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EE4-24
EE9CS1-3

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2012

MSc and EEE/ISE PART IV: MEng and ACGI

DISCRETE-TIME SYSTEMS AND COMPUTER CONTROL

Thursday, 3 May 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : A. Astolfi
Second Marker(s) : E.C. Kerrigan

DTS AND COMPUTER CONTROL

Information for candidates:

- $Z\left(\frac{1}{s}\right) = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$

- $Z\left(\frac{1}{s+a}\right) = \frac{z}{z-e^{-aT}} = \frac{1}{1-z^{-1}e^{-aT}}$

- $Z\left(\frac{1}{s^2}\right) = T\frac{z}{(z-1)^2} = T\frac{z^{-1}}{(1-z^{-1})^2}$

- $Z\left(\frac{1}{s^3}\right) = \frac{T^2}{2}\frac{z(z+1)}{(z-1)^3} = \frac{T^2}{2}\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$

- $Z\left(\frac{b}{(s+a)^2+b^2}\right) = \frac{ze^{-aT}\sin bT}{z^2 - 2ze^{-aT}\cos bT + e^{-2aT}}$

- Transfer function of the ZOH: $H_0(s) = \frac{1-e^{-sT}}{s}$

- Definition of the w-plane: $z = \frac{1+\frac{wT}{2}}{1-\frac{wT}{2}}$, $w = \frac{2}{T}\frac{z-1}{z+1}$

- Tustin transformation: $s = \frac{2}{T}\frac{z-1}{z+1}$

- Forward Euler: $s = \frac{z-1}{T}$

- Note that, for a given signal r, or $r(t)$, $R(z)$ denotes its Z-transform.

1. Consider the digital control system in Figure 1.

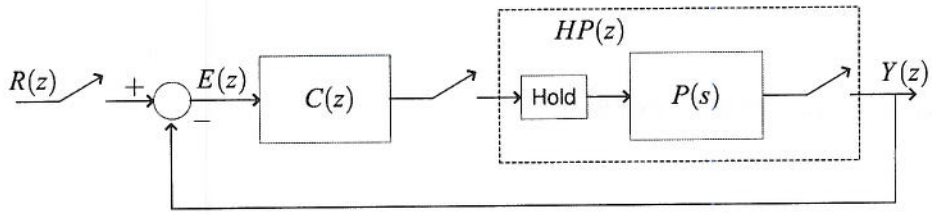

Figure 1: Block diagram for question 1.

Let

$$P(s) = \frac{1}{s^2}.$$

Assume the hold is a ZOH and let the sampling period be $T = 1$.

a) Suppose that the plant $P(s)$ is controlled using the controller

$$C(s) = \frac{1}{5}\frac{3s+1}{s+1}$$

in a unity feedback configuration. Show that the continuous-time closed-loop system is asymptotically stable. [4 marks]

b) Discuss why the selection $T = 1$ is adequate for the design of a sampled-data controller. [2 marks]

c) Compute the equivalent discrete-time model $HP(z)$ for the plant interconnected to the hold and the sampler. [4 marks]

d) Discretize the controller $C(s)$ in part a) using the forward Euler method. Compute explicitly the resulting discrete-time controller. [2 marks]

e) Using the results of parts c) and d) compute the closed-loop transfer function from the input $R(z)$ to the output $Y(z)$. [2 marks]

f) Study the stability properties of the discrete-time closed-loop transfer function computed in part e). [6 marks]

2. Consider the digital control system in Figure 2.

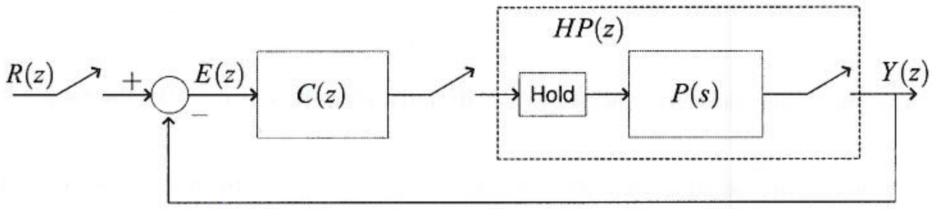

Figure 2: Block diagram for question 2.

Let

$$P(s) = \frac{1}{s+1},$$

Assume the hold is a ZOH and let the sampling period be $T = 1$.

a) Suppose that the plant $P(s)$ is controlled using the controller

$$C_1(s) = k_p\left(1 + \frac{1}{T_i s}\right),$$

with $k_p > 0$ and $T_i = 1/2$, in a unity feedback configuration. Determine for which values of k_p the continuous-time closed-loop system is asymptotically stable. [4 marks]

b) Compute the equivalent discrete-time model $HP(z)$ for the plant interconnected to the hold and the sampler. [2 marks]

c) Discretize the controller $C_1(s)$ using Tustin method. Compute the resulting controller $C_1(z)$, and determine for which values of k_p the closed-loop system is asymptotically stable. Compare the results with those obtained in part a). [6 marks]

d) Let

$$C_2(z) = k\frac{1}{z-\alpha}.$$

i) Determine values of k and α such that the discrete-time closed-loop system has all poles at $z = 0$. [4 marks]

ii) Using the inverse of Tustin transformation, determine the continuous time-controller $C_2(s)$, the discretization of which is $C_2(z)$. [2 marks]

iii) Suppose that the plant $P(s)$ is controlled using the controller $C_2(s)$ in a unity feedback configuration. Study the stability properties of the resulting continuous-time closed-loop system. [2 marks]

3. The transfer function of a simple oscillator is given by

$$P(s) = \frac{s}{s^2+1}.$$

Assume the system is interconnected to a ZOH and a sampler. Let $T > 0$ be the sampling time.

a) Compute the equivalent discrete-time model $HP(z)$ for the plant interconnected to the hold and the sampler. [4 marks]

b) Using the definition of the w-plane, determine the transfer function $HP(w)$. [4 marks]

c) Let $C(w) = K$. Consider the closed-loop system resulting from the unity feedback interconnection of the controller $C(w)$ with the transfer function $HP(w)$.

i) Determine the characteristic polynomial of the closed-loop system and write conditions on T and K such that the closed-loop system is asymptotically stable. [4 marks]

ii) Assume $T > 0$ and sufficiently small. Write the approximation of the conditions determined in part c.i) for small T. Using these conditions show that the closed-loop system is asymptotically stable for $0 < K < T/2$. [4 marks]

iii) Let $T = 2\pi$. Show that there is no selection of K which renders the closed-loop system asymptotically stable. Interpret this result. (Hint: evaluate $HP(z)$ for $T = 2\pi$.) [2 marks]

iv) Interpret the result in part c.iii) on the basis of the sampling period. (Hint: evaluate the angular frequency of the poles of $P(s)$ and relate this frequency to the sampling frequency.) [2 marks]

4. Consider the digital control system in Figure 4.

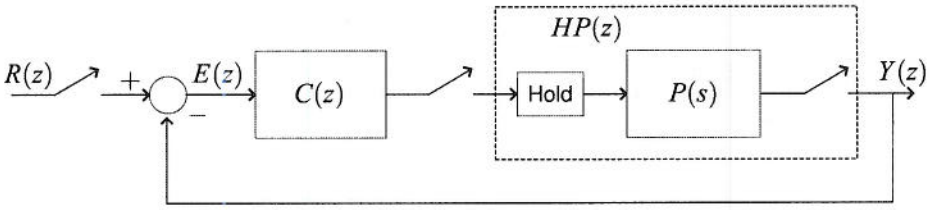

Figure 4: Block diagram for question 4.

Let

$$P(s) = \frac{s}{s^2+s+1}.$$

Assume the hold is a ZOH and let the sampling period be $T = 1/2$.

a) Compute the equivalent discrete-time model $HP(z)$ for the plant interconnected to a ZOH and a sampler. [4 marks]

b) Design a discrete-time controller $C(z)$ such that the closed-loop transfer function from the input $R(z)$ to the output $Y(z)$ is equal to

$$T(z) = \frac{z-1}{z^2}.$$

[8 marks]

c) Assume $r(k) = \alpha$, for all $k \geq 0$, with $\alpha \neq 0$. Determine the steady-state values of the output y and explain why it does not depend upon the value of α. [4 marks]

d) Let r be such that

$$r(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \geq 1 \end{cases}$$

Determine the sequence $y(k)$, for all $k \geq 0$. [4 marks]

5. Consider the digital control system in Figure 5.

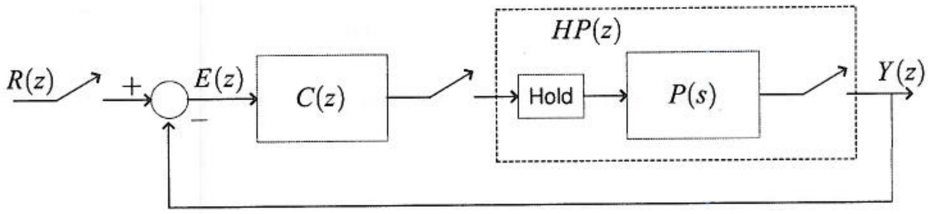

Figure 5: Block diagram for question 5.

The nominal discrete-time equivalent model is given by the transfer function

$$HP(z) = \frac{z - 1/2}{z^2(z-1)}.$$

a) Design a controller $C(z)$ such that the nominal closed-loop system is asymptotically stable and the transfer function $C(z)HP(z)$ is of type 1. [8 marks]

b) Assume that the transfer function $HP(z)$ is perturbed by the addition of a delay, that is consider the perturbed transfer function

$$HP_p(z) = \frac{1}{z}HP(z).$$

i) Determine the characteristic polynomial of the perturbed closed-loop system, that is of the closed-loop system resulting from the interconnection of the controller $C(z)$, designed in part a), with the perturbed transfer function $HP_p(z)$. [4 mark]

ii) Using the characteristic polynomial determined in part b.i) study the stability properties of the perturbed closed-loop system. [6 marks]

c) Let r be a unity ramp, that is $r(k) = k$, for $k \geq 0$. Compute the steady-state errors for the nominal closed-loop system and for the perturbed closed-loop system. Explain why these steady-state error coincide. [2 marks]

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DTS and Computer Control

Model answers 2012

Question 1

a) The characteristic polynomial of the closed-loop system is

$$p(s) = s^3 + s^2 + \frac{3}{5}s + \frac{1}{5}.$$

A simple application of the Routh test shows that all roots of the polynomial have negative real part, hence the closed-loop system is asymptotically stable.
Alternatively, one could use the root locus to prove the the roots of the polynomial $s^3 + s^2 + k(s+1)$ have negative real part for all $k > 0$.

b) The open-loop transfer function has a low-pass structure with cut-off angular frequency $\omega^\star \approx 1/2$. Setting $T = 1$ yields $\omega_s = 2\pi$, which is *significantly larger (i.e. one decade)* than $\omega^\star$.

c) Note that

$$\begin{aligned} HP(z) &= (1 - z^{-1})Z\left(\frac{1}{s^3}\right) \\ &= \frac{1}{2}\frac{z+1}{(z-1)^2}. \end{aligned}$$

d) The discretized controller is

$$C(z) = C(s)|_{s=z-1} = \frac{1}{5}\frac{3z-2}{z}.$$

(Note that, even if the forward Euler method is not a stability preserving method, in this case the discretized controller $C(z)$ is asymptotically stable.)

e) The closed-loop transfer function is

$$W(z) = \frac{C(z)HP(z)}{1 + C(z)HP(z)} = \frac{(3z-2)(z+1)}{10z^3 - 17z^2 + 11z - 2}.$$

f) The characteristic polynomial of the equivalent discrete-time closed-loop system is

$$p(z) = 10z^3 - 17z^2 + 11z - 2.$$

Using the bilinear transformation yields the polynomial

$$q(w) = 40w^3 + 30w^2 + 8w + 2.$$

A simple application of the Routh test shows that all roots of the polynomial have negative real part, hence the closed-loop system is asymptotically stable.

Question 2

a) The closed-loop characteristic polynomial is

$$s^2 + (1 + k_p)s + 2k_p,$$

hence the closed-loop system is asymptotically stable for all $k_p > 0$.

b) Note that

$$\begin{aligned} HP(z) &= (1 - z^{-1})Z\left(\frac{1}{s(s+1)}\right) \\ &= (1 - z^{-1})Z\left(\frac{1}{s} - \frac{1}{s+1}\right) \\ &= \frac{1 - e^{-1}}{z - e^{-1}}. \end{aligned}$$

c) The discretized controller is

$$C(z) = C(s)|_{s=2\frac{z-1}{z+1}} = k_p \frac{2z}{z-1}.$$

The closed-loop characteristic polynomial of the discrete-time equivalent system is

$$z^2 + (2k_p(1 - e^{-1}) - 1 - e^{-1})z + e^{-1} \approx z^2 + (1.26k_p - 1.36)z + 0.36.$$

All roots of the polynomial are inside the unity disc for $0 < k_p < \dfrac{1 + e^{-1}}{1 - e^{-1}} \approx 2.16$. Note that, for the continuous-time closed-loop system, any $k_p > 0$ is stabilizing. The condition on k_p in the sampled-data system is therefore a consequence of the digital implementation.

i) The characteristic polynomial of the closed-loop system with the controller $C_2(z)$ is

$$z^2 + (-e^{-1} - \alpha)z + (\alpha e^{-1} - ke^{-1} + k).$$

Selecting

$$a = -e^{-1} \qquad k = \frac{e^{-2}}{1 - e^{-1}}$$

yields the characteristic polynomial z^2, as requested.

ii) Using the inverse of the Tustin transformation yields

$$C_2(s) = \frac{e^{-2}}{1 - e^{-1}} \frac{s - 2}{(e^{-1} - 1)s - 2(e^{-1} + 1)}.$$

(Note that the controller $C_2(s)$ is stable, but non-minimum phase.)

iii) The characteristic polynomial of the continuous-time closed-loop system with the controller $C_2(s)$ is

$$0.3995s^2 + 1.9935s + 2,$$

hence the closed-loop system is asymptotically stable.

Question 3

a) Note that

$$\begin{aligned} HP(z) &= (1-z^{-1})Z\left(\frac{1}{s^2+1}\right) \\ &= \sin T \frac{z-1}{z^2-2z\cos T+1}. \end{aligned}$$

b) The transfer function in the w-plane is given by

$$HP(w) = HP(z)\Big|_{z=\frac{1+wT/2}{1-wT/2}} = \sin T \frac{wT(2-wT)}{T^2w^2(1+\cos T)+4(1-\cos T)}.$$

c) i) Setting $C(w) = K$ yields a closed-loop system with characteristic polynomial

$$T^2(1+\cos T - K\sin T)w^2 + 2TK\sin T w + 4(1-\cos T).$$

The roots of this polynomial are in the left part of the complex plane for all T and K such that

$$T^2(1+\cos T - K\sin T) > 0, \qquad 2TK\sin T > 0, \qquad 4(1-\cos T) > 0.$$

ii) If $T > 0$ and small, then the stability conditions become

$$-KT^3 + 2T^2 > 0 \qquad 2KT^2 > 0 \qquad 2T^2 > 0,$$

yielding $0 < K < \frac{2}{T}$.

iii) If $T = 2\pi$ then the characteristic polynomial is

$$8\pi^2 w^2,$$

i.e. the roots of the polynomial are equal to 0, for any K, hence the closed-loop system cannot be rendered asymptotically stable by any selection of K. This is consistent with the fact that, for $T = 2\pi$, the discrete-time equivalent model becomes $HP(z) = 0$.

iv) Note that the transfer function $P(s)$ has poles at $j\omega^\star = j$, *i.e.* the poles have angular frequency $\omega^\star = 1$. The associated period is $T^\star = 2\pi$. When this frequency coincides with the sampling frequency we should expect some loss of information in the construction of the discrete-time equivalent model, which explains why $HP(z)$ vanishes for $T = 2\pi$.

Question 4

a) Note that

$$
\begin{aligned}
HP(z) &= (1 - z^{-1}) Z \left(2/\sqrt{3} \frac{\sqrt{3}/2}{(s+1/2)^2 + (\sqrt{3}/2)^2} \right) \\
&= 0.377 \frac{z-1}{z^2 - 1.413z + 0.606}
\end{aligned}
$$

b) One possible selection is to design a controller which cancels the poles of $HP(z)$ with two zeros, that is

$$
C(z) = \frac{1}{0.377} \frac{z^2 - 1.413z + 0.606}{z^2 + d_1 z + d_0},
$$

where d_0 and d_1 are design parameters. The resulting closed-loop system has transfer function

$$
\frac{C(z)HP(z)}{1 + C(z)HP(z)} = \frac{z-1}{z^2 + (d_1+1)z + (d_0 - 1)}.
$$

Hence, the selection $d_0 = 1$ and $d_1 = -1$ yields the desired closed-loop transfer function.

c) If $r(k) = \alpha$, for all $k \geq 0$, then

$$
Y(z) = \alpha \frac{z-1}{z^2} \frac{z}{z-1} = \alpha \frac{1}{z},
$$

hence $\lim_{k \to \infty} y(k) = 0$, for any α. This is justified by the presence of the zero at $z = 1$, which is related to the zero at $s = 0$ of $P(s)$.

d) If $r(k) = 1$, for $k = 0$ and $r(k) = 0$, for all $k > 0$, then

$$
Y(z) = \frac{z-1}{z^2} 1 = \frac{1}{z} - \frac{1}{z^2}.
$$

Hence

$$
y(0) = 0 \qquad y(1) = 1 \qquad y(2) = -1 \qquad y(k) = 0, \quad \text{for all } k > 2.
$$

Question 5

a) A possible selection is

$$C(z) = k\frac{z}{z-1/2},$$

yielding the closed-loop characteristic polynomial

$$z^2 - z + k.$$

This polynomial has all roots inside the unity disk for $k \in (0,1)$. Selecting, for example, $k = 1/4$ yields two roots at $z = 1/2$.

b) Let

$$C(z) = \frac{1}{4}\frac{z}{z-1/2}, \qquad HP_p(z) = \frac{1}{z}HP(z).$$

i) The perturbed closed-loop characteristic polynomial is

$$4z^3 - 4z^2 + 1.$$

ii) Using the bilinear transformation yields the polynomial

$$7w^3 + 19w^2 + 5w + 1,$$

hence the perturbed closed-loop system is asymptotically stable.

c) The velocity constants of the nominal and perturbed closed-loop systems are

$$\lim_{z\to 1} HP(z)C(z)\frac{z-1}{zT} = \frac{1}{4T} \qquad \lim_{z\to 1} HP_p(z)C(z)\frac{z-1}{zT} = \frac{1}{4T}.$$

These constants are the same since the perturbation does not alter the type of the system nor its gain at $z = 1$. The steady state error for a unity-ramp reference is, for both system,

$$\frac{1}{\frac{1}{4T}} = 4T.$$

(Note that if the design of $C(z)$ yields an unstable perturbed closed-loop system, then the steady-state error is not defined.)