Final.

EE4-29
EE9CS3-2

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

MSc and EEE/ISE PART IV: MEng and ACGI

OPTIMIZATION

Friday, 4 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

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OPTIMISATION

1. Consider the function

$$f(x_1,x_2) = \sin(x_1^2 + x_2^2).$$

a) Sketch on the (x_1,x_2)-plane the level lines of the function f. [4 marks]

b) Compute the stationary points of the function. [4 marks]

c) Explain why second order sufficient conditions of optimality are inadequate to classify some of the stationary points of the functions. [6 marks]

d) Consider the change of variable

$$x_1 = \rho\cos\theta, \qquad x_2 = \rho\sin\theta,$$

with $\rho \geq 0$ and $\theta \in (-\pi,\pi]$.

i) Rewrite the function f in the new variables. Note that the function depends only upon the variable ρ. [1 marks]

ii) Compute the stationary points of the function f as a function of ρ and classify these stationary points. [1 marks]

iii) Exploiting the results in part d.ii) classify the stationary points of the function f. [4 marks]

2. Consider, on the (x_1,x_2)-plane the parabola described by the equation $x_1^2 = 4x_2$, and a point on the x_2-axis given by $(0,b)$, with $b > 0$. We wish to find the minimum distance from the point to the parabola.

a) Show that the considered problem can be posed as the constrained optimization problem

$$\begin{cases} \min\limits_{x_1,x_2} x_1^2 + (x_2 - b)^2, \\ 4x_2 - x_1^2 = 0. \end{cases}$$

[4 marks]

b) Solve the problem in part a) using the method of constraint elimination. In particular, use the constraint to eliminate the variable x_1. Solve the resulting unconstrained optimization problem in the variable x_2, hence determine a solution of the problem in part a). Show that such a solution exists for all $b \geq 2$, whereas it does not exist for $b \in (0,2)$.
Explain why the method of constrained elimination fails to provide a solution for all $b > 0$. [8 marks]

c) Write first order necessary conditions of optimality for the problem in part a). [2 marks]

d) Using the conditions in part c) determine candidate optimal solutions. In particular show that for $b \in (0,2]$ there is only one candidate optimal solution, whereas for $b > 2$ there are three candidate optimal solution. [4 marks]

e) Using the results of part d) determine the solution of the considered optimization problem and write the optimal cost as a function of $b > 0$. [2 marks]

3. A nonlinear least-squares problem is an unconstrained optimization problem of the form

$$\min_x \frac{1}{2}\sum_{i=1}^{m} r_i^2(x),$$

where $x \in I\!R^n$. The functions $r_1, r_2, \cdots, r_m$ are called residuals, and the objective function can be rewritten as $\frac{1}{2}r'(x)r(x)$, with

$$r(x) = \begin{bmatrix} r_1(x) \\ \cdots \\ r_m(x) \end{bmatrix}.$$

a) Write Newton's iteration for the solution of the considered least-square problem. [2 marks]

b) Gauss-Newton's iteration for the solution of the considered least-square problem is given by

$$x_{(k+1)} = x_{(k)} - [J'(x_{(k)})J(x_{(k)})]^{-1}J'(x_{(k)})r(x_{(k)}),$$

where

$$J(x) = \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_m}{\partial x_1} & \cdots & \frac{\partial r_m}{\partial x_n} \end{bmatrix}$$

and $x_{(k)} = \begin{bmatrix} x_{k,1}, & \cdots & x_{k,n} \end{bmatrix}$. Discuss the differences between Newton's iteration and Gauss-Newton's iteration.
(Hint: consider the *difference* between the Hessian of $\frac{1}{2}r'(x)r(x)$ and the matrix $J'(x)J(x)$.)
Discuss under what conditions the Gauss-Newton direction

$$d_{GN} = -[J'(x)J(x)]^{-1}J'(x)r(x)$$

is a descent direction. [4 marks]

c) Assume $m = 2$, $x = (x_1, x_2)$ and

$$r_1(x) = x_1 + x_2 - x_1x_2 + 2, \qquad r_2(x) = x_1 - e^{x_2}.$$

i) Sketch on the (x_1, x_2)-plane the set of points $r_1(x) = 0$ and $r_2(x) = 0$, hence argue that the considered least-square problem has two (global) solutions. Find an approximation of these global solutions using graphical considerations. [4 marks]

ii) Write explicitly Gauss-Newton's iteration for the considered problem. [4 marks]

iii) Compute three iterations of Gauss-Newton's methods from the initial conditions $(0,0)$. Evaluate the residuals at $(0,0)$ and at the last iteration. [4 marks]

iv) Comment on the convergence speed and complexity of Gauss-Newton's method. [2 marks]

4. Consider an individual living in a world with only two consumer goods. The only decision facing this individual is how to divide his/her budget. The goods are called X and Y. The variables $x \geq 0$ and $y \geq 0$ denote the number of units of each good that he/she buys. The utility function of the individual is

$$U(x,y)$$

which is subject to the constraint

$$p_1 x + p_2 y - B = 0,$$

where $p_1 > 0$ and $p_2 > 0$ are unit prices of X and Y, respectively, and $B > 0$ is the budget of the individual.

Assume that the utility function U is such that

$$\frac{\partial U}{\partial x} > 0, \qquad \frac{\partial U}{\partial y} > 0,$$

for all $x \geq 0$ and $y \geq 0$.

The consumer's goal is to maximize the utility function.

a) Write the *consumer's goal* as a constrained optimization problem, disregarding the non-negativity constraints on x and y. [2 marks]

b) Write first order necessary conditions of optimality for the problem formulated in part a). Show that the optimal multiplier is positive. [4 marks]

c) Assume $U(x,y) = xy$.

i) Determine candidate optimal solutions for the considered constrained optimization problem. [4 marks]

ii) Using second order sufficient conditions of optimality, solve the constrained optimization problem. [4 marks]

iii) Let $U^\star$ be the optimal utility. Show that

$$\frac{\partial U^\star}{\partial B}$$

equals the optimal multiplier $\lambda^\star$. [2 marks]

iv) Determine the so-called *comparative statistics*: expressions that show the effect of an increase of p_1, p_2 or B, on the optimal solutions $x^\star$ and $y^\star$. Briefly interprete the results. [4 marks]

5. Consider the optimisation problem

$$\begin{cases} \min\limits_{x_1,x_2} x_1 + x_2, \\ x_1^2 + x_2^2 \leq 1. \end{cases}$$

a) Write first order necessary conditions of optimality for the considered optimization problem. [2 marks]

b) Using the conditions derived in part a) compute candidate optimal solutions. [4 marks]

c) A barrier function for the considered problem is the function

$$B_\tau(x_1,x_2) = x_1 + x_2 - \frac{1}{\tau} \log(1 - x_1^2 - x_2^2),$$

with $\tau > 0$.

i) Compute the stationary point of B_τ. [4 marks]

ii) Show that only one stationary point of B_τ is inside the admissible set and that this point converges, as $\tau \to \infty$, to the solution of the optimization problem. [4 marks]

iii) Show that the stationary point of B inside the admissible set is a minimizer of B. [6 marks]

6. Consider the optimisation problem

$$\begin{cases} \min\limits_{x_1,x_2} -x_1 x_2, \\ x_1 + 2x_2 \leq 40, \qquad x_1 + x_2 \leq 24, \\ x_1 \geq 0, \qquad x_2 \geq 0. \end{cases}$$

a) Sketch in the (x_1, x_2)-plane the admissible set and the level lines of the objective function. Show that all admissible points are regular points for the constraints. Hence, using only graphical considerations determine the optimal solutions of the considered problem. [4 marks]

b) Write first order necessary conditions of optimality for the considered optimization problem. [2 marks]

c) Using the conditions derived in part b) compute candidate optimal solutions. (Hint: assume that the optimal solution is such that $x_1 > 0$ and $x_2 > 0$.) [6 marks]

d) Determine the solution of the optimization problem. [2 marks]

e) Replace the inequality constrains with the constraints

$$\begin{cases} x_1 + 2x_2 \leq 40, \qquad x_1 + x_2 \leq T, \\ x_1 \geq 0, \qquad x_2 \geq 0. \end{cases}$$

with $T > 0$. Determine values of T such that the solution of the optimization problem is at the point in which the first two constraints are active. [6 marks]

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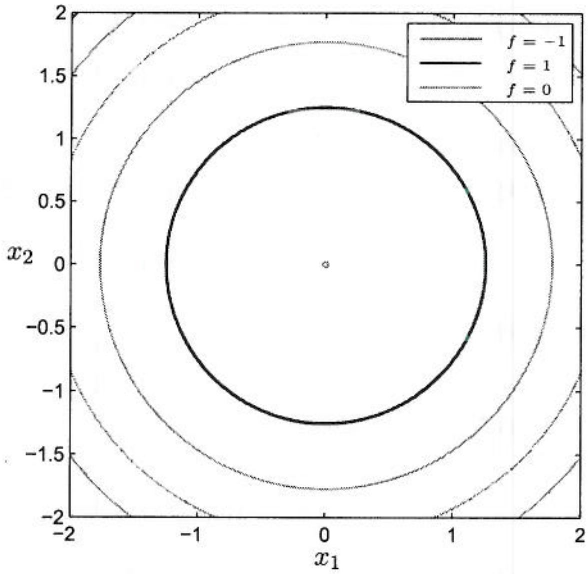
Optimisation - Model answers 2012

(Note to external examiners: all questions involve mostly applications of standard methods and concepts to unseen examples.)

Question 1

a) Note that the function is constant on any circle centered at the origin, *i.e.* on any set of the form $x_1^2 + x_2^2 = R^2$. A sketch of the level lines is therefore as in the figure below.



b) The stationary points of the function f are computed by solving the equations

$$0 = \nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 \cos(x_1^2 + x_2^2) \\ 2x_2 \cos(x_1^2 + x_2^2) \end{bmatrix}.$$

Hence, the point $(0,0)$ is a stationary point and all points such that

$$x_1^2 + x_2^2 = \frac{\pi}{2} + k\pi,$$

with k integer, are stationary points.

c) The Hessian matrix of the function f is

$$\nabla^2 f(x) = 2\cos(x_1^2 + x_2^2) I - 4\sin(x_1^2 + x_2^2) \begin{bmatrix} x_1^2 & x_1 x_2 \\ x_1 x_2 & x_2^2 \end{bmatrix}.$$

Note that

$$\nabla^2 f(0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

is positive definite, hence the point $(0, 0)$ is a local minimizer.
To classify the stationary points such that $x_1^2 + x_2^2 = \frac{\pi}{2} + k\pi$, note that at such points P_k

$$\nabla^2 f(P_k) = \mp 4 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix},$$

hence $\nabla^2 f(P_k)$ is singular, and this does not enable the use of second order sufficient conditions of optimality (which require the Hessian to be non-singular).

d) i) The function f in the new variables is given by

$$f(\rho, \theta) = \sin \rho^2,$$

hence it is a function of ρ only.

ii) The stationary points of the function $\sin \rho^2$ are all points such that

$$\frac{df}{d\rho} = 2\rho \cos \rho^2 = 0.$$

These are given by

$$\rho = 0 \qquad\qquad \rho^2 = \frac{\pi}{2} + k\pi,$$

with k any non-negative integer.

iii) Note that

$$\frac{d^2 f}{d\rho^2} = 2 \cos \rho^2 - 4\rho^2 \sin \rho^2,$$

hence the point $\rho = 0$ is a local minimizer, the points

$$\rho^2 = \frac{\pi}{2} + 2k\pi,$$

with k any non-negative integer, are local maximizers, and the points

$$\rho^2 = \frac{\pi}{2} + (2k + 1)\pi,$$

with k any non-negative integer, are local minimizers.
This implies that the point $(x_1, x_2) = (0, 0)$ is a local strict minimizer, the points (x_1, x_2) such that

$$x_1^2 + x_2^2 = \frac{\pi}{2} + 2k\pi,$$

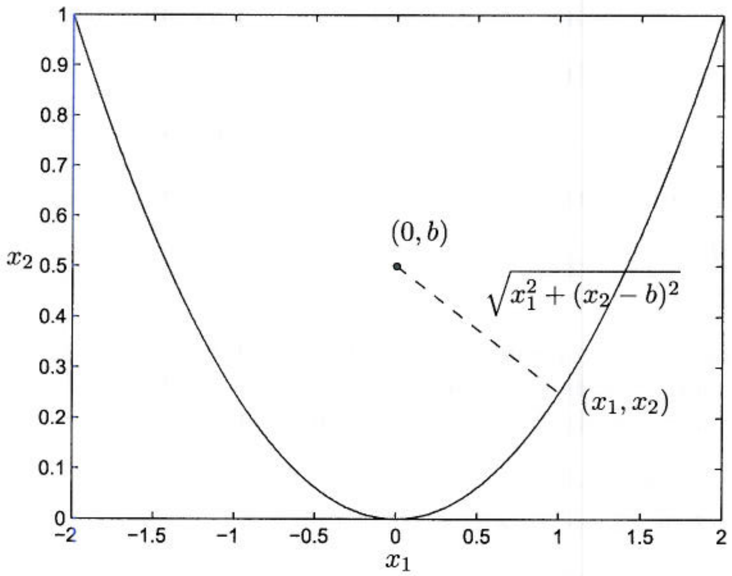
with k any non-negative integer, are local non-strict maximizers, and the points (x_1, x_2) such that

$$x_1^2 + x_2^2 = \frac{\pi}{2} + (2k + 1)\pi,$$

with k any non-negative integer, are local non-strict minimizers.

Question 2

a) Consider the graph in the figure below.



The distance between the point $(0, b)$ and the point (x_1, x_2) is given by

$$\sqrt{x_1^2 + (x_2 - b)^2},$$

with (x_1, x_2) such that $x_1^2 - 4x_2$, *i.e.* the point (x_1, x_2) belongs to the parabola. As a result, the problem of minimizing the distance from the point to the parabola can be cast as

$$\begin{cases} \min\limits_{x_1, x_2} \; x_1^2 + (x_2 - b)^2, \\ 4x_2 - x_1^2 = 0. \end{cases}$$

(Note that minimizing the distance is equivalent to minimizing the square of the distance.)

b) The constraint yields $x_1^2 = 4x_2$, which replaced in the cost function gives

$$4x_2 + (x_2 - b)^2.$$

The (global) minimum of this function is attained at $x_2 = b - 2$, yielding

$$x_1 = \pm 2\sqrt{b - 2}.$$

Clearly, for $b \geq 2$ the optimal solution of the optimization problem is attained at the points

$$(\pm 2\sqrt{b - 2}, b - 2).$$

(Note that there are two solutions because of the symmetry of the problem with respect to the x_2 variable.) For $b \in (0,2)$ the solutions above do not exist. This is however an incorrect conclusion since for any $b > 0$ it is possible to find a solution of the problem. Note that, for $b = 2$ there is only one optimal solution, namely the point $(0,0)$. At this point the constrain elimination procedure is not applicable, since the equation $4x_2 - x_1^2 = 0$ does not have a unique solution in x_1 in the neighborhood of it.

c) The Lagrangian of the problem is

$$L(x_1, x_2, \lambda) = x_1^2 + (x_2 - b)^2 + \lambda(4x_2 - x_1^2).$$

The first order necessary conditions of optimality are

$$0 = \frac{\partial L}{\partial x_1} = 2x_1 - 2\lambda x_1 \qquad 0 = \frac{\partial L}{\partial x_2} = 2x_2 - 2b + 4\lambda \qquad 0 = \frac{\partial L}{\partial \lambda} = 4x_2 - x_1^2$$

d) The first necessary condition of optimality yields $x_1 = 0$ or $\lambda = 1$. If $x_1 = 0$, then the last condition yields $x_2 = 0$ and $\lambda = b/2$. If $\lambda = 1$, then $x_2 = b-2$, and $x_1 = \pm 2\sqrt{b-2}$. In summary, the candidate optimal solutions are

- $(0,0)$, with $\lambda = b/2$, for all $b > 0$;
- $(\pm 2\sqrt{b-2}, b-2)$, with $\lambda = 1$, for all $b \geq 2$.

e) The solution of the optimization problem is

- $(0,0)$, if $b \in (0,2]$;
- $(\pm 2\sqrt{b-2}, b-2)$ for $b > 2$.

The optimal cost is

$$f^\star(b) = \begin{cases} b^2 & \text{if } b \in (0,2], \\ 4b - 4 & \text{if } b > 2, \end{cases}$$

Note that the optimal cost is continuous for all $b > 0$.

Question 3

a) Newton's method for the minimization of the function

$$f(x) = \frac{1}{2}r'(x)r(x)$$

is described by the iteration

$$x_{k+1} = x_k - [\nabla^2 f(x_k)]^{-1}\nabla f(x_k),$$

where

$$\nabla f = [\frac{\partial r}{\partial x}]' r = J'(x)r$$

(with $J = \frac{\partial r}{\partial x}$, as defined in the exam paper) and

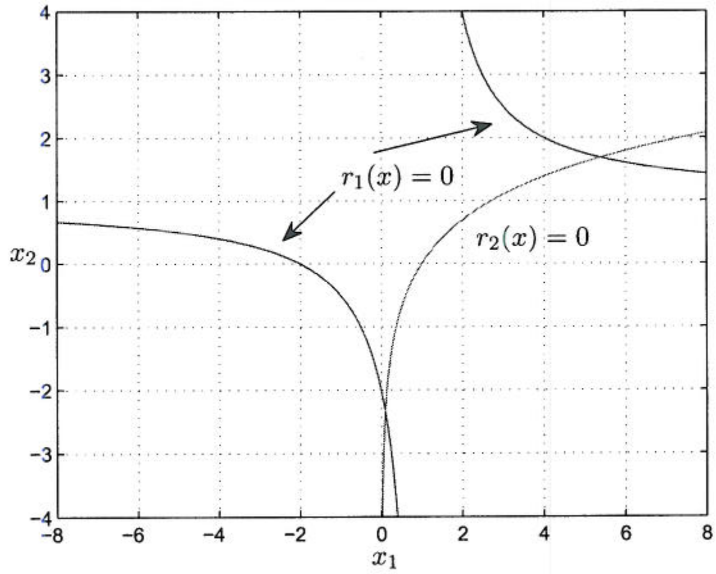
$$\nabla^2 f = [\frac{\partial r}{\partial x}]'[\frac{\partial r}{\partial x}] + \sum_{i=1}^{m} r_i \nabla^2 r_i.$$

b) The difference between Newton's method and Gauss-Newton's method is in the matrix that it is inverted. In Newton's method this is the Hessian of the function to be minimized, in Gauss-Newton's method this is one term of the Hessian, which can be computed using only first derivatives. Gauss-Newton's direction is a descent direction if

$$\nabla' f d_{GN} = -r'(x)J(x)[J'(x)J(x)]^{-1}J'(x)r(x) < 0,$$

which holds at all points in which $J(x)$ is full rank and $r(x) \neq 0$.

c) i) The sets $r_1(x) = r_2(x) = 0$ are displayed in the figure below. These sets have two intersections, hence the least square problem has only two solutions, which are both global minimizers of the function $\frac{1}{2}r'r$. From the graph, the minimizers are approximately given by the points $(0.1, -2.3)$ and $(5.4, 1.7)$.



ii) Note that

$$J(x) = \begin{bmatrix} 1 - x_2 & 1 - x_1 \\ 1 & -e^{x_2} \end{bmatrix}$$

and

$$d_{GN}(x) = -\frac{1}{(x_1 - 1) + e^{x_2}(x_2 - 1)} \begin{bmatrix} -x_2 e^{x_2} + x_1 x_2 e^{x_2} - 2 e x_2 x_1 - e^{x_2} + x_1^2 - x_1 \\ -e^{x_2} + x_2 e^{x_2} - 2 - x_2 \end{bmatrix}.$$

Hence, Gauss-Newton iteration can be written as

$$x_{k+1} = x_k + d_{GN}(x_k).$$

iii) Let $x_{(0)} = (0, 0)$. The residuals at $(0, 0)$ are $r_1(0) = 2$ and $r_2(0) = -1$. The first three elements of the sequence generated by Gauss-Newton's iteration are

$$x_{(1)} = (-0.5, -1.2), \qquad x_{(2)} = (0.0974, -2.1345), \qquad x_{(3)} = (0.096347, -2.319927),$$

and the value of the residuals after three iterations are

$$r_1(x_{(3)}) = -0.000061740, \qquad r_2(_{(3)}) = -0.001933.$$

iv) Note the fast convergence rate despite the fact that the iteration does not use second derivatives and a line search parameter.

Question 4

a) The considered problem can be posed as the optimization problem

$$\begin{cases} \max_{x,y} \; U(x,y), \\ p_1 x + p_2 y - B = 0. \end{cases}$$

b) The Lagrangian of the problem is (note the change in sign of U to transform the problem into a minimization problem)

$$L(x,y,\lambda) = -U(x,y) + \lambda(p_1 x + p_2 y - B).$$

The first order necessary conditions of optimality are

$$0 = \frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} + \lambda p_1, \qquad 0 = \frac{\partial L}{\partial y} = -\frac{\partial U}{\partial y} + \lambda p_2, \qquad 0 = \frac{\partial L}{\partial \lambda} = p_1 x + p_2 y - B.$$

Note that since $p_1 > 0$, $p_2 > 0$, $\frac{\partial U}{\partial x} > 0$, $\frac{\partial U}{\partial y} > 0$, then the first two necessary conditions of optimality may hold only for $\lambda > 0$.

c) i) If $U(x,y) = xy$, then the necessary conditions of optimality are

$$0 = -\frac{\partial L}{\partial x} = -y + \lambda p_1, \qquad 0 = -\frac{\partial L}{\partial y} = -x + \lambda p_2, \qquad 0 = \frac{\partial L}{\partial \lambda} = p_1 x + p_2 y - B.$$

Solving the above equations yields the only candidate solution

$$x^\star = \frac{1}{2}\frac{B}{p_1}, \qquad y^\star = \frac{1}{2}\frac{B}{p_2}, \qquad \lambda^\star = \frac{1}{2}\frac{B}{p_1 p_2}.$$

ii) Note that

$$\nabla^2 L = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

and

$$\frac{\partial g}{\partial(x,y)} = \begin{bmatrix} p_1 & p_2 \end{bmatrix}.$$

All nonzero vectors orthogonal to g can be written as

$$s = \alpha \begin{bmatrix} p_2 \\ -p_1 \end{bmatrix},$$

with $\alpha \neq 0$. As a result $s' \nabla^2 L s = 2\alpha^2 p_1 p_2 > 0$, which means that the candidate optimal point is indeed a solution of the optimization problem.

iii) The optimal cost is

$$U^\star = U\left(\frac{1}{2}\frac{B}{p_1}, \frac{1}{2}\frac{B}{p_2}\right) = \frac{1}{4}\frac{B^2}{p_1 p_2},$$

hence $\frac{\partial U^\star}{\partial B}$ equals the optimal multiplier $\lambda^\star$.

iv) The *comparative statistics* are

$$\frac{\partial x^\star}{\partial p_1} = -\frac{1}{2}\frac{B}{p_1^2}, \qquad \frac{\partial x^\star}{\partial p_2} = 0, \qquad \frac{\partial x^\star}{\partial B} = \frac{1}{2}\frac{1}{p_1},$$

$$\frac{\partial y^\star}{\partial p_1} = 0, \qquad \frac{\partial y^\star}{\partial p_2} = -\frac{1}{2}\frac{B}{p_2^2}, \quad \frac{\partial y^\star}{\partial B} = \frac{1}{2}\frac{1}{p_2}.$$

These *comparative statistics* can be interpreted as follows. An increase of the unit price of X yields a decrease of the optimal x, whereas it does not affect the optimal y. An increase of the unit price of Y yields a decrease of the optimal y, whereas it does not affect the optimal x. An increase of the budget yields an increase of the optimal x and y which is inversely proportional to the unit prices of the goods.

Question 5

a) The Lagrangian for the problem is

$$L(x_1, x_2, \rho) = x_1 + x_2 + \rho(x_1^2 + x_2^2 - 1).$$

The first order necessary conditions are

$$0 = \frac{\partial L}{\partial x_1} = 1 + 2\rho x_1, \qquad 0 = \frac{\partial L}{\partial x_2} = 1 + 2\rho x_2$$

$$x_1^2 + x_2^2 - 1 \le 0 \qquad \rho \ge 0 \qquad (x_1^2 + x_2^2 - 1)\rho = 0$$

b) The complementarity condition implies that either $\rho = 0$ or $x_1^2 + x_2^2 - 1 = 0$. The condition $\rho = 0$ does not give any candidate solution, whereas the condition $x_1^2+x_2^2-1 = 0$ yields the candidate optimal solution

$$x_1 = -\frac{\sqrt{2}}{2}, \qquad x_2 = -\frac{\sqrt{2}}{2}, \qquad \rho = \frac{\sqrt{2}}{2}.$$

c) i) The stationary points of the barrier functions are such that

$$0 = \frac{\partial B}{\partial x_1} = 1 + \frac{2}{\tau}\frac{x_1}{1 - x_1^2 - x_2^2}, \qquad 0 = \frac{\partial B}{\partial x_2} = 1 + \frac{2}{\tau}\frac{x_2}{1 - x_1^2 - x_2^2}.$$

The above equations imply that all stationary points are of the form $x = (\bar{x}, \bar{x})$, with $\bar{x}$ solution of

$$0 = 1 + \frac{2}{\tau}\frac{\bar{x}}{1 - 2\bar{x}^2}.$$

As a result, the stationary points are

$$P_1 = \left(\frac{1}{2}\frac{\sqrt{1+2\tau^2}}{\tau}, \frac{1}{2}\frac{\sqrt{1+2\tau^2}}{\tau}\right) \qquad P_2 = \left(\frac{1}{2}\frac{\sqrt{1-2\tau^2}}{\tau}, \frac{1}{2}\frac{\sqrt{1-2\tau^2}}{\tau}\right).$$

ii) Note that the point P_1 is outside the admissible set, since

$$(x_1^2 + x_2^2 - 1)_{P_1} = \frac{1 + \sqrt{1+2\tau^2}}{\tau^2} > 0,$$

for all $\tau > 0$, whereas the point P_2 is inside the admissible set, since

$$(x_1^2 + x_2^2 - 1)_{P_2} = \frac{1 - \sqrt{1+2\tau^2}}{\tau^2} < 0,$$

for all $\tau > 0$.

iii) Note that

$$\lim_{\tau\to\infty} P_2 = P_2(\infty) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right),$$

that is the point P_2 converges to the solution of the optimization problem as $\tau \to \infty$.

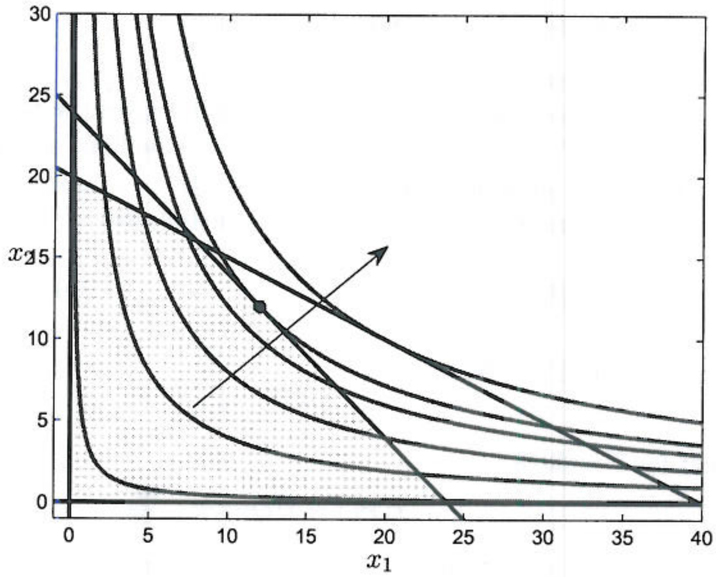
iv) To show that the point P_2 is a minimizer of the barrier function for all τ sufficiently large note that

$$\nabla^2 B(P_2) = \tau \begin{bmatrix} \dfrac{2\tau^2}{(\sqrt{1+2\tau^2}-1)^2} & 1 \\ 1 & \dfrac{2\tau^2}{(\sqrt{1+2\tau^2}-1)^2} \end{bmatrix},$$

which is positive definite for all $\tau > 0$.

Question 6

a) The admissible set and the level lines of the objective function are sketched in the figure below. Note that the function *decreases* in the direction of the arrow.



The solution of the optimization problem is at the point, denoted with a black mark, in which a level line is tangent to the admissible set.

b) The Lagrangian of the problem is

$$L(x_1, x_2, \rho_1, \rho_2, \rho_3, \rho_4) = -x_1x_2 + \rho_1(x_1 + 2x_2 - 40) + \rho_2(x_1 + x_2 - 24) + \rho_3(-x_1) + \rho_4(-x_2).$$

The first order necessary conditions of optimality are

$$0 = \frac{\partial L}{\partial x_1} = -x_2 + \rho_1 + \rho_2 - \rho_3 \qquad 0 = \frac{\partial L}{\partial x_2} = -x_1 + 2\rho_1 + \rho_2 - \rho_4$$

$$x_1 + 2x_2 - 40 \leq 0 \qquad x_1 + x_2 - 24 \leq 0 \qquad -x_1 \leq 0 \qquad -x_2 \leq 0 \qquad \rho_i \geq 0$$

$$\rho_1(x_1 + 2x_2 - 40) = 0 \qquad \rho_2(x_1 + x_2 - 24) = 0 \qquad \rho_3(-x_1) = 0 \qquad \rho_4(-x_2) = 0$$

c) Since the constraints $x_1 \geq 0$ and $x_2 \geq 0$ are assumed not active, then $\rho_3 = \rho_4 = 0$. We have therefore 4 possibilities.

- $\rho_1 = 0$ and $\rho_2 = 0$. The only candidate point is $(x_1, x_2) = (0, 0)$, which has to be discarded since we have assumed that the two constraints $x_1 \geq 0$ and $x_2 \geq 0$ are not active. Note that this point gives a (global, non-strict) maximizer, since the function is always non-negative in the admissible set.
- $\rho_1 = 0$ and $\rho_2 > 0$. The only candidate point is $(x_1, x_2) = (12, 12)$, with $\rho_2 = 12$.

- $\rho_1 > 0$ and $\rho_2 = 0$. These conditions yield $(x_1, x_2) = (20, 10)$, with $\rho_1 = 10$, which is not feasible.
- $\rho_1 > 0$ and $\rho_2 > 0$. These conditions yield $(x_1, x_2) = (8, 16)$, with $\rho_1 = -8$ and $\rho_2 = 24$, which does not satisfy the necessary conditions.

d) There is only one candidate solution, that is the point $(x_1, x_2) = (12, 12)$, which is therefore the solution of the optimization problem.

e) The first two constraints are active at the point $P_T = (x_1, x_2) = (2T-40, 40-T)$. Using this point in the necessary conditions of optimality gives the multipliers $\rho_1 = 3T - 80$ and $\rho_2 = 120 - 4T$. As a result, $T \in \left[\frac{80}{3}, 40\right]$. The modified problem has an additional candidate optimal solution, given by the point $Q_T = (x_1, x_2) = (T/2, T/2)$. The value of the objective function at the candidate optimal points is

$$f(P_T) = -(2T - 40)(40 - T) \qquad\qquad f(Q_T) = -\frac{1}{4}T^2.$$

Note that, for all $T \in \left[\frac{80}{3}, 40\right]$,

$$f(P_T) \geq f(Q_T),$$

and that for $T = 80/3$

$$f(P_T) = f(Q_T) = -\frac{40^2}{3^2}.$$

As a result, for $T = 80/3$ the optimal solution of the problem is at the point in which both constraints are active.