IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2014

EEE/EIE PART III/IV: MEng, Beng and ACGI

Corrected Copy

**DIGITAL SIGNAL PROCESSING**

Wednesday, 15 January 10:00 am

Time allowed: 3:00 hours

**There are FOUR questions on this paper.**

**Answer ALL questions.**

*All questions carry equal marks.*

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible First Marker(s) : P.A. Naylor

Second Marker(s) : W. Dai

# Digital Signal Processing

1. a) A discrete-time signal $x(n)$ is given by

$$x(n) = -2 + 2\cos\frac{n\pi}{4} + \cos\frac{n\pi}{2} + \frac{1}{2}\cos\frac{3n\pi}{4}.$$

   i) Determine the period in samples of $x(n)$. [ 3 ]

   ii) Determine $|X(k)|$, the magnitude spectrum of $x(n)$. [ 5 ]

   iii) Draw a labelled sketch of $|X(k)|$. [ 3 ]

   iv) Verify Parseval's relation for this case by computing the power in both the time and frequency domains. [ 3 ]

   b) Given a discrete-time signal $x(n)$ having Fourier transform

$$F\{x(n)\} = \frac{1}{1 - ae^{-j\omega}}$$

   find the Fourier transforms of

   i) $x(n+2)$, [ 2 ]

   ii) $x(n) \circledast x(n-2)$, [ 2 ]

   iii) $x(n) \circledast x(-n)$, [ 2 ]

   where $\circledast$ represents circular convolution.

2. The bilinear transform describing a mapping between the $s$-plane and the $z$-plane can be written

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right).$$

a) Let $z$ and $s$ be denoted

$$z = re^{j\omega}$$
$$s = \sigma + j\Omega.$$

Explain the result of the bilinear transform on $s = \sigma + j\Omega$ for the cases of $\sigma < 0$, $\sigma = 0$ and $\sigma > 0$. Include illustrative labelled sketches of the $s$-plane and $z$-plane.

[ 5 ]

b) Explain what is meant by frequency warping in the context of the bilinear transform and write an expression for the frequency $\omega$ in terms of $\Omega$.

[ 3 ]

c) Consider a continuous-time bandpass filter with system function

$$H(s) = \frac{(\Omega_u - \Omega_l)s}{s^2 + (\Omega_u - \Omega_l)s + \Omega_l\Omega_u}$$

where $\Omega_u$ and $\Omega_l$ are the upper and lower band edge frequencies respectively.

i) Apply the bilinear transform to convert $H(s)$ to a discrete-time IIR filter $H(z)$ with sampling period $T$ s. (*Hint: Do not consider frequency warping.*) [ 6 ]

ii) Write out the difference equation for the filter's output $y(n)$ given the input signal $x(n)$. [ 4 ]

iii) Draw an illustrative labelled sketch of the magnitude frequency response of $H(z)$. [ 2 ]

3. a) Consider a maximally decimated 2-band analysis filter bank directly connected in cascade to a corresponding synthesis filter bank.

i) Draw a labelled sketch of this analysis-synthesis filter bank employing analysis filters $H_0(z)$ and $H_1(z)$ and synthesis filters $F_0(z)$ and $F_1(z)$. Denote the input signal as $x(n)$ with z-transform $X(z)$, the subband signals as $y_0(n)$ and $y_1(n)$ with z-transforms $Y_0(z)$ and $Y_1(z)$ respectively, and the output of the synthesis filter bank as $\hat{x}(n)$ with z-transform $\hat{X}(z)$. [ 4 ]

ii) Derive expressions for $Y_0(z)$ and $Y_1(z)$ in terms of $X(z)$, $H_0(z)$ and $H_1(z)$.

[ 4 ]

iii) Derive an expression for $\hat{X}(z)$ in terms of $X(z)$, $H_0(z)$, $H_1(z)$, $F_0(z)$ and $F_1(z)$.

[ 4 ]

iv) Show that the expression for $\hat{X}(z)$ can be written in matrix form including the matrix term [ 2 ]

$$\mathbf{F} = \begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix}.$$

b) Consider the system shown in Fig. 3.1 for which the input signal $x(n)$ has the spectrum shown in Fig. 3.2 and $H_B(z)$ is a bandpass filter with magnitude frequency response shown in Fig. 3.3. Draw a labelled sketch of the spectrum of the signal $y(m)$ and explain your answer. [ 6 ]

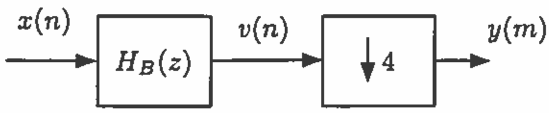


Figure 3.1 Multirate system

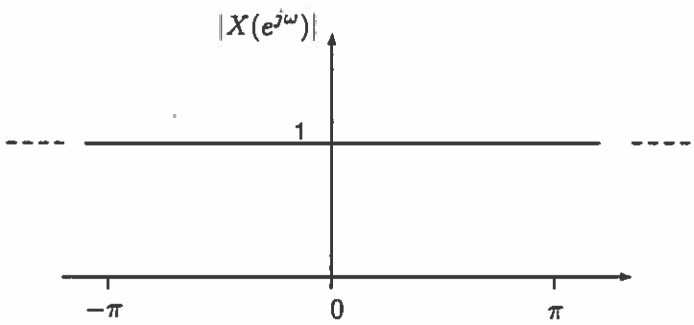


Figure 3.2 Input signal magnitude spectrum

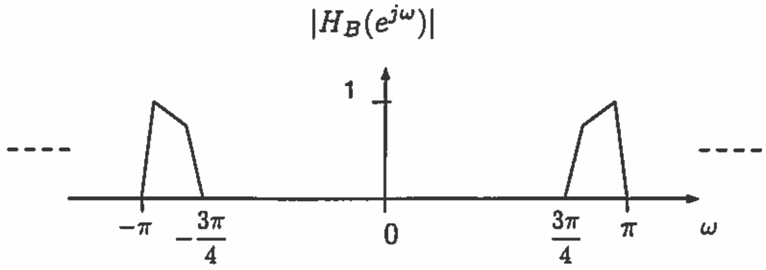


Figure 3.3 Filter magnitude frequency response

4. The Discrete Fourier Transform can be written

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} \quad k = 0, 1, \ldots, N-1.$$

a) Show that

i) $W_N^{k+N/2} = -W_N^k$

ii) $W_N^{k+N} = W_N^k$ [ 4 ]

b) i) Derive the 4-point Radix-2 Decimation-in-Time FFT algorithm and draw the signal flow graph. [ 7 ]

ii) Write a clear explanation of the terms *Radix-2* and *Decimation-in-Time* in this context. [ 2 ]

iii) Determine the number of real multiply operations required to compute the 8-point Radix-2 Decimation-in-Time FFT. Ignore multiplications by $0, +1$ and $-1$. [ 3 ]

c) For a discrete-time signal $x(n)$ of length $N$ samples with DFT $X(k)$, consider a new sequence $y(n)$ of length $2N$ such that

$$y(n) = \begin{cases} x(n/2), & \text{for} \, n \, \text{even} \\ 0, & \text{for} \, n \, \text{odd}. \end{cases}$$

Find an expression for the DFT of $y(n)$ in terms of $X(k)$. [ 4 ]