

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING  
EXAMINATIONS 2011

# DISCRETE-TIME SYSTEMS AND COMPUTER CONTROL

Time allowed: 3:00 hours

**Answer FOUR questions.**

**Any special instructions for invigilators and information for candidates are on page 1.**

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## DTS AND COMPUTER CONTROL

Information for candidates:

$$- Z\left(\frac{1}{s}\right) = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$$

$$- Z\left(\frac{1}{s+a}\right) = \frac{z}{z-e^{-aT}} = \frac{1}{1-z^{-1}e^{-aT}}$$

$$- Z\left(\frac{1}{s^2}\right) = T \frac{z}{(z-1)^2} = T \frac{z^{-1}}{(1-z^{-1})^2}$$

$$- Z\left(\frac{1}{s^3}\right) = \frac{T^2}{2} \frac{z(z+1)}{(z-1)^3} = \frac{T^2}{2} \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$$

$$- \text{Transfer function of the ZOH: } H_0(s) = \frac{1-e^{-sT}}{s}$$

$$- \text{Definition of the } w\text{-plane: } z = \frac{1 + \frac{wT}{2}}{1 - \frac{wT}{2}}, w = \frac{2}{T} \frac{z-1}{z+1}$$

- Note that, for a given signal  $r$ , or  $r(t)$ ,  $R(z)$  denotes its Z-transform.

1. Consider the digital control system in Figure 1 with  $K$  constant.

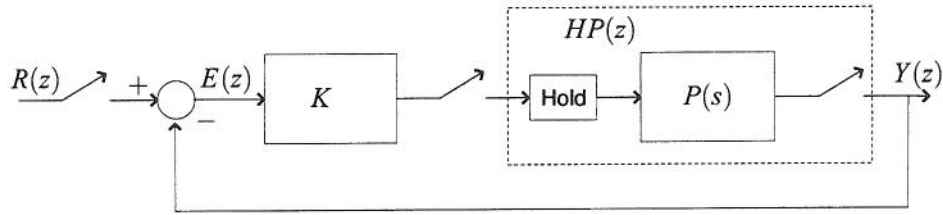


Figure 1: Block diagram for question 1.

Let

$$P(s) = \frac{1}{s(s+1)},$$

and  $K > 0$ . Assume the hold is a ZOH and let the sampling period be  $T > 0$ .

- Compute the equivalent discrete-time model  $HP(z)$  for the plant interconnected to the hold and the sampler. [ 4 marks ]
- Write the closed-loop discrete-time transfer function from the input  $r$  to the output  $y$ . [ 4 marks ]
- Show that there exists a function  $\kappa(T)$  such that the closed-loop system is asymptotically stable for all  $K \in (0, \kappa(T))$ . Show that

$$\lim_{T \rightarrow 0} \kappa(T) = \infty.$$

[ 6 marks ]

- Assume that  $r$  is a constant. Determine the steady-state values of  $e(kT)$ . Explain why this value does not depend upon  $K$ . [ 2 marks ]
- Assume that  $r$  is a unity ramp. Determine the steady-state values of  $e(kT)$ . Compare this result with the result of part d). [ 4 marks ]

2. Consider the digital control system in Figure 2.

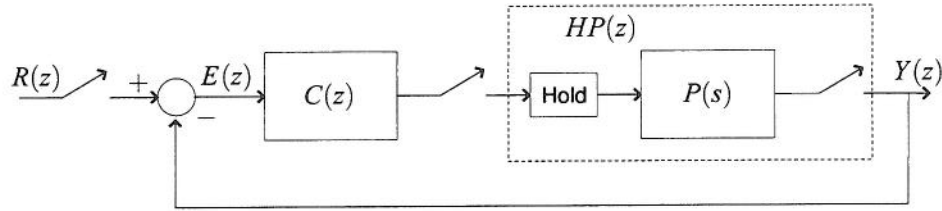


Figure 2: Block diagram for question 2.

Let

$$P(s) = \frac{K}{s} \frac{1}{\tau s + 1},$$

with  $\tau \geq 0$ . The term  $\frac{1}{\tau s + 1}$  describes unmodelled dynamics. Assume the hold is a ZOH and let the sampling period be  $T = 1$  sec.

- a) Compute the equivalent discrete-time model  $HP(z)$  for the plant interconnected to the hold and the sampler. [ 4 marks ]
- b) Assume  $\tau = 0$ .
  - i) Study the stability properties of the closed-loop system as a function of  $K$ . [ 2 marks ]
  - ii) Determine the value of  $K$  such that the closed-loop transfer function has all poles at  $z = 0$ . [ 2 marks ]
- c) Assume  $\tau > 0$  and let  $K$  be as in part b.ii).
  - i) Study the stability properties of the closed-loop system as a function of  $\tau$ . [ 4 marks ]
  - ii) Design a controller
 
$$C(z) = \alpha \frac{z - e^{-1/\tau}}{z + \beta}$$
 such that the closed-loop system has all poles at  $z = 0$ . [ 4 marks ]
  - iii) Consider the controller designed in part c.ii) in closed-loop with the discrete-time model of the system for  $\tau = 0$ . Study the stability properties of the closed-loop system as a function of  $\tau \geq 0$ . [ 2 marks ]
- d) Discuss the robustness, to variations in  $\tau$ , of the controllers designed in part b.ii) and in part c.ii). [ 2 marks ]

3. The transfer function of a simple mechanical system is given by

$$P(s) = \frac{1}{s^2}.$$

Assume the system is interconnected to a ZOH and a sampler. Let  $T = 1$  sec be the sampling time.

- a) Compute the equivalent discrete-time model  $HP(z)$  for the plant interconnected to the hold and the sampler. [ 4 marks ]
- b) Using the definition of the  $w$ -plane, determine the transfer function  $HP(w)$ . [ 2 marks ]
- c) Design, in the  $w$ -plane, a controller  $C(w)$  such that the closed-loop system is asymptotically stable. [ 8 marks ]
- d) Compute the transfer function  $C(z)$  of the discrete-time controller. Verify if the discrete-time closed-loop system resulting from the use of  $C(z)$  is asymptotically stable. [ 6 marks ]

4. Consider a system with transfer function

$$P(s) = \frac{1}{s+1}$$

and the problem of designing a feedback controller such that the closed-loop system is asymptotically stable and the system is of type 1.

- a) Show that the PI control law

$$C(s) = 1 + \frac{2}{s}$$

is such that the closed-loop system is asymptotically stable. Compute the poles of the closed-loop system and, using the transformation  $z = e^{sT}$ , with  $T = 1$ , determine the location of these poles in the complex  $z$ -plane. [ 4 marks ]

- b) Discretize the controller in part a) using the backward difference and the Tustin transformations with sampling time  $T = 1$ . Let  $C_B(z)$  and  $C_T(z)$  be the resulting controllers. [ 2 marks ]
- c) Compute the equivalent discrete-time model  $HP(z)$  for the plant interconnected to a ZOH and a sampler. Let  $T = 1$  sec be the sampling time. [ 2 marks ]
- d) Consider the discrete-time model in part c) in closed-loop with the controller  $C_B(z)$  determined in part b). Compute the poles of the closed-loop system. [ 4 marks ]
- e) Repeat part d) using the controller  $C_T(z)$ . [ 4 marks ]
- f) Using the results in parts a), d) and e) compare the three designs in terms of locations of the closed-loop poles and performance of the resulting closed-loop systems. [ 4 marks ]

5. Consider a feedback system with open-loop transfer function

$$HP(z) = k \frac{\frac{z}{10} + \frac{11}{125}}{z^2 - \frac{143}{100}z + \frac{3}{5}}$$

with  $k > 0$ .

- a) Let  $k = 1$ . Design a controller  $C(z)$  such that the closed-loop system transfer function

$$\frac{C(z)HP(z)}{1 + C(z)HP(z)}$$

has only two poles at  $z = 0$ .

[ 10 marks ]

- b) To assess the robustness of the design in the presence of variations in the gain  $k$ , consider the feedback interconnection of the discrete-time model determined in part a) with the controller determined in part b). Study the stability of the resulting closed-loop system as a function of  $k > 0$ . [ 6 marks ]
- c) Discuss briefly the results in part b). [ 4 marks ]

6. Consider a continuous-time system described by the transfer function

$$P(s) = e^{-s} \frac{1}{s-1}.$$

- a) Assume the system is connected to a ZOH and a sampler. Let  $T = 1$  sec be the sampling period. Determine the discrete-time equivalent transfer function  $HP(z)$ . [ 4 marks ]
- b) Design a digital controller  $C(z)$  such that the closed-loop system is asymptotically stable. [ 6 marks ]
- c) Suppose now that the continuous-time system is connected to a ZOH and a sampler with  $T = 1/2$  sec. Let the controller be as in part b).
- i) Determine the discrete-time equivalent transfer function  $HP(z)$ . [ 2 marks ]
  - ii) Compute the characteristic polynomial of the resulting closed-loop system. [ 4 marks ]
  - iii) Study the stability of the polynomial determined in part c.ii). [ 4 marks ]



# DTS and Computer Control

## Model answers 2011

### Question 1

a) Note that

$$\begin{aligned} H P(z) &= (1 - z^{-1}) Z \left( \frac{1}{s^2(s+1)} \right) \\ &= (1 - z^{-1}) Z \left( \frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2} \right) \\ &= \frac{z(T-1+e^{-T}) + (1-e^{-T}-Te^{-T})}{(z-1)(z-e^{-T})} \end{aligned}$$

b) The closed-loop transfer function from  $r$  to  $y$  is

$$\begin{aligned} W_{ry}(z) &= \frac{K H P(z)}{1 + K H P(z)} \\ &= K \frac{z(T-1+e^{-T}) + (1-e^{-T}-Te^{-T})}{z^2 + z(-K-1+KT+e^{-T}K-e^{-T}) + (K+e^{-T}-e^{-T}KT-e^{-T}K)}. \end{aligned}$$

c) The closed-loop system is asymptotically stable if the roots of the denominator polynomial of  $W_{ry}(z)$  are inside the unity disk. To test this condition let

$$a = -K - 1 + KT + e^{-T}K - e^{-T} \quad b = K + e^{-T} - e^{-T}KT - e^{-T}K$$

and recall that the roots of the polynomial  $z^2 + az + b$  are inside the unity disk if and only if

$$1 + a + b > 0 \quad 1 - a + b > 0 \quad |b| < 1.$$

These conditions, for the considered polynomial, are

$$\begin{aligned} KT \overbrace{(1-e^{-T})}^{>0} &> 0, \quad 2 + 2e^{-T} - \overbrace{(2e^{-T} - 2 + T + Te^{-T})}^{>0} K > 0, \\ -1 < e^{-T} + \overbrace{(1-e^{-T}-Te^{-T})}^{>0} K &< 1. \end{aligned}$$

As a result  $K$  should be such that

$$0 < K < \min \left\{ \frac{1-e^{-T}}{1-e^{-T}-Te^{-T}}, \frac{2+2e^{-T}}{2e^{-T}-2+T+Te^{-T}} \right\} = \kappa(T).$$

Note that both functions inside the min go to infinity as  $T$  goes to zero.

d) The steady-state value of  $e(kT)$  is zero, since the system is of type 1, that is the open loop transfer function has a pole at  $z = 1$ .

e) The velocity constant is  $k_v = \lim_{z \rightarrow 1} \frac{(1-z^{-1})KHP(z)}{T} = K$ , hence the steady state error is

$$e(kT) = \frac{1}{K} < \frac{1}{\kappa(T)}.$$

Note that the velocity error cannot be reduced arbitrarily, since  $K$  is upperbounded by  $\kappa(T)$  for stability.

## Question 2

a) Note that

$$\begin{aligned}
 HP(z) &= K(1 - z^{-1})Z\left(\frac{1}{s^2(\tau s + 1)}\right) \\
 &= K(1 - z^{-1})Z\left(\frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau}{s + 1/\tau}\right) \\
 &= -K \frac{z(\tau - 1 - \tau e^{-1/\tau}) + (\tau e^{-1/\tau} + e^{-1/\tau} - \tau)}{(z - 1)(z - e^{-1/\tau})}.
 \end{aligned}$$

b) For  $\tau = 0$  the discrete-time model is

$$HP_0(z) = \frac{K}{z - 1}.$$

i) The characteristic polynomial of the closed-loop system is

$$z - 1 + K,$$

hence the close-loop system is asymptotically stable for

$$0 < K < 2.$$

ii) The closed-loop poles are all at  $z = 0$  for  $K = 1$ .

c) The characteristic polynomial of the closed-loop system, with  $\tau \neq 0$  and  $K = 1$ , is

$$z^2 + z(e^{-1/\tau} - \tau - e^{-1/\tau}) + \tau(1 - e^{-1/\tau}).$$

i) The closed-loop characteristic polynomial has all roots inside the unity disk for all  $\tau > 0$ .

ii) Let

$$C(z) = \alpha \frac{z - e^{-1/\tau}}{z + \beta}.$$

The closed-loop characteristic polynomial is

$$z^2 + z(\alpha\tau e^{-1/\tau} - \alpha\tau + \alpha + \beta - 1) + (-\alpha\tau e^{-1/\tau} - \alpha e^{-1/\tau} + \alpha\tau - \beta).$$

Setting

$$\alpha = \frac{1}{1 - e^{-1/\tau}} \quad \beta = \frac{\tau e^{-1/\tau} + e^{-1/\tau} - \tau}{e^{-1/\tau} - 1}$$

yields the closed-loop characteristic polynomial  $z^2$ .

iii) Consider the controller designed in part c.ii) in closed-loop with the discrete-time model of the system with  $\tau = 0$ , that is  $HP_0(z)$ . The closed-loop characteristic polynomial is

$$z^2 + \tau z - \tau,$$

and this has all roots inside the unity circle if, and only if,  $\tau \in [0, 1/2)$ .

d) The controller designed on the model with  $\tau = 0$  stabilizes, for any  $\tau \geq 0$ , the model that includes the unmodelled dynamics. The controller designed for  $\tau \neq 0$  does not stabilize, for all  $\tau$ , the model with  $\tau = 0$ . As a result, the controller designed on the basis of the model for  $\tau = 0$  is more robust.



### Question 3

a) Note that

$$\begin{aligned} HP(z) &= (1 - z^{-1})Z\left(\frac{1}{s^3}\right) \\ &= \frac{1}{2} \frac{z+1}{(z-1)^2} \end{aligned}$$

b) The transfer function in the  $w$ -plane is given by

$$HP(w) = HP(z) \Big|_{z=\frac{1+w/2}{1-w/2}} = \frac{1}{2} \frac{2-w}{w^2}.$$

c) Let, for example,

$$C(w) = \frac{\alpha w + \beta}{w + \delta}$$

and select  $\alpha$ ,  $\beta$  and  $\delta$  such that all closed-loop poles are at  $w = -1$ . (Other designs are possible.) The closed-loop characteristic polynomial is

$$2w^3 + w^2(2\delta - \alpha) + w(2\alpha - \beta) + 2\beta.$$

Selecting

$$\alpha = \frac{7}{2} \qquad \beta = 1 \qquad \delta = \frac{19}{4}$$

yields the polynomial  $2(w+1)^3$ . The resulting controller is

$$C(w) = \frac{\frac{7}{2}w + 1}{w + \frac{19}{4}}.$$

d) The discrete-time controller is

$$C(z) = C(w) \Big|_{w=2\frac{z-1}{z+1}} = 8 \frac{4z-3}{27z+11}.$$

The characteristic polynomial of the discrete-time closed-loop system is

$$27z^3 - 27z^2 + 9z - 1 = (3z-1)^3,$$

hence the discrete-time closed-loop system is asymptotically stable.

## Question 4

- a) The closed-loop characteristic polynomial is

$$s^2 + 2s + 2$$

with roots  $s = -1 \pm I$ , hence the closed-loop system is stable. These roots are mapped in the  $z$ -plane to

$$e^{-1}e^{\pm I} \approx 0.198 \pm 0.31I$$

Finally, the system is of type 1 because of the integrator in  $C(s)$ .

- b) The controllers are given by

$$C_B(z) = C(s) \Big|_{s=1-\frac{1}{z}} = \frac{3z-1}{z-1}$$

and

$$C_T(z) = C(s) \Big|_{s=2\frac{z-1}{z+1}} = 2\frac{z}{z-1}.$$

Both controllers have a pole at  $z = 1$ , hence the resulting closed-loop systems are of type 1.

- c) The equivalent discrete-time model is

$$\begin{aligned} HP(z) &= (1 - z^{-1})Z\left(\frac{1}{s(s+1)}\right) \\ &= \frac{1 - e^{-1}}{z - e^{-1}}. \end{aligned}$$

- d) The poles of the closed-loop system are the roots of the characteristic polynomial

$$z^2 + z(2 - 4e^{-1}) + 2e^{-1} - 1,$$

namely

$$z_{B1} \approx -0.842 \quad z_{B2} \approx 0.313.$$

- e) The poles of the closed-loop system are the roots of the characteristic polynomial

$$z^2 + z(1 - 3e^{-1}) + e^{-1},$$

namely

$$z_T \approx 0.05 \pm 0.604I.$$

- f) All three designs yields stabilizing controllers. The discrete-time equivalent poles of the continuous-time design are complex conjugate with imaginary part 1.5 times the real part, giving a good damping coefficient. The backward difference based controller yields two real poles, one positive and one negative. The negative pole is *slower* than the equivalent poles of the continuous-time design and introduces *spurious* oscillations. The Tustin based controller yields two complex conjugate poles with imaginary part twelve times bigger than the real part. These poles are slower than the equivalent continuous-time poles and have a worse damping coefficient.

## Question 5

a) Let

$$C(z) = \frac{\alpha z + \beta}{z + \delta}$$

and note that since the desired closed-loop system has only two poles, the controller has to cancel the zero of  $HP(z)$ . This is achieved setting  $\delta = 22/25$ . The closed-loop system is then given by

$$\frac{C(z)HP(z)}{1 + C(z)HP(z)} = 10 \frac{\alpha z + \beta}{100z^2 + z(10\alpha - 143) + (60 + 10\beta)}.$$

Selecting  $\alpha = 143/10$  and  $\beta = -6$  yields

$$\frac{C(z)HP(z)}{1 + C(z)HP(z)} = \frac{143z - 60}{100z^2}$$

as requested.

b) The characteristic polynomial of the closed-loop system is

$$100z^2 + 143z(k - 1) + 60(k - 1).$$

and this has all roots inside the unity disk provided  $k \in [0, 303/203)$ .

c) The results in part c) shows that the proposed design yields some robustness in terms of *gain margin*, but it is not *very* robust: if the gain increases by 50% the closed-loop system is unstable.

## Question 6

a) The discrete-time equivalent transfer function is

$$HP(z) = Z \left( \frac{1 - e^{-s}}{s} e^{-s} \frac{1}{s-1} \right) = (1 - z^{-1}) \frac{1}{z} Z \left( \frac{1}{s(s-1)} \right) = \frac{e-1}{z(z-e)}.$$

b) Let (other selections are possible)

$$C(z) = k \frac{z}{z+e}.$$

The closed-loop characteristic polynomial is

$$z^2 - e^2 + k(e-1),$$

hence selecting

$$k = \frac{e^2}{e-1}$$

yields a closed-loop system with two poles at  $z = 0$ .

c) If  $T = 1/2$  sec, then the delay  $e^{-s}$  yields to poles at  $z = 0$ .

i) The discrete-time equivalent model is

$$HP(z) = Z \left( \frac{1 - e^{-s/2}}{s} e^{-s} \frac{1}{s-1} \right) = (1 - z^{-1}) \frac{1}{z^2} Z \left( \frac{1}{s(s-1)} \right) = \frac{e^{1/2} - 1}{z^2(z - e^{1/2})}.$$

ii) The characteristic polynomial of the resulting closed-loop system is

$$p(z) \approx 1.718z^3 + 1.83z^2 - 7.7z + 4.79.$$

iii) Applying the bilinear transformation to  $p(z)$  yields the polynomial

$$p(s) \approx 12.61w^3 - 25.39w^2 - 0.31w - 0.648.$$

Since all coefficients of the polynomial  $p(s)$  do not have the same sign, this polynomial is not stable. As a result, the closed-loop system is unstable.