

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

EEE/ISE PART II: MEng, BEng and ACGI

Corrected Copy

SIGNALS AND LINEAR SYSTEMS

Monday, 28 May 2:00 pm

Time allowed: 1:30 hours

There are THREE questions on this paper.

Answer ALL questions.

Question One carries 40% of the marks. The other 2 questions each carry 30%.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.L. Dragotti
Second Marker(s) : P.T. Stathaki

Special Information for the Invigilators: none

Information for Candidates

Some Fourier Transforms

$$\text{rect}\left(\frac{t}{\tau}\right) \iff \tau \text{sinc}\left(\frac{\omega\tau}{2}\right)$$

$$\frac{W}{\pi} \text{sinc}(Wt) \iff \text{rect}\left(\frac{\omega}{2W}\right)$$

Time-integration property of the Fourier transform

$$\int_{-\infty}^t x(\tau) d\tau \iff \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

Time-shifting property of the Fourier transform

$$x(t - t_d) \iff X(\omega)e^{-j\omega t_d}$$

The sum of the first M terms of a geometric series is:

$$\sum_{n=0}^{M-1} \rho^n = \frac{1 - \rho^M}{1 - \rho}$$

A useful Laplace transform

$$e^{\lambda t}u(t) \iff \frac{1}{s - \lambda}$$

The Questions

1. This question carries 40% of the mark.

(a) Given the signal:

$$x(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

sketch and dimension each of the following signals:

i. $x_1(t) = x(t - 2)$

[2]

ii. $x_2(t) = x(-2t + 3)$

[2]

(b) State with a brief explanation if the systems with the following input/output relationships are linear/non-linear, time-invariant/time-varying.

i. $y(t) = 3 + 2x(t)$

[2]

ii. $y(t) = x(t) \sin(3t + \pi/4)$

[2]

(c) Given the following two signals:

$$f_1(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 2 - t, & 1 < t \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_2(t) = \delta(t + 2) + \delta(t + 1),$$

sketch and dimension $c(t) = f_1(t) * f_2(t)$.

[4]

Question 1 continues on next page

- (d) Using the definition of the Laplace transform, compute the Laplace transform of $x(t) = u(t)$ where

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and state clearly the Region of Convergence (ROC) of the transform.

[2]

- (e) Using the time-integration property of the Fourier transform, determine the Fourier transform of $x(t) = u(t)$.

[2]

- (f) Compare $X(\omega)$ with $X(s)$ for $s = j\omega$. Explain why they are different.

[2]

- (g) Consider the electric circuit shown in Fig. 1.

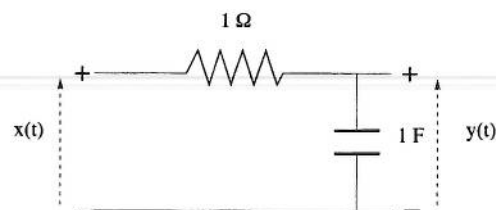


Figure 1: An RC circuit.

- i. Find the linear differential equation that relates the input $x(t)$ to the output $y(t) = \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau$.
- ii. Find the characteristic polynomial, characteristic roots and characteristic modes of this system.
- iii. Find the zero-input component of the response $y(t)$ for $t \geq 0$, if the initial condition is $y(0) = 2$.

[4]

[4]

[4]

Question 1 continues on next page

- (h) Consider the signal $x(t) = 4000\text{sinc}(4000\pi t)$
- i. Sketch and dimension the Fourier transform of $x(t)$ [2]
 - ii. Determine the Nyquist sampling rate for $x(t)$ [2]
 - iii. Determine the Nyquist sampling rate for $x^2(t)$. [2]
- (i) Using the definition of the z-transform, compute the z-transform of
- $$x[n] = a^n u[n] - a^n u[n - 5],$$
- where $a = 1/2$. [4]

2. Consider the system connected in parallel as depicted in Fig. 2. Here the linear

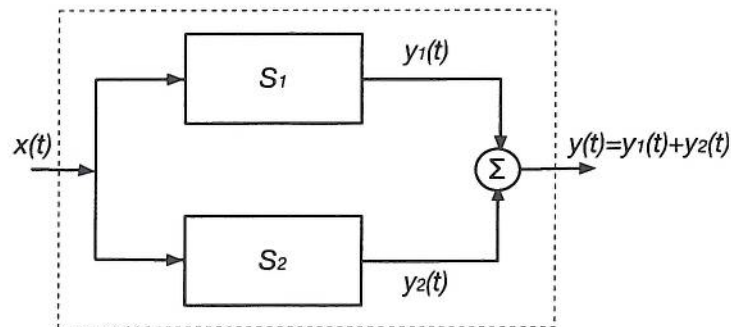


Figure 2: A parallel system.

system S_1 has the following input/output relationship:

$$\frac{d^2 y_1}{dt^2} + 2 \frac{dy_1}{dt} - 3y_1(t) = \frac{dx}{dt}$$

and the system S_2 has the following input/output relationship:

$$y_2(t) = 2x(t).$$

(a) Find the transfer function of S_1 and S_2 .

[10]

(b) Find the transfer function of the parallel connected system.

[6]

(c) Assume the system was at rest when it was excited by the input $x(t) = e^{-2t}u(t)$, determine the exact expression of the output $y(t)$ for $t \geq 0$.

[14]

3. Consider the system $h(t)$ shown in Fig. 3. This system is obtained by cascading two 'zero-order hold' systems.

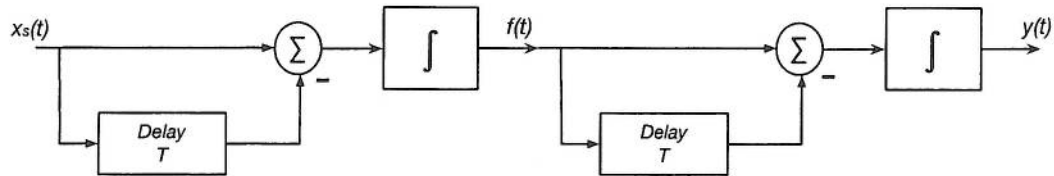


Figure 3: A cascade of two 'zero-order hold' systems .

- (a) Find the unit impulse response $h(t)$ of the system.

[10]

- (b) Find the frequency response $H(\omega)$ of the system.

[10]

- (c) The system $h(t)$ is now used to reconstruct a sampled signal. Therefore the incoming signal is $x_s(t) = \sum_{n=-\infty}^{\infty} x_n \delta(t - nT)$, where $x_n = x(nT)$ are the samples. We assume the delay T is equal to the sampling period and $T = 1$. Assume

$$x(t) = \begin{cases} t^2 & \text{for } 0 \leq t \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

- i. Derive the exact values of the samples $x_n = x(nT)$.

[5]

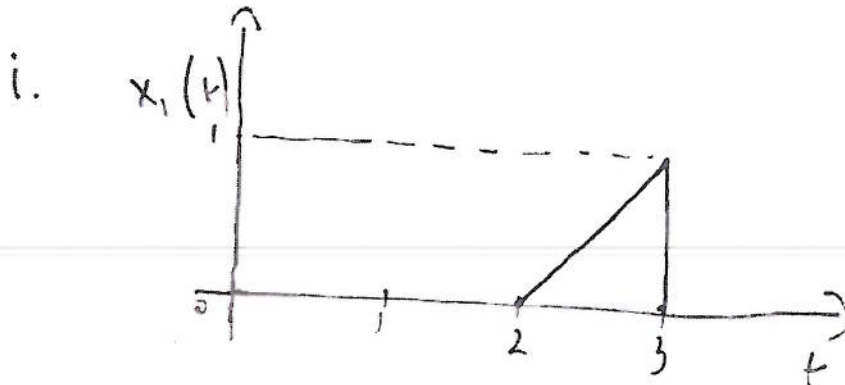
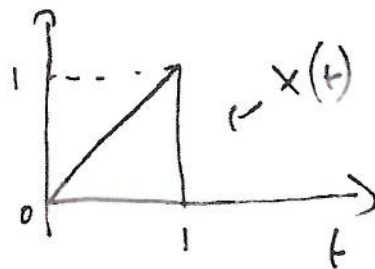
- ii. Sketch and dimension the reconstructed signal $y(t)$.

[5]

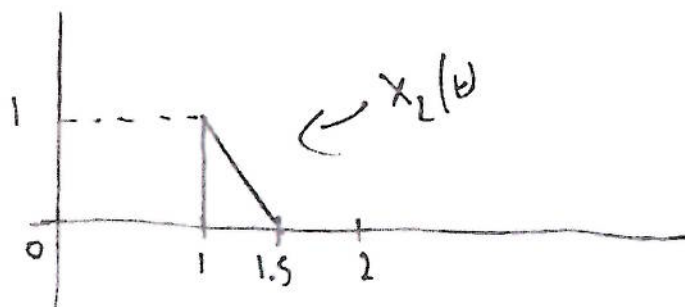
ANSWERS

QUESTION 1

1.(a)



$$x_2(t) = x(-2t + 3) = x(-2(t - \frac{3}{2}))$$



(b) LINEARITY MEANS THAT

2

IF

$$x_1(t) \rightarrow y_1(t)$$

AND

$$x_2(t) \rightarrow y_2(t)$$

THEN

$$2x_1(t) + \beta x_2(t) \rightarrow 2y_1(t) + \beta y_2(t)$$

THEREFORE

i. IS NON/LINEAR DUE TO THE DC TERM: \leftarrow

$$y(t) = 3 + 2(2x_1(t) + \beta x_2(t)) \neq 2y_1(t) + \beta y_2(t) = (2+\beta)3 + 2(2x_1 + \beta x_2)$$

ii. IS INSTEAD LINEAR

$$y(t) = (2x_1(t) + \beta x_2(t)) \sin(3t + \pi/4) = 2y_1(t) + \beta y_2(t)$$

TIME INVARIANCE MEANS THAT

IF

$$x(t) \rightarrow y(t)$$

THEN

$$x(t-T) \rightarrow y(t-T)$$

THEREFORE

i. IS TIME-INVARIANT

ii. IS TIME/VARIING

$$3 + x(t-T) = y(t-T)$$
$$x(t-T) \sin(3t + \pi/4) \neq y(t-T)$$

FEEDBACK:

NOTE THAT MOST STUDENT GOT THIS WRONG BECAUSE THEY DID NOT REALIZE THAT A CONSTANT TERM LEADS TO NON-LINEAR SYSTEMS

(C) FIRST NOTICE THAT

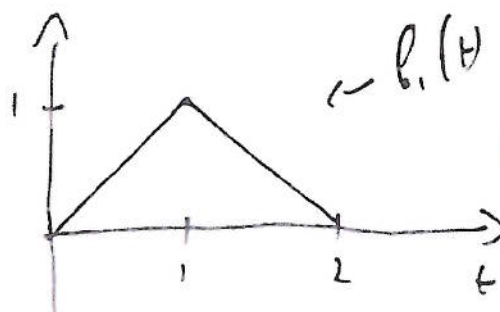
3

$$f(t) * \delta(t-2) = f(t-2)$$

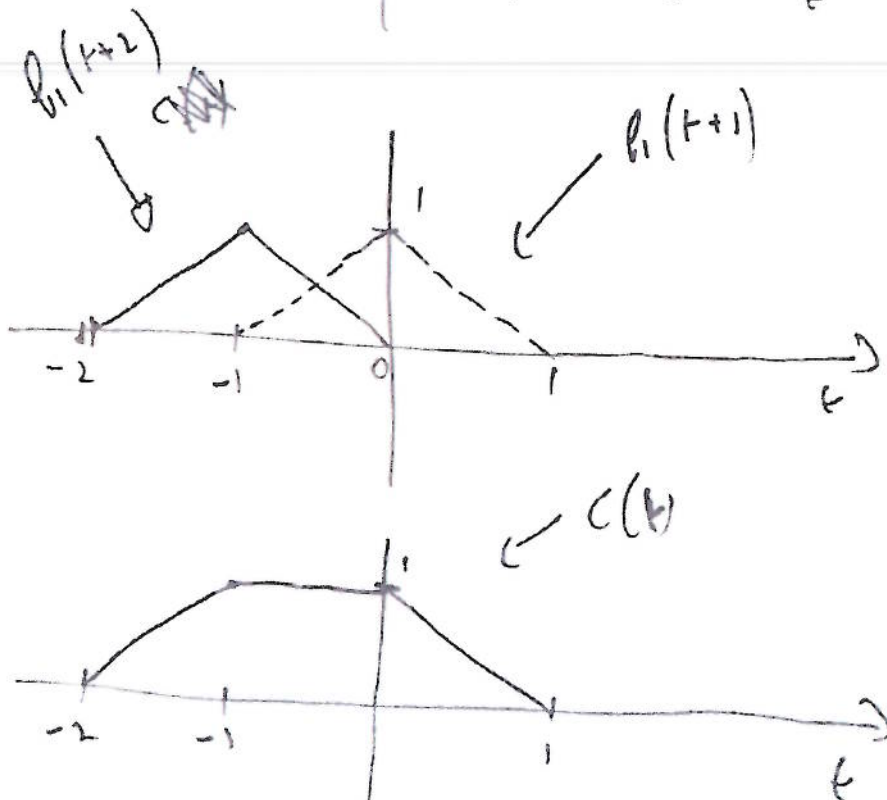
← FEEDBACK: SOME STUDENTS DID NOT SEEM TO BE AWARE OF THIS BASIC FACT

THEREFORE, BECAUSE OF LINEARITY OF CONVOLUTION, WE HAVE THAT

$$c(t) = f_1(t) + f_2(t) = f_1(t+2) + f_1(t+1)$$



← FEEDBACK: MANY STUDENTS DID NOT SHIFTED SIGNAL PROPERLY TO THE 'LEFT' OF THE PLOT



(d)

$$X(s) = \int_0^{\infty} u(t) e^{-st} dt = \int_0^{\infty} e^{-st} dt = \frac{e^{-st}}{-s} \Big|_0^{\infty}$$

$$= \frac{1}{s}$$

ROC is $(\operatorname{Re} s) > 0$

FEEDBACK: s IS A COMPLEX NUMBER
THUS ROC IS $(\operatorname{Re} s) > 0$. MOST STUDENTS
JUST WRITE $s > 0$ WHICH IS INCORRECT

(e)

NOTE THAT

$$\frac{d u(t)}{dt} = \delta(t) \quad \text{AND} \quad \int_{-\infty}^t \delta(\tau) d\tau = u(t)$$

~~AND~~ MOREOVER

$$\delta(t) \iff 1$$

~~USING~~ THEREFORE USING TIME-INTEGRATION PROPERTY

$$\int_{-\infty}^t x(\tau) d\tau \iff \frac{X(\omega)}{j\omega} + \pi X(0) \delta(\omega)$$

WE OBTAIN

$$u(t) \iff \frac{1}{j\omega} + \pi \delta(\omega)$$

(f) THEY ARE DIFFERENT SINCE

5

$$\int_{-\infty}^{\infty} u(t) dt = +\infty$$

OR ROC IN (d) DOES NOT CONTAIN
THE AXIS $s = j\omega$

(g)

$$1. \quad y(t) = \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau \quad (1)$$

$$\frac{dy}{dt} = \frac{1}{C} i(t) \quad (2)$$

$$y(t) + Ri(t) = x(t) \quad (3)$$

BY ~~REPLACING~~ (2) IN (3) WE OBTAIN

$$CR \frac{dy}{dt} + y(t) = x(t)$$

$$CR=1 \Rightarrow \frac{dy}{dt} + y(t) = x(t)$$

CHARACTERISTIC POLYNOMIAL

$$(D+1)$$

CHARACTERISTIC ROOT $\lambda_1 = -1$

CHARACTERISTIC MODE $y(t) = C e^{\lambda_1 t} = C e^{-t}$

$y(0) = 2 \Rightarrow$ ZERO-INPUT RESPONSE:

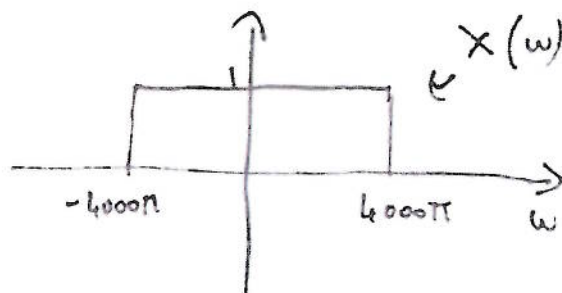
$$y(t) = 2 e^{-t}$$

(h)

i $x(t) = 4000 \sin(4000\pi t)$

USING THE TABLE

$$X(\omega) = \text{RECT}\left(\frac{\omega}{8000\pi}\right)$$



(ii) MAXIMUM NON-ZERO FREQUENCY OF $x(t)$

IS $f_m = \frac{4000\pi}{2\pi} = 2000 \text{ Hz}$. ~~thus~~ ~~fs~~

THE NYQUIST ~~REQUIREMENT~~ SAMPLING RATE

$$\text{IS } f_s = 2 \cdot f_m = 4000 \text{ Hz} = 4 \text{ kHz}$$

(iii)

USING THE FACT THAT

$$X(t) = X(t) \circ X(t) \quad (\Rightarrow) \quad \frac{1}{2\pi} X(\omega) * X(\omega)$$

WE IMMEDIATELY REALIZE THAT

THE NEW SIGNAL HAS A BANDWIDTH
WHICH IS TWICE THE ORIGINAL BANDWIDTH
AND WHICH IS STILL CENTRED IN ZERO,
THEREFORE

$$f_s = 8 \text{ kHz}$$

$$(i) \quad X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^n (u[n] - u[n-5])$$

$$= \sum_{n=0}^4 a^n z^n = \underline{a^5 z^5} \frac{(az)^5 - 1}{az - 1}$$

QUESTION 2

8

(a) TRANSFER FUNCTION:

$$H(s) = \frac{Y(s)}{X(s)}$$

FEEDBACK: MOST STUDENT
ANSWERED
PROPERLY TO
THIS QUESTION.

THEREFORE, FOR S_1

$$\frac{d^2 y_1}{dt^2} + 2 \frac{dy_1}{dt} - 3y_1(t) = \frac{dx}{dt}$$

\Downarrow

$$(s^2 + 2s - 3)Y(s) = sX(s)$$

AND

$$H_1(s) = \frac{s}{(s^2 + 2s - 3)} = \frac{s}{(s+3)(s-1)}$$

FOR S_2

$$y_2(t) = 2x(t)$$

THUS $H_2(s) = 2$

$$(b) \quad H(s) = H_1(s) + H_2(s) = 2 + \frac{s}{(s+3)(s-1)} = \frac{2s^2 + 5s - 6}{(s+3)(s-1)}$$

(c) $x(t) = e^{-2t} u(t) \Rightarrow X(s) = \frac{1}{s+2}$

$$Y(s) = H(s) \cdot X(s) = \frac{2s^2 + 5s - 6}{(s+3)(s-1)(s+2)}$$

USING PARTIAL FRACTION

$$Y(s) = \frac{A}{s+3} + \frac{B}{s-1} + \frac{C}{s+2} = \frac{2s^2 + 5s - 6}{(s+3)(s-1)(s+2)}$$

WITH $A = -\frac{3}{4}$, $B = \frac{1}{12}$, $C = \frac{8}{3}$

so

$$Y(s) = -\frac{3}{4} \frac{1}{s+3} + \frac{1}{12} \frac{1}{s-1} + \frac{8}{3} \frac{1}{s+2}$$

now

$$y(t) = -\frac{3}{4} e^{-3t} u(t) + \frac{1}{12} e^t u(t) + \frac{8}{3} e^{-2t} u(t)$$

FEEDBACK: REMEMBER THE TERM ' $u(t)$ '.

QUESTION 3

10

(a)

UNIT IMPULSE RESPONSE OBTAINED
BY SETTING $x_s(t) = \delta(t)$
THEREFORE,

$$\cancel{f(t)} \quad f(t) = \int_{-\infty}^t \delta(\tau) - \delta(\tau - T) d\tau$$

$$= u(t) - u(t - T)$$

$$y(t) = h(t) = \int_{-\infty}^t f(\tau) - f(\tau - T) d\tau$$

$$= \int_{-\infty}^t u(\tau) - u(\tau - T) - u(\tau - T) + u(\tau - 2T) d\tau$$

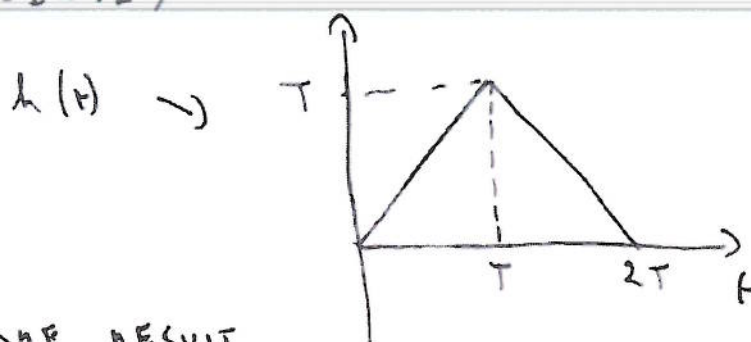
SINCE

$$\int_{-\infty}^t u(\tau) d\tau = t u(t)$$

WE OBTAIN

$$\cancel{h(t)} \quad h(t) = t u(t) - 2(t - T) u(t - T) + (t - 2T) u(t - 2T)$$

CONSEQUENTLY

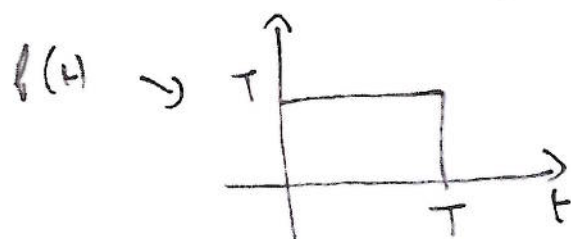


THE SAME RESULT

COULD ~~BE~~ HAVE BEEN OBTAINED

BY NOTICING THAT $h(t) = f(t) * f(t)$

AND



(b)

$H(\omega)$ IS THE FOURIER TRANSFORM OF $h(t)$.

SINCE $h(t) = f(t) * f(t)$ THEN


$$H(\omega) = F(\omega) \cdot F(\omega) \text{ AND}$$

USING THE -SHIFTING PROPERTY WE HAVE

$$f(t) = \text{RECT}\left(\frac{t}{T} - 0.5\right) \Leftrightarrow F(\omega) = T \text{sinc}\left(\frac{\omega T}{2}\right) e^{-j\frac{\omega T}{2}}$$

AND

$$H(\omega) = F^2(\omega) = T^2 \text{sinc}^2\left(\frac{\omega T}{2}\right) e^{-j\omega T}$$

FEEDBACK: MOST STUDENTS DID NOT REALIZE THAT $H(\omega) = F(\omega) \cdot F(\omega)$. THEY TRIED TO USE THE DEFINITION OF THE FOURIER TRANSFORM (FT) TO COMPUTE THE FT OF  THIS IS POSSIBLE BUT MUCH HARDER.

(c)

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$$x_n = x(nT) = \begin{cases} 1 & \text{FOR } n=1 \\ 4 & \text{FOR } n=2 \\ 9 & \text{FOR } n=3 \\ 0 & \text{OTHERWISE} \end{cases}$$

