IMPERIAL COLLEGE LONDON

E3.09
C1.0
ISE3.9

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2008

EEE/ISE PART III/IV: MEng, BEng and ACGI

CONTROL ENGINEERING

Corrected Copy

Monday, 12 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	A. Astolfi
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CONTROL ENGINEERING

1. Consider a linear, single-input, single-output, continuous-time system described by the equations

$$\dot{x} = Ax + Bu \qquad y = Cx \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$, $n \geq 2$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, and A, B, and C are matrices of appropriate dimensions.

Consider another linear, single-input, single-output, continuous-time system described by the equations

$$\dot{\xi} = F\xi + Gv \qquad \eta = H\xi \tag{1.2}$$

where $\xi(t) \in \mathbb{R}^2$, $v(t) \in \mathbb{R}$, $\eta(t) \in \mathbb{R}$, and F, G, and H are matrices of appropriate dimensions.

System (1.2) is said to *match* system (1.1) at the points s_1 and s_2, with $s_1 \neq s_2$, if (I denotes the identity matrix of appropriate dimension)

$$H(s_1 I - F)^{-1} G = C(s_1 I - A)^{-1} B \qquad H(s_2 I - F)^{-1} G = C(s_2 I - A)^{-1} B.$$

Let

$$F = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} - \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}$$

$$H = \begin{bmatrix} C(s_1 I - A)^{-1} B & C(s_2 I - A)^{-1} B \end{bmatrix}.$$

a) Show that, with the above selection of F, G and H, system (1.2) matches system (1.1) at s_1 and s_2 for all Δ_1 and Δ_2. [8 marks]

b) Show that, with the above selection of F, G and H, system (1.2) is reachable for any Δ_1 and Δ_2 such that $\Delta_1 \Delta_2 \neq 0$. (Recall that $s_1 \neq s_2$.) [4 marks]

c) Assume that $(C(s_1 I - A)^{-1} B) = (C(s_2 I - A)^{-1} B) = \kappa$. Show that, with the above selection of F, G and H, system (1.2) is observable if and only if $\kappa \neq 0$. (Recall that $s_1 \neq s_2$.) [4 marks]

d) Assume that $s_1 = 0$ and $s_2 = 1$. Consider the above selection of F. Select Δ_1 and Δ_2 such that system (1.2) has two eigenvalues at -1. [4 marks]

2. Consider a linear, discrete-time system described by the equations

$$x_1(k+1) = x_2(k), \qquad x_2(k+1) = Gx_1(k) + Bu(k), \tag{2.1}$$

where $x(k) = [x_1'(k)\ x_2'(k)]'$, with $x_1(k) \in \mathbb{R}^n$ and $x_2(k) \in \mathbb{R}^n$ for some $n \geq 1$, is the state, $u(k) \in \mathbb{R}^m$, for some $m \leq n$, is the input and G and B are matrices of appropriate dimensions.

a) Show that the system (2.1) is reachable if and only if the system

$$\xi(k+1) = G\xi(k) + Bv(k),$$

with $\xi(k) \in \mathbb{R}^n$ and $v(k) \in \mathbb{R}^m$ is reachable. [6 marks]

b) Assume $m = n$ and $B = I$. Show, using the result of part a), that the system (2.1) is reachable. [2 marks]

c) Assume $m = n$ and $B = I$. To design a state-feedback control law which asymptotically stabilizes system (2.1) one could proceed in steps, as detailed below.

i) Consider the system

$$x_1(k+1) = v(k).$$

Let $v(k) = Kx_1(k)$, and determine one $K = K'$ such that the system

$$x_1(k+1) = Kx_1(k)$$

is asymptotically stable. [2 marks]

ii) Consider the signal

$$e(k) = x_2(k) - Kx_1(k),$$

with K as selected in part c.i), and the system (2.1). Write an expression for $e(k+1)$ in terms of $x_1(k)$, $x_2(k)$ and $u(k)$. [4 marks]

iii) Determine $u(k)$ such that $e(k) = 0$ for all $k \geq 1$. [2 marks]

iv) Argue that the state feedback control law determined in part c.iii) asymptotically stabilizes the discrete-time system (2.1). [4 marks]

3. Consider the linear, single-input, continuous-time system described by the equations

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

The input $u(t)$ of the system is connected to a zero-order-hold and the state $x(t)$ is measured with a sampler. Let $T > 0$ be the sampling time.

We wish to design a state feedback control law to asymptotically stabilize the system.

a) Write the approximate Euler model of the considered system. [2 marks]

b) Show that the approximate Euler model determined in part a) is reachable for any $T > 0$. [2 marks]

c) Consider the approximate Euler model determined in part a) and design a state feedback control law $u = Kx = K_1x_1 + K_2x_2$ that assigns both eigenvalues of the resulting closed-loop system at $s = 0$. (Recall that, since the Euler approximate model is a discrete-time system, the state feedback control law is such that the closed-loop system is asymptotically stable.) [4 marks]

d) To assess the efficacy of the state feedback control law designed in part c) on the system, consider the exact model of the continuous-time system in the presence of the hold and the sampler. This model is given by

$$x(k+1) = A_d x(k) + B_d u(k),$$

where $A_d = e^{AT}$ and $B_d = \int_0^T e^{A(T-\tau)} B \, d\tau$.

i) Compute explicitly the matrices A_d and B_d for the considered system. [6 marks]

ii) Consider the closed-loop system

$$x(k+1) = A_d x(k) + B_d K x(k),$$

where K is as in part c), and A_d and B_d are as in part d.i). Discuss the stability properties of the resulting closed-loop system and discuss if the design based on the Euler model is effective. [6 marks]

4. Consider a nonlinear, single-input, continuous-time system described by the equations

$$\dot{x}_1 = x_1^2 + x_2 \qquad \dot{x}_2 = x_1 x_2 + u$$

where $x(t) = [x_1(t) \;\; x_2(t)]' \in I\!R^2$ is the state and $u(t) \in I\!R$ is the control input, and the problem of designing a state feedback control law which asymptotically stabilizes the equilibrium $x = 0$ of the system.

This problem can be solved using two different approaches, detailed in parts a) and b) below.

a) i) Write the linearization of the system at $x = 0$. [2 marks]

ii) Verify that the linearized system is reachable. [2 marks]

iii) Design a linear state feedback control law $u = K_a x$ which assigns both eigenvalues of the closed-loop linearized system at -1. [2 marks]

iv) Argue that the zero equilibrium of the nonlinear system in closed loop with the control law $u = K_a x$ is locally asymptotically stable. [2 marks]

b) i) Let $y(t) = x_1(t)$. Show that $\ddot{y} + \alpha(x_1, x_2) = u$, for some function $\alpha(\cdot)$, which should be specified. [4 marks]

ii) Design a nonlinear state feedback control law $u = K_b(x)$ such that $\ddot{y} + 2\dot{y} + y = 0$. [2 marks]

iii) Argue that the zero equilibrium of the nonlinear system in closed loop with the control law $u = K_b(x)$ is globally asymptotically stable. [4 marks]

c) Discuss briefly advantages and disadvantages of the control laws $u = K_a x$ and $u = K_b(x)$, designed in parts a) and b), respectively, in terms of the stability properties of the zero equilibrium of the associated closed-loop system and in terms of their complexity. [2 marks]

5. Consider a linear, single-input, single-output, discrete-time system described by the equations

$$x(k+1) = Ax(k) + Bu(k) \qquad y(k) = Cx(k)$$

where $x(k) \in I\!R^3$ is the state, $u(k) \in I\!R$ is the input, $y(k) \in I\!R$ is the output,

$$A = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 1 \\ 0 & 0 & -1/2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

and $\alpha \in I\!R$ is a constant parameter.

Recall that a discrete-time system is said to be reconstructable if all unobservable modes are at $s = 0$.

a) Study the observability, detectability and reconstructability properties of the system as a function of α. [8 marks]

b) Determine for which values of α it is possible to design an observer such that the state estimation error $e(k) = x(k) - \hat{x}(k)$, where $\hat{x}(k)$ is the estimate of $x(k)$, is identically equal to zero for all $k \geq 3$.
(Do not design the observer.) [6 marks]

c) Assume $\alpha = 0$. Determine the unobservable subspace and write the system in the canonical decomposition for unobservable systems. [6 marks]

6. Consider a linear, single-input, single-output, continuous-time system described by the equations

$$\dot{x} = Ax + Bu + Pd \qquad\qquad y = Cx$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}$ is the input, $d(t) \in \mathbb{R}$ is the disturbance, and $y(t) \in \mathbb{R}$ is the output.

Consider the problem of *decoupling* the effect of the disturbance from the output by means of a suitably designed state feedback control law. This problem is solvable if the condition (C) below holds.

> (C) There exists a non-negative integer $\kappa \leq n$ such that
>
> $$CB = 0 \quad CAB = 0 \quad \cdots \quad CA^{\kappa-2}B = 0 \quad CA^{\kappa-1}B \neq 0,$$
>
> $$CP = 0 \quad CAP = 0 \quad \cdots \quad CA^{\kappa-2}P = 0 \quad CA^{\kappa-1}P = 0,$$
>
> and
>
> $$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\kappa-1} \end{bmatrix} = \kappa.$$

Let

$$A = \begin{bmatrix} a_{11} & 1 & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 1 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ 0 \\ 1 \end{bmatrix} \qquad P = \begin{bmatrix} p_1 \\ p_2 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}.$$

a) Show that condition (C) holds for some κ. [4 marks]

b) Show that the system can be written in the form

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & S \end{bmatrix} x + B(u + Lx) + \begin{bmatrix} P_1 \\ 0 \end{bmatrix} d,$$

$$y = \begin{bmatrix} 0 & C_2 \end{bmatrix} x$$

with $x_2(t) \in \mathbb{R}^\kappa$,

$$S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and A_{11}, A_{12} L, P_1 and C_2 matrices of appropriate dimensions, which should be specified. [10 marks]

c) Consider the state feedback control law

$$u = -Lx + K_2 x_2$$

with $K_2 \in \mathbb{R}^{1\times\kappa}$.
Write equations for the closed-loop system and argue that the control law solves the considered disturbance decoupling problem.
Finally, show that for the closed-loop system one has

$$y(t) = C_2 e^{Ft} x_2(0),$$

for some matrix $F \in \mathbb{R}^{\kappa\times\kappa}$. [6 marks]