EE4-29

Optimisation

1. The company XYZ has invested £20000 to develop a new product. The product can be manufactured for £2 per unit. The company then performs a marketing research. The conclusion of the research is that if the company spends £a on advertising then it can sell the product at price £p per unit and it will sell

$$2000 + 4\sqrt{a} - 20p$$

units.

a) Compute the revenue for sales as a function of a and p.

[2 marks]

b) Compute the overall costs associated to the production and commercialization of the product, that is the development cost plus the production cost and the advertising cost, as a function of a and p.

[2 marks]

c) Compute the company's profit as a function of a and p.

[2 marks]

d) The company wishes to select a and p to maximize the profit. Pose this problem as an unconstrained optimization problem (disregard the non-negativity conditions on a and p).

[2 marks]

e) Compute the unique stationary point of the profit. Using second order sufficient conditions of optimality show that the stationary point is a local maximizer.

[4 marks]

f) Assume that the company is forced to fix the sale price of the product to $p = \tilde{p}$, with $\tilde{p} > 2$.

i) Determine the optimal advertising cost as a function of $\tilde{p}$.

[4 marks]

ii) Determine the optimal profit as a function of $\tilde{p}$.

[2 marks]

iii) Plot the optimal profit as a function of the fixed price $\tilde{p}$ and show that, as $\tilde{p}$ increases the profit becomes negative. [2 marks]

2. A man launches his boat from point A (see Figure 2) on a bank of a straight river, 3 km wide, and wants to reach the point B which is on the opposite bank and it is a km, with $a > 0$, distant from the point C (which is, clearly, 3 km from point A), as quickly as possible. He can row 6 km/h and run 8 km/h.

He has three options:

- row the boat accross the river to point C and then run to point B;
- row directly to point B;
- row to some point D between C and B and then run to point B.

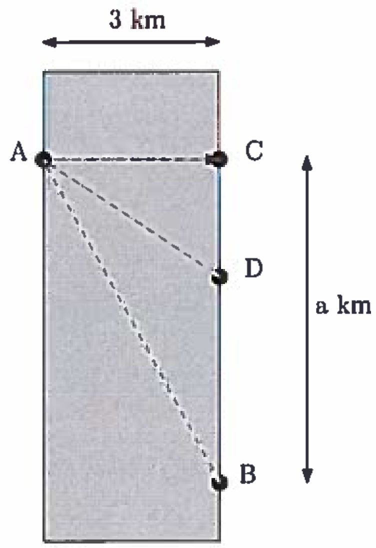

Figure 2

Determine, following the steps described below, the minimum time to reach point B.

a) Let x be the distance from C to D. Note that $x \in [0,a]$. Express the rowing distance D_{row}, that is the distance A to D, and the running distance D_{run}, that is the distance D to B, as a function of x and a. [2 marks]

b) Recall that, at constant speed, time $= \dfrac{\text{distance}}{\text{speed}}$. Determine the time T from A to B as a function of x and a, that is express the time from A to B as a function $T(x,a)$. [2 marks]

c) Assume $a = 8$ and solve the optimization problem

$$\min T(x,a)|_{a=8},$$
$$0 \leq x \leq a = 8.$$

[4 marks]

d) Consider again the problem

$$\min T(x,a),$$
$$0 \leq x \leq a,$$

with $a \geq 0$. Determine the optimal solution of the problem, as a function of a, and plot the optimal value of x as a function of a. [6 marks]

e) Consider now the problem of maximizing the distance travelled, by rowing and running, in one hour, that is consider the optimization problem

$$\max D_{run}(x,a) + D_{row}(x,a),$$
$$T(x,a) = 1.$$

Solve the problem using the constraint elimination method. Comment on the obtained result. [6 marks]

3. Two sets of two positive numbers $\{p_1, p_2\}$ and $\{q_1, q_2\}$ are given. The numbers describe probability distributions, that is

$$p_1 + p_2 = 1 \qquad\qquad q_1 + q_2 = 1.$$

In addition

$$0 < p_1 < \frac{1}{2} < p_2 < 1.$$

Consider the optimization problem

$$\begin{aligned} &\min_{x_1, x_2} 1 - (x_1 q_1 + x_2 q_2), \\ &x_1 p_1 + x_2 p_2 = \frac{1}{2}, \\ &0 \le x_1 \le 1, \\ &0 \le x_2 \le 1. \end{aligned}$$

a) Sketch on the (x_1, x_2)-plane the admissible set. [2 marks]

b) Using the sketch in part a), show that there are three types of admissible points:

$$Type_1 = (0, \star), \qquad Type_2 = (1, \star), \qquad Type_3 = (\star, \star),$$

where $\star$ denotes some numbers such that $0 < \star < 1$. [4 marks]

c) Write first order necessary conditions of optimality for the problem. [4 marks]

d) Exploiting the results in part b) and the conditions in part c) show that the following holds.

i) Assume $q_1 p_2 - q_2 p_1 < 0$. Show that $Type_1$ points are optimal and compute explicitly the optimal solution. [4 marks]

ii) Assume $q_1 p_2 - q_2 p_1 > 0$. Show that $Type_2$ points are optimal and compute explicitly the optimal solution. [2 marks]

iii) Assume $q_1 p_2 - q_2 p_1 = 0$. Show that all admissible points are optimal and explain why. [4 marks]

4. Consider the optimization problem

$$\min_{x,y} x^2 + y^2,$$

$$6 - x - 2y \leq 0.$$

a) Sketch in the (x,y)-plane the admissible set and the level lines of the objective function. Hence, using only graphical considerations determine the optimal solutions of the considered problem. [4 marks]

b) Consider the so-called log-barrier function of the problem defined as

$$B_\varepsilon(x,y) = x^2 + y^2 - \varepsilon \ln\Big(-(6 - x - 2y)\Big),$$

with $\varepsilon > 0$ and small.

i) Compute the stationary points of the function $B_\varepsilon(x,y)$. Show that $B_\varepsilon(x,y)$ has two stationary points: one admissible for all $\varepsilon \geq 0$ and one not admissible for all $\varepsilon \geq 0$. [8 marks]

ii) Show that the admissible stationary point determined in part b.i) is a local minimizer of the function $B_\varepsilon(x,y)$. [6 marks]

iii) Show that the admissible stationary point determined in part b.i) converges, as ε converges, to zero to the graphical solution determined in part a). [2 marks]

