

1. a) i) $P(S_1) = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}$ EJ.20

(ii) $S_1 \cup S_2 = \{a,b,c\}$

(iii) $S_1 \cap S_2 = \{a,b\}$

(iv) $S_2 - S_1 = \{ \emptyset \}$

(v) $|S_1 \cup S_2| = 3$ [6 MARKS]

b) Finite: S_1 from (a)

Infinite - Countable: \mathbb{N}

Infinite - Uncountable: \mathbb{R} [3 MARKS]

c) (i) No, e.g. $(2,1) \in R$ but $(1,2) \notin R$.

(ii) No, e.g. $(2,1) \in R$ and $(4,2) \in R$ but $(4,1) \notin R$.

(iii) No, e.g. $(1,1) \notin R$.

(iv) No, e.g. there is no element $(1,x)$ for any x .

(v) No - (ii) implies this. [6 MARKS]

d) (i) $\forall x (J(x) \rightarrow L(x))$

(ii) $\exists x (L(x) \wedge \neg J(x))$

(iii) $\forall x (J(x) \rightarrow A(x, \text{Jones}))$

(iv) $\forall x \forall y (A(x,y) \wedge J(x) \rightarrow J(y))$ [5 MARKS]

e) Simplify (i): b

Modus ~~Tollens~~ ^{Tollens} w/ (ii): $\neg C$

[2 MARKS]

1. f) (i) From the theorem, $f(x)$ is $O(x^3)$.

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$$|f(x)| = |x^3 + 2x^2 + 1| \geq |x^3| + |2x^2| + 1 \quad \text{for } x > 0$$

$$\geq |x^3|$$

So with any κ , e.g. $\kappa = 1$ Δ with $c = 1$,
 $\forall x ((x > \kappa) \rightarrow |f(x)| > c|x^3|) \quad \square$

(ii) `proc1 (int x) {`
 for $i = 1$ to $x * x * x + 2 * x * x + 1 - 4$
 $avar = avar * 2$;
`}`

(iii) `proc2 (int x) {`
 if $x = 1$
 return $2 * x * x$;
 else
 return $proc2(x-1) * proc2(x-1)$; [9 MARKS]
`}`

g) let $a > 1$ be a real number, $b > 1$ be an integer,
 $c > 0$ be a real number and $d > 0$ be a real number

let f be an increasing function s.t.
 $f(n) = a f(n/b) + cn^d$ whenever $n = b^k$
 for positive integer k .

- (i) If $a < b^d$, $f(n)$ is $O(n^d)$
- (ii) If $a = b^d$, $f(n)$ is $O(n^d \log n)$
- (iii) If $a > b^d$, $f(n)$ is $O(n^{\log_b a})$.

[9 MARKS]

2. a) $\text{ratapprox}(0, \sqrt{2}) = (1, 1)$

$\text{ratapprox}(1, \sqrt{2}) = (3, 2)$

$\text{ratapprox}(2, \sqrt{2}) = (7, 5)$ [6 MARKS]

b) $a_n = 1 + a_{n-1}, \quad n \geq 1$

$a_n = n$

This is $O(n)$, as are all other factors, so
exec time is $O(n)$. [6 MARKS]

c) $\forall x (\neg R(x) \rightarrow \forall y (R(y) \rightarrow (y < x) \vee (y > x)))$

This is true. [6 MARKS]

d) $r_d = \{p \mid p \in \mathbb{Q} \wedge p <_Q r\}$

(i) $[r_d \subset \mathbb{Q}]$

$r_d \subseteq \mathbb{Q}$ from defn

$r_d \neq \mathbb{Q}$ as, for example, $r_d(r+1) \in \mathbb{Q}$ but
 $r+1 \not<_Q r \Rightarrow (r+1) \notin r_d$.

(ii) $r_d \neq \emptyset$ as, for example, $(r-1) \in \mathbb{Q}$ ~~but~~

Also $(r-1) <_Q r \Rightarrow (r-1) \in r_d$.

(iii) $q \in S \Rightarrow q \in \mathbb{Q} \wedge q <_Q r$

$p <_Q q \Rightarrow p <_Q q <_Q r$

$\Rightarrow p <_Q r$

Also $p \in \mathbb{Q}$. Thus $p \in S$.

Q

2. (d) (iv) Take $q = \frac{1}{2}(p+r)$.

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Then $q \in Q$ and $p <_Q q <_Q r$

$$\Rightarrow p <_Q q.$$

[6 MARKS]

e) (i) For any set x , $x \subseteq x \Rightarrow x \leq_d x$.

$$(ii) (x \leq_d y) \wedge (y \leq_d x)$$

$$\Rightarrow (x \subseteq y) \wedge (y \subseteq x)$$

$$\equiv x = y \quad (\text{by definition of set equality})$$

$$(iii) (x \leq_d y) \wedge (y \leq_d z)$$

$$\Rightarrow (x \subseteq y) \wedge (y \subseteq z)$$

$$\Rightarrow x \subseteq z$$

$$\Rightarrow x \leq_d z.$$

(iv) We wish to show $(x \leq_d y) \vee (y \leq_d x)$

i.e. $(x \subseteq y) \vee (y \subseteq x)$.

If $x = y$, this is true.

Otherwise $\exists q \in Q$ s.t. $q \in x$ but $q \notin y$ (*)
or $q \in y$ but $q \notin x$.

Take (*), w.l.o.g.

Then $\forall p \in y (p <_Q q)$ (from behind defn).

$$\Rightarrow \forall p \in y (p \in x) \quad \text{i.e.} \quad y \subseteq x.$$

[6 MARKS]

2. ~~f)~~ ~~g)~~ $T = \{p \mid p \in \mathbb{Q} \wedge (p^2 \leq_{\mathbb{Q}} 2)\} \vee (p < 0)\}.$

(i) $T \subseteq \mathbb{Q}$ by defn.

$T \neq \mathbb{Q}$ as, for example, $2 \in \mathbb{Q}$ but $2^2 \not\leq_{\mathbb{Q}} 2$
 $\Rightarrow 2 \notin T.$

(ii) ~~g)~~ $T \neq \emptyset$ as, for example, $1 \in T$ since $1 \in \mathbb{Q}$
 $\& 1^2 = 1 \leq_{\mathbb{Q}} 2.$

(iii) $q \in T \Rightarrow q \in \mathbb{Q}$
 and $q^2 \leq_{\mathbb{Q}} 2$ or $q <_{\mathbb{Q}} 0.$

Let $q^2 <_{\mathbb{Q}} 0.$ Then $p <_{\mathbb{Q}} q$

$\Rightarrow p <_{\mathbb{Q}} q <_{\mathbb{Q}} 0 \& p <_{\mathbb{Q}} 0$

$\& p \in T.$

Let $0 \leq_{\mathbb{Q}} q^2 <_{\mathbb{Q}} 2.$ Then $p <_{\mathbb{Q}} q$

~~$p <_{\mathbb{Q}} 0$~~ $\Rightarrow p \in T$, or $p = 0 \Rightarrow p \in T$

or $0 <_{\mathbb{Q}} p.$

Now $p^2 <_{\mathbb{Q}} pq$ (since $0 <_{\mathbb{Q}} p$)

$<_{\mathbb{Q}} q^2$ (since $p <_{\mathbb{Q}} q$)

$<_{\mathbb{Q}} 2$ (since $q^2 <_{\mathbb{Q}} 2$).

$\Rightarrow p \in T.$

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2. (†) (iv) Choose $q = p + \frac{1}{n}$ with n as defined in "hint".
Then q is rational, since p is rational.

Now $q^2 = p^2 + \frac{1}{n} (2p + \frac{1}{n}) <_Q p^2 + \frac{1}{n} (2p + 1)$
as $n >_Q 1$.

$$n >_Q \frac{2p+1}{2-p^2} \Rightarrow \frac{1}{n} <_Q \frac{2-p^2}{2p+1}$$

$$\Rightarrow q^2 <_Q p^2 + 2 - p^2 \\ = 2 \quad (*)$$

Also $n >_Q 1 \Rightarrow q = p + \frac{1}{n} >_Q p \quad (+)$

So (i) $q \in S$ for $(*)$

(ii) $p <_Q q$ for $(+)$ \square

As a result, q

[10 MARKS]

3. a) (i) Let $x \in B = \mathbb{R}$.

Then $2^x \in \mathbb{R}$ and $2^x > 0 \Rightarrow 2^x \in \mathbb{R}_+$.

So $f(2^x, 0) = x \quad \square$. [3 MARKS]

(ii) Choose $A = \mathbb{R}_+ \times \{0\}$

Proof above holds for surjectivity.

For injectivity,

$$f(x_1, y_1) = f(x_2, y_2)$$

But $y_1 = y_2 = 0$.

$$\text{Also } \log_2(x_1 + 0) = \log_2(x_2 + 0)$$

$$\Rightarrow x_1 = x_2 \quad \square. \quad [6 \text{ MARKS}]$$

(iii) It is possible to obtain any ~~positive~~ non-negative value $v \in \mathbb{R}_+ \cup \{0\}$ from

$$f(2^v, 0) = v \quad \& \quad 2^v \geq 1$$

~~Also zero is possible iff~~

Negative values are not possible - we would require $f(x, y) < 0$

$$\Rightarrow \log_2(x + y) < 0$$

$$\text{So either } \log_2 x < 0 \Rightarrow x < 1 \quad x$$

$$\text{or } \log_2(x+1) < 0 \Rightarrow x+1 < 1$$

i.e. $x < 0 \quad x$.

So image is $\mathbb{R}_+ \cup \{0\}$. [6 MARKS]

3. (b)(i) $R^n \subseteq R \Rightarrow R$ is transitive :

$R^n \subseteq R \Rightarrow R^2 \subseteq R$. $(a,b) \in R$ and $(b,c) \in R$
 then $(a,c) \in R^2$. But $R^2 \subseteq R \Rightarrow (a,c) \in R$.
 So R is transitive.

R is transitive $\Rightarrow R^n \subseteq R$

True for $n=1$. Use induction to show $R^{n+1} \subseteq R$
 assuming $R^n \subseteq R$.

Consider $(a,b) \in R^{n+1} = R \cdot R^n$

$\Rightarrow \exists x ((a,x) \in R \wedge (x,b) \in R^n)$

$R^n \subseteq R \Rightarrow (x,b) \in R$. R is transitive

$\Rightarrow (a,b) \in R$. So $R^{n+1} \subseteq R$. [6 MARKS]

(ii) $\forall a \in A \exists b \in B ((a,b) \in R)$

$\wedge \forall a \in A \forall b \in B \forall c \in B ((a,b) \in R \wedge (a,c) \in R$
 $\Rightarrow b=c)$. [6 MARKS]

(iii) Trivially, $f: \{0\} \rightarrow \{0\}$
 $f(0)=0$ is transitive.

[3 MARKS]

4. a) (i) $T(\text{Steven}) \wedge I(\text{Steven})$

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(ii) $G(\text{Steven}) \wedge \forall x (I(x) \rightarrow G(x))$

(iii) $\exists x (T(x) \wedge I(x))$

$\wedge \forall x \forall y (T(x) \wedge I(x) \wedge T(y) \wedge I(y) \rightarrow x = y)$

(iv) $\forall x (T(x) \wedge I(x) \rightarrow x = \text{Steven})$

(v) $T(\text{Amanda}) \wedge \neg T(\text{James})$

(vi) $\neg I(\text{Amanda})$

[12 MARKS]

b) Simplify (iii) $\Rightarrow \forall x \forall y (T(x) \wedge I(x) \wedge T(y) \wedge I(y) \rightarrow x = y)$

Universal Instantiation

$\forall x (T(x) \wedge I(x) \wedge T(\text{Steven}) \wedge I(\text{Steven}) \rightarrow x = \text{Steven})$

Hypothesis $\Rightarrow \forall x (T(x) \wedge I(x) \rightarrow x = \text{Steven})$

[9 MARKS]

c) Universal Instantiation on (iv)

$T(\text{Amanda}) \wedge I(\text{Amanda}) \rightarrow \text{Amanda} = \text{Steven}$

Modus Ponens

$\neg (T(\text{Amanda}) \wedge I(\text{Amanda}))$

$\equiv \neg T(\text{Amanda}) \vee \neg I(\text{Amanda}) \quad (*)$

Simplify (v) $T(\text{Amanda}) \quad (+)$

Disjunctive Syllogism $(*) \wedge (+)$

$\Rightarrow \neg I(\text{Amanda})$

[9 MARKS]

5. a) $f(x)$ is $O(g(x)) \equiv \exists c \in \mathbb{R}^+ \exists \kappa \in \mathbb{R}^+ \forall x (x > \kappa) \rightarrow (|f(x)| \leq c |g(x)|)$
 $f(x)$ is $\Omega(g(x)) \equiv \exists c \in \mathbb{R}^+ \exists \kappa \in \mathbb{R}^+ \forall x (x > \kappa) \rightarrow (|f(x)| \geq c |g(x)|)$
 $f(x)$ is $\Theta(g(x)) \equiv [f(x) \text{ is } O(g(x))] \wedge [f(x) \text{ is } \Omega(g(x))]$.

[6 MARKS]

- b) $\exists \kappa_1, \kappa_2, c_1, c_2$ s.t.

$$|f_1(x)| \leq c_1 |g_1(x)|, \quad x > \kappa_1$$

$$\& \quad |f_2(x)| \leq c_2 |g_2(x)|, \quad x > \kappa_2$$

By the triangle inequality,

$$|f_1(x) + f_2(x)| \leq |f_1(x)| + |f_2(x)|$$

$$\leq c_1 |g_1(x)| + c_2 |g_2(x)|, \quad x > \max(\kappa_1, \kappa_2)$$

$$\leq c_1 \max(|g_1(x)|, |g_2(x)|) +$$

$$c_2 \max(|g_1(x)|, |g_2(x)|)$$

$$= (c_1 + c_2) \max(|g_1(x)|, |g_2(x)|)$$

So with $c = c_1 + c_2$, $\kappa = \max(\kappa_1, \kappa_2)$,

$$f_1(x) + f_2(x) \text{ is } O(\max(|g_1(x)|, |g_2(x)|)).$$

[6 MARKS]

c)

```
proc1(int n) {
    total := 1;
    for i = 1 to 2*n
        total := total * n;
}
```

(Assuming loop
term. calc.
evaluated once).
[6 MARKS]

d)

```
proc2(int n) {
    if n = 1
        return 3*n;
    else if n = 2
        return 6*n*n*n;
```

else
return $n * \text{proc2}(n \text{ div } 3)$;
}

[6 MARKS]

5. (e) Proc 1 has $O(n)$ exec time (#mults)

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Proc 2 has $O(\log n)$ exec time (#mults).

Proc therefore has $O(\max(n, \log n))$

$$= O(n)$$

$$\underline{\underline{K=1}}$$

[6 MARKS]