DISCRETE MATHS - SOLUTIONS 2008 E2.17 ~~1/11~~ E2.17 ~~E3.20~~ 1/7

1\. a)(i) $P(S_1) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$.

(ii) $S_1 \cup S_2 = \{a, b, c\}$

(iii) $S_1 \cap S_2 = \{a, b\}$

(iv) $S_2 - S_1 = \phi$

(v) $|S_1 \cup S_2| = 3$ [6 MARKS]

b) Finite : S_1 from (a)
Infinite - Countable : $\mathbb{N}$
Infinite - Uncountable : $\mathbb{R}$ [3 MARKS]

c) (i) No, e.g. $(2,1) \in R$ but $(1,2) \notin R$.
(ii) No, e.g. $(2,1) \in R$ and $(4,2) \in R$ but $(4,1) \notin R$.
(iii) No, e.g. $(1,1) \notin R$.
(iv) No, e.g. there is no element $(1, x)$ for any x.
(v) No — (ii) implies this. [6 MARKS]

d) (i) $\forall x (J(x) \rightarrow L(x))$
(ii) $\exists x (L(x) \land \neg J(x))$
(iii) $\forall x (J(x) \rightarrow A(x, \text{Jones}))$
(iv) $\forall x \forall y (A(x,y) \land J(x) \rightarrow J(y))$. [5 MARKS]

e) Simplify (i) : b
Modus ~~Ponens~~ Tollens w/ (ii) : $\neg c$ [2 MARKS]

1. f) (i) From the theorem, $f(x)$ is $O(x^3)$.

$$|f(x)| = |x^3 + 2x^2 + 1| \geq |x^3| + |2x^2| + 1 \quad \text{for } x > 0$$

$$\geq |x^3|$$

So with any K, e.g. $K = 1$ & with $c = 1$,

$$\forall x \left((x > K) \rightarrow |f(x)| \geq c|x^3| \right) \square$$

(ii)

```
proc1 (int x) {
    for i = 1 to x*x*x + 2*x*x + 1 - 4
        avar = aver * 2;
}
```

(iii)

```
proc2 (int x) {
    if x = 1
        return 2 * x * x;
    else
        return proc2(x-1) * proc2(x-1);
}
```

[9 MARKS]

g) Let $a \geq 1$ be a real number, $b > 1$ be an integer, $c > 0$ be a real number and $d \geq 0$ be a real number.

Let f be an increasing function s.t.

$$f(n) = a f(n/b) + c n^d \quad \text{whenever } n = b^k$$

for positive integer k.

(i) If $a < b^d$, $f(n)$ is $O(n^d)$

(ii) If $a = b^d$, $f(n)$ is $O(n^d \log n)$

(iii) If $a > b^d$, $f(n)$ is $O(n^{\log_b a})$.

[9 MARKS]

~~3.~~ 2. a) (i) Let $x \in B = \mathbb{R}$.

Then $2^x \in \mathbb{R}$ and $2^x > 0 \Rightarrow 2^x \in R_+$.

So $f(2^x, 0) = x$ $\square$. [3 MARKS]

(ii) Choose $A = R_+ \times \{0\}$

Proof above holds for surjectivity.

For injectivity,

$$f(x_1, y_1) = f(x_2, y_2)$$

But $y_1 = y_2 = 0$.

Also $\log_2(x_1 + 0) = \log_2(x_2 + 0)$

$\Rightarrow x_1 = x_2$ $\square$. [6 MARKS]

(iii) It is possible to obtain any ~~positive~~ non-negative value $v \in R_+ \cup \{0\}$ from

$$f(2^v, 0) = v \quad \& \quad 2^v \geq 1$$

~~Also zero is possible if~~

Negative values are not possible - we would require $f(x, y) < 0$

$\Rightarrow \log_2(x + y) < 0$

So either $\log_2 x < 0 \Rightarrow x < 1$ ×

or $\log_2(x+1) < 0 \Rightarrow x + 1 < 1$

i.e. $x < 0$ ×.

So image is $\underline{R_+ \cup \{0\}}$. [6 MARKS]

~~3.~~ 2. (b)(i) $R^n \subseteq R \Rightarrow R$ is transitive:

$R^n \subseteq R \Rightarrow R^2 \subseteq R$. $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R^2$. But $R^2 \subseteq R \Rightarrow (a,c) \in R$.
So R is transitive.

R is transitive $\Rightarrow R^n \subseteq R$

True for $n=1$. Use induction to show $R^{n+1} \subseteq R$ assuming $R^n \subseteq R$.

Consider $(a,b) \in R^{n+1} = R \cdot R^n$

$\Rightarrow \exists x \left((a,x) \in R \wedge (x,b) \in R^n \right)$.

$R^n \subseteq R \Rightarrow (x,b) \in R$. R is transitive $\Rightarrow (a,b) \in R$. So $R^{n+1} \subseteq R$ $\square$. [6 MARKS]

(ii) $\forall a \in A \; \exists b \in B \left((a,b) \in R \right)$

$\wedge \; \forall a \in A \; \forall b \in B \; \forall c \in B \left((a,b) \in R \wedge (a,c) \in R \rightarrow b = c \right)$. [6 MARKS]

(iii) Trivially, $f : \{0\} \rightarrow \{0\}$, $f(0) = 0$ is transitive.

[3 MARKS]

~~4.~~ 3. a) (i) $T(\text{Steven}) \wedge I(\text{Steven})$

(ii) $G(\text{Steven}) \wedge \forall x\,(I(x) \rightarrow G(x))$

(iii) $\exists x\,(T(x) \wedge I(x))$

$\wedge \forall x \forall y\,(T(x) \wedge I(x) \wedge T(y) \wedge I(y) \rightarrow x = y)$

(iv) $\forall x\,(T(x) \wedge I(x) \rightarrow x = \text{Steven})$

(v) $T(\text{Amanda}) \wedge \neg T(\text{James})$

(vi) $\neg I(\text{Amanda})$

[12 MARKS]

b) Simplify (iii) $\Rightarrow \forall x \forall y\,(T(x) \wedge I(x) \wedge T(y) \wedge I(y) \rightarrow x = y)$

Universal Instantiation

$\forall x\,(T(x) \wedge I(x) \wedge T(\text{Steven}) \wedge I(\text{Steven}) \rightarrow x = \text{Steven})$

Hypothesis $\Rightarrow$ $\forall x\,(T(x) \wedge I(x) \rightarrow x = \text{Steven})$

[9 MARKS]

c) Universal Instantiation on (iv)

$T(\text{Amanda}) \wedge I(\text{Amanda}) \rightarrow \text{Amanda} = \text{Steven}$

Modus Ponens

$\neg(T(\text{Amanda}) \wedge I(\text{Amanda}))$

$\equiv \neg T(\text{Amanda}) \vee \neg I(\text{Amanda})$ (*)

Simplify (v) $T(\text{Amanda})$ (+)

Disjunctive Syllogism (*) & (+)

$\Rightarrow$ $\neg I(\text{Amanda})$

[9 MARKS]

~~10~~ ~~11~~ 6/7

~~8.~~ 4.

a) $f(x)$ is $O(g(x)) \equiv \exists c \in R^+ \exists k \in R^+ \forall x \left((x > k) \rightarrow (|f(x)| \leq c\,|g(x)|)\right)$

$f(x)$ is $\Omega(g(x)) \equiv \exists c \in R^+ \exists k \in R^+ \forall x \left((x > k) \rightarrow (|f(x)| \geq c\,|g(x)|)\right)$

$f(x)$ is $\Theta(g(x)) \equiv [f(x)$ is $O(g(x))] \wedge [f(x)$ is $\Omega(g(x))]$.

[6 MARKS]

b) $\exists\, k_1, k_2, c_1, c_2$ s.t.

$$|f_1(x)| \leq c_1 |g_1(x)|, \quad x > k_1$$

$$\& \quad |f_2(x)| \leq c_2 |g_2(x)|, \quad x > k_2$$

By the triangle inequality,

$$
\begin{aligned}
|f_1(x) + f_2(x)| &\leq |f_1(x)| + |f_2(x)| \\
&\leq c_1 |g_1(x)| + c_2 |g_2(x)|, \quad x > \max(k_1, k_2) \\
&\leq c_1 \max(|g_1(x)|, |g_2(x)|) + \\
&\qquad c_2 \max(|g_1(x)|, |g_2(x)|) \\
&= (c_1 + c_2) \left| \max(|g_1(x)|, |g_2(x)|) \right|.
\end{aligned}
$$

So with $c = c_1 + c_2$, $k = \max(k_1, k_2)$,

$f_1(x) + f_2(x)$ is $O(\max(|g_1(x)|, |g_2(x)|))$.

[6 MARKS]

c)
```
proc1(int n) {
    total := 1;
    for i = 1 to 2*n
        total := total * n;
}
```

(Assuming loop term condition evaluated once)

[6 MARKS]

d)
```
proc2(int n) {
    if n = 1
        return 3*n;
    else if n = 2
        return n*n*n;
    else
        return n * proc2(n div 3);
}
```

[6 MARKS]

~~8.~~ 4. (e) Proc1 has $O(n)$ exec time (#mults)

Proc2 has $O(\log n)$ exec time (#mults).

Proc therefore has $O(\max(n, \log n))$

$= O(n)$ ~~ws~~

$\underline{\underline{K = 1}}$.

[6 MARKS]