IMPERIAL COLLEGE LONDON

EE3-10
EE9CS3-1

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

MSc and EEE/ISE PART III/IV: MEng, BEng and ACGI

Corrected Copy

MATHEMATICS FOR SIGNALS AND SYSTEMS

Friday, 4 May 2:30 pm

Time allowed: 3:00 hours

There are THREE questions on this paper.

Answer ALL questions. All questions carry equal marks.

15:20 → correction (page 1/3 question 1c)

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : M.M. Draief

Second Marker(s) : D. Angeli

1. We consider the following systems of linear equations

$$\begin{aligned} 2x_1 - x_2 &= 3 \\ -x_1 + 2x_2 - x_3 &= -5 \\ -x_2 + 2x_3 &= 5 \end{aligned} \qquad (1.1)$$

a) i) Write the system (1.1) in matrix form, i.e. $Ax = y$ where $A \in \mathbb{R}^{3\times 3}$ and $x, y \in \mathbb{R}^3$. [1]

ii) Compute the determinant of A. [1]

iii) Determine x^* the solution of the system (1.1) and justify that it is the unique solution of the system (1.1). [1]

b) We now study the matrix $J = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix}$.

i) Write the system (1.1) in the form $x = Jx + z$ where $x, z \in \mathbb{R}^3$. [2]

ii) Find an orthogonal matrix P, i.e. $P^T P = I$ where I is the identity matrix, such that $J = PDP^T$ where

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & -1/\sqrt{2} \end{pmatrix}.$$

[2]

iii) Compute J^k the kth power of J for all non-negative integers k.

Hint: Distinguish odd and even values of k. [2]

c) Let $x^{(0)} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and define the sequence of vectors $x^{(0)}, x^{(1)}, \ldots$ as follows. For $k \geq 0$,

$$x^{(k+1)} = Jx^{(k)} + z,$$

where z is defined in b) i) and let $\delta^{(k)} = x^{(k)} - x^*$, where x^* is defined in a) ~~iv)~~ iii).

i) Compute $x^{(1)}$ and $\delta^{(1)}$. [1]

ii) Show that $\delta^{(k)} = PD^kP^T\delta^{(0)}$, P and D defined in b) ii). [1]

iii) Show that $||D^k x|| \leq \frac{1}{2^{k/2}}||x||$ for all $x \in \mathbb{R}^3$. [3]

iv) Show that for U an orthogonal matrix $||Ux|| = ||x||$. [1]

v) Show that $||\delta^{(k)}|| \leq \sqrt{\frac{13}{2^k}}$. [3]

vi) Show that, for $k \geq k_0$ where $k_0 = \frac{\log(13)+6\log(10)}{\log(2)}$, we have

$$||x^{(k)} - x^*|| \leq 10^{-3}.$$

[2]

2. The aim of this problem is to derive an algorithm for performing the QR decomposition using orthogonal matrices known as *Givens rotators*.

a) We start by considering rotators in $\mathbb{R}^2$ given by $Q = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$, where $\theta \in [0, 2\pi)$.

i) Show that Q is an orthogonal matrix, i.e. $Q^T Q = I$. [1]

ii) Let $x \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Find a rotator Q such that $Q^T x = \begin{pmatrix} ||x|| \\ 0 \end{pmatrix}$. [2]

iii) For $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$. Find a rotator Q such that $Q^T A = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}$. [1]

iv) For $A \in \mathbb{R}^{2\times 2}$ non-singular, find a rotator Q such that $Q^T A = R$, R upper triangular. [2]

b) We now examine the general case. Let $A = (a_{ij})_{i,j=1,\dots,n} \in \mathbb{R}^{n\times n}$ be a non-singular matrix and define **Givens rotators** as follows.

$$Q^{(ij)} = \begin{pmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & c & 0 & \dots & 0 & -s & & & \\ & & & 0 & 1 & \ddots & & 0 & & & \\ & & & \vdots & \ddots & \ddots & \ddots & \vdots & & & \\ & & & 0 & & \ddots & 1 & 0 & & & \\ & & & s & 0 & \dots & 0 & c & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{pmatrix} \quad (2.1)$$

The matrix $Q^{(ij)}$ is such that all the entries are equal to 0 but the diagonal entries that are equal to 1 except entries $(i,i)$and (j,j) both equal to $c = \cos(\theta)$, and entry (i,j) equals $-s$ and enty (j,i) equals s where $s = \sin(\theta)$.

i) Find Q a Givens rotator such that Q^T transforms $x = (x_1, \dots, x_n)^T$ into a vector whose jth coordinate is equal to 0. [2]

ii) Show that $Q^{(ij)}A$ and ${Q^{(ij)}}^T A$ only alter the ith and jth rows of A. [1]

iii) Show that $AQ^{(ij)}$ and $A{Q^{(ij)}}^T$ only alter ith and jth column of A. [1]

c) We now describe how to perform the QR-decomposition using Givens rotators.

i) Find a Givens rotator $Q^{(21)}$ such that $Q^{(21)}(a_{11} \dots a_{n1})^T = (\star, 0, a_{31}, \dots, a_{n1})^T$ where $\star$ is some real number. [2]

ii) Show that there are rotators of the form $Q^{(21)}, \dots, Q^{(n1)}$ such that $\left(Q^{(n1)}\right)^T \dots \left(Q^{(21)}\right)^T A$ has its first column of the form $(\bullet, 0, \dots, 0)^T$ where $\bullet$ is some real number. [3]

iii) Describe a method for deriving the QR decomposition using Givens rotators and derive its complexity. [5]

3. We define the family of *Hermite polynomials* $(H_n(x))_{n\geq 0}$ by

$$H_0(x) = 1 \quad \text{and} \quad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right),$$

where $\frac{d^n}{dx^n}\left(e^{-x^2}\right)$ is the nth derivative of e^{-x^2}.

a) i) Compute H_1, H_2, H_3. [2]

ii) Show that for all non-negative integer n we have

$$\frac{dH_n}{dx}(x) = 2xH_n(x) - H_{n+1}(x).$$

[2]

iii) Show that for $k < n$, $\int_{-\infty}^{+\infty} x^k H_n(x) e^{-x^2} dx = 0$,

Hint: Perform successive integrations by part and use the fact that for all non-negative integers k and l,

$$\lim_{x\to\infty} x^k \frac{d^l}{dx^l}\left(e^{-x^2}\right) = \lim_{x\to-\infty} x^k \frac{d^l}{dx^l}\left(e^{-x^2}\right) = 0$$

[2]

iv) Show that $\int_{-\infty}^{+\infty} x^n H_n(x) e^{-x^2} dx = n!\sqrt{\pi}$.

Hint: Use, without justification, the identity $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$. [2]

v) Show that the family of polynomials $(H_n(x))_{n\geq 0}$ forms a family of orthogonal polynomials for the inner product $\langle f,g\rangle = \int_{-\infty}^{+\infty} e^{-x^2} f(x)g(x)dx$. [2]

b) We now study the solutions of the differential equation

$$-\frac{d^2 f}{dx^2}(x) + x^2 f(x) = \lambda f(x), \tag{3.1}$$

for $\lambda \in \mathbb{R}$ a parameter.

i) By decomposing the polynomial xH_n in the basis $(H_0, \ldots, H_{n+1})$ of $\mathbb{R}_{n+1}[X]$, the space of polynomials of degree less or equal to $n+1$, and using a) iii) and v), show that

$$xH_n(x) = \frac{\langle xH_n, H_{n+1}\rangle}{\langle H_{n+1}, H_{n+1}\rangle} H_{n+1}(x) + \frac{\langle xH_n, H_n\rangle}{\langle H_n, H_n\rangle} H_n(x) + \frac{\langle xH_n, H_{n-1}\rangle}{\langle H_{n-1}, H_{n-1}\rangle} H_{n-1}(x).$$

[2]

ii) Using a) ii), iii) and v), show that $\langle xH_n, H_n\rangle = 0$. [2]

In fact, one can show that

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0. \tag{3.2}$$

In the remainder, we assume that this holds and no justification is required

iii) Using question a) ii) and b) i) and the identity (3.2), prove that

$$\frac{d^2 H_n}{dx^2}(x) - 2x\frac{dH_n}{dx}(x) + 2nH_n(x) = 0. \tag{3.3}$$

[3]

iv) Using identity (3.3), show that the function $f_n(x) = e^{-x^2/2} H_n(x)$ is solution of the differential equation (3.1) for $\lambda = 2n+1$. [3]

Q1.

$$\begin{cases} 2x_1 - x_2 = 3 \\ -x_1 + 2x_2 - x_3 = -5 \\ -x_2 + 2x_3 = 5 \end{cases}$$

a)

i) $$\underbrace{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 3 \\ -5 \\ 5 \end{bmatrix}}_{y}.$$ [1]

ii) $\det(A) = 8 - 2 - 2 = 4.$ [1]

iii) $$A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$ [1]

$$x^* = y = A^{-1}x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

A is non-singular, $\det(A) \neq 0$, by (ii), so x^* is the unique solution of (1.1).

b) $$J = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix}.$$

i) $$Jx = \begin{bmatrix} x_2/2 \\ x_1/2 + x_3/2 \\ x_2/2 \end{bmatrix};$$ [2]

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = y \Rightarrow x = Jx + \underbrace{\frac{1}{2} \begin{pmatrix} 3 \\ -5 \\ 5 \end{pmatrix}}_{c}$$

ii) $P = \begin{bmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix}$. 2/9

Q1 2/3

eigenvectors of J associated
to eigenvalue $0, 1/\sqrt{2}, -1/\sqrt{2}$ respectively.

$P^{-1} = P^T$ since P orthogonal

$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & -1/\sqrt{2} \end{bmatrix}$ [2]

iii) $\forall k \quad J^k = P D^k P^T$

[2]

$k = 0 \quad J^{2k} = I$

$k \geq 1 \quad J^{2k} = \begin{bmatrix} 1/2^{k+1} & 0 & 1/2^{k+1} \\ 0 & 1/2^k & 0 \\ 1/2^{k+1} & 0 & 1/2^{k+1} \end{bmatrix}$

$k \geq 0 \quad J^{2k+1} = \begin{bmatrix} 0 & 1/2^{k+1} & 0 \\ 1/2^{k+1} & 0 & 1/2^{k+1} \\ 0 & 1/2^{k+1} & 0 \end{bmatrix}$

c) i) $x^{(1)} = \frac{1}{2}\begin{pmatrix} 5 \\ -4 \\ 7 \end{pmatrix}$; $\delta^{(1)} = \frac{1}{2}\begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$. [1]

ii) $\delta^{(k+1)} = x^{(k+1)} - x^* = (Jx^{(k)} + 2) - (Jx^* + 2)$

$= J(x^{(k)} - x^*) = J\,\delta^{(k)}$.

By induction $\delta^{(k)} = J^k \delta^{(0)} = P D^k P^T \delta^{(0)}$ [1]

$\frac{I}{9}$

Q1 $\frac{3}{3}$

iii)

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$D^k x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2^{-k/2} & 0 \\ 0 & 0 & (-1)^k 2^{-k/2} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\|D^k x\|^2 = \|(0, 2^{-k/2} x_2, 2^{-k/2}(-1)^k x_3)^T\|^2$$

$$= \frac{1}{2^k}(x_2^2 + x_3^2) \leq \left(\frac{1}{2^{k/2}} \|x\|\right)^2$$ [3]

iv) $\|Ux\|^2 = (Ux)^T Ux = x^T \underbrace{(U^T U)}_{I} x = x^T x = \|x\|^2$ [1]

v)

$$\|\delta^{(k)}\| = \|P D^k P^T \delta^{(0)}\|$$

$$= \|D^k P^T \delta^{(0)}\|$$ P orthogonal & by (iv).

$$\leq \frac{1}{2^{k/2}} \|P^T \delta^{(0)}\|$$ by (iii)

$$= \frac{1}{2^{k/2}} \|\delta^0\|$$ by (iv); P^T orthogo.

It remains to compute $\|\delta^0\|$

$$\|\delta^0\| = \left\| \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \right\| = \sqrt{13}.$$ [3]

Hence $\|\delta^{(k)}\| \leq \sqrt{\frac{13}{2^k}}$.

vi) $\|\delta^{(k)}\| = \|x^{(k)} - x^*\| \leq \sqrt{\frac{13}{2^k}}$. Let k_0 the smallest integer such that $\sqrt{\frac{13}{2^{k_0}}} \leq 10^{-3}$.

$$\Rightarrow k_0 = \frac{\log(13) + 6\log(10)}{\log(2)} \approx 24.$$ [2]

Q2

a) $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

i) $Q^T Q = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Using $\cos^2\theta + \sin^2\theta = 1$

[1]

ii).

$Q^T x = \begin{pmatrix} \cos\theta \; x_1 + \sin\theta \; x_2 \\ -\sin\theta \; x_1 + \cos\theta \; x_2 \end{pmatrix}$

1st condition $-\sin\theta \, x_1 + \cos\theta \, x_2 = 0$.

$\tan\theta = x_2 / x_1$ (*) [2]

Let $\theta \in [0, 2\pi)$ such that $\cos\theta = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$

& $\sin\theta = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}$ then condition (*) satisfied

and $Q^T x = \begin{pmatrix} x_1^2 + x_2^2 / \sqrt{x_1^2 + x_2^2} \\ 0 \end{pmatrix} = \begin{pmatrix} \|x\| \\ 0 \end{pmatrix}$

iii) Let such θ such that.

$\cos\theta = 1/\sqrt{2}$ $\sin\theta = 1/\sqrt{2}$; $\theta = \pi/4$.

Hence, for $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $Q^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$.

and $Q^T A = 1/\sqrt{2} \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}$ [1]

iv) For general A_j 5/9 [2]

Q2 2/3 Let θ such that $\cos\theta = \frac{a_{11}}{\sqrt{a_{11}^2 + a_{21}^2}}$; $\sin\theta = \frac{a_{21}}{\sqrt{a_{11}^2 + a_{21}^2}}$

b) i) $\left(Q^{(ij)}\right)^T \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{array}{cl} \begin{bmatrix} x_1 \\ \vdots \\ x_{i-1} \\ c x_i + s x_j \\ x_{i+1} \\ \vdots \\ x_{j-1} \\ -s x_i + c x_j \\ x_{j+1} \\ \vdots \\ x_n \end{bmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \leftarrow j \\ \\ \\ \\ \end{array} \end{array}$ (**).

we want $-\sin\theta\, x_i + \cos\theta\, x_j = 0$. [2]

Let $\theta \in [0, 2\pi)$ such that $\cos\theta = \frac{x_i}{\sqrt{x_i^2 + x_j^2}}$

$\sin\theta = \frac{x_j}{\sqrt{x_i^2 + x_j^2}}$. In this case the

jth coordinate is equal to 0 whereas the ith coordinate is given by $\sqrt{x_i^2 + x_j^2}$.

ii) see (**) where the same transform applies to each column of A. [1]
which is altered in row i & row j only

iii) by the transpose of ii) columns become rows and vice-versa. So the only alterations take place in the columns i&j of A. [1]

c) i) $Q^{(21)}$ Givens rotator of the form (2.1) [2]
where $\cos\theta = \frac{a_{11}}{\sqrt{a_{11}^2 + a_{21}^2}}$; $\sin\theta = \frac{a_{21}}{\sqrt{a_{11}^2 + a_{21}^2}}$.

Let $* = a'_{11} \left(= \sqrt{a_{11}^2 + a_{21}^2}\right)$.

ii) $Q^{(31)}$ such that $\cos\theta = \frac{a'_{11}}{\sqrt{a'^2_{11} + a_{31}^2}}$; $\sin\theta = \frac{a_{31}}{\sqrt{a'^2_{11} + a_{31}^2}}$

More generally $Q^T_{(k+1)1}$ such that if $a^{(k)}_{11}$ the value in entry $(1,1)$ after application of $Q^T_{k1} \cdots Q^T_{21}$

then $\cos(\theta_{(k+1),1}) = (a^{(k)}_{11})^2 \Big/ \sqrt{(a^{(k)}_{11})^2 + a^2_{(k+1)1}}$

& $\sin(\theta_{(k+1),1}) = \dfrac{a_{(k+1),1}}{\sqrt{(a^{(k)}_{11})^2 + a^2_{(k+1),1}}}$. $\underline{Q}^2$ 3/3

Once we apply $R^{(1)} = (Q^T_{n1} \cdots Q^T_{21})A$ as above the first column of $R^{(1)}$ is of the form $\begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. [3]

iii) Once we resolved the 1st column following ii). We can easily construct Givens rotators.

$Q_{n2} \cdots Q_{32}$ such that [5]

$R^{(2)} = Q^T_{n2} \cdots Q^T_{32} Q^T_{n1} \cdots Q^T_{21} A$ has zeros in columns 1 & 2 below the main diagonal

In a similar fashion we can find rotators of the form $Q_{21} \cdots Q_{nn-1}$ such that

$R = Q^T_{nn-1} Q^T_{nn-2} \cdots Q^T_{21} A$ is upper triangular

A "rough" calculation gives $O(n^3)$ (more precisely $2n^3 + O(n^2)$).

~~For the first column. (n-1) Given matrices. for each of them we have 5 operations for cos θ & one additional for sin θ and then. 2~~ $\boxed{\text{1 BONUS POINTS}}$ For proper justification of Complexity.

Mathematics for signals & systems do 11/12/12.

Q3 — Q3 7/9 — 7/9

a) i) $H_1(x) = (-1)e^{x^2}\frac{d}{dx}e^{-x^2} = 2x$ [2]

$H_2(x) = e^{x^2}\frac{d}{dx}\left(-2xe^{-x^2}\right) = 4x^2-2.$

$H_3(x) = -e^{x^2}\frac{d}{dx}\left(\left(4x^2-2\right)e^{-x^2}\right) = 8x^3-12x$

ii) $\frac{d}{dx}H_n(x) = (-1)^n\frac{d}{dx}\,e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$

$= (-1)^n\left[2xe^{x^2}\frac{d^n}{dx^n}e^{-x^2} + e^{x^2}\frac{d^{n+1}}{dx^{n+1}}e^{-x^2}\right.$

$H_n'(x) = 2x\,H_n(x) - H_{n+1}(x)$

iii) $\int_{-\infty}^{+\infty} e^{-x^2}x^k H_n(x)\,dx = (-1)^n\int_{-\infty}^{+\infty} x^k\frac{d^n}{dx^n}e^{-x^2}dx.$

IP $= \underbrace{\left[(-1)^n x^k\frac{d^{n-1}}{dx^{n-1}}e^{x^2}\right]_{-\infty}^{+\infty}}_{=0 \text{ by hint}} - (-1)^n\int_{-\infty}^{+\infty} kx^{k-1}\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}dx$

Repeating step above k-times using hint $= \dots = (-1)^{n+k}\,k!\int_{-\infty}^{+\infty}\frac{d^{n-k}}{dx^{n-k}}e^{-x^2}dx$

$= (-1)^{n+k}\,k!\left[\frac{d^{n-k+1}}{dx^{n-k+1}}e^{-x^2}\right]_{-\infty}^{+\infty} = 0$ by hint

[2]

iv) $\int_{-\infty}^{+\infty} x^n H_n(x)\,e^{x^2} \overset{\text{as above}}{=} n!\int_{-\infty}^{+\infty}e^{-x^2}dx = \underline{\underline{n!\sqrt{\pi}}}$ by hint

[2]

v)

$k<n$ $\int_{-\infty}^{+\infty}H_k(x)\,H_n(x)\,e^{-x^2}dx = \int\sum_{\ell=0}^{k} h_\ell a_\ell x^\ell H_n(x)e^{-x^2}dx$

$=0$ by linearity of integral & 3) a) iii)

[2]

b)

i)

Q3 2/3

$$x H_n(x) = \sum_{k=0}^{n+1} \frac{\langle x H_n, H_k \rangle}{\langle H_k, H_k \rangle} H_k . \quad (*)$$

8/9

note that $\langle x H_n, H_k \rangle = \langle H_n, x H_k \rangle$

and by a) iii) $\langle H_n, x H_k \rangle$ is equal 0 whenever the degree of $x H_k$ is smaller tha n. that is to say $k+1 < n$; $k \leq n-2$.

Hence $(*) \Rightarrow$

$$x H_n = \frac{\langle x H_n, H_{n+1} \rangle}{\langle H_{n+1}, H_{n+1} \rangle} H_{n+1} + \frac{\langle x H_n, H_n \rangle}{\langle H_n, H_n \rangle} H_n + \frac{\langle x H_n, H_{n-1} \rangle}{\langle H_{n-1}, H_{n-1} \rangle} H_{n-1}$$

[2]

$$\cancel{\langle x H_n, H_n \rangle = \int_{-\infty}^{+\infty} e^{x^2} \left[\frac{d^n}{dx^n} e^{-x^2} \right]^2 dx}$$

by ii) $\langle x H_n, H_n \rangle = \langle \frac{1}{2} \frac{dH_n}{dx} + \frac{1}{2} H_{n+1}, H_n \rangle$

$$= \frac{1}{2} \underbrace{\langle \frac{dH_n}{dx}, H_n \rangle}_{=0 \text{ by a) iii)}} + \frac{1}{2} \underbrace{\langle H_{n+1}, H_n \rangle}_{=0 \text{ by a) v)}}.$$

$\langle x H_n, H_n \rangle = 0$.

[2].

iii) $Q3\ 3/3$

$$\underbrace{H_{n+1} - 2x\, H_n}_{-H_n'} + 2n\, H_{n-1} = 0.$$

9/9

$$-H_n' + 2n\, H_{n-1} = 0 \text{ by a) ii)} \Rightarrow H_n' = 2n H_{n-1} \quad (**)$$

$$\Rightarrow \quad H_{n+1} - 2x H_n + H_n' = 0.$$

differentiating $\quad H_{n+1}' - 2H_n - 2x H_n' + H_n'' = 0.$

By $(**)$ $\quad H_{n+1}' = (2n+2)\, H_n.$

Hence $\quad (2n+2)\, H_n - 2H_n - 2x H_n' + H_n'' = 0.$

$$\Rightarrow \quad H_n'' - 2x H_n' + 2n\, H_n = 0.$$

[3]

iv) $\quad f_n'(x) = -x f_n(x) + e^{-x^2/2} H_n'.$

$$f_n''(x) = -f_n(x) - x f_n'(x) - x e^{-x^2/2} H_n' + e^{-x^2/2} H_n''.$$

$$-f_n'' + x^2 f_n = f_n + x f_n' + x e^{-x^2/2} H_n' - e^{-x^2} H_n'' + x^2 f_n.$$

$$= f_n - \underbrace{\cancel{x^2 f_n} + x e^{-x^2/2} H_n'}_{x f_n'} + x e^{-x^2/2} H_n' - e^{-x^2/2} H_n'' + \cancel{x^2 f_n}$$

$$= f_n + \Big[\underbrace{2x\, H_n' - H_n''}_{2n\, H_n \text{ by b) iii)}}\Big]\, e^{-x^2/2}.$$

$$= f_n + 2n\, f_n = (2n+1)\, f_n. \quad [3].$$