E1.11 - ISE1 MATHS

SOLUTIONS 2008

ISE 1 1

Solution 1. We have

$$\int_0^{2\pi} \exp(in\theta)\mathrm{d}\theta = [\frac{\exp(in\theta)}{in}]_0^{2\pi} = \frac{\exp(2\pi i n) - 1}{in}.$$ 5

If n is a non-zero integer, the numerator vanishes. If $n = 0$, the integrand is 1, so the integral is clearly equal to 2π. 1

Now $\cos(\theta) = \frac{1}{2}(\exp(i\theta) + \exp(-i\theta))$. Hence the integral we need is

$$\int_0^{2\pi} \cos^{2n}(\theta)\mathrm{d}\theta =$$ 8

$$\int_0^{2\pi} \sum_{r=0}^{2n} \frac{1}{2^{2n}} \exp(ir\theta - (2n-r)i\theta) \frac{(2n)!}{(r!)(n-r)!}\mathrm{d}\theta.$$

Every term in this sum integrates to zero, except for $r = n$, when the argument of the exponential vanishes: 4

$$\exp(in\theta - i(2n-n)\theta) = 1.$$

Hence 2

$$\int_0^{2\pi} \cos^{2n}(\theta)\mathrm{d}\theta = \frac{2\pi}{2^{2n}} \frac{(2n)!}{(n!)^2}.$$

(Total 20)

Setter JG

Checker MWL

ISE 1 2

Solution 2. (i) (a)

$$\sum_{n=1}^{\infty} \frac{n+3}{(n+2)(n+1)},$$

4

Here we have $\frac{n+3}{(n+2)(n+1)} > \frac{1}{n+2}$, and

$$\sum_{n=1}^{\infty} \frac{1}{n+2}$$

is divergent. Hence this series diverges too, by the comparison test.

(b)

4

$$\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{(n+2)(n+3)}.$$

Here the terms of the series are alternating in sign, and their magnitudes are monotonically decreasing. Hence the series converges, by the alternating series test.

(ii) The radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is given by

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$$R = \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}|,$$

if this exists. The series converges for $|z| < R$, diverges for $|z| > R$.

(a) For

$$\sum_{n=0}^{\infty} \frac{2n+1}{\sqrt{n^2+1}} z^n,$$

the radius of convergence is

4

$$R = \lim_{n\to\infty} |\frac{2n+1}{\sqrt{n^2+1}} \frac{\sqrt{(n+1)^2+1}}{2n+3}|$$

$$= \lim_{n\to\infty} \frac{2n+1}{2n+3} \lim_{n\to\infty} \frac{\sqrt{(n+1)^2+1}}{\sqrt{n^2+1}} = 1.$$

(b) For

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n,$$

the radius of convergence is

4

$$R = \lim_{n\to\infty} \frac{(n!)^2}{(2n)!} \frac{(2n+2)!}{((n+1)!)^2}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2} = 4.$$

Setter JG

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ISE 1 (3)

Solution 3. (a) (i) Here the numerator and denominator are continuous and the denominator non-zero, so

$$\lim_{x\to 0}\frac{\exp(2x)}{\cosh(x)} = e^0 = 1,$$

3

(ii) Here we may use L'Hôpital's rule, as numerator and denominator vanish together:

$$\lim_{x\to\pi/4}\frac{2\sin^2(x)-1}{\tan(x)-1} = \lim_{x\to\pi/4}\frac{2\sin(x)\cos(x)}{\sec^2(x)} = \frac{1}{1} = 1,$$

3

(iii) Extract a factor of n from each fractional power:

$$\lim_{n\to\infty}[n((n^2+3)^{1/2}-(n^3+n)^{1/3})] =$$

$$\lim_{n\to\infty}[n^2((1+\frac{3}{n^2})^{1/2}-(1+\frac{1}{n^2})^{1/3})] =$$

$$\lim_{n\to\infty}[n^2((1+\frac{3}{2n^2}+\mathrm{O}(n^{-4}))-(1+\frac{1}{3n^2}+\mathrm{O}(n^{-4})))] = \frac{3}{2}-\frac{1}{3} = \frac{7}{6}.$$

4

(b) The Maclaurin series for these two functions are:

$$\exp(x^2) = \sum_{n=0}^{\infty}\frac{x^{2n}}{n!},$$

3

and

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}.$$

3

Hence the first three non-zero terms of the Maclaurin series for the product

$$\exp(x^2)\cos(x)$$

are given by

$$\begin{aligned}\exp(x^2)\cos(x) &= (1+x^2+\frac{x^4}{2}...)(1-\frac{x^2}{2}+\frac{x^4}{24}...)\\ &= 1+\frac{x^2}{2}+(\frac{1}{24}-\frac{1}{2}+\frac{1}{2})x^4...\\ &= 1+\frac{1}{2}x^2+\frac{1}{24}x^4....\end{aligned}$$

4

ISE 1 4

Solution 4. Evaluate the integrals

(i) Integrate by parts:

$$\int_0^{\frac{\pi}{2}} x\cos(x)\mathrm{d}x,$$

$$= [x\sin(x)]_0^{\pi/2} - \int_0^{\pi/2} \sin(x)\,\mathrm{d}x = \frac{\pi}{2} - 1.$$

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(ii) Substitute $x = \exp(u)$:

$$\int_0^1 \ln(x)x^2\mathrm{d}x = \int_{-\infty}^0 u\exp(2u)\ \exp(u)\,\mathrm{d}u =$$

$$[u\exp(3u)/3]_{-\infty}^0 - \int_{-\infty}^0 \frac{\exp 3u}{3}\,\mathrm{d}u = 0 - \left[\frac{\exp(3u)}{9}\right]_{-\infty}^0 = -1/9.$$

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(iii) Split into partial fractions:

$$\int_1^\infty \frac{1}{x(x+1)(x+2)}\,\mathrm{d}x = \int_1^\infty \left[\frac{\frac{1}{2}}{x} - \frac{1}{x+1} + \frac{\frac{1}{2}}{x+2}\right]\mathrm{d}x =$$

$$\frac{1}{2}\left[\ln(\frac{x(x+2)}{(x+1)^2}\right]_1^\infty = \frac{1}{2}\ln(\frac{4}{3}).$$

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Setter SG

Checker MWL

ISE 1 S01 5

Solution 5. (i)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+2y}{2x+y};$$

This equation is homogeneous, so set $y = xv(x)$; then

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$$x\frac{\mathrm{d}v}{\mathrm{d}x} + v = \frac{1+2v}{2+v},$$

or, rearranging,

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+2v}{2+v} - \frac{2v+v^2}{2+v}$$
$$= \frac{1-v^2}{2+v}.$$

3

This equation is separable,

$$\int^x \frac{\mathrm{d}x'}{x'} = \int^v \frac{2+v'}{1-v'^2}\,\mathrm{d}v'$$

3

$$= \int^{y/x} \frac{3/2}{1-v'} + \frac{1/2}{v'+1}\,\mathrm{d}v',$$

so

$$\ln(x/x_0) = -3/2\ln(1-y/x) + 1/2\ln(1+y/x).$$

Here x_0 is an undetermined arbitrary constant.

(ii)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2y = \exp(-x^3), \quad \text{with} \quad y(0) = 0;$$

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This equation has an integrating factor $\exp(x^3)$, multiplying by this, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\exp(x^3)\right) = 1.$$

3

Hence

$$y\,\exp(x^3) = x - x_0;$$

but we are given $y(0) = 0$, so $x_0 = 0$. Hence $y = x\exp(-x^3)$.

(iii)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \exp(-x), \text{with} \quad y(0) = 1, \text{and} \quad y'(0) = 1.$$

2

Here the complementary function is a sum of exponentials, $\exp(\lambda x)$, where $\lambda^2 + 3\lambda + 2 = 0$, so $\lambda = -1$ or $\lambda = -2$. So the complementary function is

$$y_{CF} = A\exp(-x) + B\exp(-2x).$$

The particular integral cannot be just $\exp(-x)$, for this appears in the complementary function. Try

$$y_{PI} = \alpha x\exp(-x).$$

Then

$$y''_{PI} + 3y'_{PI} + 2y_{PI} = \alpha(-2\exp(-x) + 3\exp(-x)) = \exp(-x),$$

3

so $\alpha = 1$. Hence

$$y = A\exp(-x) + B\exp(-2x) + x\exp(-x).$$

To find the constants A and B, solve

$$y(0) = A + B = 1,$$

$$y'(0) = -A - 2B + 1 = 1,$$

2

hence $A = 2$, $B = -1$, and so

$$y = (2 + x)\exp(-x) - \exp(-2x).$$

Question 6

Marks & seen/unseen

Parts

Solution

a) $y^2 + 2xy\frac{dy}{dx} - 2\sin y - 2x\cos y\frac{dy}{dx} = 0$

so $y^2 - 2\sin y = (2x\cos y - 2xy)\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{y^2 - 2\sin y}{2x(\cos y - y)}$ 5

b) $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = 2x . 2r\cos\theta + 2y . \sin 2\theta$

$= \underline{4xr\cos\theta + 2y\sin 2\theta} = \underline{4r^3\cos^2\theta + 2r\sin^2 2\theta}$ 2

$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = 2x . (-r^2\sin\theta) + 2y . 2r\cos 2\theta$

$= \underline{-2xr^2\sin\theta + 4yr\cos 2\theta}$. 2

$= \underline{-2r^4\cos\theta\sin\theta + 4r^2\sin 2\theta\cos 2\theta}$

(c) $f_x = x^2 - 2y, \quad f_y = 2y - 2x$

Both 0 when $y = x$, $x^2 - 2x = 0$, so $x = 0$ or 2, $y = 0$ or 2. Stationary points

$\underline{(0,0), \quad (2,2)}$. 5

Let $A = f_{xx} = 2x, \quad B = f_{xy} = -2, \quad C = f_{yy} = 2$.

At $(0,0)$, $AC - B^2 = -4 < 0 \therefore$ saddle 3

At $(2,2)$, $AC - B^2 = 4 > 0$ and $A > 0 \therefore$ minimum 3

EXAMINATION QUESTIONS/SOLUTIONS 2007-08

Course: ISE 1

Question: 7

Marks & seen/unseen

Parts

Solution

Fourier series is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left(\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,dx\right)$$

$$= \frac{2}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^{\pi} = \begin{cases} -\frac{4}{n^2\pi}, & n \text{ odd} \\ 0, & n \text{ even}\end{cases}$$

So Fourier series:

$$\underline{\frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right)}$$ 14

Put $x = 0$:

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

Hence

$$\underline{1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}}$$ 6

Setter's initials: MLR

Checker's initials: AGK

Question 8

Marks & seen/unseen

Parts

Soln, continued

(b) Take Laplace transforms:

$$(1) \quad t\,L(y) + t\,L(z) + L(y) = 0$$

$$(2) \quad t\,L(y) + 2t\,L(z) - L(y) = \frac{1}{t+1}$$

2

(1) ×2 − (2) gives

$$(t+3)\,L(y) = -\frac{1}{t+1}$$

so $\quad L(y) = -\dfrac{1}{(t+1)(t+3)} = \dfrac{1}{2}\left(\dfrac{1}{t+3} - \dfrac{1}{t+1}\right)$

Hence

$$y = \frac{1}{2}\left(e^{-3x} - e^{-x}\right)$$

From (1), $\quad L(z) = -\dfrac{(t+1)}{t}\,L(y) = \dfrac{1}{t(t+3)}$

$$= \frac{1}{3}\left(\frac{1}{t} - \frac{1}{t+3}\right).$$

So

$$z = \frac{1}{3}\left(1 - e^{-3x}\right)$$

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Question

8

Marks & seen/unseen

Parts

Solution

(a) Take Laplace transforms of both sides:

$$-1 + tL(y) + 2L(y) = \frac{5t}{t^2+1}$$

$$\therefore \quad L(y) = \frac{5t}{(t^2+1)(t+2)} + \frac{1}{t+2}$$ 3

By Partial Fractions

$$\frac{5t}{(t^2+1)(t+2)} = \frac{at+b}{t^2+1} + \frac{c}{t+2}$$

where

$$(a+c)t^2 + (2a+b)t + 2b+c \equiv 5t$$

so $a = 2, b = 1, c = -2$.

So

$$y = L^{-1}\left(\frac{2t+1}{t^2+1} - \frac{1}{t+2}\right)$$

$$= \underline{2\cos x + \sin x - e^{-2x}}$$ 5

[As stated in question, no credit for methods not using Laplace transforms.]

Parts

Solutions

a) System of linear eqns is

$$x + y + z = 1$$
$$2x + y + az = -1$$
$$x - y + z = b$$

Augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & a & -1 \\ 1 & -1 & 1 & b \end{array}\right)$$

Reduce to echelon form:

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & a-2 & -3 \\ 0 & -2 & 0 & b-1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & a-2 & -3 \\ 0 & 0 & 4-2a & b+5 \end{pmatrix}$$

Last eqn is

$$(4-2a)z = b+5.$$

So

(i) one soln. if $a \neq 2$

(ii) a line if $a = 2$, $b = -5$

(iii) no solns. if $a = 2$, $b \neq -5$. 10

Parts

(b) Characteristic poly. of A is

$$\begin{vmatrix} 1-x & -7 \\ 0 & 8-x \end{vmatrix} = (x-1)(x-8).$$

So eigenvalues are 1, 8.

$\underline{\lambda = 1}$ Eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\underline{\lambda = 8}$ Eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

So take $P = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ (many other possible P's of course). Then 5

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} = D.$$

So if

$$B = P \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} P^{-1}$$

then

$$B^3 = \left(P \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} P^{-1}\right)^3 = P \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} P^{-1} = A.$$

So take

$$B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}}}$$ 5