E4.23
C1.4
ISE4.27

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2008

MSc and EEE/ISE PART IV: MEng and ACGI

**STABILITY AND CONTROL OF NON-LINEAR SYSTEMS**

Monday, 12 May 10:00 am

Time allowed: 3:00 hours

Corrected Copy

**There are SIX questions on this paper.**

**Answer FOUR questions.**

*All questions carry equal marks*

**Any special instructions for invigilators and information for candidates are on page 1.**


Examiners responsible First Marker(s) : J.C. Allwright

Second Marker(s) : E.C. Kerrigan

**Information for invigilators:** none

**Information for candidates:**

$\|x\|$ denotes $\sqrt{x^{\mathrm{T}}x}$ for $x \in \mathbb{R}^n$

$\|x\|_{\mathcal{L}_2}$ denotes $\sqrt{\int_0^\infty x(t)^{\mathrm{T}}x(t)dt}$ for $x \in \mathcal{L}_2$

$x_\tau$ denotes the truncation of $x$ onto $[0,\tau]$

$\mathcal{L}_{2e}$ denotes the extension of $\mathcal{L}_2$

Unless clear from the context: $\varepsilon(t)$, $y(t)$, $u(t)$ and $r(t)$ are all scalar valued

For a symmetric matrix $P$: $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the minimum and maximum eigenvalues of $P$, respectively

* denotes complex conjugate

$\triangleq$ means 'is defined equal to'

$I_n$ is the $n \times n$ identity matrix.

# The Questions

1. (a) Here $x \in \mathbb{R}$.

(i) Consider the systems

$$\ddot{x} = f_1(x, \dot{x}) \triangleq -x + 1; \qquad \ddot{x} = f_2(x, \dot{x}) \triangleq -x.$$

By adapting the general form of solution for such systems, and without using isoclines, sketch typical trajectories for these systems in the $(x, \dot{x})$ phase-space. [3]

(ii) Consider also the systems

$$\ddot{x} = f_3(x, \dot{x}) \triangleq -\dot{x}; \qquad \ddot{x} = f_4(x, \dot{x}) \triangleq \dot{x}.$$

Determine the corresponding phase-plane differential equations and use them to sketch typical trajectories for each system in the $(x, \dot{x})$ phase-plane. [3]

(iii) Use the results of parts (a-i) and (a-ii) above to sketch the complete closed trajectory (starting and returning later to the starting point) that has the initial condition $x(0) = 0, \dot{x}(0) = 3$ for the system

$$\ddot{x} = \begin{cases} f_1(x, \dot{x}) & \text{if } x > 1 \\ f_2(x, \dot{x}) & \text{if } x < 0. \\ f_3(x, \dot{x}) & \text{if } x \in [0,1] \text{ and } \dot{x} \geq 0 \\ f_4(x, \dot{x}) & \text{if } x \in [0,1] \text{ and } \dot{x} < 0. \end{cases}$$

Here the $f_i$ are those of parts (a-i) and (a-ii) above. [4]

What is the value of $t$ at which $x$ becomes equal to 1 for the second time?.

**Hint:** $\int \frac{1}{\gamma x + \delta} dx = \frac{1}{\gamma} \ln(\gamma x + \delta)$. [3]

(b) In Figures 1.1 and 1.2 below: $n$ is a skew-symmetric function.

(i) Consider Figure 1.1. Suppose $e(t) = a\sin(\omega t)$ for scalar $a \geq 0$ and all $t \geq 0$. Outline the way in which a Fourier series representation of $u$ over one period of $\sin(\omega t)$ can be used to obtain an approximation to $u$ that has the form $\overline{a}\sin(\omega t + \phi)$ for values of $\overline{a}$ and $\phi$ which should be specified. Hence derive briefly a general formula for the describing-function $N(a)$ of $n$. [4]

(ii) Consider Figure 1.2. State the harmonic balance equation. Outline a graphical method for predicting whether $e$ will oscillate. How can you predict the amplitude and frequency of such an oscillation? [3]

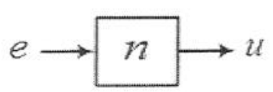


Figure 1.1

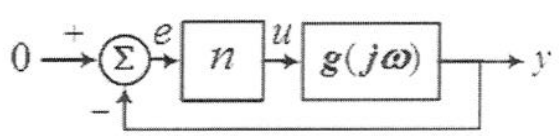


Figure 1.2

2. (a) Consider the function $v(x,t) = x^{\mathrm{T}}Px$ where $x \in \mathbb{R}^n$ and $P^{\mathrm{T}} = P > 0$.

Show that $v$ is radially-unbounded positive-definite and decrescent on $\mathbb{R}^n$. **[4]**

(b) (i) Consider the system

$$\begin{aligned} \dot{x}_1 &= 2x_2 - x_1 \\ \dot{x}_2 &= -x_1 - 3x_2 \end{aligned} \tag{2.1}$$

and the function

$$v(x,t) = x_1^2 + 2x_2^2. \tag{2.2}$$

Why is the origin an equilibrium state for this system?

Write the strongest stability result for the origin that can be obtained using $v$, the result of part (a) and Lyapunov theory. Justify your application of the relevant theorem. **[4]**

(ii) Consider $v$ of (2.2) and the following modified version of system (2.1):

$$\begin{aligned} \dot{x}_1 &= 2x_2 - x_1^3 \\ \dot{x}_2 &= -x_1 - 3x_2^5. \end{aligned} \tag{2.3}$$

What can be said about the stability properties of the origin for (2.3) using the Lyapunov Linearization Theorem? Justify your answer. **[2]**

For (2.3), show that $-\dot{v}$ is positive definite on the set

$$G_1 = \{x \in \mathbb{R}^2 : \|x\|_2 \leq 1\}.$$ **[2]**

Obtain the strongest stability result for the origin regarding system (2.3) that can be proved using part (a) and the above property of $-\dot{v}$. **[1]**

(c) This part concerns the application of sliding mode control to the system

$$\dot{x}(t) = Ax(t) + bu(t) + d : x(0) = x_0$$

where $d, x \in \mathbb{R}^2$ and $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with constant $d$.

Suppose it is desired that $x_1(t) = \exp(-2t)$ when $t \geq 0$.

Let

$$e(t) = x_1(t) - \exp(-2t)$$

and

$$s(t) = \dot{e}(t) + 3e(t).$$

What useful outcome is obtained by arranging that $s(t) = 0$ for all $t$ greater than a particular time $\tau$? Justify your answer. **[3]**

Show that there are control laws that will reduce $s$ to zero and keep it at zero. **[4]**

3. Here $A \in \mathbb{R}^{n\times n}$ is a stability matrix if the real part of each eigenvalue of $A$ is strictly negative.

(a) Consider the model of a system given by

$$\dot{x}(t) = Ax(t) + bu(t) : x(0) = x_o. \tag{3.1}$$

Here $A$ is a stability matrix and $b,\ x \in \mathbb{R}^n$. The control $u$ is scalar-valued.

(i) Suppose $P \in \mathbb{R}^{n\times n}$ satisfies the equation

$$A^{\mathrm{T}}P + PA + Q - Pbb^{\mathrm{T}}P = 0 \tag{3.2}$$

where $P = P^{\mathrm{T}} > 0$ and $Q \in \mathbb{R}^{n\times n}$ with $Q = Q^{\mathrm{T}} > 0$.

Use the radially-unbounded positive-definite decrescent function

$$v(x,t) = x^{\mathrm{T}}Px \tag{3.3}$$

to show that, when the control law

$$u(t) = -b^{\mathrm{T}}Px(t) \tag{3.4}$$

is applied, the origin is globally asymptotically stable for system (3.1). **[5]**

(ii) Now suppose that, owing to a modelling error, the system is actually

$$\dot{x}(t) = (A+\delta A)x(t) + bu(t) : x(0) = x_o \tag{3.5}$$

where $\delta\mathrm{A} \in \mathbb{R}^{n\times n}$.

Use the $P$. $v$ and $u$ defined in part (a-i) to show that the origin is globally asymptotically stable for the actual system of (3.5) if $\|\delta A\| < \Delta$ for an appropriate strictly positive $\Delta$, which should be specified. **[8]**

(b) Suppose the matrix $A \in \mathbb{R}^{n\times n}$ has the real Schur factorization $A = ZMZ^{\mathrm{T}}$ where $Z$ is orthogonal.

Apply the method used by Bartels and Stuart to exploit this factorization in order to transform the matrix Lyapunov equation

$$A^{\mathrm{T}}P + PA = -Q \tag{3.6}$$

into an equation which is more easy to solve. Here $A,\ P,\ Q \in \mathbb{R}^{n\times n}$ with $P^{\mathrm{T}} = P > 0$ and $Q^{\mathrm{T}} = Q > 0$.

State the way in which you would determine the solution $P$ of (3.6) from the solution of the transformed equation. **[5]**

Use this method to determine $P$ when $M$ is a non-singular diagonal stability matrix and $Q = qI_n$ with $q$ a strictly positive scalar and $I_n$ the identity matrix from $R^{n\times n}$. **[2]**

4. Consider the system

$$x_{k+1} = Ax_k + bu_k : \quad x_0 = x_o$$

where $b,\ x_k \in \mathbb{R}^n$ and where $A \in \mathbb{R}^{n\times n}$ is a discrete-time stability matrix in that the modulus of each eigenvalue of $A$ is strictly less than 1.

For symmetric positive-definite $Q \in \mathbb{R}^{n\times n}$, let

$$P = Q + A^{\mathrm{T}}QA + (A^{\mathrm{T}})^2QA^2 + (A^{\mathrm{T}})^3QA^3 + \ldots$$

and

$$v(x) = x^{\mathrm{T}}Px.$$

(a) Suppose $n > 2$ and $Q = I_n$ and $A^2 = 0$, where $I_n$ is the $n \times n$ identity matrix. You may assume without proof that $v$ is radially-unbounded positive-definite and decrescent on $\mathbb{R}^n$.

(i) Determine the value of $P$. Show that if the $u_k$ are all zero then

$$v(x_{k+1}) - v(x_k) \le -\|x_k\|^2,\ \forall k \ge 0. \qquad [4]$$

Hence prove global asymptotic stability of the origin using a Lyapunov theorem. [1]

(ii) Now suppose that each $u_k$ is chosen to be $\widehat{u}_k(x_k)$ where this minimizes $v(Ax_k + bu_k)$ with respect to $u_k \in [-1, 2]$.

Why might one wish to use such an optimal control? [1]

Derive a simple method for finding $\widehat{u}_k(x_k)$ that takes advantage of the simple form of $v$. [4]

Modify the analysis of part (a-i) to show that the origin is globally asymptotically stable for the optimally controlled system. [5]

(b) Suppose $n = 2$ and the matrix Lyapunov equation

$$A^{\mathrm{T}}PA - P = -Q$$

has a unique symmetric solution $P$.

Define the vec of a matrix $M \in \mathbb{R}^{2\times 2}$ and the Kronecker product of two matrices from $\mathbb{R}^{2\times 2}$.

Use vecs and a Kronecker product to determine $P$ in terms of the solution of a linear equation $Mf = g$ for appropriate $M$, $f$ and $g$, which should be specified.

Why is your $M$ non-singular? [5]

5. (a) Consider the system $H$ of Figure 5.1 where $u$ and $y$ are scalar valued.

(i) Define strict input passivity for $H$.

Suppose the output $y$ of $H$ is given by $y(t) = \phi(u(t), t)$ where $\phi \in$ sector[$\alpha$,$\beta$] with $0 < \alpha < \beta < \infty$.

Show that $H$ is strictly input passive and that

$$\|y_T\|_{\mathcal{L}_2} \leq \beta \|u_T\|_{\mathcal{L}_2}, \forall T \in [0, \infty), \forall u \in \mathcal{L}_{2e}.$$ **[5]**

(ii) Define strict output passivity for $H$.

Suppose $H$ consists of a time-invariant system

$$\dot{x} = Ax + bu : x(0) = x_0$$
$$y = c^{\mathrm{T}} x$$

where the real part of each eigenvalue of $A$ is strictly negative.

Suppose $P \in \mathbb{R}^{n \times n}$ is symmetric and positive definite and solves

$$A^T P + PA = -cc^{\mathrm{T}}$$

and suppose that

$$c = Pb.$$

By considering $\int_0^T \frac{d}{dt} x(t)^{\mathrm{T}} P x(t) dt$, show that $H$ is strictly output passive. **[7]**

(b) Consider the system of Figure 5.2 below, where $H_1$ is passive and $H_2$ is strictly output passive and the system equations have a unique solution whenever $r \in \mathcal{L}_{2e}$.

The subsystem $H_1$ does not contain any dynamics and $x_o$ represents the initial condition for $H_2$.

Suppose $r(t) = 0$ for all $t$.

Show that $\|y_T\|_{\mathcal{L}_2} < \infty$ for all finite $T$.

**Hint**: start by considering $\int_0^T u(t) r(t) dt$ for general $r$. **[8]**

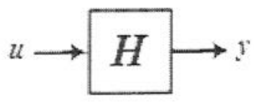


Figure 5.1

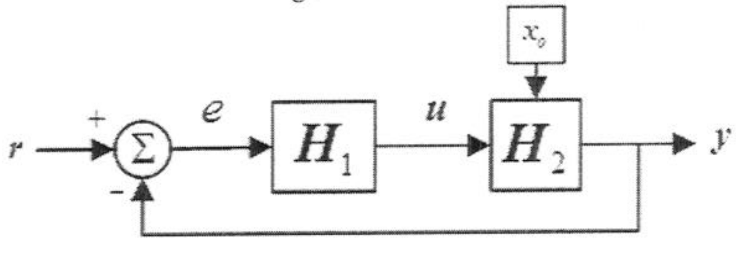


Figure 5.2

6. (a) Consider a SISO system with input $u(t)$ and output $y(t)$.

Show that the system is passive if $y(t)u(t) = \alpha f(t)\dot{f}(t)$ for scalar $\alpha > 0$.

Now suppose $y(t)u(t) = (1-\alpha\gamma(t))p(t)$ for scalar $\alpha$, $\gamma(t)$, $p(t)$ with $\alpha > 0$.

Write $(1-\alpha\gamma(t))p(t)$ as $\alpha f(t)\dot{f}(t)$ with $f(t) = \frac{\alpha^{-1}-\gamma(t)}{g}$ for scalar $g > 0$ and use the above to show that the system is passive if $\gamma$ is chosen so that $\dot{\gamma}(t) = -g^2 p(t)$. **[5]**

(b) Consider a plant modelled by

$$\dot{x}(t) = Ax(t) + \alpha bu(t)$$

where $\alpha$ is a strictly positive scalar with unknown value.

Suppose you would like the plant to behave like the reference model

$$\dot{\overline{x}}(t) = \overline{A}\overline{x}(t) + br(t)$$

where $\overline{A} \triangleq A - bf^{\mathrm{T}}$ and $f$ has been chosen so that each eigenvalue of $\overline{A}$ has real part that is strictly negative.

The control law

$$u(t) = \gamma(t)[r(t) - f^{\mathrm{T}}x(t)]$$

is to be applied to the plant with $\gamma$ chosen adaptively.

Show that perfect model following is possible by suitable choice of $\gamma$. **[2]**

Let

$$e = \overline{x} - x.$$

Show that

$$\dot{e}(t) = \overline{A}e(t) + w(t)$$

where

$$w(t) = \{\alpha\gamma(t) - 1\}b\{f^{\mathrm{T}}x(t) - r(t)\}.$$

Determine an error subsystem, with a suitable output $z$, that is strictly output passive.

There is no need to solve any Lyapunov equations that are involved. **[6]**

Then, possibly using a result from part (a) above, derive an adaptation law for $\gamma(t)$ such that $\|\overline{x} - x\|_{\mathcal{L}_2} < \infty$.

Give sufficient detail to make your method clear. **[7]**