Imperial College
London

[E2.11 (Maths) ISE 2011]

B.ENG. and M.ENG. EXAMINATIONS 2011

MATHEMATICS (INFORMATION SYSTEMS ENGINEERING E2.11)

Date Thursday 9th June 2011 2.00 - 4.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

Answer FOUR questions, to include at least one from Section B.

Answers to questions from Section A and Section B should be written in different answer books.

Mathematical and statistics formulae sheets are provided.

[Before starting, please make sure that the paper is complete; there should be seven pages, with a total of SIX questions. Ask the invigilator for a replacement if your copy is faulty.]

Section A

1. The Fourier transform $\widehat{f}(\omega)$ of $f(t)$ is defined by

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t)\, dt \ .$$

(i) Calculate the Fourier transform of $g(t)$ where $g(t) = f(t + \tau)$, and τ is an arbitrary real constant.

(ii) Calculate the Fourier transform $\widehat{f}(\omega, a, b)$ of $f(t, a, b)$ where $f(t, a, b)$ is defined by

$$f(t, a, b) = 1 \qquad a < t < b \ ,$$

$$f(t, a, b) = 0 \qquad \text{otherwise} \ .$$

Verify that the transform $\widehat{f}(\omega, a\, b)$ is consistent with the shift theorem proved in Part (i).

(iii) Define the *convolution* $f * g(t)$ of two functions $f(t)$ and $g(t)$, and state the Convolution Theorem.

(iv) Calculate the convolution of $f(t, -1, 1)$, defined in part (ii), with itself, and write down the Fourier Transform of this convolution.

Hence, using Parseval's Theorem,

$$\int_{-\infty}^{\infty} |\, f(t)\,|^2\, dt = 2\pi \int_{-\infty}^{\infty} |\, \widehat{f}(\omega)\,|^2\, d\omega \ ,$$

calculate

$$\int_{-\infty}^{\infty} \frac{\sin^4(\omega)}{\omega^4}\, d\omega \ .$$

PLEASE TURN OVER

2. (i) Sketch the region of integration for the integral

$$\int_{x=0}^{1} \left(\int_{y=0}^{x^2} xy\, dy \right) dx \ ,$$

and then evaluate the integral in two ways: as it is written here, and after first changing the order of integration. Verify that the results are the same.

(ii) State Green's Theorem for a path integral of the form

$$\oint_C \left[P(x,\, y)\, dx \ + \ Q(x,\, y)\, dy \right] \ ,$$

where you may assume that the path C is closed and encircles a simply connected region R once anticlockwise.

(iii) Hence evaluate

$$\oint_C \left[\frac{xy^2}{2}\, dx \ + \ x^2 y\, dy \right] \ ,$$

where C is a closed path from the origin along the x-axis to $(1,\, 0)$, then along $x = 1$ as far as $(1,\, 1)$, then returning along the parabolic arc $y = x^2$ back to the origin.

3. (i) An analytic function $f(z)$ has real and imaginary parts $u(x,\, y)$ and $v(x,\, y)$ respectively.

Write down the Cauchy-Riemann equations which relate these two functions.

Show that both $u(x,\, y)$ and $v(x,\, y)$ satisfy Laplace's equation.

An analytic function $f(z)$ has real part $u(x,\, y) = x\exp(x)\,\cos(y) - y\exp(x)\,\sin(y)$.

Find $v(x,\, y)$, given that $v(0,\, 0) = 0$. Hence find $f(z)$.

(ii) Find the poles of the function

$$f(z) = \frac{\exp(iz)}{z^4 + 5z^2 + 4}\,,$$

and the residue at each pole.

Hence evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^4 + 5x^2 + 4}\, dx\,,$$

by integrating $f(z)$ around a suitable *closed* contour.

PLEASE TURN OVER

4. (i) Sketch the graph of the function

$$f(t) = H(t - T),$$

where H denotes the usual Heaviside function, for $t \geq 0$, where $T > 0$.

Calculate the Laplace transform $F(s)$ of $f(t)$ with respect to t.

(ii) Solve the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = f(t),$$

in the region $x > 0$, $t > 0$, where $f(t)$ is defined in part (i). The initial conditions are

$$u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0,$$

and the boundary conditions are such that

$$u(0, t) = 0,$$

and $u(x, t)$ is bounded as $x \to \infty$.

Hint: Take the Laplace transform of the whole equation with respect to t.

5. (i) In a binary symmetric channel, where X denotes the digit transmitted and Y denotes the digit received, the following transmissions probabilities hold, with all transmissions independent.

$$\begin{aligned} P(Y=1 \mid X=1) &= 0.9 & \qquad P(Y=0 \mid X=0) &= 0.9 \\ P(Y=1 \mid X=0) &= 0.1 & \qquad P(Y=0 \mid X=1) &= 0.1 \end{aligned}$$

The probability of a 1 being transmitted is 0.8.

(a) Find the probability that a 1 is received.

(b) If a 1 is received, find the probability that a 1 was transmitted.

(c) If a 3 bit string of all ones is transmitted, what is the probability that the received string will contain at most one error?

(ii) In a study to design a web crawler to search for adverts for academic jobs on webpages of job sites, the following events are defined

V: webpage advertises an academic job
A_1: webpage contains the string "Professor"
A_2: webpage contains the string "university"
A_3: webpage contains the string "academic"

It is found that,

$$\begin{aligned} P(A_1 \mid V) &= 0.2 & P(A_2 \mid V) &= 0.9 & P(A_3 \mid V) &= 0.6 \\ P(A_1 \mid \overline{V}) &= 0.01 & P(A_2 \mid \overline{V}) &= 0.5 & P(A_3 \mid \overline{V}) &= 0.1 \end{aligned}$$

Conditional on V assume that A_1, A_2 and A_3 are independent.
Given that $P(V) = 0.1$, find the probability that the webpage advertises an academic job if

(a) A_1 occurs ($= p_1$, say).

(b) both A_1 and A_2 occur ($= p_2$, say).

(c) A_1, A_2 and A_3 occur ($= p_3$, say).

(d) Explain why $p_3 > p_2 > p_1$.

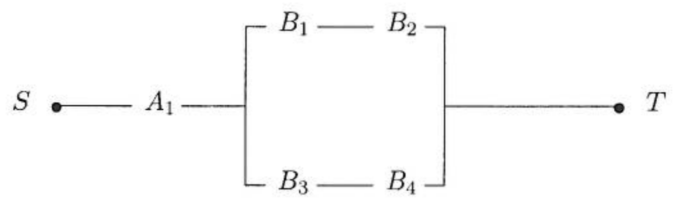
PLEASE TURN OVER

6. The lifetimes, T_A and T_B of components of type A and B, in hours, follow normal distributions with known standard deviations $\sigma = 5/4$ and 1 respectively.

   (i) The lifetimes of a sample of size $n_A = 20$ components of type A have a sample mean of $\bar{x}_A = 15$. Calculate the 95% confidence interval for the mean lifetime of a component of type A.

   (ii) Assuming that the mean lifetimes of components A and B are 15 and 16 hours respectively, find the reliabilities at time t (hours) of a component of type A and B in terms of the cdf of a normal distribution. Evaluate the reliabilities at $t = 14$ hours.

   The following network is constructed using one component, A_1, of type A and four components, B_1, B_2, B_3 and B_4, of type B, all operating independently. The network functions if there is a path of functioning components between S and T.



   (iii) Assuming the reliabilities found in part (ii), determine the reliability of the network at 14 hours.

   (iv) Now assume that the reliability at 14 hours of A_1 is R_A and the reliability at 14 hours of a component of type B is 0.9, find the smallest value of R_A that ensures that the overall reliability of the system exceeds 0.9 at 14 hours.

END OF PAPER

M A T H E M A T I C S   D E P A R T M E N T

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\, \mathbf{b},\, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b} \times \mathbf{c} = \mathbf{b}\,.\,\mathbf{c} \times \mathbf{a} = \mathbf{c}\,.\,\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha\,x + \frac{\alpha(\alpha-1)}{2!}\,x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\,x^3 + \dots \quad (\alpha \text{ arbitrary},\ |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots\ ,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots\ ,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots\ ,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{(n+1)} + \dots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\ ;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\ ; \quad \cosh iz = \cos z\ ; \quad \sin iz = i\sinh z\ ; \quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1}\,Df\,D^{n-1}\,g + \ldots + \tbinom{n}{r}\,D^r f\,D^{n-r} g + \ldots + D^n f\;g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\quad \epsilon_n(h) = h^{n+1} f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x,\,y)$ about $(a,\,b)$:

$$f(a+h,\,b+k) = f(a,\,b) + [hf_x + kf_y]_{a.b} + 1/2!\,\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a.b} + \cdots$$

(d) Partial differentiation of $f(x,\,y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u,\,v)$, $y = y(u,\,v)$, then $f = F(u,\,v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\,y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let $(a,\,b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D > 0$ and $f_{xx}(a,\,b) < 0$, then $(a,\,b)$ is a maximum;
If $D > 0$ and $f_{xx}(a,\,b) > 0$, then $(a,\,b)$ is a minimum;
If $D < 0$ then $(a,\,b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $\quad I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\,y)dx + Q(x,\,y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2\,t/(1+t^2)\,, \quad \cos\theta = (1-t^2)/(1+t^2)\,, \quad d\theta = 2\,dt/(1+t^2)\,.$

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a\,.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]\,, \quad n = 0, 1, 2\ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh\,, \quad y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)\,[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)\,[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let $I_1,\ I_2$ be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15\,,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t) dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t, \alpha)$	$(\partial/\partial\alpha)F(s, \alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin \omega t$	$\omega/(s^2 + \omega^2)$, $(s > 0)$
$\cos \omega t$	$s/(s^2 + \omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

1. $\boxed{\text{Probabilities for events}}$

For events A, B, and C $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of A $\quad P(A) \,/\, P(\overline{A})$

Conditional probability $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\quad P(A \cap B \cap C) = P(A)\, P(B \mid A)\, P(C \mid A \cap B)$

Bayes' rule $\quad P(A \mid B) = \dfrac{P(A)\, P(B \mid A)}{P(A)\, P(B \mid A) + P(\overline{A})\, P(B \mid \overline{A})}$

A and B are independent if $\quad P(B \mid A) = P(B)$

A, B, and C are independent if $\quad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

2. $\boxed{\text{Probability distribution, expectation and variance}}$

The probability distribution for a discrete random variable X is called the probability mass function (pmf) and is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) = \mu = \sum_x x p_x$

For function $g(x)$ of x, $\quad E\{g(X)\} = \sum_x g(x) p_x$, so $E(X^2) = \sum_x x^2 p_x$

Sample mean $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ estimates μ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \operatorname{var}(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$

Sample variance $\quad s^2 = \dfrac{1}{n-1} \left\{ \sum_k x_k^2 - \dfrac{1}{n} \left(\sum_j x_j \right)^2 \right\}$ estimates σ^2

Standard deviation $\quad \operatorname{sd}(X) = \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness $\quad \beta_1 = E\left(\dfrac{X - \mu}{\sigma}\right)^3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^3$

Kurtosis $\quad \beta_2 = E\left(\dfrac{X - \mu}{\sigma}\right)^4 - 3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^4 - 3$

Sample median $\quad \widetilde{x}$ or x_{med}. Half the sample values are smaller and half larger

If the sample values $x_1, \ldots, x_n$ are ordered as $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$, then $\widetilde{x} = x_{(\frac{n+1}{2})}$ if n is odd, and $\widetilde{x} = \frac{1}{2}\left(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})}\right)$ if n is even

α-quantile $Q(\alpha)$ is such that $P(X \leq Q(\alpha)) = \alpha$

Sample α-quantile $\widehat{Q}(\alpha)$ Proportion α of the data values are smaller

Lower quartile Q1 $= \widehat{Q}(0.25)$ one quarter are smaller

Upper quartile Q3 $= \widehat{Q}(0.75)$ three quarters are smaller

Sample median $\widetilde{x} = \widehat{Q}(0.5)$ estimates the population median $Q(0.5)$

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf) $F(x) = P(X \leq x) = \int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$

The probability density function (pdf) $f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x$, $\operatorname{var}(X) = \sigma^2 = E(X^2) - \mu^2$, where $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\mathrm{d}x$

4. Discrete probability distributions

Discrete Uniform *Uniform* (n)

$$p_x = \frac{1}{n} \quad (x = 1, 2, \ldots, n) \qquad \mu = (n+1)/2, \ \sigma^2 = (n^2-1)/12$$

Binomial distribution *Binomial* (n, θ)

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n) \qquad \mu = n\theta, \ \sigma^2 = n\theta(1-\theta)$$

Poisson distribution *Poisson* (λ)

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots) \ \text{(with } \lambda > 0) \qquad \mu = \lambda, \ \sigma^2 = \lambda$$

Geometric distribution *Geometric* (θ)

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots) \qquad \mu = \frac{1}{\theta}, \ \sigma^2 = \frac{1-\theta}{\theta^2}$$

5. Continuous probability distributions

Uniform distribution *Uniform* (α, β)

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & \text{(otherwise).} \end{cases} \qquad \mu = (\alpha+\beta)/2, \ \sigma^2 = (\beta-\alpha)^2/12$$

Exponential distribution *Exponential* (λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \leq 0). \end{cases} \qquad \mu = 1/\lambda, \ \sigma^2 = 1/\lambda^2$$

Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\} \quad (-\infty < x < \infty), \quad E(X) = \mu, \quad \mathrm{var}\,(X) = \sigma^2$$

Standard normal distribution $N(0,1)$

If X is $N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$

6. Reliability

For a device in continuous operation with failure time random variable T having pdf $f(t)$ $(t > 0)$

The reliability function at time t	$R(t)$	$=$	$P(T > t)$		
The failure rate or hazard function	$h(t)$	$=$	$f(t)/R(t)$		
The cumulative hazard function	$H(t)$	$=$	$\int_0^t h(t_0)\,\mathrm{d}t_0$	$=$	$-\ln\{R(t)\}$
The Weibull(α, β) distribution has	$H(t)$	$=$	βt^{α}		

7. System reliability

For a system of k devices, which operate independently, let

$R_i = P(D_i) = P($"device i operates"$)$

The system reliability, R, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$$

A system of devices in parallel operates if any device operates

$$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1 - R_1)(1 - R_2) \cdots (1 - R_k)$$

8. Covariance and correlation

The covariance of X and Y $\quad \mathrm{cov}\,(X,Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample covariance	s_{xy}	$=$	$\dfrac{1}{n-1} S_{xy}$	estimates $\mathrm{cov}\,(X,Y)$	
Correlation coefficient	ρ	$=$	$\mathrm{corr}\,(X,Y)$	$=$	$\dfrac{\mathrm{cov}\,(X,Y)}{\mathrm{sd}\,(X) \cdot \mathrm{sd}\,(Y)}$
Sample correlation coefficient	r	$=$	$\dfrac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$	estimates ρ	

9. Sums of random variables

$$E(X+Y) = E(X)+E(Y)$$

$$\text{var}(X+Y) = \text{var}(X)+\text{var}(Y)+2\,\text{cov}(X,Y)$$

$$\text{cov}(aX+bY,\ cX+dY) = (ac)\,\text{var}(X)+(bd)\,\text{var}(Y)+(ad+bc)\,\text{cov}(X,Y)$$

If X is $N(\mu_1,\sigma_1^2)$, Y is $N(\mu_2,\sigma_2^2)$, and $\text{cov}(X,Y)=c$, then $X+Y$ is $N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2+2c)$

10. Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

Bias of t	$\text{bias}(t)$	$= E(T)-\theta$
Standard error of t	$\text{se}(t)$	$= \text{sd}(T)$
Mean square error of t	$\text{MSE}(t)$	$= E\{(T-\theta)^2\} = \{\text{se}(t)\}^2+\{\text{bias}(t)\}^2$

If $\overline{x}$ estimates μ, then $\text{bias}(\overline{x})=0$, $\text{se}(\overline{x})=\sigma/\sqrt{n}$, $\text{MSE}(\overline{x})=\sigma^2/n$, $\widehat{\text{se}}(\overline{x})=s/\sqrt{n}$

Central limit property If n is fairly large, $\overline{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ.

For a random sample $x_1,x_2,\ldots,x_n$

$$\ell(\theta;\,x_1,x_2,\ldots,x_n) = P(X_1=x_1\,|\,\theta)\cdots P(X_n=x_n\,|\,\theta) \qquad \text{(discrete distribution)}$$

$$\ell(\theta;\,x_1,x_2,\ldots,x_n) = f(x_1\,|\,\theta)\,f(x_2\,|\,\theta)\cdots f(x_n\,|\,\theta) \qquad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\widehat{\theta}$ for which the likelihood is a maximum

12. Confidence intervals

If $x_1,x_2,\ldots,x_n$ are a random sample from $N(\mu,\sigma^2)$ and σ^2 is known, then the 95% confidence interval for μ is $(\overline{x}-1.96\frac{\sigma}{\sqrt{n}},\ \overline{x}+1.96\frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0=t_{n-1,0.05}$

The 95% confidence interval for μ is $(\overline{x}-t_0\frac{s}{\sqrt{n}},\ \overline{x}+t_0\frac{s}{\sqrt{n}})$

13. Standard normal table — Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table — Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

m	$p=$ 0.10	0.05	0.02	0.01	m	$p=$ 0.10	0.05	0.02	0.01
1	6.31	12.71	31.82	63.66	9	1.83	2.26	2.82	3.25
2	2.92	4.30	6.96	9.92	10	1.81	2.23	2.76	3.17
3	2.35	3.18	4.54	5.84	12	1.78	2.18	2.68	3.05
4	2.13	2.78	3.75	4.60	15	1.75	2.13	2.60	2.95
5	2.02	2.57	3.36	4.03	20	1.72	2.09	2.53	2.85
6	1.94	2.45	3.14	3.71	25	1.71	2.06	2.48	2.78
7	1.89	2.36	3.00	3.50	40	1.68	2.02	2.42	2.70
8	1.86	2.31	2.90	3.36	∞	1.645	1.96	2.326	2.576

15. Chi-squared table — Values $\chi^2_{k,p}$ of x for which $P(X > x) = p$, when X is χ^2_k and $p = .995$, .975, *etc*

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum_y \dfrac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of χ^2_k with significance point p,

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ

17. Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$$p_{x\bullet} = \sum_y p_{xy} \quad \text{and} \quad P(X = x \,|\, Y = y) = \frac{p_{xy}}{p_{\bullet y}}$$

Continuous distribution

Joint cdf $\quad F(x,y) = P(\{X \le x\} \cap \{Y \le y\}) = \displaystyle\int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0, y_0)\, \mathrm{d}x_0\, \mathrm{d}y_0$

Joint pdf $\quad f(x,y) = \dfrac{\mathrm{d}^2 F(x,y)}{\mathrm{d}x\, \mathrm{d}y}$

Marginal pdf of X $\quad f_X(x) = \displaystyle\int_{-\infty}^{\infty} f(x, y_0)\, \mathrm{d}y_0$

Conditional pdf of X given $Y = y$ $\quad f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. Linear regression

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is $\quad \widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}, \quad \widehat{\beta} = \dfrac{S_{xy}}{S_{xx}}$

The residual sum of squares $\quad \text{RSS} = S_{yy} - \dfrac{S_{xy}^2}{S_{xx}}$

$\widehat{\sigma^2} = \dfrac{\text{RSS}}{n-2} \qquad \dfrac{n-2}{\sigma^2}\widehat{\sigma^2}$ is from χ^2_{n-2}

$$E(\widehat{\alpha}) = \alpha, \quad E(\widehat{\beta}) = \beta,$$

$$\text{var}(\widehat{\alpha}) = \frac{\sum x_i^2}{n S_{xx}}\sigma^2, \quad \text{var}(\widehat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \text{cov}(\widehat{\alpha}, \widehat{\beta}) = -\frac{\overline{x}}{S_{xx}}\sigma^2$$

$$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x, \qquad E(\widehat{y}_x) = \alpha + \beta x, \qquad \text{var}(\widehat{y}_x) = \left\{\frac{1}{n} + \frac{(x - \overline{x})^2}{S_{xx}}\right\}\sigma^2$$

$\dfrac{\widehat{\alpha} - \alpha}{\widehat{\text{se}}(\widehat{\alpha})}, \quad \dfrac{\widehat{\beta} - \beta}{\widehat{\text{se}}(\widehat{\beta})}, \quad \dfrac{\widehat{y}_x - \alpha - \beta x}{\widehat{\text{se}}(\widehat{y}_x)}$ are each from t_{n-2}

E2.11 (Maths) ISE

(1)

Solutions - 2011

A1. The Fourier transform $\hat{f}(\omega)$ of $f(t)$ is defined by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t)\, dt.$$

(i) The Fourier transform of $g(t)$ where $g(t) = f(t+\tau)$, and τ is an arbitrary real constant, is

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} f(t+\tau) \exp(-i\omega t)\, dt$$
$$= \int_{-\infty}^{\infty} f(s) \exp(-i\omega s + i\omega\tau)\, ds$$
$$= \exp(i\omega\tau)\hat{f}(\omega).$$

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(ii) If $f(t,a,b)$ is defined by

$$f(t,a,b) = 1 \qquad a < t < b,$$

$$f(t,a,b) = 0 \qquad \text{otherwise},$$

then

$$\hat{f}(\omega;a,b) = \int_a^b \exp(-i\omega t)\, dt =$$
$$\frac{(\exp(-i\omega b) - \exp(-i\omega a))}{-i\omega}.$$

4

We see that $f(t+\tau,a,b) = f(t,a-\tau,b-\tau)$, and its FT is thus

$$\frac{(\exp(-i\omega(b-\tau)) - \exp(-i\omega(a-\tau)))}{-i\omega} =$$

$$\frac{(\exp(-i\omega b) - \exp(-i\omega a))}{-i\omega} \exp(i\omega\tau),$$

2

as required.

(iii) The *convolution* $f * g(t)$ of two functions $f(t)$ and $g(t)$ is defined as

$$f * g(t) = \int_{-\infty}^{\infty} f(s) g(t-s)\, ds.$$

The Convolution Theorem states that

$$f \hat{*} g(\omega) = \hat{f}(\omega)\hat{g}(\omega).$$

3

(iv) The convolution of $f(t,-1,1)$ with itself is

$$f * f(t) = \int_{-\infty}^{\infty} f(s,-1,1) f(t-s,-1,1)\, ds.$$

The integrand is nonzero where $-1 < s < 1$ and $-1 < t-s < 1$; that is, where $-1 < s < 1$ and $t-1 < s < t+1$. We have 4 cases to consider:

RLU

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1. If $t > 2$ then these two ranges of s do not overlap and the integrand is zero: $f * f(t) = 0$ if $t > 0$.
2. If $0 < t < 2$ the integral becomes $\int_{t-1}^{1} ds = 2 - t$.
3. If $-2 < t < 0$ then the integral is $\int_{-1}^{t+1} ds = 2 + t$.
4. If $t < -2$ we again get zero.

4

So $f * f(t) = 2 - |t|$, if $|t < 2$, and zero otherwise.

The Fourier Transform of this convolution is the square of the Fourier transform $\hat{f}(\omega, -1, 1)$. So

$$\widehat{f * f} = \left(\frac{2\sin(\omega)}{\omega}\right)^2 = 4\frac{\sin^2(\omega)}{\omega^2}.$$

Hence, using Parseval's Theorem,

$$\int_{-\infty}^{\infty} 16\frac{\sin^4(\omega)}{\omega^4}\, d\omega = 2\pi \int_{-2}^{2} (2 - |t|)^2\, dt$$

$$= 4\pi \int_{0}^{2} t^2 dt = 4\pi 8/3.$$

3

Dividing though, we get

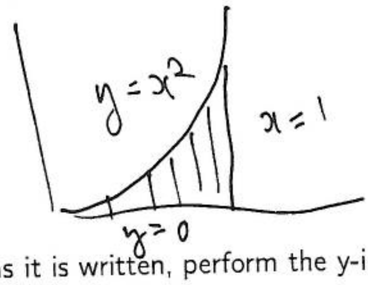
$$\int_{-\infty}^{\infty} \frac{\sin^4(\omega)}{\omega^4}\, d\omega = \frac{2\pi}{3}.$$

RLJ

A2. (i) The region of integration for the integral

$$\int_{x=0}^{1}\left(\int_{y=0}^{x^2} xy\,dy\right)\,dx,$$

is bounded below by the x-axis $y=0$, above by the parabola $y=x^2$, and on the right by the line $x=1$.



To evaluate the integral as it is written, perform the y-integration first:

$$\int_{x=0}^{1}\left(\int_{y=0}^{x^2} xy\,dy\right)\,dx$$

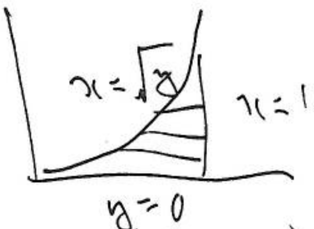
$$=\int_{x=0}^{1} dx\left(x\int_{y=0}^{x^2} y\,dy\right)$$

$$=\int_{x=0}^{1} dx\left(x[y^2/2]_0^{x^2}\right)$$

$$=\int_{x=0}^{1} x^5/2\,dx=[x^6/12]_0^1=1/12.$$

6

Alternatively to change the order of integration, we write the boundaries as $x=\sqrt{y}$ on the left, $x=1$ on the right; $y=0$ below, $y=1$ above.



We get

$$\int_{y=0}^{1}\left(\int_{x=\sqrt{y}}^{1} xy\,dx\right)\,dy$$

$$=\int_{y=0}^{1} dy\left(y[x^2/2]_{\sqrt{y}}^{1}\right)$$

$$=\int_{y=0}^{1} y(1-y)/2\,dy=[y^2/4-y^3/6]_0^1=1/12;$$

6

we see that the result is the same in each case.

(ii) Green's Theorem states that, for a path integral of the form

$$\oint_C P(x,y)\,dx+Q(x,y)\,dy,$$

RLW

(2)

where the path C is closed and encircles a simply connected region R once anticlockwise,

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy = \int\int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy.$$

4

(iii) To evaluate

$$\oint_C \frac{xy^2}{2}\,dx + x^2 y\,dy,$$

where C is a closed path from the origin along the x-axis to $(1,0)$, then along $x=1$ as far as $(1,1)$, then returning along the parabolic arc $y=x^2$ back to the origin, use Green's Theorem.

4

$$\oint_C \frac{xy^2}{2}\,dx + xy^2\,dy = \int\int_R (2xy - xy)\,dx\,dy,$$

and R is the same region as in Part (i). By that result, the path integral we need equals $1/12$.

RLJ

A3. (i) If an analytic function $f(z)$ has real and imaginary parts $u(x,y)$ and $v(x,y)$ respectively, then the Cauchy-Riemann equations which relate these two functions are:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Differentiating the first wrt x, the second wrt y, and adding, we see that $u(x,y)$ satisfies Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

and similarly for v.

4

If an analytic function has real part $u(x,y) = x\exp(x)\cos(y) - y\exp(x)\sin(y)$, then $v(x,y)$ must satisfy the CR equations

$$(x+1)\exp(x)\cos(y) - y\exp(x)\sin(y) = \frac{\partial v}{\partial y},$$

$$(x+1)\exp(x)\sin(y) + y\exp(x)\cos(y) = \frac{\partial v}{\partial x}.$$

Integrating to (x,y) from $(0,0)$ along any path, we find

$$v = y\exp(x)\cos(y) + x\exp(x)\sin(y).$$

The function $f(z) = u(x,y) + iv(x,y)$, so we find

4

$$f(z) = (x+iy)\exp(x)\cos(y) + i(x+iy)\exp(x)\sin(y)$$

$$= z\exp(z).$$

(ii) The poles of the function

$$f(z) = \frac{\exp(iz)}{z^4 + 5z^2 + 4},$$

are found at the zeroes of the denominator. Hence they are where

$$z^4 + 5z^2 + 4 = (z^2+4)(z^2+1) = 0,$$

giving $z = \pm i$, $z = \pm 2i$. The residue at $z = z_i$ is

$$\operatorname*{res}_{z=z_i} f(z) = \lim_{z\to z_i}(z - z_i)f(z);$$

thus

5

$$\operatorname*{res}_{z=i} f(z) = \lim_{z\to i}(z-i)\frac{\exp(iz)}{(z^2+4)(z^2+1)}$$

$$= \frac{e^{-1}}{3.2i} = -\frac{ie^{-1}}{6},$$

$$\operatorname*{res}_{z=-i} f(z) = \lim_{z\to -i}(z+i)\frac{\exp(iz)}{(z^2+4)(z^2+1)}$$

RLJ

3

$$= \frac{e}{-3.2i} = \frac{ie}{6},$$

$$\operatorname*{res}_{z=2i} f(z) = \lim_{z \to 2i} (z - 2i) \frac{\exp(iz)}{(z^2+4)(z^2+1)}$$
$$= \frac{e^{-2}}{4i.(-3)} = \frac{ie^{-2}}{12},$$

and

$$\operatorname*{res}_{z=-2i} f(z) = \lim_{z \to -2i} (z + 2i) \frac{\exp(iz)}{(z^2+4)(z^2+1)}$$
$$= \frac{e^{2}}{-4i.(-3)} = -\frac{ie^{2}}{12}.$$

To evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{\cos(x)}{x^4 + 5x^2 + 4} \, dx,$$

we integrate $f(z)$ around a *closed* contour C, consisting of the real axis, closed with a large semicircle in the *upper* half plane, where $\exp(iz)$ is exponentially small. I will be the real part of this integral.

$$\oint_C \frac{\exp(iz)}{z^4 + 5z^2 + 4} \, dz = 2\pi i \sum_{poles} \operatorname{res} z = z_i f(z),$$

7

where the sum is taken over poles inside C, here $z = i$ and $z = 2i$. Thus

$$I = \Re\left(2\pi i (\operatorname*{res}_{z=i} f(z) + \operatorname*{res}_{z=2i} f(z))\right)$$
$$= \Re\left(2\pi i (-\frac{ie^{-1}}{6} + \frac{ie^{-2}}{12})\right)$$
$$= 2\pi(\frac{e^{-1}}{6} - \frac{e^{-2}}{12}).$$

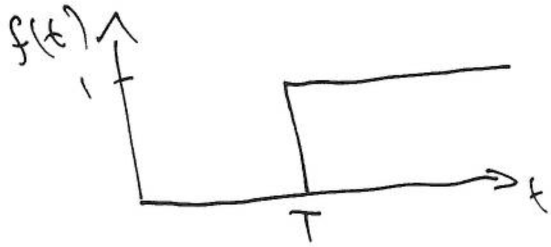
RLW

4

A4. (i) The graph of the function

$$f(t) = H(t - T),$$

for $t \geq 0$, where $T > 0$, has the form



2

The Laplace transform $F(s)$ of $f(t)$ with respect to t is, by definition,

$$F(s) = \int_0^\infty f(t) \exp(-st)\, dt$$

$$= \int_T^\infty \exp(-st)\, dt = \frac{\exp(-sT)}{s}.$$

We could also get this result by the shift theorem. 4

(ii) To solve the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = f(t),$$

Seen similar

in the region $x > 0$, $t > 0$, with the boundary and initial conditions

$$u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0,$$

$$u(0, t) = 0,$$

$$u(x, t) \quad \text{is bounded as} \quad x \to \infty,$$

4

we take the Laplace transform of the whole equation with respect to t.

This gives, where $U(x, s)$ is the Laplace transform of $u(x, t)$ with respect to t,

$$\frac{\mathrm{d^2U}}{\mathrm{dx^2}} - (s^2 U(x, s) - su(x, 0) - u_t(x, 0)) = \frac{\exp(-sT)}{s}.$$

Note that the right hand side is independent of x, and that the two terms in the initial conditions vanish:

$$\frac{\mathrm{d^2U}}{\mathrm{dx^2}} - s^2 U(x, s) = \frac{\exp(-sT)}{s}.$$

Note that this is an *ordinary* differential equation for $U(x, s)$ wrt x. It follows that

$$U = A(s) \exp(sx) + B(s) \exp(-sx) - \frac{\exp(-sT)}{s^3}.$$

4

Now if u is bounded as $x \to \infty$, so is its Laplace transform for positive s, and so we must have $A(s) = 0$.

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4

If $u(0,t)=0$, then $U(0,s)=0$, so that we find

$$B(s)=\frac{\exp(-sT)}{s^3}.$$

Thus

$$U(x,s)=\frac{\exp(-sT)}{s^3}\left(\exp(-sx)-1\right).$$ 3

Inverting, we find

$$u(x,t)=-\mathcal{L}^{-1}\left(\frac{\exp(-sT)}{s^3}\right)+\mathcal{L}^{-1}\left(\frac{\exp(-sT)}{s^3}\exp(-sx)\right).$$

Now the inverse Laplace Transform of $1/s^3$ is $t^2/2$. By the shift theorem, therefore, we get:

$$u(x,t)=\frac{(t-x-T)^2}{2}H(t-x-T)-\frac{(t-T)^2}{2}H(t-T).$$ 3

RLV

5. (i) (a)

$$\begin{aligned} P(Y=1) &= P(Y=1 \mid X=0)P(X=0) + P(Y=1 \mid X=1)P(X=1) \\ &= 0.1 \times 0.2 + 0.9 \times 0.8 = 0.74 \end{aligned} \quad \boxed{2}$$

(b)

$$P(X=1 \mid Y=1) = \frac{P(Y=1 \mid X=1)P(X=1)}{P(Y=1)} = \frac{0.9 \times 0.8}{0.74} = 0.9730 \quad \boxed{3}$$

(c) Let N = number of digits in error, then $N \sim Binomial(3, 0.1)$ $\boxed{2}$
and $P(N=n) = \binom{3}{n}(0.1)^n(0.9)^{3-n}$.

$$P(N \le 1) = P(N=0) + P(N=1) = (0.9)^3 + 3(0.1)(0.9)^2 = 0.972 \quad \boxed{2}$$

(ii) (a)

$$\begin{aligned} P(V \mid A_1) &= \frac{P(A_1 \mid V)P(V)}{P(A_1)} = \frac{P(A_1 \mid V)P(V)}{P(A_1 \mid V)P(V) + P(A_1 \mid \overline{V})P(\overline{V})} \\ &= \frac{0.2 \times 0.1}{0.2 \times 0.1 + 0.01 \times 0.9} = 0.690 \end{aligned} \quad \boxed{2}$$

(b)

$$\begin{aligned} P(V \mid A_1 \cap A_2) &= \frac{P(A_1 \cap A_2 \mid V)P(V)}{P(A_1 \cap A_2)} \\ &= \frac{P(A_1 \mid V)P(A_2 \mid V)P(V)}{P(A_1 \cap A_2 \mid V)P(V) + P(A_1 \cap A_2 \mid \overline{V})P(\overline{V})} \\ &= \frac{P(A_1 \mid V)P(A_2 \mid V)P(V)}{P(A_1 \mid V)P(A_2 \mid V)P(V) + P(A_1 \mid \overline{V})P(A_2 \mid \overline{V})P(\overline{V})} \\ &= \frac{0.2 \times 0.9 \times 0.1}{0.2 \times 0.9 \times 0.1 + 0.01 \times 0.5 \times 0.9} = 0.8 \end{aligned} \quad \boxed{3}$$

(c)

$$\begin{aligned} P(V \mid A_1 \cap A_2 \cap A_3) &= \frac{P(A_1 \cap A_2 \cap A_3 \mid V)P(V)}{P(A_1 \cap A_2 \cap A_3)} \\ &= \frac{P(A_1 \mid V)P(A_2 \mid V)P(A_3 \mid V)P(V)}{P(A_1 \cap A_2 \cap A_3 \mid V)P(V) + P(A_1 \cap A_2 \cap A_3 \mid \overline{V})P(\overline{V})} \\ &= \frac{P(A_1 \mid V)P(A_2 \mid V)P(A_3 \mid V)P(V)}{P(A_1 \mid V)P(A_2 \mid V)P(A_3 \mid V)P(V) + P(A_1 \mid \overline{V})P(A_2 \mid \overline{V})P(A_3 \mid \overline{V})P(\overline{V})} \\ &= \frac{0.2 \times 0.9 \times 0.6 \times 0.1}{0.2 \times 0.9 \times 0.6 \times 0.1 + 0.01 \times 0.5 \times 0.1 \times 0.9} = 0.96 \end{aligned} \quad \boxed{4}$$

(d) The probabilities increase as we have $P(A_i \mid V) > P(A_i \mid \overline{V})$, so as we include more terms, we introduce more information about whether the advert is for an academic job. $\boxed{2}$

6. (i) Let μ_A and μ_B be the mean lifetime of components of types A and B respectively.
The 95% CI for μ_A is

$$(\bar{x}_A \pm 1.96\sigma_A/\sqrt{n_A}) = (15 \pm 1.96(5/4)/\sqrt{20}) = (14.452, 15.548).$$ $\boxed{3}$

(ii) Reliability:

$$R(t) = \mathrm{P}(T > t) = \mathrm{P}\left(\frac{T-\mu}{\sigma} > \frac{t-\mu}{\sigma}\right) = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right)$$ $\boxed{3}$

$$R_A(14) = 1 - \Phi\left(\frac{14-15}{5/4}\right) = 1 - \Phi(-4/5) = \Phi(0.8) = 0.788.$$ $\boxed{2}$

$$R_B(14) = 1 - \Phi\left(\frac{14-16}{1}\right) = 1 - \Phi(-2) = \Phi(2) = 0.977.$$ $\boxed{2}$

(iii) Let N = event that the network is functioning after 14 hours, A and B_1, B_2, B_3, B_4 be the events that individual components are functioning after 14 hours, and $R_N(14)$ = the reliability of the network at $t = 14$ hours:

$$\begin{aligned} N &= A \cap ((B_1 \cap B_2) \cup (B_3 \cap B_4)) \\ \Rightarrow R_N(14) &= R_A(14)\left(R_B^2(14) + R_B^2(14) - R_B^4(14)\right) \\ &= 0.788\left(2 \times 0.977^2 - 0.977^4\right) \\ &= 0.7864. \end{aligned}$$ $\boxed{6}$

(iv) We have

$$\begin{aligned} R_N(14) &= R_A\left(R_B^2(14) + R_B^2(14) - R_B^4(14)\right) \\ &= R_A\left(2 \times 0.9^2 - 0.9^4\right) \\ \Rightarrow 0.9 &< R_A\left(2 \times 0.9^2 - 0.9^4\right) \\ R_A &> 0.9337. \end{aligned}$$ $\boxed{4}$