EE4-26
EE9CS5-3

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

MSc and EEE PART IV: MEng and ACGI

ESTIMATION AND FAULT DETECTION

Friday, 20 May 10:00 am

Corrected Copy

Time allowed: 3:00 hours

11.15

There are FIVE questions on this paper.

Q3 p4

Answer FOUR questions.

Q4 p5

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : R.B. Vinter

Second Marker(s) : D. Angeli

Information for candidates:

Some formulae relevant to the questions.

The normal $N(m, \sigma^2)$ density:

$$N(m, \sigma^2)(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-m)^2}{2\sigma^2}\right)$$

System equations:

$$\begin{aligned} x_k &= F x_{k-1} + u^s + w_k \\ y_k &= H x_k + u^o + v_k \,. \end{aligned}$$

Here, w_k and v_k are white noise sequences with covariances Q^s and Q^0 respectively.

The Kalman filter equations are

$$\begin{aligned} P_{k|k-1} &= F P_{k-1} F^T + Q^s \\ P_k &= P_{k|k-1} - P_{k|k-1} H^T (H P_{k|k-1} H^T + Q^o)^{-1} H P_{k|k-1} \,, \\ K_k &= P_{k|k-1} H^T (H P_{k|k-1} H^T + Q^o)^{-1} \,, \\ \hat{x}_k &= \hat{x}_{k|k-1} + K_k (y_k - \hat{y}_{k|k-1}) \,, \end{aligned}$$

in which $\hat{x}_{k|k-1} = F\hat{x}_{k-1} + u^s$ and $\hat{y}_{k|k-1} = H\hat{x}_{k|k-1} + u^o$

1. (i) A stationary, scalar, zero-mean stochastic process y_k is described by the state space equations:

$$\begin{cases} \mathbf{x}_k = F\mathbf{x}_{k-1} + \mathbf{w}_k \\ y_k = \mathbf{c}^T\mathbf{x}_k , \end{cases} \tag{1}$$

in which $\mathbf{c}$ is a given n-vector, F is a given $n \times n$ matrix, and $\mathbf{w}_k$ is a white noise process, with covariance Q. Derive equations for the covariance function of y_k,

$$R_y(l) = E[y_k y_{k-l}] \qquad \text{for } l = \ldots, -1, 0, 1, \ldots;,,$$

in terms of the Lyapunov equation for the state equation. [6]

(ii) The populations of two species, $x^1(t)$ and $x^2(t)$, in a bioreactor have different death rates but the same random food supply. The total population is denoted by $y(t)$. The processes $x^1(t)$, $x^2(t)$ and $y(t)$ are assumed to satisfy the following continuous time model

$$\begin{cases} dx^1(t)/dt = -\alpha_1 x^1(t) + w(t) \\ dx^2(t)/dt = -\alpha_2 x^2(t) + w(t) \\ y(t) = x^1(t) + x^2(t) \end{cases}$$

in which α_1 and α_2 are given positive constants and $w(t)$ is continuous time white noise with covariance function $r_w(\tau) = 1 \times \delta(\tau)$.

Measurements y_k are taken of $y(t)$ at times $t = kh$, $k = \ldots, -1, 0, +1, \ldots$, thus:

$$y_k = y(kh) .$$

Here, the sample period h is a given positive constant.

Show that y_k satisfies the state space equations (1), and determine the values taken by F, $\mathbf{c}$ and Q in this case. [8]

Determine the covariance function $R_y(k)$ of the measurement process y_k for $k = 0$ and 1. [4]

Explain why $R_y(0)$ does not depend on the sampling period h. [2]

2. (i): A measurement $\mathbf{y}$ is made of a scalar random variable x. The measurement $\mathbf{y}$ is a vector random variable model as

$$\mathbf{y} \ = \ x\mathbf{d} + \mathbf{v} \ ,$$

in which $\mathbf{d}$ is a given vector, and x and $\mathbf{v}$ are independent random variables with probability densities $x \sim \mathcal{N}(m, p_0)$ and $\mathbf{v} \sim \mathcal{N}(0, Q)$.

By applying Bayes' Rule, or otherwise, show that the least squares estimate $\hat{x}$ of x given $\mathbf{y}$ and the error variance p satisfy

$$p^{-1} \ = \ p_0^{-1} + \mathbf{d}^T Q^{-1} \mathbf{d} \qquad \text{and} \qquad \hat{x} \ = \ m + \frac{1}{\mathbf{d}^T Q^{-1} \mathbf{d} + p_0^{-1}} \mathbf{d}^T Q^{-1} \left[\mathbf{y} - m\mathbf{d} \right] \ .$$

(Hint: Show that [12]

$$p(x|y) \ = \ (\ldots) \exp \left\{ -\frac{1}{2} \left(p^{-1} x^2 - 2p^{-1} \hat{x} x + (\ldots) \right) \right\}$$

where $(\ldots)$ denotes terms depending only on y.)

(ii): A cheap sensor, whose accuracy deteriorates as time passes, provides measurements $y_1, \ldots, y_N$ of a scalar random variable x at successive times. It is assumed that

$$y_k \ = \ x + v_k \ ,$$

$k = 1, \ldots, N$, in which the v_k's are random variables. x and $v_1, \ldots, v_N$ are independent with $x \sim \mathcal{N}(m, p_0)$ and $v_k \sim \mathcal{N}(0, q_k)$ for each k. Here $p_0 > 0$ and $q_1, \ldots, q_N$ is an increasing sequence of numbers.

By using the results of Part (i), or otherwise, show that the least squares estimate $\hat{x}$ of x given $y_1, \ldots, y_N$ is

$$\hat{x} \ = \ m + \left(p_0^{-1} + \sum_{i=k}^{N} q_k^{-1} \right)^{-1} \sum_{k=1}^{N} q_k^{-1} (y_k - m)$$

What is the error variance q_N, based on N measurements? [4]

[2]

Now assume that $p_0 = 1$ and $\sqrt{q_k} = k$ for $k = 1, 2, \ldots$ Show that the error variance cannot be less than 2.64^{-1}, however many measurements are taken. [2]

(You can use the fact that $\sum_{k=0}^{\infty} 1/k^2 \ \approx 1.64$)

3. A sensor generates measurements $\mathbf{y}_1, \mathbf{y}_2, \ldots$ of a signal $\mathbf{x}(t)$ at times $t = kh$, $k = 1, 2, \ldots$, in which h is the sample period. The measurements $\mathbf{y}_k$ are related to the sampled states $\mathbf{x}_k = \mathbf{x}(kh)$ by the equations

$$\begin{cases} \mathbf{x}_k &= F\mathbf{x}_{k-1} + \mathbf{w}_k \\ \mathbf{y}_k &= H\mathbf{x}_k + \mathbf{v}_k \,, \end{cases} \tag{2}$$

in which F and H are given matrices, and $\{\mathbf{w}_k\}$ and $\{\mathbf{v}_k\}$ are Gaussian white noise processes, independent of each other and of $\mathbf{x}_0$, and with covariances Q^s and Q^m respectively.

To reduce energy consumption of the sensor, the sampling rate is halved, i.e. only measurements at even times

$$\mathbf{y}_k, \quad k = 2, 4, 6, \ldots$$

are captured.

Show that if k is an even time then the conditional mean and covariance of $\mathbf{x}_k$ given measurements at even times up to and including k, namely

$$\hat{\mathbf{x}}_k := E[\mathbf{x}_k \,|\, \mathbf{y}_2, \mathbf{y}_4, \ldots, \mathbf{y}_k] \quad \text{and} \quad P_k := \operatorname{cov}\{\mathbf{x}_k \,|\, \mathbf{y}_2, \mathbf{y}_4, \ldots, \mathbf{y}_k\} \,,$$

are given by

$$\hat{\mathbf{x}}_{k+2} = F^2\hat{\mathbf{x}}_k + K_{k+2}[\mathbf{y}_{k+2} - HF^2\hat{\mathbf{y}}_k] \,.$$

$\hat{x}_k$

Determine the gain matrix K_{k+2}. Determine also a formula ~~for the~~ conditional covariance P_{k+2} of x_{k+2} given the even measurements up to time $k+2$. You may quote the standard Kalman filter equations. [14]

Hint: calculate the Kalman filter over two time periods, starting at an even sampling time k, for a time-varying output matrix H_k, which takes value 0 for time $k+1$ and value H for $k+2$.

Now consider the special case of system (2) when x_k and y_k are scalar processes governed by the equations:

$$x_k = w_k \quad \text{and} \quad y_k = x_k + v_k.$$

Obtain recursive formulae, in this case, for the estimates $\hat{x}_k$ of the state x_k at even times, conditioned on measurements at even times up to time k. [4]

Explain why, in this case, the estimates of x_k given measurements at even times up to time k are the same as the estimates for x_k given measurements at *all* times. [2]

4. The output y_k of a stochastic control system is modelled as a stationary scalar stochastic process satisfying the equation

k

$$y_k + ay_{k-1} = u_k + e_t + ce_{k-1} ,$$

in which a and c are constants ($|a| < 1$) and $\{e_k\}$ is a white noise process with unit variance, and $\{u_k\}$ is a sequence of control actions.

For $u_k = 0$, and for general values of the parameters a and c, derive formulae for the covariance function $R_y(k)$, $k = 1, 2, \ldots$ of the process y_k. [10]

Feedback control, with the strucure

$$u_k = Ky_{k-1}$$

is now applied, to reduce the variance of the output y_k. Notice the one-step delay in the feedback relation, which is required for measuring the state and implementing the control.

Using the results of earlier calculations, derive a formula for the output variance $r(K)$.

$$r(K) = E[y_k^2] .$$

[4]

Show that K^*, the value of the K which minizes the variance, is given by

$$K^* = a - c$$

[4]

Finally, suppose that the control feedback $u_k = K^* y_{k-1}$ is implemented with a small error δ, i.e. the control signal actually applied is

$$u_k = (K^* + \delta)y_{k-1} .$$

Show that the control feedback no longer minimizes the output variance, not even approximately. [2]

Hint: consider cancellation of poles and zeros in the difference equation for y_k under closed loop control.

5. (i): Consider the vector signal process $\{\mathbf{x}_k\}$ and scalar measurement process $\{y_k\}$ described by the equations:

$$\begin{aligned} \mathbf{x}_k &= F\mathbf{x}_{k-1} + w_k \\ y_k &= h(\mathbf{x}_k) + v_k \,, \end{aligned}$$

in which F is a given $n \times n$ matrix and $h(\mathbf{x})$ is a given nonlinear function of the n-vector $\mathbf{x}$. $\mathbf{w}_k$ and v_k are white noise sequences with covariances Q^s and q^0 respectively, independent of each other and of x_0.

Describe the extended Kalman filter algorithm for the recursive estimation of $\mathbf{x}_k$ given $\{y_1, \ldots, y_k\}$, expressed in terms of the gradient $\nabla h(\mathbf{x})$ of the nonlinear function in the measurement equation. Explain the nature of the approximations involved. [8]

Calculate $\nabla h(x)$ when the sensor is a nonlinear range sensor in $2D$ space, for which $n = 2$ and $h(x)$ is the function

$$h(x_1, x_2) = (x_1^2 + x_2^2)^{3/2} \,.$$

[2]

(ii): An optical sensor takes a measurement θ (in radians) of the horizontal angular location of a target. It is possible, however, that the measurement originates, not from the target location, but from 'clutter' (i.e. some random atmospheric disturbance effect). Two hypothesis should be considered:

(H_0): The measurement originates from the target, in which case it is a sample of the probability density $p_0(\theta) = N(0, \sigma^2)(\theta)$.

(H_1): The measurement originates from clutter in which case it is a sample of $p_1(\theta)$.

$$p_1(\theta) = \begin{cases} (2\pi)^{-1} & \text{for } \theta \in [-\pi, \pi] \\ 0 & \text{otherwise} \end{cases}$$

(Here σ^2 is a known small positive constant.)

Regarding (H_0) as the null hypothesis, design a Neyman-Pearson test of the proposition 'the measurement originates from the target', at the 1% significance level, i.e. under the constraint that the probability that the decision rule accepts (H_1) when (H_0) is true, is 0.01. [8]

Derive a formula for the power of the test [2]

Data: if $x \sim N(0,1)$, *then* $P(x^2 \geq 6.635) = 0.01$.

[END]

ESTIMATION and FAULT DETECTION
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1.(i) We have $E\{x_k x_k^T\} = E\{(Fx_{k-1}+w_k)(Fx_{k-1}+w_k)^T\}$
whence $R_x(0) = F R_x(0) F^T + 0 + Q$, since w_k and x_{k-1} are indep.
Then $E\{x_k x_{k-l}\} = E\{F x_{k-1} x_{k-l}\} + 0 \Rightarrow R_x(l) = F R_x(l-1)$, ...
$y_k = c^T x_k$, So $R_y(l) = c^T R_x(l) c$ for all l.
Hence $\underline{R_y(l) = c^T F^{|l|} R_x(0) c}$, $l = \dots, -1, 0, +1$
where $R_x(0)$ solves the Lyapunov eqn. $R_x(0) = F R_x(0) F^T + Q$.

(ii) From the variation of constants formula, $x(kh) = e^{Fh} x((k-1)h) + \int_{(k-1)h}^{kh} e^{F(kh-s)} \begin{bmatrix}1\\1\end{bmatrix} w(s)\,ds$
But $e^{Fh} = I + \begin{bmatrix}-\alpha_1 & 0\\ 0 & -\alpha_2\end{bmatrix} + \frac{1}{2!}[\dots]^2 + \dots = \begin{bmatrix}e^{-\alpha_1 h} & 0\\ 0 & e^{-\alpha_2 h}\end{bmatrix}$
Also, $\text{cov}\left\{\int_{(k-1)h}^{kh} e^{F(kh-s)}\begin{bmatrix}1\\1\end{bmatrix} w(s)\,ds\right\} = \int_0^h \begin{bmatrix}e^{-\alpha_1 s} & 0\\ 0 & e^{-\alpha_2 s}\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}e^{-\alpha_1 s} & 0\\ 0 & e^{-\alpha_2 s}\end{bmatrix} ds$

$$= \int_0^h \begin{bmatrix} e^{-2\alpha_1 s} & e^{-(\alpha_1+\alpha_2)s} \\ e^{-(\alpha_1+\alpha_2)s} & e^{-\alpha_2 s}\end{bmatrix} ds = \begin{bmatrix} \frac{1}{2\alpha_1}(1-e^{-2\alpha_1 h}) & \frac{1}{\alpha_1+\alpha_2}(1-e^{-(\alpha_1+\alpha_2)h}) \\ \frac{1}{(\alpha_1+\alpha_2)}(1-e^{-(\alpha_1+\alpha_2)h}) & \frac{1}{2\alpha_2}(1-e^{-(\alpha_1+\alpha_2)}) \end{bmatrix} =: \tilde{Q}$$

We have shown
$\underline{x_k} = \underline{F x_{k-1}} + \underline{\tilde{w}_k}$, where $F = \text{diag}\{e^{-\alpha_i h}\}$ and $\text{cov}\{\tilde{w}_k\} = \tilde{Q}$.
(the $\tilde{w}_k$'s are independent, by properties of the stochastic integral.)
By part (i), $R_y(0) = c^T P c$ where P solves

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22}\end{bmatrix} = \begin{bmatrix} e^{-\alpha_1 h} & 0 \\ 0 & e^{-\alpha_2 h}\end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22}\end{bmatrix}\begin{bmatrix} e^{-\alpha_1 h} & 0 \\ 0 & e^{-\alpha_2 h}\end{bmatrix} + \tilde{Q}$$

$$= \begin{bmatrix} e^{-2\alpha_1 h} P_{11} & e^{-(\alpha_1+\alpha_2)h} P_{12} \\ e^{-(\alpha_1+\alpha_2)h} P_{12} & e^{-2\alpha_2 h} P_{22}\end{bmatrix} + \tilde{Q}$$

Equating terms $\Rightarrow$ $P_{11}(1-e^{-2\alpha_1 h}) = \frac{1}{2\alpha_1}(1-e^{-2\alpha_1 h})$
$P_{12}(1-e^{-(\alpha_1+\alpha_2)h}) = \frac{1}{(\alpha_1+\alpha_2)}(1-e^{-(\alpha_1+\alpha_2)h})$, $P_{22}(1-e^{-2\alpha_2 h}) = \frac{1}{2\alpha_2}(1-e^{-2\alpha_2 h})$,
Whence $P = \begin{bmatrix} \frac{1}{2\alpha_1} & \frac{1}{(\alpha_1+\alpha_2)} \\ 1/(\alpha_1+\alpha_2) & 1/2\alpha_2 \end{bmatrix}$.

Then, by part (i),
$R_y(0) = \frac{1}{2}\alpha_1^{-1} + \frac{1}{2}\alpha_2^{-1} + 2(\alpha_1+\alpha_2)^{-1}$
$R_y(1) = \frac{1}{2\alpha_1} e^{-\alpha_1 h} + \frac{1}{(\alpha_1+\alpha_2)}(e^{-\alpha_1 h} + e^{-\alpha_2 h}) + \frac{1}{2\alpha_2} e^{-\alpha_2 h}$

Since y_k 'coincides' with $y(t)$ at $t = kh$, and both are stationary processes, $R_y(0)$ is simply $E\{y^2(t)\}$ (for any t).
But $E\{y^2(t)\}$ does not depend on the chosen sampling period h.

2.(i) We have $p(y|x) = \text{const.} \exp\{-\frac{1}{2}[y - dx]^T Q^{-1}[y - dx]\}$
and $p(x) = \text{const.} \exp\{-\frac{1}{2}(x-m)^2/P_0\}$
Bayes rule tell us

$$p(x|y) = (\dots) p(y|x)p(x) = (\dots)\exp\{-\frac{1}{2}\left((y-dx)^T Q^{-1}(y-dx) + \frac{(x-m)^2}{P_0}\right.$$

$$= (\dots)\exp\{-\frac{1}{2}\left(x^2 d^T Q^{-1} d + x^2 P_0^{-1} - 2x^T(d^T Q^{-1} y + P_0^{-1} m) + (\dots\right.$$

The exponent is

$$-\frac{1}{2}\left(x^2(P_0^{-1} + d^T Q^{-1} d) - 2x(P_0^{-1} + d^T Q^{-1} d)\,\frac{d^T Q^{-1} y + P_0^{-1} m + (\dots)}{P_0^{-1} + d^T Q^{-1} d}\right) - (\dots$$

But $P_0^{-1}/(P_0^{-1} + d^T Q^{-1} d) = 1 - \dfrac{d^T Q^{-1} d}{P_0^{-1} + d^T Q^{-1} d}$
So last term in (*) is

$$\frac{d^T Q^{-1} y + P_0^{-1} m}{P_0^{-1} + d^T Q^{-1} d} = \frac{d^T Q^{-1}(y - \underline{d} m)}{P_0^{-1} + d^T Q^{-1} d} + m$$

Writing $\hat{x} = (P_0^{-1} + d^T Q^{-1} d)^{-1} d^T Q^{-1}(y - \underline{d} m) + m$, $P^{-1} = P_0^{-1} + d^T Q$
we have

$$p(x|y) = (\dots)\exp\{-\frac{1}{2}\left(x^2 P^{-1} - 2x P^{-1}\hat{x} + (\dots)\right)\}$$

Since x, y are jointly Gaussian, it follows

$$p(x,y) = \frac{1}{(2\pi)^{\frac{1}{2}} P^{-\frac{1}{2}}} \exp\left\{-\frac{1}{2}\frac{|x-\hat{x}|^2}{P}\right\}$$

So, $\hat{x}$ and P are the conditional mean and variance of x resp

(ii) Write $\underline{y} = [y_1, \dots, y_N]^T$. Then

$$\underline{y} = \underline{1}\,x + \underline{v}$$

where $\underline{1}^T = [1, \dots, 1]$, and $\text{cov}\{v\} = \text{diag}\{q_1, \dots, q_N\}$.
From (i)

$$\hat{x} = m + \frac{1}{P_0^{-1} + \sum_{k=1}^N q_k^{-1}} \sum_{k=1}^{N} q_k^{-1}(y_k - m), \quad P = \frac{1}{P_0^{-1} + \sum_{k=1}^N q_k^{-1}}$$

We see that

$$P_N^{-1} = P_0^{-1} + \sum_{k=1}^N q_k^{-1} \to P_0^{-1} + \sum_{k=1}^{\infty} q_k^{-1} \text{ as } N \to \infty$$

Since P_N decreases with N

$$P_N^{-1} \le 1 + \sum_{k=1}^{\infty} \frac{1}{k^2} = \quad , \forall N$$

So

$$P_N \ge \frac{1}{1 + \sum_{k=1}^{\infty} 1/k^2} = \frac{1}{1 + 1.64} = \underline{\frac{1}{2.64}}$$

3. Take an 'even' time k. Assume we know

$$\hat{x}_k = E[x_k \mid y_2, y_4, \ldots y_k], \quad P_k = \mathrm{cov}\{x_k \mid y_2, y_4, \ldots\}$$

Because there is no measurement at time k,

$$\hat{x}_{k+1} = \hat{x}_{k+1|k} = F\hat{x}_k \quad \text{and}$$
$$P_{k+1} = P_{k+1|k} = FP_kF^T + Q^s$$

Then

$$\hat{x}_{k+2|k+1} = F(F\hat{x}_k) = F^2\hat{x}_k, \quad \hat{y}_{k+2|k+1} = HF^2\hat{x}_k$$

and

$$P_{k+2|k+1} = F(FP_kF^T + Q^s)F^T + Q^s$$

From the standard Kalman Filter equations, applied over the interval $[(k+1)h, (k+2)h]$ we have (noting that a measurement is taken at time $(k+2)h$)

$$P_{k+2} = S - SH^T[HSH^T + Q^m]^{-1}HS \quad (S \text{ given by } (*))$$

and

$$\hat{x}_{k+2} = F^2\hat{x}_k + K_{k+2}(y_{k+2} - HF^2\hat{x}_k)$$

where $K_{k+2} = SH^T[HSH^T + Q^m]^{-1}$
and $S\ (= P_{k+2|k+1}) = F(FP_kF^T + Q^s)F^T + Q^s$

In the special case to be considered, $F = 0$ and $H = 1$.
Then $S = P_{k+2|k+1} = Q^s$, $P_{k+2} = Q^s - \dfrac{(Q^s)^2}{(Q^s + Q^m)}$

and $K_{k+2} = \dfrac{Q^s}{Q^s + Q^m}$ for all even k. So, for even k,

$$\hat{x}_{k+2} = \frac{Q^s}{Q^s + Q^m} y_{k+2} \quad \text{and} \quad P_{k+2} = \frac{Q^s}{Q^s + Q^m} \cdot Q^m$$

But the standard Kalman filter (based on measurements at all k's) is: $\hat{x}_{k+2} = F\hat{x}_{k+1} + P_{t|t-1}H(HP_{t|t-1}H^T + Q^m)^{-1}(y_{k+2} - HF\hat{x}_{k+1})$
in which $P_{t|t-1} = FP_{t-1}F^T + Q^s$.
We have $P_{k|k-1} = Q^s$ and $\hat{x}_{k+2} = 0 + \dfrac{Q^s}{Q^s + Q^m}(y_{k+2} - 0)$

which is the same as before.
The reason is that x_{k+2} is independent of $y_{1:k-1}$, so the LS estimate of x_{k+2} given $\{\ldots, y_{k-2}, y_k\}$ is the same as that given $\{\ldots, y_{k-1}, y_k\}$.

4. We have $y_t = -a y_{t-1} + e_t + c e_{t-1}$

Squaring both sides and taking expectations gives

$$R_y(0) = a^2 R_y(0) - 2ac R_{ye}(0) + 1 + c^2$$

$E\{\times e_t\} \Rightarrow R_{ye}(0) = 0 + 1 + 0$. Hence $R_{ye}(0) = 1$

Then $(1-a^2) R_y(0) = c^2 + 1 - 2ac$. So $R_y(0) = \dfrac{c^2+1-2ac}{1-a^2}$ (*)

$E\{\times y_{t-1}\} \Rightarrow R_y(1) = -a R_y(0) + c R_{ye}(0)$

So $R_y(1) = \dfrac{-a(c^2+1-2ac)}{(1-a^2)} + c$

Shouldn't this be $(-a)^{l-1}$?

$E\{\times y_{t-2}\} \Rightarrow R_y(2) = -a R_y(1)$. Similarly $R_y(l) = -a R_y(l-1)$ for $l \geq 2$

So

$$R_y(l) = (-a)^{l-2} \left(\frac{-a(c^2+1-2ac)}{(1-a^2)} + c \right) \quad \text{for } l \geq 2$$

Since the process is a scalar process, $R_y(l) = R_y(-l)$ for all l.

Now consider

$$y_t + a y_{t-1} = e_t + c e_{t-1} + u_t$$

If $u_t = +k y_{t-1}$, closed loop system becomes

$$y_t + (a-k) y_{t-1} = e_t + c e_{t-1}$$

(*) gives the variance of $\{y_t\}$ as

$$v(k) = \frac{c^2 + 1 - 2(a-k)c}{1-(a-k)^2}$$

To find minimizing $k = k^*$:

$$\frac{d}{dk} v(k) \Big|_{k=k^*} = \frac{+2c\left[1-(a-k)^2\right] - 2(a-k)\left[c^2+1-2(a-k)c\right]}{(\dots)}$$

When $a - k = c$, L.H.S. $= \left(2c(1-c^2) - 2c(1-c^2)\right) / (\dots) = 0$

Hence $v(k)$ is minimized at $k = k^* = a - c$.

The closed loop system (for $k = k^*$) is $(1 + c z^{-1}) y_t = (1 + c z^{-1}) e_t$.

If $u_t = k^* y_{t-1}$ is implemented exactly, the system is $y_t = e_t$, which is stable and achieves the minimum variance $\sigma_y^2 = 1$.

If $u_t = (k^* + \delta) y_{t-1}$ however and $|c| > 0$, the cl system is

$$(1 + (c-\delta) z^{-1}) y_t = (1 + c z^{-1}) e_t.$$

For δ small, this is an unstable system, and so the asymptotic output variance is ∞.

5 (i) The EKF algorithm is based on the assumption that x_k will close to its predicted mean $\hat{x}_{k|k-1} = F\hat{x}_{k-1}$, in which case the signal and measurement equations can be approximated by

$$x_k = Fx_{k-1} + \omega_k, \quad y_k = h(\hat{x}_{k|k-1}) + \nabla h(\hat{x}_{k|k-1}) \times (x_k - \hat{x}_{k-1})$$

The standard Kalman filter eqns are now applied to these approximate linear equations. This gives

$$\hat{x}_k = \hat{x}_{k|k-1} + K_k [y_k - h(\hat{x}_{k|k-1})]$$

and $P_k = P_{k|k-1} - P_{k|k-1} H^T (H P_{k|k-1} H^T + q^v)^{-1} H$, | $\hat{x}_{t-1}$, P_{t-1} are prior esti + error cov resp.

$$P_{k|k-1} = F P_{k-1} F^T + Q^w$$

$\hookrightarrow P_{t|t-1}$

in which $H = \nabla h(\hat{x}_{k|k-1})$.

If $h(x) = (x_1^2 + x_2^2)^{3/2}$, then by the chain rule, for $i = 1, 2$

$\frac{\partial h}{\partial x_i} = (x_1^2 + x_2^2)^{\frac{1}{2}} \cdot 3/2 \times 2x_2$. Hence $\nabla h(x)$ is 2-row vector

$$\nabla h(x) = \frac{3}{(x_1^2 + x_2^2)^{1/2}} [x_1, x_2]$$

$\nabla h(x) = 3(x_1^2 + x_2^2)^{1/2} \cdot [x_1, x_2]$

(ii) The log-likelihood function, for $-\pi/2 \leq \theta < \pi/2$ is

$$\log_e \left(\frac{P_1(\theta)}{P_0(\theta)} \right) = -\log_e \left\{ \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{ -\frac{\theta^2}{2\sigma^2} \right\} \right\} + \ln(2\pi)^{-1}$$

$$= c + \frac{1}{2} l(\theta), \text{ where } l(\theta) = \frac{\theta^2}{\sigma^2}, \ c = \text{constant}$$

The N-P test says:

accept H_0 if $\log_e\left(\frac{p_1(\theta)}{p_0(\theta)}\right) \geq \alpha \equiv l(\theta) \geq \bar{\alpha}$

for some modified threshold $\bar{\alpha}$.

The "significance level constraint requires

$$P(l(\theta) \geq \bar{\alpha}) = 0.01, \text{ for } \theta \sim p_0(\theta)$$

i.e $\int_{\bar{\alpha}}^{\infty} \left(\frac{\theta}{\sigma}\right)^2 d\theta = 0.01$

But $\theta/\sigma \sim N(0,1)$, so $\bar{\alpha} = 6.635$ (from data)

So test is: Accept H0 (H1) if $(\theta/\sigma)^2 > 6.635$ (< 6.635)

The power of the test is

$$P[(\tfrac{\theta}{\sigma})^2 \geq \bar{\alpha} \mid H_1] = P[|\theta| \geq \sigma \bar{\alpha}^{1/2} \mid H1]$$

$$= (2\pi)^{-1} \times 2 (\pi - \sigma\sqrt{\bar{\alpha}})$$

$-\pi \quad \sigma\bar{\alpha}^{-1/2} \quad +\pi$