1. a) i) Let $z(t)$ be the position of the box. The force equations are

$$f = K_1 z + D\dot{z} + K_2(z-y), \qquad M\ddot{y} + K_2(y-z) = 0.$$

Taking Laplace transforms, substituting and eliminating z gives

$$G(s) = \frac{1}{s^3 + (1+K_1)s^2 + s + K_1}.$$

ii) The Routh array is:

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 1+K_1 & K_1 \\ s & \frac{1}{1+K_1} & \\ 1 & K_1 & \end{array}$$

So $K_1 > 0$ for positive signs for the first column and therefore stability.

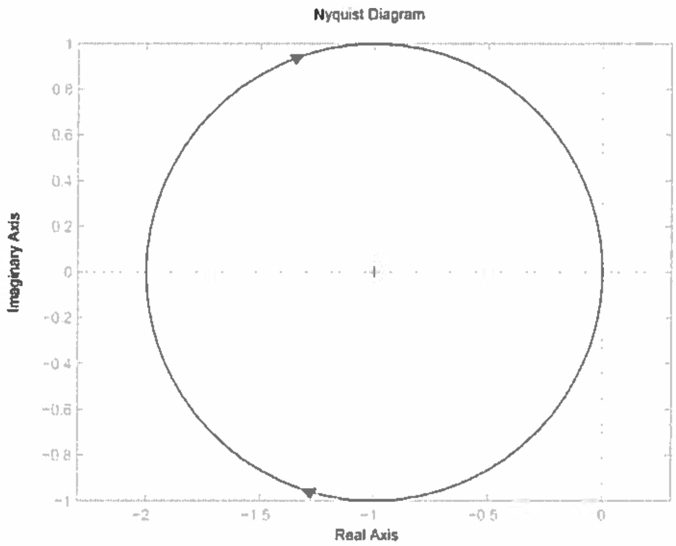
iii) When $K_1 = 0$ the array has a zero entry in the first column corresponding to marginal stability. Substituting $K_1 = 0$ into $G(s)$ gives the poles as the roots of $s(s^2+s+1)$ which are $0, -0.5 \pm j0.5\sqrt{3}$.

iv) Using the final value theorem and the fact that $f(s) = 1/s$,

$$y_{ss} := \lim_{t\to\infty} y(t) = \lim_{s\to 0} sy(s) = \lim_{s\to 0} sG(s)f(s) = \lim_{s\to 0} \frac{sG(s)}{s} = G(0) = \frac{1}{K_1}.$$

So for $y_{ss} = 2$, $K_1 = 0.5$.

b) i) The Nyquist diagram is shown below.



ii) The Nyquist criterion states that $N = Z - P$, where N is the number of clockwise encirclements by the Nyquist diagram of the point $-k^{-1}$, P is the number of unstable open–loop poles and Z is the number of unstable closed–loop poles. Since $G(s)$ has one unstable pole, $P = 1$.

- When $-\infty < k < 0.5$, $N = 0$ so $Z = 1$.
- When $0.5 < k < \infty$, $N = 1$ so $Z = 2$.
- When $k = 0.5$, the closed–loop is $\dfrac{0.5G(s)}{1+0.5G(s)} = \dfrac{1}{s^3}$ and so there are three closed-loop poles at the origin.

iii) A PD compensator has the form $K(s) = k(s+z)$. The characteristic equation for the closed-loop is

$$1+K(s)G(s) = 1+\frac{2k(s+z)}{s^3-1} = 0 \Rightarrow s^3+2ks+2kz-1 = 0$$

Since the coefficient of s^2 is zero, the closed-loop cannot be stabilised.

iv) Since $G(s) = \frac{2}{(s-1)(s^2+s+1)}$ then $G(s)K(s) = \frac{2k}{(s-1)(s^2+2s+3)}$. The characteristic equation for the closed-loop is

$$1+K(s)G(s) = 1+\frac{2k}{(s-1)(s^2+2s+3)} = 0 \Rightarrow s^3+s^2+s+2k-3 = 0$$

The Routh array:
$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 1 & 2k-3 \\ s & 2(2-k) & \\ 1 & 2k-3 & \end{array}$$

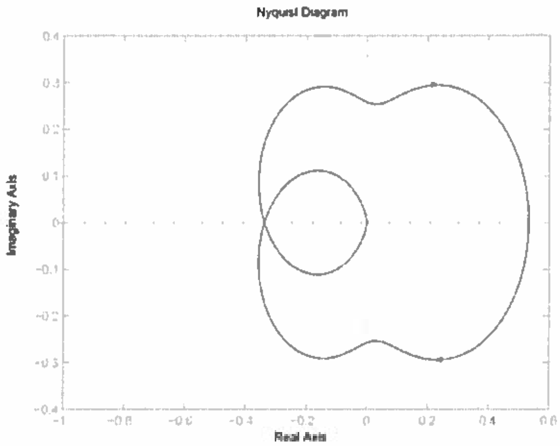
For stability, we need $1.5 < k < 2$ which is clearly possible.

2. a) The characteristic equation for the closed-loop is

$$1+KG(s)=1+\frac{K}{s^3+as^2+bs+c}=0 \Rightarrow s^3+as^2+bs+c+K=0$$

The Routh array:
$$\begin{array}{c|cc} s^3 & 1 & b \\ s^2 & a & c+K \\ s & b-\frac{c+K}{a} & \\ 1 & c+K & \end{array}$$

The real-axis intercepts: $0, -\frac{1}{ab-c}, 1/c$. A typical Nyquist diagram is:



b) We have $N=Z-P$, where N is the number of clockwise encirclements by the Nyquist diagram of $-K^{-1}$, P is the number of unstable poles of G and Z is the number of unstable closed-loop poles. To find P, the Routh array for $G(s)$:

$$\begin{array}{c|cc} s^3 & 1 & b \\ s^2 & a & c \\ s & b-\frac{c}{a} & \\ 1 & c & \end{array}$$

Since $a>0$, $c>0$ and $ab>c$ then $G(s)$ is always stable and so $P=0$.

- When $-\infty<K<-c$, $N=1$ so $Z=1$.
- When $-c<K<ab-c$, $N=0$ so $Z=0$.
- When $ab-c<K<\infty$, $N=2$ so $Z=2$.
- When $K=-c$, the closed-loop is marginally stable (a pole at 0)
- When $K=ab-c$: the closed-loop is marginally stable (poles at $\pm j\sqrt{b}$)

c) The gain margin is $ab-c$ since the real-axis intercept is at $\frac{1}{ab-c}$.

d) Since $ab-c\geq 2$, the gain margin is at least 2 for all parameter values.

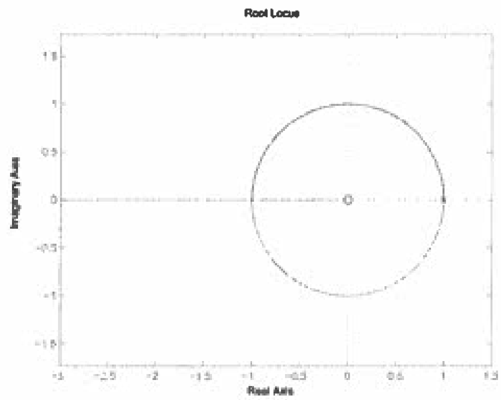
e) The gain and phase margins are adequate and we expect good transient responses. However, the DC gain $G(0)=1/c$ is less than 1 and so we need to improve the steady state performance. Since phase-lag compensation increases low frequency gain, and hence improve steady-state tracking it follows that the system requires phase-lag compensation.

3. a) For a maximum overshoot of 5% and a settling time of 4 seconds the closed–loop poles must be placed at $s_0, \bar{s}_0 = -1 \pm j$.

b) For $z=0$, the closed–loop characteristic equation is

$$1+G(s)=0 \Rightarrow 1-\frac{2s}{s^2+ks+1}=0 \Rightarrow s^2-2s+1+ks=0 \Rightarrow 1+k\overbrace{\frac{s}{(s-1)^2}}^{\hat{G}(s)}=0$$

i) We plot the root locus of $\hat{G}$:



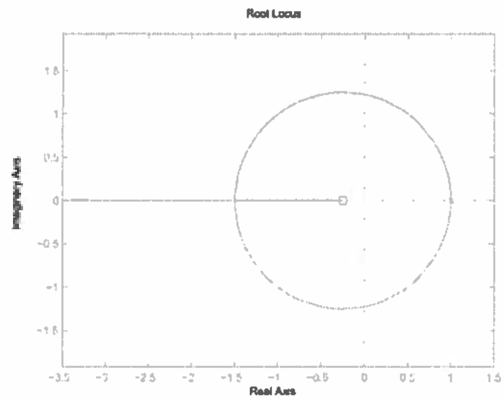
ii) Thus a settling time of 4s is only achievable with the closed–loop poles set at -1 and so the response is not oscillatory. The corresponding k is obtained from the gain criterion as $k=-1/\hat{G}(-1)=4$.

c) For general z, proceeding as before, the closed–loop characteristic equation is

$$1+G(s)=0 \Rightarrow 1-\frac{2s}{s^2+k(s+z)+1}=0 \Rightarrow s^2-2s+1+k(s+z)=0 \Rightarrow 1+k\overbrace{\frac{s+z}{(s-1)^2}}^{\hat{G}(s)}=0$$

i) $\hat{G}(s)$ has two poles at 1 and a zero at $-z$. Let the angle from $s_0=-1+j$ to $-z$ be θ and to 1 be θ_1. The angle criterion requires $\theta=2\theta_1-180^\circ$ and after some trigonometry this gives $z=0.25$.

ii) For $z=0.25$, the root–locus of $\hat{G}(s)$ is shown below.



iii) The gain criterion requires $k=-1/\hat{G}(s_0)$ and so $k=4$.