E3.08
ISE3.17

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2008

EEE/ISE PART III/IV: MEng, BEng and ACGI

ADVANCED SIGNAL PROCESSING

Corrected Copy
Q5

Wednesday, 7 May 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer TWO of questions 1, 2, 3 and ONE of questions 4, 5.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : D.P. Mandic, D.P. Mandic
Second Marker(s) : M. Petrou, M. Petrou

1) Consider the problem of parametric autoregressive moving average (ARMA) modelling. Write down the equation of a general AR(p) model. [1]

a) Consider the first order autoregressive (Markov) process.

i) Derive the expression for the autocorrelation function for this process. [3]

ii) Write down and plot the autocorrelation function for an AR(1) process for the cases when the parameter $a = 0.9$ and $a = -0.9$. [3]

iii) What are the variance and spectrum of such a process? What can we say about the spectrum of an AR(1) process for a negative value of the parameter a? [3]

iv) Define the partial autocorrelation function and explain how the partial autocorrelation coefficients are calculated. Can the values of partial autocorrelation coefficients suggest the order of the AR model of a given process? [3]

b) Consider the process

$$z[n] = a_1 z[n-1] + w[n] + b_1 w[n-1]$$

where $z[n]$ is the ARMA process, $w[n]$ is white noise, and a_1 and b_1 are the model parameters. State the conditions for which the process $z[n]$ is stationary and invertible.

[3]

For $z[n]$, find the expressions for

i) the first two autocorrelation coefficients, ρ_0 and ρ_1.

[2]

ii) the autocorrelation coefficients ρ_k for $k \geq 2$.

[2]

2) It frequently occurs in practice that the minimum variance unbiased estimator (MVU), even if it exists, cannot be found. Common solutions to this problem are suboptimal estimators, such as BLUE, MLE, or LS estimator.

Consider the problem of estimating the value of a DC level A in white Gaussian noise $\{w\}$, where the noisy data are given by

$$x[n] = A + w[n], \qquad n = 0, \ldots, N-1, \qquad w[n] \sim \mathcal{N}(0, \sigma_w^2)$$

a) Estimate the value of A using the best linear unbiased estimator (BLUE). [4]

   i) Explain the constraints involved in the definition of the BLUE. [1]

   ii) Do we need the knowledge of the probability density function (pdf) of noise? [1]

b) Estimate the value of A using maximum likelihood estimation (MLE). [4]

   i) What is the prior knowledge needed in order to apply MLE? [1]

   ii) When is the maximum likelihood estimation optimal? [1]

c) Estimate the value of A using the method of least squares (LS). [3]

   i) Are there any assumptions needed in order to conduct LS estimation? [1]

   ii) State one problem associated with the LS estimation. [1]

d) Determine the Cramer Rao Lower Bound (CRLB) for the unknown parameter A. Compare this solution with solutions from a), b) and c). [3]

3) Consider the problem of least squares (LS) estimation.

a) Sketch the block diagram of the data model and state the optimisation problem of the least squares estimation. [4]

b) Given a wide sense stationary random process $x(n)$, design a "linear" predictor that will predict the value $x(n+1)$ using a linear combination of $x(n)$ and $x(n-1)$. Thus the predictor for $x(n+1)$ is of the form

$$\hat{x}(n+1) = ax(n) + bx(n-1)$$

where a and b are constants. Assume that the process has zero mean $E\{x(n)\} = 0$ and that we want to minimise the mean square error

$$\xi = E\left\{[x(n+1) - \hat{x}(n+1)]^2\right\}$$

i) With $r_x(k)$ the autocorrelation of $x(n)$, determine the optimum predictor of $x(n)$ by finding the values of a and b that minimise the mean square error. [4]

ii) If $x(n+1)$ is uncorrelated with $x(n)$, what form does the predictor take? If $x(n+1)$ is uncorrelated with both $x(n)$ and $x(n-1)$, what form does the predictor take? [4]

c) It is often convenient to perform prediction through autoregressive modelling. If $x(n+1)$ is an $AR(2)$ process driven by white Gaussian noise $w \sim \mathcal{N}(0,1)$, derive the coefficients of such a predictor. [4]

i) Based on the estimation of $E\{\hat{x}(n+1)\}$ from its $AR(2)$ representation, state the expression for the prediction error. [4]

4)

a) State the problem of sequential least squares. [2]

b) Explain the difference between the method of least squares and sequential least squares. [2]

c) What are the advantages of using a sequential estimator? [3]

d) A least squares estimator for signal $\{x\}$ given by

$$x(n) = A + w(n)$$

that is, DC level A in white Gaussian noise $\{w\}$, is given by

$$\hat{A}(N-1) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)$$

i) Derive the sequential least squares estimator for this case. [6]

ii) Describe each of the terms that comprise the sequential estimator in i). [2]

iii) Derive the minimum least square error for the estimator in i). [2]

e) Consider a filter given by

$$y(n) = \frac{1}{4}\left[x(n) + x(n-1) + x(n-2) + x(n-3)\right]$$

Can such a filter be used to estimate the DC level in noise? Explain the difference between this estimator and the sequential least squares estimator. [3]

5) Consider the linear optimum filtering problem (Wiener filtering problem). The input–output relation of the filter is described by

$$y = \sum_{n=1}^{N} w_n x_n$$

If d denotes the desired response for the filter, and the error signal is $e = d - y$, then the mean squared error is defined as

$$J = \frac{1}{2} E\{e^2\}$$

a) If the weights w assume a time varying form, derive the steepest descent algorithm for an iterative solution of the Wiener filtering problem. [6]

b) Derive the least mean square (LMS) algorithm based on the use of instantaneous estimates of the statistical quantities used in ~~b)~~. a) [4]

   i) What kind of error is introduced by using the LMS? [2]

   ii) Explain the difference between the LMS and least squares solution. Which algorithm uses the "deterministic" cost function, and which the "stochastic" cost function? [2]

c) Two sets of measurements $\{x_1\}$ and $\{x_2\}$ are combined to give an estimate

$$y(n) = w(n)x_1(n) + (1 - w(n))\, x_2(n)$$

Derive the LMS update for the weight coefficient w. Explain in your own words the operation of this adaptive filter. [3]

   i) For $0 \leq w \leq 1$, we have a convex combination of measurements $x_1(n)$ and $x_2(n)$. Explain how the convexity constraint can help the adaptive filtering operation. In your own words, explain the operation for $w \in \{0, 1\}$. [3]