

MSc and EEE/EIE PART IV: MEng and ACGI

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Special Information for the Invigilators: NONE

Information for Candidates:

Causal spline $\beta_n^+(t)$

The causal spline $\beta_n^+(t)$ of order n is obtained from the $(n + 1)$ -fold convolution of the causal box function $\beta_0^+(t)$. Specifically,

$$\beta_n^+(t) = \underbrace{\beta_0^+(t) * \beta_0^+(t) \dots * \beta_0^+(t)}_{n+1 \text{ times}}$$

with

$$\beta_0^+(t) = \begin{cases} 1 & t \in [0, 1) \\ 0 & \text{otherwise.} \end{cases}$$

Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(t - n) = \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k) e^{j2\pi kt}.$$

The Questions

1. (a) Find the overall transfer function $Y(z)/X(z)$ of the system in Figure 1a.

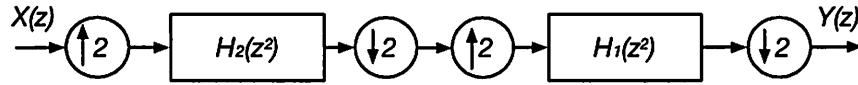


Figure 1a: Multirate system one.

[6]

- (b) In the system in Figure 1b, if $H(z) = H_0(z^2) + z^{-1}H_1(z^2)$, prove that $Y(z) = X(z)H_0(z)$.

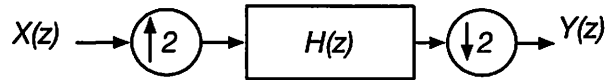


Figure 1b: Multirate system two.

[6]

- (c) Let $H(z)$, $F(z)$ and $G(z)$ be filters satisfying

$$H(z)G(z) + H(-z)G(-z) = 2,$$

$$H(z)F(z) + H(-z)F(-z) = 0.$$

Prove that for one of the systems in Figure 1c $Y(z)/X(z) = 1$, while for the other $Y(z)/X(z) = 0$.

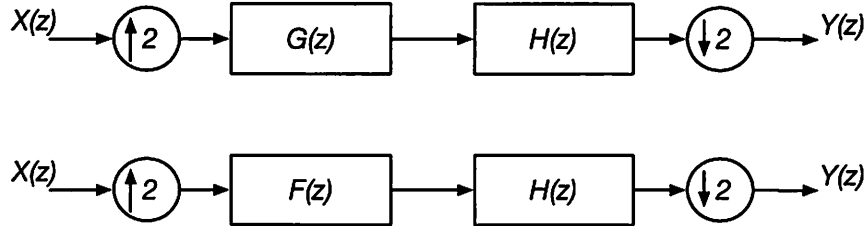


Figure 1c: Two examples of multirate systems.

[6]

- (d) Show that the filter $H(z) = (1 - z)^2$ annihilates discrete-time polynomial of maximum degree 1. Specifically, show that the convolution $h[n] * x[n] = 0$ when $x[n]$ is a discrete-time polynomial of maximum degree 1.

[7]

2. Rudin-Shapiro Polynomial

Rudin-Shapiro polynomials are defined by the following recursive equations

$$\begin{aligned} P_0(z) &= Q_0(z) = 1 \\ P_{n+1}(z) &= P_n(z) + z^k Q_n(z) \\ Q_{n+1}(z) &= P_n(z) - z^k Q_n(z) \end{aligned}$$

where $k = 2^n$.

- (a) Derive the Rudin-Shapiro polynomial pair (P, Q) of degree 3.

[6]

- (b) Prove that for $n > 0$, P_n and Q_n lead to a two-channel perfect reconstruction (PR) orthogonal filter bank, that is, show the following:

$$\begin{aligned} P_n(z)P_n(z^{-1}) + Q_n(z)Q_n(z^{-1}) &= k_n, \\ P_n(z)P_n(-z^{-1}) + Q_n(z)Q_n(-z^{-1}) &= 0. \end{aligned}$$

Determine the constant k_n .

[9]

- (c) Consider now the 2-channel filter bank of Fig. 2 with $G_0(z) = P_2(z)$ and $G_1(z) = Q_2(z)$.

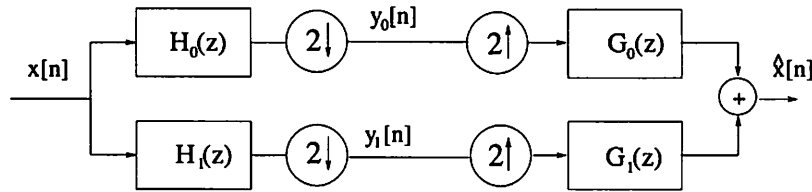


Figure 2: Two-channel filter bank.

- i. Assume $x[n] = 1$ and ignore any boundary effect. Which of the signals $y_0[n]$, $y_1[n]$ is nonzero? (Justify your answer)
- ii. Can you design a 2-channel perfect reconstruction orthogonal filter bank with filters of the same length as $P_2(z)$ and $Q_2(z)$ such that $y_1[n] = 0$ when $x[n] = n$? (Justify your answer)

[5]

[5]

3. Denote with $V_n = \text{span}(\{\beta_n^+(t-k)\}_{n \in \mathbb{Z}})$ the shift-invariant space generated by the causal spline $\beta_n^+(t)$ and its integer shifts. Given the signal $x(t) \in V_n$ with

$$x(t) = \sum_k \alpha_k \beta_n^+(t-k),$$

we want to show that

$$\frac{dx(t)}{dt} = \sum_k \tilde{\alpha}_k \beta_{n-1}^+(t-k)$$

with $\tilde{\alpha}_k = \alpha_k - \alpha_{k-1}$. Thus the computation of continuous-time derivative of $x(t)$ only requires computing the finite difference of the sequence α_k .

- (a) Begin by showing that

$$\frac{d\beta_n^+(t)}{dt} = \beta_{n-1}^+(t) - \beta_{n-1}^+(t-1).$$

[Hint: use the fact that $\frac{d\beta_0^+(t)}{dt} = \delta(t) - \delta(t-1)$.]

[5]

- (b) Now show that

$$\frac{dx(t)}{dt} = \sum_k \tilde{\alpha}_k \beta_{n-1}^+(t-k)$$

with $\tilde{\alpha}_k = \alpha_k - \alpha_{k-1}$.

[5]

- (c) Using the above fact compute the derivative of $x(t)$ with

$$x(t) = \sum_{k=-1}^1 \alpha_k \beta_1^+(t-k)$$

and $\alpha_{-1} = \alpha_0 = \alpha_1 = 2$.

[5]

- (d) Sketch and dimension $dx(t)/dt$ of part (c).

[5]

- (e) Compute the orthogonal projection of $x(t)$ of part (c) onto $V_0 = \text{span}(\{\beta_0^+(t-k)\}_{n \in \mathbb{Z}})$.

[5]

4. Consider the symmetric linear spline given by $\varphi(t) = \beta_0(t) * \beta_0(t)$, with

$$\beta_0(t) = \begin{cases} 1, & |t| < 1/2 \\ 1/2 & |t| = 1/2 \\ 0, & \text{otherwise.} \end{cases}$$

We know that $\varphi(t)$ is a valid scaling function. However, the linear spline is not orthogonal. It is our aim to orthogonalize it.

(a) Compute the deterministic autocorrelation function

$$a[n] = \langle \varphi(t), \varphi(t - n) \rangle.$$

Denote with $\hat{\varphi}(\omega)$ the Fourier transform of $\varphi(t)$ and with $A(e^{j\omega})$ the discrete-time Fourier transform of $a[n]$. Show that the new function $\phi(t)$ with Fourier transform

$$\hat{\phi}(\omega) = \frac{\hat{\varphi}(\omega)}{\sqrt{A(e^{j\omega})}}$$

is an orthogonal basis of the subspace $V_0 = \text{span} \{ \phi(t - n) \}_{n \in \mathbf{Z}}$. (Hint: Show that the Riesz basis criterion $A \leq \sum_{n \in \mathbf{Z}} |\hat{\phi}(\omega + 2\pi n)|^2 \leq B$ is satisfied with $A = B = 1$).

[9]

(b) Find the z -domain expression of the filter $H_0(z)$ that leads to the two-scale equation:

$$\phi(t) = \sqrt{2} \sum_n h_0[n] \phi(2t - n).$$

(Hint: Use the fact that $\hat{\varphi}(\omega) = \frac{G_0(e^{j\omega/2})}{\sqrt{2}} \hat{\varphi}(\omega/2)$ and the fact that $\hat{\phi}(\omega) = \frac{\hat{\varphi}(\omega)}{\sqrt{A(e^{j\omega})}}$.)

[6]

(c) Now consider the function

$$\varphi(t) = \beta_1(t) + \frac{1}{2} \frac{d\beta_1(t)}{dt},$$

where $\beta_1(t) = \beta_0(t) * \beta_0(t)$. Is $\varphi(t)$ a valid scaling function? (Justify your answer).

[10]

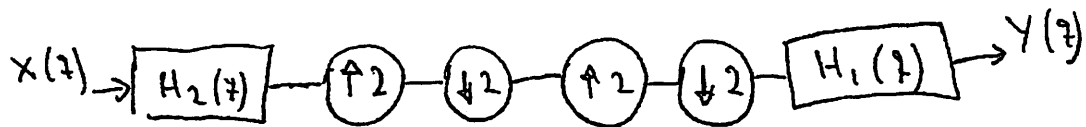
SOLUTIONS

1.

(a) WE USE THE FACT THAT



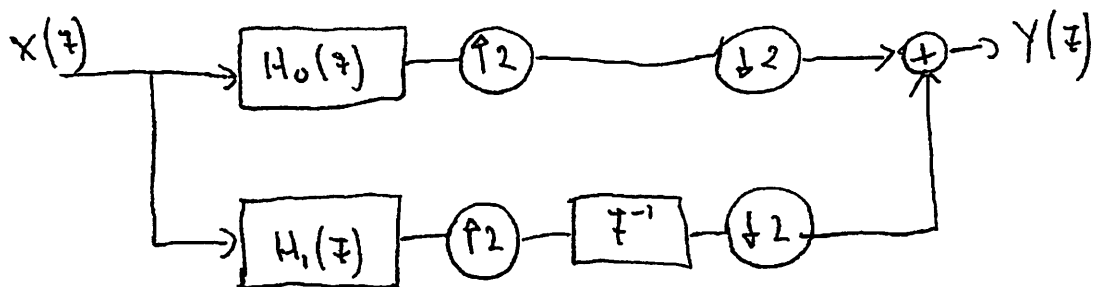
TO OBTAIN



CONSEQUENTLY

$$\frac{Y(z)}{X(z)} = H_1(z)H_2(z)$$

(b) THE EASIEST WAY TO SHOW THIS IS BY NOTING THAT WE CAN RE-DRAW THE SYSTEM AS FOLLOWS



THE LOWER BRANCH CONTAINS AN UPSAMPLER FOLLOWED BY A DELAY AND A DOWN SAMPLER. THE OUTPUT OF SUCH A SYSTEM IS ZERO.

THEREFORE $\frac{Y(z)}{X(z)} = H_0(z)$

2

(C) FOR THE FIRST SYSTEM, THE INPUT/OUTPUT RELATIONSHIP IS

$$\frac{Y(z)}{X(z)} = \frac{1}{2} \left[H\left(z^{1/2}\right) G\left(z^{1/2}\right) + H\left(z^{-1/2}\right) G\left(z^{-1/2}\right) \right] \stackrel{(i)}{=} 1$$

~~WHERE~~ WHERE (i) FOLLOWS FROM:

$$H(z) G(z) + H(z^{-1}) G(z^{-1}) = 2$$

FOR THE SECOND SYSTEM, THE INPUT/OUTPUT RELATIONSHIP IS

$$\frac{Y(z)}{X(z)} = \frac{1}{2} \left[H\left(z^{1/2}\right) F\left(z^{1/2}\right) + H\left(z^{-1/2}\right) F\left(z^{-1/2}\right) \right] \stackrel{(i)}{=} 0$$

WHERE (i) FOLLOWS FROM:

$$H(z) F(z) + H(z^{-1}) F(z^{-1}) = 0$$

(d) USING THE DEFINITION OF THE Z-TRANSFORM

WE HAVE THAT

$$H(z) = (1-z)^2 = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

WE NOTE THE FOLLOWING

$$\left. \frac{dH(t)}{dt} \right|_{t=1} = \sum_{k=-\infty}^{\infty} h[k] = 0 \quad (1)$$

ALSO IF WE EVALUATE THE DERIVATIVE OF $H(t)$ AT $t=1$ WE OBTAIN

$$\left. \frac{dH(t)}{dt} \right|_{t=1} = - \sum_{k=-\infty}^{\infty} k h[k] t^{-k-1} \Big|_{t=1} \Rightarrow \sum_{k=-\infty}^{\infty} k h[k] = 2(1-t) \Big|_{t=1} = 0 \quad (2)$$

~~ANALY~~

~~CONSEQUENTLY~~

CONSEQUENTLY, IF $x[n] = a_0$

THEN

$$h[n] * x[n] = \sum_k h[k] x[n-k] = a_0 \underbrace{\sum_k h[k]}_{=0} = 0$$

BECAUSE OF (1)

IF $x[n] = a_0 + a_1 n$

THEN

$$h[n] * x[n] = \sum_k h[k] x[n-k] = \sum_k h[k] (a_0 + a_1 (n-k))$$

$$= a_0 \underbrace{\sum_k h[k]}_{=0} + a_1 \sum_k (n-k) h[k]$$

$$= a_1 n \underbrace{\sum_k h[k]}_{=0 \text{ BECAUSE OF (1)}} - a_1 \underbrace{\sum_k k h[k]}_{=0 \text{ BECAUSE OF (2)}} = 0$$

QUESTION 2

4

(a)

$$P_1(z) = P_0(z) + z^2 Q_0(z) = 1 + z$$

$$Q_1(z) = P_0(z) - z^2 Q_0(z) = 1 - z$$

$$P_2(z) = P_1(z) + z^2 Q_1(z) = 1 + z + z^2 - z^3$$

$$Q_2(z) = P_1(z) - z^2 Q_1(z) = 1 + z - z^2 + z^3 //$$

(b) WE PROVE THIS BY INDUCTION :

$$P_1(z)P_1(z^{-1}) + Q_1(z)Q_1(z^{-1}) = (1+z)(1+z^{-1}) + (1-z)(1-z^{-1}) = 4$$

$$P_1(z)P_1(-z^{-1}) + Q_1(z)Q_1(-z^{-1}) = (1+z)(1-z^{-1}) + (1-z)(1+z^{-1}) = 0$$

WE NOW ASSUME THAT PR CONDITIONS ARE SATISFIED BY $P_n(z)$, $Q_n(z)$ AND SHOW THAT THIS IMPLIES THAT THE PR CONDITIONS ARE ALSO SATISFIED BY $P_{n+1}(z)$, $Q_{n+1}(z)$:

$$\begin{aligned} P_{n+1}(z)P_{n+1}(z^{-1}) + Q_{n+1}(z)Q_{n+1}(z^{-1}) &= \\ &= (P_n(z) + z^{2^n} Q_n(z))(P_n(z^{-1}) + z^{-2^n} Q_n(z^{-1})) + \\ &+ (P_n(z) - z^{2^n} Q_n(z))(P_n(z^{-1}) - z^{-2^n} Q_n(z^{-1})) = \end{aligned}$$

$$\begin{aligned}
 &= P_m(z)P_m(z^{-1}) + Q_m(z)Q_m(z^{-1}) + z^{-2m}P_m(z)Q_m(z^{-1}) + \\
 &+ z^{2m}Q_m(z)P_m(z^{-1}) - z^{-2m}P_m(z)Q_m(z^{-1}) - z^{2m}Q_m(z)P_m(z^{-1}) + \\
 &+ P_m(z)P_m(z^{-1}) + Q_m(z)Q_m(z^{-1}) = \\
 &= 2 \left[P_m(z)P_m(z^{-1}) + Q_m(z)Q_m(z^{-1}) \right] = 2K_m. \\
 &\quad \quad \quad = K_m \quad \text{(BY HYPOTHESIS)}
 \end{aligned}$$

SIMILARLY BY INDUCTION WE CAN SHOW THAT

$$P_{m+1}(z)P_{m+1}(z^{-1}) + Q_{m+1}(z)Q_{m+1}(z^{-1}) = 0$$

FINALLY, SINCE $K_{m+1} = 2K_m$ AND $K_1 = 4$

WE CONCLUDE THAT

$$K_m = 2^{(m+1)}$$

(c) i. $y_0[n]$ OR $y_1[n]$ ARE ZERO ONLY IF $P_2(z)$ OR $Q_2(z)$ HAVE A ZERO AT $\omega=0$ ($z=1$).

HOWEVER

$$\left. P_2(z) \right|_{z=1} = 4 \quad \text{AND} \quad \left. Q_2(z) \right|_{z=1} = 2$$

THEREFORE BOTH $y_0[n]$ AND $y_1[n]$
WILL BE NON-ZERO

(i).

DAUBESCHIES FILTERS ARE THE SHORTEST
FILTERS WITH THE MAXIMUM NUMBER OF
ZEROS AT $\omega=0$ LEADING TO
PN ORTHOGONAL FILTER BANKS.

WE ALSO KNOW FROM THE COURSE
THAT IF WE WANT P ZEROS AT
 $\omega=0$ THEN THE 4 FILTERS HAVE
LENGTH $2 \cdot P$. CONSEQUENTLY,

SINCE OUR FILTERS HAVE LENGTH 4
WE CAN HAVE AT MOST $P=2$

ZEROS AT $\omega=0$. THIS IS

ENOUGH TO ANNIHILATE POLYNOMIAL

OF MAXIMUM DEGREE ONE.

THEREFORE THE ANSWER IS YES.

THE FILTER WILL BE DAUBESCHIES 2.

QUESTION 3

7

(a)

$$\beta_m^+(t) = \beta_{m-1}^+(t) * \beta_0^+(t);$$

$$\frac{d \beta_m^+(t)}{dt} = \frac{d}{dt} [\beta_{m-1}^+(t) * \beta_0^+(t)]$$

$$= \beta_{m-1}^+(t) * \frac{d \beta_0^+(t)}{dt}$$

$$= \beta_{m-1}^+(t) * [\delta(t) - \delta(t-1)]$$

$$= \beta_{m-1}^+(t) - \beta_{m-1}^+(t-1).$$

//

(b) $x(t) = \sum_{k=-\infty}^{\infty} a_k \beta_m^+(t-k);$

$$\frac{dx(t)}{dt} = \frac{d}{dt} \left[\sum_k a_k \beta_m^+(t-k) \right]$$

$$= \sum_k a_k \frac{d \beta_m^+(t-k)}{dt}$$

$$= \sum_k a_k [\beta_{m-1}^+(t-k) - \beta_{m-1}^+(t-k-1)]$$

$$= \sum_k a_k \beta_{m-1}^+(t-k) - \sum_k a_k \beta_{m-1}^+(t-k-1),$$

8

REPLACING $l+1=l$ IN THE SECOND SUM
YIELDS :

$$\begin{aligned}\frac{d}{dt} x(t) &= \sum_l d_{ll} \beta_{m-1}^+(t-l) - \sum_l d_{l-1} \beta_{m-1}^+(t-l) \\ &= \sum_k (d_k - d_{k-1}) \beta_{m-1}^+(t-k) \\ &= \sum_k \tilde{d}_k \beta_{m-1}^+(t-k)\end{aligned}$$

□

(c)

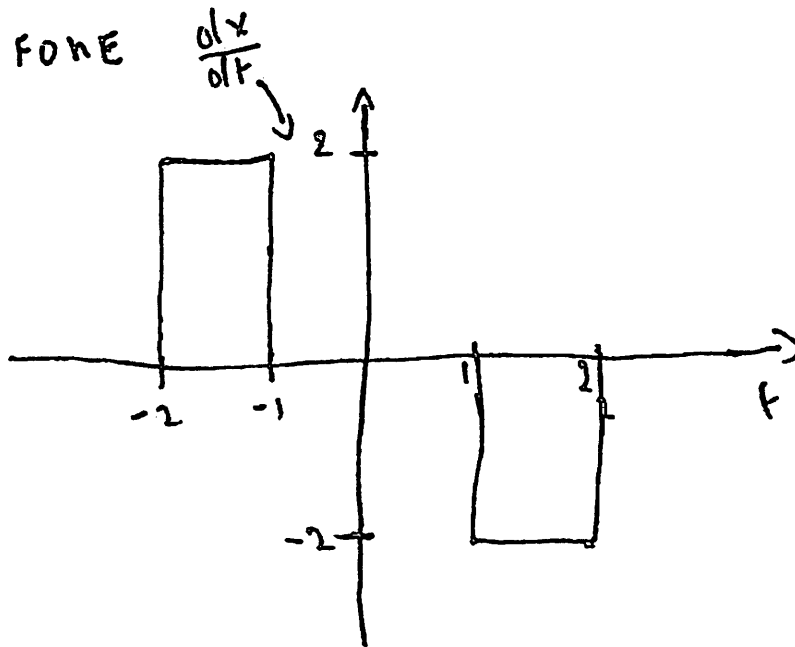
$$\frac{d}{dt} x(t) = \sum_{k=-\infty}^{\infty} \tilde{d}_k \beta_0^+(t-k)$$

WITH $\tilde{d}_{-1} = 2$, $\tilde{d}_1 = -2$ AND
 $\tilde{d}_k = 0$ OTHERWISE

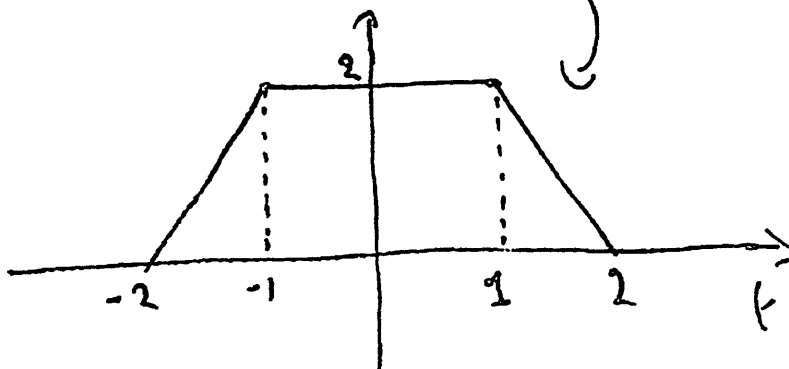
$$(b) \quad \frac{dx}{dt} = \sum_k \tilde{a}_k \beta_0^+(t - kL) \quad (1)$$

$$\text{WITH } \tilde{a}_k = \begin{cases} 2 & \text{FOR } k = -1 \\ -2 & \text{FOR } k = 1 \\ 0 & \text{OTHERWISE} \end{cases}$$

THEREFORE



WE ALSO NOTE THAT $x(t)$



SO $\frac{dx}{dt}$ GIVEN BY (1) IS THE CORRECT
DERIVATIVE OF $x(t)$

(2) CONTRARY TO $\{\beta_1^+(t-1k)\}_{k \in \mathbb{Z}}$,

THE SET $\{\beta_0^+(t-1k)\}_{k \in \mathbb{Z}}$ LEADS

TO AN ORTHONORMAL BASIS.

THEREFORE, THE ORTHOGONAL
PROJECTION OF $x(t)$ ONTO V_0

IS GIVEN BY

$$x_{\perp}(t) = \sum_k c_{1k} \beta_0^+(t-1k)$$

WITH

$$c_{1k} = \langle x(t), \beta_0^+(t-1k) \rangle$$

SINCE

$$x(t) = \sum_{\ell=-1}^1 a_{\ell} \beta_1^+(t-\ell) = 2 \sum_{\ell=-1}^1 \beta_1^+(t-\ell)$$

WE HAVE THAT:

$$\begin{aligned} c_{1k} &= \langle 2 \sum_{\ell=-1}^1 \beta_1^+(t-\ell), \beta_0^+(t-1k) \rangle \\ &= 2 \sum_{\ell=-1}^1 \langle \beta_1^+(t-\ell), \beta_0^+(t-1k) \rangle. \end{aligned}$$

//

WE NOTE THAT :

$$\langle \beta_i^+(t), \beta_0^+(t-k) \rangle = \begin{cases} \frac{1}{2} & \text{For } k=0, 1 \\ 0 & \text{OTHERWISE} \end{cases}$$

CONSEQUENTLY

$$C_k = \begin{cases} 1 & k = -2 \\ 2 & k = -1, 0, \\ 1 & k = 1 \\ 0 & \text{OTHERWISE} \end{cases}$$

//

QUESTION 4

12

$$(a) \quad a[n] = \begin{cases} 2/3 & \text{For } n=0 \\ 1/6 & \text{For } n=\pm 1 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$A(e^{j\omega}) = \sum_{k=-\infty}^{\infty} |\hat{\psi}(\omega + 2\pi k)|^2 \quad (1)$$

THUS, IF

$$\hat{\phi}(\omega) = \frac{\hat{\psi}(\omega)}{\sqrt{A(e^{j\omega})}} \quad (2)$$

WE HAVE THAT

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2 = \sum_{k=-\infty}^{\infty} \left| \frac{\hat{\psi}(\omega + 2\pi k)}{\sqrt{A(e^{j\omega})}} \right|^2$$

$$= \frac{1}{A(e^{j\omega})} \sum_k |\hat{\psi}(\omega + 2\pi k)|^2 \stackrel{(b)}{=} 1$$

WHERE (a) FOLLOWS FROM EQ. (2) AND

FROM THE FACT THAT $A(e^{j\omega})$ IS PERIODIC OF PERIOD 2π , AND (b) FROM (1).

(b)

3

$$\hat{\varphi}(\omega) = \frac{G_0(e^{j\omega/2})}{\sqrt{2}} \hat{\varphi}\left(\frac{\omega}{2}\right) \Rightarrow$$

$$\hat{\phi}(\omega) = \frac{\hat{\varphi}(\omega)}{\sqrt{A(e^{j\omega})}} = \frac{G_0(e^{j\omega/2})}{\sqrt{2} A(e^{j\omega})} \hat{\varphi}\left(\frac{\omega}{2}\right) \Rightarrow$$

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2}} G_0(e^{j\omega/2}) \sqrt{\frac{A(e^{j\omega/2})}{A(e^{j\omega})}} \hat{\phi}\left(\frac{\omega}{2}\right)$$

Thus

$$H_0(e^{j\omega/2}) = G_0(e^{j\omega/2}) \sqrt{\frac{A(e^{j\omega/2})}{A(e^{j\omega})}}$$

Ans

$$H_0(z) = G_0(z) \sqrt{\frac{A(z)}{A(z^2)}}$$

#

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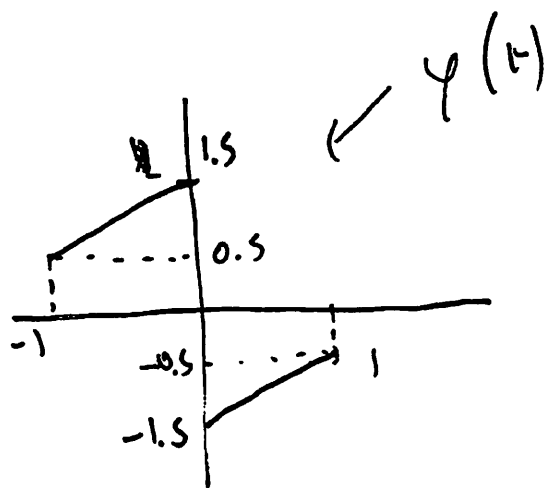
(c) NO, BECAUSE ^{ONE OF} THE
THREE CRITERIA OF A VALID
SCALING FUNCTION ~~ARE~~ IS NOT
SATISFIED

GRAPHICALLY WE SEE THAT

(i) PARTITION OF UNITY IS
SATISFIED :

$$\sum_n \varphi(t-n) = 1$$

AS



(ii) BUT THE
TWO SCALE EQUATION

$$\varphi(t) = \sqrt{2} \sum_n \varphi_0[n] \varphi(2t-n)$$

~~THIS~~ IS NOT SATISFIED

(iii)

NESTED BOUND CRITERION IS
SATISFIED WITH

$$A = 1 \quad \text{AND} \quad B = \frac{4}{3}$$