Fields 2012 – Solutions

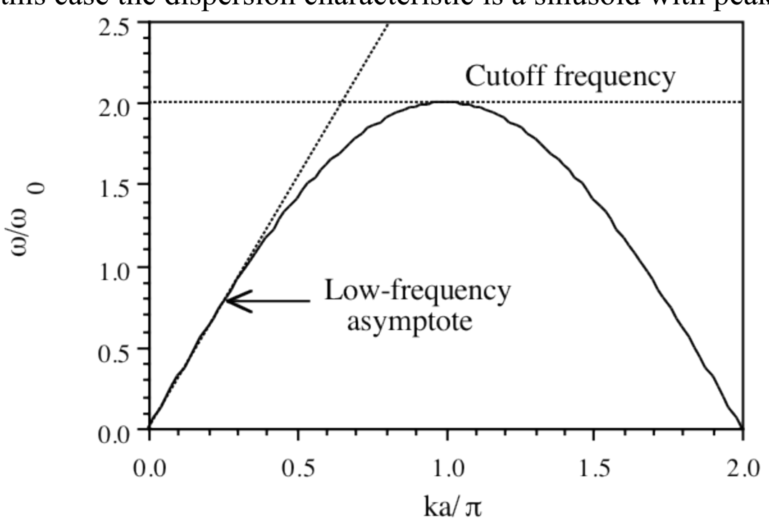
1a) Answer two parts out of three.

i) The characteristic impedance Z_0 is the ratio between voltage and current (for waves travelling in a transmission line) or between electric field and magnetic field (for electromagnetic waves).
[2]

Most candidates answered this correctly.
For example for a transmission line, the governing equations are;
$dV/dz = -j\omega L_p I$ and $dI/dz = -j\omega C_p V$
Hence, $d^2V/dz^2 = - j\omega L_p dI/dz = -\omega^2 L_p C_p V$
Assuming the travelling wave solutions $V(z) = V_0 \exp(-jkz)$ and $I(z) = I_0 \exp(-jkz)$
We get $-k^2V = -\omega^2 L_p C_p V$, so that $k = \omega(L_p C_p)^{1/2}$
We also get $-jkV_0 = -j\omega L_p I_0$
Hence $Z_0 = V_0/I_0 = \omega L_p/k = \sqrt{(L_p/C_p)}$
Most common mistake: failure to give sufficient detail, e.g derive a formula.
[3]

ii) A dispersion diagram is a plot of angular frequency (ω) against propagation constant (k). It is very useful way of displaying the important features of wave propagation, since it allows the phase velocity $v_p = \omega/k$, the group velocity $v_g = d\omega/dk$ and the frequency range of propagating waves to be read off.
Most candidates answered this correctly.
[2]

For example, the relation between w and k for a low pass ladder network is $\omega/\omega_0 = 2 \sin(ka/2)$. In this case the dispersion characteristic is a sinusoid with peak amplitude $2\omega_0$ as shown below



Most common mistake: failure to illustrate the answer with a dispersion graph.
[3]

iii) The phase velocity v_{ph} gives the speed of a single wave. Unfortunately a single wave cannot carry any information, since it never varies. To send some data, a carrier must be modulated. The envelope (which contains the information) then travels at a slightly different speed, the group velocity v_g (which represents the velocity of a group of waves).

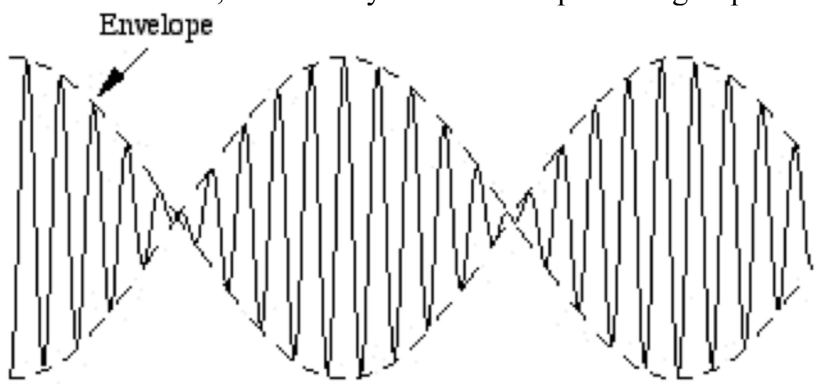
Most candidates answered this correctly.

[2]

To calculate v_g, consider the simplest possible AM signal, formed by beating together two signals of different angular frequencies $\omega + d\omega$ and $\omega - d\omega$. The corresponding k-values at these frequencies are $k + dk$ and $k - dk$. For equal amplitudes, the combined voltage is:

$$V = V_0 [\exp\{j((\omega + d\omega)t - (k + dk)z)\} + \exp\{j((\omega - d\omega)t - (k - dk)z)\}]$$

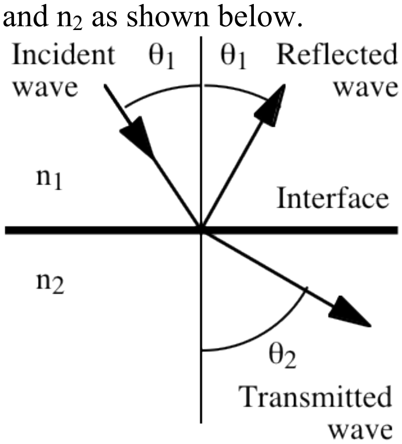
This result can be written alternatively as $V = 2V_0 \exp\{j\omega t - kz)\} \cos\{d\omega t - dk z\}$. Hence, the wave is an amplitude-modulated carrier as shown below. The velocity of the carrier is $v_p = \omega/k$ as before. However, the velocity of the envelope is the group velocity $v_g = d\omega/dk$.



Most common mistake: failure to illustrate the answer with mathematics and/or a diagram showing the information-carrying envelope.

[3]

b) Snell's law $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$ relates the angles of incidence and refraction when an electromagnetic wave strikes an interface between two dielectric media with refractive indices n_1 and n_2 as shown below.



Most candidates got the formula correct. Most common mistake: failure to include a diagram.

[2]

At a critical angle $\theta_1 = \theta_C = \sin^{-1}(n_2/n_1)$, we obtain $\theta_2 = \pi/2$. In this case, the transmitted wave is parallel to the interface, and there is no real solution for θ_2 for $\theta_1 > \theta_c$,. Under these conditions,

total internal reflection occurs and no energy can cross the interface. Note however that we must have $n_2 < n_1$ for total internal reflection, so that incidence must be from the high-index side.
Most candidates answered this correctly.

[2]

Assuming that $n_1 = 1.5$ (glass) and $n_2 = 1$ (air) we obtain $\theta_C = \sin^{-1}(1/1.5) = 41.81^o$.
Most candidates answered this correctly.

[1]

c) The electric field due to one plane wave travelling at an angle $+\theta$ to the z-axis is:
$E_y = E_{y0}$ {exp[-jk_0(z cos(θ) + x sin(θ))]
Here $k_0 = 2\pi/\lambda$
Most candidates answered this correctly.

[1]

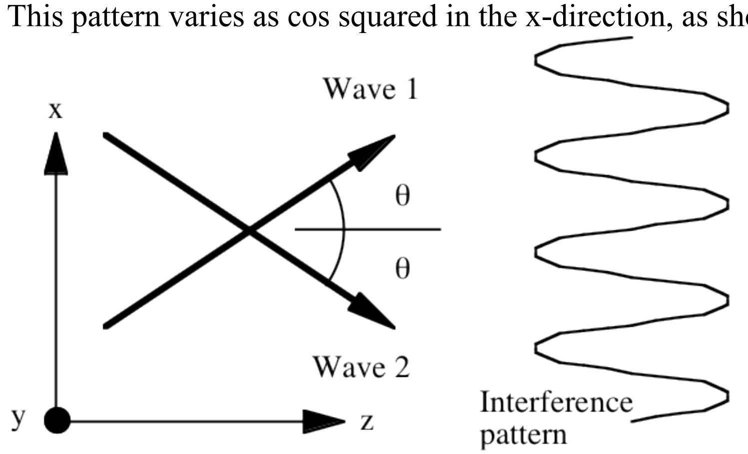
For two waves travelling at angles $\pm\theta$ to the z-axis, the combined field can be written as the sum:
$E_y = E_{y0}$ {exp[-jk_0(z cos(θ) + x sin(θ))] + exp[-jk_0(z cos(θ) - x sin(θ))}
Most candidates answered this correctly.

[1]

This field can be written alternatively as
$E_y = 2E_{y0}$ exp[-jk_0z cos(θ)] cos[k_0x sin(θ)]$\underline{j}$
Most common mistake: failure to re-arrange the plane wave expression correctly.

[1]

The time averaged power density is $\underline{S} = 1/2\ (\underline{E} \times \underline{H}^*)$
In this case, $S_z = -1/2\ E_yH_x^*$
Since H_x is proportional to E_y, $S_z = C \cos^2[k_0x \sin(\theta)]$
This pattern varies as cos squared in the x-direction, as shown in the RH figure.



Most common mistake: failure to draw a picture.

[2]

2a) A driven antenna will generate a characteristic pattern of time-averaged electric and magnetic field $\underline{E}$ and $\underline{H}$. These fields in turn will give rise to a power flow $\underline{S} = 1/2\ \mathrm{Re}(\underline{E} \times \underline{H}^*)$. In the far field, $\underline{S}$ is radial, so $\underline{S} = S\ \underline{r}$. At a given radius, S is a function only of the angular coordinates θ and ϕ, and is known as a radiation pattern. The normalised radiation pattern F is the ratio of S to its maximum value. The directivity D is then the ratio of the maximum of F to its average over 4π space.
Most common mistake: failure to explain directivity correctly, even though the expression is in the formula sheet.

[3]

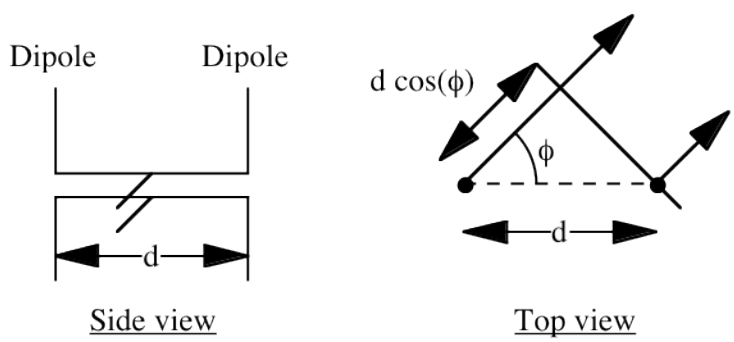
The effective area A_e is the equivalent area from which an antenna can gather power, and deliver it to matched load.
Most candidates answered this correctly.

[2]

b) Assuming an average power P, the power density at radius r is $S = P/4\pi r^2$
Assuming an effective area A, the received power is $P_R = AS = P\ A/4\pi r^2$
If $P = 20 \times 10^3$ W, $R = 10 \times 10^3$ m and $A = 10\ \mathrm{m}^2$, then:
$P_R = 10^4 \times 10 /(4 \times \pi \times 10^8)\ \mathrm{W} = 7.96 \times 10^{-5}\ \mathrm{W} = 79.6\ \mu\mathrm{W}$
Most candidates answered this correctly.

[5]

c) The broadside antenna consists of two dipoles arranged thus:



Most candidates answered this correctly.

[5]

d) In the far field, suppose each antenna generates an electric field $E_\theta(\phi)$. A receiver will detect contributions from one, with phases that depend on the distance travelled. Signals from the LH dipole travel a distance $d \cos(\phi)$ further than those from the RH dipole.

Ignoring the common path, the combined field $E_\theta'(\phi)$ is:
$E_\theta'(\phi) = E_\theta(\phi)\ \{1 + \exp[-jk_0 d \cos(\phi)]\}$ where $k_0 = 2\pi/\lambda$.

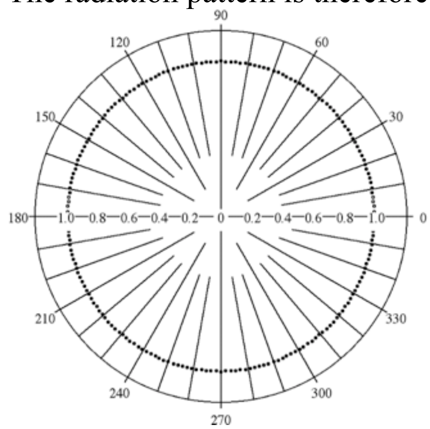
[3]

If the antennae are individually isotropic, $E_\theta(\phi) = 1$. We then obtain:
$E_\theta'(\phi) = \exp[-jk_0 d \cos(\phi)/2]\ \{\exp[+jk_0 d \cos(\phi)/2] + \exp[-jk_0 d \cos(\phi)/2]\}$, or
$E_\theta'(\phi) = 2E_\theta(\phi) \exp[-jk_0 d \cos(\phi)/2] \cos[k_0 d \cos(\phi)/2]$
The normalised radiation pattern is $F'(\phi) = |E_\theta'(\phi)|^2$ divided by its maximum value
In this case, $F'(\phi) = \cos^2[\cos(\phi)\ k_0 d/2] = \cos^2[\cos(\phi)\ \pi d/\lambda]$

Most common mistake: failure to work out $|E_\theta'(\phi)|^2$ correctly.

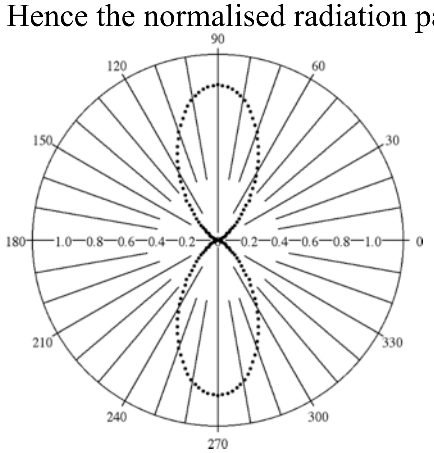
[4]

e) If the frequency is f = 1.5 MHz, the wavelength is $\lambda = c/f = 3 \times 10^8 / (1.5 \times 10^6) = 200$ m
If d = 1 m, then $d/\lambda = 0.005$ and $\pi d/\lambda = 0.005\,\pi$
Whatever the value of ϕ, $\cos(\phi)\,\pi d/\lambda$ is then very small, so $\cos^2[\cos(\phi)\,\pi d/\lambda] \approx 1$
The radiation pattern is therefore approximately isotropic, as shown below



[4]

If the frequency is f = 150 MHz, $\lambda = 3 \times 10^8 / (150 \times 10^6) = 2$ m
If d = 1 m, then $d/\lambda = 0.5$ and $\pi d/\lambda = \pi/2$
The normalised radiation pattern is then $F'(\phi) = \cos^2[\cos(\phi)\,\pi/2]$
When $\phi = 0$, $F'(\phi) = \cos^2(\pi/2) = 0$; when $\phi = \pi/2$, $F'(\phi) = \cos^2(0) = 1$ etc
Hence the normalised radiation pattern is as shown below.



[4]

Most common mistake: most candidates who failed to work out $|E_\theta'(\phi)|^2$ correctly could not draw the radiation patterns, even though they are from the lecture notes.