EE4-47
EE9CS6-1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

MSc and EEE PART IV: MEng and ACGI

MODELLING AND CONTROL OF MULTI-BODY MECHANICAL SYSTEMS

Friday, 4 May 2:30 pm

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks.

This is an OPEN BOOK examination.

Any special instructions for invigilators and information for candidates are on page 1.

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Second Marker(s) : A. Astolfi, A. Astolfi

1. The body-fixed axes of a rigid body are initially aligned with an earth-fixed set of axes. The rotation of this body is represented by three Euler angles ψ, ϕ and θ in the yaw-roll-pitch convention. In this convention the body is first rotated from its nominal configuration by an angle ψ about the z-axis, then by an angle ϕ about the intermediate x-axis of the body and finally by an angle θ about the new y-axis of the body.

   a) By making use of the standard single-axis-rotation transformation matrices, show that the complete transformation from earth-fixed coordinates to body-fixed coordinates is given by the matrix

$$\begin{bmatrix} \cos\theta\cos\psi - \sin\phi\sin\theta\sin\psi & \cos\theta\sin\psi + \sin\phi\sin\theta\cos\psi & -\cos\phi\sin\theta \\ -\cos\phi\sin\psi & \cos\phi\cos\psi & \sin\phi \\ \sin\theta\cos\psi + \sin\phi\cos\theta\sin\psi & \sin\theta\sin\psi - \sin\phi\cos\theta\cos\psi & \cos\phi\cos\theta \end{bmatrix}$$

[10 marks]

   b) Write the body angular velocity vector, $\boldsymbol{\Omega}$, in terms of the Euler angles, in the body-fixed coordinate system. [10 marks]

2. Consider a mass m suspended from a spring of negligible mass as shown in Figure 2.1. Assume that m is free to move in a vertical plane under the action of gravity, the spring and two controlling forces $\boldsymbol{F}_r$ and $\boldsymbol{F}_\theta$. The unstretched length of the spring is r_0 and its stiffness is c.

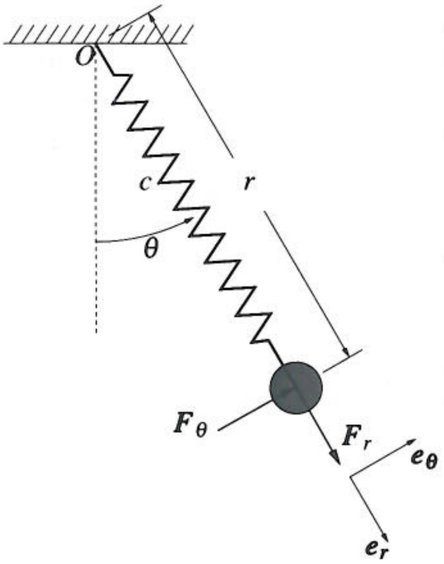

Figure 2.1 Springy pendulum.

A moving Cartesian coordinate system with unit vectors $\boldsymbol{e_r}$ and $\boldsymbol{e_\theta}$ is used to analyse the motion of the mass. This coordinate system has a fixed origin O but it rotates by an angle θ.

a) State the number of degrees of freedom of the system, and the associated generalised coordinates. [1]

b) Write the position vector, $\boldsymbol{r}$, of the mass in the moving coordinate system. [1]

c) Determine the velocity vector of the mass. [1]

d) Compute the total kinetic energy and potential energy of the mass, and hence determine the Lagrangian function. [3]

e) Use the Lagrangian approach to derive the equations of motion of the mass. [6]

f) Assume that the control force F_r is given by

$$F_r = -mr\dot{\theta}^2 - mg\cos\theta,$$

and that at time $t = 0$ the radial distance and radial velocity of the mass are given by $r = r_0 + \varepsilon$ and $\dot{r} = 0$, in which ε is a small quantity.

i) Determine the motion of the mass, $r(t)$, in the radial direction $\boldsymbol{e_r}$. [4]

ii) An extra term is added to the control force F_r to provide damping for the oscillations in the radial motion of the mass. Calculate this term. Hence determine an expression for the control force F_θ which will force the springy pendulum to behave like a simple pendulum. [4]

3. A uniform wheel of radius r_2 rolls without slipping on a circular surface (fixed to earth) of radius r_1 under the influence of gravity, as shown in Figure 3.1; the wheel is constrained to always touch the circular surface by a massless link (not shown) joining the centres of the wheel and the circular surface. The mass and spin inertia of the wheel are m and I respectively. A driving torque T_d is applied on the wheel and reacts on earth.

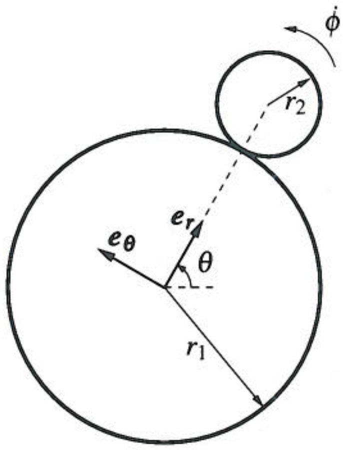

Figure 3.1 Wheel rolling on circular surface.

Polar unit vectors $\boldsymbol{e_r}$ and $\boldsymbol{e_\theta}$ are used to analyse the motion of the wheel. This coordinate system has a fixed origin at the centre of the circular surface but it rotates with the line joining the centres of the surface and the wheel, by an angle θ. The wheel rotates by an angle ϕ.

a) Write the position vector, $\boldsymbol{r}$, of the centre of mass of the wheel in the moving coordinate system. [1]

b) Determine the velocity vector of the centre of mass of the wheel. [1]

c) Compute the total kinetic energy and potential energy of the wheel, and hence determine the Lagrangian function. [3]

d) By considering the velocity of the instantaneous contact point between the wheel and the surface derive the equation of the rolling constraint. [3]

e) Use the Lagrangian approach to derive the equation of motion of the wheel in terms of the generalised coordinate θ. [8]

f) If the driving torque T_d reacts on the massless link holding the wheel on the circular surface (e.g. a motor is attached on the link and drives the wheel) then show that the equation of motion of the wheel is

$$\left(mr_2{}^2 + I\right)\ddot{\theta} + \frac{mgr_2{}^2}{r_1 + r_2}\cos\theta = \frac{r_2(r_1 + 2r_2)}{(r_1 + r_2)^2}T_d.$$

[4]

4. A uniform wheel of radius r_2 rolls without slipping on a circular surface (fixed to earth) of radius r_1 under the influence of gravity, as shown in Figure 4.1; the wheel is constrained to always touch the circular surface by a massless link (not shown) joining the centres of the wheel and the circular surface. The mass and spin inertia of the wheel are m and I respectively. A driving torque T_d is applied on the wheel and reacts on earth.

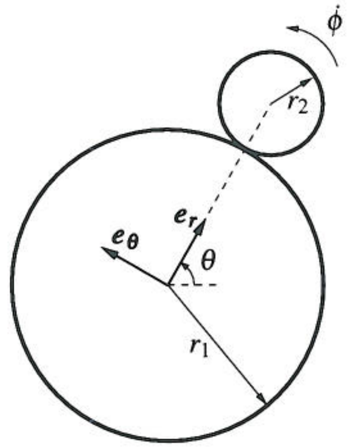

Figure 4.1 Wheel rolling on circular surface.

Polar unit vectors $\boldsymbol{e_r}$ and $\boldsymbol{e_\theta}$ are used to analyse the motion of the wheel. This coordinate system has a fixed origin at the centre of the circular surface but it rotates with the line joining the centres of the surface and the wheel, by an angle θ. The wheel rotates by an angle ϕ.

a) By considering the velocity of the instantaneous contact point between the wheel and the circular surface derive the equation of the rolling constraint. [3]

b) Determine the acceleration vector of the centre of mass of the wheel. [2]

c) Use the Newtonian approach to derive for the wheel:

i) The equation of motion with respect to the generalised coordinate θ. [6]

ii) The force of constraint which prevents the wheel from moving away from the circular surface. [3]

iii) The force of constraint which maintains the rolling constraint. [3]

d) Specify the type of motion the wheel will perform for small perturbations from the $\theta = -90°$ position when $T_d = 0$. [3]

5. Two particles of mass m each are attached at the two ends of a rigid rod of length l and of negligible mass that is free to rotate by an angle ψ about the vertical axis and by an angle θ about a horizontal axis which is perpendicular to the rod, as shown in Figure 5.1. Both axes of rotation pass through the centre of the rod. A moving Cartesian coordinate system attached to the rod with fixed origin O and with unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ is used to analyse the motion of the system. The $\boldsymbol{i}$ vector has a direction into the page at the instant shown and it is along the axis of the θ rotation. A moment N is applied on the rod in the $\boldsymbol{k}$ direction. The effect of gravity is neglected.

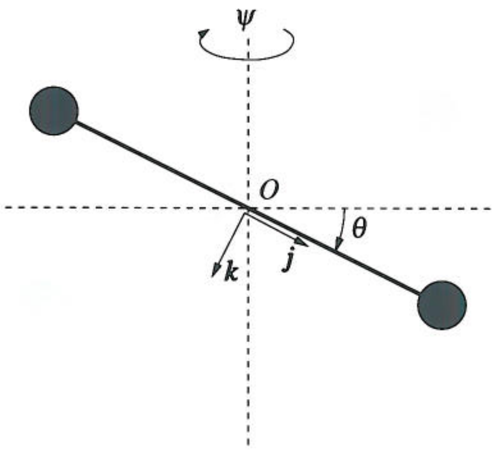

Figure 5.1 Two masses on a massless link.

Consider the two masses and the rod as one rigid system.

a) Write an expression for the angular velocity vector of the system in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$. [3]

b) Calculate the inertia matrix of the system. [3]

c) Compute the angular momentum vector of the system. [2]

d) Use the <u>vectorial</u> approach to derive the equations of motion. [12]

6. Two particles of mass m each are attached at the two ends of a rigid rod of length l and of negligible mass that is free to rotate by an angle ψ about the vertical axis and by an angle θ about a horizontal axis which is perpendicular to the rod, as shown in Figure 6.1. Both axes of rotation pass through the centre of the rod. A moving Cartesian coordinate system attached to the rod with fixed origin O and with unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ is used to analyse the motion of the system. The $\boldsymbol{i}$ vector has a direction into the page at the instant shown and it is along the axis of the θ rotation. A moment N is applied on the rod in the $\boldsymbol{k}$ direction. The effect of gravity is neglected.

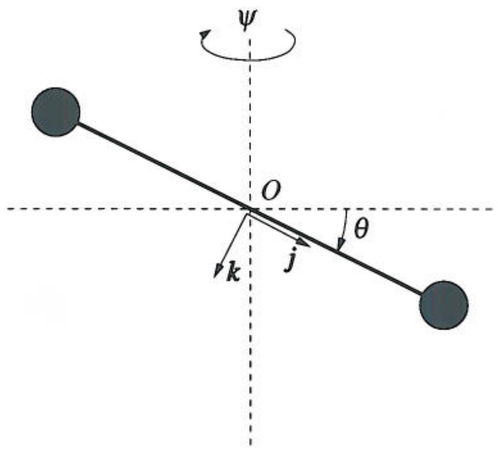

Figure 6.1 Two masses on a massless link.

a) Compute the total kinetic energy of the system. [4]

b) Use the Lagrangian approach to derive the equations of motion. [12]

c) Assume the angular velocity $\dot{\psi}$ about the vertical axis is constant.

i) Calculate the moment N which is required to impose this constraint. [2]

ii) Determine the equation of motion of the system when θ is a small angle and hence write the angular frequency of the oscillations of the motion. [2]

EE4-47 / EE9CS6-1

Modelling and control of multibody mechanical systems

Model answers 2012

Question 1

a) Three single-axis-rotation transformation matrices are needed.

$$D_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the rotation matrix by angle ψ about a z axis.

$$C_\theta = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix},$$

which is the rotation matrix by angle θ about a y axis.

$$B_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix},$$

which is the rotation matrix by angle ϕ about an x axis.

The complete transformation from earth-fixed coordinates to body-fixed coordinates is $A = C_\theta B_\phi D_\psi$ and it amounts to

$$\begin{bmatrix} \cos\theta\cos\psi - \sin\phi\sin\theta\sin\psi & \cos\theta\sin\psi + \sin\phi\sin\theta\cos\psi & -\cos\phi\sin\theta \\ -\cos\phi\sin\psi & \cos\phi\cos\psi & \sin\phi \\ \sin\theta\cos\psi + \sin\phi\cos\theta\sin\psi & \sin\theta\sin\psi - \sin\phi\cos\theta\cos\psi & \cos\phi\cos\theta \end{bmatrix}$$

b)

$$\mathbf{\Omega} = C_\theta B_\phi \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix} + C_\theta \begin{bmatrix} \dot\phi \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot\theta \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\dot\psi\cos\phi\sin\theta + \dot\phi\cos\theta \\ \dot\psi\sin\phi + \dot\theta \\ \dot\psi\cos\phi\cos\theta + \dot\phi\sin\theta \end{bmatrix}$$

Question 2

a) 2 degrees of freedom. Generalised coordinates: r, θ.

b) $\boldsymbol{r} = r\boldsymbol{e_r}$.

c) $\dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e_r} + r\dot{\theta}\boldsymbol{e_\theta}$.

d) The kinetic energy is $T = \frac{1}{2}m\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$.
The potential energy is $V = -mgr\cos\theta + \frac{1}{2}c(r - r_0)^2$, with the level of point O corresponding to zero gravitational potential energy.
The Lagrangian is $L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta - \frac{1}{2}c(r - r_0)^2$.

e) The Lagrangian equation with respect to the generalised coordinate r is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = F_r,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) - mr\dot{\theta}^2 - mg\cos\theta + c(r - r_0) = F_r,$$

or

$$m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + c(r - r_0) = F_r,$$

or

$$\ddot{r} - r\dot{\theta}^2 - g\cos\theta + \frac{c}{m}(r - r_0) = \frac{F_r}{m}. \tag{1}$$

The Lagrangian equation with respect to the generalised coordinate θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = rF_\theta,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta}\right) + mgr\sin\theta = rF_\theta,$$

or

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta = \frac{F_\theta}{m}.$$

f) i) Substitute in Equation (1) $F_r = -mr\dot{\theta}^2 - mg\cos\theta$. Therefore

$$\ddot{r} + \frac{c}{m}(r - r_0) = 0.$$

Therefore

$$r - r_0 = A\cos\left(\sqrt{\frac{c}{m}}t\right) + B\sin\left(\sqrt{\frac{c}{m}}t\right).$$

But at $t = 0$, $r = r_0 + \epsilon$ and therefore $A = \epsilon$. Also, $\dot{r} = 0$ at $t = 0$ therefore $B = 0$. Then

$$r(t) = r_0 + \epsilon\cos\left(\sqrt{\frac{c}{m}}t\right).$$

ii) The extra term is $-\mu(r - r_0)$ in which $\mu > 0$ is the damping constant. Once the radial oscillations die r will be constant at r_0 and $\dot{r} = 0$. Therefore if $F_\theta = 0$ it can be seen from the second equation of motion that this equation reduces to the same equation as that of a simple pendulum of length r.

Question 3

a) The position vector is $\boldsymbol{r} = (r_1 + r_2)\boldsymbol{e_r}$.

b) The velocity vector is $\dot{\boldsymbol{r}} = (r_1 + r_2)\dot{\theta}\boldsymbol{e_\theta}$.

c) The kinetic energy is $T = \frac{1}{2}m\left(r_1 + r_2\right)^2 \dot{\theta}^2 + \frac{1}{2}I\dot{\phi}^2$.
The potential energy is $V = mg(r_1 + r_2)\sin\theta$ with the centre of the circular surface the zero potential energy level.
Hence the Lagrangian is $L = T - V = T = \frac{1}{2}m\left(r_1 + r_2\right)^2 \dot{\theta}^2 + \frac{1}{2}I\dot{\phi}^2 - mg(r_1 + r_2)\sin\theta$.

d) The velocity of the centre of mass of the wheel is $\dot{\boldsymbol{r}} = (r_1 + r_2)\dot{\theta}\boldsymbol{e_\theta}$ and the velocity of the contact point caused by the rotation of the wheel is $-r_2\dot{\phi}\boldsymbol{e_\theta}$. The total velocity of the material contact point is zero, therefore the constraint equation is

$$(r_1 + r_2)\dot{\theta} - r_2\dot{\phi} = 0.$$

e) The Lagrangian equation with respect to the generalised coordinate θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \lambda(r_1 + r_2) = 0,$$

in which λ is the Lagrange multiplier corresponding to the rolling constraint. The RHS of the equation is zero because the torque T_d reacts on earth. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m(r_1 + r_2)^2\dot{\theta}\right) + mg(r_1 + r_2)\cos\theta + \lambda(r_1 + r_2) = 0,$$

or

$$m(r_1 + r_2)^2\ddot{\theta} + mg(r_1 + r_2)\cos\theta + \lambda(r_1 + r_2) = 0.$$

The Lagrangian equation with respect to the generalised coordinate ϕ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} - \lambda r_2 = T_d,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(I\dot{\phi}\right) - \lambda r_2 = T_d,$$

or

$$I\ddot{\phi} - \lambda r_2 = T_d,$$

which implies that

$$\lambda = \frac{I}{r_2}\ddot{\phi} - \frac{T_d}{r_2},$$

and after making use of the rolling constraint equation

$$\lambda = \frac{I(r_1 + r_2)}{{r_2}^2}\ddot{\theta} - \frac{T_d}{r_2}.$$

After substitution of λ into the first Lagrangian equation we obtain the equation of motion

$$\left(m{r_2}^2 + I\right)\ddot{\theta} + \frac{mg{r_2}^2}{r_1 + r_2}\cos\theta = \frac{r_2}{r_1 + r_2}T_d.$$

f) The RHS of the first Lagrangian equation is T_d so that

$$m(r_1+r_2)^2\ddot{\theta}+mg(r_1+r_2)\cos\theta+\lambda(r_1+r_2)=T_d.$$

This is because the torque reacting on the massless link causes this link to apply a force on the centre of the wheel of magnitude $T_d/(r_1+r_2)$ (consider a free body diagram of the link). The next steps are the same as previously and give

$$\left(m{r_2}^2+I\right)\ddot{\theta}+\frac{mg{r_2}^2}{r_1+r_2}\cos\theta=\frac{r_2(r_1+2r_2)}{(r_1+r_2)^2}T_d.$$

Question 4

a) The velocity of the centre of mass of the wheel is $\dot{\boldsymbol{r}} = (r_1 + r_2)\dot{\theta}\boldsymbol{e_\theta}$ and the velocity of the contact point caused by the rotation of the wheel is $-r_2\dot{\phi}\boldsymbol{e_\theta}$. The total velocity of the material contact point is zero, therefore the constraint equation is

$$(r_1 + r_2)\dot{\theta} - r_2\dot{\phi} = 0.$$

b) By differentiating twice the position vector $\boldsymbol{r} = (r_1 + r_2)\boldsymbol{e_r}$ the acceleration vector is given by $\ddot{r} = -(r_1 + r_2)\dot{\theta}^2\boldsymbol{e_r} + (r_1 + r_2)\ddot{\theta}\boldsymbol{e_\theta}$.

c) Newton's second law of motion gives $\boldsymbol{F} = m\ddot{\boldsymbol{r}}$ and therefore

$$F_r\boldsymbol{e_r} + (F_\theta + F_{\theta roll})\boldsymbol{e_\theta} + mg\boldsymbol{k} = m\left(-(r_1 + r_2)\dot{\theta}^2\boldsymbol{e_r} + (r_1 + r_2)\ddot{\theta}\boldsymbol{e_\theta}\right),$$

in which F_r is the radial force of constraint applied by the massless link to hold the wheel on the surface, F_θ is the force applied by the massless link on the wheel in the $\boldsymbol{e_\theta}$ direction, and $F_{\theta roll}$ is the force from the surface on the wheel which maintains the rolling constraint. $\boldsymbol{k}$ is a unit vector in the vertical downwards direction. When T_d reacts on earth $F_\theta = 0$ and therefore

$$F_r = mg\sin\theta - m(r_1 + r_2)\dot{\theta}^2,$$

$$F_{\theta roll} = mg\cos\theta + m(r_1 + r_2)\ddot{\theta}.$$

By considering the motion of the wheel about its centre of mass, $\mathrm{d}\boldsymbol{H}/\mathrm{d}t = \boldsymbol{N}$ in which the angular momentum is $H = I\dot{\phi}$. Therefore

$$I\ddot{\phi} = -F_{\theta roll}r_2 + T_d,$$

or after substitution of $F_{\theta roll}$

$$I\ddot{\phi} = -mgr_2\cos\theta - m(r_1 + r_2)r_2\ddot{\theta} + T_d.$$

By making use of the rolling constraint equation and substituting $\ddot{\phi}$ we obtain the equation of motion

$$\left(m{r_2}^2 + I\right)\ddot{\theta} + \frac{mg{r_2}^2}{r_1 + r_2}\cos\theta = \frac{r_2}{r_1 + r_2}T_d.$$

d) For $T_d = 0$ and $\theta = -\pi/2 + \epsilon$ in which ϵ is a small perturbation, the equation of motion becomes

$$\left(m{r_2}^2 + I\right)\ddot{\epsilon} + \frac{mg{r_2}^2}{r_1 + r_2}\epsilon = 0,$$

which describes simple harmonic motion with angular frequency of oscillation

$$\omega = \sqrt{\frac{\frac{mg{r_2}^2}{r_1+r_2}}{m{r_2}^2 + I}}.$$

Question 5

a) The angular velocity of the system about the vertical axis is $\dot{\psi}$ and in the $\boldsymbol{i}$ direction it is $\dot{\theta}$. All together it is

$$\boldsymbol{\Omega} = \dot{\theta}\boldsymbol{i} + \dot{\psi}\sin\theta\boldsymbol{j} + \dot{\psi}\cos\theta\boldsymbol{k}.$$

b) The moment of inertia about the axis in the $\boldsymbol{i}$ direction is

$$I_{xx} = m\left(\frac{l}{2}\right)^2 + m\left(\frac{l}{2}\right)^2 = \frac{1}{2}ml^2.$$

The moment of inertia about the axis in the $\boldsymbol{j}$ direction is zero and the moment of inertia about the axis in the $\boldsymbol{k}$ direction is the same as I_{xx}. These three axes are principal and therefore the inertia matrix is

$$I = \begin{bmatrix} \frac{1}{2}ml^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}ml^2 \end{bmatrix}.$$

c) The angular momentum is given by $H = I\Omega$ which in vector form is

$$\boldsymbol{H} = \frac{1}{2}ml^2\dot{\theta}\boldsymbol{i} + \frac{1}{2}ml^2\dot{\psi}\cos\theta\boldsymbol{k}.$$

d) The motion about the centre of mass is given by

$$\frac{\mathrm{d}'\boldsymbol{H}}{\mathrm{d}t} + \boldsymbol{\Omega}\times\boldsymbol{H} = \boldsymbol{N},$$

or

$$\frac{1}{2}ml^2\ddot{\theta}\boldsymbol{i} + \frac{1}{2}ml^2(\ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta)\boldsymbol{k} + \frac{1}{2}ml^2(\dot{\theta}\boldsymbol{i} + \dot{\psi}\sin\theta\boldsymbol{j} + \dot{\psi}\cos\theta\boldsymbol{k})\times(\dot{\theta}\boldsymbol{i} + \dot{\psi}\cos\theta\boldsymbol{k}) = N\boldsymbol{k},$$

or

$$\frac{1}{2}ml^2\left(\left(\dot{\psi}^2\sin\theta\cos\theta + \ddot{\theta}\right)\boldsymbol{i} + \left(-\dot{\theta}\dot{\psi}\cos\theta + \dot{\theta}\dot{\psi}\cos\theta\right)\boldsymbol{j} + \left(-2\dot{\psi}\dot{\theta}\sin\theta + \ddot{\psi}\cos\theta\right)\boldsymbol{k}\right) = N\boldsymbol{k}.$$

Therefore the two equations of motion are

$$\ddot{\theta} + \dot{\psi}^2\sin\theta\cos\theta = 0,$$

and

$$\frac{1}{2}ml^2\left(\ddot{\psi}\cos\theta - 2\dot{\psi}\dot{\theta}\sin\theta\right) = N.$$

Question 6

a) The velocity vector of each mass is given by

$$-\frac{l}{2}\dot{\psi}\cos\theta\boldsymbol{i} + \frac{l}{2}\dot{\theta}\boldsymbol{k}.$$

Therefore the total kinetic energy of the system is

$$T = \frac{1}{2}m\left(\left(-\frac{l}{2}\dot{\psi}\cos\theta\right)^2 + \left(\frac{l}{2}\dot{\theta}\right)^2\right) \times 2,$$

or

$$T = \frac{1}{4}ml^2\dot{\theta}^2 + \frac{1}{4}ml^2\dot{\psi}^2\cos^2\theta.$$

b) Since gravity is neglected the potential energy is zero and $L = T$. The Lagrangian equation with respect to the generalised coordinate ψ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\psi}}\right) - \frac{\partial L}{\partial\psi} = N\cos\theta,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}ml^2\dot{\psi}\cos^2\theta\right) = N\cos\theta,$$

or

$$\frac{1}{2}ml^2\left(\ddot{\psi}\cos^2\theta - 2\dot{\psi}\dot{\theta}\cos\theta\sin\theta\right) = N\cos\theta.$$

Therefore the first equation of motion is

$$\frac{1}{2}ml^2\left(\ddot{\psi}\cos\theta - 2\dot{\psi}\dot{\theta}\sin\theta\right) = N.$$

The Lagrangian equation with respect to the generalised coordinate θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}ml^2\dot{\theta}\right) + \frac{1}{2}ml^2\dot{\psi}^2\cos\theta\sin\theta = 0.$$

Therefore the second equation of motion is

$$\ddot{\theta} + \dot{\psi}^2\sin\theta\cos\theta = 0.$$

c) i) If $\dot{\psi}$ is constant then $\ddot{\psi} = 0$ and from the first equation of motion

$$N = -ml^2\dot{\psi}\dot{\theta}\sin\theta.$$

ii) When θ is small, from the second equation of motion

$$\ddot{\theta} + \dot{\psi}^2\theta = 0,$$

which is simple harmonic motion in θ with angular frequency of the oscillations equal to $\dot{\psi}$.