IMPERIAL COLLEGE LONDON

E4.10
CS5.1
SC4

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

MSc and EEE PART IV: MEng and ACGI

Corrections.
Q1 (a)
Q2 (b)(iii)
Q5 (b)(iv)

PROBABILITY AND STOCHASTIC PROCESSES

Thursday, 7 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : M.M. Draief

Second Marker(s) : R.B. Vinter

PROBABILITY AND STOCHASTIC PROCESSES

1. Let X and Y be two random variables, where m_X and m_Y denote their respective means, σ_X and σ_Y denote their respective variances.

a) Give the definition of the ~~covariance~~ ρ_{XY} and prove that $\rho_{XY} \in [-1, 1]$. [5]

Hint: Let $U = \frac{X - m_X}{\sigma_X}$ *and* $V = \frac{Y - m_Y}{\sigma_Y}$. *Use the fact that*

$$\mathbf{E}\left((U+V)^2\right) \geq 0 \quad \text{and} \quad \mathbf{E}\left((U-V)^2\right) \geq 0\,.$$

Correlation

Variance of X is

Variance of Y is

b) We say that X and Y are linearly dependent if there exist two constants $a \neq 0$ and b such that $Y = aX + b$.

(*i*) Show that if X and Y are linearly dependent then $|\rho_{XY}| = 1$. [2]

(*ii*) Show that if $\rho_{XY} = 1$ then X and Y are linearly dependent. [3]

Hint: In (ii), show that $\mathbf{E}\left((U-V)^2\right) = 0$, *for* $U = \frac{X - m_X}{\sigma_X}$ *and* $V = \frac{Y - m_Y}{\sigma_Y}$.

c) We consider two dependent (correlated) random variables X and Y. The best linear estimator of Y given X is given by $\hat{Y} = aX + b$ which minimises $\mathbf{E}\left[(Y - \hat{Y})^2\right]$.

(*i*) Show that

$$\hat{Y} = \rho_{XY} \frac{\sigma_Y}{\sigma_X}(X - m_X) + m_Y\,.$$

[5]

(*ii*) Prove that

$$\mathbf{E}\left[(Y - \hat{Y})^2\right] = (1 - \rho_{XY}^2)\sigma_Y^2\,.$$

[3]

(*iii*) For which value of ρ_{XY} is the estimation exact? [2]

2. Let $(X_t, t \geq 0)$ be a random telegraph process with parameter λ, i.e. a $\{-1, 1\}$-valued continuous time process such that the number of 0 crossings in the interval $(0,t)$ is described by a Poisson process with parameter λt. Assume that X_0 is such that

$$\mathbf{P}(X_0 = 1) = \mathbf{P}(X_0 = -1) = \frac{1}{2}$$

a) Let $t, \tau \geq 0$

(*i*) Show that

$$\mathbf{P}\,(\text{There are } n \text{ crossings between } t \text{ and } t+\tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^n}{n!}.$$

[1]

(*ii*) Prove that

$$\begin{aligned}\mathbf{P}(X_{t+\tau} = 1 \mid X_t = 1) &= \mathbf{P}(X_{t+\tau} = -1 \mid X_t = -1) \\ &= e^{-\lambda\tau}[1 + \frac{(\lambda\tau)^2}{2!} + \frac{(\lambda\tau)^4}{4!} + \ldots].\end{aligned}$$

[3]

(*iii*) Prove that

$$\begin{aligned}\mathbf{P}(X_{t+\tau} = 1 \mid X_t = -1) &= \mathbf{P}(X_{t+\tau} = -1 \mid X_t = 1) \\ &= e^{-\lambda\tau}[\lambda\tau + \frac{(\lambda\tau)^3}{3!} + \frac{(\lambda\tau)^5}{5!} + \ldots].\end{aligned}$$

[3]

(*iv*) Show that $\mathbf{E}(X_\tau) = 0$ for all $\tau \geq 0$. [3]

Hint: Compute $\mathbf{E}(X_{t+\tau} \mid X_t = 1)$ *and* $\mathbf{E}(X_{t+\tau} \mid X_t = -1)$ *then use Bayes's rule.*

b) We now focus on the autocorrelation function $R_X(\tau)$ of the process $(X_t, t \geq 0)$.

(*i*) Give the definition of $R_X(\tau)$. [1]

(*ii*) Show that

$$R_X(\tau) = \frac{1}{2}\mathbf{E}(X_{t+\tau} \mid X_t = 1) - \frac{1}{2}\mathbf{E}(X_{t+\tau} \mid X_t = -1).$$

[3]

(*iii*) Prove that $R_X(\tau) = e^{-2\lambda\tau}$. [4]

c) Conclude that X_t is wide-sense stationary process. [2]

3. We perform n tosses of a fair coin. The variable X_i describes the outcome of the i-th toss: $X_i = 1$ if Heads shows and $X_i = 0$ if Tail shows. Let $X = \sum_{i=1}^{n} X_i$.

a) State the distribution of X, and compute its expectation and its variance. [2]

b) We now examine the probability that X deviates from its mean.

(i) Show that

$$\mathbf{P}\left(X \geq \frac{3n}{4}\right) \leq \mathbf{P}\left(|X - \frac{n}{2}| \geq \frac{n}{4}\right).$$

[2]

(ii) Using Chebyshev's inequality, prove that

$$\mathbf{P}\left(X \geq \frac{3n}{4}\right) \leq \frac{4}{n}.$$

[4]

(iii) Show that $\lim_{n\to\infty} \mathbf{P}(X \geq \frac{3n}{4}) = 0$ and comment. [2]

c) We now derive a tighter bound for the convergence of $\mathbf{P}(X \geq \frac{3n}{4})$ to 0 as n goes to ∞.

(i) Let $x, \theta \geq 0$. Combining Markov's inequality and the fact that

$$\{X \geq x\} = \{e^{\theta X} \geq e^{\theta x}\},$$

prove that

$$\mathbf{P}(X \geq x) \leq \exp\left(\frac{n}{2}(e^{\theta} - 1) - \theta x\right).$$

[5]

Hint: Use the following inequality $1 + \alpha \leq e^{\alpha}$, *for* $\alpha \geq 0$.

(ii) Choose θ so that $\frac{1}{2}(e^{\theta} - 1) - \frac{3}{4}\theta \leq -0.01$. [1]

(iii) Prove that for the choice of θ in the previous question

$$\mathbf{P}\left(X \geq \frac{3n}{4}\right) \leq e^{-0.01n}.$$

[4]

4. Let $Y_1, Y_3, Y_5, \ldots$ be a sequence of independent and identically distributed random variables such that $\mathbf{P}(Y_{2k+1} = -1) = \mathbf{P}(Y_{2k+1} = 1) = \frac{1}{2}$, for $k = 0, 1, 2, \ldots$. Let $Y_{2k} = Y_{2k-1} Y_{2k+1}$, for $k = 1, 2, \ldots$.

a) Show that $\mathbf{P}(Y_{2k} = \alpha, Y_{2k+2} = \beta) = 1/4$, for $\alpha, \beta \in \{-1, 1\}$. Conclude that $Y_2, Y_4, Y_6, \ldots$ is a sequence of independent and identically distributed random variables and give their joint distribution. [3]

b) Show that $\mathbf{P}(Y_{2k} = \alpha, Y_{2k+1} = \beta) = 1/4$, for $\alpha, \beta \in \{-1, 1\}$. Is the sequence $Y_1, Y_2, Y_3, \ldots$ independent and identically distributed? [3]

c) Compute $\mathbf{P}(Y_{2k+1} = 1 \mid Y_{2k} = -1)$ and $\mathbf{P}(Y_{2k+1} = 1 \mid Y_{2k} = -1, Y_{2k-1} = 1)$. Is the process $Y_1, Y_2, Y_3, \ldots$ a Markov chain? [4]

d) Let $Z_n = (Y_n, Y_{n+1})$ be a process in $\{0, 1\}^2$.

(i) Show that

$$\mathbf{P}(Z_{n+1} = (1,1) \mid Z_n = (1,1)) = \begin{cases} \frac{1}{2}, & \text{if } n \text{ even}, \\ 1, & \text{if } n \text{ odd}, \end{cases}$$

[4]

(ii) Show that $(Z_n, n \geq 0)$ is a (non-homogeneous) Markov chain and give its transition probabilities. [6]

5. Consider the weather chain $(X_n, n \geq 0)$ set to $X_n = 1$ if it rains on day n and $X_n = 0$ otherwise. Suppose that X_n evolves as a Markov chain with transition matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

where $\alpha, \beta \in [0,1]$.

a) Sketch the diagram of the evolution of the above Markov chain and briefly describe the dynamics in the following cases (i) $\alpha = \beta = 0$, (ii) $\alpha = \beta = 1$, (iii) $\alpha = 1, \beta = 0$ and (iv) $\alpha = 0, \beta = 1$. [2]

b) In what follows we suppose that $\alpha, \beta \in (0,1)$.

(i) Let $p_{00}(n) = \mathbf{P}(X_n = 0 \mid X_0 = 0)$. Show that

$$p_{00}(n+1) = (1-\alpha-\beta)p_{00}(n) + \beta, \quad \text{for } n \geq 0.$$

[4]

(ii) Prove that

$$p_{00}(n) = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n.$$

[2]

(iii) Derive the expressions for $p_{01}(n) = \mathbf{P}(X_n = 1 \mid X_0 = 0)$, $p_{10}(n) = \mathbf{P}(X_n = 0 \mid X_0 = 1)$ and $p_{11}(n) = \mathbf{P}(X_n = 1 \mid X_0 = 1)$. [4]

(iv) Compute the limits $p_{11}(n), p_{\cancel{12}}(n), p_{\cancel{21}}(n), p_{\cancel{22}}(n)$ when n goes to infinity.
10 01 00
[2]

c) Using two different methods, compute the stationary distribution of the weather chain. [6]

6. Consider the Markov chain on $\{1,2,3,4\}$ with the following transition matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We define the probability of absorption by

$$h_i = \mathbf{P}(X_n \text{ is absorbed in state } 4 \mid X_0 = i)$$

and the expected time to absorption by

$$k_i = \mathbf{E}(\text{ time for } X_n \text{ to be absorbed in state } 1 \text{ or } 4 \mid X_0 = i)$$

where $i = 1,2,3,4$.

In what follows, **carefully justify** your results.

a) First compute the probabilities of absorption h_i :

(i) Show that

$$h_2 = \frac{1}{2}h_1 + \frac{1}{2}h_3 \quad \text{and} \quad h_3 = \frac{1}{2}h_2 + \frac{1}{2}h_4 .$$

[6]

(ii) After deriving the values of h_1 and h_4, show that

$$h_2 = \frac{1}{3} \quad \text{and} \quad h_3 = \frac{2}{3} .$$

[4]

b) Now compute the expected times to absorption k_i :

(i) Show that

$$k_2 = 1 + \frac{1}{2}k_1 + \frac{1}{2}k_3 \quad \text{and} \quad k_3 = 1 + \frac{1}{2}k_2 + \frac{1}{2}k_4$$

[6]

(ii) After deriving the values of k_1 and k_4, show that

$$k_2 = 2 \quad \text{and} \quad k_3 = 2 .$$

[4]

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PROBABILITY & STOCHASTIC PROCESSES.

1

Q1

a) $\rho_{XY} = \dfrac{\mathbb{E}\big((X-m_X)(Y-m_Y)\big)}{\sigma_X \sigma_Y} = \dfrac{\mathbb{E}(XY) - m_X m_Y}{\sigma_X \sigma_Y}.$

We immediately check that

$\mathbb{E}(U) = \mathbb{E}(V) = 0, \quad \sigma_U = \sigma_V = 1 \quad \& \quad \mathbb{E}(UV) = \rho_{XY}.$

$(U+V)^2$ & $(U-V^2)$ are non-negative so

(1) $\mathbb{E}\big((U+V)^2\big) \geq 0$ & (2) $\mathbb{E}\big((U-V)^2\big) \geq 0.$

(1) $\Rightarrow \rho_{XY} \geq -1$; (2) $\Rightarrow \rho_{XY} \leq 1.$

b/ (i) $Y = aX + b.$

$$\sigma^2_Y = \mathbb{E}\big((Y-m_Y)^2\big) = a^2 \sigma_X^2 \Rightarrow \sigma_Y = |a| \sigma_X$$

$$\rho_{XY} = \mathbb{E}\left(\frac{(X-m_X)(aX+b-am_X-b)}{\sigma_X \sigma_Y}\right)$$

$$= \frac{a\,\mathbb{E}\big((X-m_X)^2\big)}{|a|\sigma_X} =$$

$|\rho_{XY}| = 1.$

(ii) $\rho_{XY} = 1$. $\mathbb{E}\big((U-V)^2\big) = \mathbb{E}(U^2) - 2\mathbb{E}(UV) + \mathbb{E}(V^2) = 0$

$\Rightarrow U = V$ (with probability 1).

$\Rightarrow \dfrac{X-m_X}{\sigma_X} = \dfrac{Y-m_Y}{\sigma_Y}$ & X & Y are linearly dependent

(2)

Q1)

(c) (i) $\hat{Y} = aX + b$. Let a^*, b^* best a & b to minimize

$$\mathbb{E}\left(\left(Y-\hat{Y}\right)\right)^2 = f(a,b) = \mathbb{E}\left(\left(Y-aX-b\right)^2\right).$$

$$\frac{\partial f(a,b)}{\partial b} = -2\,\mathbb{E}\left(Y-aX-b\right)$$

$$\Rightarrow \quad b^* = m_Y - a\,m_X.$$

$$f(a,b^*) = \mathbb{E}\left(\left[\left(Y-m_Y\right) - a\left(X-m_X\right)\right]^2\right)$$

$$\frac{\partial f(a,b^*)}{\partial a} = -2\,\mathbb{E}\left[\left(Y-m_Y\right)\left(X-m_X\right)\right] + 2a\,\mathbb{E}\left(\left(X-m_X\right)^2\right).$$

$$\Rightarrow \quad a^* = \frac{\mathbb{E}\left(\left(Y-m_Y\right)\left(X-m_X\right)\right)}{\mathbb{E}\left(\left(X-m_X\right)^2\right)} = \rho_{XY}\frac{\sigma_Y}{\sigma_X}.$$

$$\boxed{\hat{Y} = \rho_{XY}\frac{\sigma_Y}{\sigma_X}\left(X-m_X\right) + m_Y}$$

(ii) $$\mathbb{E}\left(\left(Y-\hat{Y}\right)^2\right) = \mathbb{E}\left(\left[\left(Y-m_Y\right) - \rho_{XY}\frac{\sigma_Y}{\sigma_X}\left(X-m_X\right)\right]^2\right).$$

$$= \left(1-\rho_{XY}^2\right)\sigma_Y^2$$

(iii) The above implies that the estimator $\hat{Y}$ determines Y exactly if $\rho_{XY}^2 = 1$.

Q2 Similar to problem solved in lecture.

a)

(i) Direct translation of the definitions

(ii)

$\mathbb{P}(X_{t+\tau}=1 \mid X_t=+1) = \mathbb{P}(X_{t+\tau}=-1 \mid X_t=-1)$

$= \mathbb{P}(\exists$ even number of crossings between t & $t+\tau)$

$$= e^{-\lambda\tau} \sum_{k\geq 0} \frac{(\lambda\tau)^{2k}}{2k!} \left(= e^{-\lambda\tau}\cosh(\lambda\tau)\right).$$

(iii) Similarly

$\mathbb{P}(X_{t+\tau}=1 \mid X_t=-1) = \mathbb{P}(X_{t+\tau}=-1 \mid X_t=1)$

$= \mathbb{P}(\exists$ odd number of crossings in $(t,t+\tau))$

$$= e^{-\lambda\tau} \sum_{k\geq 0} \frac{(\lambda\tau)^{2k+1}}{(2k+1)!} \left(= e^{-\lambda\tau}\sinh(\lambda\tau)\right)$$

(iv) $\mathbb{E}(X_t) = \mathbb{E}(X_t \mid X_0=1)\,\mathbb{P}(X_0=1) + \mathbb{E}(X_t \mid X_0=-1)\,\mathbb{P}(X_0=-1)$

$= 1/2\left(\mathbb{E}(X_t \mid X_0=1) + \mathbb{E}(X_t \mid X_0=-1)\right).$

$\mathbb{E}(X_t \mid X_0=1) = \mathbb{P}(X_t=1 \mid X_0=1) - \mathbb{P}(X_t=-1 \mid X_0=1)$

$$= e^{-\lambda t}\sum_{k\geq 0} (\lambda t)^{2k}/2k! - e^{-\lambda t}\sum_{k\geq 0} \frac{(\lambda t)^{2k+1}}{(2k+1)!}$$

$$= e^{-\lambda t}\left(\sum_{k\geq 0} (-1)^k (\lambda t)^k/k!\right)$$

$$= e^{-2\lambda t}.$$

Q2

(4)

a) (1)

$$\mathbb{E}(X_t \mid X_0=-1) = \mathbb{P}(X_t=1 \mid X_0=-1) - \mathbb{P}(X_t=-1 \mid X_0=-1)$$

$$= e^{-\lambda t}\sum_{k\geq 0}\frac{(\lambda t)^{2k+1}}{(2k+1)!} - e^{-\lambda t}\sum_{k\geq 0}\frac{(\lambda t)^{2k}}{(2k)!}$$

$$= -e^{-2\lambda t}.$$

$$\Rightarrow \quad \mathbb{E}(X_t)=0 \quad . \quad \forall t$$

b)

(i) $R_X(\tau) = \mathbb{E}(X_{t+\tau}\, X_t)$.

(ii) $$R_X(\tau) = \mathbb{E}(X_{t+\tau} X_t \mid X_t=1)\,\mathbb{P}(X_t=1) + \mathbb{E}(X_{t+\tau} X_t \mid X_t=-1)\,\mathbb{P}(X_t=-1)$$

$$= \mathbb{E}(X_{t+\tau} \mid X_t=1)\,\mathbb{P}(X_t=1) - \mathbb{E}(X_{t+\tau} \mid X_t=-1)\,\mathbb{P}(X_t=-1)$$

As previously $= e^{-2\lambda\tau}\,\mathbb{P}(X_t=1) + e^{-2\lambda\tau}\,\mathbb{P}(X_t=-1)$

(iii) X_t is $\{1,-1\}$-valued $\Rightarrow \mathbb{P}(X_t=1)+\mathbb{P}(X_t=-1)=1$

$$\Rightarrow \quad R_X(\tau) = e^{-2\lambda\tau}.$$

c) X_t is such that $\mathbb{E}(X_t)=0$ &
$\mathbb{E}(X_{t+\tau} X_t) = e^{-2\lambda\tau} \quad \forall t,\tau$

$\Rightarrow$ X_t is a wide sense stationary process.

Q3

a) X is Binomial with parameters n & $1/2$

$$\mathbb{P}(X=k) = \binom{n}{k} \left(1/2\right)^n.$$

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_i X_i\right) = n\,\mathbb{E}(X_1) = np = n/2$$

$$\mathrm{Var}(X) = n\,\mathrm{Var}(X_1) = np(1-p) \quad (X_i \text{ independent})$$
$$= n/4$$

b).

(i) The event $\{|X - n/2| \geq n/4\} = \{X \geq \frac{3n}{4}\} \cup \{X \leq n/4\}$

which contains the event $\{X \geq 3n/4\}$

$$\Rightarrow \mathbb{P}(|X - n/2| \geq n/4) \geq \mathbb{P}(X \geq \tfrac{3n}{4}).$$

(ii) By Chebyshev,

$$\mathbb{P}(X \geq \tfrac{3n}{4}) \leq \mathbb{P}(|X - n/2| \geq \tfrac{n}{4})$$
$$\leq \mathrm{Var}(X) \Big/ (n/4)^2$$
$$= \frac{4}{n}.$$

(iii) $$\lim_{n\to\infty} \mathbb{P}(X \geq \tfrac{3n}{4}) \leq \lim_{n\to\infty} 4/n \to 0$$

c) By Markov

$$\mathbb{P}(X \geq x) = \mathbb{P}(e^{\theta X} \geq e^{\theta x}) \leq \mathbb{E}(e^{\theta X})\, e^{-\theta x}.$$

$$\mathbb{E}(e^{\theta X}) = \left(1/2 + 1/2\, e^{\theta}\right)^n = \left(1 + 1/2\,(e^{\theta}-1)\right)^n$$

$$\leq \exp\left\{ \frac{n}{2}(e^{\theta}-1) \right\}.$$

Hence, $\mathbb{P}(X \geq x) \leq \exp\left(\frac{n}{2}(e^{\theta}-1) - \theta x \right).$

In our example $x = 3n/4$.

$$\mathbb{P}\left(X \geq \frac{3n}{4}\right) \leq \exp\left\{ \frac{n}{2}(e^{\theta}-1) - \frac{3n}{4}\theta \right\}.$$

(ii) For $\theta = \log 2$ $\quad \frac{1}{2}(e^{\theta}-1) - \frac{3}{4}\theta \simeq -0.019 \leq -0.01.$

(iii) Putting everything together we get

$$\mathbb{P}\left(X \geq \frac{3n}{4}\right) \leq \exp\{-0.01\, n\}. \xrightarrow[n\to\infty]{} 0$$

Q4

a)

Joint distribution.

Y_{2k} \ Y_{2k+2}	1	-1
1	1/4	1/4
-1	1/4	1/4

Marginal distribution. $\mathbb{P}(Y_{2k}=1) = \mathbb{P}(Y_{2k}=-1) = 1/2$
& $\mathbb{P}(Y_{2k+2}=1) = \mathbb{P}(Y_{2k+2}=-1) = 1/2$.

(Y_{2k}) is iid.

b)

Joint distribution

Y_{2k} \ Y_{2k+1}	1	-1
1	1/4	1/4
-1	1/4	1/4

& $\mathbb{P}(Y_{2k}=1) = \mathbb{P}(Y_{2k}=-1) = 1/2$

and $\mathbb{P}(Y_{2k+1}=1) = \mathbb{P}(Y_{2k+1}=-1) = 1/2$.

The sequence is pairwise independent but not iid.

as Y_{2k}, Y_{2k+1} & Y_{2k-1} are not independent by construction.

Q4

c) $\mathbb{P}(Y_{2k+1}=1 / Y_{2k}=-1)=1/2$

$\mathbb{P}(Y_{2k+1} \mid Y_{2k}=1 \ \& \ Y_{2k-1}=-1)=0.$

Which ensures that Y_k is not a Markov chain.

(d) $Z_n=(Y_n, Y_{n+1})$; $S=\{-1,+1\}^2$

We need to distinguish between n even and n odd.

(1) $n=2k$ $\quad Z_{2k}=(Y_{2k}, Y_{2k+1})$

$Z_{2k+1}=(Y_{2k+1}, Y_{2k+2}).$

* $Z_{2k}=(-1,-1) \Rightarrow Y_{2k}=-1$; $Y_{2k+1}=-1.$

Hence Z_{2k+1} must be of the form $(-1, Y_{2k+2})$.

& $Y_{2k+2}=Y_{2k+1}Y_{2k+3}=-Y_{2k+3}=\begin{cases} 1 & \text{w. p } 1/2 \\ -1 & \text{w. p. } 1/2. \end{cases}$

• $\mathbb{P}(Z_{2k+1}=(-1,1) \mid Z_{2k}=(-1,-1))=\mathbb{P}(Z_{2k+1}=(-1,-1) \mid Z_k=(-1,-1))$

$=1/2.$

Q 4

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* $Z_{2k} = (-1, 1) \Rightarrow Y_{2k} = -1 \quad , \quad Y_{2k+1} = 1.$

$Z_{2k+1} = (1, Y_{2k+2})$

$$Y_{2k+2} = Y_{2k+1} Y_{2k+3} = Y_{2k+3} = \begin{cases} 1 & \text{w.p } 1/2 \\ -1 & \text{w.p } 1/2. \end{cases}$$

. $\mathbb{P}(Z_{2k+1} = (+1, 1) \mid Z_{2k} = (-1, 1)) = \mathbb{P}(Z_{2k+1} = (1, -1) \mid Z_{2k} = (-1$

$= 1/2.$

* $Z_{2k} = (1, -1) \Rightarrow Y_{2k} = 1, \quad Y_{2k+1} = -1.$

$Z_{2k+1} = (-1, Y_{2k+2}).$

$\mathbb{P}(Z_{2k+1} = (-1, +1) \mid Z_{2k} = (1, -1)) = \mathbb{P}(Z_{2k+1} = (-1, -1) \mid Z_{2k} = (1, -1))$

$= 1/2.$

* $Z_{2k} = (1, 1) \Rightarrow Y_{2k} = 1, \quad Y_{2k+1} = 1.$

$Z_{2k+1} = (1, Y_{2k+2}).$

$\mathbb{P}(Z_{2k+1} = (1, 1) \mid Z_{2k} = (1, 1)) = \mathbb{P}(Z_{2k+1} = (1, -1) \mid Z_{2k} = (1, 1))$

$= 1/2.$

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Q4

$n = 2k+1$

$$Z_{2k+1} = (Y_{2k+1}, Y_{2k+2}) \qquad Z_{2k+2} = (Y_{2k+2}, Y_{2k+3})$$

Recall $Y_{2k+2} = Y_{2k+1} Y_{2k+3}$.

given Z_{2k+1}, Z_{2k+2} is completely determined.

$$\mathbb{P}(Z_{2k+2} = (-1,1) \mid Z_{2k+1} = (-1,1)) = 1$$

$$\mathbb{P}(Z_{2k+2} = (1,-1) \mid Z_{2k+1} = (-1,1)) = 1$$

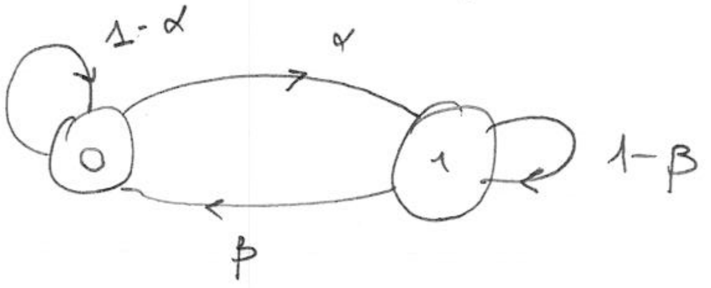
$$\mathbb{P}(Z_{2k+2} = (-1,-1) \mid Z_{2k+1} = (1,-1)) = 1$$

$$\mathbb{P}(Z_{2k+2} = (1,1) \mid Z_{2k+1} = (1,1)) = 1.$$

This completely determins the transition probabilitie of Z_n.

Q5.

a)



b)

(i) $p_{00}(n+1) = \mathbb{P}(X_{n+1} = 0 \mid X_0 = 0)$

$$= p_{01}^{(n)} \, p_{10} + p_{00}^{(n)} \, p_{00}$$

$$= \beta \, p_{01}^{(n)} + (1-\alpha) \, p_{00}^{(n)}$$

Since $p_{00}^{(n)} + p_{01}^{(n)} = 1.$

$$p_{00}(n+1) = \left(1 - p_{00}^{(n)}\right)\beta + (1-\alpha) \, p_{00}(n)$$

$$= (1-\alpha-\beta) \, p_{00}(n). \quad + \beta$$

(ii) By induction

(iii) $p_{01}^{(n)} = 1 - p_{00}(n) = \dfrac{\alpha}{\alpha+\beta} + \dfrac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n.$

By symmetry

$$p_{11}(n) = \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^n$$

$$p_{10}(n) = \frac{\beta}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^n.$$

Q5 (12)

$\alpha \,|\, 1-(\alpha+\beta) \,|$

b) (iv)

$$\lim_{n} p_{00}(n) = \lim_{n\to\infty} p_{10}(n) = \frac{\beta}{\alpha+\beta}$$

$$\lim_{n} p_{11}(n) = \lim_{n\to\infty} p_{01}(n) = \frac{\alpha}{\alpha+\beta}.$$

c) Use the result b) (iv) which gives stationary distribution $\Rightarrow \pi_1 = \beta/\alpha+\beta$, $\pi_2 = \frac{\alpha}{\alpha+\beta}$

or solve the invariant distribution equation

$$\pi P = \pi \qquad \pi_1 + \pi_2 = 1$$

$$\Rightarrow \begin{cases} \pi_1(1-\alpha) + \beta\pi_2 = \pi_1 \\ \alpha\pi_1 + (1-\beta)\pi_2 = \pi_2. \\ \pi_1 + \pi_2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \pi_2 = \frac{\alpha}{\beta}\pi_1 \\ \pi_1 + \pi_2 = 1 \end{cases} \Rightarrow \begin{cases} \pi_1 = \beta/\alpha+\beta \\ \pi_2 = \alpha/\alpha+\beta \end{cases}$$

Q6 Similar Problem solved in lecture.

a)

(i) starting from 2 we jump to 1 w.p. 1/2 and to 2. w.p. 1/2.

$$h_2 = 1/2\, h_1 + 1/2\, h_3.$$

Similarly for a start at 3

$$h_3 = 1/2\, h_2 + 1/2\, h_4.$$

(ii) as 1 is an absorbing state if we start from 1 we stay there

& $h_1 = 0$.

It is clear that $h_4 = 1$.

$$h_2 = 1/2\, h_3 = 1/2\left(1/2\, h_2 + \frac{1}{2}\right)$$

$$\Rightarrow \quad h_2 = 1/3.$$

& $h_3 = 2/3$.

Q6

b)

(i) Starting from 2 after one jump we jump to 1 w.p. 1/2 or to 3 w.p. 1

k_2 The average time to be absorbed at 1 or 4 starting from 2 is therefore given by

$$k_2 = 1 + 1/2\, k_3 + 1/2\, k_1$$

Similarly

$$k_3 = 1 + 1/2\, k_2 + 1/2\, k_4$$

(ii) It is easily seen that $k_1 = k_4 = 0$.

$$k_2 = (1 + 1/2\, k_3) = 1 + 1/2\,(1 + 1/2\, k_2)$$

$$\Rightarrow \quad k_2 = 2$$

& so $k_3 = 2$