IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2013

EEE/EIE PART I: MEng, Beng and ACGI

INTRODUCTION TO SIGNALS AND COMMUNICATIONS

Wednesday, 12 June 10:00 am

Time allowed: 2:00 hours

There are THREE questions on this paper.

Answer ALL questions.
Q1 carries 40% of the marks. Questions 2 and 3 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : K.K. Leung
Second Marker(s) : M.K. Gurcan

Special Instructions for Invigilator: **None**

Information for Students:

Some Fourier Transforms

$\cos \omega_o t$ <=> $\pi[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)]$

Some useful trigonometric identities

$$\cos x \cos y = \tfrac{1}{2}\cos(x - y) + \tfrac{1}{2}\cos(x + y)$$
$$\sin(x - y) = \sin x \cos y - \sin y \cos x$$
$$a \cos x + b \sin x = c \cos(x + \theta)$$

where $c = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(-b/a)$

Complex exponential

$$e^{jx} = \cos x + j \sin x$$

1. This is a general question. (40%)

a. Given a time signal $f(t)$ and its Fourier transform $F(\omega)$, let $g(t) = \frac{df(t)}{dt}$. That is, $g(t)$ is the first derivative of $f(t)$. Further, we use $G(\omega)$ to denote the Fourier transform of $g(t)$.

i. Express $f(t)$ in terms of $F(\omega)$ by the definition of inverse Fourier transform. [2]

ii. By differentiating both sides of the expression obtained in part i, obtain an expression for $G(\omega)$ in terms of $F(\omega)$. [3]

iii. Now assume that $f(t)$ is given by the following diagram.

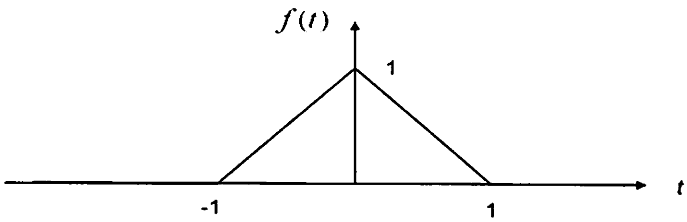


Figure 1. Signal $f(t)$.

Sketch the signal $g(t)$. [1]

iv. Derive the Fourier transform $G(\omega)$ of $g(t)$ obtained in part iii. [3]

v. Use results in parts ii and iv to obtain the Fourier transform $F(\omega)$ for signal $f(t)$. [1]

b. Let a signal $x(t)$ be the sum of three signal components, $a(t)$, $b(t)$ and $c(t)$, as $x(t) = a(t) + b(t) + c(t)$. Further, let P_x, P_a, P_b and P_c be the power of $x(t)$, $a(t)$, $b(t)$ and $c(t)$, respectively. We assume that all powers are finite.

i. Derive an expression for P_x in terms of $a(t)$, $b(t)$ and $c(t)$. [2]

ii. Identify three sufficient mathematical conditions for $P_x = P_a + P_b + P_c$; that is, the power of the signal equal to the sum of the powers of the individual signal components. What is the commonly used term for these relationships among $a(t)$, $b(t)$ and $c(t)$? [4]

iii. Assume that the mathematical conditions (relationships) identified in part ii are valid. For an arbitrary signal $y(t)$, is it always possible to express $y(t)$ as $y(t) = \alpha a(t) + \beta b(t) + \gamma c(t)$ where α, β and γ are some constants? Explain why or why not. [2]

iv. Following part iii, if it turns out that for any given signal $y(t)$, we can always express $y(t)$ as $y(t) = \alpha a(t) + \beta b(t) + \gamma c(t) + \lambda d(t)$ where λ is another constant. Given that, what can be said about the relationships between $d(t)$ and the other signal components $a(t)$, $b(t)$ and $c(t)$? [2]

1. This is a general question. (Continued)

c. Consider two forms of amplitude modulation (AM), namely, the double-sideband with suppressed carrier (DSB-SC) and the single-sideband (SSB) signal. For both forms of AM, let ω_c be the carrier angular frequency in radians/second and $m(t) = A\cos(\omega_m t)$ be the modulating signal where A and ω_m are the amplitude and the angular frequency of the modulating signal, respectively. We use $\phi_{DSB}(t)$ to denote the DSB-SC signal.

   i. Sketch the spectrum of the modulating signal $m(t)$. [2]
   ii. Give an expression for $\phi_{DSB}(t)$. [2]
   iii. Sketch the spectrum of $\phi_{DSB}(t)$. [2]
   iv. Based on result in part iii, sketch the spectrum for the upper-side-band (USB) signal. [2]
   v. Write the expression of the USB signal. [2]
   vi. Name two advantages of using SSB over DSB transmission. [2]

d. Consider a phase modulation (PM) signal, $\phi_{PM}(t)$, with $m(t)$ as the modulating signal, f_c denoting the carrier frequency, and k_P as the proportionality constant.

   i. Give an expression for $\phi_{PM}(t)$. [2]
   ii. Determine the instantaneous frequency for the PM signal as a function of time. [2]
   iii. Assume that $m(t)$ is given by the following diagram:

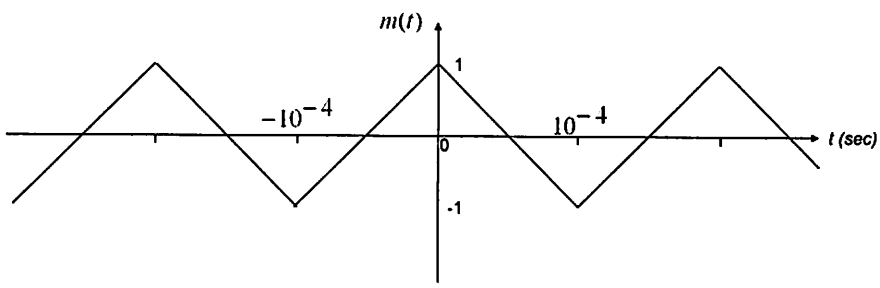


Figure 2. The modulating signal $m(t)$.

   Furthermore, let $f_c = 100MHz$ and $k_P = 10\pi$. Determine the maximum and minimum instantaneous frequencies for $\phi_{PM}(t)$. [2]
   iv. Based on results in part iii, sketch the signal $\phi_{PM}(t)$. [2]

2. Signals. (30%)

a. Consider two linear time-invariant (LTI) systems, A and B, for which the unit impulse responses are given by $h_A(t)$ and $h_B(t)$ with their Fourier transforms denoted by $H_A(\omega)$ and $H_B(\omega)$, respectively.

i. Let $y(t)$ be the output signal of system A when $x(t)$ is input to the system. Express $y(t)$ in terms of $x(t)$ and $h_A(t)$. [3]

ii. For $i = 1$ and 2, let $y_i(t)$ denote the output signal of system A when $x_i(t)$ is input to the system. Consider that the input signal $x(t)$ is actually a weighted sum of two signals, $x_1(t)$ and $x_2(t)$. That is, $x(t) = ax_1(t) + bx_2(t)$ where a and b are constants. Use the expression obtained in part i to obtain an expression for the output signal $y(t)$ in terms of $y_1(t)$ and $y_2(t)$ when $x(t)$ is the input of the system A. [3]

iii. Now consider that the input signal is $x(t - T)$ where T is constant. Use the expression obtained in part i to express the output of system A in terms of $y(t)$ and T. [3]

iv. Assume that systems A and B are now "cascaded". That is, the output from system A is input to system B. Let $x(t)$ and $z(t)$ be the input and output of the cascaded system and their Fourier transforms be denoted by $X(\omega)$ and $Z(\omega)$, respectively. Derive the relationship between $X(\omega)$ and $Z(\omega)$ in terms of $H_A(\omega)$ and $H_B(\omega)$. [4]

v. Treat the cascaded system A and B as one single system. Derive the unit impulse response function for the combined system in terms of $h_A(t)$ and $h_B(t)$. [4]

b. Consider the following periodic signal $x(t)$.

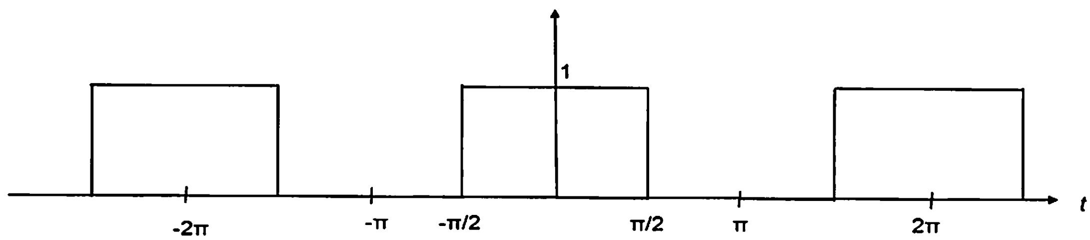


Figure 3. The periodic signal $x(t)$.

i. What is the fundamental frequency ω_0 of $x(t)$ in radians/second? [2]

ii. Determine the complex Fourier series coefficients D_n for $n = -\infty$ to ∞ for $x(t)$ where $x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$. [7]

iii. Sketch the spectrum for the signal $x(t)$. [4]

3. Communications techniques. (30%)

   a. Let us design a frequency converter (also known as a frequency mixer) to change the carrier frequency of an amplitude-modulated signal $m(t)\cos(\omega_c t)$ from ω_c to another frequency ω_I. That is, the input and output of the converter are $m(t)\cos(\omega_c t)$ and $m(t)\cos(\omega_I t)$, respectively. Note that ω_c and ω_I are in unit of radians/second.

      i. Draw a block diagram for the frequency converter, which includes the use of one single sinusoidal signal at an appropriate frequency. [6]
      ii. Provide a mathematical justification for why the converter design works properly. [6]
      iii. Draw the frequency spectrum diagram to illustrate the frequency of the signal shifted by the converter. [4]
      iv. If the signal $m(t)$ has a bandwidth of B Hz, give two conditions for ω_I and ω_c that are required in order for the converter to work properly. Explain why. [4]

   b. A television signal including video and audio has a bandwidth of 4.5 MHz. This signal is sampled, quantized, and binary coded to obtain a sequence of binary pulses (each of which represents one bit).

      i. Determine the sample rate if the signal is to be sampled at a rate 20% above the Nyquist sampling rate. [3]
      ii. If the samples are quantized into 1,024 levels, determine the number of binary pulses required to encode each sample. [2]
      iii. Determine the binary pulse rate (bits per second) of the binary-coded signal. [2]
      iv. Assuming that the communication channel is perfect (e.g., without any noise or other kinds of impairments), determine the minimum bandwidth required to transmit this signal. Explain why. [3]

Model answers.

1. a. i) $$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

ii) $$\frac{df(t)}{dt} = \frac{1}{2\pi}\frac{d}{dt}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega\, F(\omega)\, e^{j\omega t}\, d\omega$$

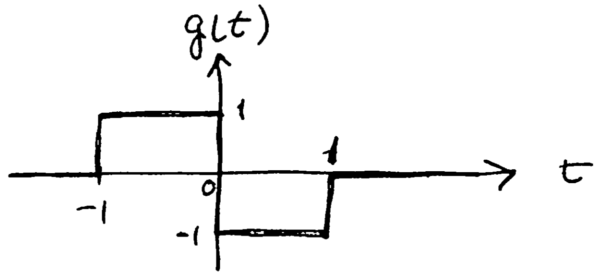
$$= \frac{j\omega}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

$$\Rightarrow \quad g(t) = \frac{df(t)}{dt} = \frac{j\omega}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

$$\Rightarrow \quad g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega F(\omega)\, e^{j\omega t}\, d\omega$$

Therefore, $\mathcal{F}(g(t)) = G(\omega) = j\omega F(\omega)$

iii)



iv) By definition $$G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow \quad G(\omega) = \int_{-1}^{0} 1 \cdot e^{-j\omega t}\, dt + \int_{0}^{1} (-1) \cdot e^{-j\omega t}\, dt$$

$$\Rightarrow \quad G(\omega) = \frac{-1}{j\omega} e^{-j\omega t}\Big|_{-1}^{0} + \frac{1}{j\omega} e^{-j\omega t}\Big|_{0}^{1}$$

$$= \frac{-1}{j\omega}\left[1 - e^{j\omega}\right] + \frac{1}{j\omega}\left[e^{-j\omega} - 1\right]$$

$$\Rightarrow \quad G(\omega) = \frac{1}{j\omega}\left[e^{j\omega} - 1 + e^{-j\omega} - 1\right]$$

$$\Rightarrow \quad G(\omega) = \frac{1}{j\omega}\Big[-2 + \cos\omega + j\sin\omega$$
$$+ \cos\omega - j\sin\omega\Big]$$

$$\Rightarrow \quad G(\omega) = \frac{1}{j\omega}\left[-2 + 2\cos\omega\right]$$

$$\Rightarrow \quad G(\omega) = \frac{-2}{j\omega}\left[1 - \cos\omega\right]$$

v) $\quad G(\omega) = j\omega F(\omega)$

$$\Rightarrow \quad F(\omega) = \frac{G(\omega}{j\omega} = \frac{-2}{(j\omega)^2}\left[1 - \cos\omega\right]$$

$$\Rightarrow \quad F(\omega) = \frac{2}{\omega^2}\left[1 - \cos\omega\right].$$

1.b. i) $$P_x = \int_{-\infty}^{\infty} x^2(t)\, dt$$

$$\Rightarrow P_x = \int_{-\infty}^{\infty} [a(t)+b(t)+c(t)]^2\, dt$$

ii) $$P_x = \int_{-\infty}^{\infty} a^2(t)\, dt + \int_{-\infty}^{\infty} b^2(t)\, dt + \int_{-\infty}^{\infty} c^2(t)\, dt$$

$$+ 2\int_{-\infty}^{\infty} a(t)\, b(t)\, dt + 2\int_{-\infty}^{\infty} b(t)\, c(t)\, dt$$

$$+ 2\int_{-\infty}^{\infty} a(t)\, c(t)\, dt$$

$$\Rightarrow P_x = P_a + P_b + P_c$$

$$+ 2\left[\int_{-\infty}^{\infty} a(t)\, b(t)\, dt + \int_{-\infty}^{\infty} b(t)\, c(t)\, dt \right.$$

$$\left. + \int_{-\infty}^{\infty} a(t)\, c(t)\, dt \right]$$

Therefore, in order for $P_x = P_a + P_b + P_c$

$$\int_{-\infty}^{\infty} a(t)\, b(t)\, dt = 0$$

$$\int_{-\infty}^{\infty} b(t)\, c(t)\, dt = 0$$

and $$\int_{-\infty}^{\infty} c(t)\, a(t)\, dt = 0$$ are sufficient conditions.

That is, $a(t)$, $b(t)$ and $c(t)$ are mutually orthogonal to each other.

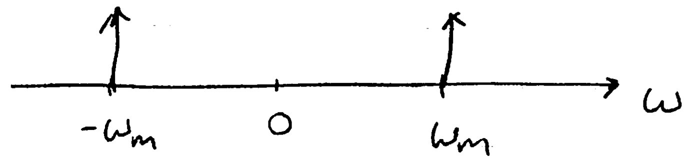
1.b. iii) No, it is not always possible to express any arbitrary signal $y(t)$ as

$$y(t) = \alpha\, a(t) + \beta b(t) + \gamma\, c(t)$$

because $a(t)$, $b(t)$ & $c(t)$ may not sufficient form the complete basis for the signal space.

iv) If $y(t) = \alpha\, a(t) + \beta\, b(t) + \gamma\, c(t) + \lambda d(t)$ for any signal $y(t)$, then $a(t)$, $b(t)$, $c(t)$ must be orthogonal to $d(t)$. (In fact, $a(t)$, $b(t)$, $c(t)$, and $d(t)$ must also form the complete basis for the signal space.)

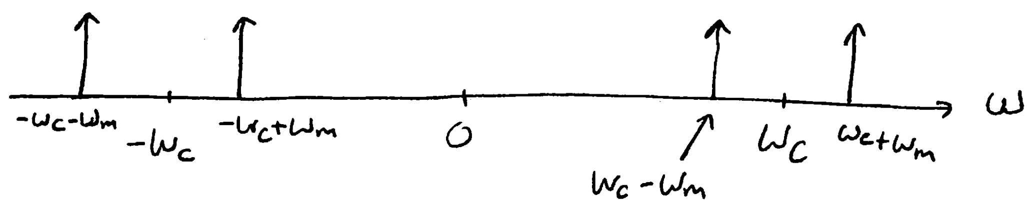
1 c i)



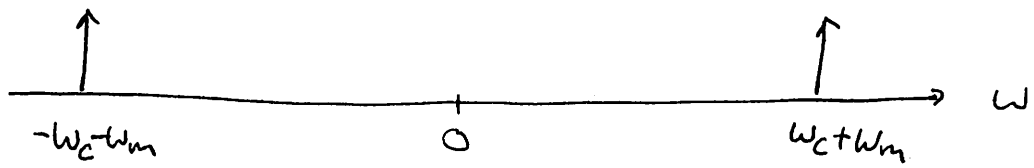
ii)

$$\phi_{DSB}(t) = A\cos(\omega_m t)\cdot\cos(\omega_c t)$$

iii)



iv)



v)

$$\phi_{USB}(t) = \cos\left[(\omega_c+\omega_m)t\right]$$

vi) SSB uses ½ of the bandwidth as for DSB
$\Rightarrow$ cheeper as bandwidth is normally expensive

As using ½ of the bandwidth, it may potentially pick up less noise for SSB when compared with DSB.

1 d. i) $\phi_{pm}(t) = A\cos[\omega_c t + k_p m(t)]$

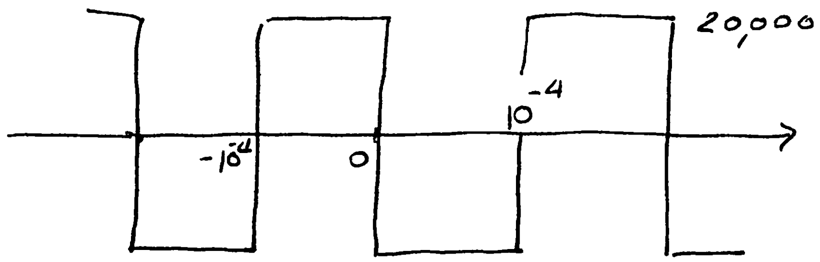
ii) The instantaneous ~~frequency~~ angle is

$$\theta(t) = \omega_c t + k_p m(t)$$

Thus, the instantaneous frequency is

$$\omega_i(t) = \frac{d\theta(t)}{dt} = \omega_c + k_p \dot{m}(t)$$

iii) For the given $m(t)$, $\dot{m}(t)$ is



$$f_i(t) = f_c + \frac{k_p}{2\pi}\dot{m}(t) \quad (\text{in Hz})$$

$$f_i(t)\Big|_{min} = f_c + \frac{k_p}{2\pi}(-20{,}000)$$

$$= 100\,M + \frac{10\pi}{2\pi}(-2\times10^4)$$

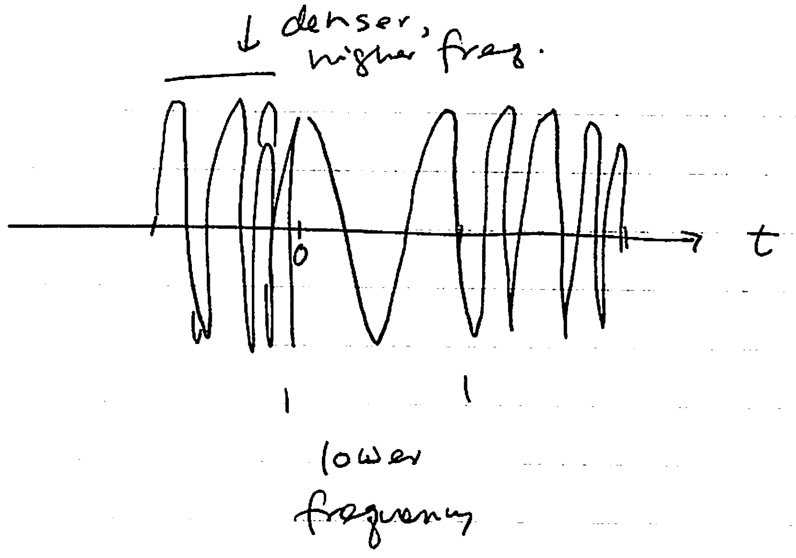
$$\Rightarrow f_i(t)\Big|_{min} = 100\,M - 10^5$$

$$= (100 - 0.1)\times10^6$$

$$f_i(t)\Big|_{min} = 99.9\text{ MHz}$$

Similarly $f_i(t)_{max} = 100.1\text{ MHz}$

1.d. iv)



2. a. i)

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau) \cdot h_A(\tau)\, d\tau$$

ii) or $$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

ii)

$$y(t) = \int_{-\infty}^{\infty} \left[a x_1(\tau) + b x_2(\tau)\right] h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} a\, x_1(\tau)\, h(t-\tau)\, d\tau$$

$$+ \int_{-\infty}^{\infty} b x_2(\tau)\, h(t-\tau)\, d\tau$$

$$\Rightarrow \quad y(t) = a \int_{-\infty}^{\infty} x_1(\tau)\, h(t-\tau)\, d\tau$$

$$+ b \int_{-\infty}^{\infty} x_2(\tau)\, h(t-\tau)\, d\tau$$

$$\Rightarrow \quad y(t) = a y_1(t) + b\, y_2(t)$$

iii) Let $z(t)$ be the output when the input signal is $x(t-T)$. So,

$$z(t) = \int_{-\infty}^{\infty} x(\tau - T)\, h(t-\tau)\, d\tau$$

Let $u = \tau - T \quad \Rightarrow \quad du = d\tau \quad \because T$ is a constant

2.a. iii) Further, as $u = \tau - T$

$$\Rightarrow \quad t - \tau = t - u - T$$

Therefore,

$$z(t) = \int_{-\infty}^{\infty} x(\tau - T)\, h(t - \tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(u)\, h(t - T - u)\, du$$

$$\Rightarrow z(t) = g(t - T).$$

iv)

$$X(\omega) \longrightarrow \boxed{H_A(\omega)} \xrightarrow{Y(\omega)} \boxed{H_B(\omega)} \longrightarrow Z(\omega)$$

$$Y(\omega) = X(\omega)\, H_A(\omega)$$

$$Z(\omega) = Y(\omega) \cdot H_B(\omega)$$

$$\Rightarrow \quad Z(\omega) = X(\omega)\, H_A(\omega) \cdot H_B(\omega)$$

v) From part iv), the transfer function of the "cascaded" system is $H_A(\omega) \cdot H_B(\omega)$.

Therefore, the unit impulse response for the combined system is simply the convolution of $h_A(t)$ and $h_B(t)$:

$$h(t) = \int_{-\infty}^{\infty} h_A(\tau)\, h_B(t - \tau)\, d\tau$$

2b. i) $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2\pi} = 1$ rad/sec

ii) $D_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{2}$

For $n \neq 0$

$$D_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1 \cdot e^{-jn\omega_0 t}\,dt$$

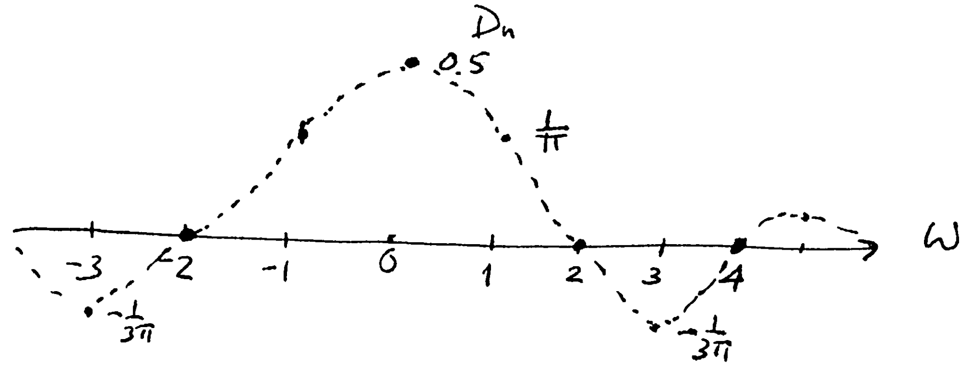
$$\Rightarrow D_n = \frac{1}{2\pi}\cdot\frac{1}{-jn\omega_0}\cdot e^{-jn\omega_0 t}\Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2\pi}\cdot\frac{-1}{jn}\cdot\left(e^{-jn.\pi/2} - e^{+jn.\pi/2}\right)$$

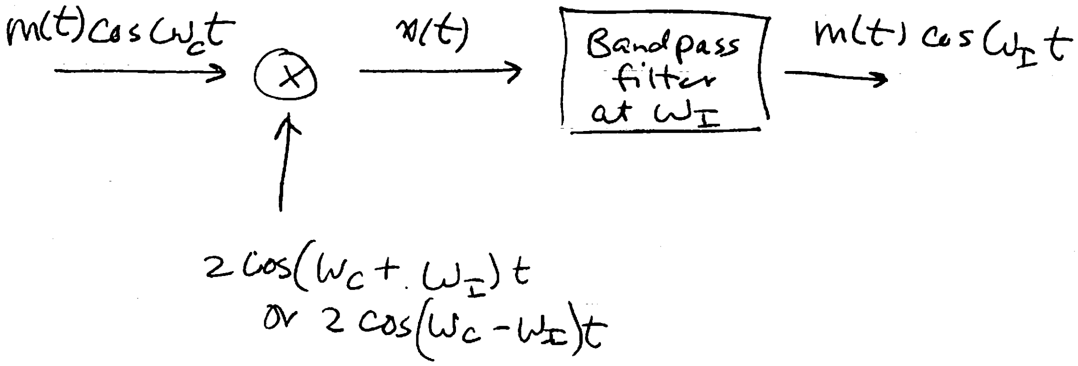
$$\Rightarrow D_n = \frac{1}{2\pi}\cdot\frac{-1}{jn}\Big[\cos\left(n\tfrac{\pi}{2}\right) - j\sin\left(n.\tfrac{\pi}{2}\right) - \cos\left(n\tfrac{\pi}{2}\right) - j\sin\left(n\tfrac{\pi}{2}\right)\Big]$$

$$\Rightarrow D_n = \frac{1}{n\pi}\cdot\sin\left(\frac{n\pi}{2}\right) \quad \text{for } n \neq 0$$

iii)



3.a. i)



ii) The product $x(t)$ is

$$x(t) = 2m(t)\cos\omega_c t\cos(\omega_c+\omega_I)t$$

$$\Rightarrow x(t) = m(t)\left[\cos\omega_I t + \cos(2\omega_c+\omega_I)t\right]$$

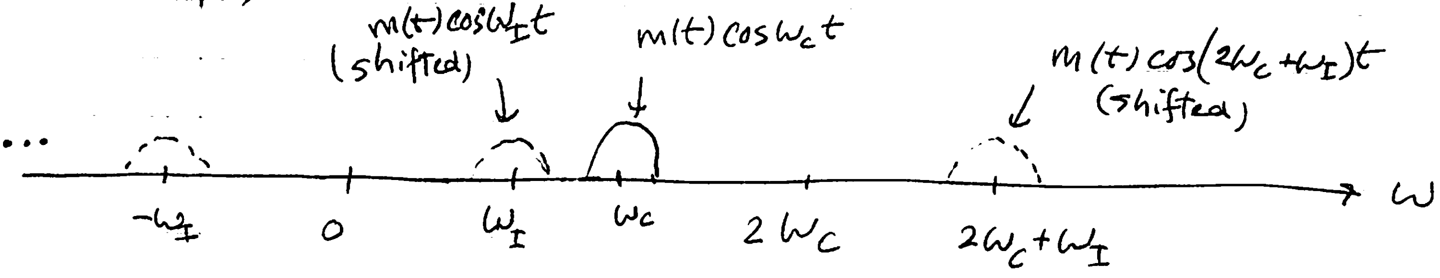
Since the bandpass filter is tuned around ω_c, the output of the filter is $m(t)\cos\omega_I t$, as required.

Similar is true when the mixer frequency is chosen to be $\omega_c - \omega_I$. That is,

$$x(t) = 2m(t)\cos\omega_c t\cos(\omega_c-\omega_I)t$$

$$\Rightarrow x(t) = m(t)\left[\cos\omega_I t + \underbrace{\cos(2\omega_c-\omega_I)t}_{\text{filtered out by the bandpass filter.}}\right]$$

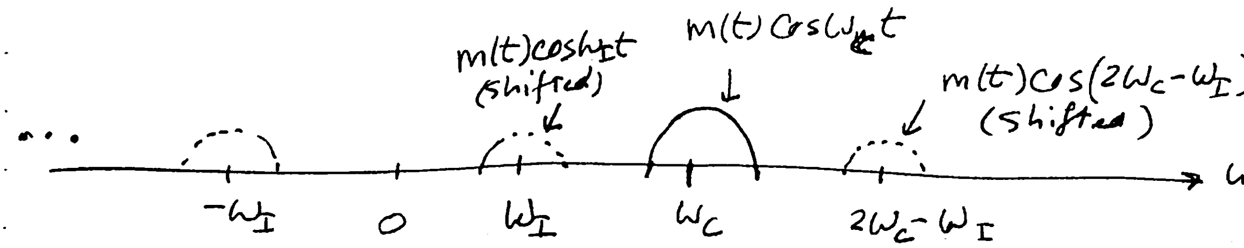
3.a. iii) for the mixer frequency : $\omega_c + \omega_I$
iv)



In this case, if $m(t)$ has a bandwidth of B Hz.

$$\omega_I \geq 2\pi B$$

For the mixer frequency : $\omega_c - \omega_I$



In this case, we require the frequency band between ω_I and $2\omega_c - \omega_I$ be bigger than $2\times 2\pi B$ in order to avoid overlapping of the shifted spectra.

That is, we require

$$2\omega_c - \omega_I - \omega_I \geq (2\pi B)\times 2$$

$$\Rightarrow \quad \omega_c - \omega_I \geq 2\pi B$$

Of course, we still require that

$$\omega_I \geq 2\pi B$$ (otherwise the spectra on the -ve freq can overlap that on the +ve ω_I freq.

(Implicitly, we assume $\omega_c \geq 2\pi B$.)

3.b. i) Given the signal with 4.5 MHz, the Nyquist sampling rate is 9 MHz.

That is, the signal is sampled at

$$1.2 \times 9 = 10.8 \text{ MHz rate.}$$

ii) Let N be the ~~total~~ number of required binary pulses

$$2^N = 1024$$

$$\Rightarrow N = 10 \qquad \text{i.e., } 10 \text{ bits/sample}$$

iii) Binary pulse rate

$$10.8 \text{ M samples/sec} \times 10 \text{ bits/sample}$$

$$= 108 \text{ Mbps}$$

iv) Minimum bandwidth $= \frac{108}{2} = 54$ MHz

This is so because at best, each two pieces of info (bits) can be sent for each Hz of bandwidth per second.