MATHEMATICS FOR SIGNALS AND SYSTEMS

1. Let $A \in \mathbb{R}^{n\times m}$ be a matrix with n rows and m columns whose entries are real numbers. We assume that $n \geq m$ and that A has rank m.

We assume that we know the QR-decomposition of the matrix A, i.e. there exist two matrices $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{n\times m}$ such that $A = QR$, Q is orthogonal ($Q^T Q = I$ where I is the identity matrix), and R is an upper triangular matrix (the upper $m\times m$ block of R is upper triangular and the rest of the entries of R are equal to zero). More precisely

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} & \dots & r_{1m} \\ 0 & r_{22} & r_{23} & \ddots & \vdots \\ \vdots & \ddots & r_{33} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & r_{mm} \\ 0 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

In this problem the goal is to devise an algorithm to derive the QR-decomposition of the matrix $\tilde{A} \in \mathbb{R}^{(n+1)\times m}$ obtained by adding a row to the matrix A. More precisely,

$$\tilde{A} = \begin{pmatrix} A_1 \\ z^T \\ A_2 \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

with $A_1 \in \mathbb{R}^{n_1\times m}$, $A_2 \in \mathbb{R}^{n_2\times m}$, $n_1 + n_2 = n$, and $z \in \mathbb{R}^m$.

a) Let us also assume that

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R,$$

where Q_1 and Q_2 have the same number of rows as A_1 and A_2 respectively.

i) Show that

$$\tilde{A} = \begin{pmatrix} \mathbf{0} & Q_1 \\ 1 & \mathbf{0}^T \\ \mathbf{0} & Q_2 \end{pmatrix} \begin{pmatrix} z^T \\ R \end{pmatrix}, \tag{1.1}$$

where $\mathbf{0}$ represent zero vectors of the appropriate dimensions. [2]

ii) Show that $\hat{Q} = \begin{pmatrix} \mathbf{0} & Q_1 \\ 1 & \mathbf{0}^T \\ \mathbf{0} & Q_2 \end{pmatrix}$ is an orthogonal matrix. [3]

iii) Is (1.1) a QR decomposition? Justify your answer. [2]

b) We define the matrix U_1 as follows, for $\theta \in \mathbb{R}$

$$U_1 = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 & \dots & 0 \\ \sin(\theta) & \cos(\theta) & 0 & \ddots & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

i) Show that U_1 is orthogonal. [1]

ii) Find a value of θ such that $U_1^T \begin{pmatrix} z^T \\ R \end{pmatrix}$ has 0 in the entry on its second row and first column. [2]

iii) More generally show that there is a sequence of orthogonal matrices $U_1, \dots, U_m$ such that $\tilde{R} = U_m^T \dots U_1^T \begin{pmatrix} z^T \\ R \end{pmatrix} \in \mathbb{R}^{(n+1)\times m}$ is upper triangular. [2]

iv) Show that $\tilde{Q} = \hat{Q} U_1 \dots U_m$ is an orthogonal matrix. [2]

v) Describe an algorithm that updates the QR of a matrix if a row is added to it. [2]

vi) Is the procedure described in b) v) more efficient than performing the QR decomposition of $\tilde{A}$ from scratch, i.e. without relying on an update of the QR decomposition of A? Justify your answer carefully. [4]

2. Define $||x|| = \sqrt{x^T x}$.

a) Given a vector x and an orthogonal projection P, i.e. $P^2 = P$ and $P^T = P$.

i) Show that Px and $x - Px$ are orthogonal. [2]

ii) Show that $||x||^2 = ||Px||^2 + ||x - Px||^2$. [2]

b) Let U a subspace of $\mathbb{R}^d$ of dimension k and let $u_1, \ldots, u_k$ an orthonormal basis of U, i.e. $u_i^T u_i = 1$, and for all $i \neq j$ $u_i^T u_j = 0$. Consider the matrix

$$P_U = \sum_{i=1}^{k} u_i u_i^T = u_1 u_1^T + u_2 u_2^T + \cdots + u_k u_k^T$$

i) Show that P_U is an orthogonal projection and derive its range and null-space. [3]

ii) Let $a_1, \ldots, a_n$ be n vectors in $\mathbb{R}^d$. Show that

$$\sum_{j=1}^{n} ||P_U a_j||^2 = \sum_{j=1}^{n} \sum_{i=1}^{k} (u_i^T a_j)^2.$$

Hint: start by showing that for a given vector $x \in \mathbb{R}^d$ we have that $||P_U x||^2 = \sum_{i=1}^{k} (u_i^T x)^2$. [2]

c) We say that a subspace U of dimension k is the *best-fit k-dimensional subspace* if it maximizes the sum of the squared lengths of the orthogonal projections onto it, i.e. the subspace U such

$$\sum_{j=1}^{n} ||P_U a_j||^2 = \max_{\substack{\text{V subspace of } \mathbb{R}^d \\ dim(V)=k}} \sum_{i=1}^{n} ||P_V a_i||^2 .$$

Using questions a.ii show that the subset U that maximize the sum of the squared lengths of the projections onto the subspace does also minimize the sum of squared distances to the subspace. [2]

d) Let $A \in \mathbb{R}^{n \times n}$. In what follows we will describe a procedure for deriving the singular vectors of A. Let $a_1^T, \ldots a_n^T$ the vectors representing the rows of A (a_i^T being the ith row of A).

i) Show that for any vector $v \in \mathbb{R}^n$ we have

$$\sum_{i=1}^{n} (a_i^T v)^2 = ||Av||^2.$$

[2]

ii) Let $v_1 \in \mathbb{R}^n$ be such that $Av_1 = \max_{||v||=1} ||Av||$. Using c and d.i, show that for all $v \in \mathbb{R}^n$

$$\sum_{i=1}^{n} ||a_i - (v^T a_i)v|| \geq \sum_{i=1}^{n} ||a_i - (v_1^T a_i)v_1||.$$

[2]

e) We now define a greedy procedure for deriving the singular vectors of A. Recall that

$$Av_1 = \max_{||v||=1} ||Av||.$$

Let $v_2 \in \mathbb{R}^n$ such that

$$Av_2 = \max_{||v||=1, v^T v_1 = 0} ||Av||.$$

Similarly lt $v_3 \in \mathbb{R}^n$ such that

$$Av_3 = \max_{||v||=1,\, v^T v_1 = 0,\, v^T v_2 = 0} ||Av||.$$

We stop the process when we found vectors $v_1, v_2 \ldots, v_r$ as singular vectors and

$$\max_{||v||=1, v^T v_1 = 0, \ldots, v^T v_r = 0} ||Av|| = 0.$$

Using an induction show that the subspace Span$\{v_1, \ldots, v_r\}$ spanned by $\{v_1, \ldots, v_r\}$ is the best-fit k-dimensional subspace for the vectors $a_1, \ldots a_n$. [5]

3. Let $\mathbb{R}[X]$ be the vector space of polynomials with real coefficients, and $\mathbb{R}_n[X]$ be the subspace of polynomials with degree smaller or equal to n. For P and Q in $\mathbb{R}[X]$, we define

$$\langle P,Q\rangle = \int_{-1}^{1} P(x)Q(x)\frac{1}{\sqrt{1-x^2}}dx\,.$$

a) Define $T_k(x)$ the polynomials such that, for $k \geq 1$ and $\theta \in (0,\pi)$

$$T_k(\cos(\theta)) = \cos(k\theta)\,, \quad T_0 = 1\,,$$

known as Chebyshev's polynomials.

i) Give the expressions of T_1, T_2 and T_3. [3]

ii) Show that, for $k \geq 1$, we have

$$T_{k+1} = 2XT_k - T_{k-1}\,.$$

[3]

iii) Using the change of variable $\theta = \arccos(x)$, compute $\langle T_n, T_m\rangle$, when $n = m$ and $n \neq m$. [3]

iv) Derive an orthonormal basis for $\mathbb{R}_3[X]$. Justify your answer. [2]

b) Consider the application on $\mathbb{R}_n[X]$ defined by

$$D(P) = (1-X^2)P'' - XP'$$

where $P \in \mathbb{R}_n[X]$, P' and P'' are its first and second derivatives respectively.

i) Using that the fact that $T_k(\cos(\theta)) = \cos(k\theta)$, show that

$$-\cos(\theta)T_k'(\cos(\theta)) + \sin(\theta)^2 T_k''(\cos(\theta)) = -k^2 T_k(\cos(\theta))\,.$$

[3]

ii) Show that $D(T_k) = -k^2 T_k$. [3]

iii) Derive the eigenvalues and eigenvectors of the transformation D on $\mathbb{R}_n[X]$. [3]