IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2014

EEE/EIE PART II: MEng, BEng and ACGI

Corrected Copy

COMMUNICATION SYSTEMS

Q3a i) corrected
ii) 15.20

Tuesday, 27 May 2:00 pm

Time allowed: 2:00 hours

There are THREE questions on this paper.

Answer ALL questions.
Q1 carries 40% of the marks. Questions 2 and 3 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : D. Gunduz
Second Marker(s) : J.A. Barria

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Information for Students

Fourier Transform Pairs

Pair Number	$x(t)$	$X(f)$
1.	$\Pi\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc} \tau f$
2.	$2W \operatorname{sinc} 2Wt$	$\Pi\left(\frac{f}{2W}\right)$
3.	$\Lambda\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}^2 \tau f$
4.	$\exp(-\alpha t)u(t),\ \alpha > 0$	$\frac{1}{\alpha + j2\pi f}$
5.	$t\exp(-\alpha t)u(t),\ \alpha > 0$	$\frac{1}{(\alpha + j2\pi f)^2}$
6.	$\exp(-\alpha\lvert t\rvert),\ \alpha > 0$	$\frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
7.	$e^{-\pi(t/\tau)^2}$	$\tau e^{-\pi(f\tau)^2}$
8.	$\delta(t)$	1
9.	1	$\delta(f)$
10.	$\delta(t - t_0)$	$\exp(-j2\pi f t_0)$
11.	$\exp(j2\pi f_0 t)$	$\delta(f - f_0)$
12.	$\cos 2\pi f_0 t$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$
13.	$\sin 2\pi f_0 t$	$\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$
14.	$u(t)$	$(j2\pi f)^{-1} + \frac{1}{2}\delta(f)$
15.	$\operatorname{sgn} t$	$(j\pi f)^{-1}$
16.	$\frac{1}{\pi t}$	$-j \operatorname{sgn}(f)$
17.	$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\lambda)}{t - \lambda}\, d\lambda$	$-j \operatorname{sgn}(f) X(f)$
18.	$\sum_{m=-\infty}^{\infty} \delta(t - mT_s)$	$f_s \sum_{m=-\infty}^{\infty} \delta(f - mf_s),$ $f_s = T_s^{-1}$

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Useful Relations and Formulas

Sum and Difference Identities:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Double Angle Identities:

$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{2\cot x}{\cot^2 x - 1} = \frac{2}{\cot x - \tan x}$$

Half Angle Identities:

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$
$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$
$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

Product-Sum Identities:

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$
$$\cos x \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$
$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$
$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

Sum-Product Identities:

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$
$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

Hilbert Transform of a signal $g(t)$ is given by

$$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(\tau)}{t-\tau}d\tau$$

Differentiation Rule of Leibnitz

Let $F(z) = \int_{a(z)}^{b(z)} f(x,z)dx$. Then we have

$$\frac{dF(z)}{dz} = \frac{db(z)}{dz}f(b(z),z) - \frac{da(z)}{dz}f(a(z),z) + \int_{a(z)}^{b(z)}\frac{\partial f(x,z)}{\partial z}dx$$

Q-function is defined as

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty}\exp -z^2/2dz$$

For large x, we have

$$Q(x) \approx \frac{1}{\sqrt{2\pi}x}\exp -x^2/2$$

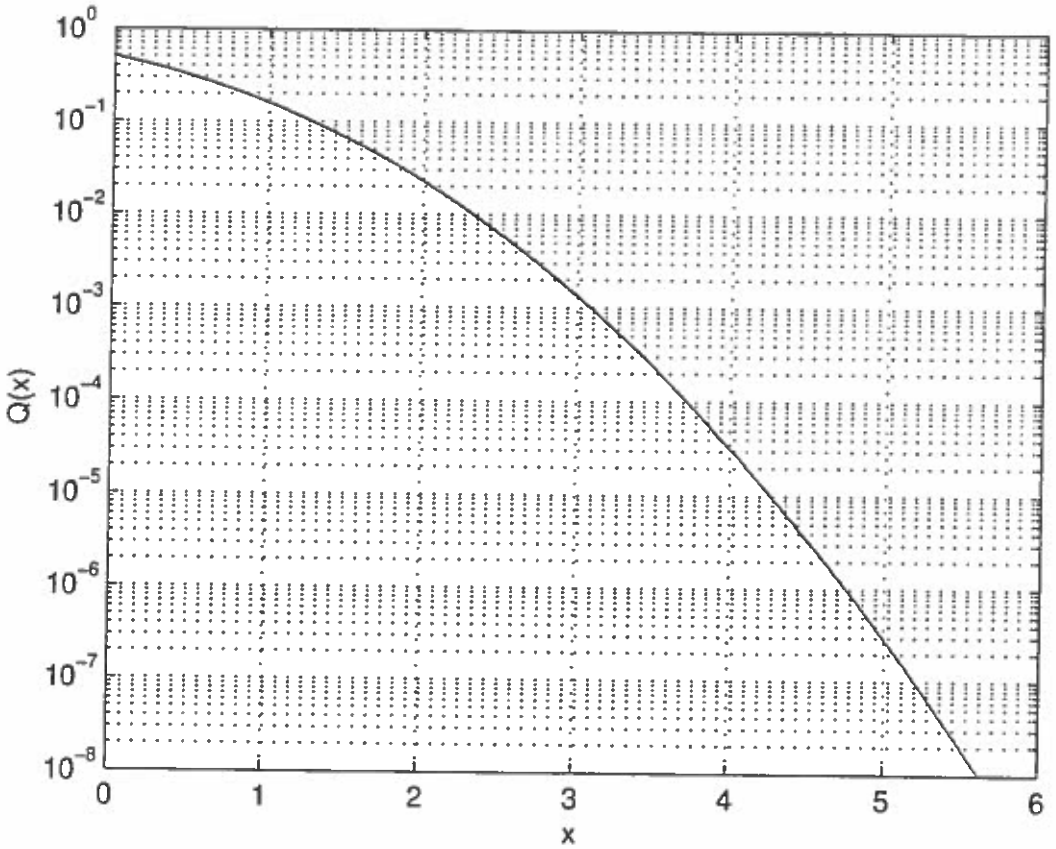

Figure 0.1 The graph of the Q-function, where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$.

1. a) i) Explain the notion of white noise. [2]

ii) Explain whether white noise exists in reality. If yes, given an example. If no, argue why the white noise assumption can still be reasonable to model noise in a communication system. [2]

iii) Write down the expressions for the power spectral density and the autocorrelation function of white noise. [2]

iv) Explain the notion of Gaussian noise. State the theoretical justification for modelling channel noise with a Gaussian distribution. [3]

v) Let $X(t)$ be a white Gaussian noise process. Consider two distinct samples of this process, say $X(t_1)$ and $X(t_2)$, $t_1 \neq t_2$. Explain whether they are correlated, independent, or both. [2]

b) Consider a frequency modulation (FM) communication system in the presence of additive white noise. The block diagram of an FM receiver is given in Fig. 1.1.

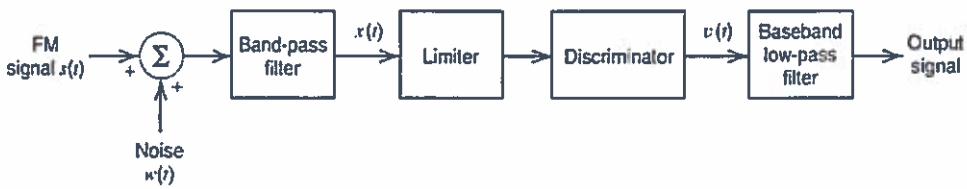

Figure 1.1 Block diagram of an FM receiver.

The FM modulated signal $s(t)$ is given by

$$s(t) = A\cos[2\pi f_c t + \phi(t)],$$

where the instantaneous phase function is defined as

$$\phi(t) \triangleq 2\pi k_f \int_0^t m(\tau)d\tau,$$

and $m(t)$ is the message signal. Consider a wideband FM transmitter in which the message bandwidth W is much smaller compared to the FM signal bandwidth B_T.

i) Explain the function of the band-pass filter. [2]

ii) Write down the expression for the signal at the output of the bandpass filter, $x(t)$, and draw the corresponding phasor diagram. (Hint: Denote the phase of $x(t)$ by $\theta(t)$, and use the narrowband noise representation in terms of its envelope and phase components.) [3]

iii) Explain the function of the limiter. [2]

iv) Write down the signal $v(t)$ at the output of the discriminator. [3]

v) What is the power spectral density (PSD) of the noise component in $v(t)$ when the signal-to-noise ratio (SNR) is sufficiently high? [2]

vi) Explain the function of the low-pass filter (the rightmost block in Fig. 1.1), and plot the PSD of the noise component at the output of the low-pass filter. [2]

c) i) Let S be a random variable defined over the set $\mathscr{S} = \{s_1, s_2, \ldots, s_K\}$, where $P\{S = s_k\} = p_k$, for $k = 1, \ldots, K$, such that $\sum_{k=1}^{K} p_k = 1$. Let T be another random variable defined over the set $\mathscr{T} = \{t_1, t_2, \ldots, t_M\}$, where $P\{T = t_m\} = q_m$, for $m = 1, \ldots, M$, such that $\sum_{m=1}^{M} q_m = 1$. Recall that the entropy of a random variable S is defined as

$$H(S) = \sum_{k=1}^{K} p_k \log_2 \left(\frac{1}{p_k}\right).$$

The *joint entropy* of S and T is defined similarly by considering (S,T) as a single random variable that takes its values over the set $\{(s_k, t_m) : k = 1, \ldots, K, m = 1, \ldots, M\}$. If the joint distribution of S and T is given by $P\{(S,T) = (s_k, t_m)\} = r_{k,m}$, where $\sum_{k=1}^{K} \sum_{m=1}^{M} r_{k,m} = 1$; then the joint entropy of S and T is given by:

$$H(S,T) = \sum_{k=1}^{K} \sum_{m=1}^{M} r_{k,m} \log_2 \left(\frac{1}{r_{k,m}}\right).$$

Prove that $H(S,T) = H(S) + H(T)$, if T is independent of S. [3]

ii) We have a biased coin for which the outcomes are *heads* (H) with probability 0.2 and *tails* (T) with probability 0.8. Let X denote the vector obtained by 8 tosses of this coin. For example, one possible outcome for X is $HTTHHHHH$, which occurs with probability $0.2^6 \times 0.8^2$. What is the entropy of X? [4]

d) Suppose you sample a signal at 10 KHz, and quantize the sampled signal values with a uniform quantizer at n bits per sample. Assume that the signal samples are Gaussian distributed with mean zero and standard deviation σ.

i) How many megabytes of memory do you need to store the quantized version of a two-hour long signal if you use a 10-bit quantizer, i.e., $n = 10$? [2]

ii) If the quantizer range is required to be $\pm 3\sigma$, what is the quantization signal-to-noise ratio (in dB) in terms of n? [3]

iii) What is the probability that the input signal will overload the quantizer (i.e., the probability of the input signal falling out of the quantizer range)? [3]

2. a) Consider two random variables X and Y.

i) Let $Z = 3 \cdot X$ and $T = Y + 5$ be two new random variables. Prove that if X and Y are uncorrelated, then Z and T are also uncorrelated. [2]

ii) Let $U = X + Y$ and $V = X - Y$ be two new random variables. If X and Y are independent, can you claim that U and V are always independent? Explain your reasoning. [4]

iii) Prove that if the variance of X is equal to the variance of Y, then U and V (defined above) are uncorrelated. [4]

b) i) Consider the random process

$$Y(t) = A\cos(2\pi f_c t),$$

where $f_c = 100$ Hz, and A is a random variable uniformly distributed in the range $[-2, 2)$. Find the mean and autocorrelation function of $Y(t)$. Is $Y(t)$ a wide sense stationary (WSS) process? [4]

ii) Suppose

$$Z(t) = A\cos(2\pi f_c t + \Theta),$$

where $f_c = 100$ Hz, Θ is a uniform random variable between $-\pi$ and π, and A is another random variable uniformly distributed in the range $[-2, 2)$. A and Θ are independent. Find the mean and autocorrelation function of $Z(t)$. Is $Z(t)$ a wide sense stationary (WSS) process? [6]

c) Let $Y(t) = X(t+a) - X(t-a)$, where $X(t)$ is a wide sense stationary (WSS) process.

i) Explain whether $Y(t)$ is also WSS. [4]

ii) Find the power spectral density (PSD) of $Y(t)$ in terms of the PSD of $X(t)$, $S_X(f)$. [6]

3. a) Consider a binary digital communication channel. Bit "0" is transmitted with a pulse of amplitude 0, and bit "1" is transmitted with a pulse of amplitude $A > 0$. The noise in the channel is additive white Gaussian noise; however, the noise distribution is **dependent** on the channel input. The sampled value of the received signal can thus be written as:

$$Y = \begin{cases} N_0, & \text{bit 0 was sent,} \\ A + N_1, & \text{bit 1 was sent,} \end{cases}$$

where N_0 is a zero-mean Gaussian random variable with standard deviation σ_0, and N_1 is a zero-mean Gaussian random variable with standard deviation σ_1. Assume that the detection threshold is T at the receiver, where $T \in (0, A)$. In other words, if $Y \geq T$, the transmitted bit is estimated as 1, while if $Y < T$, it is estimated as 0.

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i) Given that a bit 0 was sent, derive the error probability P_{e0} in terms of T ~~A~~ and σ_0, and write it in the form of the Q-function, $Q(x)$. [4]

ii) Given that a bit 1 was sent, derive the error probability P_{e1} in terms of T A and σ_1, and write it using the Q-function. [2]

iii) If a bit 0 is sent with probability p_0 and a bit 1 is sent with probability p_1, obtain the total error probability P_e in terms of p_1, P_{e0} and P_{e1}. [2]

iv) Assume $p_1 = 2/3$ and $\sigma_1 = 2\sigma_0$. Find the detection threshold T that minimizes P_e. [10]

b) Consider the sinusoidal message signal $m(t) = A_m \sin(2\pi f_m t)$, where $A_m = 10$ V and $f_m = 10$ kHz. Assume that this message is transmitted over a channel which attenuates the transmitted signal by 20 dB, and adds noise with the following power spectral density:

$$S(f) = \begin{cases} N_o/2, & |f| < 10^5, \\ N_o/4, & 10^5 \leq |f| < 10^{10}, \\ 0, & otherwise, \end{cases}$$

where $N_o = 10^{-14}$ Watts/Hz.

Assuming that a suitable ideal filter is used at the receiver to limit the out-of-band noise, followed by coherent detection, what is the signal-to-noise ratio (SNR) in dB at the receiver output, and the minimum required bandwidth for each of the following modulation schemes?

i) Baseband transmission. [6]

ii) DSB-SC with a carrier frequency of 1 MHz and a carrier amplitude of $A_c = 1$ V (with coherent detection). [6]