E4.10
CS5.1
SC4

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2010

MSc and EEE PART IV: MEng and ACGI

Corrected Copy
4) a) ii)

PROBABILITY AND STOCHASTIC PROCESSES

Thursday, 6 May 10:00 am

Time allowed: 3:00 hours

Q4,a,ii Correction announced at the start.

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

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PROBABILITY AND STOCHASTIC PROCESSES

1. We consider two random variables X and Y such that the joint distribution of the random vector (X,Y) is given by, for $1 \le j \le i \le 5$,

$$p_{ij} = \mathbf{P}(X = i, Y = j) = \frac{1}{15}.$$

a) i) Show that $(p_{ij})_{i,j \in \{1,\dots,5\}}$ is a probability distribution. [1]

ii) Compute the marginal distributions of X and Y. [2]

iii) For $1 \le j \le i \le 5$, compute the conditional distribution

$$\mathbf{P}(X = i \mid Y = j).$$

[2]

b) i) Compute $\mathbf{E}(X \mid Y = j)$, $j = 1, \dots, 5$. Show that $\mathbf{E}(X \mid Y) = \frac{Y+5}{2}$. [3]

ii) Compute $\mathbf{E}(Y \mid X = i)$, $i = 1, \dots, 5$, and $\mathbf{E}(Y \mid X)$. [3]

iii) Using questions 1. b) i) and ii), show that

$$\mathbf{E}(X) = \frac{1}{2}\mathbf{E}(Y) + \frac{5}{2}, \quad \mathbf{E}(Y) = \frac{1}{2}\mathbf{E}(X) + \frac{1}{2}.$$

[3]

iv) Using the previous question, compute $\mathbf{E}(X)$ and $\mathbf{E}(Y)$. [2]

c) By analogy to the argument above, derive $\mathbf{E}(X)$ and $\mathbf{E}(Y)$ for X and Y having joint distribution

$$\mathbf{P}(X = i, Y = j) = \frac{2}{n(n+1)},$$

where n is a positive integer and $1 \le j \le i \le n$. [4]

2. We consider an urn containing n balls numbered $1, \dots, n$. We uniformly draw a ball from the urn, we write down its number and put it back in the urn. We suppose that the draws are independent from each other. We stop the process when we see each ball (number) at least once. This process is known as the *coupon collector problem.*

Let τ be the (random) time by which we complete the coupon collection.

a) i) Let τ_i be the number of draws before you see i different balls, $i = 1, \dots, n$. We have $\tau_n = \tau$.

Derive the distribution of $X_i = \tau_i - \tau_{i-1}$, $i = 2, \dots, n$, and compute $\mathbf{E}(X_i)$. [3]

ii) Show that $\mathbf{E}(\tau) \sim n \log(n)$, for n large. [3]

Hint: Use the fact that $\sum_{i=1}^{n} \frac{1}{i} \sim \log(n)$, for n large.

b) For $j = 1, \dots, n$, let $Y_{k,j}$ be the indicator function that the ball j is **not** chosen in one of the first k draws.

i) Compute $\mathbf{E}(Y_{k,j})$ and $\mathbf{E}(Y_{k,i}Y_{k,j})$, for $i \neq j$. [3]

ii) Compute the covariance of $Y_{k,i}$ and $Y_{k,j}$. Note that $\mathrm{Cov}(Y_{k,i}Y_{k,j})$ is negative. Comment. [3]

c) Let $U_k = \sum_{i=1}^{n} Y_{k,i}$.

i) Using 2. b) ii), show that $\mathrm{Var}(U_k) \leq \mathbf{E}(U_k)$. [3]

ii) Let $k = cn$, for $c > 0$, show that $\lim_{n \to \infty} \mathbf{E}(U_k)/n = e^{-c}$. [1]

Hint: Use the fact that $\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^{cn} = e^{-c}$.

iii) Using Chebyshev's inequality, show that, for any $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbf{P}\left(|U_n - \mathbf{E}(U_n)| \geq n\varepsilon/2\right) = 0\,.$$

Comment. [4]

Chebyshev's inequality: Let X with mean $\mathbf{E}(X)$ and variance $\mathrm{Var}(X)$. Then, for $x > 0$,

$$\mathbf{P}\left(|X - \mathbf{E}(X)| \geq x\right) \leq \frac{\mathrm{Var}(X)}{x^2}\,.$$

3. Let T be an integer-valued random variable distributed according to the following distribution, for $k = 0, 1, 2, \ldots$,

$$\mathbf{P}(T = k) = (1-a)a^k ,$$

where $a \in (0,1)$.

a) The *characteristic function* of a random variable X is given by, for $|z| \leq 1$,

$$\phi_X(z) = \mathbf{E}(z^X) .$$

i) Compute $\phi_T(z)$ and deduce the mean and the variance of T. [3]

ii) Compute $\mathbf{P}(T \geq n)$, for $n \geq 1$ and show that, for $k \geq 0$,

$$\mathbf{P}(T \geq n+k \mid T \geq n) = \mathbf{P}(T \geq k) .$$

Comment. [3]

b) Let $T_1, T_2, \ldots$ be a sequence of independent and identically distributed random variables following the same distribution as T above.

Derive the characteristic function, mean and variance of the random variable $S_n = T_1 + \cdots + T_n$, $n \geq 1$. [2]

c) Let ν be an integer-valued random variable with distribution, for $n = 1, 2, \ldots$,

$$\mathbf{P}(\nu = n) = (1-b)b^{n-1} ,$$

where $b \in (0,1)$. We assume that ν is independent of the random variables $T_1, T_2, \ldots$ in question b).

i) Show that

$$\mathbf{P}(S_\nu = k) = \sum_{n \geq 1} \mathbf{P}(\nu = n)\mathbf{P}(S_n = k) .$$

[2]

Hint: S_ν is equal to S_n when $\nu = n$.

ii) Show that

$$\mathbf{E}(S_\nu) = \mathbf{E}(T)\mathbf{E}(\nu) \quad \text{and} \quad \mathrm{Var}(S_\nu) = \mathrm{Var}(\nu)\mathbf{E}(T)^2 + \mathbf{E}(\nu)\mathrm{Var}(T) .$$

[5]

iii) Show that the characteristic distribution of S_ν is given by

$$\phi_{S_\nu}(z) = \frac{1-\alpha}{1-\alpha z} ,$$

where $\alpha = \frac{a}{1-b(1-a)}$. [4]

iv) What is the distribution of S_ν? [1]

4. Let $X_1, X_2 \dots$ be a sequence of independent and identically distributed random variables such that

$$\mathbf{P}(X_i = -1) = 1 - P(X_i = 1) = p \quad \text{for all } i \geq 1\,,$$

where $p \in (0,1)$. Define the following sequence, for $n \geq 1$,

$$Z_n = X_1 X_2 \dots X_n\,.$$

a) i) Compute the conditional probabilities $\mathbf{P}(Z_n = y \mid Z_{n-1} = x)$, for $x, y \in \{-1, 1\}$ and $n \geq 1$. [2]

$n \geq 1$

ii) Show that the random process $(Z_n)_{n\geq 0}$ is a homogeneous Markov process. Carefully justify your answer. [4]

iii) Derive a necessary and sufficient condition on the parameter p so that the random variables $Z_1, Z_2 \dots$ are independent. [2]

b) i) Show that, for $n, k \geq 1$, the transition probabilities

$$\mathbf{P}(Z_{n+k} = y \mid Z_n = x), \quad x, y \in \{-1, 1\}$$

are given by the matrix

$$\begin{pmatrix} 1 - p_k & p_k \\ p_k & 1 - p_k \end{pmatrix}$$

where $1 - 2p_k = (1 - 2p)^k$. [5]

ii) Compute $\lim_{n\to\infty} \mathbf{P}(Z_n = x)$, $x \in \{-1, 1\}$ using two different methods: (1) by question 4. b) i), and (2) by solving the equation for the invariant distribution. [5]

iii) For $x, y \in \{-1, 1\}$, compute

$$\lim_{n\to\infty} \mathbf{P}(Z_{n+1} = x, Z_n = y)\,.$$

[2]

5. In this problem, we analyse the *Acceptance-Rejection Method.* More precisely, we assume that we know how to generate random variables with probability density function g and we wish to find a method to generate a random variable with probability density function f.

We suppose that there exists a constant $c > 1$ such that, for all $x \in \mathbb{R}$, we have

$$f(x) \leq cg(x)\,.$$

We also assume that we can generate uniformly distributed random variables on the interval $[0,1]$.

a) Let U be a uniformly distributed random variable on $[0,1]$.

i) Give the expression of the probability density function of U and compute its mean and its variance. [2]

ii) Let $x \in (0,1)$, compute $\mathbf{P}(U \leq x)$. [1]

b) Let $U_1, U_2, \ldots$ be a sequence of independent and identically distributed random variables uniformly distributed on $[0,1]$ and let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed random variables with probability density function g. We assume that the two sequences are independent.

We are interested in the sequence of events, for $n \geq 1$,

$$A_n = \{U_n > \frac{f(Y_n)}{cg(Y_n)}\}\,.$$

i) Compute $\mathbf{P}(A_n^c)$, where A_n^c in the complement of A_n. [3]

ii) Show that, for $x \in \mathbb{R}$,

$$\mathbf{P}(A_n^c, Y_n \leq x) = \frac{1}{c}\int_{-\infty}^{x} f(y)dy\,.$$

[2]

iii) Using the two previous questions, compute $\mathbf{P}(A_1, \ldots A_{n-1}, A_n^c, Y_n \leq x)$. [4]

c) To generate a random variable according to f, we generate the random variables Y_n and U_n until the instant τ which is the first (random) instant n where we have

$$U_n \leq \frac{f(Y_n)}{cg(Y_n)}\,.$$

In what follows we will show that $Z = Y_\tau$ is distributed according to f.

i) Show that

$$\mathbf{P}(Z \leq x) = \sum_{n \geq 1} \mathbf{P}(A_1, \ldots A_{n-1}, A_n^c, Y_n \leq x)\,.$$

[3]

ii) Conclude that Z has f as probability density function. [5]

6. a) Let us assume that $(X_t)_{t\geq 0}$ is a given continuous-time Markov chain on the state space S with rate matrix $Q=(q_{ij})_{i,j\in S}$. Show that if there exists a probability distribution $\pi=(\pi_i)_{i\in S}$ such that, for all $i,j\in S$, we have

$$\pi_i q_{ij} = \pi_j q_{ji} ,$$

then π is the invariant distribution of $(X_t)_{t\geq 0}$. In this case, the Markov chain is said to be *reversible*. [3]

Hint: Use the fact that if Q is a rate matrix then $\sum_{j\in S} q_{ij} = 0$, for all $i \in S$.

b) We consider the dynamics of a Markovian single server queue in continuous time. Customers join the queue and are served on a first-in-first-out basis, i.e., according to the order in which they join the queue. We suppose that the time between two successive arrivals is exponentially distributed with parameter $\lambda > 0$ and that each customer requires a service time that is exponentially distributed with parameter $\mu > 0$.

Let X_t, $t \geq 0$, be the random process that describes the number of customers waiting in the queue including the one being served.

i) Derive the transition matrix Q of $(X_t)_{t\geq 0}$. [2]

ii) Using question 6. a), show that, if $\rho = \frac{\lambda}{\mu} < 1$, the stationary (invariant) distribution of $(X_t)_{t\geq 0}$ is given by

$$\pi_i = (1-\rho)\rho^i .$$

Comment on the *stability condition* $\rho < 1$. [4]

iii) Compute the average number of customers in the queue in the stationary regime. [2]

c) Let us now assume that we have a post office with two cashiers. The two queues at the cashiers run (separately) in parallel. Each of these queues operates following the dynamics in 6) b) with the same parameters λ and μ.

i) Derive the average number of customers in the post office. [1]

ii) A clever employee suggests merging the two queues so that customers arrive at rate 2λ wait until one of the two cashiers is available and is then served. Describe the underlying continuous-time Markov chain and discuss its stability. [2]

iii) Using similar arguments as in 6. b), show that the average number of customers in the post office in stationary regime is given by

$$\frac{2\mu\lambda}{(\mu+\lambda)(\mu-\lambda)} .$$

[4]

iv) Is the suggestion of the employee better than running the two queues separately? Justify your answer. [2]

PROBABILITY & STOCHASTIC PROCESSES.

SOLUTIONS 2010

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1/

a/

i/ $\displaystyle\sum_{1\le j\le i\le 5} \frac{1}{15} = \sum_{i=1}^{5}\sum_{j=1}^{i} \frac{1}{15} = \sum_{i=1}^{5} \frac{i}{15} = \frac{6\times 5}{2}\times\frac{1}{15} = 1$

ii) $i,j = 1..5$ $\displaystyle \mathbb{P}(X=i) = \sum_{j=1}^{i} \mathbb{P}(X=i, Y=j) = \frac{i}{15}$

$$\mathbb{P}(Y=j) = \sum_{i=j}^{5} \mathbb{P}(X=i, Y=j) = \frac{6-j}{15}$$

iii/ $\displaystyle \mathbb{P}(X=i \mid Y=j) = \frac{\mathbb{P}(X=i, Y=j)}{\mathbb{P}(Y=j)}$ $\quad 1 \le j \le i \le 5$

$$= \frac{1/15}{\frac{6-j}{15}} = \frac{1}{6-j}$$

b/

i/
$$\mathbb{E}(X \mid Y=j) = \sum_{i=j}^{5} i\, \mathbb{P}(X=i \mid Y=j)$$

$$= \frac{1}{6-j}\sum_{i=j}^{5} i = \frac{1}{6-j}\,\frac{30-j(j-1)}{2}$$

$$= \frac{j+5}{2}$$

$$\mathbb{E}(X \mid Y) = \frac{Y+5}{2}.$$

1/

b/ ii) $\mathbb{P}(\cancel{Y}=j \mid X=i) = \frac{1}{i}$ $\qquad 1 \le j \le i \le 5$

$$\mathbb{E}(Y \mid X=i) = \frac{1}{i} \sum_{j=1}^{i} j = \frac{i+1}{2}.$$

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$$\mathbb{E}(Y \mid X) = \frac{X+1}{2}.$$

iii) $$\mathbb{E}(X) = \mathbb{E}\left[\mathbb{E}(X \mid Y)\right] = \frac{\mathbb{E}(Y)}{2} + 5/2.$$

$$\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y \mid X)) = \frac{\mathbb{E}(X)}{2} + 1/2.$$

iv) Solving the system $\Rightarrow$ $\begin{cases} \mathbb{E}(X) = \frac{11}{3} \\ \mathbb{E}(Y) = \frac{7}{3} \end{cases}$

c). By analogy: $\mathbb{P}(X=i) = \frac{2i}{n(n+1)}$.

$\mathbb{P}(X=j) = \frac{2(n+1-j)}{n(n+1)}$.

$\mathbb{P}(X=i \mid Y=j) = \frac{1}{n+1-j}$ $\qquad 1 \le j \le i \le n$.

$\mathbb{E}(X \mid Y) = \frac{Y}{2} + n/2$; $\quad \mathbb{E}(Y \mid X) = \frac{X}{2} + \frac{1}{2}$.

$\Rightarrow$ $\mathbb{E}(X) = \frac{\mathbb{E}(Y)}{2} + \frac{n}{2}$, $\mathbb{E}(Y) = \frac{\mathbb{E}(X)}{2} + 1/2 \Rightarrow \begin{cases} \mathbb{E}\,X = \frac{2n+1}{3} \\ \mathbb{E}(Y) = n+2/3 \end{cases}$

2/

a)

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i) $X_i \sim \mathrm{Geo}\left(1 - \frac{i-1}{n}\right)$.

$$\mathbb{E}(X_i) = \frac{n}{n-i+1}.$$

Detailed derivation provided in solutions of Coursework I.

ii) $\mathbb{E}(Z) = \mathbb{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_i \mathbb{E}(X_i) = n \sum_{i=1}^{n} \frac{1}{n-i+1}$

$$= n \sum_{i=1}^{n} \frac{1}{i}$$

For n large $\sim n \log(n)$.

b) $\mathbb{P}(Y_{k,j} = 1) = \mathbb{P}(j \text{ does not appear in draws } 1 \ldots k)$

$= \mathbb{P}(j \text{ does not appear in draw } 1)^k$

$= \left(\frac{n-1}{n}\right)^k.$

$$\mathbb{E}(Y_{k,j}) = \left(\frac{n-1}{n}\right)^k$$

$$\mathbb{E}(Y_{k,i} Y_{k,j}) = \left(\frac{n-2}{n}\right)^k$$

$k \leftarrow$ # draws.

neither i or j appear in a draw

Since

$$\mathbb{P}(Y_{k,i} Y_{k,j} = 1) = \mathbb{P}(Y_{k,i} = 1 \ \& \ Y_{k,j} = 1).$$

2/
b/
ii).

$$\mathrm{Cov}(Y_{k,i}, Y_{k,j}) = \left(1-\frac{2}{n}\right)^k - \left(1-\frac{1}{k}\right)^{2k} < 0.$$

$Y_{k,i}$ & $Y_{k,j}$ are negatively correlated. This not surprising as if ball j is not chosen then it is more likely to that ball i will be chosen.

NOT REQUIRED:

Moreover the $\mathrm{Var}(U_n)$ to be defined in Q b is smaller than the variance of the sum of n independent iden. dis Bernoulli r.v.

c/
i/

$$\mathbb{E}(U_k) = n\left(1-\frac{1}{n}\right)^k$$

$$\mathrm{Var}(U_k) = n\,\mathbb{E}(Y_1)\left[1-\mathbb{E}(Y_1)\right] + \sum_j \sum_{k \neq j} \mathrm{Cov}(Y_j, Y_k).$$

$$\leq n\,\mathbb{E}(Y_1)\left(1-\mathbb{E}(Y_1)\right)$$

$$\leq n\,\mathbb{E}(Y_1) = \mathbb{E}(U_k).$$

ii/ $$\frac{\mathbb{E}(U_k)}{n} = \left(1 - \frac{1}{n}\right)^{cn} \xrightarrow[n\to\infty]{} e^{-c}.$$

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iii) $$\mathbb{P}\left(\left|U_k - \mathbb{E}(U_k)\right| \geq \frac{n\varepsilon}{2}\right) \leq 4\,\frac{\mathrm{var}(U_k)}{n^2\varepsilon^2} \leq 4\,\frac{\mathbb{E}(U_k)}{k^2\varepsilon^2} \xrightarrow[n\to\infty]{} 0.$$

This means $\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{U_k}{n} - e^{-c}\right| \geq \varepsilon\right) = 0.$

The above limit shows that it is not enough to draw cn times to have a reasonable chance of seeing all n ball. In fact, with cn draws, there is a high probability that ne^{-c} balls will not be seen.

NOT REQUIRED

In fact, we can see from the above that.

if we let $k = \alpha n \log(n)$

$$\begin{cases} \lim_{n\to\infty} \mathbb{P}(U_k = 0) = 1 & \text{if } \alpha > 1 \\ \lim_{n\to\infty} \mathbb{P}(U_k = 0) = 0. & \alpha < 1 \end{cases}$$

3/

a/ i/ $\phi_T(z) = \mathbb{E}(z^T) = \sum_{k\geq 0} (1-a)\, a^k z^k$

$$= \frac{(1-a)}{1-az} \qquad |z| < 1$$

$$\mathbb{E}(T) = \phi_T'(1) = \frac{a}{1-a}.$$

$$\mathbb{E}(T(T-1)) = \phi''(1) = 2\left(\frac{a}{1-a}\right)^2$$

$$\mathrm{Var}(T) = \mathbb{E}(T(T-1)) + \mathbb{E}(T) - \mathbb{E}(T)^2$$

$$= \frac{a}{(1-a)^2}$$

ii/

$$\mathbb{P}(T \geq n) = \sum_{k \geq n} (1-a)\, a^k = a^n.$$

$$\mathbb{P}(T \geq n+k \mid T \geq n) = \frac{\mathbb{P}(T \geq n+k,\, T \geq n)}{\mathbb{P}(T \geq n)}$$

$$= \frac{\mathbb{P}(T \geq n+k)}{\mathbb{P}(T \geq n)}$$

$$= \frac{a^{n+k}}{a^n} = a^k.$$

$$= \mathbb{P}(T \geq k).$$

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3/

/ii/ Memoriless property of the geometric distribution

b/ $\phi_{S_n}(z) = \left(\phi_T(z)\right)^n$ by i.i.d property

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$\mathbb{E}(S_n) = n\,\mathbb{E}(T)$

$\mathrm{Var}(S_n) = n\,\mathrm{Var}(T)$.

c/ i/ $\mathbb{P}(S_\nu = k) = \sum_{n \geq 1} \mathbb{P}(\nu = n)\,\mathbb{P}(S_\nu = k \mid \nu = n)$

$= \sum_{n \geq 1} \mathbb{P}(\nu = n)\,\mathbb{P}(S_n = k)$

ii/ $\mathbb{E}(S_\nu) = \sum_{k \geq 0} k\,\mathbb{P}(S_\nu = k)$

$= \sum_{k \geq 0} k \sum_{n \geq 1} \mathbb{P}(\nu = n)\,\mathbb{P}(S_n = k)$

$= \sum_{n \geq 1} \mathbb{P}(\nu = n) \underbrace{\sum_{k \geq 0} k\,\mathbb{P}(S_n = k)}_{\mathbb{E}(S_n)}$

$= \sum_{n \geq 0} \mathbb{P}(\nu = n)\, n\,\mathbb{E}(T)$

$= \mathbb{E}(T)\,\mathbb{E}(\nu)$.

ii/ $\mathrm{Var}(S_\nu) = \mathbb{E}(S_\nu^2) - \mathbb{E}(S_\nu)^2$

from above

$$\mathbb{E}(S_n^2) = n^2\, \mathbb{E}(T)^2 + n\, \mathrm{Var}(T).$$

$$\mathrm{Var}(S_\nu) = \mathrm{Var}(\nu)\, \mathbb{E}(T)^2 + \mathbb{E}(\nu)\, \mathrm{Var}(T).$$

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iii) $\mathbb{E}(2^{S_\nu}) = \phi_\nu(\phi_T(2))$

$$= \sum_{n \geq 1} \mathbb{E}(\cancel{z}^{S_n} \mid \nu = n)\, \mathbb{P}(\nu = n)$$

$$= \sum_{n \geq 1} \phi_T(2)^n\, \mathbb{P}(\nu = n)$$

$$= \mathbb{E}(\phi_T(2)^\nu) = \phi_\nu(\phi_T(2)).$$

$$= \phi_\nu\left(\frac{1-a}{1-a2}\right)$$

& $\phi_\nu(z) = \dfrac{(1-b)\,z}{1-bz}$

$$\mathbb{E}(2^{S_\nu}) = \frac{(1-b)\,\frac{1-a}{1-a2}}{1-b\,\frac{1-a}{1-a2}} = \frac{(1-b)(1-a)}{1-a2-b+ab}$$

$$= 1-\alpha \,/\, 1-\alpha 2$$

$$\alpha = \frac{a}{1-b(1-a)}.$$

3/ c/ iv) S_v is a geometric distribut

$$\mathbb{P}(S_v \geq k) = \alpha^k$$

as T.

7/19

4/ $X_i = \begin{cases} +1 & \text{w.p } 1-p \\ -1 & \text{w.p } p \end{cases}$ $p \in (0,1)$.

$$Z_n = X_1 \dots X_n \qquad Z_0 = 1.$$

$\frac{10}{19}$

a) i) $\mathbb{P}(Z_n = y \mid Z_{n-1} = x)$

$$\begin{array}{c|cc} & -1 & 1 \\ -1 & 1-p & p \\ 1 & p & 1-p \end{array} = Q$$

ii/ $\mathbb{P}(Z_{n+1} = y \mid Z_0 = z_0 \dots Z_n = z_n)$

$= \mathbb{P}(X_{n+1} = x \mid Z_0, \dots Z_n = z_n)$

$= \mathbb{P}(X_{t+1} = x) = p$ or $1-p$ independently $\forall t$.

$Z_n \sim$ homogeneous MC with trans. matrix Q.

iii/ ~~in the~~ above Z_n are independent

if $p = 1-p = 1/2$.

if $p = 1/2 \Rightarrow \mathbb{P}(Z_n = z_n \dots Z_0 = 1) = \left(\frac{1}{2}\right)^n$

$\Rightarrow Z_i$ are independent.

4/

b/

i/ $\mathbb{P}(Z_{n+k} = y \mid Z_n = x) = (Q^k)_{xy}$

$$\det Q^k = (1-p_k)^2 - p_k^2$$
$$= 1 - 2p_k$$

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$$\det Q^k = [\det(Q)]^k = (1-2p)^k.$$
$$\Rightarrow 1 - 2p_k = (1-2p)^k$$

Could use induction.

p

ii/ $\lim_{n\to\infty} \mathbb{P}(Z_n = \cancel{k}) = \,?$

$$\lim_{n\to\infty} \mathbb{P}(Z_n = -1) = \lim_{n\to\infty} p_k = 1/2$$
$$= \lim_{n\to\infty} \mathbb{P}(Z_n = +1).$$

~~Alternatively stationary dist $\pi P = \pi$~~
~~$\Rightarrow \pi_{-1}(1-p) + \pi_1 p = \pi_{-1}$~~
~~$\pi_{-1}(1-p)$~~

4/

b/ ii/

2nd method: Using the stationary distribution

$$\pi P = \pi \qquad \begin{cases} \pi_{-1}(1-p) + \pi_1 \, p = \pi_{-1} \\ \pi_{-1} p + \pi_1 (1-p) = \pi_1 \end{cases}$$

$$\Rightarrow \quad \pi_{-1} = \pi_1 = 1/2$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}(Z_n = \pm 1) = 1/2 .$$

iii/

$$\lim_{n\to\infty} \mathbb{P}(Z_{n+1} = x \, , \, Z_n = y)$$

$$= \lim_{n\to\infty} \mathbb{P}(Z_{n+1} = x \mid Z_n = y) \; \mathbb{P}(Z_n = y)$$

$x \backslash y$	-1	1
-1	$1/2\,(1-p)$	$1/2 \; p$
1	$1/2\, p$	$1/2\,(1-p)$

$\frac{12}{13}$

5/

a/

i/ $$f_U(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}(U) = \int x f_U(x) = \int_0^1 x\,dx = 1/2 .$$

$$\mathbb{E}(U^2) = \int_0^1 x^2\,dx = 1/3 .$$

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$$Var(U) = 1/3 - 1/4 = \frac{1}{12} .$$

ii/. $$\mathbb{P}(U \leq x) = \int_0^x du = x .$$

b/ i/ $$\mathbb{P}(A_n^c) = \mathbb{P}\left(U_n \leq \frac{f(Y_n)}{cg(Y_n)} \right)$$

$$= \int_{\mathbb{R}} \mathbb{P}\left(U_n \leq \frac{f(y)}{cg(y)} \right) g(y)\,dy$$

$0 \leq \frac{f(y)}{cg(y)} \leq 1$ by assumption.

$$= \int_{\mathbb{R}} \frac{f(y)}{cg(y)} g(y)\,dy$$

$$= \int_{\mathbb{R}} f(y)\,dy \frac{1}{c} = 1/c$$

5/

b/ By previous question.

ii/ $\mathbb{P}(A_n^c, Y_n \le x) = \int_{-\infty}^{x} \frac{1}{c} f(y)\,dy$

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iii/ $\mathbb{P}(A_1 \dots A_{n-1}, A_n^c, Y_n \le x)$

$= \mathbb{P}(A_1) \dots \mathbb{P}(A_{n-1})\, \mathbb{P}(A_n^c, Y_n \le x)$

by independence

$= \left(1 - \frac{1}{c}\right)^{n-1} \frac{1}{c} \int_{-\infty}^{x} \frac{1}{c} f(y)\,dy$

5/

c/ $\tau = \min\{ n \geq 1 \; ; \; U_n \le \frac{f(Y_n)}{c\,g(Y_n)} \}$

i) $\mathbb{P}(Z \le x) = \mathbb{P}(Y_\tau \le x)$

$= \sum_{n \geq 1} \mathbb{P}(\tau = n \; ; \; Y_n \le x).$

$= \sum_{n \geq 1} \mathbb{P}(A_1 \dots A_{n-1}, A_n^c, Y_n \le x)$

ii) . $\mathbb{P}(Z \le x) = \sum_{n \geq 1} \left(1 - 1/c\right)^{n-1} 1/c \int_{-\infty}^{x} f(y)\,dy = \int_{-\infty}^{x} f(y)\,dy$

6)

a) if (π_i) is such that

$$\pi_i q_{ij} = \pi_j q_{ji}$$

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then $(\pi Q)_i = \sum_j \pi_j q_{ij} = \sum_j \pi_i q_{ij}$

$$= \pi_i \sum_j q_{ij}$$

$$= 0$$

since Q is a rate matrix.

b) i) $\cancel{q_{ij} =}$ $Q = \begin{pmatrix} -(\lambda+\mu) & \lambda & 0 & \\ \mu & -(\lambda+\mu) & \lambda & \\ 0 & \mu & -(\lambda+\mu) & \lambda \\ & & \ddots & \ddots \end{pmatrix}$

ii)

$$\pi_i q_{ij} = \pi_j q_{ji}$$

$$\pi_i q_{i\,i+1} = \pi_{i+1} q_{i+1,i}$$

$$\pi_{i+1} = \pi_i \frac{\lambda}{\mu}.$$

$$\Rightarrow \pi_i = \pi_0 \left(\frac{\lambda}{\mu}\right)^i$$

$$\Rightarrow \sum \pi_i = 1$$

if $\rho = \frac{\lambda}{\mu} < 1 \Rightarrow \pi_i = \rho^i (1-\rho).$

$i \geq 0.$

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b)

ii) $\rho < 1$ the rate of arrival λ is smaller then rate of departure μ hence the queue is stabl

if $\rho > 1$ then the queue will build up until becoming infinitely large.

iii) $$\mathbb{E}(N_1) = \sum_i i \pi_i = \sum_{i \geq 0} i \rho^i (1-\rho) = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}.$$

c) i) $$\mathbb{E}(N_2) = 2\mathbb{E}(N_1) = \frac{2\lambda}{\mu - \lambda}$$

ii). M/M/2 queue.

$$\begin{bmatrix} -\lambda & \lambda & & & 0 \\ \mu & -(\lambda+\mu) & \lambda & & \\ & 2\mu & -(\lambda+2\mu) & \lambda & \\ 0 & & 2\mu & -(\lambda+2\mu) & \lambda \\ & & & & \ddots \end{bmatrix}$$

6) c).

ii)

if $\lambda < \mu \Rightarrow 2\lambda < 2\mu \Rightarrow \lambda < 2\mu$

if the initial system is stable so is the M/M/2

iii) by $\pi_i q_{ij} = \pi_j q_{ji}$; $s = 2$

$$\Rightarrow \pi_i = \begin{cases} \rho^n / n! \ \pi_0 & n = 0, 1, 2 \\ \dfrac{\rho^n}{s! \, s^{n-s}} \pi_0 & \forall n \geq s. \end{cases}$$

$$\Rightarrow \begin{aligned} \pi_0 &= \tfrac{1}{2} (\mu/\lambda)^2 \pi_2 \\ \pi_1 &= (\mu/\lambda) \pi_2 \\ \pi_n &= (\lambda/\mu)^{n-2} \pi_2 \quad n \geq 2. \end{aligned}$$

with $\sum_i \pi_i = 1 \Rightarrow \pi_2 = \dfrac{2\lambda^2(\mu - \lambda)}{\mu^2(\mu + \lambda)}$

6)

c) iii) $\mathbb{E}(N_2) = \sum_i i\,\Pi_i = \Pi_1 + \Pi_2 \sum_{i \geq 2} i \left(\frac{\lambda}{\mu}\right)^{i-2}$

$$= \frac{2\mu\lambda}{(\mu+\lambda)(\mu-\lambda)}$$

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iv)

It is not difficult to see that

since $\frac{\cancel{\mu\lambda}}{\mu+\lambda} < 1 \Rightarrow \mathbb{E}(N_2) \leq \mathbb{E}(N_1)$

From the post office view point, their place is less crowded with the suggested strategy.

NOT REQUIRED

Once could also look at the customers view point; it is not difficult to show that the av. waiting time in strategy 1 is $\mathbb{E}(W_1) = \frac{\lambda}{\mu(\mu-\lambda)}$ & for the crossstrategy is $\mathbb{E}(W_2) = \frac{\lambda^2}{\mu(\mu+\lambda)(\mu-\lambda)} < \mathbb{E}(W_1)$ with a similar conclusion.

Finally; if we look at activity rate of employers in the two systems.

Then $\rho_1 = 1 - \Pi_0 = \frac{\lambda}{\mu}$. 1^{st} system.

$\rho_2 = 1 - (\Pi_0 - \Pi_1/2)$ 2^{nd} system

$= \frac{\lambda}{\mu} = \rho_1$

In the cases strategy 2 does not improve on strategy 1.