IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2014

EEE PART I: MEng, BEng and ACGI

Corrected Copy

MATHEMATICS 1B (E-STREAM AND I-STREAM)

Friday, 30 May 10:00 am

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

Please answer questions from Section A and Section B in separate answer books.

All questions carry equal marks (25% each)

Any special instructions for invigilators and information for candidates are on page 1.

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Section A

1. a) Determine whether the following series converge. Justify your answer carefully.

   i) $\sum_{n\geq 2} \dfrac{1}{\sqrt{n^2-3}}$ [2]

   ii) $\sum_{n\geq 0} (-1)^n \dfrac{3^n}{5^n}$ [2]

   iii) $\sum_{n\geq 1} \dfrac{5^n}{n^n}$ [3]

   b) Derive the first four terms of the Taylor series expansion of $\ln(1+x)$ about 0. [8]

   c) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}$$

   Hint: Check that 2 is an eigenvalue, and provide two linearly independent eigenvectors associated with it. [10]

2. Let P be the plane defined by

$$x+y+z=10$$

and L be the line through the point $(-1,-3,4)$ whose direction is given by the vector $(1,0,0)$.

   a) Find the point of intersection of L and P. [5]

   b) Compute the minimum distance between the point $(1,0,0)$ and the plane P. [10]

   c) Find the equation of the plane Q containing the line L and orthogonal to P. [10]

Section B

3. a) Consider the following differential equation:

$$\frac{d^2y}{dx^2} - y = 2e^x - 1.$$

i) Find the complementary function. [3]

ii) Find a particular integral. [3]

iii) Find a solution $y(x)$ that satisfies the initial conditions

$$y(0) = 0, \qquad \frac{dy(0)}{dx} = 0.$$

[3]

b) Consider the following differential equation:

$$(\lambda_1 xy + \cos x \cos y)\, dx + \left(x^2 - \frac{1}{2}\lambda_2 \sin x \sin y\right) dy = 0.$$

i) Find the values of the constants λ_1 and λ_2 such that the differential equation is exact. [2]

ii) Find $f(x,y)$ such that the LHS of the differential equation is equal to df. [3]

iii) Hence find the solution of the differential equation. [3]

c) Consider the following differential equation:

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{2}{x^2}.$$

i) Find an integrating factor $\mu(x)$ that solves the equation

$$\mu(x)\frac{dy(x)}{dx} + \mu(x)\frac{3}{x}y(x) = \frac{d}{dx}(\mu(x)y(x)).$$

[4]

ii) By multiplying by $\mu(x)$, find the general solution of the differential equation. [4]

4. a) Consider the partial differential equation

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = x^2 + y^2. \tag{4.1}$$

Assume that $f(x,y)$ is radially symmetric.

i) By considering the change of coordinates

$$\rho = \sqrt{x^2+y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

and using the chain rule

$$\begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \\ \dfrac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \rho}{\partial x} & \dfrac{\partial \phi}{\partial x} \\ \\ \dfrac{\partial \rho}{\partial y} & \dfrac{\partial \phi}{\partial y} \end{bmatrix} \begin{bmatrix} \dfrac{\partial f}{\partial \rho} \\ \\ \dfrac{\partial f}{\partial \phi} \end{bmatrix}$$

show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial \rho}\right)^2 + \frac{a}{\rho^2}\left(\frac{\partial f}{\partial \phi}\right)^2 \tag{4.2}$$

for some $a > 0$. What is the value of a? [6]

ii) Use equation (4.2) to transform equation (4.1) into an ordinary differential equation and obtain the general solution $f(x,y)$. [6]

b) Suppose that the function $z(x,y)$ is implicitly defined by

$$F(x,y,z) = x^2 + y^2 - \frac{z^2}{2} + 2 = 0, \qquad z > 0.$$

i) Use the fact that $dF = 0$ to derive expressions for $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. [4]

ii) By using the answer to Part (i) above, or by expressing z explicitly as a function of x and y, find the stationary points of $z(x,y)$. [5]

iii) Classify the stationary points by evaluating the Hessian. [4]