EE1-10A

MATHEMATICS 1

********* Solutions *********

Information for Candidates:

********* Questions and Solutions *********

1. a) i) A function $s(x)$ has 4 real roots and has a continuous derivative. If $g(x) = s'(x)$, what is the minimum number of real roots of $g(x)$? Justify your answer. [4 marks]

By Rolle's theorem we know that there must be at least one x between any two $f(x)$ of equal value where $f'(x) = 0$, thus between two real roots of $f(x)$ there must be a real root of $g(x)$ so minimum number of roots of $g(x) = 3$.

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ii) If the order of a polynomial is known and its roots are known, when is this enough information to specify the polynomial uniquely – always, sometimes or never? Justify your answer. [4 marks]

A polynomial of order n has exactly $n-1$ roots, but has n parameters which can only be determined by n values so knowing the roots is never enough.

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iii) If $f(x)$ and $g(x)$ are 4^{th} and 3^{rd} order polynomials respectively, then solving $f(x) = g(x)$ is equivalent to finding the roots of a polynomial of what order? Justify your answer. [4 marks]

This is equivalent to $h(x) = f(x) - g(x) = 0$. Since $h(x)$ will have the largest order of f and g, the answer is 4^{th} order.

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b) i) What is the name of the type of function which describes the locus of a point which is always equidistant from a given line and a given reference point? [2 marks]

This is a parabola.

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ii) Find the function $y(x)$ which is equidistant from the line $y = 2$ and the point $(x, y) = (0, 4)$. [4 marks]

The distances to the line and point, r_1 and r_2, are $r_1 = \sqrt{x^2 + (y-4)^2}$ and $r_2 = |y-2|$.

Setting $r_1^2 = r_2^2$ *gives* $x^2 - 4y + 12 = 0$ *or* $y = \frac{x^2}{4} + 3$.

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c) i) For a complex number X, where

$$X = \frac{a+ib}{a-ib}$$

and a and b are real, find expressions for the modulus and argument of

X. [4 marks]

The modulus of the fraction equals the ratio of the numerator and denominator moduli, and since they are equal the modulus of X is 1. The angle is the numerator angle minus the denominator angle, so $\phi = 2\arctan\left(\frac{b}{a}\right)$?

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ii) For X as in (i) above, if $a = b$, find the value of X. [4 marks]

Here $\phi = 2\arctan(1) = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$. *So* $X = \exp(\frac{i\pi}{2}) = i$.

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d) i) Euler's equation gives $e^{i\theta}$ in terms of trigonometric functions. Write Euler's equation. [2 marks]

$e^{i\theta} = \cos\theta + i\sin\theta$.

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ii) Using Euler's equation, show that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. [2 marks]

$\cos\theta + i\sin\theta = e^{i\theta}$, *therefore* $(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{in\theta}$. *But* $e^{in\theta} = e^{i(n\theta)} = \cos n\theta + i\sin n\theta$

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iii) Using (i) and (ii) above, derive trigonometric identities for $\sin 3\theta$ and $\cos 3\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$. [4 marks]

Using (ii),

$$\begin{aligned}
\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\
&= \left(\cos^2\theta - \sin^2\theta + i2\cos\theta\sin\theta\right)(\cos\theta + i\sin\theta) \\
&= \left(\cos^3\theta - \sin^2\theta\cos\theta - 2\cos\theta\sin^2\theta\right) \\
&\quad - i\left(\sin^3\theta - \cos^2\theta\sin\theta - 2\sin\theta\cos^2\theta\right)
\end{aligned}$$

Equating real and imaginary parts respectively gives

$$\begin{aligned}
\cos 3\theta &= \cos^3\theta - \sin^2\theta\cos\theta - 2\cos\theta\sin^2\theta \\
\sin 3\theta &= -\left(\sin^3\theta - \cos^2\theta\sin\theta - 2\sin\theta\cos^2\theta\right)
\end{aligned}$$

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2. Consider the function

$$y = f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x,$$

defined for $x > 0$.

a) Compute the first and second derivatives of the function. Hence determine the only stationary point of the function. Show that $\frac{d^2 f}{dx^2} > 0$, for all $x > 0$, and hence that the stationary point is a minimum. [6 marks]

The first and second derivatives of f are

$$\frac{df}{dx} = \frac{1}{2}\frac{(x-1)(x+1)}{x} \qquad \frac{d^2 f}{dx^2} = \frac{1}{2}\frac{x^2+1}{x^2}.$$

Note that $\frac{d^2 f}{dx^2}$ is clearly positive and that $\frac{df}{dx} = 0$ implies $x = 1$. As a result, $x = 1$ is the only stationary point of the function (for $x > 0$) and it is a minimum.

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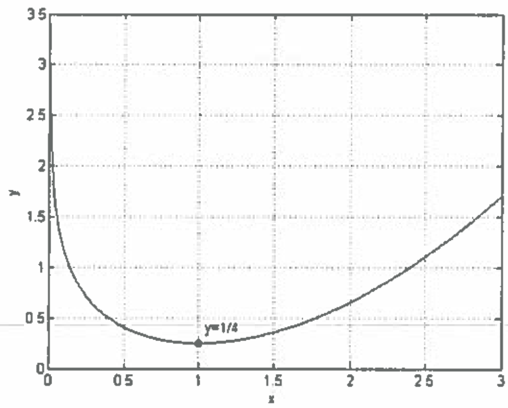
b) Plot the graph of the function for $x \in [0,3]$. Clearly indicate the stationary point and the value of the function for $x \to 0$. Note that the function f takes positive value for all $x > 0$. [4 marks]

The graph of the function is indicated in the figure below. Note that as $x \to 0$, the function goes to $+\infty$, the stationary point is the point $x = 1$ and $y = 1/4$, and the function is



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c) Compute the indefinite integral

$$I = \int f(x)dx.$$

[5 marks]

The indefinite integral is

$$I = \frac{1}{12}x^3 - \frac{1}{2}x\ln x + \frac{1}{2}x + c$$

where c is an integration constant.

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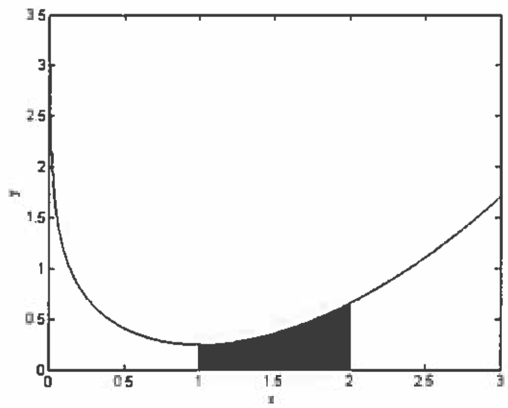
d) Consider the region A in the (x,y)-plane defined as follows. The region is bounded from above by the graph of the function f and from below by the x-axis. The region is bounded from the left by the line $x = 1$ and from the right by the line $x = 2$.

The region A is the shaded area in the figure below.



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i) Compute the area of the region A. [4 marks]

The area of the region is

$$A = \int_1^2 f(x)dx = \frac{13}{12} - \ln 2 \approx 0.39$$

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ii) Compute the length of the perimeter of the region A. Note that the perimeter is composed of four curves, the lengths of which must be computed separately. [6 marks]

The length of the perimeter is given by

$$P = 1 + f(1) + f(2) + \int_1^2 \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx = 3.$$

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iii) Compute the volume of the solid of revolution obtained by rotating the region A around the x-axis. [8 marks]

The volume of the surface of revolution is given by

$$
\begin{aligned}
V &= \pi \int_1^2 (f(x))^2 dx \\
&= \pi \left[\frac{1}{80} x^5 - \frac{1}{12} x^3 \ln x + \frac{1}{36} x^3 + \frac{1}{4} x (\ln x)^2 - \frac{1}{2} x \ln x + \frac{1}{2} x \right]_1^2 \\
&= \pi \left(\frac{779}{720} - \frac{5}{3} \ln 2 + \frac{1}{2} (\ln 2)^2 \right) \approx 0.1669\pi
\end{aligned}
$$

3. The complex Fourier series for a periodic function, $u(t)$, with period $T = \frac{1}{F}$ is given by

$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{j2\pi nFt}.$$

a) Show that, if m and n are integers, then

$$\int_0^T e^{j2\pi mFt} e^{j2\pi nFt}\, dt = \begin{cases} T & \text{if } m = -n \\ 0 & \text{otherwise} \end{cases}.$$

[6 marks]

$$I = \int_0^T e^{j2\pi mFt} e^{j2\pi nFt}\, dt = \int_0^T e^{j2\pi(m+n)Ft} dt$$
$$= \frac{1}{j2\pi(m+n)F}\left[e^{j2\pi(m+n)Ft}\right]_0^T$$
$$= \frac{1}{j2\pi(m+n)F}\left(e^{j2\pi(m+n)FT} - 1\right)$$

However $FT = 1$ and, since $m+n$ is an integer, $e^{j2\pi(m+n)FT} = e^{j2\pi(m+n)} = 1$.

Hence $I = \frac{0}{j2\pi(m+n)F}$ which is zero unless the denominator is also zero which happens when $m+n = 0$.

For this special case, $I = \int_0^T e^{j2\pi(m+n)Ft} dt = \int_0^T e^{j0} dt = T$.

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b) Hence show that

$$\frac{1}{T}\int_0^T u(t) e^{-j2\pi mFt} dt = U_m.$$

State clearly any assumptions you make. [8 marks]

$$\frac{1}{T}\int_0^T u(t)e^{-j2\pi mFt} dt = \frac{1}{T}\int_0^T \sum_{n=-\infty}^{\infty} U_n e^{j2\pi nFt} e^{-j2\pi mFt} dt$$
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} U_n \int_0^T e^{j2\pi nFt} e^{-j2\pi mFt} dt$$
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} U_n T \delta_{n-m}$$
$$= \sum_{n=-\infty}^{\infty} U_n \delta_{n-m} = U_m$$

where $\delta_k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$.

In the second line, we assume that we can interchange the order of the summation and integral. In the final line we are summing an infinite number of terms but they are all zero except for the term with $m = n$.

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c) Suppose $u(t)$ has period $T_u = 4$ and $u(t) = e^{-0.3t}$ for $0 \le t < 4$.

By evaluating the integral in part b), determine an expression for the complex Fourier coefficients U_n. Simplify the expression where possible. [10 marks]

$$
\begin{aligned}
U_n &= \frac{1}{T}\int_0^T u(t)e^{-j2\pi nFt}dt \\
&= \frac{1}{4}\int_0^4 e^{-0.3t}e^{-j0.5\pi nt}dt \\
&= \frac{1}{4}\int_0^4 e^{(-0.3-j0.5\pi n)t}dt \\
&= \frac{1}{4(-0.3-j0.5\pi n)}\left[e^{(-0.3-j0.5\pi n)t}\right]_0^4 \\
&= \frac{1}{-1.2-j2\pi n}\left(e^{-1.2-j2\pi n}-1\right)
\end{aligned}
$$

Since n is an integer, $e^{-j2\pi n} \equiv 1$ *and so* $U_n = \frac{e^{-1.2}-1}{-1.2-j2\pi n} = \frac{1-e^{-1.2}}{1.2+j2\pi n}$.

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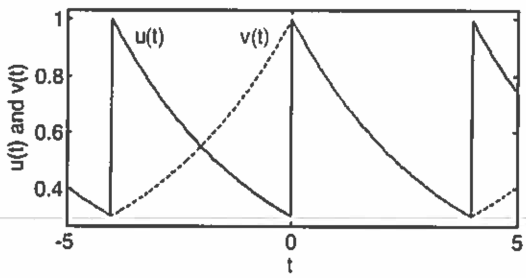
d) Suppose $v(t)$ has period $T_v = 8$ and $v(t) = \begin{cases} u(t) & \text{for } 0 \le t < 4 \\ u(-t) & \text{for } -4 \le t < 0 \end{cases}$.

i) Sketch a graph showing both $u(t)$ and $v(t)$ on the same set of axes over the range $-5 \le t \le 5$. [4 marks]

$u(t)$ and $v(t)$ are identical for $0 \le t < 4$ but $v(t)$ is an even function. So the graphs look like



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ii) The partial Fourier series of order N are defined by

$$u_N(t) = \sum_{n=-N}^{N} U_n e^{j2\pi nFt}$$

$$v_N(t) = \sum_{n=-N}^{N} V_n e^{j2\pi nFt}$$

where U_n and V_n are the complex Fourier coefficients of $u(t)$ and $v(t)$ respectively.

Explain why, for any fixed value of N, $v_N(t)$ will generally be a better approximation of $u(t)$ over the range $0 \le t \le 4$ than $u_N(t)$. You are <u>not</u> required to determine expressions for $u_N(t)$ and $v_N(t)$. [5 marks]

$u(t)$ has a discontinuity whenever t is a multiple of 4 whereas $v(t)$ has no discontinuities. This has two consequences: (i) $u_N(t)$ will suffer from Gibbs' phenomenon and will overshoot $u(t)$ by about 9% either side of the discontinuity and (ii) for large n we will have $U_n \propto n^{-1}$ whereas we will have $V_n \propto n^{-2}$. This means that the energy of the error will decrease faster with N for $v_N(t)$ than for $u_N(t)$. Although not requested from candidates, graphs of $u_6(t)$ and $v_6(t)$ are shown below.