Master

E4.54

CS4.1

ISE4.61

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EXAMINATIONS 2009

MSc and EEE/ISE PART IV: MEng and ACGI

PREDICTIVE CONTROL

Monday, 18 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : E.C. Kerrigan

Second Marker(s) : A. Astolfi

PREDICTIVE CONTROL 2009

1. Consider the following finite-horizon discrete-time optimal control problem:

$$\min_{u_0,u_1,\ldots,u_{N-1}} \sum_{k=0}^{N-1} \|Qx_k + Ru_k\|_\infty$$

where the system dynamics are given by

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0,1,\ldots,N-1,$$

where the state $x_k \in \mathbb{R}^n$, input $u_k \in \mathbb{R}^m$ and weighting matrices $Q \in \mathbb{R}^{p\times n}$ and $R \in \mathbb{R}^{p\times m}$.

a) Can the solution to the above problem be found by differentiating the cost function and setting it to zero? Motivate your answer. [2]

b) Show that the above problem can be solved by setting up and solving a linear program (LP) of the form

$$\min_{\theta} h'\theta$$

subject to the constraints

$$L\theta \leq s.$$

In other words, derive expressions for θ, h, s and L such that the solution of the optimal control problem is easily found from the solution of the LP. [14]

c) What are the sizes of the vectors θ, h, s and the matrix L in terms of N, m, n and p? [4]

2. Consider the following discrete-time system:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k, \end{aligned}$$

where the state $x_k \in \mathbb{R}^n$, input $u_k \in \mathbb{R}^m$ and output $y_k \in \mathbb{R}^p$.

The following constraints are imposed on the inputs and outputs:

$$\begin{aligned} u_\ell \leq u_k \leq u_h, \\ y_\ell \leq y_k \leq y_h. \end{aligned}$$

In addition, it is required that the output track a constant reference $r \in \mathbb{R}^p$ in steady-state.

a) Which equality and inequality constraints need to be satisfied for an equilibrium state-input pair (x_∞, u_∞) to exist such that the output at the equilibrium satisfies the above inequality constraints while ensuring that the output is equal to the reference? [4]

b) Show that the problem of determining whether or not the constraints in part a) can be satisfied can be solved by setting up and solving a linear program (LP) of the form

$$f := \min_{\theta} h'\theta$$

subject to the constraints

$$\begin{aligned} L\theta &\leq s, \\ M\theta &= t. \end{aligned}$$

In other words, derive expressions for θ, h, s, t, L and M such that the constraints in part a) are satisfied if and only if $f = 0$. [10]

c) What are the sizes of θ, h, s, t, L and M in terms of m, n and p? [6]

3. We are interested in solving the following optimal control problem :

$$\min_{u_0,\ldots,u_{N-1}} \|Px_N\|_2^2 + \sum_{k=0}^{N-1} \|Qy_k + Ru_k\|_2^2$$

subject to the constraints

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \quad k = 0,1,\ldots,N-1 \\ y_k &= Cx_k, \quad k = 0,1,\ldots,N \\ u_\ell \leq u_k &\leq u_h, \quad k = 0,1,\ldots,N-1 \\ y_\ell \leq y_k &\leq y_h, \quad k = 1,2,\ldots,N \end{aligned}$$

where the state $x_k \in \mathbb{R}^n$, input $u_k \in \mathbb{R}^m$, output $y_k \in \mathbb{R}^p$ and weights $P \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{q\times p}$ and $R \in \mathbb{R}^{q\times m}$ are such that $Q'R = 0$.

a) If we define the decision variable

$$\theta := \begin{pmatrix} u_0' & x_1' & u_1' & x_2' & u_2' & \cdots & x_{N-1}' & u_{N-1}' & x_N' \end{pmatrix}',$$

show that the above problem can be solved by setting up and solving a quadratic program (QP) of the form

$$\min_\theta \theta' H \theta + h'\theta$$

subject to the constraints

$$\begin{aligned} L\theta &\leq s, \\ M\theta &= t. \end{aligned}$$

In other words, derive expressions for h, s, t, H, L and M such that the solution of the optimal control problem is easily found from the solution of the QP. [10]

b) What are the sizes of θ, h, s, t, H, L and M in terms of N, m, n, p and q? [7]

c) Give a sufficient condition for which one can guarantee that the optimal input sequence to the above optimal control problem is unique, if a solution exists. Justify your answer. [3]

4. a) List a number of potential advantages and disadvantages of predictive control, compared to some of the other control synthesis methods you may have encountered in your studies so far. [4]

b) Give a description and graphical illustration of the receding horizon principle. [6]

c) Consider the following finite horizon optimal control problem:

$$\min_{u_0,u_1,\ldots,u_{N-1}} \quad \|Px_N\|_2^2 + \sum_{k=0}^{N-1} \left(\|Qx_k\|_2^2 + \|Ru_k\|_2^2\right)$$

where the system dynamics are given by

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0,1,\ldots,N-1,$$

the state $x_k \in \mathbb{R}^n$, input $u_k \in \mathbb{R}^m$ and the weights $P \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are all full column rank.

Prove that there exists a reachable or controllable system (A,B) and a choice of N, P, Q and R such that the system in closed-loop with the receding horizon control law, derived from the solution to the above control problem, is unstable. [10]

5. a) Consider the quadratic form

$$\ell(z,u) := z'Qz + u'Ru + 2u'Sz,$$

where R is positive definite. Show that $\ell(z,u) \geq 0$ for all (z,u) if and only if $Q - S'R^{-1}S$ is positive semidefinite.

Hint: You may wish to use the fact that $\ell(z,u) \geq 0$ for all (z,u) if and only if the function $L(z) := \min_u \ell(z,u) \geq 0$ for all z. [6]

b) A popular cost function in predictive control applications is the following:

$$J := y_N'My_N + \sum_{k=0}^{N-1} \left(y_k'My_k + u_k'Vu_k + (\Delta u_k)'W\Delta u_k \right),$$

where M, V and W are symmetric matrices, the discrete-time dynamics are given by

$$x_{k+1} = Ax_k + Bu_k, \qquad y_k = Cx_k,$$

and the change in control input at time k is defined as

$$\Delta u_k := u_k - u_{k-1}.$$

Show that, by defining the augmented state vector

$$z_k := (x_k' \ u_{k-1}')',$$

one can rewrite the cost function in the form

$$J = z_N'Qz_N + \sum_{k=0}^{N-1} \left(z_k'Qz_k + u_k'Ru_k + 2u_k'Sz_k \right)$$

where the augmented discrete-time dynamics are given by

$$z_{k+1} = \bar{A}z_k + \bar{B}u_k$$

with $\bar{A}$, $\bar{B}$, Q, R and S suitably defined. Please write out explicit expressions for $\bar{A}$, $\bar{B}$, Q, R and S. [12]

c) Give sufficient conditions on M, V and W such that R is positive definite and $Q - S'R^{-1}S$ is positive semi-definite, with Q, R and S as in part b). [2]

6. We are interested in solving the following optimal control problem :

$$\min_{u_0,\ldots,u_{N-1}} \|Px_N\|_2^2 + \sum_{k=0}^{N-1} \left(\|Qx_k\|_2^2 + \|Ru_k\|_2^2\right)$$

subject to the constraints

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \quad k = 0,1,\ldots,N-1 \\ y_k &= Cx_k, \quad k = 0,1,\ldots,N \\ u_\ell \leq u_k &\leq u_h, \quad k = 0,1,\ldots,N-1 \\ y_\ell \leq y_k &\leq y_h, \quad k = 1,2,\ldots,N \end{aligned}$$

where the state $x_k \in \mathbb{R}^n$, input $u_k \in \mathbb{R}^m$, output $y_k \in \mathbb{R}^p$ and the weights $P \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$.

a) What is meant with 'hard' and 'soft' constraints? [2]

b) Why might one need to distinguish been hard and soft constraints? In other words, why is it not possible to keep all constraints hard or soft? [2]

c) Give an example of a hard constraint and an example of a soft constraints. [2]

d) Suppose that there is a constant, unmeasurable output disturbance d_k, i.e.

$$y_k = Cx_k + d_k$$

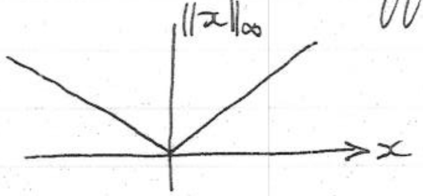
where $d_{k+1} = d_k$. Derive necessary and sufficient conditions that will enable one to construct an observer to estimate the state and disturbance. [8]

e) Suppose now that you have an estimate of the current state of the system $\hat{x}$ and an estimate of the value of disturbance $\hat{d}$. Suppose the disturbance is such that it is not possible to compute an input sequence that would ensure all the constraints are satisfied over the control horizon.

How would you soften the output constraints and compute an input sequence that would minimize the worst-case output constraint violation? You need only state the optimal control problem that you would solve — please do *not* transform it into an equivalent LP or QP. [6]

Question 1 (Application of theory to new problem)

(a) No. The cost function is not differentiable, e.g.



[2]

(b) Let $\bar{u} := \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} \Rightarrow \bar{x} := \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{pmatrix} = \underbrace{\begin{pmatrix} I \\ A \\ A^2 \\ \vdots \\ A^{N-1} \end{pmatrix}}_{\Phi} x_0 + \underbrace{\begin{pmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & B & & \vdots \\ \vdots & & \ddots & \\ A^{N-2}B & \cdots & & B \end{pmatrix}}_{\Gamma} \bar{u}$

$$\Rightarrow \bar{x} = \Phi x_0 + \Gamma \bar{u}$$

Note that $\sum_{k=0}^{N-1} \|Qx_k + Ru_k\|_\infty = \min_{\varepsilon} 1_N' \varepsilon$

s.t. $\|Qx_k + Ru_k\|_\infty \leq \varepsilon_k, \; k=0,\ldots,N-1$

$$= \min_{\varepsilon} 1_N' \varepsilon \text{ s.t. } -1_p \varepsilon_k \leq Qx_k + Ru_k \leq 1_p \varepsilon_k, \; k=0,\ldots,N-1$$

where $\varepsilon := \begin{pmatrix} \varepsilon_0 \\ \varepsilon_1 \\ \vdots \\ \varepsilon_{N-1} \end{pmatrix}$ and $1_p := \left.\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}\right\}$ of size p.

The inequality constraints can be written as:

$$\begin{aligned}
Qx_0 + Ru_0 - 1_p \varepsilon_0 &\leq 0 \\
-Qx_0 - Ru_0 - 1_p \varepsilon_0 &\leq 0 \\
Qx_1 + Ru_1 - 1_p \varepsilon_1 &\leq 0 \\
-Qx_1 - Ru_1 - 1_p \varepsilon_1 &\leq 0 \\
\vdots \quad & \quad \vdots \\
Qx_{N-1} + Ru_{N-1} - 1_p \varepsilon_{N-1} &\leq 0 \\
-Qx_{N-1} - Ru_{N-1} - 1_p \varepsilon_{N-1} &\leq 0
\end{aligned}$$

Question 1 (contd.)

$$I_N := \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{pmatrix} \quad \begin{pmatrix} N \times N \text{ identity} \\ \text{matrix} \end{pmatrix}$$ (2)

or, equivalently,

$$\underbrace{\begin{pmatrix} I_N \otimes Q \\ -I_N \otimes Q \end{pmatrix}}_{\bar{Q}} \bar{x} + \underbrace{\begin{pmatrix} I_N \otimes R \\ -I_N \otimes R \end{pmatrix}}_{\bar{R}} \bar{u} + \underbrace{(-I_N \otimes 1_{2p})}_{\bar{S}} \varepsilon \leq 0$$

$$\iff \bar{Q}\left(\Phi x_0 + \Gamma \bar{u}\right) + \bar{R}\,\bar{u} + \bar{S}\,\varepsilon \leq 0$$

$$\iff \begin{pmatrix} \bar{Q}\Gamma + \bar{R} & \bar{S} \end{pmatrix} \begin{pmatrix} \bar{u} \\ \varepsilon \end{pmatrix} \leq -\bar{Q}\,\Phi\, x_0$$

$$\Rightarrow \min_{\bar{u}} \sum_{k=0}^{N-1} \| Q x_k + R u_k \|_\infty$$

$$= \min_{(\bar{u},\varepsilon)} \underbrace{\begin{pmatrix} 0 & 1_N' \end{pmatrix}}_{h'} \underbrace{\begin{pmatrix} \bar{u} \\ \varepsilon \end{pmatrix}}_{\theta} \text{ s.t. } \underbrace{\begin{pmatrix} \bar{Q}\Gamma + \bar{R} & \bar{S} \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} \bar{u} \\ \varepsilon \end{pmatrix}}_{\theta} \leq \underbrace{-\bar{Q}\,\Phi\, x_0}_{s}$$

$\Longrightarrow$ [14]

(c) h is a vector of length $Nm + N$
θ " " " " $Nm + N$
s " " " " $2pN$
L " " matrix with $2pN$ rows and $N(m+1)$ columns.

(note: none of the sizes are dependent on n) [4]

$\Longrightarrow$ QED.

Question 2 (Application of theory to new problem)

(a)
$$\begin{cases} x_\infty = Ax_\infty + Bu_\infty \iff (I-A)x_\infty - Bu_\infty = 0 \\ Cx_\infty = r \\ u_l \le u_\infty \le u_h \\ y_l \le Cx_\infty \le y_h \end{cases}$$

[4]

~~(b)~~ $\cancel{\begin{pmatrix} I-A & -B \\ C & 0 \end{pmatrix}\begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}}$

(b) Equality constraints equivalent to $\begin{pmatrix} I-A & -B \\ C & 0 \end{pmatrix}\begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}$

Inequality constraints equivalent to $\begin{pmatrix} C & 0 \\ -C & 0 \\ 0 & I \\ 0 & -I \end{pmatrix}\begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} \le \begin{pmatrix} y_h \\ -y_l \\ u_h \\ -u_l \end{pmatrix}$

To find a feasible solution, we seek to minimise the ε violation of the inequality constraints, i.e.

$f = \min\limits_{x_\infty, u_\infty, \varepsilon} \varepsilon$ s.t. Equality constraints above and

$$\begin{pmatrix} C & 0 \\ -C & 0 \\ 0 & I_m \\ 0 & -I_m \end{pmatrix}\begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} \le \begin{pmatrix} y_h \\ -y_l \\ u_h \\ -u_l \end{pmatrix} + \begin{pmatrix} 1_p \\ 1_p \\ 1_m \\ 1_m \end{pmatrix} \varepsilon$$

$\varepsilon \ge 0$, where $\varepsilon \in \mathbb{R}$

note!!

~~which is equivalent.~~

where ~~each~~ $1_p := \left.\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\right\}$ of size p.

$I_m :=$ $m \times m$ identity matrix

equivalently,

$$f = \min_{(x_\infty, u_\infty, \varepsilon)} \underbrace{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}}_{h}{}' \underbrace{\begin{pmatrix} x_\infty \\ u_\infty \\ \varepsilon \end{pmatrix}}_{\theta}$$

s.t.
$$\underbrace{\begin{pmatrix} C & 0 & -1_p \\ -C & 0 & -1_p \\ 0 & I_m & -1_m \\ 0 & -I_m & -1_m \\ 0 & 0 & -1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} x_\infty \\ u_\infty \\ \varepsilon \end{pmatrix}}_{\theta} \leq \underbrace{\begin{pmatrix} y_h \\ -y_l \\ u_h \\ -u_l \\ 0 \end{pmatrix}}_{s}$$

$$\underbrace{\begin{pmatrix} I-A & -B & 0 \\ C & 0 & 0 \end{pmatrix}}_{M} \begin{pmatrix} x_\infty \\ u_\infty \\ \varepsilon \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ r \end{pmatrix}}_{t}$$

$\Longrightarrow$ [10]

(c) θ and h are vectors of length $n+m+1$ ✓✓
s is a vector of length $2p+2m+1$ ✓
t " " " " $n+p$ ✓
L is a matrix with $2p+2m+1$ rows and $n+m+1$ columns ✓
M " " " " $n+p$ " " $n+m+1$ " ✓

[6]

QED.

Question 3 (Application of Theory to new problem)

(a) The cost function is equivalent to

$$x_N' P' P x_N + \sum_{k=0}^{N-1} u_k' R' R u_k + x_k' C' Q' Q C x_k + \underbrace{2 u_k' R' Q C x_k}_{=0}$$

$$= \begin{pmatrix} u_0 \\ x_1 \\ u_1 \\ x_1 \\ \vdots \\ u_{N-1} \\ x_N \end{pmatrix}' \underbrace{\begin{pmatrix} R'R & & & & & & 0 \\ & C'Q'QC & & & & & \\ & & R'R & & & & \\ & & & \ddots & & & \\ & 0 & & & C'Q'QC & & \\ & & & & & R'R & \\ & & & & & & P'P \end{pmatrix}}_{H} \begin{pmatrix} u_0 \\ x_1 \\ u_1 \\ \vdots \\ u_{N-1} \\ x_N \end{pmatrix}$$

$+ \ x_0' C' Q' Q x_0$ — constant — can be omitted

$\Rightarrow \ h = 0$

The equality constraints are equivalent to:

$$\begin{aligned}
-Bu_0 + x_1 \qquad\qquad\qquad\qquad & = Ax_0 \\
-Ax_1 - Bu_1 + x_2 \qquad\qquad & = 0 \\
-Ax_2 - Bu_2 + x_3 \qquad & = 0 \\
\ddots \quad x_{N-1} & = 0 \\
-Ax_{N-1} - Bu_{N-1} + x_N & = 0
\end{aligned}$$

note.

~~equiv~~ which then is equivalent to:

$$\underbrace{\begin{pmatrix} -B & I & 0 & \cdots & - & - & - & & 0 \\ 0 & -A & -B & I & 0 & & - & \cdots & 0 \\ 0 & 0 & 0 & -A & -B & I & 0 & \cdots & 0 \\ \vdots & & & & & \ddots & & & \\ & & & & & & I & 0 & 0 \\ 0 & - & \cdots & & & & -A & -B & I \end{pmatrix}}_{M} \begin{pmatrix} u_0 \\ x_1 \\ u_1 \\ \vdots \\ x_{N-1} \\ u_{N-1} \\ x_{N-1} \end{pmatrix} = \underbrace{\begin{pmatrix} Ax_0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}}_{t}$$

Question 3 (contd.)

The inequality constraints are equivalent to:

$$\left.\begin{array}{rl} u_0 & \le u_h \\ -u_0 & \le -u_l \\ Cx_1 & \le y_h \\ -Cx_1 & \le -y_l \\ u_1 & \le u_h \\ -u_1 & \le -u_l \\ & \vdots \\ Cx_N & \le y_h \\ -Cx_N & \le y_l \end{array}\right\} \Longleftrightarrow \underbrace{\begin{pmatrix} I_m & 0 & 0 & \cdots & 0 \\ -I_m & 0 & 0 & & \\ 0 & C & 0 & & \\ 0 & -C & 0 & 0 & \\ 0 & 0 & I_m & & \\ 0 & 0 & -I_m & & \\ \vdots & & & \ddots & \\ & 0 & & & C \\ 0 & & & & -C \end{pmatrix}}_{L} \theta \le \underbrace{\begin{pmatrix} u_h \\ -u_l \\ y_h \\ -y_l \\ u_h \\ -u_l \\ \vdots \\ y_h \\ -y_l \end{pmatrix}}_{s}$$

[10]

(b) θ and h are column vectors of ~~s~~ length $N(m+n)$ ✓✓

s is a vector of length $2Nm + 2Np = 2N(m+p)$ ✓

t " " " " " Nn ✓

L " " matrix with $2N(m+p)$ rows and $N(m+n)$ columns ✓

M " " " " Nn rows " $N(m+n)$ " ✓

H " " " " $N(m+n)$ " " $N(m+n)$ " ✓

[7]

(c) If R ~~needs to be~~ is full column rank. Note that we do not require H to be positive definite. ($R'R > 0$ ~~⇔~~)

One can turn the optimization problem into an equivalent ~~Q problem~~ QP, but where the decision variables are only over the inputs (i.e. the equality constraints can be explicitly ~~removed~~ solved), i.e. a QP with inequality constraints only (no equality)

This was done during lectures, where ~~is~~ it is then easy to show that the Hessian (of the cost) is positive definite if R is full column rank. $\Rightarrow$ QP has a unique ~~~~ optimal input sequence, if a solution exists.

[3]

Question 4

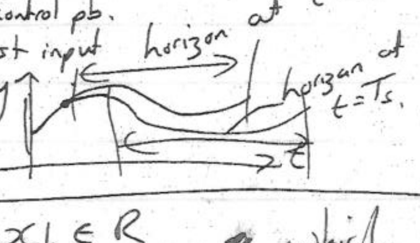
	Advantages	Disadvantages
	Constraints	Comp. expensive
	Nonlinearities	Difficult to analyse (nonlinear control law)

a) Bookwork [4]

b) Bookwork [6]

1. Sample.
2. Solve optimal control pb.
3. Implement first input
4. Go to 1

Keep horizon constant.



c) ~~Bookwork~~ Done during lectures.

Let $N=1$ and $x_{k+1} = a x_k + u_k$, $u_k, x_k \in \mathbb{R}$, which is clearly reachable.

Since solution depends on ratio of Q to R, set $R=1$ and $Q \neq 0$, $\rho \neq 0$ — full column rank condition is satisfied.

problem becomes

$$\Rightarrow \min_{u_0} \; \rho^2 x_1^2 + Q^2 x_0^2 + u_0^2$$

$$= \min_{u_0} \underbrace{\rho^2 (a x_0 + u_0)^2 + Q^2 x_0^2 + u_0^2}_{V}$$

$$\frac{\partial V}{\partial u_0} = 2u_0 + 2\rho^2 (a x_0 + u_0) = 0$$

$$\Leftrightarrow \quad (1+\rho^2) u_0 = -a\rho^2 x_0$$

$$\Leftrightarrow \quad u_0 = \underbrace{\frac{-a\rho^2}{1+\rho^2}}_{K_{rhc}} x_0$$

$$\Rightarrow \text{closed-loop is } x_{k+1} = \frac{a(1+\rho^2) - a\rho^2}{1+\rho^2} x_k$$

$$= \frac{a}{1+\rho^2} x_k$$

which is unstable if $\left| \frac{a}{1+\rho^2} \right| > 1 \Leftrightarrow |a| > |1+\rho^2|$

QED. [10]

8

Question 5

(a) Application of theory. ~~R>~~ $\frac{\partial l}{\partial u} = 2Ru + 2S'z = 0 \Leftrightarrow u = -R^{-1}S'z$ $\because R>0$

$$\Rightarrow L(z) = z'Qz + z'SR^{-1}RR^{-1}S'z - 2z'SR^{-1}S'z$$
$$= z'(Q - SR^{-1}S')z$$

$$L(z) \geq 0 \Leftarrow Q - SR^{-1}S' \geq 0$$ [6]

(b) (In prescribed textbook):

$$z_{k+1} = \begin{pmatrix} x_{k+1} \\ u_k \end{pmatrix} = \begin{pmatrix} Ax_k + Bu_k \\ u_k \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_k \\ u_{k-1} \end{pmatrix} + \begin{pmatrix} B \\ I \end{pmatrix} u_k$$

$= \bar{A} z_k + \bar{B} u_k$, where $\underline{\bar{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}}$, $\underline{\bar{B} = \begin{pmatrix} B \\ I \end{pmatrix}}$

$$\Rightarrow y_k = Cx_k = (C \;\; 0)\, z_k.$$

(Application of ~~new~~ theory): ~~*~~

The stage cost can be written as

$$z_k' \begin{pmatrix} C' \\ 0 \end{pmatrix} M (C \;\; 0) z_k + u_k' V u_k + u_k' W u_k + u_{k-1}' W u_{k-1} - 2 u_k' W u_{k-1}$$

$$= z_k' \begin{pmatrix} C'MC & 0 \\ 0 & W \end{pmatrix} z_k + u_k'(V+W)u_k + 2u_k'(0 \;\; -W)\begin{pmatrix} x_k \\ u_{k-1} \end{pmatrix}$$

$$\Rightarrow \underline{Q = \begin{pmatrix} C'MC & 0 \\ 0 & W \end{pmatrix}}, \;\; \underline{R = V + W}, \;\; \underline{S = (0 \;\; -W)}$$

[12]

(c) $V \geq 0$, $W \geq 0$ and either $V > 0$ or $W > 0$

$\Rightarrow R > 0$.

If, in addition $M \geq 0 \Rightarrow$ stage cost is positive definite

$\Rightarrow Q - S'R^{-1}S \geq 0$ [2]

(application of theory)

QED

Question 6

(a) (Bookwork.) Hard constraints are / cannot be violated, whereas soft constraints can / are allowed to be violated. [2].

(b) (Bookwork). Hard constraints may represent actual, physical constraints whereas soft constraints are often performance constraints that may be violated for periods of time if a disturbance comes along. [2]

(c) (Bookwork).
Hard: Rudder limits on an aeroplane
Soft: Temperature limits in a room [2].

(d) Form the augmented system:

$$\begin{pmatrix} x_{k+1} \\ d_{k+1} \end{pmatrix} = \underbrace{\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}}_{\tilde{A}} \begin{pmatrix} x_k \\ d_k \end{pmatrix} + \underbrace{\begin{pmatrix} B \\ 0 \end{pmatrix}}_{\tilde{B}} u_k .$$

$$y_k = \underbrace{\begin{pmatrix} C & I \end{pmatrix}}_{\tilde{C}} \begin{pmatrix} x_k \\ d_k \end{pmatrix}$$

Can construct an observer $\iff (\tilde{C}, \tilde{A})$ detectable.

From Hautus test: $(\tilde{C}, \tilde{A})$ detectable

$$\iff \begin{pmatrix} C & I \\ \lambda I - A & 0 \\ 0 & \lambda I - I \end{pmatrix} \text{ full column rank } \forall \lambda .$$

~~The first columns~~ From the Hautus test the first n columns are ~~full column rank~~ linearly independent $\iff (C, A)$ detectable.
The second set of colums are linearly independent from the first n colums except possibly for $\lambda = 1$.

If $\lambda = 1 \Rightarrow \begin{pmatrix} C & I \\ I-A & 0 \\ 0 & 0 \end{pmatrix}$ full column rank

$\iff \begin{pmatrix} C & I \\ I-A & 0 \end{pmatrix}$ full colum rank

$\Rightarrow (\tilde{C}, \tilde{A})$ detectable if and only if (C, A) detectable and $\begin{pmatrix} I-A & 0 \\ C & I \end{pmatrix}$ full column rank.

[8]

(e) $\displaystyle \min_{u_0, u_1, .., u_{N-1}, \varepsilon} \quad \varepsilon$

subject to:

$$x_0 = \hat{x}$$
$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, .., N-1$$
$$u_l \le u_k \le u_h \quad k = 0, 1, .., N-1$$
$$y_l - 1_p \varepsilon \le Cx_k + \hat{d} \le y_h + 1_p \varepsilon$$
$$k = 1, 2, .., N$$

where ~~exra~~ $\varepsilon \in \mathbb{R}$ (note no need to add $\varepsilon \ge 0$ constraint).

$1_p = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ } of size p.

[6].

QED.