Imperial College
London

[E1.10 (Maths 1) 2013]

B.ENG. and M.ENG. EXAMINATIONS 2013

PART I : MATHEMATICS 1 (ELECTRICAL ENGINEERING)

Date Thursday 30th May 2013 10.00 - 12.00

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer Question 1 and THREE of the remaining five

Answer Section A and Section B in different answerbooks.

Question 1 carries twice the marks of each of the other questions.

CALCULATORS MAY **NOT** BE USED.

A mathematical formulae sheet is provided.

[*Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of SIX questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. (i) Find all roots of

$$z^4 + 2z^2 + 2 = 0$$

in polar form, i.e. $z = r\ \exp(i\varphi)$ with $r, \varphi \in \mathbb{R}$.

(ii) Express

$$\exp(i\pi/3) + \exp(-i2\pi/3)$$

in Cartesian form, i.e. $z = x + iy$ with $x, y \in \mathbb{R}$.

(iii) Find q so that the limit

$$\lim_{x\to\infty} x^q \left(\sqrt{x} - \sqrt{x-1}\right)$$

is finite and non-zero.
Do **not** *use L'Hôpital's rule.*

(iv) Find the limit

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$$

You **can** *use L'Hôpital's rule.*
You can use $\lim_{x\to 0} \sin(x)/x = 1$.

(v) Differentiate

$$(\tan(x))^{\exp(x)}$$

(vi) Integrate

$$\int \tan(x)\ \mathrm{d}x$$

(vii) Integrate

$$\int_0^1 \tan(x)^2\ \mathrm{d}x$$

Q1 CONTINUES ON THE NEXT PAGE

(viii) Find the Taylor expansion of

$$\frac{1}{\exp(x)+1}$$

about $x = 0$ to first order (up to and including the term linear in x) and state the remainder term $R_2(x)$.

(ix) Find the general solution of the following first order ODE:

$$2y'(x) = \frac{y(x)^2}{x^2} + 1$$

(x) Find the general solutions of the following second order ODE:

$$y''(x) + y(x) = \sin(2x)$$

2. Find $\dfrac{dy}{dx}$ as a function of x in each of the following cases :

(i) $$y(x) \;=\; \frac{\exp(2x)}{\sin(3x^2)} \;;$$

(ii) $$y(x) \;=\; (1+x^2)\tan^{-1}(x) \;;$$

Note: $\tan^{-1}(x)$ *denotes the inverse* tan *function*

(iii) $$y(x) \;=\; \frac{x+3}{(x+1)(x-2)} \;;$$

(iv) $$y(x) \;=\; x^{(x^2)} \;.$$

Note: This is x *raised to the power* x^2*.*

(v) Find the following nth derivative

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x\ln(x)\right)$$

for $n \geq 2$.

Note: Simplify the result as much as possible.

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3. Evaluate the following limits *without using L'Hôpital's rule unless specified*:

(i) $$\lim_{x\to 1} \frac{x}{\sin(x)} \;;$$

Do not use L'Hôpital's rule.

(ii) $$\lim_{x\to 1} \frac{x-1}{\sin(x\pi)} \;;$$

Do not use L'Hôpital's rule.

(iii) $$\lim_{x\to\infty} \; x^{2/3}\left((1+x)^{1/3} + (1-x)^{1/3}\right) \;;$$

Consider only real roots. Do not use L'Hôpital's rule.

(iv) $$\lim_{x\to 1} \frac{x^n - 1}{x^m - 1} \quad \text{for integer } n, m > 0;$$

You **may** *use L'Hôpital's rule.*

(v) $$\lim_{x\to 1} \frac{\exp(x) - \exp(1)(1+x^2)/2}{(x-1)^3} \;;$$

Do *use L'Hôpital's rule.*

4. (i) Integrate $\displaystyle\int_0^{2\pi} \sin(mx)\sin(nx)\,dx$ for positive integers $0 < n, m \in \mathbb{N}$.

Note: The case $n = m$ requires special attention.

(ii) Show that $\displaystyle\int_0^{\infty} x^n \exp(-x^2)\;\mathrm{d}x \;=\; \frac{n-1}{2}\int_0^{\infty} x^{n-2}\exp(-x^2)\;\mathrm{d}x$ for any positive integer n, i.e. $0 < n \in \mathbb{N}$.

(iii) Integrate $$\int \frac{(x-1)(x^2 + \frac{5}{2}x + 1)}{2x^2 - 3x + 1}\;\mathrm{d}x \;.$$

(iv) Integrate $$\int \frac{\mathrm{d}x}{2+\cos(x)} \;.$$

5. (i) Express in polar form

$$(1-i)^3 \qquad \text{and} \qquad (\sqrt{3}i+1)^{1/2}\ .$$

(ii) Find the real part of $\left(\dfrac{1}{\sin(\theta)+i\cos(\theta)}\right)^4$ in terms of $\cos(2\theta)$.

(iii) Find all solutions of the equation

$$\cos(z) = \frac{5}{3}$$

in the Cartesian form, i.e. $z = x + iy$ with $x, y \in \mathbb{R}$.

6. (i) Find the solution $y(x)$ of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -xy(x)\ .$$

(ii) Find the general solution $y(x)$ of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{2}y(x) = \sin(x)\ .$$

(iii) Find the solution $y(x)$ of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -xy(x)$$

by expressing $y(x)$ in a MacLaurin series, $\quad y(x) = \displaystyle\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$,

and determining the recurrence relation for a_n. Determine the value of a_2.

Note: Do **not** *attempt to solve the recurrence relation.*

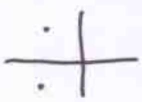
END OF PAPER

EXAMINATION QUESTIONS/SOLUTIONS 2012-13 | Course

Question 1 | TOPIC *General* | Marks & seen/unseen

Parts

(i)

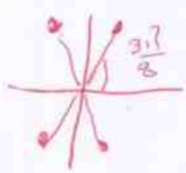


$$z^4 + 2z^2 + 2 = 0$$

$$\Rightarrow (z^2+1)^2 = -1 \Rightarrow z^2 = \pm i - 1$$
$$= \sqrt{2}\, e^{i(\pi \pm \frac{\pi}{4})}$$

$$\Rightarrow z = 2^{\frac{1}{4}} e^{i\varphi} \quad \text{with } \varphi = \tfrac{3}{8}\pi, \tfrac{5}{8}\pi, \tfrac{11}{8}\pi, \tfrac{13}{8}\pi$$

or $z = \pm\, 2^{\frac{1}{4}} e^{i(\frac{\pi}{2} \pm \frac{\pi}{8})}$ — 4



(ii)

$$e^{i\frac{\pi}{3}} + e^{-i\frac{2\pi}{3}} = \tfrac{1}{2}\left(\cos\tfrac{\pi}{3} + \cos\tfrac{2\pi}{3}\right)$$
$$+ \tfrac{1}{2} i \left(\sin\tfrac{\pi}{3} - \sin\tfrac{2\pi}{3}\right)$$
$$= \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{2}\right)$$
$$+ \tfrac{1}{2} i \left(\sqrt{\tfrac{3}{4}} - \sqrt{\tfrac{3}{4}}\right) = 0$$

or observe $-e^{i\frac{\pi}{3}} = e^{i\frac{\pi}{3} - i\pi} = e^{-i\frac{2\pi}{3}}$ — 4

(iii)

$$\sqrt{x} - \sqrt{x-1} = \frac{x-(x-1)}{\sqrt{x}+\sqrt{x-1}} = \frac{1}{\sqrt{x}}\ \frac{1}{1+\sqrt{1-\frac{1}{x}}}$$

$$\Rightarrow \lim_{x\to\infty} x^q\left(\sqrt{x} - \sqrt{x-1}\right) = \tfrac{1}{2} \quad \text{for } q = \tfrac{1}{2}$$ — 4

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Question 1 TOPIC General | Marks & seen/unseen

Parts

(iv)
$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} = \lim_{x\to 0} \frac{\cos(x)-1}{3x^2}$$
$$= \lim_{x\to 0} \frac{-\sin(x)}{6x} = -\frac{1}{6}$$
4

(v)
$$\frac{d}{dx}\left(\tan(x)\right)^{\exp(x)} = \frac{d}{dx}\exp\left(\ln(\tan(x))\exp(x)\right)$$
$$= \left\{\frac{1}{\tan(x)}\frac{1}{\cos^2(x)}\exp(x) + \ln(\tan(x))\exp(x)\right\} \tan(x)^{\exp(x)}$$
4
$$= \left\{\frac{1}{\sin(x)}\frac{1}{\cos(x)}\exp(x) + \ln(\tan(x))\exp(x)\right\} \tan(x)^{\exp(x)}$$

(vi)
$$\int \tan(x)\, dx = -\int \frac{d}{dx}\ln(\cos(x))\, dx$$
$$= -\ln(\cos(x)) + C'$$
4

Question 1 TOPIC General | Marks & seen/unseen

Parts

(vii) $\int_0^1 \tan^2(x)\,dx = -\int \left(1 - \frac{1}{\cos^2(x)}\right) dx$

$= -x + C' + \int \frac{d}{dx}\tan x \; dx \; = \tan x - x + C'$ 4

$= \tan(1) - 1$

(viii) $f(x) = \left(\exp(x) + 1\right)^{-1}$

$f'(x) = -\left(\exp(x)+1\right)^{-2} \exp(x)$

$f''(x) = 2\left(\exp(x)+1\right)^{-3} \exp(2x) + f'(x)$

$f(0) = 1/2$

$f'(0) = -1/4$

$f''(0) = 0$ (not needed)

$\Rightarrow f(x) = \frac{1}{2} - \frac{1}{4}x + R_2(x)$ 4

with $R_2(x) = \frac{x^2}{2} f''(\xi) \quad \xi \in [0, x]$

EXAMINATION QUESTIONS/SOLUTIONS 2012-13

Course

Question 1

TOPIC *General*

Marks & seen/unseen

Parts

(ix)

$$2y' = \frac{y^2}{x^2} + 1 \quad \textit{homogeneous}$$

$$= 2f(v) \qquad v = \frac{y}{x}\ ,\ f(v) = \tfrac{1}{2}(v^2+1)$$

$$\Rightarrow \ln x + c_1' = \int \frac{dv}{f(v) - v} = 2\int \frac{dv}{(v-1)^2}$$

$$= \frac{-2}{v-1} = \frac{2}{1-v}$$

$$\Rightarrow \frac{2}{\ln x + c_1'} = 1 - \frac{y}{x} \Rightarrow \underline{y = x - \frac{2x}{\ln x + c_1'}}$$ 4

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	EXAMINATION QUESTIONS/SOLUTIONS 2012-13	Course
Question	TOPIC General	Marks & seen/unseen

Parts

(x) $y''(x) + y(x) = \sin(2x)$

Complementary function, oscillatory case

$$y_{CF}(x) = A\sin(x) + B\cos(x)$$

Particular integral, try $y_{PI}(x) = C\sin(2x) + D\cos(2x)$

$\Rightarrow (C-4C)\sin(2x) + (D-4D)\cos(2x) = \sin(2x)$

$\Rightarrow D = 0, \quad C = -\frac{1}{3}$

General solution

$$y(x) = A\sin(x) + B\cos(x) - \frac{1}{3}\sin(2x)$$ 4

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EXAMINATION QUESTIONS/SOLUTIONS 2012-13 | Course

Question 2 | TOPIC Differentiation | Marks & seen/unseen

Parts

(i) $y(x) = \dfrac{e^{2x}}{\sin(3x^2)} \Rightarrow y'(x) = \dfrac{2e^{2x}}{\sin(3x^2)} - 6x\dfrac{\cos(3x^2)e^{2x}}{\sin^2(3x^2)}$ 4

$= \dfrac{2e^{2x}}{\sin(3x^2)}\left(1 - 3x\cot(3x^2)\right)$

(ii) $y(x) = (1+x^2)\tan^{-1}(x)$

$\Rightarrow y'(x) = (1+x^2)\dfrac{1}{1+x^2} + 2x\tan^{-1}(x)$

$= 1 + 2x\tan^{-1}(x)$ 4

(iii) $y(x) = \dfrac{2x^2-2x-4}{(x+1)(x-2)}$ Separate sheet

Either note that $y(x) = 2 \Rightarrow y'(x) = 0$

or brute force:

$y'(x) = \dfrac{4x-2}{(x+1)(x-2)} - \dfrac{2x^2-2x-4}{(x+1)^2(x-2)^2}\overbrace{\left((x+1)+(x-2)\right)}^{2x-1}$ 4

$= 2\dfrac{2x-1}{(x+1)(x-2)}\underbrace{\left\{1 - \dfrac{x^2-x-2}{(x+1)(x-2)}\right\}}_{=0} = 0$

(iv) $y(x) = x^{(x^2)} = \exp\left(x^2\ln(x)\right)$

$\dfrac{d}{dx}y(x) = \left(2x\ln(x) + x\right)x^{(x^2)}$ 4

(v) $\dfrac{d^n}{dx^n}x\ln x = \sum_{k=0}^{n}\binom{n}{k}x^{(k)}(\ln x)^{(n-k)} = x\dfrac{(n-1)!}{x^n}(-)^{n-1}$

$+ n\dfrac{(n-2)!}{x^{n-1}}(-)^{n-2} = \dfrac{(n-2)!}{x^{n-1}}(-)^{n-1}\left\{n-1-n\right\}$ 4

$= \dfrac{(n-2)!}{x^{n-1}}(-)^n$ using $\dfrac{d^n}{dx^n}\ln x = \dfrac{(n-1)!}{x^n}(-)^{n-1}$

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Course

Question
2

Marks & seen/unseen

Parts
(iii)

$$y(x) = \frac{x+3}{(x+1)(x-2)}$$

$$\frac{dy}{dx} = \frac{1}{(x+1)(x-2)} - \frac{(x+3)}{(x+1)^2(x-2)} - \frac{(x+3)}{(x+1)(x-2)^2}$$

$$= \frac{1}{(x+1)^2(x-2)^2}\left\{(x+1)(x-2) - (x+3)(x-2) - (x+3)(x+1)\right\}$$

$$= \frac{1}{(x+1)^2(x-2)^2}\left\{-2x+4 - x^2-4x-3\right\} = -(x^2+6x-1)$$

$$= -\frac{x^2+6x-1}{(x+1)^2(x-2)^2}$$

<u>or</u>

$$y(x) = \frac{x+3}{(x+1)(x-2)} = \frac{1}{x-2} + \frac{2}{(x+1)(x-2)}$$

$$\frac{dy}{dx} = \frac{-1}{(x-2)^2} - \frac{2(x-2+x+1)}{(x+1)^2(x-2)^2}$$

$$= -\frac{x^2+2x+1+4x-2}{(x+1)^2(x-2)^2} = -\frac{x^2+6x-1}{(x+1)^2(x-2)^2}$$

4

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	EXAMINATION QUESTIONS/SOLUTIONS 2012-13	Course
Question 3	TOPIC *Limits*	Marks & seen/unseen

Parts

(i) $$\lim_{x\to 1} \frac{x}{\sin(x)} = \frac{1}{\sin(1)}$$ 3 seen similar for all

(ii) $$\lim_{x\to 1} \frac{x-1}{\sin(x\pi)} = \lim_{y\to 0} \frac{y}{\sin(\pi+y\pi)}$$ 5

$$= \lim_{u\to 0} \frac{u/\pi}{-\sin(u)} = -\frac{1}{\pi}$$

(iii) $$\lim_{x\to\infty} x^{2/3}\left((1+x)^{1/3} + (1-x)^{1/3}\right)$$

$$= \lim_{x\to\infty} x^{2/3} \frac{(1+x)+(1-x)}{(1+x)^{2/3} - (1+x)^{1/3}(1-x)^{1/3} + (1-x)^{2/3}}$$ 5

$$= 2 \lim_{x\to\infty} \frac{1}{(1+\frac{1}{x})^{2/3} + (1+\frac{1}{x})^{1/3}(1-\frac{1}{x})^{1/3} + (1-\frac{1}{x})^{2/3}}$$

$$= \frac{2}{3}$$

(iv) $$\lim_{x\to 1} \frac{x^n-1}{x^m-1} = \lim_{x\to 1} \frac{n\,x^{n-1}}{m\,x^{m-1}} = \frac{n}{m}$$ with L'Hopital

or

$$\lim_{x\to 1} \frac{x^n-1}{x^m-1} = \lim_{x\to 1} \frac{(x-1)\overbrace{(x^{n-1}+x^{n-2}+\dots+1)}^{n\text{ terms}}}{(x-1)\underbrace{(x^{m-1}+x^{m-2}+\dots+1)}_{m\text{ terms}}}$$ 3

$$= \frac{n}{m}$$

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EXAMINATION QUESTIONS/SOLUTIONS 2012-13

Course

Question 3

TOPIC Limits

Marks & seen/unseen

Parts

(v)

$$\lim_{x \to 1} \frac{e^x - e(1+x^2)/2}{(x-1)^3}$$

4

$$= \lim_{x \to 1} \frac{e^x - ex}{3(x-1)^2} = \lim_{x \to 1} \frac{e^x - e}{6(x-1)}$$

$$= \lim_{x \to 1} \frac{e^x}{6} = \frac{e}{6}$$

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EXAMINATION QUESTIONS/SOLUTIONS 2012-13

Course

Question 4

TOPIC Integration

Marks & seen/unseen

Parts

i

$$\int_0^{2\pi} \sin(nx)\sin(mx)\,dx = \frac{1}{2}\int_0^{2\pi} \cos((n-m)x) - \cos((n+m)x)\,dx$$

$$\left[\text{using} \quad \begin{aligned} \cos((n+m)x) &= \cos(nx)\cos(mx) - \sin(nx)\sin(mx) \\ \cos((n-m)x) &= \cos(nx)\cos(mx) + \sin(nx)\sin(mx) \end{aligned}\right.$$

$$\begin{cases} = 0 \text{ for } & n \neq m \quad (\text{note } n, m > 0) \\ = \pi \text{ for } & n = m \end{cases}$$

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EXAMINATION QUESTIONS/SOLUTIONS 2012-13

Course

Question 4

TOPIC Integration

Marks & seen/unseen

Parts ii

$$\int_0^\infty x^n \exp(-x^2)\,dx = \lim_{b\to\infty} \int_0^b x^n \exp(-x^2)\,dx$$

Seen Similar for all

$$\int_0^b x^n e^{-x^2} dx = \frac{1}{2}\int_0^{b^2} y^{\frac{n-1}{2}} e^{-y}\,dy$$

$$= -\left[\frac{1}{2} y^{\frac{n-1}{2}} e^{-y}\right]_0^{b^2} + \frac{1}{2}\int_0^{b^2} \frac{n-1}{2}\, y^{\frac{n-3}{2}} e^{-y}\,dy$$

$$= -[\dots] + \frac{n-1}{2}\int_0^b x^{n-2} e^{-x^2} dx$$

for $n>1$

Take $b\to\infty$ to remove $[\dots]$, so that

5

$$\int_0^\infty x^n \exp(-x^2)\,dx = \frac{n-1}{2}\int_0^\infty x^{n-2} e^{-x^2} dx$$

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EXAMINATION QUESTIONS/SOLUTIONS 2012-13

Course

Question 4

TOPIC Integration

Marks & seen/unseen

Parts

(iii) $I = \int \frac{(x-1)(x^2+\frac{5}{2}x+1)}{2x^2-3x+1}\,dx$

Denominator $2x^2-3x+1 = (x-1)(2x-1)$

$\Rightarrow I = \int \frac{x^2+\frac{5}{2}x+1}{2x-1}\,dx = \int \frac{\frac{1}{2}x(2x-1)+3x+1}{2x-1}\,dx$

$= \int \frac{1}{2}x + \frac{3}{2} + \frac{\frac{5}{2}}{2x-1}\,dx = \frac{1}{4}x^2 + \frac{3}{2}x + \frac{5}{4}\ln(2x-1) + C$ 5

(iv) $\int \frac{dx}{2+\cos(x)}$ use $t = \tan(\frac{x}{2})$, $\cos(x) = \frac{1-t^2}{1+t^2}$

$dx = \frac{2}{1+t^2}\,dt$

$\Rightarrow \int \frac{dx}{2+\cos(x)} = \int \frac{1}{2+\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2}\,dt = \int \frac{2}{3+t^2}\,dt$

$= \frac{2}{\sqrt{3}} \int \frac{\sqrt{3}}{3+t^2}\,dt = \frac{2}{\sqrt{3}} \arctan\left(\frac{t}{\sqrt{3}}\right) + C'$

$= \frac{2}{\sqrt{3}} \arctan\left(\frac{\tan(\frac{x}{2})}{\sqrt{3}}\right) + C'$ 5

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EXAMINATION QUESTIONS/SOLUTIONS 2012-13 | Course

Question 5 | TOPIC *Complex numbers* | Marks & seen/unseen

Parts

(i) $(1-i)^3 = 2^{3/2}\, e^{-i\frac{\pi}{4}3}$ also allow $e^{i\frac{5}{4}\pi}$ 

$\left(1+\sqrt{3}\, i\right)^{1/2} = \sqrt{2}\, e^{i\frac{\pi}{3}\frac{1}{2}},\ \sqrt{2}\, e^{i\frac{7}{6}\pi}$  4

(ii) $\left(\dfrac{1}{\sin(\theta)+i\cos(\theta)}\right)^4 = \left(\dfrac{-i}{e^{-i\theta}}\right)^4$

$= e^{4i\theta} \left(= \cos(4\theta) + i\sin(4\theta)\right)$

$= \left(e^{2i\theta}\right)^2 = \left(\cos(2\theta) + i\sin(2\theta)\right)^2$

$= \left(\cos(2\theta)\right)^2 - \left(\sin(2\theta)\right)^2 + 2i\sin(2\theta)\cos(2\theta)$

$= 2\left(\cos(2\theta)\right)^2 - 1 + 2i\sin(2\theta)\cos(2\theta)$

$\Rightarrow \mathrm{Re}\left(\left(\dfrac{1}{\sin\theta + i\cos\theta}\right)^4\right) = 2\cos^2 2\theta - 1$ 6

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EXAMINATION QUESTIONS/SOLUTIONS 2012-13 | Course

Question 5

TOPIC Complex numbers

Marks & seen/unseen

Parts

(iii)

$$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

$$z = y - iX$$

$$\frac{5}{3} = \frac{1}{2}\left(e^{x}\left(\cos(y) + i\sin(y)\right) + e^{-x}\left(\cos(y) - i\sin(y)\right)\right)$$

$$\Rightarrow \quad \frac{5}{3} = \frac{1}{2}\cos(y)\left(e^{x} + e^{-x}\right)$$

$$0 = \frac{1}{2}\sin(y)\left(e^{x} - e^{-x}\right)$$

from the second line $e^{x} = e^{-x} = 1$ seems a solution but then there is no $y \in \mathbb{R}$ with $\cos(y) = \frac{5}{3}$ $\Rightarrow y = n\pi,\ n \in \mathbb{Z}$

$\Rightarrow \cos(y) = (-)^{n} \Rightarrow$

$$\frac{5}{3} = \frac{1}{2}(-)^{n}\left(e^{x} + e^{-x}\right) \quad \rightarrow n \text{ even}$$

$$\Rightarrow e^{2x} - \frac{10}{3}e^{x} + 1 = 0$$

$$\Rightarrow e^{x} = \frac{\frac{10}{3} \pm \sqrt{\frac{100}{9} - 4}}{2} = \frac{5}{3} \pm \frac{4}{3}$$

$$\Rightarrow x = \pm \ln(3)$$

$$\Rightarrow z = \pm i\ln(3) + 2k\pi, \quad k \in \mathbb{Z}$$

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Question 6 | TOPIC ODEs | Marks & seen/unseen

Parts

(i) $y' = -xy$ separable

$$\int \frac{dy}{y} = -\int x\,dx \Rightarrow \ln y + C' = -\tfrac{1}{2}x^2$$

$$y(x) = A\,e^{-\frac{1}{2}x^2}$$

4

(ii) ↑ (iii)

(ii) next page

$$y = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

$$y' = \sum_{n=1}^{\infty} a_n\, n \frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{n!}$$

$$y'' = \sum_{n=0}^{\infty} a_{n+2} \frac{x^n}{n!}$$

$$y'' = -xy \Rightarrow \sum_{n=0}^{\infty} a_{n+2} \frac{x^n}{n!} = -\sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n!}$$

$$= -\sum_{n=1}^{\infty} a_{n-1} \frac{x^n}{(n-1)!}$$

$$\Rightarrow \frac{a_{n+2}}{n!} = -\frac{a_{n-1}}{(n-1)!} \Rightarrow a_{n+2} = -n\, a_{n-1}, \; n \geq 1$$

$\underline{a_2 = 0}$ and $\underline{a_m = -a_{m-3}\,(m-2)}$

$a_0 = ?$

$a_1 = ?$

$a_3 = -a_0$

$a_4 = -2a_1$

...

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EXAMINATION ~~QUESTIONS~~/SOLUTIONS 2012-2013

Course

Question 6

TOPIC ODEs

Marks & seen/unseen

Parts

ii

$$y'' + y' + \frac{1}{2} y = \sin(x)$$

$a=1, \ b=1, \ c=\frac{1}{2} \qquad b^2-4ac=-1<0$ osc'l

$$y_{CF} = \left(A \sin\frac{x}{2} + B\cos\frac{x}{2}\right)e^{-\frac{x}{2}} \qquad \Omega = \frac{1}{2}, \ -\frac{b}{2a} = -\frac{1}{2}$$

$$y_{p1} = C\sin x + D\cos x$$

$$y_{p1}' = C\cos x - D\sin x$$

$$y_{p1}'' = -y_{p1}$$

$$\Rightarrow \quad y_{p1}' - \frac{1}{2}y_{p1} = \left(C - \frac{D}{2}\right)\cos x - \left(D + \frac{C}{2}\right)\sin x = \sin x$$

$$\Rightarrow C = \frac{D}{2} \Rightarrow -D\frac{5}{4} = 1 \Rightarrow D = -\frac{4}{5}, \ C = -\frac{2}{5}$$

$$y_{p1} = -\frac{2}{5}\sin x - \frac{4}{5}\cos x$$

General soln:

$$y(x) = \left(A\sin\frac{x}{2} + B\cos\frac{x}{2}\right)e^{-\frac{x}{2}} - \frac{2}{5}\sin x - \frac{4}{5}\cos x$$

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Setter's initials HJ

Checker's initials GP

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