SOLUTIONS: COMPLEX CALCULUS
EE2L

1. Exercise

a) Differentiating u with respect to x and y yields:

$$\frac{\partial u}{\partial x} = 2y - \frac{\mathrm{e}^{y}\sin(x)}{2} - \frac{\mathrm{e}^{-y}\sin(x)}{2}$$
$$\frac{\partial u}{\partial y} = 2x + \frac{\mathrm{e}^{y}\cos(x)}{2} - \frac{\mathrm{e}^{-y}\cos(x)}{2}.$$

Taking second derivatives with respect to x and y, yields:

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\mathrm{e}^{y}\cos(x)}{2} - \frac{\cos(x)}{2\,\mathrm{e}^{y}},$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{\mathrm{e}^{y}\cos(x)}{2} + \frac{\cos(x)}{2\,\mathrm{e}^{y}}.$$

Notice that:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence, u is harmonic.

b) From Cauchy-Riemann equations we have that $u_x = v_y$, hence:

$$\frac{\partial v}{\partial y} = 2y - \frac{\mathrm{e}^{y}\sin(x)}{2} - \frac{\mathrm{e}^{-y}\sin(x)}{2}.$$

Taking indefinite integrals of the above equation with respect to y yields:

$$v = y^2 - \sin(x)\,\frac{e^{y} - e^{-y}}{2} + c_1(x).$$

Moreover, $v_x = -u_y$, hence:

$$\frac{\partial v}{\partial x} = -2x - \frac{\mathrm{e}^{y}\cos(x)}{2} + \frac{\mathrm{e}^{-y}\cos(x)}{2}.$$

Taking indefinite integrals of the above equation with respect to x yields:

$$v = -x^2 - \sin(x)\frac{e^{y} - e^{-y}}{2} + c_2(y).$$

Equating the two expressions we see that:

$$c_1(x) + y^2 = c_2(y) - x^2$$

and therefore, for some complex $c \in \mathbb{C}$:

$$c_1(x) = c - x^2$$
$$c_2(y) = c + y^2.$$

Therefore, v is determined up to an additive constant as follows:

$$v = y^2 - x^2 - \sin(x)\frac{e^{y} - e^{-y}}{2} + c.$$

c) Notice that, for $z = x + i0$, it holds:

$$f(z) = u(x,0) + iv(x,0) = \cos(x) + i(c - x^2)$$

Hence, we may take:

$$f(z) = \cos(z) + i(c - z^2).$$

a) The function is a ratio of polynomials. The denominator vanishes for $z \in \{\pm i, \pm 2i, \pm 3i\}$. The numerator only vanishes for $z = 0$. Hence, the function is holomorphic in $\mathbb{C} \setminus \{\pm i, \pm 2i, \pm 3i\}$.

b) The function has poles in $\pm i, \pm 2i$ and $\pm 3i$. Computation of the Residues yields:

$$Res(i) = \lim_{z \to i} (z - i) f(z) = \lim_{z \to i} \frac{z}{(z+i)(z^2+4)(z^2+9)} = \frac{1}{48}$$

$$Res(2i) = \lim_{z \to 2i} (z - 2i) f(z) = \lim_{z \to 2i} \frac{z}{(z^2+1)(z+2i)(z^2+9)} = -\frac{1}{30}$$

$$Res(3i) = \lim_{z \to 3i} (z - 3i) f(z) = \lim_{z \to 3i} \frac{z}{(z^2+1)(z^2+4)(z+3i)} = \frac{1}{80}$$

c) In order to compute the improper integral, it is enough to realize that:

$$\int_{-\infty}^{+\infty} f(x) dx = \lim_{R \to +\infty} \int_{-R}^{R} f(x) dx = \lim_{R \to +\infty} \left[\int_{\Gamma_R} f(z)\, dz - \int_{\gamma_R} f(z)\, dz \right],$$

where $\gamma_R : [0, \pi] \to \mathbb{C}$, is the curve $\gamma_R(t) = Re^{it}$, while Γ_R is the concatenation of the segment $[-R, R]$ with the curve γ_R. Notice that Γ_R is a simple closed curve. By Cauchy's formula, for all $R > 3$ we have:

$$\int_{\Gamma_R} f(z)\, dz = 2\pi i (1/48 + 1/80 - 1/30) = 0$$

Notice that the function $|f(z)|$ decreases as $|z|^6$ when $|z| \to +\infty$. Therefore, $|z||f(z)|$ also approaches zero as $|z| \to +\infty$. As a consequence, $\lim_{R \to +\infty} \int_{\gamma_R} f(z)\, dz = 0$. Combining the previous considerations we see:

$$\int_{-\infty}^{+\infty} f(x) dx = 0.$$

d) Notice that $f(x) = -f(-x)$. Hence, since the integral converges absolutely, we have:

$$\int_{-\infty}^{+\infty} f(x) dx = \int_{-\infty}^{0} f(x) dx + \int_{0}^{+\infty} f(x) dx = -\int_{0}^{+\infty} f(x) dx + \int_{0}^{+\infty} f(x) dx = 0.$$

2. Exercise

a) Let $X(s)$ denote $\mathscr{L}[x]$. Then $\mathscr{L}[\dot{x}] = sX(s) - 1$ and $\mathscr{L}[\ddot{x}] = s^2X(s) - s$. Finally $\mathscr{L}[x^{(3)}] = s^3X(s) - s^2$. Substituting in the differential equations we see that:

$$s^3X(s) - s^2 = -s^2X(s) + s - sX(s) + 1 - X(s).$$

Solving with respect to $X(s)$ yields:

$$X(s) = \frac{s^2 + s + 1}{s^3 + s^2 + s + 1}$$

Notice that: $(s^3+s^2+s+1) = (s+1)(s^2+1)$, therefore:

$$X(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

with:

$$A = \lim_{s\to -1} \frac{s^2+s+1}{s^2+1} = 1/2$$

and $B = C = 1/2$. Taking inverse Laplace's transforms yields:

$$x(t) = \frac{1}{2}e^{-t} + \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t).$$