IMPERIAL COLLEGE LONDON

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C1.6

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2008

MSc and EEE PART IV: MEng and ACGI

MODELLING AND CONTROL OF MULTI-BODY MECHANICAL SYSTEMS

Wednesday, 14 May 10:00 am

Time allowed: 3:00 hours

Corrected Copy

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks.

This is an OPEN BOOK examination.

Any special instructions for invigilators and information for candidates are on page 1.

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MODELLING AND CONTROL OF MULTIBODY MECHANICAL SYSTEMS

1. The body-fixed axes of a rigid body are initially aligned with an earth-fixed set of axes. The rotation of this body is represented by three Euler angles ψ, ϕ and θ in the 3-1-3 convention. In this convention the body is first rotated from its nominal configuration by an angle ψ about the z-axis, then by an angle ϕ about the intermediate x-axis of the body and finally by an angle θ about the new z-axis of the body.

    a) By making use of the standard single-axis-rotation transformation matrices, show that the complete transformation from body-fixed coordinates to earth-fixed coordinates is given by the matrix

$$\begin{bmatrix} \cos\psi\cos\theta - \sin\psi\cos\phi\sin\theta & -\cos\psi\sin\theta - \sin\psi\cos\phi\cos\theta & \sin\psi\sin\phi \\ \sin\psi\cos\theta + \cos\psi\cos\phi\sin\theta & -\sin\psi\sin\theta + \cos\psi\cos\phi\cos\theta & -\cos\psi\sin\phi \\ \sin\phi\sin\theta & \sin\phi\cos\theta & \cos\phi \end{bmatrix}$$

[10 marks]

    b) Show that the body angular velocity vector, $\boldsymbol{\Omega}$, in the earth-fixed coordinate system is given by

$$\begin{bmatrix} \dot{\phi}\cos\psi + \dot{\theta}\sin\psi\sin\phi \\ \dot{\phi}\sin\psi - \dot{\theta}\cos\psi\sin\phi \\ \dot{\psi} + \dot{\theta}\cos\phi \end{bmatrix}$$

[10 marks]

2. A thin, rigid, uniform circular disc rolls without slipping on a horizontal surface. This idealised mechanical system is sometimes referred to as the "rolling disc".

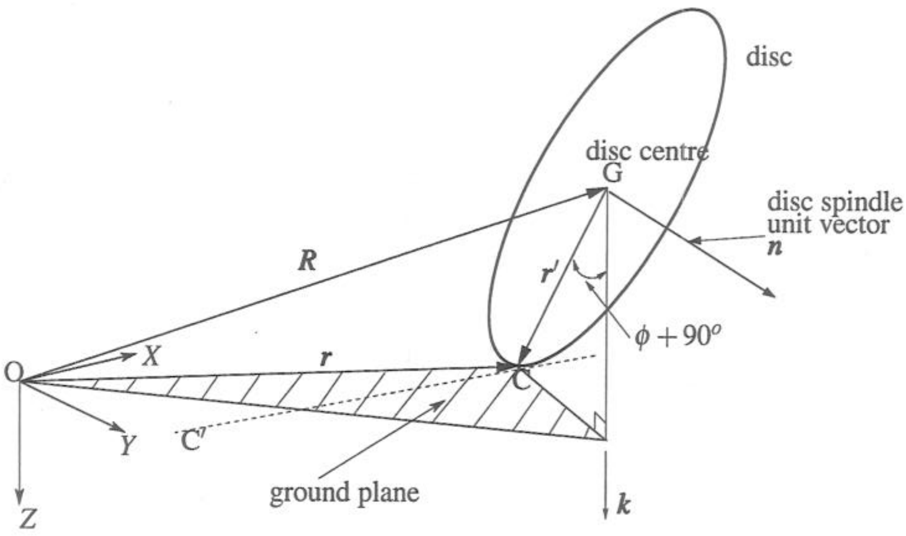

Figure 2.1 Rolling disc.

Unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are associated with the earth-fixed axis X, Y and Z. In its nominal (initial) configuration the disc lies flat on the ground plane (X-Y plane) with its spindle vector, $\boldsymbol{n}$, coinciding with the $\boldsymbol{k}$ direction. The rotation of the disc is represented by three Euler angles ψ, ϕ and θ in the 3-1-3 convention. In this convention the disc is first rotated from its nominal configuration by an angle ψ about the Z-axis, then by an angle ϕ about the intermediate X-axis of the disc and finally by an angle θ about the new Z-axis of the disc.

The complete transformation from disc-fixed coordinates to earth-fixed coordinates is given by the matrix

$$\begin{bmatrix} \cos\psi\cos\theta-\sin\psi\cos\phi\sin\theta & -\cos\psi\sin\theta-\sin\psi\cos\phi\cos\theta & \sin\psi\sin\phi \\ \sin\psi\cos\theta+\cos\psi\cos\phi\sin\theta & -\sin\psi\sin\theta+\cos\psi\cos\phi\cos\theta & -\cos\psi\sin\phi \\ \sin\phi\sin\theta & \sin\phi\cos\theta & \cos\phi \end{bmatrix},$$

and the disc angular velocity vector, $\boldsymbol{\Omega}$, in the earth-fixed coordinate system is given by

$$\boldsymbol{\Omega} = \begin{bmatrix} \dot{\phi}\cos\psi+\dot{\theta}\sin\psi\sin\phi \\ \dot{\phi}\sin\psi-\dot{\theta}\cos\psi\sin\phi \\ \dot{\psi}+\dot{\theta}\cos\phi \end{bmatrix}.$$

a) The translation of the centre of the disc is defined according to the position vector

$$\boldsymbol{R} = x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}.$$

The full motion of the disc is thus defined by three translational and three rotational coordinates which are subject to three kinematic constraints. Describe briefly the physical origins of these constraints. [1 mark]

b) At some arbitrary disc configuration, the plane of the disc intersects the ground along some line C′C (see Figure 2.1). Looking at the disc along this line, both the disc and the ground appear as straight lines as shown in Figure 2.2. The centre of the disc is G and the instantaneous point of contact is C.

i) Express the position vector $\boldsymbol{r}$ of point C, in terms of $\boldsymbol{R}$ and $\boldsymbol{r}'$, where $\boldsymbol{r}'$ is the radial vector from the disc centre to the contact point. [2 marks]

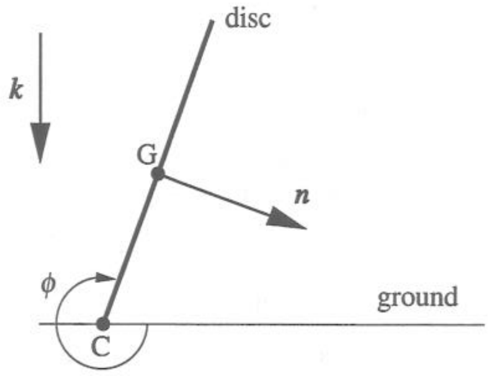

Figure 2.2 Side view of disc.

ii) Hence write the velocity vector of C in terms of $\boldsymbol{R}$, $\boldsymbol{r}'$ and their derivatives and the angular velocity vector of the disc, $\boldsymbol{\Omega}$. [3 marks]

iii) Use a vector operation between unit vectors $\boldsymbol{k}$ and $\boldsymbol{n}$ to define a vector $\boldsymbol{p}$ which is in the direction of the line C′C. Hence make use of the transformation matrix given earlier to express $\boldsymbol{n}$ in the earth-fixed coordinate system and show that

$$\boldsymbol{p} = -\cos\psi\sin\phi\,\boldsymbol{i} - \sin\psi\sin\phi\,\boldsymbol{j}.$$

[4 marks]

iv) Use vectors $\boldsymbol{p}$ and $\boldsymbol{n}$ to define vector $\boldsymbol{r}'$. The radius of the disc is a. Hence show that

$$\boldsymbol{r}' = a\sin\psi\cos\phi\,\boldsymbol{i} - a\cos\psi\cos\phi\,\boldsymbol{j} - a\sin\phi\,\boldsymbol{k}$$

[4 marks]

c) The constraints on the motion of the disc imply that the velocity of the contact point, C, is zero. By considering the three components of this velocity vector along the earth-fixed axes, derive the constraint equations. Show that one of the constraints is indeed a holonomic constraint. [6 marks]

3. A symmetric top which is not subject to any net forces or moments is in motion about its centre of mass. Its angular configuration is given by three Euler angles in the 3-1-3 convention. In this convention the body is first rotated from its nominal configuration by an angle ψ about the z-axis, then by an angle ϕ about the intermediate x-axis of the body and finally by an angle θ about the new z-axis of the body. In the nominal configuration the body's axis of symmetry is aligned with the z-axis.

The angular velocity vector of the body in the body-fixed coordinate system is given by

$$\begin{bmatrix} \dot{\psi} \sin\theta \sin\phi + \dot{\phi} \cos\theta \\ \dot{\psi} \cos\theta \sin\phi - \dot{\phi} \sin\theta \\ \dot{\psi} \cos\phi + \dot{\theta} \end{bmatrix}$$

a) Write the kinetic energy of the rigid body. [5 marks]

b) Using the Lagrangian approach show that

$$\dot{\psi} \cos\phi + \dot{\theta} = C_1,$$

$$\dot{\psi} = \frac{C_2 - I_3 C_1 \cos\phi}{I_1 \sin^2\phi},$$

where I_3 is the body moment of inertia about the axis of symmetry, I_1 is the moment of inertia of the rigid body about an axis which is perpendicular to the axis of symmetry and which passes through the centre of mass of the body; C_1 and C_2 are two first integrals (constants) of the motion. [8 marks]

c) A third constant of the motion is the kinetic energy since there are no external forces or moments applied on the system. Use the kinetic energy equation to derive a differential equation involving ϕ alone. Hence argue that *regular precession*, with the body symmetry axis rotating about the z-axis, can be a consequence of the derived equations. [7 marks]

4. The Foucault pendulum, shown in Figure 4.1, consists of a point mass suspended by a long wire, which is free to swing in <u>any</u> direction. Assume that the wire is massless and inextensible.

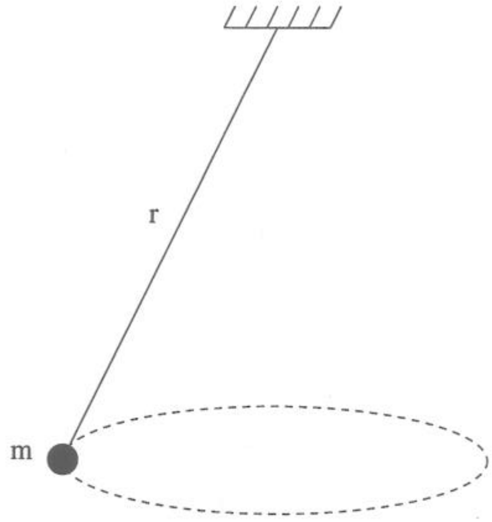

Figure 4.1 Foucault pendulum.

a) Derive the equations which describe the motion of the pendulum. (Hint: use spherical coordinates). [14 marks]

b) What is the force of constraint provided by the wire? [6 marks]

5. a) Find the inertia matrix of a uniform cube of mass m and side a with respect to an axis system which has its origin, O, at one corner of the cube and its axes along the three edges of the cube, as shown in Figure 5.1. [7 marks]

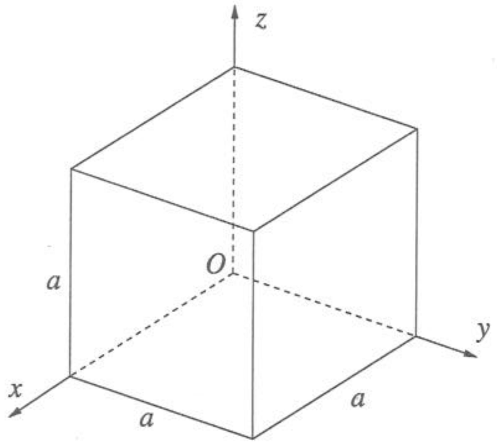

Figure 5.1 Uniform cube.

b) Use your result in part (a) to find the inertia matrix about a set of axes which is parallel to the one in (a) and which passes through the centre of mass of the cube. [7 marks]

c) Hence find the moment of inertia about an axis along the cube diagonal. [6 marks]

6. Consider a pendulum, as depicted in Figure 6.1. Assume that the pendulum can be modelled as a massless rod of length $l = 1$ with a mass M attached at its end. Assume the pendulum is controlled by an input torque u and the angle q is as in the figure.

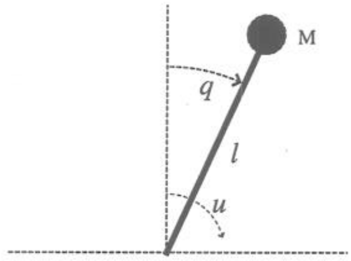

Figure 6.1 The pendulum.

a) Compute the kinetic and potential energy of the system, and the internal Hamiltonian $H_0(q, p)$. [4 marks]

b) Assume that the coupling Hamiltonian $H_1(q)$ is

$$H_1(q) = q.$$

Write the system in Hamiltonian form. [2 marks]

c) Assume that the torque u is constant. Compute the equilibrium points of the system as a function of u. [4 marks]

d) Show that the system with constant input torque can be written as a Hamiltonian system without input and with a modified potential energy. [2 marks]

e) Assume $u = k(q)$. Show that the system can be written as a Hamiltonian system with a modified internal Hamiltonian $\tilde{H}_0(q, p)$. Determine $k(q)$ such that the potential energy has a minimum for $q = q_\star$. [4 marks]

f) Consider the system with $u = k(q) + v$, where $k(q)$ is as in part e). Let $\tilde{H}_0(q, p)$ be the internal Hamiltonian determined in part e). Argue that, for $v = 0$, one has $\dot{\tilde{H}}_0 = 0$. Discuss how to select v to achieve the condition

$$\dot{\tilde{H}}_0 = -rp^2,$$

for some $r > 0$. Argue that this implies that all trajectories starting close to the equilibrium $(q_\star, 0)$ converge to this equilibrium. [4 marks]