	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course
	E2.11 - ISE2 MATHS - SOLUTIONS 2008	ISE2.
Question 1		Marks & seen/unseen

Parts

(i) $$\hat{u}_a(\omega) = \int_{-\infty}^{\infty} u_a(t) e^{-i\omega t}\, dt = \int_{-a}^{a} e^{-i\omega t}\, dt$$
$$= -\frac{1}{i\omega}\left[e^{-i\omega a} - e^{i\omega a}\right] = \frac{2}{\omega}\sin(\omega a)$$ 4

(ii) Since $\frac{dh}{dt} = g$ we have $\hat{g} = i\omega\hat{h}$

& also $g(t) = f(t+a) - f(t-a)$
$$\Rightarrow \hat{g}(\omega) = \int_{-\infty}^{\infty} f(t+a) e^{-i\omega t}\, dt - \int_{-\infty}^{\infty} f(t-a) e^{-i\omega t}\, dt$$
$$= \int_{-\infty}^{\infty} f(s) e^{-i\omega(s-a)}\, ds - \int_{-\infty}^{\infty} f(s) e^{-i\omega(s+a)}\, ds$$
$$= e^{i\omega a}\hat{f}(\omega) - e^{-i\omega a}\hat{f}(\omega)$$ [or can just quote shift rule]
$$= 2i\sin(\omega a)\hat{f}(\omega)$$

Thus we have $\hat{h} = \frac{2}{\omega}\sin(\omega a)\hat{f}$ as the req'd relation. 6 mostly seen, but phrased differently

(iii) Using convolution:
$$h(t) = (FT)^{-1}\left\{ \underbrace{\frac{2}{\omega}\sin(\omega a)}_{\equiv \hat{g},\ \text{say}}\ \hat{f} \right\}$$
$$= \int_{-\infty}^{\infty} f(t-s) g(s)\, ds.$$

but, from part (i) we see that $g(t) = u_a(t)$.
$$\therefore h(t) = \int_{-\infty}^{\infty} f(t-s) u_a(s)\, ds$$
$$= \int_{-a}^{a} f(t-s)\, ds$$ [N.B. this is also equal to $\int_{-a}^{a} f(t+s)\, ds$] 6

(iv) By differentiating this last expression we have
$$dh/dt = \int_{-a}^{a} f'(t-s)\, ds$$ [or use $\int_{-a}^{a} f(t+s) ds$]
$$= \int_{t-a}^{t+a} f'(q)\, dq \quad (q = t-s)$$
$$= f(t+a) - f(t-a)$$ which is g, so OK ✓ 4

(Total 20)

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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course ISE 2.
Question 2.		Marks & seen/unseen

Parts

(i)
$$\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2 d\omega = \int_{-\infty}^{\infty}\hat{f}(\omega)\hat{f}^*(\omega)\,d\omega$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\right)\left(\int_{-\infty}^{\infty} f^*(u)e^{i\omega u}du\right)d\omega \quad 2$$

Changing the order of integration:
$$= \int_{t=-\infty}^{\infty} f(t)\int_{u=-\infty}^{\infty} f^*(u)\underbrace{\left(\int_{\omega=-\infty}^{\infty} e^{-i\omega(t-u)}d\omega\right)}_{2\pi\delta(t-u)\ \text{(given)}}du\,dt \quad 2$$

$$= 2\pi\int_{t=-\infty}^{\infty} f(t)f^*(t)\,dt \quad 2$$

$$= \underline{\underline{2\pi\int_{-\infty}^{\infty}|f(t)|^2\,dt}} \quad 2$$

Hence result. 2 — 10

[Alternatively, this may be proved using the convolution theorem]

(i) is seen
(ii) is unseen

(ii) We are given $f(t) = \begin{cases} 0 & t<0 \\ e^{-t} & t\geq 0\end{cases}$

LHS of Parseval formula is $\int_{-\infty}^{\infty}|f(t)|^2dt = \int_0^{\infty}e^{-2t}dt$ 2

$$= \left[-\tfrac{1}{2}e^{-2t}\right]_0^{\infty} = \tfrac{1}{2} \quad 1$$

To evaluate RHS, first need $\hat{f}(\omega)$.

$$\hat{f}(\omega) = \int_0^{\infty}e^{-t}e^{-i\omega t}dt = \int_0^{\infty}e^{-(1+i\omega)t}dt$$

$$= \left[-\frac{1}{1+i\omega}e^{-(1+i\omega)t}\right]_0^{\infty}$$

$$= \underline{1/(1+i\omega)} \quad 3$$

Then $|\hat{f}|^2 = \hat{f}\hat{f}^* = \dfrac{1}{1+i\omega}\cdot\dfrac{1}{1-i\omega}$

$$= \underline{1/(1+\omega^2)} \quad 2$$

Thus, RHS of Parseval is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{1+\omega^2} \underset{\text{(formula sheet)}}{=} \frac{1}{2\pi}\left[\tan^{-1}(\omega)\right]_{-\infty}^{\infty} = \frac{1}{2\pi}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = 1/2 \quad 2$$

10 (Total 20)

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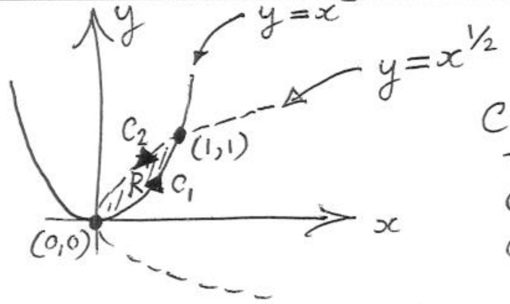
	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course ISE2
Question SOLUTION TO Q3		Marks & seen/unseen

Parts

(i)



C is formed from the union of C_1 & C_2.

C_1 goes from $(0,0)$ to $(1,1)$

C_2 goes from $(1,1)$ to $(0,0)$ — 5

(ii) For $n=1$ we have $y=x^2$ & hence $dy=2x\,dx$.

Thus, writing I_1 in terms of $\cancel{y}$ x we have

$$I_1 = \int_0^1 \frac{2x^8}{2x^2}dx - \frac{x^7 2x\,dx}{2x^2} = \underline{\underline{0}}$$ — 3

For $n=2$ we have $y^2=x \Rightarrow 2y\,dy=dx$

Writing I_2 in terms of y we have

$$I_2 = \int_1^0 \frac{y^4}{(y^2)^2}\cdot 2y\,dy - \frac{1}{2}\frac{y^3}{y^2}dy$$

$$= \int_1^0 (2y - \tfrac{1}{2}y)\,dy = \frac{3}{2}\left[\frac{y^2}{2}\right]_1^0 = \underline{\underline{-\frac{3}{4}}}$$ — 4

(iii) We have $P = y^4/x^2$, $Q = -\frac{1}{2}y^3/x$

Thus $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{2}\frac{y^3}{x^2} - 4\frac{y^3}{x^2} = -\frac{7}{2}\frac{y^3}{x^2}$.

unseen but similar problems done.

$\therefore$ Applying Green's Theorem:

$$\int_C P\,dx + Q\,dy = \iint_R -\frac{7}{2}\frac{y^3}{x^2}\,dx\,dy$$

$$= \int_{x=0}^{x=1}\int_{y=x^2}^{y=x^{1/2}} -\frac{7}{2}\frac{y^3}{x^2}\,dy\,dx$$

$$= -\frac{7}{8}\int_{x=0}^{x=1}\frac{1}{x^2}\left[y^4\right]_{x^2}^{x^{1/2}}dx$$

$$= -\frac{7}{8}\int_0^1 (1-x^6)\,dx = -\frac{3}{4}$$ — 6

Total 20

(iv) Since $\int_C \equiv \int_{C_1} + \int_{C_2}$, we have $-\frac{3}{4} = 0 - \frac{3}{4}$ ✓ so results are consistent. — 2

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	EXAMINATION QUESTIONS/SOLUTIONS 2007-08	Course ISE2
Question 4		Marks & seen/unseen
Parts (i)	Poles of $f(z)$ occur when denominator is zero. i.e. $z(3z^2+13z+4)=0$ $\Rightarrow z(3z+1)(z+4)=0$ $\Rightarrow \underline{z=0,\ z=-1/3,\ z=-4}$	1 3
	To find order of poles: At $z=0$: Consider $\lim\limits_{z\to 0} z^\alpha f(z) = \lim\limits_{z\to 0}\frac{z^{\alpha-1}}{3z^2+13z+4}$ $\neq 0$ & finite only if $\alpha=1$. i.e. order of pole is 1 Similarly: At $z=-\frac{1}{3}$: $\lim\limits_{z\to 0}(z+\frac{1}{3})^\alpha f(z)\neq 0$ iff $\alpha=1$ $z=-4$: $\lim\limits_{z\to -4}(z+4)^\alpha f(z)\neq 0$ iff $\alpha=1$ So all poles are simple. [Alternatively just state that each zero of the denominator occurs to multiplicity one]	unseen but a similar problem studied in class 3
	$\operatorname*{Res}\limits_{z=0} f(z) = \lim\limits_{z\to 0} z f(z) = \underline{\underline{\frac{1}{4}}}$	2
	$\operatorname*{Res}\limits_{z=-1/3} f(z) = \lim\limits_{z\to -1/3}\cancel{(z+\frac{1}{3})}\frac{1}{3z\cancel{(z+1/3)}(z+4)} = \underline{\underline{-\frac{3}{11}}}$	2
	$\operatorname*{Res}\limits_{z=-4} f(z) = \lim\limits_{z\to -4}\cancel{(z+4)}\frac{1}{z(3z+1)\cancel{(z+4)}} = \underline{\underline{\frac{1}{44}}}$	2
(ii)	Hence: $\mathcal{L}^{-1}(F(s)) =$ Sum of residues of $F(s)e^{st}$ at poles of $F(s)$	1
	$\operatorname*{Res}\limits_{s=0} F(s)e^{st} = \frac{1}{4}$; $\operatorname*{Res}\limits_{s=-1/3} F(s)e^{st} = -\frac{3}{11}e^{-t/3}$ $\operatorname*{Res}\limits_{s=-4} F(s)e^{st} = \frac{1}{44}e^{-4t}$ using results obtained in (i) Thus $\underline{\underline{\mathcal{L}^{-1}(F(s)) = \frac{1}{4} - \frac{3}{11}e^{-t/3} + \frac{1}{44}e^{-4t}}}$	6
	[Otherwise: P.F.'s: $F(s) = \frac{1}{4s} + \frac{1/44}{s+4} - \frac{9/11}{(3s+1)}$ Invert, using tables: $f(t) = \frac{1}{4} + \frac{1}{44}e^{-4t} - \frac{3}{11}e^{-t/3}$]	Total 20
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5. Let

T_1 - event first test passes (no memory errors detected)
$\overline{T_1}$ - event first test fails (memory errors detected)
T_2 - event second test passes (no memory errors detected)
$\overline{T_2}$ - event second test fails (memory errors detected)
E - event that there are memory errors
$\overline{E}$ - event that there are no memory errors

(i)

$$\begin{aligned} \mathrm{P}(T_1 \mid \overline{E}) &= 1 & \mathrm{P}(\overline{T_1} \mid \overline{E}) &= 0 \\ \mathrm{P}(T_1 \mid E) &= 0.2 & \mathrm{P}(\overline{T_1} \mid E) &= 0.8 \\ \mathrm{P}(T_2 \mid \overline{E}) &= 1 & \mathrm{P}(\overline{T_2} \mid \overline{E}) &= 0 \\ \mathrm{P}(T_2 \mid E) &= 0.01 & \mathrm{P}(\overline{T_2} \mid E) &= 0.99 \\ \mathrm{P}(E) &= 0.02 & \mathrm{P}(\overline{E}) &= 0.98 \end{aligned}$$

(a)

$$\begin{aligned} \mathrm{P}(\overline{T_1}) &= \mathrm{P}(\overline{T_1} \mid E)\mathrm{P}(E) + \mathrm{P}(\overline{T_1} \mid \overline{E})\mathrm{P}(\overline{E}) \\ &= 0.8 \times 0.02 + 0 \times 0.98 = 0.016. \end{aligned}$$

$\boxed{2}$

(b)

$$\begin{aligned} \mathrm{P}(E \mid \overline{T_1}) &= \frac{\mathrm{P}(\overline{T_1} \mid E)\mathrm{P}(E)}{\mathrm{P}(\overline{T_1})} \\ &= \frac{0.8 \times 0.02}{0.016} = 1. \end{aligned}$$

$\boxed{2}$

(as expected!)

(c)

$$\begin{aligned} \mathrm{P}(T_1 \cap T_2) &= \mathrm{P}(T_1 \cap T_2 \mid E)\mathrm{P}(E) + \mathrm{P}(T_1 \cap T_2 \mid \overline{E})\mathrm{P}(\overline{E}) \\ &= \mathrm{P}(T_1 \mid E)\mathrm{P}(T_2 \mid E)\mathrm{P}(E) + \mathrm{P}(T_1 \mid \overline{E})\mathrm{P}(T_2 \mid \overline{E})\mathrm{P}(\overline{E}) \\ &= 0.2 \times 0.01 \times 0.02 + 1 \times 1 \times 0.98 = 0.98004. \\ \mathrm{P}(E \mid T_1 \cap T_2) &= \frac{\mathrm{P}(T_1 \cap T_2 \mid E)\mathrm{P}(E)}{\mathrm{P}(T_1 \cap T_2)} \\ &= \frac{\mathrm{P}(T_1 \mid E)\mathrm{P}(T_2 \mid E)\mathrm{P}(E)}{\mathrm{P}(T_1 \cap T_2)} \\ &= \frac{0.2 \times 0.01 \times 0.02}{0.98004} = 4.08 \times 10^{-5}. \end{aligned}$$

$\boxed{6}$

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(ii) Let R_1 = running time of first test
R_2 = running time of second test

$$R_1 \sim N(5, 2^2) \qquad R_2 \sim N(60, 10^2) \qquad Z \sim N(0,1)$$

(a)

$$\begin{aligned} P(R_1 > 6) &= P\left(\frac{R_1 - 5}{2} > \frac{6-5}{2}\right) = P\left(Z > \frac{1}{2}\right) \\ &= 1 - \Phi(0.5) = 1 - 0.691 = 0.309. \end{aligned} \qquad \boxed{3}$$

(b)

$$\begin{aligned} P(R_1 < 3) &= P\left(\frac{R_1 - 5}{2} < \frac{3-5}{2}\right) = P(Z < -1) \\ &= \Phi(-1) = 1 - \Phi(1) = 1 - 0.841 = 0.159. \end{aligned} \qquad \boxed{2}$$

(c) Total test time $R = R_1 + R_2$.

$$R \sim N(5 + 60, 2^2 + 10^2) = N(65, 104) \qquad \boxed{2}$$

(d) Given

$$\overline{x} = 5.24, s = 2.12, t_0 = t_{n-1,0.05} = 2.26.$$

95% CI for mean:

$$\begin{aligned} \left(\overline{x} - t_0\frac{s}{\sqrt{n}}, \overline{x} + t_0\frac{s}{\sqrt{n}}\right) &= \left(5.24 - 2.26 \times \frac{2.12}{\sqrt{10}}, 5.24 + 2.26 \times \frac{2.12}{\sqrt{10}}\right) \\ &= (3.725, 6.755) \end{aligned}$$

this interval contains the reported mean value of 5 minutes, so we would fail to reject a hypothesis that the mean is 5 minutes at the 5% level. $\boxed{3}$

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6. (i) (a)

$$\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t = \int_0^{\infty} \lambda\beta(\lambda t)^{\beta-1} e^{-(\lambda t)^\beta}\,\mathrm{d}t = \left[-e^{-(\lambda t)^\beta}\right]_0^\infty = 1,$$

and $f(t) \geq 0 \; \forall t \geq 0$ as $\lambda, \beta \geq 0$. $\boxed{3}$

(b) Reliability:

$$\begin{aligned} R(t) &= \mathrm{P}(T > t) = \int_t^\infty \lambda\beta(\lambda t_0)^{\beta-1} e^{-(\lambda t_0)^\beta}\,\mathrm{d}t_0 \\ &= \left[-e^{-(\lambda t_0)^\beta}\right]_t^\infty = e^{-(\lambda t)^\beta} \end{aligned}$$ $\boxed{2}$

Hazard:

$$h(t) = \frac{f(t)}{R(t)} = \frac{\lambda\beta(\lambda t)^{\beta-1} e^{-(\lambda t)^\beta}}{e^{-(\lambda t)^\beta}} = \lambda\beta(\lambda t)^{\beta-1}$$ $\boxed{2}$

(c)

$$\begin{aligned} \mathrm{P}(T > t_0 + t \mid T > t_0) &= \frac{\mathrm{P}(T > t_0 + t \cap T > t_0)}{\mathrm{P}(T > t_0)} \\ &= \frac{\mathrm{P}(T > t_0 + t)}{\mathrm{P}(T > t_0)} = \frac{e^{-(\lambda(t+t_0))^\beta}}{e^{-(\lambda t_0)^\beta}} \\ &= e^{(\lambda t_0)^\beta - (\lambda(t+t_0))^\beta} \end{aligned}$$ $\boxed{3}$

(d) Now, $\mathrm{P}(T > t) = e^{-(\lambda t)^\beta}$.
"memoryless" when $\mathrm{P}(T > t_0 + t \mid T > t_0) = \mathrm{P}(T > t)$. From part (c), this occurs when $\beta = 1$, (and, of course trivially when $t_0 = 0$!) giving $T \sim \mathit{Exponential}(\lambda)$. $\boxed{2}$

(ii) Let R be the reliability at 30 minutes (=0.5 hours).
Let A, B_1, B_2 be the events that the corresponding component is operating at 30 minutes.

$$\mathrm{P}(T > t) = e^{-(\lambda t)^\beta} \Rightarrow \mathrm{P}\left(T > \frac{1}{2}\right) = e^{-(\frac{\lambda}{2})^\beta}$$ $\boxed{2}$

Giving,

$$\mathrm{P}(A_1) = e^{-(\frac{0.5}{2})^{0.8}} = 0.7190; \quad \mathrm{P}(B_1) = \mathrm{P}(B_2) = e^{-(\frac{0.5}{2})^{0.5}} = 0.6065.$$ $\boxed{2}$

$$\begin{aligned} R &= \mathrm{P}(A_1 \cap (B_1 \cup B_2)) = \mathrm{P}(A_1)\mathrm{P}(B_1 \cup B_2) \\ &= \mathrm{P}(A_1)(\mathrm{P}(B_1) + \mathrm{P}(B_2) - \mathrm{P}(B_1 \cap B_2)) \\ &= \mathrm{P}(A_1)(\mathrm{P}(B_1) + \mathrm{P}(B_2) - \mathrm{P}(B_1)\mathrm{P}(B_2)) \\ &= 0.7190 \times (0.6065 + 0.6065 - 0.6065^2) = 0.6077. \end{aligned}$$ $\boxed{4}$

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