are

$$w_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 & 1 & \cdots & 0 \end{bmatrix} \quad \dots \quad w_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}.$$

There is a left eigenvector of A orthogonal to B if and only if there is a $B_i = 0$. Hence, the system is reachable if and only if

$$B_1 B_2 \dots B_n \neq 0.$$

ii) The right eigenvectors of A are

$$v_1 = w_1' \qquad v_2 = w_2' \quad \dots \quad v_n = w_n'.$$

There is a right eigenvector of A orthogonal to C if and only if there is a $C_i = 0$. Hence, the system is observable if and only if

$$C_1 C_2 \dots C_n \neq 0.$$

d) The left eigenvectors of the given A are

$$w_1 = \begin{bmatrix} \alpha & \beta & 0 \end{bmatrix} \qquad w_2 = \begin{bmatrix} \alpha & 0 & \gamma \end{bmatrix}$$

for any α, β and γ such that $|\alpha| + |\beta| > 0$ and $|\alpha| + |\gamma| > 0$. Note that, for example,

$$w_1 B = \alpha B_1 + \beta B_2,$$

and this can be rendered zero selecting $\alpha = B_2$ and $\beta = -B_1$, if $B_1 \neq 0$ or $B_2 \neq 0$, or selecting any nonzero α and β is $B_1 =$ and $B_2 = 0$. As a result, there is (always) a left eigenvector of A orthogonal to B, hence the system is not reachable.