IMPERIAL COLLEGE LONDON

Final

E4.05
CS7.22
SO7

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2010

MSc and EEE PART IV: MEng and ACGI

TRAFFIC THEORY & QUEUEING SYSTEMS

Thursday, 29 April 2:30 pm

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : J.A. Barria
Second Marker(s) : M.M. Draief

© Imperial College London

Special instructions for students

1. Erlang Loss formula recursive evaluation:

$$E_N(\rho) = \frac{\rho E_{N-1}(\rho)}{N + \rho E_{N-1}(\rho)}$$
$$E_0(\rho) = 1.$$

2. Engset Loss formula recursive evaluation (for a fixed M and $p = \alpha/1+\alpha$):

$$e_N = \frac{(M-N+1)\alpha e_{N-1}}{N + (M-N+1)\alpha e_{N-1}}$$
$$e_0 = 1.$$
$$\alpha = \lambda/\mu.$$

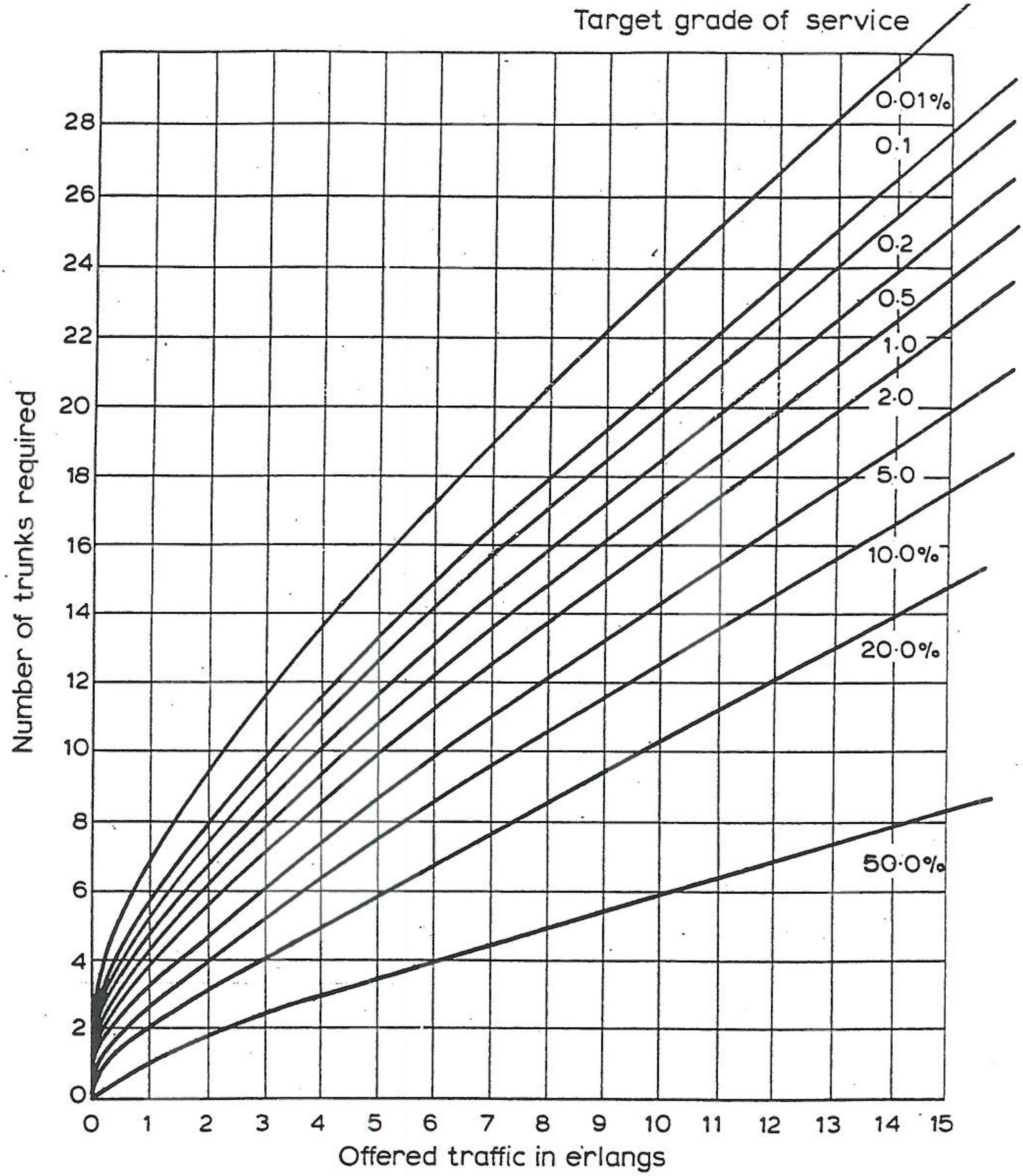
3. Traffic capacity on basis of Erlang B formula (next page).

Note: for large ρ, N is approximately linear: $N \approx 1.33\rho + 5$

4. Expected residual time

$$E[R] = \frac{1}{2}\sum_{k=1}^{m} \lambda_k E[S_k^2]$$

© Imperial College London



Traffic capacity on basis of Erlang B. formula.

The Questions

1.

a) For the Erlang model with infinite number of channels ($N = \infty$).

i) Show step by step the derivation of mean carried traffic $E(N_t)$.

[5]

ii) Show step by step the derivation of the variance of the carried traffic $Var(N_t)$.

[5]

b) For the Engset models.

i) Show step by step that the Engset distribution is a truncated binomial distribution with parameters M and p.

[8]

State clearly the meaning of M and p.

[2]

© Imperial College London

2.

a) For the M/G/1 queueing system:

i) Define and derive the expected residual time *E(R)* and show clearly all the steps in the derivation.

[5]

ii) Show step by step that the expected waiting time is $E(W) = \dfrac{E(R)}{1-\rho}$

[5]

b) Pure chance traffic is offered to a first choice communication link which can overflow into a second choice link.

- Calls arrive at a rate of $\lambda = 3$ $[calls\,/\,m]$ and their mean holding time is $1/\mu = 3$ $[m]$.

- The overflow link is composed of a single channel link:

- The average time that the first choice link is saturated is $1/\beta = 0.14124[m]$, and

- The average time that the first choice link is not saturated is $1/\alpha = 6.896[m]$.

i) Derive an approximated IPP model that describes the overflow link dynamics. State clearly the state space and state transition diagram.

[5]

ii) Write the balance equation for the IPP model defined in 2. b) i).

[5]

3.

a) A Poisson stream of messages with rate 450 [m / s] is fed to a communication channel via a large buffer.

Assume:

- One message is composed of one or more packets

- Message length from one (1) packet to ten (10) packets are equally likely to occur.

- The packet length is 80 [bits].

- Transmission rate of the channel 2048 [Mbits / s].

- Queue discipline FIFO.

Determine the overall mean message waiting time.

[10]

b) For an M/M/K/N system:

Obtain the probability that the system is empty. Explain clearly all the steps of your derivation.

[10]

© Imperial College London

4.

a) In the queueing system shown in Fig. 4.1 tokens arrive at a rate 1/D [s^{-1}] and the token buffer is of size *M*.

The arrival of cells to the system is λ [*cells / s*], and if a cell upon arrival does not find a token, that cell is lost.

i) Derive an approximate value of P_L (cell loss probability).

ii) Derive that value of the carried traffic λ^*

[10]

b) Consider a large number *N* of multiplexed ON-OFF traffic sources with parameter *p*.

i) Obtain an approximation to: $P_L = \sum_{i=J0}^{N} \frac{(i-C)\pi_i}{m}$ as $N \to \infty$ and $p \to 0$.

[5]

ii) Obtain an approximation to: $\varepsilon = \sum_{i=J0}^{N} \pi_i$ as $N \to \infty$ and $p \to 0$.

[5]

Note: $J0$ is the overload state

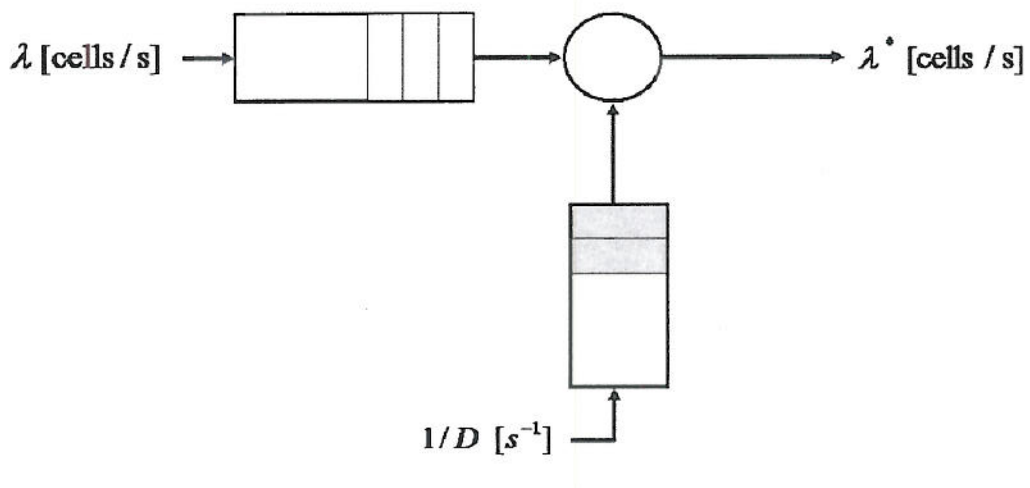


Figure 4.1:

5.

a) For the N-source stochastic fluid model represented in Fig. 5.1 the stationary probability $F(x)$ that the buffer occupancy is less than or equal to x can be written in the matrix form as:

$$\frac{dF(x)}{dx}D = F(x)M$$

i) For $N = 2$ construct the matrices D and M.

[4]

ii) Obtain the dominant eigenvalue for $N = 1$. Explain the importance of this eigenvalue.

[4]

iii) For $N = 1$ obtain the condition for the system to be stable.

[4]

b) For an M/M/K/N system. Prove that the distribution of queue length seen by delay arrivals is a truncated geometrical distribution. Explain clearly all the steps of your derivation.

[8]

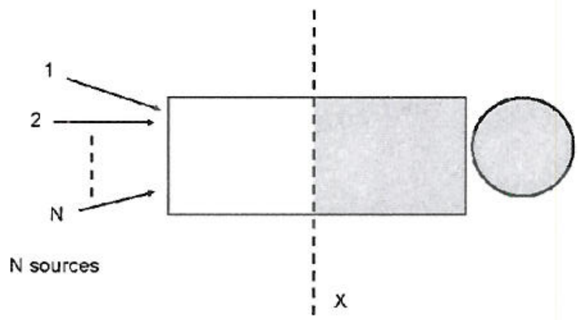


Figure 5.1:

6.

a) Computer system under observation contains four (4) statistically identical processors and three (3) statistically identical buses. The system can detect when a processor or bus has failed, and it can reconfigure so that the failed component is removed.

The system is operational if at least 2 processors and 2 buses are working in order. Otherwise is considered a failure.

The coverage factor depends on the state of the system. If a component (processor or bus) failure rate is x, there are k operational copies of the component, and the reconfiguration rate α, then the coverage factor is given by:

$$c(x,k) = \frac{\alpha}{\alpha + (k-1)x}$$

Processor failure rate = λ [1/s]

Buses failure rate = μ [1/s]

Note: the coverage factor is defined by the probability of no near-coincident fault (probability that reconfiguration is successful)

i) Define the state space of the system.
Clearly identify the operational states. [5]

ii) Construct the Markov chain. [5]

iii) Clearly show the transition rates. [10]

E4.05/C57.22/S07

Traffic Theory & Queueing Systems

Examinations : 2010 Session — *Confidential*

MODEL ANSWER and MARKING SCHEME

First Examiner	Paper Code
Second Examiner	Question — Page 1 out of 10

Question labels in left margin — Marks allocations in right margin

Q1 a) i)

$$E(N_t) = \sum_{i=0}^{\infty} i\pi_i$$

$$= \sum i \frac{\rho^i}{i!}\pi_0 = \frac{1}{e^\rho}\sum i\frac{\rho^i}{i!}$$

$$= \frac{1}{e^\rho}\left(\frac{1\rho^1}{1!} + \frac{2\rho^2}{2!} + \frac{3\rho^3}{3!} + \dots\right) = \frac{1}{e^\rho}\rho\left(1 + \frac{\rho}{1!} + \frac{\rho^2}{2!} + \dots\right)$$

$$= \frac{1}{e^\rho}\rho e^\rho = \rho$$ 5

ii) $\text{Var} = E[(x-\mu)^2] = E(x^2 - 2x\mu + \mu^2) = E(x^2) - (E(x))^2$

$$E(N_t^2) = \sum_{i=0}^{\infty} i^2\pi_i = \frac{1}{e^\rho}\sum i^2\frac{\rho^i}{i!} = \frac{1}{e^\rho}\left(\frac{1\rho^1}{1!} + \frac{4\rho^2}{2!} + \frac{9\rho^3}{3!} + \dots\right)$$

$$= \frac{1}{e^\rho}\left(\rho + \frac{2\rho^2}{2!} + \frac{3\rho^3}{3!} + \frac{4\rho^4}{4!} + \dots\right) + \frac{1}{e^\rho}\left(\frac{2\rho^2}{2!} + \frac{6\rho^3}{3!} + \frac{12\rho^4}{4!} + \dots\right)$$

$$= \frac{1}{e^\rho}\rho e^\rho + \frac{1}{e^\rho}\rho^2\left(\frac{2}{2!} + \rho + \frac{\rho^2}{2!} + \dots\right)$$

$$= \rho + \frac{1}{e^\rho}\rho^2 e^\rho = \rho + \rho^2$$

$$\text{variance} = \rho + \rho^2 - \rho^2 = \rho$$ 5

Q1 b) i) $\boxed{N \geq M}$, $i_{max} = M$

$$\pi_i = \left(\frac{\lambda_{i-1}}{\mu_i}\right)\pi_{i-1} = \left(\frac{(M-(i-1))\lambda}{i\mu}\right)\pi_{i-1}$$

$$\pi_i = \binom{M}{i}\alpha^i \pi_0 \qquad i = 1, 2, \ldots, M$$

$$\pi_0 = \left[\sum_{j=0}^{M}\binom{M}{j}\alpha^j\right]^{-1} \qquad \alpha = \frac{\lambda}{\mu}$$

$$p = \frac{\alpha}{1+\alpha}, \quad \alpha = \frac{p}{1-p}, \quad M = \text{no of sources}$$ 2

$$\pi_i = \frac{\binom{M}{i}p^i(1-p)^{M-i}}{\sum_{j=0}^{M}\binom{M}{j}p^j(1-p)^{M-j}} \qquad i = 0, 1, \ldots M$$

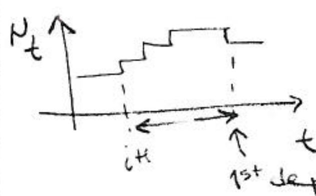
π_i is binomial (M, p)

$\boxed{M > N}$, $i_{max} = N$

$$\pi_i = \frac{\binom{M}{i}p^i(1-p)^{M-i}}{\sum_{j=0}^{N}\binom{M}{j}p^j(1-p)^{M-j}} \qquad i = 0, 1, \ldots N$$ 8

(truncated binomial distribution)

Q2 i) M|G|1



$$W_i = R_i + \sum_{j=1}^{Q_i} S_{i-j}$$

S_i = service time

W_i = waiting time

Q_i = queue length found on arrival

$$E[W_i] = E[R_i] + E\left[\sum_{j=1}^{Q} S_{i-j}\right]$$

$$= E[R_i] + E[Q_i]E[S]$$

explain this step

$$E[W] = E[R] + E[Q]E[S]$$

using Little's $E[Q] = \lambda E[W]$

$$E[W] = E[R] + \rho E[W] \qquad \rho = \lambda E(S)$$

$$E[W] = \frac{E[R]}{1-\rho}$$

5

ii)

$$E[R_t] = \lim_{T\to\infty} \frac{1}{T}\int_0^T R_t\, dt = \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{M_T}\left(\frac{1}{2}S_i^2\right)$$

$$= \lim_{T\to\infty} \frac{1}{2}\underbrace{\left(\frac{M_T}{T}\right)}_{\lambda}\underbrace{\left[\frac{1}{M_T}\sum_{i=1}^{M_T} S_i^2\right]}_{E[S^2]}$$

$$E[R_t] = \frac{1}{2}\lambda E[S^2]$$

5

Examinations : - Session *Confidential*

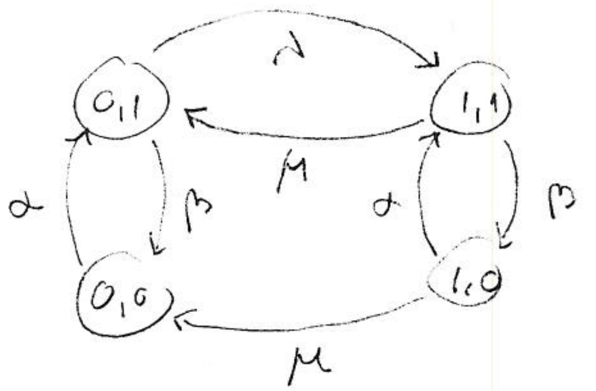
MODEL ANSWER and MARKING SCHEME

First Examiner | Paper Code

Second Examiner | Question | Page 4 *out of*

Question labels in left margin | Marks allocations in right margin

Q2 b)



$\pi_0 = P(N_t = 0, Y_t = 0)$ $\quad \sigma_0$

$\sigma_0 = P(N_t = 0, Y_t = 1)$ 5

...

global balance equations

$$\alpha \pi_0 = \beta \sigma_0 + \mu \pi_1$$

$$(\mu + \lambda)\, \sigma_0 = \alpha \pi_0 + \mu \sigma_1$$

$$(\alpha + \mu)\, \pi_1 = \beta \sigma_1$$

$$(\mu + \mu)\, \sigma_1 = \alpha \pi_1 + \lambda \sigma_0$$

$\alpha = 0.1450$

$\beta = 7.080$

$\lambda = 2$ 5

$\mu = 0.33$

Q3 a) $\lambda = 450$ m/s

$$E(S) = \frac{80}{2048} \cdot \underbrace{\frac{(1+2+\dots+10)}{10}}_{5.5} \mu s = 0.2148 \text{ [ms]}$$ 3

$$\rho = \lambda E(S) = 0.09666$$

$$E(S^2) = \underbrace{\left(\frac{80}{2048}\right)^2}_{1525.87} \underbrace{\frac{(1^2+2^2+\dots+10^2)}{10}}_{38.5} \mu s = 5.874 \times 10^{-8} \text{ [s]}$$ 3

$$\text{Mean Waity Time} = \frac{\lambda E(S^2)}{2(1-\rho)} = \frac{450 \times 5.874 \times 10^{-8}}{2(1-0.966)} = 0.388 \text{ ms}$$ 4

b) $M|M|K|N \quad (N = K + B)$

$$\pi_i = \left(\frac{A^i}{i!}\right)\pi_0 \quad 0 \le i \le K$$

$$= \left(\frac{A^K}{K!}\right)\rho^{i-K}\pi_0 \quad K \le i \le K+B$$

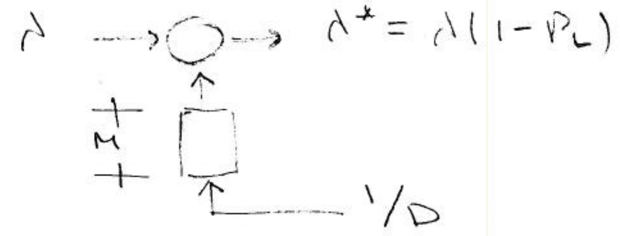
Explain 5

$$\pi_0 = S^{-1}$$

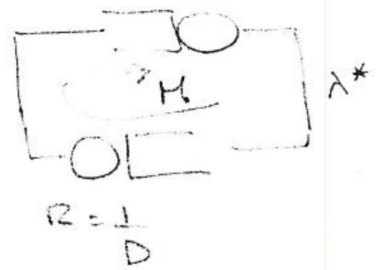
$$S = \frac{A^K}{K!}\left[E_K^{-1}(A). + \frac{\rho(1-\rho^B)}{1-\rho}\right] \quad \rho \ne 1$$

$$\pi_0 = \frac{1}{(A^K/K!)}\left[\frac{(1-\rho)E_K(A)}{(1-\rho)+\rho(1-\rho^B)E_K(A)}\right]$$ 5

Q4 a)



Model



$\lambda^* = \lambda(1-P_L) = \lambda(1-P_e)$

P_L = probability 'upper' queue empty
= probability 'lower' queue full 2

M|M|1|M

$$P_L = \frac{\rho^M(1-\rho)}{1-\rho^{M+1}}$$ 5

$$\rho = \frac{\lambda}{R} = \lambda D$$

$$\lambda^* = \lambda\left[\frac{1-\rho^M}{1-\rho^{M+1}}\right]$$ 3

Q4b) Large number of ON-OFF sources multiplexed

$N \gg 1$, $p \ll 1$

$\pi_i = \binom{N}{i} p^i (1-p)^{N-i}$ (binomial)

Is approximated quite closely by the normal distribution ($m = Np$, $\sigma^2 = Np(1-p)$) 5

$$P_L = \frac{1}{m} \int_{c}^{\infty} \frac{e^{-(x-m)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} (x - C)\, dx$$

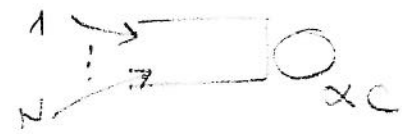
$$\varepsilon = \int_{c}^{\infty} \frac{e^{-(x-m)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\, dx$$ 5

if $(c-m) > 3\sqrt{2}\sigma$

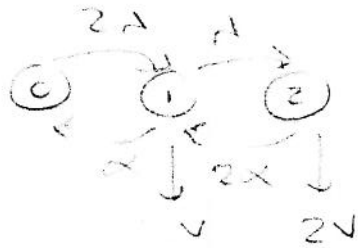
$$\varepsilon = \frac{\sigma e^{-(c-m)^2/2\sigma^2}}{\sqrt{2\pi}\,(c-m)}$$ 5

$$P_L = \frac{1-p}{c-m}\, \varepsilon$$ 5

Q5 a) i)



$N = 2$



$$M = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \alpha & -(\alpha+\lambda) & \lambda \\ 0 & 2\alpha & -2\alpha \end{bmatrix}$$ 2

$$D = \begin{bmatrix} -c\alpha & c & 0 \\ c & (1-c)\alpha & 0 \\ 0 & 0 & (2-c)\alpha \end{bmatrix}$$ 2

ii) $\frac{dF(x)}{dx} = F(x)\, M' \qquad M' = M D^{-1}$

$$M = \begin{bmatrix} -\lambda & \lambda \\ \alpha & -\alpha \end{bmatrix} = \begin{bmatrix} -\gamma & \gamma \\ 1 & -1 \end{bmatrix} \qquad \gamma = \frac{\lambda}{\alpha}$$

$$D = \begin{bmatrix} -c\alpha & c \\ 0 & (1-c)\alpha \end{bmatrix} \rightarrow D^{-1} = \begin{bmatrix} \frac{1}{-c\alpha} & 0 \\ 0 & \frac{1}{(1-c)\alpha} \end{bmatrix}$$

$$M' = \begin{bmatrix} \frac{\gamma}{c} & \frac{\gamma}{1-c} \\ -\frac{1}{c} & -\frac{1}{1-c} \end{bmatrix} \quad \alpha = 1 \qquad zI - M' = 0$$

$$z = \frac{\gamma}{c} - \frac{1}{1-c}.$$ 4

system is stable if $\rho \equiv \left(\frac{\gamma}{1+\gamma}\right)\frac{1}{c} < 1$ 4

$$z = \frac{-(1-\rho)(1+\gamma)}{1-c}$$

Q5 b)

$$P(Q_t = i \mid \text{delay}) = P(Q_t = i \mid \text{all server busy})$$

$N = \infty$

$$P(Q_t = i \mid N_t \geq K) = \frac{P(N_t = K+i)}{\sum_{j=0}^{\infty} P(N_t = K+j)}$$

$$= \frac{\pi_K \rho^i}{\sum_{j=0}^{\infty} \pi_K \rho^j} = \frac{\rho^i}{\sum_{j=0}^{\infty} \rho^j}$$

4

$$= (1-\rho)\rho^i \quad i = 0, 1, \ldots$$

$N < \infty = K + B$

$$P(Q_t = i \mid \text{delay}) = P(Q_t = i \mid K \leq N_t < K+B)$$

$$= \frac{P(N_t = K+i)}{P(\text{delay})}$$

$$= \frac{\pi_K \rho^i}{\pi_K \left(\frac{1-\rho^B}{1-\rho}\right)} \quad i = 0, \ldots, B-1$$

$$= \rho^i \left(\frac{1-\rho}{1-\rho^B}\right) \quad i = 0, \ldots B-1$$

4

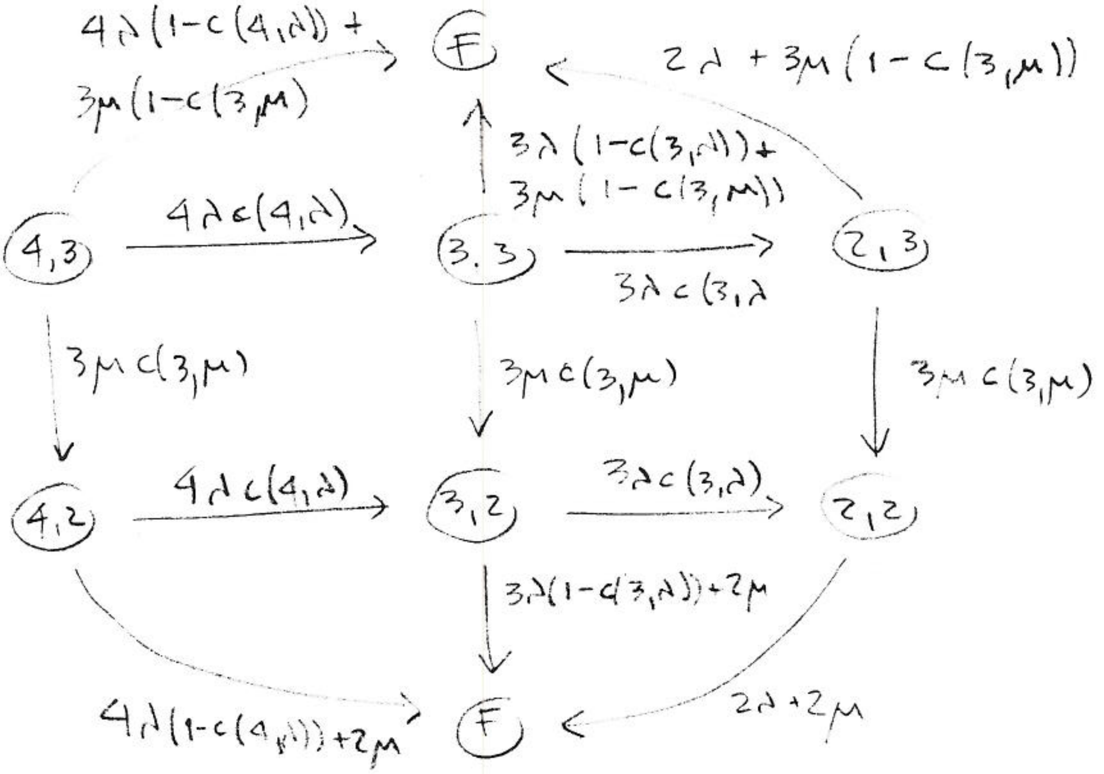
(truncated geometric distribution)

Q6 a) i) (# processor up, # buses up)

(4,3) (4,2) (3,3) (3,2) (2,3) (2,2) operational states 5

all other Failure states

ii) 5



10