

The Answers

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1. (a) {unseen modification of standard result}

The q -formulation of the difference equation is $qx_k = Ax_k + bq^2u_k$.

By taking Z -transforms we obtain

$z(x^Z(z) - x_o) = Ax^Z(z) + bz^2(u^Z(z) - u_0 - u_1z^{-1})$. Hence

$(zI - A)x^Z(z) = bz^2u^Z(z) - bz^2u_0 - bzu_1 + zx_0$, so

$x^Z(z) = (zI - A)^{-1}bz^2u^Z(z) - (zI - A)^{-1}bz^2u_0 - (zI - A)^{-1}bzu_1 + (zI - A)^{-1}zx_0$ and

$$y^Z(z) = z^2c'(zI - A)^{-1}b + zc'(zI - A)^{-1}x_o - c'(zI - A)^{-1}b\{z^2u_0 + zu_1\}, \quad [3]$$

(b) {unseen}

$$(i) \quad y^Z(z) = c'(zI - A)^{-1}bu^Z(z) + zc'(zI - A)^{-1}x_o. \quad [1]$$

$$(ii) \quad A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & \alpha \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & \alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 2\alpha \\ 0 & 0 & \alpha^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 2\alpha \\ 0 & 0 & \alpha^2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & \alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2\alpha \\ 0 & 0 & 2\alpha^2 \\ 0 & 0 & \alpha^3 \end{bmatrix} = \alpha \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 2\alpha \\ 0 & 0 & \alpha^2 \end{bmatrix} = \alpha A^2$$

$$A^4 = A^3A = (\alpha A^2)A = \alpha A^3 = \alpha^2 A^2$$

....

$$A^k = \alpha^{k-2}A^2$$

$$\text{so } p(\alpha) = \alpha^{k-2}. \quad [5]$$

(iii) Since all the u_k are zero, $y_k = c'A^kx_o$ so

$$y^Z(z) = z^0c'x_o + z^{-1}c'Ax_o + z^{-2}c'A^2x_o + z^{-3}c'A^3x_o + z^{-4}c'A^4x_o + \dots$$

$$= z^0c'x_o + z^{-1}c'Ax_o + z^{-2}c'A^2x_o + \alpha z^{-3}c'A^2x_o + \alpha^2 z^{-4}c'A^2x_o + \dots$$

$$= z^0c'x_o + z^{-1}c'Ax_o + z^{-2}c'A^2x_o(1 + \alpha z^{-1} + \alpha^2 z^{-2} + \dots)$$

$$= z^0c'x_o + z^{-1}c'Ax_o + \frac{z^{-2}}{(1 - \alpha z^{-1})} c'A^2x_o.$$

$$\text{Hence } H^Z(z) = z^0c' + z^{-1}c'A + \frac{z^{-2}}{(1 - \alpha z^{-1})} c'A^2. \quad [5]$$

(iv) From part (b-i): the pulse Z -transfer function is $G^Z(z) = c'(zI - A)^{-1}b$

and the contribution to $y^Z(z)$ from x_o is $zc'(zI - A)^{-1}x_o$ so $H^Z(z)$ of part (b-ii)

equals $zc'(zI - A)^{-1}$. Therefore we obtain $G^Z(z) = z^{-1}H^Z(z)b$. [3]

(c) {small modification of a standard result}

The Variation of Constants formula gives the difference equation

$$\begin{aligned} x(t_{k+1}) &= e^{A(t_{k+1}-t_k)}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}bu(\tau)d\tau \\ &= e^{AT}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}b(\tau-t_k)u_kd\tau \\ &= \bar{A}x(t_k) + \bar{b}u_k \end{aligned}$$

where $\bar{A} = e^{AT}$ and (after making the change of variable $\theta = \tau - t_k$)

$\bar{b} = \int_0^T e^{A(T-\theta)}\theta d\theta b$. Hence the required difference equation is

$$x_{k+1} = \bar{A}x_k + \bar{b}u_k \text{ where } x_k = x(t_k) \text{ for all } k. \quad [3]$$

2. (a) $\{bookwork\}$

Now $G^D(\gamma) \triangleq TG^Z(1+\gamma T)$ so the poles p_i^D of $G^D(\gamma)$ are the values of γ for which $G^Z(1+\gamma T)$ is infinite. Hence $1+p_i^D T = p_i^Z$ for all i , i.e. $p_i^D = \frac{1}{T}(p_i^Z - 1)$ for all i . [3]

The system is BIBO-stable iff $|p_i^Z| < 1$, i.e. iff $p_i^D \in \{\frac{1}{T}(z-1) : |z| < 1\} = \{z - \frac{1}{T} : |z| < \frac{1}{T}\}$. Hence for BIBO-stability the poles of $G^D(\gamma)$ should all belong to the hatched area shown in Figure A2.1 below. [2]

(b) $\{unseen\}$

$$\begin{aligned} \text{(i) Now } x^Z(z) &= (zI - \{I + AT\})^{-1} \{Tbu^Z(z) + zx_o\} \text{ so} \\ x^D(\gamma) &= Tx^Z(1+\gamma T) = T(I + \gamma TI - I - AT)^{-1} \{Tbu^Z(1+\gamma T) + (1+\gamma T)x_o\} \\ &= T(T(\gamma I - A))^{-1} \{Tbu^Z(1+\gamma T) + (1+\gamma T)x_o\} \\ &= (\gamma I - A)^{-1} \{bTu^Z(1+\gamma T) + (1+\gamma T)x_o\} \\ &= (\gamma I - A)^{-1} \{bu^D(\gamma) + (1+\gamma T)x_o\} \\ &= (\gamma I - A)^{-1} bu^D(\gamma) + (1+\gamma T)(\gamma I - A)^{-1} x_o. \end{aligned}$$

Hence $y^D(\gamma) = c'(\gamma I - A)^{-1} bu^D(\gamma) + (1+\gamma T)c'(\gamma I - A)^{-1} x_o$.

Therefore $G^D(\gamma) = c'(\gamma I - A)^{-1} b = c'(sI - A)^{-1} b|_{s=\gamma} = G^L(\gamma)$, which is the connection required between $G^D(\gamma)$ and $G^L(s)$. [5]

(ii) The system is BIBO-stable if each eigenvalue of $I + AT$ has modulus smaller than 1,

Now $I + AT = VV^{-1} + V\Lambda V^{-1}T = V(I + \Lambda T)V^{-1}$ so the eigenvalues of $I + AT$ are $1 + \lambda_i T$ and $|1 + \lambda_i T| = \sqrt{(1 + \Re(\lambda_i)T)^2 + \Im(\Lambda T)^2}$ which is not necessarily smaller than one even if $\Re(\lambda_i) < 0$. Hence the discrete-time approximation is not necessarily BIBO-stable even if the original continuous-time system is (continuous-time) BIBO-stable. [4]

(iii) The standard exact relationship is that $x_{k+1} = \bar{A}x_k + \bar{B}u_k$ where $\bar{A} = e^{AT}$ and

$$\bar{B} = \int_0^T e^{A\tau} d\tau B.$$

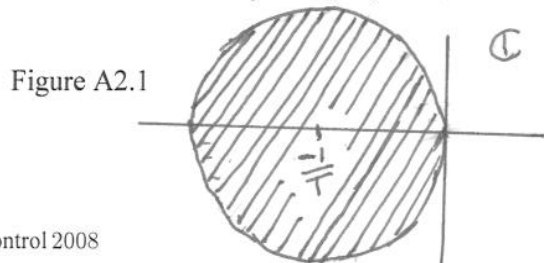
We can write e^{AT} as $e^{AT/2}e^{AT/2}$ i.e. as $e^{AT/2}(e^{-AT/2})^{-1}$. Approximating $e^{AT/2}$ by $(I + AT/2)$ and $e^{-AT/2}$ by $(I - AT/2)$ gives the approximation $(I + AT/2)(I - AT/2)^{-1}$ to \bar{A} .

Hence (2.3) is dt BIBO-stable iff each eigenvalue of $(I + AT/2)(I - AT/2)^{-1}$ has modulus smaller than one. Now

$$\begin{aligned} (I + AT/2)(I - AT/2)^{-1} &= (VV^{-1} + V\Lambda V^{-1}T/2)(VV^{-1} - V\Lambda V^{-1}T/2)^{-1} \\ &= V(I + \Lambda T/2)V^{-1}[V(I - \Lambda T/2)V^{-1}]^{-1} = V(I + \Lambda T/2)(I - \Lambda T/2)^{-1}V^{-1} \\ &= V \text{diag}\left\{\dots \frac{1 + \lambda_i T/2}{1 - \lambda_i T/2} \dots\right\} V^{-1}. \end{aligned}$$

Consequently system (2.3) is BIBO-stable iff for all i $|\frac{1 + \lambda_i T/2}{1 - \lambda_i T/2}| < 1$. Now $|\frac{1 + \lambda_i T/2}{1 - \lambda_i T/2}| = \frac{|1 + \lambda_i T/2|}{|1 - \lambda_i T/2|} = \frac{\sqrt{(1 + \Re(\lambda_i)T/2)^2 + (\Im(\lambda_i)T/2)^2}}{\sqrt{(1 - \Re(\lambda_i)T/2)^2 + (\Im(\lambda_i)T/2)^2}} < 1$ if $\Re(\lambda_i) < 0$. Hence

system (2.3) is discrete-time BIBO-stable if system (2.1) is continuous-time BIBO-stable. [6]



3. (a) {modified bookwork}

$$\begin{aligned}
 \text{Now } u_k &= \sum_{m=0}^2 h_m u_{k-m} = \sum_{m=0}^2 h_m \cos(\omega(k-m)T) = \sum_{m=0}^2 h_m \frac{e^{j\omega(k-m)T} + e^{-j\omega(k-m)T}}{2} \\
 &= \sum_{m=0}^2 h_m \frac{e^{j\omega(k-m)T}}{2} + \sum_{m=0}^2 h_m \frac{e^{-j\omega(k-m)T}}{2} \\
 &= e^{j\omega kT} \sum_{m=0}^2 h_m \frac{e^{-j\omega mT}}{2} + e^{-j\omega kT} \sum_{m=0}^2 h_m \frac{e^{j\omega mT}}{2} \\
 &= e^{j\omega kT} \sum_{m=0}^{\infty} h_m \frac{e^{-j\omega mT}}{2} + e^{-j\omega kT} \sum_{m=0}^{\infty} h_m \frac{e^{j\omega mT}}{2} \text{ for } k \geq 3 \\
 &\quad (\text{since we have been told that } h_k = 0 \text{ for } k > 2) \\
 &= \frac{1}{2} [e^{j\omega kT} h^F(\omega T) + e^{-j\omega kT} h^F(-\omega T)] \quad [5] \\
 &= \frac{1}{2} [e^{j\omega kT} h^F(\omega T) + e^{-j\omega kT} h^F(\omega T)^*] \text{ (since the } h_k \text{ are real)} \\
 &= \frac{1}{2} [e^{j\omega kT} |h^F(\omega T)| e^{j\angle h^F(\omega T)} + e^{-j\omega kT} |h^F(\omega T)| e^{-j\angle h^F(\omega T)}] \\
 &= \frac{1}{2} |h^F(\omega T)| [e^{j\{\omega kT + \angle h^F(\omega T)\}} + e^{-j\{\omega kT + \angle h^F(\omega T)\}}] \\
 &= |h^F(\omega T)| \cos(\omega kT + \angle h^F(\omega T)) \quad (A3.1)
 \end{aligned}$$

as required. [4]

(b) {new example}

$$\text{Now } C^Z(z) = \frac{(z-j)(z+j)}{z^2} = \frac{z^2+1}{z^2} = 1 + 0.z + 1.z^{-2} \text{ so } h_0 = 1, h_1 = 0, h_2 = 1. \quad [2]$$

The pole-zero configuration is shown in Figure A3.1 below and, since $T = 1$,

$$C^F(\omega T) = C^Z(e^{j\omega T}) = 1 + e^{-2j\omega T} \text{ so } C^F(\omega T)|_{\omega=0} = 2 \text{ and } C^F(\omega T)|_{\omega=\pi/2} = 1 - 1 = 0.$$

This indicates that the frequency response ($|C^F(\omega T)| - v - \omega$) is that shown in Figure A3.2.

Consequently (A3.1) indicates that $u_k = 0$ for all $k \geq 3$. [3]

In detail, $u^Z(z) = C^Z(z)e^Z(z)$ where $C^Z(z) = \frac{(z-j)(z+j)}{z^2} = \frac{z^2+1}{z^2}$ and

$$e^Z(z) = \frac{z(z - \cos(\omega T))}{z^2 - 2z\cos(\omega T) + 1} = \frac{z^2}{z^2 + 1}.$$

Hence $u^Z(z) = \frac{z^2+1}{z^2} \frac{z^2}{z^2+1} = 1$ and consequently $\{u_k\} = \{1, 0, 0, 0, \dots\}$,

which is consistent with the result predicted from the pole-zero pattern above. [3]

(c) {new example} Here

$$u^Z(z) = \frac{z^2+1}{z^2+z-2} e^Z(z) = \frac{1+z^{-2}}{1+z^{-1}-2z^{-2}} e^Z(z) = (1+z^{-2})w^Z(z) \quad (A3.2)$$

where $w^Z(z) = \frac{e^Z(z)}{1+z^{-1}-2z^{-2}}$. Hence $(1+z^{-1}-2z^{-2})w^Z(z) = e^Z(z)$ so we have

$$w^Z(z) = e^Z(z) - z^{-1}w^Z(z) + 2z^{-2}w^Z(z). \quad (A3.3)$$

Taking the inverse Z -transforms of (A3.2) and (A3.3), we obtain the canonical direct realization

$$u_k = w_k + w_{k-2}; w_k = e_k - w_{k-1} + 2w_{k-2}.$$

Figure A3.1

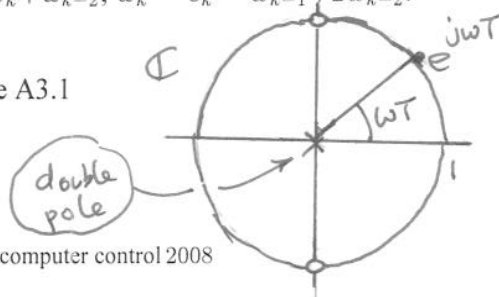
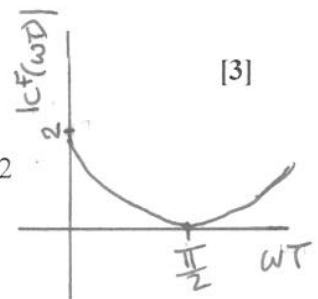


Figure A3.2



4. {all parts unseen}

- (a) The pulse Z -transfer function for the minor loop in the forward path is $\frac{K}{1+0.75\frac{K}{z-1}} = \frac{K(z-1)}{z-1+0.75K}$.

$$\begin{aligned} \text{The value of } H^Z(z) \text{ is } c'(zI-A)^{-1}b &= [0.25 \quad 0.25] \begin{bmatrix} z-2.5 & 0.5 \\ -4.5 & z+0.5 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \\ &= [0.25 \quad 0.25] \begin{bmatrix} z+0.5 & -0.5 \\ 4.5 & z-2.5 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} / \{(z+0.5)(z-2.5) + 2.25\} \\ &= [0.25 \quad 0.25] \begin{bmatrix} z+0.5 & -0.5 \\ 4.5 & z-2.5 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} / \{z^2 - 2z + 1\} \\ &= [0.25 \quad 0.25] \begin{bmatrix} 0.5z \\ 2.25 + 0.5z - 1.25 \end{bmatrix} / \{(z-1)^2\} = \frac{0.25(z+1)}{(z-1)^2}. \end{aligned}$$

$$\text{The complete forward path is } \frac{K(z-1)}{z-1+0.75K} \times \frac{0.25(z+1)}{(z-1)^2} = \frac{0.25K(z+1)}{(z-1+0.75K)(z-1)}.$$

$$\text{Hence the closed-loop transfer function is } \frac{\frac{0.25K(z+1)}{(z-1+0.75K)(z-1)}}{1 + \frac{0.25K(z+1)}{(z-1+0.75K)(z-1)}}$$

$$= \frac{0.25K(z+1)}{(z-1+0.75K)(z-1) + 0.25K(z+1)} = \frac{0.25K(z+1)}{(z-1)^2 + K(z-0.5)}. \quad [4]$$

- (b) Let the closed-loop denominator be denoted $d(z) = (z-1)^2 + K(z-0.5)$
 $= z^2 + z(K-2) + 1 - 0.5K$.

Then $d(1) = 0.5K > 0$ iff $K > 0$ and $d(-1) = 4 - 1.5K > 0$ iff $K < \frac{8}{3}$. The Jury array is:

$$\begin{bmatrix} 1 - 0.5K & K-2 & 1 \end{bmatrix}$$

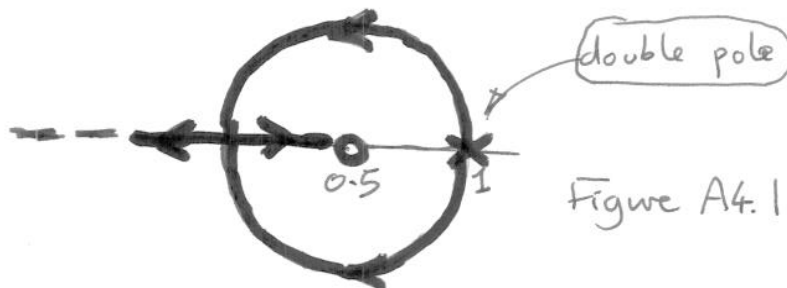
and the corresponding stability condition is $|1 - 0.5K| < 1$, i.e. $K \in (0, 4)$.

$$\text{Hence } K_{max} = \frac{8}{3}. \quad [5]$$

- (c) Since the closed-loop denominator is $d(z) = (z-1)^2 + K(z-0.5)$ the corresponding root-locus is that for $G^Z(z) = \frac{z-0.5}{(z-1)^2}$. The break points σ_b for this root locus are amongst the solutions of $\frac{1}{\sigma_b-0.5} = \frac{1}{\sigma_b-1} + \frac{1}{\sigma_b-1}$ i.e. $\sigma_b-1 = 2(\sigma_b-0.5)$. Hence $\sigma_b = 0$. Therefore the root-locus is that shown in Figure A4.1 below. The root-locus shows there is a gain that locates both closed-loop poles at the origin and the gain required is $K = \frac{-1}{G^Z(0)} = 2$. [5]

- (d) The Initial Value Theorem gives $e_0 = \lim_{|z| \rightarrow \infty} z e^Z(z) = \lim_{|z| \rightarrow \infty} \frac{(z+0.5)}{z(z-1)} = 0$. The inversion integral involved gives $e_k = \frac{1}{2\pi j} \oint_{\Gamma_1} e^Z(z) z^{k-1} dz$ where Γ_1 is the circle in the complex plane that is centred on the origin and has radius 1. For $k \geq 2$, $\frac{1}{2\pi j} \oint_{\Gamma_1} e^Z(z) z^{k-1} dz = \frac{1}{2\pi j} \oint_{\Gamma_1} \frac{(z+0.5)}{z(z-1)} z^{k-1} dz$
 $= \frac{1}{2\pi j} \oint_{\Gamma_1} \frac{(z+0.5)}{(z-1)} z^{k-2} dz = \text{residue of } \frac{(z+0.5)}{(z-1)} z^{k-2} @ 1 = (z-1) \frac{(z+0.5)z^{k-2}}{(z-1)} \Big|_{z=1} = 1.5$. [3]

- (e) The cancellation of the forward path pole at $z = 1$ leaves the the A -matrix of the controlled system with 1 as an eigenvalue and this, in the presence of any noise and errors in parameter values, might well cause at least one state to diverge to ∞ . [3]



5. (a) {bookwork}

Let p' be the last row of the controllability matrix M and let $V = \begin{bmatrix} p' \\ p'A \\ \dots \\ p'A^{n-1} \end{bmatrix}$.

Then $VV^{-1} = I$ so $\begin{bmatrix} p' \\ p'A \\ \dots \\ p'A^{n-1} \end{bmatrix} V^{-1} = I$ and consequently $\begin{bmatrix} p'V^{-1} \\ p'AV^{-1} \\ \dots \\ p'A^{n-1}V^{-1} \end{bmatrix} = I$.

Therefore

$$\begin{aligned} p'V^{-1} &= [1 \ 0 \ 0 \ \dots \ 0] \\ p'AV^{-1} &= [0 \ 1 \ 0 \ \dots \ 0] \\ &\dots \\ p'A^{n-1}V^{-1} &= [0 \ 0 \ 0 \ \dots \ 1]. \end{aligned}$$

Hence

$$VAV^{-1} = \begin{bmatrix} p' \\ p'A \\ \dots \\ p'A^{n-2} \\ p'A^{n-1} \end{bmatrix} (AV^{-1}) = \begin{bmatrix} p'AV^{-1} \\ p'A^2V^{-1} \\ \dots \\ p'A^{n-1}V^{-1} \\ p'A^nV^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ * & * & * & * & * \end{bmatrix}$$

which is a companion matrix C . Consequently $A = V^{-1}CV$. [8]

(i) {unseen case}

The root-locus is that of Figure A5.1 so there is no gain K that locates the closed-loop poles (i.e. eigenvalues) at 0.2. [2]

Figure A5.1



(ii) {Application of standard method to unseen case}

$$Ab = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \text{ so } M = [b \ Ab] = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}.$$

$|M| = 4 \neq 0$ so the system is reachable.

$$M^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{bmatrix}. \text{ Therefore } p' = \text{last row of } M^{-1} = [0.5 \ 1].$$

$$p'A = [0.5 \ 1] \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = [0.5 \ 0.5].$$

$$V = \begin{bmatrix} p' \\ p'A \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0.5 \end{bmatrix} \text{ so } V^{-1} = \frac{1}{(-0.25)} \begin{bmatrix} 0.5 & -1 \\ -0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 2 & -2 \end{bmatrix}.$$

The corresponding companion matrix is

$$C_\alpha = VAV^{-1} = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where $\alpha' = \text{last row of } C_\alpha = [1 \ 0]$.

The desired characteristic polynomial $= (\lambda - 0.2)(\lambda - 0.2) = \lambda^2 - 0.4\lambda + 0.04$
 $= \lambda^2 - \phi_2\lambda - \phi_1$ so the coefficient vector $\phi' = [-0.04 \ 0.4]$.

The required feedback vector f is therefore

$$f = V'(\alpha - \phi) = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.5 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -0.04 \\ 0.4 \end{bmatrix} \right) = \begin{bmatrix} 0.32 \\ 0.84 \end{bmatrix}. \quad [10]$$

6. (a) *{bookwork}*

A and M are similar implies that $A = PMP^{-1}$ for a suitable P . Then

$$\begin{aligned} |\lambda I - A| &= |\lambda I - PMP^{-1}| = |\lambda PP^{-1} - PMP^{-1}| = |P(\lambda I - M)P^{-1}| = |P||\lambda I - M||P^{-1}| \\ &= |\lambda I - M|. \end{aligned}$$

Therefore the eigenvalues of A (which are the values of λ such that $|\lambda I - A| = 0$) are the eigenvalues of M . [3]

(b) (i) *{a way of analysing the situation that is different from that used in the lectures}*

$$\begin{aligned} \text{Now } \begin{bmatrix} x_{k+1} \\ \hat{x}_{k+1} \end{bmatrix} &= \begin{bmatrix} Ax_k + b(r_k - f'\hat{x}_k) \\ (A - lc')\hat{x}_k + ly_k + b(r_k - f'\hat{x}_k) \end{bmatrix} = \begin{bmatrix} Ax_k - bf'\hat{x}_k \\ (A^{ob}\hat{x}_k + lc'x_k - bf'\hat{x}_k) \end{bmatrix} + \begin{bmatrix} br_k \\ br_k \end{bmatrix} \\ &= \begin{bmatrix} A & -bf' \\ lc' & A^{ob} - bf' \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} + \begin{bmatrix} b \\ b \end{bmatrix} r_k = \bar{A} \bar{x}_k + \bar{b} r_k \text{ where } \bar{A} = \begin{bmatrix} A & -bf' \\ lc' & A^{ob} - bf' \end{bmatrix}, \bar{b} = \begin{bmatrix} b \\ b \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} \text{Further } P\bar{A}P^{-1} &= \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A & -bf' \\ lc' & A^{ob} - bf' \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A - bf' & bf' \\ lc' + A^{ob} - bf' & -A^{ob} + bf' \end{bmatrix} = \begin{bmatrix} A^f & bf' \\ A^f - lc' - A^{ob} + bf' & A^{ob} \end{bmatrix} = \begin{bmatrix} A^f & bf' \\ 0 & A^{ob} \end{bmatrix} \end{aligned}$$

so, by a property of block upper-triangular matrices, the eigenvalues of \bar{A} are the eigenvalues of $A^f = A - bf'$ together with the eigenvalues of $A^{ob} = A - lc'$. The eigenvalues of \bar{A} determine overall performance and the above result reveals that f and l can be designed separately without interaction. This makes such design relatively simple. [9]

(ii) *{extension of bookwork}* If f and l are chosen so the eigenvalues of $A - bf$ and $A - lc'$ are all zero then it follows from above that all the eigenvalues of \bar{A} are zero.

\bar{A} is similar to a companion matrix for which the bottom row contains only zeros

(since all its eigenvalues are zero).. Therefore $\bar{A} = \bar{P} \bar{C} \bar{P}^{-1}$ for an appropriate P

$$\begin{aligned} \text{where } \bar{C} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \text{ Since } r_k \equiv 0, \text{ it follows that } \bar{x}_k = \bar{A}^k \bar{x}_o \\ &= (\bar{P} \bar{C} \bar{P}^{-1}) \dots (\bar{P} \bar{C} \bar{P}^{-1}) \bar{x}_o = \bar{P} \bar{C}^k \bar{P}^{-1} \bar{x}_o. \text{ Note that} \\ \bar{C}^2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \bar{C}^3 = \bar{C}^2 \bar{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \bar{C}^4 = \bar{C}^3 \bar{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and} \\ \bar{C}^5 &= \bar{C}^4 \bar{C} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \text{ Consequently } \bar{C}^k = 0 \text{ for all} \end{aligned}$$

$k \geq 5$. Therefore $\bar{x}_k = 0$, for all $k \geq 5$. Hence both x_k and \hat{x}_k become zero by time t_5 , whatever the initial conditions of the plant and observer, which seems attractive.

A downside is that large control actions might be needed. [8]