Exam Solutions

1. a) **This question is straightforward bookwork. Most students have done well.**

i) [2]

White noise is a random process which has a flat power spectral density spanning the entire spectrum.

ii) [2]

White noise cannot be realized since it has infinite average power. It is a reasonable modelling assumption as long as the bandwidth of the noise process is larger than the bandwidth of the communication system itself.

iii) [2]

$$S_N(f) = \frac{N_o}{2},\ \forall\, f.$$

$$R_N(\tau) = \frac{N_o}{2}\delta(\tau).$$

iv) [3]

A process is Gaussian when the probability density function of its samples follow a Gaussian (Normal) distribution. Gaussian distribution for noise is motivated based on the *central limit theorem*, which states that the arithmetic mean of a sufficiently large number of realizations of independent random variables is approximately Gaussian distributed.

v) [2]

Since it is white, its samples are uncorrelated:

$$E[X(t_1)X(t_2)] = R_N(t_2 - t_1) = 0, \text{ for } t_1 \neq t_2.$$

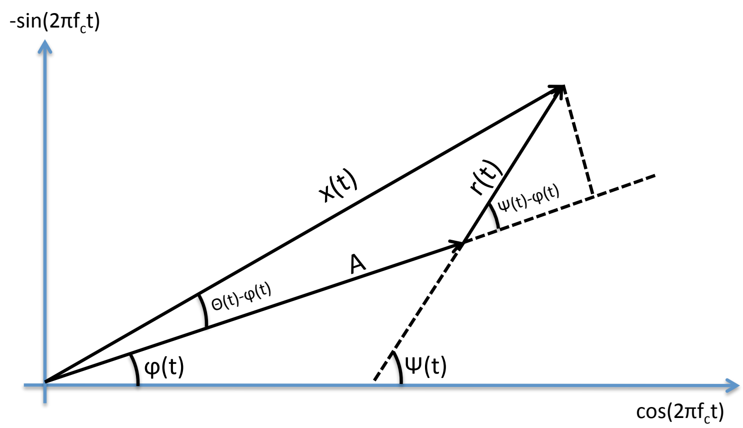
Since it is also Gaussian, samples are independent.

b) i) [2]

Bandpass filter at the input of the receiver removes the noise components out of the pass-band of the transmitted FM signal $f_c \pm \frac{B_T}{2}$ and $-f_c \pm \frac{B_T}{2}$

ii) [3]

$$\begin{aligned} x(t) =& s(t) + n(t), \\ =& A\cos\left[2\pi f_c t + \phi(t)\right] + r(t)\cos\left[2\pi f_c t + \psi(t)\right]. \end{aligned}$$



Please be careful with the naming of the axes in the phasor diagram.

iii) [2]

Since the FM modulated signal has a constant envelope, limiter removes the amplitude variations.

iv) [3]

$$v(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt} \simeq k_f m(t) + n_d(t).$$

v) [2]

$$n_d(t) = \frac{1}{2\pi A}\frac{dn_Q(t)}{dt}.$$

From this, we can find

$$S_{N_d}(f) = \frac{f^2}{A^2} S_{N_Q}(f).$$

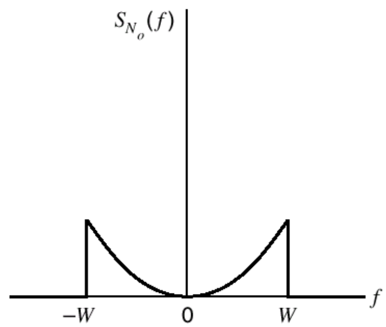
Using the PSD of the quadrature component of a narrowband white noise signal, we get

$$S_{N_d}(f) = \begin{cases} \frac{N_o f^2}{A^2}, & \text{if } |f| \le \frac{B_T}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

vi) [2]

We have

$$S_{N_o}(f) = \begin{cases} \frac{N_o f^2}{A^2}, & \text{if } |f| \le W, \\ 0, & \text{otherwise.} \end{cases}$$



c) **This question requires understanding the definition of entropy (provided here) and applying it to a joint distribution.**

i) [3]

If S and T are independent, then

$$P\{(S,T)=(s_k,t_m)\}=P\{S=s_k\}P\{T=t_m\},$$

that is, $r_{k,m}=p_kq_m$ for any $k=1,\ldots,K$ and $m=1,\ldots,M$. Then we can write:

$$\begin{aligned}
H(S,T) &= \sum_{k=1}^{K}\sum_{m=1}^{M} r_{k,m}\log_2\left(\frac{1}{r_{k,m}}\right)\\
&= \sum_{k=1}^{K}\sum_{m=1}^{M} p_kq_m\log_2\left(\frac{1}{p_kq_m}\right)\\
&= \sum_{k=1}^{K}\sum_{m=1}^{M} p_kq_m\left[\log_2\left(\frac{1}{p_k}\right)+\log_2\left(\frac{1}{q_m}\right)\right]\\
&= \sum_{k=1}^{K}\sum_{m=1}^{M} p_kq_m\log_2\left(\frac{1}{p_k}\right)+\sum_{k=1}^{K}\sum_{m=1}^{M} p_kq_m\log_2\left(\frac{1}{q_m}\right)\\
&= \sum_{k=1}^{K} p_k\log_2\left(\frac{1}{p_k}\right)+\sum_{m=1}^{M} q_m\log_2\left(\frac{1}{q_m}\right)\\
&= H(S)+H(T).
\end{aligned}$$

ii) [4]

The second part requires the use of the first part, since it asks for nothing but the joint entropy of 8 indepedent events. Very few students were able to see this. Some other tried to find the joint probability distribution of 8 coin tosses, which gives the correct answer, albeit with significantly more effort.

Let X_i denote the random variable corresponding to the outcome of the ith coin toss. We have

$$\begin{aligned}
H(X_i) &= 0.2\times\log_2\left(\frac{1}{0.2}\right)+0.8\times\log_2\left(\frac{1}{0.8}\right)\\
&\approx 0.72.
\end{aligned}$$

Note that the entropy of X is nothing but the joint entropy of the outcome of the 8 coin tosses. Since the coin tosses are independent of each other, it follows from the previous question that the joint entropy is simply the sum of the individual entropies. Hence, we have

$$H(X)=\sum_{i=1}^{8}H(X_i)=8\times 0.72=5.76$$

d) **A very similar problem to this one was solved in the class. Most students have correctly answered it.**

i) [2]

For $n = 10$ we have 10^5 bits per second. This corresponds to

$$2 \times 60 \times 60 \times 10^5 = 72 \times 10^7$$

bits in 2 hours. This is equivalent to 90 Mbytes of memory space.

ii) [3]

Since the signal is Gaussian with standard deviation, its average power is $P_S = \sigma^2$. The quantization step is chosen such that a range of 6σ can be covered with 2^n quantization levels. This corresponds to $\Delta = \frac{6\sigma}{2^n}$.

The mean square error of a uniform quantizer is given by

$$P_N = \frac{\Delta^2}{12} = \frac{36\sigma^2}{12 \times 2^{2n}}.$$

The signal to noise ratio is found to be

$$SNR = \frac{P_S}{P_N} = \frac{\sigma^2}{3\sigma^2 2^{-2n}} = \frac{2^{2n}}{3}.$$

Changing to the dB scale, we obtain

$$\begin{aligned} SNRdB =& 10\log_{10}\frac{1}{3} + 20n\log_{10} 2 \\ =& -4.77 + 6.02n. \end{aligned}$$

iii) [3]

Let X denote a zero-mean Gaussian random variable with standard deviation σ. The overload probability is found as follows:

$$\begin{aligned} P_{\text{overload}} =& P\{X < 3\sigma\} + P\{X > 3\sigma\} \\ =& 2\int_{3\sigma}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\ =& 2 \cdot Q(3) \approx 2 \times 10^{-3}. \end{aligned}$$

2. a) **This problem requires basic knowledge of probability and stochastic processes; in particular, definitions of variance, uncorrelated, independent, wide-sense stationary, power spectral density.**

i) [2]

We have

$$\begin{aligned} E[ZT] &= E[3XY + 15X] \\ &= 3E[XY] + 15E[X] \\ &= 3E[X]E[Y] + 15E[X] \\ &= 3E[X](E[Y] + 5) \\ &= E[3X]E[Y + 5] = E[Z]E[T]. \end{aligned}$$

ii) [4]

No. To show that U and V are not always independent, we just need one counter-example. For example, consider $Y = 0$ independent of X. In this case, we have $U = V = X$, which are not independent.

iii) [4]

$$\begin{aligned} E[UV] &= E[(X+Y)(X-Y)] = E[X^2 - Y^2] \\ &= E[X^2] - E[Y^2] \\ &= \mathrm{Var}(X) + E[X]^2 - \mathrm{Var}(Y) - E[Y]^2 \\ &= E[X]^2 - E[Y]^2 \\ &= (E[X] + E[Y])(E[X] - E[Y]) = E[U]E[V]. \end{aligned}$$

b) **Although we have solved a very similar problem in the exercises, some students had difficulty with this second part. The most common mistake stemmed from not identifying the random and deterministic parameters in part i). Some students treated f_c as random as well, rather than taking the expectations with respect to a only.**

i) [4]

$$\begin{aligned} E[Y(t)] &= \int_{a=-\infty}^{\infty} a\cos(2\pi f_c t) \cdot f_A(a)\, da \\ &= \frac{1}{4}\cos(2\pi f_c t)\int_{a=-2}^{2} a\, da \\ &= 0. \end{aligned}$$

$$\begin{aligned} E[Y(t_1)Y(t_2)] &= \int_{a=-2}^{2} a^2 \cos(2\pi f_c t_1)\cos(2\pi f_c t_2) f_A(a) da \\ &= \frac{1}{4}\cdot\frac{1}{2}\left[\cos(2\pi f_c(t_2 - t_1)) + \cos(2\pi f_c(t_1 + t_2))\right]\int_{a=-2}^{2} a^2 da \\ &= \frac{2}{3}\cdot\frac{1}{2}\left[\cos(2\pi f_c(t_2 - t_1)) + \cos(2\pi f_c(t_1 + t_2))\right] \end{aligned}$$

Since the autocorrelation function is not only a function of $(t_2 - t_1)$, $Y(t)$ is not a wide-sense stationary process.

ii) [6]

Most common difficulty in the second part of the problem was due to not separating the two expectations, which follows due to the independence of the two random variables.

$$\begin{aligned} E[Z(t)] =& E[A\cos(2\pi f_c t + \Theta)] \\ =& E[A] \cdot E[\cos(2\pi f_c t + \Theta)] \\ =& 0 \end{aligned}$$

where the first line follows from the independence of A and Θ, and the second line follows since both expectations are equal to 0.

$$\begin{aligned} E[Z(t_1)Z(t_2)] =& E[A^2\cos(2\pi f_c t_1 + \Theta)\cos(2\pi f_c t_2 + \Theta)] \\ =& E[A^2]E[\cos(2\pi f_c t_1 + \Theta)\cos(2\pi f_c t_2 + \Theta)] \end{aligned}$$

due to independence. We have

$$\begin{aligned} E[A^2] =& \int_{a=-2}^{2} a^2 f_A(a)\, da \\ =& \frac{1}{4}\frac{a^3}{3}|_{a=-2}^{2} = \frac{4}{3} \end{aligned}$$

and

$$\begin{aligned} & E[\cos(2\pi f_c t_1 + \Theta)\cos(2\pi f_c t_2 + \Theta)] \\ & = \frac{1}{2\pi}\int_{\theta=-pi}^{\pi} \cos(2\pi f_c t_1 + \Theta)\cos(2\pi f_c t_2 + \Theta) \\ & = \frac{1}{4\pi}\int_{\theta=-pi}^{\pi} [\cos(2\pi f_c(t_2 - t_1)) + \cos(2\pi f_c(t_1 + t_2) + 2\Theta)]\, d\theta \\ & = \frac{1}{2}\cos(2\pi f_c(t_2 - t_1)) \end{aligned}$$

Since $E[Z(t)]$ is independent of time t, and the autocorrelation function is a function of $(t_2 - t_1)$, $Z(t)$ is a wide-sense stationary process.

c) **While the first part of this problem was correctly answered by a fair number of students, most had difficulty in the second part. It seems that most students did not remember the relation between the autocorrelation function and the power spectral density.**

i) [4]

We have

$$E[Y(t)] = E[X(t+a)] - E[X(t-a)] = 0,$$

since $E[X(t)]$ is independent of time t. We can also write

$$\begin{aligned}
E[Y(t+\tau)Y(t)] =& E\left[(X(t+\tau+a)-X(t+\tau-a))(X(t+a)-X(t-a))\right] \\
=& E[X(t+\tau+a)X(t+a)] - E[X(t+\tau+a)X(t-a)] \\
& - E[X(t+\tau-a)X(t+a)] + E[X(t+\tau-a)X(t-a)] \\
=& R_X(\tau) - R_X(\tau+2a) - R_X(\tau-2a) + R_X(\tau) \\
=& 2R_X(\tau) - R_X(\tau+2a) - R_X(\tau-2a).
\end{aligned}$$

Since the expectation of $Y(t)$ is independent of time t, and its autocorrelation function is a function of τ, $Y(t)$ is a wide-sense stationary process.

ii) [6]

Remember that the power spectral density for a wide-sense stationary process is obtained by taking the Fourier transform of its autocorrelation function. We have

$$\begin{aligned}
S_Y(f) =& 2S_X(f) - S_X(f)e^{j2\pi f(2a)} - S_X(f)e^{-j2\pi f(2a)} \\
=& S_X(f)\left[2 - e^{j4\pi fa} - e^{-j4\pi fa}\right] \\
=& S_X(f)\left[2 - \cos(4\pi fa) - j\sin(4\pi fa) - \cos(-4\pi fa) - j\sin(-4\pi fa)\right] \\
=& S_X(f)\left[2 - 2\cos(4\pi fa)\right] \\
=& S_X(f)\left[4 - 4\cos^2(2\pi fa)\right] \\
=& 4S_X(f)\sin^2(2\pi fa).
\end{aligned}$$

Alternative Solution:

We can consider $Y(t)$ as $X(t)$ going through a linear time-invariant (LTI) system with impulse response $h(t) = \delta(t+a) - \delta(t-a)$. We know that when a WSS process is applied to an LTI system, the output is also a WSS process.

Moreover, it is known that $S_Y(f) = |H(f)|^2 S_X(f)$. We have

$$H(f) = e^{j2\pi fa} - e^{-j2\pi fa} = j \times 2\sin(2\pi fa).$$

Hence,

$$S_Y(f) = 4\sin^2(2\pi fa)S_X(f).$$

3. a) **First three parts of this problem are standard steps that have been done in the lecture. The solution expects the students to apply the basic ideas in signal detection in digital transmission, in a slightly modified setup with message-dependent noise. The last part requires the use of the Leibnitz rule, which has been done in the lecture and class multiple times. The common mistake stemmed from not correctly applying the Leibnitz rule and manipulating the resultant equation.**

i) [4]

When a 0 is transmitted, the receiver makes an error if the received signal is above the threshold T. Then we can write the error probability as

$$\begin{aligned} P_{e0} &= \int_T^\infty \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{x^2}{2\sigma_0^2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{T/\sigma_0}^\infty e^{-\frac{x^2}{2}} dx \\ &= Q\left(\frac{T}{\sigma_0}\right). \end{aligned}$$

ii) [2]

When a 1 is transmitted, the receiver makes an error if the received signal is below the threshold T. Then the corresponding error probability is

$$\begin{aligned} P_{e1} &= P\{A + N_1 < T\} = P\{N_1 < T - A\} = P\{N_1 > A - T\} \\ &= \int_{A-T}^\infty \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{x^2}{2\sigma_1^2}} \\ &= Q\left(\frac{A-T}{\sigma_1}\right). \end{aligned}$$

iii) [2]

$$P_e = p_0 P_{e0} + p_1 P_{e1}.$$

iv) [10]

$$P_e = p_0 \frac{1}{\sqrt{2\pi\sigma_0^2}} \int_T^\infty e^{-\frac{x^2}{2\sigma_0^2}} dx + p_1 \int_{A-T}^\infty \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{x^2}{2\sigma_1^2}}.$$

To find the optimal threshold, we find T which satisfies $\frac{dP_e}{dT} = 0$. We first need to take the derivative of P_e with respect to T. We use the Leibnitz rule for this. We have

$$\frac{dP_e}{dT} = -p_0 \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{T^2}{2\sigma_0^2}} + p_1 \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(A-T)^2}{2\sigma_1^2}} = 0.$$

This is equivalent to

$$\frac{1}{3}\frac{1}{\sqrt{2\pi\sigma_0^2}}e^{-\frac{T^2}{2\sigma_0^2}} = \frac{2}{3}\frac{1}{\sqrt{2\pi 4\sigma_0^2}}e^{-\frac{(A-T)^2}{8\sigma_0^2}},$$

which reduces to

$$4T^2 = (A-T)^2,$$

and, finally, we get $T = \frac{A}{3}$.

b) **This problem requires direct application of the noise analysis for baseband and AM systems, which have been done in the lecture and class. A very similar problem (with non-Gaussian noise) was done in class. Students were relatively successful in getting the solution right.**

i) [6]

The message signal is a sinusoidal wave with amplitude $A_m = 10$. Its average power is found as $P_S = \frac{1}{2}A_m^2 = 50$ W.

Due to the 20 dB attenuation in the channel, the received signal power is $P_R = 10^{-2}$ times the transmitted signal power.

Note that the message bandwidth is $W = 10$ KHz. This is also the minimum required bandwidth for baseband transmission.

The PSD of noise within the message bandwidth is $N_0/2$. Hence, the noise power is $2W \times \frac{N_o}{2} = 1 \times 10^{-10}$.

Hence, the SNR at the receiver output is

$$SNR_{baseband} = \frac{P_R}{P_N} = \frac{10^{-2} \times 50}{1 \times 10^{-10}} = 0.5 \times 10^{10}.$$

In the dB scale, we have $SNR_{baseband,dB} = 10\log_{10}(0.5) + 100 = 97$ dB.

ii) [6]

Remember that, when there is no attenuation, the recovered signal at the output of a coherent $DSB-SC$ receiver is $A_c m(t) + n_I(t)$, where $n_I(t)$ is the in-phase component of the band-pass noise.

Then the signal power at the receiver output is $A_c^2 P_S = 50$ W. With attenuation, we have $P_R = 50 \times 10^{-2}$ W.

Required bandwidth for DSB-SC is double the bandwidth of the message: 20 KHz.

The noise PSD in the transmission bandwidth around the carrier frequency of 1 MHz is $N_o/4$, and the PSD of the in-phase noise component is $N_o/2$. Then the average noise power at the receiver output is $2W \times \frac{N_o}{2} = 1 \times 10^{-10}$.

Hence, the SNR at the receiver output is

$$SNR_{DSB-SC} = \frac{P_R}{P_N} = \frac{10^{-2} \times 50}{1 \times 10^{-10}} = 0.5 \times 10^{10}.$$

In the dB scale, we have $SNR_{DSB-SC,dB} = 10\log_{10}(0.5) + 100 = 97$ dB.

Note that we get the same SNR despite the fact that we have used half the transmit power $P_T = \frac{1}{2}A_c^2 P_S$, compared to baseband modulation. This is due to the reduced noise PSD within the transmitted bandwidth.