Answers EE1-06

1. This is a general question. (40%)

a. We have $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$.

i. $\omega_0 = \frac{2\pi}{T_0}$ [1]

ii. $\cos(m\omega_0 t)$ and $\cos(n\omega_0 t)$ are orthogonal for $m \neq n$ as they have 0 correlation:

$$\frac{1}{T_0}\int_{T_0} \cos(m\omega_0 t)\cos(n\omega_0 t)dt = \frac{1}{T_0}\int_{T_0} \{\tfrac{1}{2}\cos[(m-n)\omega_0 t] + \tfrac{1}{2}\cos[(m+n)\omega_0 t]\}dt = 0$$

[2]

iii. Coefficients b_n=0 for all n [1]

as $\sin(n\omega_0 t)$'s are odd functions and cannot be cancelled by cosine components. [1]

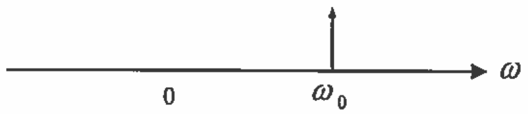
iv. The power of $f(t)$ is $a_0^2 + \sum_{n=1}^{\infty} \frac{a_n}{2}$ [2]

due to the Parseval's theorem. [1]

b. Given $\phi(t) = g(t)e^{j\omega_0 t}$.

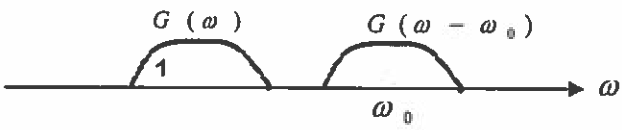
i. The Fourier transform of $e^{j\omega_0 t}$ is $2\pi\delta(\omega - \omega_0)$ because the inverse Fourier transform of the latter is $e^{j\omega_0 t}$. [2]

ii. The spectrum for $e^{j\omega_0 t}$ is [2]



iii. $\Phi(\omega) = \int_{-\infty}^{\infty} g(t)e^{j\omega_0 t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} g(t)e^{-j(\omega-\omega_0)t}dt = G(\omega - \omega_0)$. [2]

iv.



[2]

v. The multiplication of $e^{j\omega_0 t}$ with $g(t)$ to obtain $\phi(t)$ corresponds to the convolution of $2\pi\delta(\omega - \omega_0)$ with $G(\omega)$, which in turn corresponds to shifting $G(\omega)$ to the angular frequency of ω_0. [2]

0

1c. i. $$s(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$ where $\omega_0 = \frac{2\pi}{T_0}$

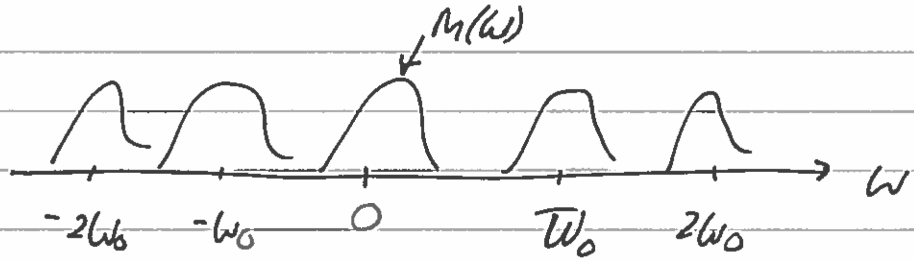
ii. $$D_n = \frac{1}{T_0} \int_{T_0} s(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T_0} \cdot \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jn\omega_0 t} dt$$

$$\Rightarrow D_n = 1/T_0$$

iii. $$s(t) m(t) = m(t) \cdot \sum_{n=-\infty}^{\infty} \frac{1}{T_0} \cdot e^{+jn\omega_0 t}$$

iv.



v. Use a BPF to obtain the $M(\omega \pm \omega_0)$

The relationship is $\omega_0 = \omega_c = \frac{2\pi}{T_0}$

1\. d. i. $\phi_{FM}(t) = A\cos\left[\omega_c t + k_f \int m(\tau)d\tau\right]$

ii. The instantaneous frequency $\omega_i(t)$:

$$\omega_i(t) = \frac{d}{dt}\left[\omega_c t + k_f \int m(\tau)d\tau\right]$$

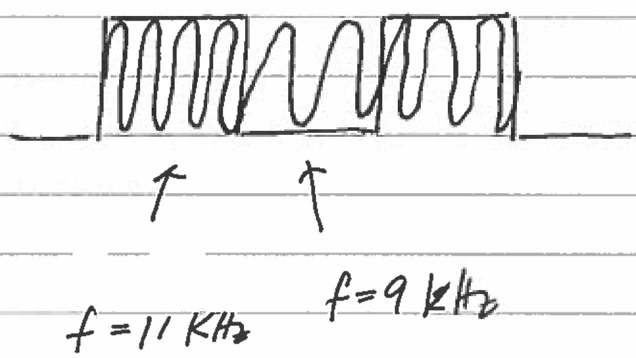
$$\Rightarrow \omega_i(t) = \omega_c + k_f m(t)$$

iii. $$m_p = \max_t |m(t)| = 1$$

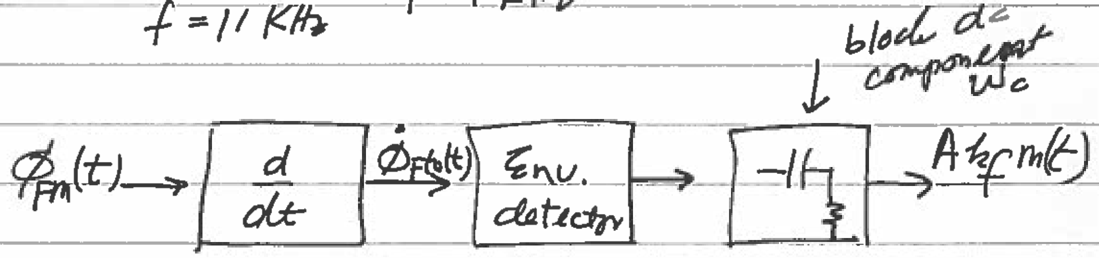
$$\Delta f = k_f \cdot m_p = k_f = 1\ kHz$$

$$\Rightarrow k_f = 1{,}000$$

iv.



V.



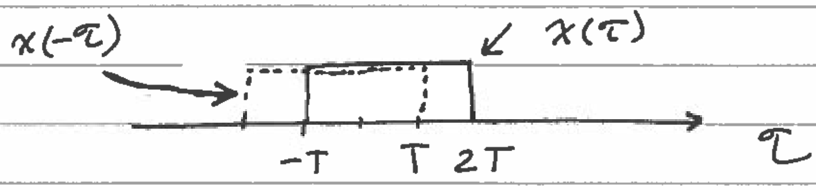
$$\phi_{FM}(t) = A\cos\left[\omega_c t + k_f \int m(\tau)d\tau\right]$$

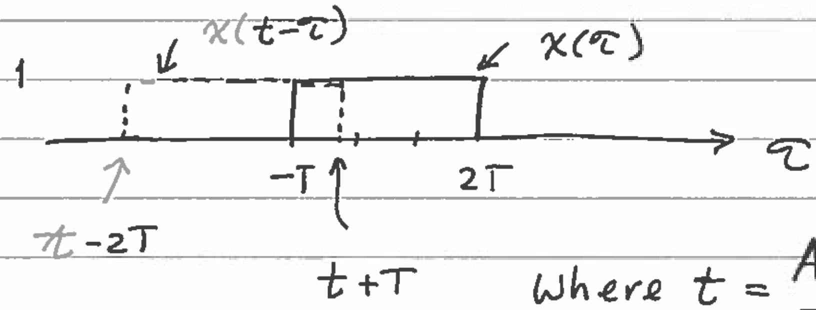
$$\dot{\phi}_{FM}(t) = \frac{d\phi_{FM}(t)}{dt} = \underline{A\left[\omega_c t + k_f\ m(t)\right]} \cdot \sin\left[\omega_c t + k_f \int m(\tau)d\tau\right]$$

envelop which can be extracted by envelop detector.

2a i. $y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, x(t-\tau)\, d\tau$

ii.





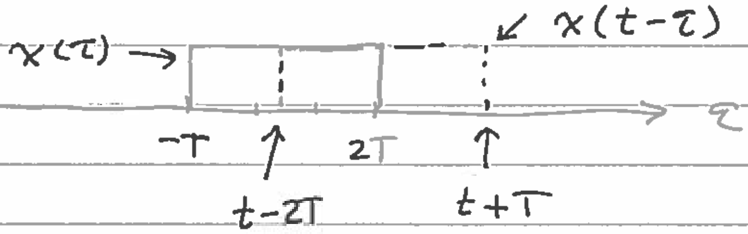
where t = Amount of time-shift

When $t+T < -T \Rightarrow t < -2T$,
the two curves don't overlap.
That is, $y(t) = 0$

when $-T \leq t+T \leq 2T \Rightarrow -2T \leq t \leq T$,

$$y(t) = \int_{\tau=-T}^{t+T} 1 \cdot d\tau = t + 2T.$$

When $-T \leq t-2T \leq 2T \Rightarrow T \leq t \leq 4T$

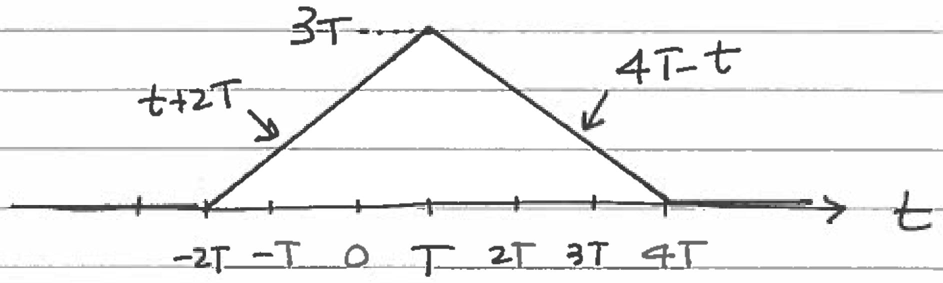


$$y(t) = \int_{\tau=t-2T}^{2T} 1 \cdot d\tau = 2T - (t-2T)$$

$$y(t) = 4T - t$$

when $t > 4T$, no overlap, $y(t) = 0$.

2 a. iii.



2 b. i.

$$R_g(\tau) = \int_{-\infty}^{\infty} g(t)\,g(t+\tau)\,dt \qquad (*)$$

ii. Let us consider $R_g(-\tau)$.

Use the above, we have

$$R_g(-\tau) = \int_{-\infty}^{\infty} g(t)\,g(t-\tau)\,dt$$

Let $u = t-\tau$. $\Rightarrow$ $t = u+\tau$

$$\Rightarrow \quad R_g(-\tau) = \int_{u=-\infty}^{\infty} g(u+\tau)\,g(u)\,du$$

$$\Rightarrow \quad R_g(-\tau) = R_g(\tau) \quad \text{from}(*)$$

iii.

$$S_g(\omega) = \int_{\tau=-\infty}^{\infty} R_g(\tau)\,e^{-j\omega\tau}\,d\tau$$

$$\Rightarrow \quad S_g(\omega) = \int_{\tau=-\infty}^{\infty}\int_{t=-\infty}^{\infty} g(t)\,g(t+\tau)\,e^{-j\omega\tau}\,dt\,d\tau$$

$$\Rightarrow \quad S_g(\omega) = \int_{t=-\infty}^{\infty} g(t)\int_{\tau=-\infty}^{\infty} g(t+\tau)\,e^{-j\omega\tau}\,d\tau\,dt$$

$$\Rightarrow \quad S_g(\omega) = \int_{t=-\infty}^{\infty} g(t)\int_{\tau'=-\infty}^{\infty} g(\tau')\,e^{-j\omega(\tau'-t)}\,d\tau'\,dt$$

2 b i. $$R_g(\tau) = \int_{-\infty}^{\infty} g(t)\, g(t+\tau)\, dt$$

ii. Consider
$$R_g(-\tau) = \int_{-\infty}^{\infty} g(t)\, g(t-\tau)\, dt$$

Let $u = t - \tau \Rightarrow t = u + \tau$

$$\Rightarrow R_g(-\tau) = \int_{u=-\infty}^{\infty} g(u+\tau)\, g(u)\, du$$

$$\Rightarrow R_g(-\tau) = R_g(\tau)$$

iii. $$S_g(\omega) = \int_{\tau=-\infty}^{\infty} R_g(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} \int_{t=-\infty}^{\infty} g(t)\, g(t+\tau)\, e^{-j\omega\tau}\, d\tau\, dt$$

$$\Rightarrow S_g(\omega) = \int_{t=-\infty}^{\infty} g(t) \int_{\tau=-\infty}^{\infty} g(t+\tau)\, e^{-j\omega\tau}\, d\tau\, dt$$

$$= \int_{t=-\infty}^{\infty} g(t) \int_{\tau'=-\infty}^{\infty} g(\tau')\, e^{-j\omega(\tau'-t)}\, d\tau'\, dt$$

$$= \int_{t=-\infty}^{\infty} g(t) \int_{\tau'=-\infty}^{\infty} g(\tau')\, e^{-j\omega\tau'} \cdot e^{j\omega t} \cdot d\tau'\, dt$$

$$S_g(\omega) = \int_{t=-\infty}^{\infty} g(t) \cdot G(\omega)\, e^{j\omega t}\, dt$$

2 b. iii. $\Rightarrow S_g(\omega) = \int_{t=\infty}^{\infty} g(t) e^{j\omega t} \int_{\tau'=-\infty}^{\infty} g(\tau') e^{-j\omega\tau'} d\tau' dt$

$$\Rightarrow S_g(\omega) = \int_{t=-\infty}^{\infty} g(t) e^{j\omega t} G(\omega) dt$$

$$\Rightarrow S_g(\omega) = G(\omega) G(-\omega)$$

Since $g(t)$ is real, $G(-\omega) = G^*(\omega)$. conjugate

So, $S_g(\omega) = G(\omega) \cdot G^*(\omega) = |G(\omega)|^2$

iv. For the LTI system,

$$Y(\omega) = H(\omega) \cdot G(\omega) \quad \text{— ①}$$

$$\Rightarrow Y^*(\omega) = H^*(\omega) G^*(\omega) \quad \text{— ②}$$ $*$: conjugate

$$\Rightarrow |Y(\omega)|^2 = |H(\omega)|^2 |G(\omega)|^2$$

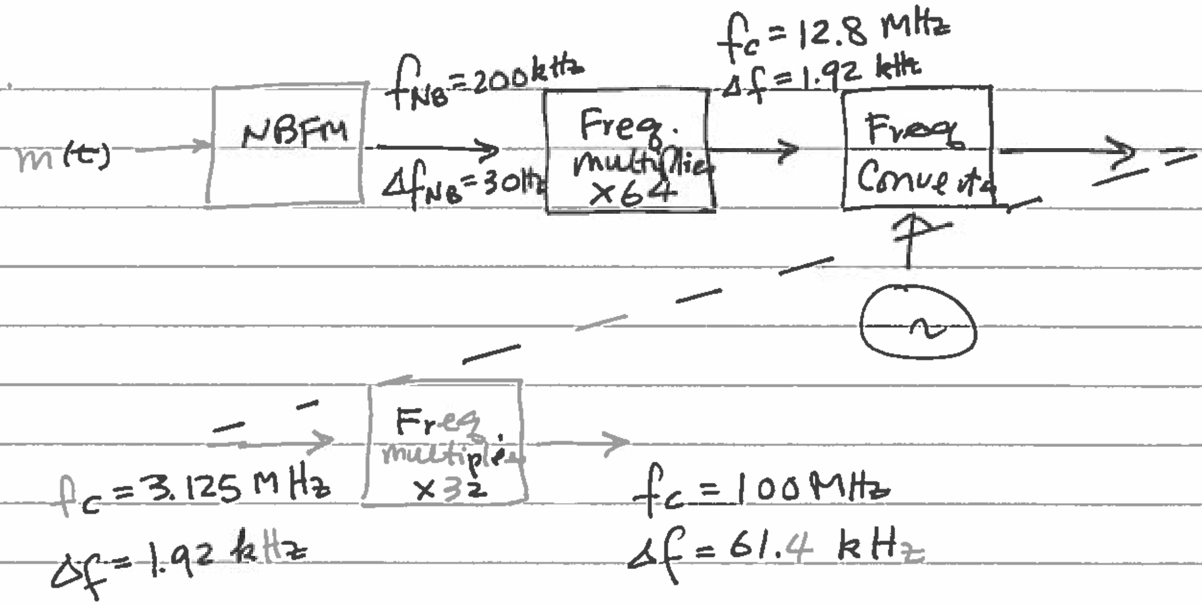
by multiplying equ. ① & ②

Since $S_g(\omega) = |G(\omega)|^2$ and by the result in part iii that $S_y(\omega) = |Y(\omega)|^2$, we

Therefore have

$$S_y(\omega) = |H(\omega)|^2 S_g(\omega).$$

3 a i.



ii. The second oscillator has a frequency of 9.675 MHz.

iii. The second oscillator is used to translate the carrier frequency from 12.8 MHz to 3.125 MHz before the last stage of frequency multiplier.

The frequency converter ~~operates~~ shifts the input carrier frequency ω_1 to $\omega_1 - \omega_2$ by using the second oscillator frequency of ω_2.

That is, $\cos(\omega_1 t) \cdot \cos(\omega_2 t)$

$$= \frac{1}{2}\left[\cos(\omega_1 - \omega_2)t + \cos(\omega_1 + \omega_2)t\right]$$

where $\omega_1 = 12.8 \times 2\pi$

$\omega_1 - \omega_2 = 3.125 \times 2\pi$

The oscillator frequency $\omega_2 = 12.8 - 3.125$

$= 9.675$ MHz.

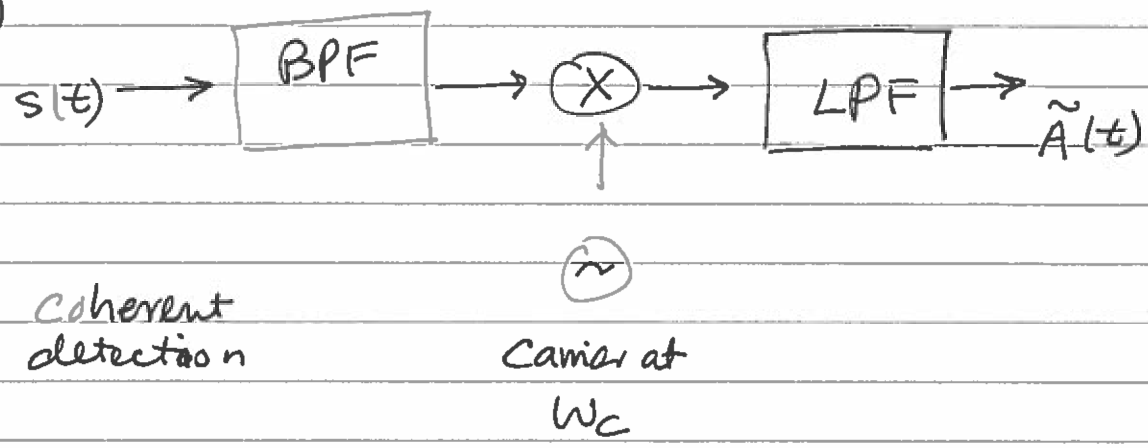
3.a. iv. The transmission system design is not unique because we can use a 32x frequency multiplier at the first stage and 64x later. In that case, the second oscillator frequency is chosen accordingly to generate the target $f_c = 100$ MHz at the end.

3.b i)

$$s(t) = A(t)\cos(\omega_c t)$$

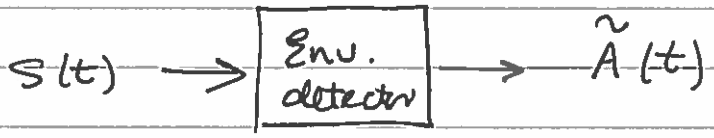
where $A(t) = \begin{cases} 0 & \text{if bit 0 is sent} \\ A & \text{otherwise} \end{cases}$

ii)



coherent detection

or



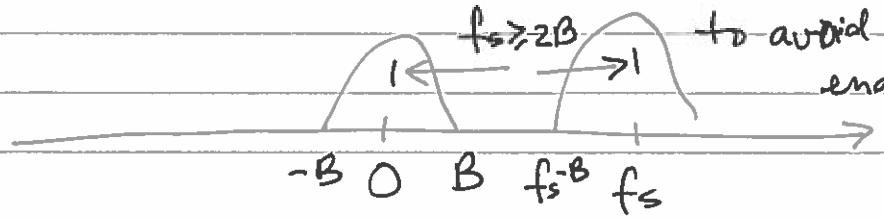
Envelop detection

iii) Given R, the maximum sampling rate supported by the link is $f_s = \frac{R}{K}$

By Nyquist sampling requirement,

$$f_s \geq 2B \quad \text{where } B \text{ is the bandwidth of } g(t)$$

$$\Rightarrow \quad B \leq \frac{R}{2K} \text{ Hz}$$



to avoid overlap & enable correct recovery.