1. a) Figure 1.1 illustrates an RLC circuit. The capacitor has capacitance C, the inductor has inductance L and the resistor resistance R. The input is the applied voltage $v_i(t)$ and the output is the voltage across the capacitor and resistor $v_o(t)$.

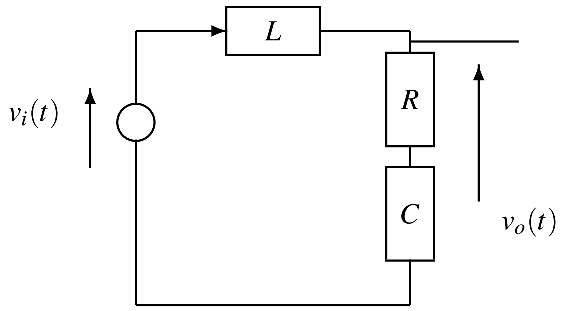


Figure 1.1

i) Determine $G(s)$, the transfer function relating v_o to v_i. [4]

ii) Let $v_i(t)$ be a unit step applied at $t = 0$. Use the final value theorem, which should be stated, to find the steady–state value of $v_o(t)$. [5]

iii) Derive the value of R so that $G(s)$ is marginally stable. What is the frequency of oscillations? Give your answer in terms of L and C. [5]

b) In Figure 1.2 below, $G(s) = \frac{s+1}{s-1}$ and K is a variable gain.

i) Sketch the locus of the closed–loop poles for $0 \leq K < \infty$. [5]

ii) Using the gain criterion, find the value of K for which the closed–loop is marginally stable. [4]

iii) Find the range of $K \geq 0$ for which the closed–loop is stable. [4]

c) In Figure 1.2 below, $G(s) = \frac{1}{s-0.5}$ and K is a variable gain.

i) Draw the Nyquist diagram of $G(s)$ indicating real-axis intercepts. [5]

ii) Take $K = 0.25$. Use the Nyquist criterion, which should be stated, to determine the number of unstable closed–loop poles. [4]

iii) Take $K = 1$. Use the Nyquist criterion to show that the closed–loop is stable. Comment on the gain margin. [4]

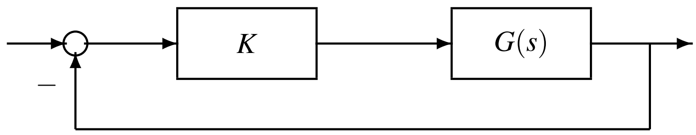


Figure 1.2

2. Let

$$G(s) = \frac{1}{s+1}$$

and consider the feedback loop shown in Figure 2 below.

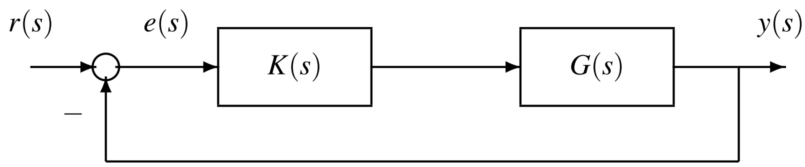


Figure 2

A feedback compensator $K(s)$ is required such that the following design specifications are satisfied:

(i) The closed–loop is stable.

(ii) The closed–loop step response is non-oscillatory and has a settling time of 2 seconds.

(iii) The DC gain of the transfer function from $e(s)$ to $y(s)$ is equal to 11.

a) Draw the root locus of $G(s)$ accurately for all $K > 0$. [5]

b) Derive the location of the closed–loop pole that satisfies the second design specification. [5]

c) Show that the design specifications cannot be satisfied using a proportional compensator. [5]

d) Design a PD compensator that achieves the specifications. [5]
Hint: Define your compensator in terms of two parameters, say K_d and z. Next, obtain algebraic relations, perhaps involving the gain criterion, to satisfy the second and third specifications.

e) Draw the root locus of the compensated system. [5]

f) Evaluate the steady–state error of the closed–loop system for a unit step reference signal. [5]

3. Consider the feedback control system in Figure 3 below. Here,

$$G(s) = \frac{6}{(s+1)^3}$$

and $K(s)$ is the transfer function of a compensator.

a) Sketch the Nyquist diagram of $G(s)$, indicating the low and high frequency portions. Also, calculate the real–axis intercepts. [7]

b) Take $K = 1$. Show that the closed–loop is stable and determine the gain and phase margins. [7]

c) Without doing any actual design, briefly describe how a phase–lead compensator,

$$K(s) = \frac{1+s/\omega_0}{1+s/\omega_p}, \qquad 0 < \omega_0 < \omega_p,$$

would affect the gain and phase margins. Your answer should emphasize the difficulties involved in the design. [8]

d) Design a stabilising phase–lead compensator $K(s)$ such that the loop gain has the same DC gain as $G(s)$ and the gain margin of $G(s)K(s)$ is infinite. Draw a rough sketch of the Nyquist diagram of $G(s)K(s)$. [8]

Hint: You may consider using a special type of phase–lead compensator that implements a pole–zero cancellation.

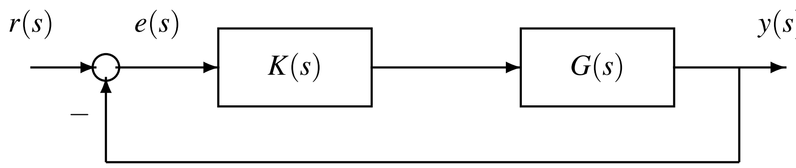


Figure 4

1. a) i) Using the potential divider rule and the impedances we have

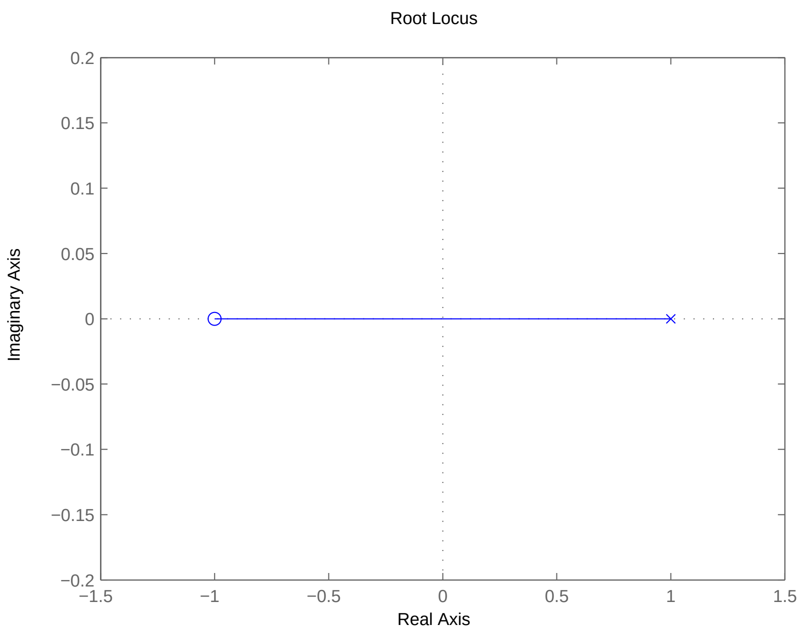
$$G(s) := \frac{v_o(s)}{v_i(s)} = \frac{sRC+1}{s^2LC+sRC+1}\,.$$

ii) Using the final value theorem and the fact that $v_i(s) = 1/s$,

$$\lim_{t\to\infty} v_o(t) = \lim_{s\to 0} s v_o(s) = \lim_{s\to 0} sG(s)v_i(s) = \lim_{s\to 0} sG(s)\frac{1}{s} = G(0) = 1.$$

iii) For marginal stability, the poles must be imaginary so $R = 0$. The frequency of oscillations isgiven by $\omega = \frac{1}{\sqrt{LC}}$.

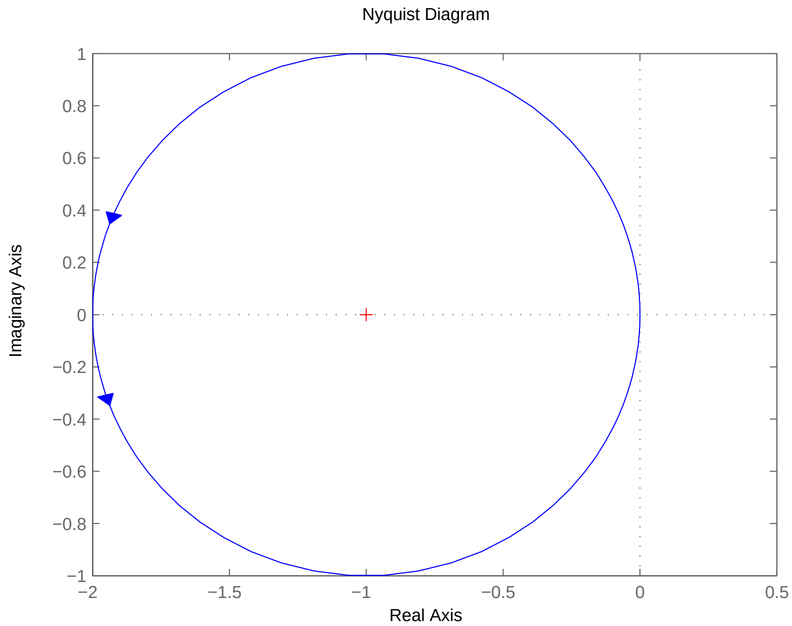
b) i) The root–locus is shown below.



ii) The closed–loop is marginally stable when at least one pole is on the imaginary axis and all others are in the left half–plane. It follows from the root–locus that the marginal pole pole is at $s = 0$. Using the gain criterion $K = -1/G(0) = 1$.

iii) It follows from the root–locus that the closed–loop is stable for all $K > 1$.

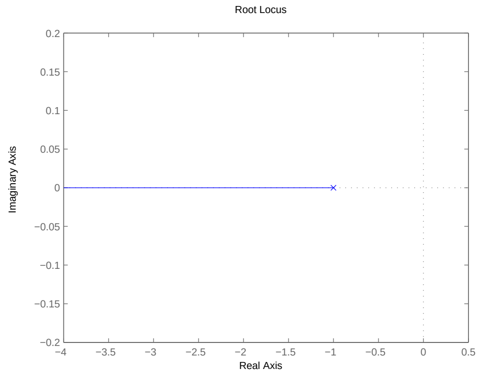
c) i) The Nyquist diagram is shown below. It is clear that the real axis intercepts are at -2 and 0.



ii) Let $K = 0.25$. The Nyquist criterion states that $N = Z - P$, where N is the number of clockwise encirclements by $G(s)$ of the point $-1/K$ as s traverses the Nyquist contour, which in this case is equal to 0; P is the number of unstable open–loop poles, which in this case is equal to 1; and Z is the number of unstable closed–loop poles. Thus there are $Z = N + P = 1$ unstable closed–loop poles.

iii) When $K = 1$, then $N = -1$, $P = 1$ and so $Z = N + P = 0$ and the closed–loop is stable. Since the gain can be increased without bound the system has infinite gain margin for increasing gain. The gain can also be decreased by 50% before losing stability.

2. a) The root–locus is shown below.

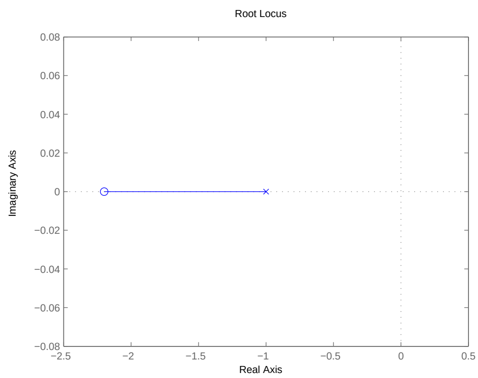


b) For a non–oscillatory response with a settling time of 2 seconds, the closed–loop pole must be located at -2.

c) Using the gain criterion, for a closed–loop pole at -2, $K = -1/G(-2) = 1$. The resulting DC gain is then equal to $KG(0) = 1$ and the third specification is not specified. Thus there does not exist a proportional compensator that satisfies the design specifications.

d) Following the hint, a PD compensator has the form $K(s) = K_d(s+z)$. To satisfy the second specification, the gain criterion requires that $1 - K_d(-2+z) = 0$. To satisfy the DC gain criterion we need $K_d z = 11$. Therefore $K_d = 5$ and $z = 2.2$. So the compensator is $K(s) = 5(s+2.2)$.

e) The root–locus is shown below.



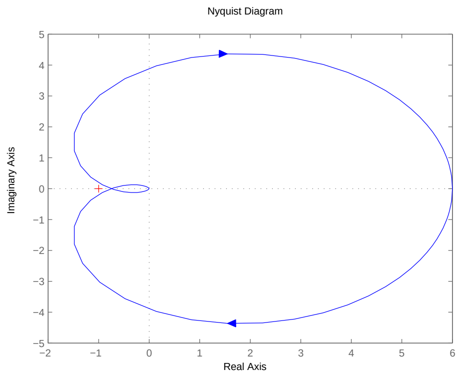
f) The error signal is given by

$$e(s) = \frac{r(s)}{1 + G(s)K(s)}$$

Using the final value theorem for $r(s) = 1/s$ gives

$$e_{ss} = \frac{1}{1 + G(0)K(0)} = \frac{1}{12}.$$

3. a) The Nyquist plot is shown below. The real-axis intercepts can be found by setting the imaginary part of $G(j\omega)$ to zero. This gives intercepts at $\boxed{\omega_i = 0, \pm\sqrt{3}, \infty}$ and so $\boxed{G(j\omega_i) = 6, -0.75, -0.75, 0.}$



b) The number of unstable closed-loop poles is determined by the number of encirclements by $G(s)$ of the point -1, which is zero. Thus the closed–loop is stable since $G(s)$ has no unstable poles. Since the real–axis intercept is at -0.75, the gain margin is $4/3$. For the phase margin, we need the intercept with the unit circle centred on the origin. We solve $|G(j\omega)| = 1$, this gives $\omega_1 \sqrt{6^{\frac{2}{3}} - 1}$ and $\arg[G(j\omega_1)] \approx -190°$. The phase margin is then $\approx 10°$.

c) The phase-lead has gain close to 1 for $\omega < \omega_0$ and close to $\frac{\omega_p}{\omega_0} > 1$ for $\omega > \omega_p$. The phase is positive and large between ω_0 and ω_p but small elsewhere. Thus the gain increase for $\omega > \omega_p$ degrades stability margins while the phase–lead increases the phase margin. It is important to balance the destabilizing increase in gain and the stabilizing increase in phase. We should place w_p and w_0 in the crossover frequency range (when $|G(j\omega)| \approx 1$).

d) One way of getting an infinite gain margin is to to reduce the order of $G(s)$ from 3 to 2. This can be done using the PD compensator $K(s) = K_d(s+1)$ (which is a special type of phase–lead compensator) since the zeros cancels one of the poles of $G(s)$. Taking $K_d = 1$ (to preserve the DC gain of $G(s)$), a sketch of the Nyquist diagram is given below.