EE3-10
EE9CS3-1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

MSc and EEE/ISE PART III/IV: MEng, BEng and ACGI

MATHEMATICS FOR SIGNALS AND SYSTEMS

Tuesday, 24 May 10:00 am

Time allowed: 3:00 hours

There are THREE questions on this paper.

Answer ALL questions. All questions carry equal marks.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : M.M. Draief

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MATHEMATICS FOR SIGNAL AND SYSTEMS

1. For a matrix A in $\mathbb{R}^{n\times n}$, we define the k-th power of A as $A^k = A^{k-1} \times A = A \times A^{k-1}$, for $k \geq 1$ and $A^0 = I$ the identity matrix. We denote by $\mathrm{Im}(A)$ and $\mathcal{N}(A)$ the range (or image) and the nullspace (or kernel) of A, respectively.

   We say that two subspaces V and W of $\mathbb{R}^n$ are complementary, denoted by $V \oplus W = \mathbb{R}^n$, if (i) $V \cap W = \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$, and (ii) any vector $x \in \mathbb{R}^n$ can be written as $x = v + w$ where $v \in V$ and $w \in W$.

   a) We let $n = 3$ and define the matrix A by

$$A = \begin{pmatrix} 4 & -1 & 5 \\ -2 & -1 & -1 \\ -4 & 1 & -5 \end{pmatrix}.$$

      i) Derive $\mathrm{Im}(A)$ and $\mathcal{N}(A)$ and determine a basis for each of them. [3]

      ii) Do we have $\mathrm{Im}(A) \oplus \mathcal{N}(A) = \mathbb{R}^3$? Justify your answer. [2]

      iii) Let $A^2 = A \times A$. Derive $\mathrm{Im}(A^2)$ and $\mathcal{N}(A^2)$ and determine a basis for each of them. [3]

      iv) Show that $\mathrm{Im}(A^2) \oplus \mathcal{N}(A^2) = \mathbb{R}^3$. [3]

   b) We now let $n = 4$ and define the matrix A_m as follows

$$A_m = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & m & 0 & 0 \\ 1 & 0 & -m & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

   where $m \in \mathbb{R}$ is a parameter.

      i) Derive bases for $\mathcal{N}(A_m)$ and $\mathrm{Im}(A_m)$. [3]

      ii) For $m \neq 0$, show that $\mathrm{Im}(A_m) \oplus \mathcal{N}(A_m) = \mathbb{R}^4$. [2]

      iii) We now fix $m = 0$. Compute A_0^3.

      Do we have $\mathrm{Im}(A_0^3) \oplus \mathcal{N}(A_0^3) = \mathbb{R}^4$?

      Justify your answer. [2]

   c) We define the following property

   For $A \in \mathbb{R}^{n\times n}$, there exists an integer $p \geq 1$ such that $\mathrm{Im}(A^p) \oplus \mathcal{N}(A^p) = \mathbb{R}^n$, $(\star)$

      i) Let A be a non-singular (invertible) matrix. Find p such that the property $(\star)$ is satisfied for A. Justify your answer. [1]

      ii) Let A be a projection. Find a value p such that $(\star)$ is satisfied.

      Explain your answer. A formal proof is not required. [1]

   In fact, the property $(\star)$ is satisfied for any matrix A.

2. For x, y two vectors in $\mathbb{R}^m$, we define the inner product $(x \mid y) = x^T y = \sum_{i=1}^m x_i y_i$ where x_i and y_i are the i-th coordinates of x and y, respectively, and ^{T} is the operation of transposing a vector or a matrix. We also let the norm of x be $||x|| = \sqrt{x^T x} = \sqrt{\sum_{i=1}^m x_i^2}$.

Let $A \in \mathbb{R}^{m \times n}$. For $v \in \mathbb{R}^m$, we define $v_0 \in \mathrm{Im}(A)$, the orthogonal projection of v on $\mathrm{Im}(A)$, i.e.,

$$(v - v_0 \mid Ax) = 0, \text{ for all } x \in \mathbb{R}^n. \qquad (\star\star)$$

a) Let $x_0 \in \mathbb{R}^n$ such that $Ax_0 = v_0$.

i) Show that for all $x \in \mathbb{R}^n$, we have

$$||Ax - v||^2 = ||v - v_0||^2 + ||Ax - v_0||^2.$$

[3]

ii) Prove that $||Ax_0 - v|| = \min_{x \in \mathbb{R}^n} ||Ax - v||$. We will refer to x_0 as a *pseudo-solution* of the equation $Ax = v$. [2]

iii) Suppose that A has zero-nullspace and let x_1 be a vector such that

$$||Ax_1 - v|| = ||v_0 - v||.$$

Show that $x_1 = x_0$. [3]

iv) By rewriting $(\star\star)$ in matrix form show that x_0 is a pseudo-solution of $Ax = v$ **if and only if** x_0 is a solution of the *normal equation*

$$A^T A x_0 = A^T v.$$

[2]

v) Assume that A has zero-null space. Describe an algorithm for solving the normal equation using the Cholesky decomposition (the description of the Cholesky decomposition is not required). [2]

vi) Ignoring the cost of the Cholesky decomposition, how many additional operations does the previous algorithm (Question 2.a)v)) perform? [2]

b) Let $n = 3$,

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Compute the pseudo-solutions of $Ax = v$. [2]

c) Let n be an integer greater or equal to 2 and define the following three real-valued vectors $(a_1, a_2, \cdots, a_n)$, $(b_1, b_2, \cdots, b_n)$ and $(c_1, c_2, \cdots, c_n)$. We would like to find two real numbers λ and μ that minimise the following sum

$$\sum_{k=1}^n (\lambda a_k + \mu b_k - c_k)^2.$$

i) Restate the above minimisation problem in terms of finding the pseudo-inverse of a linear equation $Ax = v$. [1]

ii) Derive a condition on $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ so that the matrix A, defined in Question 2.c)i), has zero null-space. [1]

iii) Under the condition of Question 2.c) ii), solve the minimisation problem. Express λ and μ in terms of inner products. [2]

3. We consider the set $\mathbb{R}_n[X]$ of polynomials with real coefficients and degrees less or equal to n endowed with the inner product $\langle P, Q\rangle = \int_{-1}^{1} P(t)Q(t)dt$.

a) Show that $\langle P, Q\rangle = \int_{-1}^{1} P(t)Q(t)dt$ is indeed an inner product on $\mathbb{R}_n[X]$. [1]

b) Give the expression of $\langle P, Q\rangle$ when P and Q are polynomials in $\mathbb{R}_2[X]$ in terms of the coefficients of both P and Q. [1]

c) Let L be the application on $\mathbb{R}_n[X]$ such that

$$L(P) = \frac{d}{dX}\left[(X^2-1)\frac{dP}{dX}\right].$$

i) Show that if $P \in \mathbb{R}_n[X]$ then $L(P) \in \mathbb{R}_n[X]$ and that L is a linear transformation on $\mathbb{R}_n[X]$. [2]

ii) Prove that, for all P, Q in $\mathbb{R}_n[X]$, we have

$$\langle L(P), Q\rangle = \langle P, L(Q)\rangle.$$

[3]

Hint: Perform integrations by parts.

d) Let $P_0 = 1$ and for $k = 1, \cdots, n$, define the polynomial P_k of degree k as follows

$$P_k = \frac{d^k}{dX^k}\left((X^2-1)^k\right),$$

the k-th derivative of $(X^2-1)^k$.

i) Compute P_1 and P_2. [1]

ii) Derive an expression of $L(P_k)$ in terms of P_k' and P_k'' the first and second derivatives of P_k, respectively. [1]

iii) Prove the following identity

$$(X^2-1)\frac{d[(X^2-1)^k]}{dX} - 2kX(X^2-1)^k = 0.$$

[1]

iv) By differentiating $(k+1)$ times the above expression, establish that

$$(X^2-1)P_k''(X) + 2XP_k'(X) = k(k+1)P_k(X).$$

[4]

Hint: Use Leibniz's formula

$$(fg)^{(k+1)} = \sum_{i=1}^{k+1}\binom{k+1}{i} f^{(i)} g^{(k+1-i)},$$

where $f^{(i)}$ is the i-th derivative of f.

v) Find the eigenvalues and eigenvectors of the transformation L. [2]

e) Let k, l two integers between 0 and n.

i) Express $\langle L(P_k), P_l\rangle$ and $\langle L(P_l), P_k\rangle$ in terms of $\langle P_k, P_l\rangle$. [2]

ii) Prove that $(P_0, P_1, \cdots, P_n)$ is an orthogonal basis of $\mathbb{R}_n[X]$ when endowed with the inner product $\int_{-1}^{1} P(t)Q(t)dt$. [2]

These polynomials are known as Legendre polynomials.

EE3-10
EE9 CS3-1

SOLUTIONS - 2011

MATHEMATICS FOR SIGNAL & SYSTEMS (121 - 2011).

Q1

1/10

(1)

a) $$A = \begin{bmatrix} 4 & -1 & 5 \\ -2 & -1 & -1 \\ -4 & 1 & -5 \end{bmatrix}$$

i) $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathcal{N}(A) \Rightarrow \begin{cases} 4x - y + 5z = 0 \\ -2x - y - z = 0 \\ -4x + y - 5z = 0 \end{cases} \Leftrightarrow \begin{cases} y = 4x + 5z \\ z = -2x - y \end{cases}$$

$$\Leftrightarrow \begin{cases} y = -x \\ z = -x \end{cases}$$

$$\mathcal{N}(A) = \{ x e_1 - x e_2 - x e_3 \,;\, x \in \mathbb{R} \} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}.$$

From lecture $\mathrm{Im}(A) = \mathrm{Span}\{\text{column vectors}\}$.

Hence $$\mathrm{Im}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 4 \\ -2 \\ -4 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \\ -5 \end{pmatrix} \right\}$$

Since $\begin{pmatrix} 5 \\ -1 \\ -5 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -4 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$, then

$$\mathrm{Im}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 4 \\ -2 \\ -4 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

ii) It is not difficult to see that

$$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{3}\left[\begin{pmatrix} 4 \\ -2 \\ -4 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right] \Rightarrow$$

$$\mathrm{Im}(A) \cap \mathcal{N}(A) \neq \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

So the answer is $\mathrm{Im}(A) \oplus \mathcal{N}(A) \neq \mathbb{R}^3$.

1) a) iii) $A^2 = \begin{bmatrix} -2 & 2 & -4 \\ -2 & 2 & -4 \\ 2 & -2 & 4 \end{bmatrix}$.

Q.1 +2

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in N(A^2) \iff 2x - 2y + 4z = 0 \iff x = y - 2z$.

$N(A^2) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$Im(A^2) = \text{Span}\left\{ \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix} \right\}$

$= \text{Span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

iv) $u \in N(A^2) \cap Im(A^2) \iff \exists\ \alpha, \beta, \gamma \in \mathbb{R}$ |

$u = \alpha(e_1 + e_2) + \beta(-2e_1 + e_3) = \gamma(-e_1 - e_2 + e_3)$

$\iff \begin{cases} \alpha - 2\beta + \gamma = 0 \\ \alpha - \gamma = 0 \\ \beta - \gamma = 0 \end{cases} \iff \alpha = \beta = \gamma = 0.$

Hence $N(A^2) \cap Im(A^2) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$.

Moreover $\dim(N(A^2)) + \dim(Im(A^2)) = 3$.

Note: One can use the fact that $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ forms a basis of $\mathbb{R}^3$.

Q1 (3)

1/b)

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = u \in \mathcal{N}(A_m) \iff \begin{cases} -y = 0 \\ my = 0 \\ x - mz - t = 0 \\ y = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ t = x - mz \end{cases}$$

$$\mathcal{N}(A_m) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -m \end{pmatrix} \right\}.$$

$$\text{Im}(A_m) = \text{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ m \\ 0 \\ 1 \end{pmatrix} \right\}.$$

ii) $m \neq 0$.

$u \in \mathcal{N}(A_m) \cap \text{Im}(A_m) \iff \exists\, \alpha, \beta, \gamma, \delta$.

$$u = \alpha(e_1 + e_4) + \beta(e_3 - me_4) = \gamma e_3 + \delta(-e_1 + me_2 + e_4)$$

$$\iff \begin{cases} \alpha + \delta = 0 \\ m\delta = 0 \\ \beta - \gamma = 0 \\ \alpha - m\beta - \delta = 0 \end{cases} \Rightarrow \alpha = \beta = \gamma = \delta = 0 \quad (m \neq 0)$$

Moreover $\dim(\text{Im}(A_m)) + \dim(\mathcal{N}(A_m)) = 4$.

iii) $m = 0$ $\quad A_0^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, Hence

trivially $\mathcal{N}(A_0) = \mathbb{R}^4$ & $\text{Im}(A_0) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\Rightarrow \mathbb{R}^4 = \text{Im}(A_0) \oplus \mathcal{N}(A_0)$.

1/c) Q1 4

i) A invertible $\Rightarrow$ $\mathcal{N}(A) = \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$

$\mathrm{Im}(A) = \mathbb{R}^n$

$\Rightarrow \mathcal{N}(A) \oplus \mathrm{Im}(A) = \mathbb{R}^n$

~~ii)~~ if A is a projection then

$\mathcal{N}(A) \oplus \mathrm{Im}(A) = \mathbb{R}^n$.

Q2

2/ a/ $(v - v_0 | Ax) = 0 \quad \forall x \in \mathbb{R}^3 \quad (*)$ (5)

i) $\|Ax - v\|^2 = \|Ax - v_0 + v_0 - v\|^2$

$= \|Ax - v_0\|^2 + \|v_0 - v\|^2 + 2(Ax - v_0 | v_0 - v)$

Remark that $(Ax - v_0 | v_0 - v) = (A(x - y_0) | v_0 - v)$

$= 0$ by $(*)$.

$\Rightarrow \|Ax - v\|^2 = \|Ax - v_0\|^2 + \|v_0 - v\|^2$

ii) $\|Ax - v\| \geq \|Ax - v_0\|$ unless $v = v_0$.

iii) $\|Ax_1 - v\|^2 = \|Ax_1 - Ax_0 + Ax_0 - v\|^2$

$= \|Ax_1 - Ax_0\|^2 + \|Ax_0 - v\|^2$

$+ 2 \underbrace{(A(x_1 - x_0) | Ax_0 - v)}_{= 0 \text{ by } (*)}$.

Whenever $\|Ax_1 - v\| = \|Ax_2 - v\|$

$\Rightarrow \|Ax_1 - Ax_2\| = 0 \Rightarrow A(x_1 - x_2) = 0$

$\Rightarrow x_1 = x_2$ since A has zero null-space.

2/ a/ Q2 6

iv) $(**) \Leftrightarrow (v-v_0)^T Ax = 0 \quad \forall x$

$\Leftrightarrow (A^T v - Ax_0)^T x = 0 \quad \forall x$

$\Leftrightarrow A^T(v - Ax_0) = 0$

$\Rightarrow A^T A x_0 = A^T v$.

v) • Cholesky decomposition of $A^T A = L^T L \in \mathbb{R}^{n \times n}$, $L \in$

$v \in \mathbb{R}^m$

$A^T \in \mathbb{R}^{n \times m}$

* $A^T v : (2m-1)n$.

* $A^T A : (2m-1) n^2$

* $L^T \underbrace{L x_0} = A^T v$.

$L^T w = A^T v \quad : \quad n^2$

$L x_0 = w \quad : \quad n^2$

vi) In total ~~$2mn + 2mn^2 + 2n^2$ flops~~.

$2mn - \cancel{2}n + 2mn^2 - n^2 + 2n^2$

$= 2mn^2 + 2mn + n^2 - n \left(= \Theta(mn^2)\right)$.

$\theta 2$ 7

2/c)iii) $A^T \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{matrix} \Sigma a_i c_i \\ \Sigma b_i c_i \end{matrix}$

$$\begin{bmatrix} \langle a,a \rangle & \langle a,b \rangle \\ \langle a,b \rangle & \langle b,b \rangle \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \mu_2 \end{pmatrix} = \begin{matrix} \langle a,c \rangle \\ \langle b,c \rangle . \end{matrix}$$

$$\langle a,a \rangle \lambda + \langle a,b \rangle \mu = \langle a,c \rangle$$
$$\langle a,b \rangle \lambda + \langle b,b \rangle \mu = \langle b,c \rangle .$$

$$\lambda = \frac{\langle a,c \rangle - \langle a,b \rangle \mu}{\langle a,a \rangle}$$

$$\frac{\langle a,b \rangle}{\langle a,a \rangle} \left[\langle a,c \rangle - \langle a,b \rangle \mu \right] + \langle b,b \rangle \mu = \langle b,c \rangle$$

$$\langle b,b \rangle - \frac{\langle a,b \rangle^2}{\langle a,a \rangle} \mu = \langle b,c \rangle - \frac{\langle a,b \rangle \langle a,c \rangle}{\langle a,a \rangle}$$

$$\begin{cases} \mu = \dfrac{\langle a,a \rangle \langle b,c \rangle - \langle a,b \rangle \langle a,c \rangle}{\langle a,a \rangle \langle b,b \rangle - \langle a,b \rangle^2} \\ \lambda = \dfrac{\langle b,b \rangle \langle a,c \rangle - \langle a,b \rangle \langle b,c \rangle}{\langle a,b \rangle \langle b,b \rangle - \langle a,b \rangle^2} \end{cases}$$

8 Q3

3/

a) $\langle P, Q\rangle = \langle Q, P\rangle$

$\langle P, P\rangle = \int_{-1}^{1} P^2(t)\,dt \geq 0$

$\langle \lambda P + \mu R, Q\rangle = \lambda\langle P, Q\rangle + \mu\langle R, Q\rangle.$

$\langle P, P\rangle = 0 \Rightarrow P^2 = 0$ on $(-1, 1)$

$\Rightarrow P = 0.$

b) $P(x) = a_0 + a_1 x + a_2 x^2$

$Q(x) = b_0 + b_1 x + b_2 x^2$

$P(x)Q(x) = b_0 a_0 + (a_1 b_0 + a_0 b_1)x$

$+ (a_1 b_1 + a_2 b_0 + b_2 a_0)x^2 + (a_1 b_2 + a_2 b_1)x^3 + a_2 b_2 x^4$

$\langle P, Q\rangle = 2 b_0 a_0 + \frac{2 a_1 b_1 + a_2 b_0 + a_0 b_2}{3} + \frac{2 a_2 b_2}{5}.$

c) i) High degree in $L(P)$ was

from $\frac{d}{dX}(X^2-1)\frac{dX^n}{dX}$

i.e $\frac{d}{dX}(X^2-1)\, n X^{n-1}$ which has

at most degree n.

Trivially we have

$$L(P + \lambda Q) = L(P) + \lambda L(Q).$$

Q3 9

ii) $$\langle L(P)\, Q \rangle = \int_{-1}^{1} \frac{d}{dt}(t^2-1)\frac{dP(t)}{dt}\, Q(t)\, dt.$$

Integration by part.

$$= \underbrace{\left[(t^2-1)\frac{dP}{dt}\, Q\right]_{-1}^{1}}_{=0} - \int_{-1}^{1} (t^2-1)\frac{dP}{dt}\frac{dQ(t)}{dt}\, dt$$

Integration by part,

$$= - \underbrace{\left[(t^2-1)\frac{dQ(t)}{dt}\, P(t)\right]_{-1}^{1}}_{=0} + \int_{-1}^{1} \frac{d}{dt}\left[(t^2-1)\frac{dQ}{dt}\right] P(t)\, dt$$

$$= \langle P, L(Q) \rangle.$$

d) i) $P_1 = 2X$, $P_2 = 12X^2 - 4$.

ii) $L(P_k) = \left((X^2-1)P_k'\right)' = 2X P_k' + (X^2-1)P_k''$.

iii) $$(X^2-1)\left((X^2-1)^k\right)' = (X^2-1) \cdot 2k\, X (X^2-1)^{k-1}$$
$$= 2k\, X (X^2-1)^k.$$

Q3 10

iv) Leibniz formula $\frac{d^{k+1}}{dx^{k+1}} f(x)g(x) = \sum_{i=0}^{k+1} \binom{k+1}{i} f^{(i)}(x) g^{(k+1-i)}(x)$

Applying this to d) iii).

$$(x^2-1) P_k''(x) + 2x(k+1) P_k'(x) + (k+1)k P_k(x)$$
$$- 2kx P_k'(x) - 2k(k+1) P_k(x) = 0.$$

$$\boxed{(x^2-1) P_k''(x) + 2x P_k'(x) = k(k+1) P_k(x)}$$

v) Eigenvalues $k(k+1)$ eigenvectors P_k's.

e)

i) $\langle L(P_k), P_\ell \rangle = k(k+1) \langle P_k, P_\ell \rangle$.

$\langle L(P_\ell), P_k \rangle = \ell(\ell+1) \langle P_\ell, P_k \rangle$

ii) i) $\Rightarrow$ $k(k+1)\langle P_k, P_\ell\rangle = \ell(\ell+1)\langle P_k, P_\ell\rangle$

together with d) ii) $k \neq \ell \Rightarrow \langle P_k, P_\ell \rangle = 0$.

ii) Since $P_0 \dots P_n$ is orthogonal of cardinality n then it is an orthogonal basis of $\mathbb{R}_n[x]$.