

## Control engineering exam paper - Model answers

2008

## Question 1

- a) To show that system (1.2) matches system (1.1) at  $s_1$  and  $s_2$  we need to verify that

$$H(s_i I - F)^{-1} G = C(s_i I - A)^{-1} B,$$

for  $i = 1, 2$ . Note that

$$F = \begin{bmatrix} s_1 - \Delta_1 & -\Delta_1 \\ -\Delta_2 & s_2 - \Delta_2 \end{bmatrix}$$

and that

$$(sI - F)^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s - s_2 + \Delta_2 & -\Delta_1 \\ -\Delta_2 & s - s_1 + \Delta_1 \end{bmatrix},$$

where

$$\Delta(s) = s^2 + s((\Delta_1 - s_1) + (\Delta_2 - s_2)) + (s_1 s_2 - s_1 \Delta_2 - s_2 \Delta_1).$$

A direct calculation shows that

$$(s_1 I - F)^{-1} G = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (s_2 I - F)^{-1} G = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which proves the claim.

- b) The reachability matrix is

$$R = \begin{bmatrix} \Delta_1 & \Delta_1(s_1 - \Delta_1) - \Delta_1 \Delta_2 \\ \Delta_2 & \Delta_2(s_2 - \Delta_2) - \Delta_1 \Delta_2 \end{bmatrix},$$

and  $\det(R) = (s_2 - s_1)\Delta_1\Delta_2$ . Hence, since  $s_1 \neq s_2$ , the reachability matrix is full rank if and only if  $\Delta_1\Delta_2 \neq 0$ .

- c) If  $C(s_1 I - A)^{-1} B = C(s_2 I - A)^{-1} B = \kappa$  then the observability matrix is

$$O = \begin{bmatrix} \kappa & \kappa \\ \kappa(s_1 - \Delta_1 - \Delta_2) & \kappa(s_2 - \Delta_2 - \Delta_1) \end{bmatrix},$$

and  $\det(O) = \kappa^2(s_2 - s_1)$ . Hence the system is observable if and only if  $\kappa \neq 0$ .

- d) Since  $s_1 = 0$  and  $s_2 = 1$  we have

$$F = \begin{bmatrix} -\Delta_1 & -\Delta_1 \\ -\Delta_2 & 1 - \Delta_2 \end{bmatrix}.$$

The characteristic polynomial of  $F$  is

$$s^2 + s(\Delta_1 + \Delta_2 - 1) - \Delta_1$$

and this should be equal to  $(s + 1)^2 = s^2 + 2s + 1$ . As a result

$$\Delta_1 = -1 \quad \Delta_2 = 4.$$

## Question 2

- a) Note that the system can be written as

$$x(k+1) = \begin{bmatrix} 0 & I \\ G & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ B \end{bmatrix} u(k).$$

The reachability matrix of this system is

$$R = \begin{bmatrix} 0 & B & 0 & GB & 0 & G^2B & \dots \\ B & 0 & GB & 0 & G^2B & 0 & \dots \end{bmatrix},$$

hence the rank of  $R$  is twice the rank of the matrix

$$\begin{bmatrix} B & GB & G^2B & \dots \end{bmatrix},$$

which is the reachability matrix of the system

$$\xi(k+1) = G\xi(k) + Bv(k).$$

Hence, system (2.1) is reachable if and only if the above system is reachable.

- b) If  $m = n$  and  $B = I$  then the system

$$\xi(k+1) = G\xi(k) + Bv(k) = G\xi(k) + v(k)$$

is reachable, hence by part a), the system (2.1) is reachable.

- c) i) A family of matrices  $K = K'$  which achieves the stabilization goal is  $K = \alpha I$ , with  $\alpha \in (-1, 1)$ .  
ii) Note that

$$e(k) = x_2(k) - \alpha x_1(k),$$

hence

$$e(k+1) = x_2(k+1) - \alpha x_1(k+1) = Gx_1(k) + u(k) - \alpha x_2(k).$$

- iii) Selecting

$$u(k) = -Gx_1(k) + \alpha x_2(k)$$

yields  $e(k+1) = 0$ , which implies that  $e(k) = 0$  for all  $k \geq 1$  (and all  $e(0)$ ).

- iv) The control law determined in part c.iii) stabilizes the discrete-time system because, for  $k \geq 1$  we have  $e(k) = 0$ , which implies  $x_2(k) = \alpha x_1(k)$ , for all  $k \geq 1$ . Therefore, for  $k \geq 1$

$$x_1(k+1) = \alpha x_1(k),$$

hence  $x_1(k)$  converges to zero as  $k \rightarrow \infty$ . This, together with the fact that  $e(k) = 0$ , for all  $k \geq 1$ , implies that  $x_2(k)$  also converges to zero as  $k \rightarrow \infty$ . As a result, the zero equilibrium of the system is attractive and, since this is a linear, time-invariant system, attractivity implies stability, *i.e.* the system is asymptotically stable.

### Question 3

- a) The approximate discrete-time Euler model is

$$x(k+1) = (I + TA)x(k) + TBu(k) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ T \end{bmatrix} u(k).$$

- b) The reachability matrix of the Euler model is

$$R = \begin{bmatrix} 0 & T^2 \\ T & T \end{bmatrix}$$

and  $R$  has full rank for all  $T > 0$ , *i.e.* the approximate model is reachable for all  $T > 0$ .

- c) Consider the matrix

$$I + TA + TBK = \begin{bmatrix} 1 & T \\ TK_1 & 1 + TK_2 \end{bmatrix}.$$

Its characteristic polynomial is

$$s^2 + s(-2 - TK_2) + (1 + TK_2 - T^2K_1),$$

and this should be equal to  $s^2$ . As a result

$$K_1 = -\frac{1}{T^2} \quad K_2 = -\frac{2}{T}.$$

- d) i) By definition of the matrix exponential we have

$$A_d = e^{AT} = I + AT + A^2 \frac{T^2}{2} + \dots$$

Since  $A^2 = A^3 = \dots = 0$ , then

$$A_d = I + AT = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

and

$$B_d = \int_0^T \begin{bmatrix} 1 & T-\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau = \int_0^T \begin{bmatrix} T-\tau \\ 1 \end{bmatrix} d\tau = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}.$$

Note that the matrix  $A_d$  coincides with the  $A$  matrix of the Euler model, whereas the matrix  $B_d$  is different.

- ii) The matrix  $A_d + B_dK$ , with  $K$  as in part c) is

$$\begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{T} & -1 \end{bmatrix}.$$

This matrix has eigenvalues equal to  $-1$  and  $1/2$ . Hence, the resulting system is stable, not asymptotically, *i.e.* the design based on the Euler approximate model is not adequate to stabilize the sampled-data system. (A more careful analysis, note requested, would reveal that the main reason why the design is not effective is the selection of the closed-loop eigenvalues.)

## Question 4

- a) i) The linearized system at  $x = 0$  is described by

$$\dot{\delta}_x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \delta_x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta_u.$$

- ii) The reachability matrix of the linearized system is

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Note that  $\text{rank} R = 2$ , hence the system is reachable.

- iii) A direct application of Ackerman formula yields

$$K_a = \begin{bmatrix} -1 & -2 \end{bmatrix}.$$

- iv) The linearized closed-loop system is asymptotically stable hence, by the principle of stability in the first approximation, the zero equilibrium of the controlled nonlinear system is locally asymptotically (exponentially) stable.

- b) i) Note that

$$\dot{y} = \dot{x}_1 = x_1^2 + x_2.$$

Hence

$$\ddot{y} = 2x_1\dot{x}_1 + \dot{x}_2 = 2x_1(x_1^2 + x_2) + x_1x_2 + u.$$

As a result

$$\ddot{y} + (-2x_1^3 - 3x_1x_2) = u.$$

- ii) Setting

$$K_b(x) = (-2x_1^3 - 3x_1x_2) - 2\dot{y} - y = (-2x_1^3 - 3x_1x_2) - 2(x_1^2 + x_2) - x_1,$$

yields

$$\ddot{y} + 2\dot{y} + y = 0,$$

as requested.

- iii) The controlled nonlinear system can be written using the coordinates  $(y, \dot{y})$ . In these coordinates the system is linear and the zero equilibrium is globally asymptotically stable.

- c) Clearly the control law  $K_b(x)$  is more complex, *e.g.* requires more computation, than the control law  $K_a(x)$ . However, while  $K_b(x)$  is such that the zero equilibrium of the closed-loop system is globally asymptotically stable,  $K_a(x)$  only guarantees a local property. (Note, in addition, that the system in closed loop with the controller  $u = K_a x$  has two equilibria:  $(0, 0)$  and  $(1, -1)$ . This proves that the zero equilibrium cannot be globally asymptotically stable.)

## Question 5

- a) The observability matrix of the system is

$$O = \begin{bmatrix} 1 & 0 & 1 \\ 0 & \alpha & -\frac{1}{2} \\ -\alpha^2 & 0 & \alpha + \frac{1}{4} \end{bmatrix}.$$

The determinant of the observability matrix is

$$\det O = \alpha^3 + \alpha^2 + \frac{1}{4}\alpha.$$

Hence the observability matrix loses rank for  $\alpha = 0$  and  $\alpha = -1/2$ . The system is therefore observable for all  $\alpha \neq 0$  and  $\alpha \neq -1/2$ .

For  $\alpha = 0$  the observability pencil is

$$\begin{bmatrix} sI - A \\ C \end{bmatrix} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & -1 \\ 0 & 0 & s + 1/2 \\ 1 & 0 & 1 \end{bmatrix}$$

and it loses rank only for  $s = 0$ . Therefore the unobservable mode is for  $s = 0$ , hence the system is detectable and reconstructable for  $\alpha = 0$ .

For  $\alpha = -1/2$  the observability pencil is

$$\begin{bmatrix} sI - A \\ C \end{bmatrix} = \begin{bmatrix} s & 1/2 & 0 \\ -1/2 & s & -1 \\ 0 & 0 & s + 1/2 \\ 1 & 0 & 1 \end{bmatrix}$$

and it loses rank only for  $s = -1/2$ . Therefore the unobservable mode is for  $s = -1/2$ , hence the system is detectable, but not reconstructable, for  $\alpha = -1/2$ .

- b) To design an observer with the requested property it is necessary to find a matrix  $L$  such that the eigenvalues of  $A + LC$  are all equal to zero. Such an  $L$  exists if (and only if) the system is observable or reconstructable. As a result, it is possible to design the observer with the requested property for all  $\alpha \neq -1/2$ .
- c) For  $\alpha = 0$  the observability matrix is

$$O = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

The matrix has rank equal to two. A basis for the unobservable subspace is

$$\ker O = \text{span} \left[ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right].$$

To write the system in observability canonical form consider the change of coordinates  $x = L\hat{x}$ , with

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that the last column of  $L$  spans the kernel of  $O$ . In the new coordinates  $\hat{x}$  the system is described by

$$\hat{x}(k+1) = L^{-1}AL\hat{x} + L^{-1}Bu = \hat{A}\hat{x} + \hat{B}u \quad y(k) = CL\hat{x} = \hat{C}\hat{x},$$

where

$$\hat{A} = \left[ \begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -1/2 & 0 \\ \hline 0 & 1 & 0 \end{array} \right] \quad \hat{B} = \left[ \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right] \quad \hat{C} = \left[ \begin{array}{cc|c} 1 & 1 & 0 \end{array} \right],$$

which clearly shows the decomposition into observable and unobservable subsystems.

## Question 6

a) Note that

$$CB = 0 \quad CAB = 1 \quad CP = 0 \quad CAP = 0$$

and

$$\text{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 2.$$

Hence  $\kappa = 2$  is such that condition (C) holds.

b) The equations of the system can be rewritten as

$$\begin{aligned} \dot{x}_1 &= \hat{A}_{11}x_1 + \hat{A}_{12}x_2 + B_1u + P_1d, \\ \dot{x}_2 &= \hat{A}_{21}x_1 + \hat{A}_{22}x_2 + B_2u, \\ y &= C_2x_2, \end{aligned}$$

hence the output equation is already in the desired form. Note now that

$$\hat{A}_{21} = B_2L_1 \quad \hat{A}_{22} = S + B_2L_2$$

for some matrices  $L_1$  and  $L_2$ . As a result

$$\dot{x}_2 = Sx_2 + B_2(u + L_1x_1 + L_2x_2),$$

which shows that the  $\dot{x}_2$  equation is in the desired form, with  $L = [L_1 \ L_2]$ . Finally

$$\begin{aligned} \dot{x}_1 &= \hat{A}_{11}x_1 + \hat{A}_{12}x_2 + B_1u + P_1d, \\ &= \hat{A}_{11}x_1 + \hat{A}_{12}x_2 + B_1(u + L_1x_1 + L_2x_2) - B_1L_1x_1 - B_1L_2x_2 + P_1d, \\ &= (\hat{A}_{11} - B_1L_1)x_1 + (\hat{A}_{12} - B_1L_2)x_2 + B_1(u + L_1x_1 + L_2x_2) + P_1d, \end{aligned}$$

which shows that the  $\dot{x}_1$  equation can be written as requested, with  $A_{11} = \hat{A}_{11} - B_1L_1$ ,  $A_{12} = \hat{A}_{12} - B_1L_2$ .

c) Setting  $u = -Lx + K_2x_2$  yields the equations

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + (A_{12} + B_1K_2)x_2 + P_1d, \\ \dot{x}_2 &= (S + B_2K_2)x_2, \\ y &= C_2x_2. \end{aligned}$$

The  $x_2$  subsystem is not directly affected by the disturbance  $d$  and by the  $x_1$  subsystem, which is not observable. This implies that  $y$  is not affected by  $d$ . In addition, the second of the above equations yields

$$x_2(t) = e^{(S+B_2K_2)t}x_2(0),$$

hence

$$y(t) = c_2e^{(S+B_2K_2)t}x_2(0),$$

which shows again that  $d$  is not acting on  $y$ , *i.e.* the control law has decoupled the effect of the disturbance from the output.

