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CS3.2
ISE4.55

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2010

MSc and EEE/ISE PART IV: MEng and ACGI

OPTIMIZATION

Monday, 26 April 2:30 pm

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	A. Astolfi
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OPTIMISATION

1. Consider the problem of minimizing a function of n variables $x_1, x_2, \cdots, x_n$, defined as

$$f(x_1, \cdots, x_n) = f_1(x_1)f_2(x_2)\cdots f_n(x_n),$$

that is the function f is the product of the n functions f_i, each of the variable x_i only.

a) Assume that all functions f_i are such that

$$f_i(x_i) > 0$$

for all x_i and that there exist unique $x_i^\star$ such that $x_i^\star$ is a stationary point of f_i.

i) Compute the stationary point $x^\star$ of the function f. [2 marks]

ii) Using second order sufficient conditions show that the stationary point $x^\star$ of the function f is a strict local minimizer if and only if all $x_i^\star$ are strict local minimizers of the functions f_i. [6 marks]

b) Assume $n = 3$, that is consider the function

$$f(x_1, x_2, x_3) = f_1(x_1)f_2(x_2)f_3(x_3).$$

Assume that the functions f_i do not have stationary points but that there exists, for $i = 1, 2, 3$, a unique point x_i° such that

$$f_i(x_i^\circ) = 0$$

and

$$f_i(x_i) \neq 0$$

for all $x_i \neq x_i^\circ$.

i) Compute all stationary points of the function f. [4 marks]

ii) Show that the Hessian matrix of f at any stationary point is either identically zero or it has positive and negative eigenvalues. Hence argue that none of the stationary point can be a strict local minimizer. (Hint: recall that a symmetric matrix has real eigenvalues and that the trace of a matrix, that is the sum of its diagonal entries, is equal to the sum of its eigenvalues.) [8 marks]

2. The methods of optimisation can be used to solve simple geometric problems. Consider the following list of problems. For each of them, formulate the problem as an optimisation problem, defining the decision variables, the cost to be optimised, and the admissible set, and provide explicit solutions.

a) Show that of all rectangles with a fixed positive area the one with the smallest perimeter is a square. [4 marks]

b) Show that of all rectangles with a fixed positive perimeter the one with the largest area is a square. [2 marks]

c) Find the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and on the parabola $y = 8 - x^2$. [4 marks]

d) A piece of wire 10 meters long is cut into two pieces. One piece is bent to form a square, the other piece is bent to form an equilater triangle. How should the wire be cut so that the total area of the square and of the triangle is a maximum or a minimum? [10 marks]

3. Consider the optimisation problem

$$\begin{cases} \min\limits_{x_1,x_2} \; -x_1 + x_2, \\ 0 \leq x_1 \leq 1, \\ x_2 \geq x_1^2. \end{cases}$$

a) State first order necessary conditions of optimality for this constrained optimisation problem. [4 marks]

b) Using the conditions in part a) determine a candidate optimal solution $x^\star$ for the considered optimisation problem. [8 marks]

c) Transform the considered optimization problem into an optimization problem with equality constrains by adding an auxiliary variable and disregarding, for simplicity, the constraints $0 \leq x_1 \leq 1$.

State first order necessary conditions of optimality for this transformed problem. Determine a candidate optimal solution and show that it coincides with the solution determined in part b). [8 marks]

4. An alternative way to introduce Newton's method for the solution of a nonlinear equation is to consider the evaluation of the integral

$$f(x) = f(x_k) + \int_{x_k}^{x} \dot{f}(t)dt,$$

where $\dot{f}$ denotes the derivative of the function f, by means of the so-called Newton-Cotes quadrature formula of order zero (the rectangular rule) yielding

$$f(x) \approx f(x_k) + (x - x_k)\dot{f}(x_k),$$

setting $x = x_{k+1}$ and replacing the $\approx$ sign with an $=$ sign , thus yielding

$$f(x_{k+1}) = f(x_k) + (x_{k+1} - x_k)\dot{f}(x_k),$$

and setting $f(x_{k+1}) = 0$, thus obtaining the iteration

$$x_{k+1} = x_k - \frac{f(x_k)}{\dot{f}(x_k)}.$$

a) Consider the evaluation of the integral by means of the Newton-Cotes quadrature formula of order one (the trapezoidal rule), that is

$$\int_{x_k}^{x} \dot{f}(t)dt \approx \frac{x - x_k}{2}\left(\dot{f}(x_k) + \dot{f}(x)\right).$$

i) Determine a new iteration for the solution of the nonlinear equation $f(x) = 0$.
(Hint: the obtained iteration is implicitly defined, that is x_{k+1} is a function of x_k and of $\dot{f}(x_{k+1})$!) [6 marks]

ii) An explicit iteration can be obtained replacing $\dot{f}(x_{k+1})$ with $\dot{f}(x^\star)$, where

$$x^\star = x_k - \frac{f(x_k)}{\dot{f}(x_k)}.$$

Write the expression of the resulting modified Newton's iteration.
[2 marks]

b) Consider the problem of determining the square root of 2.

i) Write Newton's iteration for the solution of this problem. Let $x_0 = 1$ and apply three steps of Newton's iteration, that is compute the values x_1, x_2, and x_3 resulting from the application of Newton's iteration with the given initial point. Evaluate the absolute error $e_k = |\sqrt{2} - x_k|$.
[4 marks]

ii) Write the modified Newton's iteration for the solution of this problem. Let $x_0 = 1$ and apply three steps of the modified Newton's iteration, that is compute the values x_1, x_2, and x_3 resulting from the application of the modified Newton's iteration with the given initial point. Evaluate the absolute error $e_k = |\sqrt{2} - x_k|$. [6 marks]

iii) Compare the Newton's iteration and the modified Newton's iteration in terms of convergence speed and computational complexity.
[2 marks]

5. Consider the optimisation problem

$$\begin{cases} \min\limits_{y_1,y_2} y_1y_2, \\ y_1^2+y_2^2 \leq 1. \end{cases}$$

a) Sketch in the (y_1,y_2)-plane the admissible set and the level lines of the the function y_1y_2. Hence, using only graphical considerations determine the optimal solutions of the considered problem. [4 marks]

b) State first order necessary conditions of optimality for this constrained optimisation problem. [2 marks]

c) Using the conditions derived in part b) compute candidate optimal solutions. Show that the optimal solutions derived graphically in part a) satisfy the necessary conditions of optimality. [4 marks]

d) The considered problem can be transformed into a linear programming problem using the change of variable

$$x_1 = (y_1-y_2)^2 \qquad x_2 = (y_1+y_2)^2.$$

i) Write the equations describing the transformed problem.
(Hint: note that the transformed problem has three inequality constraints.) [6 marks]

ii) Sketch in the (x_1,x_2)-plane the admissible set and the level lines of the cost function. Hence determine the optimal solution of the transformed problem. [2 marks]

iii) Show how the optimal solution of the transformed problem can be used to determine the optimal solutions of the original problem. [2 marks]

6. Consider the optimisation problem

$$\begin{cases} \min\limits_{x_1,x_2} -(x_1+x_2), \\ x_1^2+x_2^2=1. \end{cases}$$

a) State first order necessary conditions of optimality for this constrained optimization problem. [2 marks]

b) Using the conditions derived in part a) compute candidate optimal solutions. [4 marks]

c) Using second order sufficient conditions of optimality determine the solution $x^\star$ of the optimization problem and the corresponding optimal multiplier $\lambda^\star$. [4 marks]

d) Suppose that one wants to solve the considered problem using the exact penalty function method.

i) Write the exact penalty function G for the considered optimization problem. [4 marks]

ii) Verify that the optimal solution $x^\star$ determined in part c) is a stationary point of the function G. Discuss how the optimal multiplier $\lambda^\star$, determined in part c), can be obtained from the exact penalty function. [4 marks]

iii) The Hessian matrix of the function G at the optimal solution $x^\star$ determined in part c) is

$$\nabla^2 G(x^\star) = \begin{bmatrix} \frac{4}{\varepsilon} & -\sqrt{2}+\frac{4}{\varepsilon} \\ -\sqrt{2}+\frac{4}{\varepsilon} & \frac{4}{\varepsilon} \end{bmatrix}.$$

Determine a range of values for the parameter ε so that $x^\star$ is a minimizer for the function G. [2 marks]

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Optimisation - Model answers 2010

(Note to external examiners: all questions involve mostly applications of standard methods and concepts to unseen examples.)

Question 1

a) i) The stationary points of the function f are computed by solving the equations

$$0 = \nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} f_2 \cdots f_n \\ f_1 \frac{\partial f_2}{\partial x_2} f_3 \cdots f_n \\ \vdots \\ f_1 f_2 \cdots \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

Since all f_i's are positive, and have a unique stationary point, the only stationary point of f is the point

$$x^\star = (x_1^\star, x_2^\star, \cdots, x_n^\star).$$

ii) Note that, for $i \neq j$,

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial f_i}{\partial x_i} \frac{\partial f_j}{\partial x_j} M$$

where M is a positive function, hence

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x^\star) = 0.$$

As a result, the Hessian matrix of the function f at $x^\star$ is

$$\nabla^2 f(x^\star) = \operatorname{diag}\left(\frac{\partial^2 f_1}{\partial x_1^2}(x_1^\star) f_2(x_2^\star) \cdots f_n(x_n^\star), \ \cdots, \ f_1(x_1^\star) f_2(x_2^\star) \cdots \frac{\partial^2 f_n}{\partial x_n^2}(x_n^\star) \right).$$

This implies that the function f has a strict local minimizer at $x^\star$ if and only if all functions f_i have a strict local minimizer at $x_i^\star$.

b) i) The stationary points of the functions f are the solution of the equations

$$0 = \nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} f_2 f_3 \\ f_1 \frac{\partial f_2}{\partial x_2} f_3 \\ f_1 f_2 \frac{\partial f_3}{\partial x_3} \end{bmatrix}.$$

These equations admit infinitely many solutions given by

$$x_{12}^\circ = (x_1^\circ, x_2^\circ, \bar{x}_3), \qquad x_{13}^\circ = (x_1^\circ, \bar{x}_2, x_3^\circ), \qquad x_{23}^\circ = (\bar{x}_1, x_2^\circ, x_3^\circ),$$

where $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are arbitrary values.

ii) The Hessian matrix of f is

$$\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f_1}{\partial x_1^2} f_2 f_3 & \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} f_3 & \frac{\partial f_1}{\partial x_1}\frac{\partial f_3}{\partial x_3} f_2 \\ \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} f_3 & \frac{\partial^2 f_2}{\partial x_2^2} f_1 f_3 & \frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_3} f_1 \\ \frac{\partial f_1}{\partial x_1}\frac{\partial f_3}{\partial x_3} f_2 & \frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_3} f_1 & \frac{\partial^2 f_3}{\partial x_3^2} f_1 f_2 \end{bmatrix}.$$

Hence

$$\nabla^2 f(x_{12}^\circ) = \begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where

$$\alpha = \frac{\partial f_1}{\partial x_1}(x_1^\circ)\frac{\partial f_2}{\partial x_2}(x_2^\circ) f_3(\bar{x}_3).$$

The function α is zero for $\bar{x}_3 = x_3^\circ$ and it is non-zero otherwise. Hence, $\nabla^2 f(x_{12}^\circ)$ is either identically zero or has trace zero, which means that it has a positive and a negative eigenvalue. In both cases, the points x_{12}° cannot be local strict minimizers.

Similar considerations apply for x_{13}° and x_{23}°.

Question 2

a) Let x and y be the height and length of the rectangle, then we want to minimize $P = 2x + 2y$ subject to $xy = A$, $x > 0$ and $y > 0$.
Using the constraint on the area we have

$$y = \frac{A}{x},$$

hence we need to minimize

$$P = 2x + 2\frac{A}{x}$$

subject to $x > 0$. Ignoring the positivity constraint on x, the stationary points of P are the solutions of

$$0 = \frac{dP}{dx} = 2 - 2\frac{A}{x^2},$$

i.e. $x = \pm\sqrt{A}$. The only feasible solution is $x = \sqrt{A}$, which is a global minimizer, since P is convex for $x > 0$, yielding $y = \sqrt{A}$, hence the rectangle with minimum perimeter is a square with perimeter $P = 4\sqrt{A}$.

b) Let x and y be the height and length of the rectangle, then we want to maximize $A = xy$ subject to $P = 2x + 2y$, $x > 0$ and $y > 0$.
Using the constraint on the perimeter we have

$$A = x\frac{P - 2x}{2} = \frac{1}{2}Px - x^2,$$

subject to $x > 0$. Ignoring the positivity constraint on x, the stationary points of A are the solutions of

$$0 = \frac{dA}{dx} = \frac{1}{2}P - 2x,$$

i.e. $x = \frac{1}{4}P$. This solution is positive, hence feasible, and it is a global maximizer since the function is concave, hence the rectangle with maximum area is a square with area $A = \frac{P^2}{16}$.

c) The area of the rectangle is (note that to have two vertices on the given parabola, the two vertices on the x-axis should be symmetric with respect to the y-axis)

$$A = 2xy = 2x(8 - x^2) = 16x - 2x^3,$$

with $0 \leq x \leq \sqrt{8}$. The function A is continuous in the interval $[0, \sqrt{8}]$, hence its global maximum is either a stationary point or an extreme of the interval. Stationary points are the solutions of

$$0 = \frac{dA}{dx} = 16 - 6x^2,$$

i.e. $x = \pm\sqrt{8/3}$. Note now that

$$x = 0 \Rightarrow A = 0, \qquad x = \sqrt{8/3} \Rightarrow A = \frac{64}{3}\frac{\sqrt{2}}{\sqrt{3}}, \qquad x = \sqrt{8} \Rightarrow A = 0,$$

hence the optimal solution is $x = \sqrt{8/3}$.

d) Let x be the length of the wire used for the square, and $10-x$ the length used for the triangle. Each side of the square is $x/4$, and its area is $x^2/16$. Each side of the triangle is $(10-x)/3$, and its area is $\frac{\sqrt{3}}{4}\frac{(10-x)^2}{3^2}$. The total area enclosed is

$$A = \frac{1}{16}x^2 + \frac{\sqrt{3}}{36}(10-x)^2,$$

with $x \in [0,10]$. Extrema, *i.e.* minimizers and maximizers are either stationary points in $[0,10]$ or the extremes of the intervals. The function A has only a stationary point (since it is a quadratic function), namely $x = \frac{80\sqrt{3}}{18+8\sqrt{3}} \approx 4.35$. Note now that

$$x = 0 \Rightarrow A = \frac{100\sqrt{3}}{36} \approx 4.81, \qquad x \approx 4.35 \Rightarrow A \approx 2.72, \qquad x = 10 \Rightarrow A = 6.25.$$

Hence, to have the minimum area, use 4.35m of wire for the square and the rest for the triangle, to have maximum area use all the wire for the square!

Question 3

a) Define the Lagrangian

$$L(x_1, x_2, \rho_1, \rho_2, \rho_3) = -x_1 + x_2 + \rho_1(-x_1) + \rho_2(x_1 - 1) + \rho_3(x_1^2 - x_2).$$

The first order necessary conditions of optimality are

$$0 = \frac{\partial L}{\partial x_1} = -1 - \rho_1 + \rho_2 + 2\rho_3 x_1 \qquad 0 = \frac{\partial L}{\partial x_2} = 1 - \rho_3$$

$$-x_1 \le 0 \qquad x_1 - 1 \le 0 \qquad x_1^2 - x_2 \le 0 \qquad \rho_1 \ge 0 \qquad \rho_2 \ge 0 \qquad \rho_3 \ge 0$$

$$\rho_1 x_1 = 0 \qquad \rho_2(x_1 - 1) = 0 \qquad \rho_3(x_1^2 - x_2) = 0.$$

b) To begin with note that ρ_3 has to be equal to one, hence $x_1^2 = x_2$. Consider now the following four possibilities.

- $\rho_1 = 0$, $\rho_2 = 0$. This yields $x_1 = 1/2$ and $x_2 = 1/4$.
- $\rho_1 = 0$, $\rho_2 > 0$. This yields $x_1 = 1$, $x_2 = 1$, $\rho_2 = -1$, which is not admissible.
- $\rho_1 > 0$, $\rho_2 = 0$. This yields $x_1 = 0$, $x_2 = 0$, $\rho_1 = -1$, which is not admissible.
- $\rho_1 > 0$, $\rho_2 > 0$. This yields $x_1 = 0$ and $x_1 = 1$, which is meaningless.

In summary the only candidate solution is $x_1 = 1/2$, $x_2 = 1/4$, $\rho_1 = 0$, $\rho_2 = 0$, $\rho_3 = 1$.

c) The inequality constraint $x_1^2 - x_2 \le 0$ can be rewritten as

$$x_1^2 - x_2 + y^2 = 0,$$

where y is an auxiliary variable. The problem is thus transformed into the problem

$$\begin{cases} \min\limits_{x_1, x_2, y} & -x_1 + x_2, \\ x_1^2 - x_2 + y^2 = 0. & \end{cases}$$

The Lagrangian for this problem is

$$L = -x_1 + x_2 + \lambda(x_1^2 - x_2 + y^2),$$

and the necessary conditions of optimality are

$$0 = \frac{dL}{dx_1} = -1 + 2\lambda x_1, \qquad 0 = \frac{dL}{dx_2} = 1 - \lambda, \qquad 0 = \frac{dL}{dy} = 2\lambda y, \qquad x_1^2 - x_2 + y^2 = 0.$$

The only candidate solution is

$$\lambda = 1, \qquad y = 0, \qquad x_1 = 1/2, \qquad x_2 = 1/4,$$

which coincides with the one determined in part b).

Question 4

a) i) Consider the relation

$$f(x) = f(x_k) + \frac{x - x_k}{2}(\dot{f}(x) + \dot{f}(x_k)).$$

Setting $x = x_{k+1}$ and $f(x) = 0$ yields

$$0 = f(x_k) + \frac{x_{k+1} - x_k}{2}(\dot{f}(x_{k+1}) + \dot{f}(x_k)),$$

hence solving for x_{k+1} provides the iteration

$$x_{k+1} = x_k - 2\frac{f(x_k)}{\dot{f}(x_{k+1}) + \dot{f}(x_k)}$$

ii) The modified Newton's iteration is

$$x_{k+1} = x_k - 2\frac{f(x_k)}{\dot{f}(x_k - f(x_k)/\dot{f}(x_k)) + \dot{f}(x_k)}.$$

b) To determine the square root of 2 consider the equation $x^2 - 2 = 0$.

i) Newton's iteration is given by

$$x_{k+1} = x_k - \frac{1}{2}\frac{x_k^2 - 2}{x_k}.$$

The sequence generated by Newton's iteration is

$$x_0 = 1, \qquad x_1 = 1.5, \qquad x_2 = 1.416666667, \qquad x_3 = 1.414215686,$$

and this yields the sequence of the absolute error

$$e_0 = 0.414213562, \quad e_1 = 0.085786438, \quad e_2 = 0.002453105, \quad e_3 = 0.000002124.$$

ii) The modified Newton's iteration is

$$x_{k+1} = x_k - \frac{2(x_k^2 - 2)x_k}{3x_k^2 + 2}.$$

The sequence generated by the modified Newton's iteration is

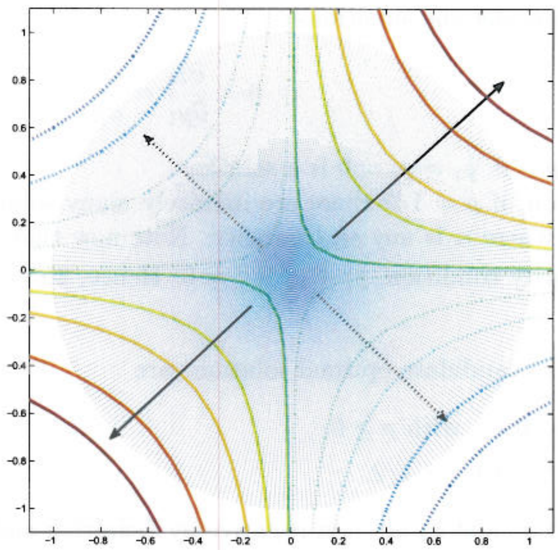
$$x_0 = 1, \qquad x_1 = 1.4, \qquad x_2 = 1.414213198, \qquad x_3 = 1.414213563,$$

and this yields the sequence of the absolute error

$$e_0 = 0.414213562, \quad e_1 = 0.014213562, \quad e_2 = 3.64 \times 10^{-7}, \quad e_3 = 1 \times 10^{-9}.$$

iii) The modified Newton's iteration is much faster (this is a general conclusion) and has similar complexity than the (classical) Newton's iteration.

Question 5



a) The admissible set is the shaded area in the figure above. The level lines are the solid (positive values of f) and dotted (negative values of f) lines. The value of the function f increases in the direction of the solid arrows, and decreases in the direction of the dotted arrows.
The solution of the problem is obtained for negative values of f at a point in which the level lines are tangent to the circle $y_1^2 + y_2^2 = 1$. At such points $y_1 = -y_2$, hence the (global) minimizers are the point

$$P_1 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \qquad P_2 = \left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).$$

The value of the function at the optimal points is $f(P_1) = f(P_2) = -\frac{1}{2}$.

b) The Lagrangian of the problem is

$$L(y_1, y_2, \rho) = y_1 y_2 + \rho(y_1^2 + y_2^2 - 1).$$

The first order necessary conditions of optimality are

$$0 = \frac{\partial L}{\partial y_1} = y_2 + 2\rho y_1, \qquad 0 = \frac{\partial L}{\partial y_2} = y_1 + 2\rho y_2,$$

$$y_1^2 + y_2^2 - 1 \leq 0, \qquad \rho \geq 0, \qquad \rho(y_1^2 + y_2^2 - 1) = 0.$$

c) Consider the two cases.

- $\rho = 0$. In this case $y_1 = y_2 = 0$.
- $\rho > 0$. Consider the equations

$$0 = \frac{\partial L}{\partial y_1} = \frac{\partial L}{\partial y_2}.$$

For any $\rho > 0$, $y_1 = y_2 = 0$ is a solution.
In addition, if $\rho = 1/2$ there are infinitely many solutions of the form $(y_1, y_2) = (\alpha, -\alpha)$, where α is any real number. Note now that $\rho > 0$ implies, by the complementarity condition, $y_1^2 + y_2^2 - 1 = 0$. Hence, $2\alpha^2 = 1$, yielding $\alpha = \pm\frac{\sqrt{2}}{2}$.

In summary the candidate optimal solutions are

- $(y_1, y_2) = (0, 0)$, with $\rho \geq 0$.
- P_1 and P_2 with $\rho = 1/2$.

d) i) Note that x_1 and x_2 are non-negative by definition and that $x_1 = y_1^2 - 2y_1y_2 + y_2^2$, and $x_2 = y_1^2 + 2y_1y_2 + y_2^2$. Hence

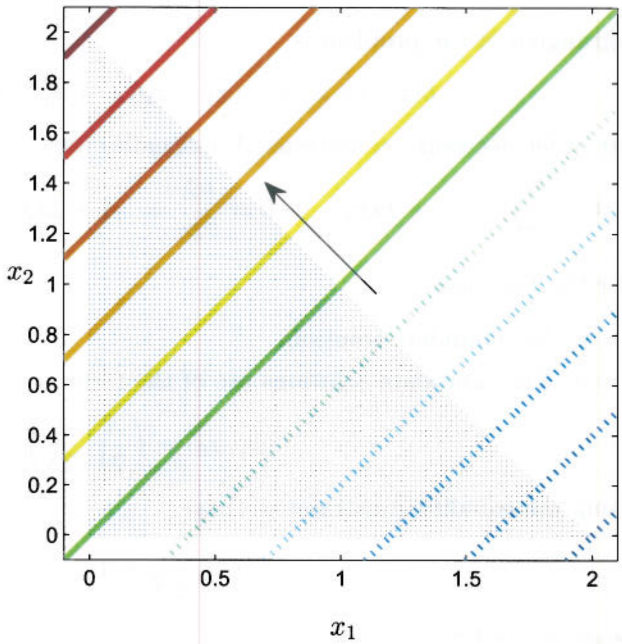
$$\frac{x_2 - x_1}{4} = y_1y_2, \qquad \frac{x_1 + x_2}{2} = y_1^2 + y_2^2.$$

As a result, in the variables x_1 and x_2 the problem is

$$\begin{cases} \displaystyle\min_{x_1, x_2} \frac{x_2 - x_1}{4}, \\ x_1 \geq 0, \\ x_2 \geq 0, \\ \dfrac{x_1 + x_2}{2} \leq 1. \end{cases}$$

ii) The admissible set is the shaded area in the figure below. The level lines are the solid (positive values of f) and dotted (negative values of f) lines. The value of the function f increases in the direction of the solid arrow. The optimal solution is the point

$$P = (2, 0).$$



iii) The point P in the (x_1, x_2)-variables is transformed into points in the (y_1, y_2)-variables solving the equations

$$(y_1 - y_2)^2 = 2 \qquad (y_1 + y_2)^2 = 0.$$

These equations have the solutions

$$(y_1, y_2) = \left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right),$$

which coincide with the points P_1 and P_2 determined in part c).

Question 6

a) The Lagrangian of the problem is

$$L(x_1, x_2, \lambda) = -(x_1 + x_2) + \lambda(x_1^2 + x_2^2 - 1).$$

The first order necessary conditions of optimality are

$$0 = \frac{\partial L}{\partial x_1} = -1 + 2\lambda x_1, \qquad 0 = \frac{\partial L}{\partial x_2} = -1 + 2\lambda x_2, \qquad 0 = x_1^2 + x_2^2 - 1.$$

b) Consider the two cases.

- $\lambda = 0$. No candidate solution.
- $\lambda \neq 0$. The candidate solutions are of the form

$$(x_1, x_2) = \left(\frac{1}{2\lambda}, \frac{1}{2\lambda}\right).$$

Using the equality constraint we have

$$\frac{1}{2\lambda^2} = 1,$$

hence $\lambda = \pm\frac{\sqrt{2}}{2}$.

As a result, the candidate optimal solutions are

$$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \qquad P_2 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

c) Note that

$$\nabla^2 H = 2\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence $\nabla^2 H$ is positive definite at the candidate optimal point with $\lambda > 0$. This implies that the point P_1, with multiplier $\lambda = \frac{\sqrt{2}}{2}$, is the solution of the considered optimization problem.

d) i) An exact penalty function for the problem is

$$G(x_1, x_2) = -(x_1 + x_2) + \lambda(x)(x_1^2 + x_2^2 - 1) + \frac{1}{\epsilon}(x_1^2 + x_2^2 - 1)^2,$$

where

$$\lambda(x) = \frac{1}{2}\frac{x_1 + x_2}{x_1^2 + x_2^2}.$$

ii) The stationary points of the function G are the solutions of the equations

$$0 = \frac{\partial G}{\partial x_1}, \qquad 0 = \frac{\partial G}{\partial x_2}.$$

A direct computation and substitution shows that the point P_1 is a stationary point of the function G. The optimal multiplier is obtained as

$$\lambda^\star = \lambda(x^\star).$$

iii) As stated in the exam paper, the Hessian of the function G at the optimal solution is

$$\nabla^2 G(P_1) = \begin{bmatrix} \dfrac{4}{\epsilon} & -\sqrt{2} + \dfrac{4}{\epsilon} \\ -\sqrt{2} + \dfrac{4}{\epsilon} & \dfrac{4}{\epsilon} \end{bmatrix}.$$

This matrix is positive definite for all $\epsilon \in (0, 4\sqrt{2})$. Hence, for any ϵ is this interval, the function G has a local minimizer at P_1.