IMPERIAL COLLEGE LONDON

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DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2013

MSc and EEE/EIE PART IV: MEng and ACGI

Corrected Copy

INFORMATION THEORY

Tuesday, 14 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer ALL questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

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Information for students

Notation:

(a) Random variables are shown in Tahoma font. X, $\mathbf{\mathsf{x}}$, $\mathbf{\mathsf{X}}$ denote a random scalar, vector and matrix respectively.

(b) The size of a set A is denoted by $|A|$.

(c) By default, the logarithm is to the base 2.

(d) $\oplus$ denotes the exclusive-or operation, or modulo-2 addition.

(e) "i.i.d." means "independent identically distributed".

(f) $C(x) = \frac{1}{2}\log_2(1+x)$ is the capacity function for the Gaussian channel in bits/channel use.

The Questions

1. Basics of information theory.

a) Given probability mass vectors **p** and **q**, both of length m,

i) Write down the entropy formula of $H(\mathbf{p})$. Given an interpretation of the entropy.

ii) What are the maximum and minimum of $H(\mathbf{p})$? When are they achieved?

iii) For $\mathbf{p} = [\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}]$ and $\mathbf{q} = [\frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{2}]$, compute the relative entropy $D(\mathbf{p}||\mathbf{q})$ and $D(\mathbf{q}||\mathbf{p})$.

iv) Give properties of a "distance" and based on iii) say if the relative entropy is a distance or not.

[12]

b) Let X be a random variable taking integer values. What can you say about the relationship between $H(X)$ and $H(Y)$ (justification is needed) if:

i) $Y = X^2$

ii) $Y = X^3$

[6]

c) If $X \to Y \to Z$ forms a Markov chain, and for Y, the alphabet size $|Y| = k$, show that $I(X; Z) \leq \log k$. What does this tell you if $k = 1$?

[7]

2. Source coding.

a) Rate-distortion of Gaussian sources. Assume $X \sim N(0,\sigma^2)$ and $E(X-\hat{X})^2 \le D$. Justify each step of the following derivations.

i) Lower bound on mutual information.

$$I(X;\hat{X}) \overset{(1)}{=} h(X) - h(X \mid \hat{X}) \overset{(2)}{=} \tfrac{1}{2}\log 2\pi e\sigma^2 - h(X-\hat{X} \mid \hat{X})$$

$$\overset{(3)}{\ge} \tfrac{1}{2}\log 2\pi e\sigma^2 - h(X-\hat{X}) \overset{(4)}{\ge} \tfrac{1}{2}\log 2\pi e\sigma^2 - \tfrac{1}{2}\log\left(2\pi e \operatorname{Var}\left(X-\hat{X}\right)\right)$$

$$\overset{(5)}{\ge} \tfrac{1}{2}\log 2\pi e\sigma^2 - \tfrac{1}{2}\log 2\pi e D$$

$$\overset{(6)}{\Rightarrow} I(X;\hat{X}) \ge \max\left(\tfrac{1}{2}\log\frac{\sigma^2}{D}, 0\right)$$

ii) Achievability. To show that we can find a distribution $p(\hat{x},x)$ that achieves the lower bound, we construct a test channel that introduces distortion $D<\sigma^2$ shown in Fig. 2.1.

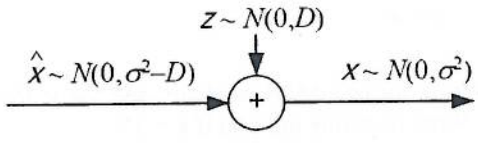


Fig. 2.1. Test channel.

$$I(X;\hat{X}) = h(X) - h(X \mid \hat{X}) = \tfrac{1}{2}\log 2\pi e\sigma^2 - h(X-\hat{X} \mid \hat{X})$$

$$\overset{(7)}{=} \tfrac{1}{2}\log 2\pi e\sigma^2 - h(Z \mid \hat{X}) \overset{(8)}{=} \tfrac{1}{2}\log\frac{\sigma^2}{D}$$

$$\overset{(9)}{\Rightarrow} I(X;\hat{X}) = \max\left(\tfrac{1}{2}\log\frac{\sigma^2}{D}, 0\right)$$

$$\overset{(10)}{\Rightarrow} D(R) = \frac{\sigma^2}{2^{2R}}$$

[10]

b) Shannon code. Given the probability mass vector $\mathbf{p} = [p_1, p_2, \dots, p_n]$ for an n-symbol source X, where $p_1 \ge p_2 \ge \cdots \ge p_n$ are sorted in decreasing order. Calculate the partial sum of the probabilities of the first $i-1$ symbols:

$$F_i = \sum_{j=1}^{i-1} p_j$$

Shannon's algorithm of source coding encodes the i-th symbol x_i into a codeword by rounding off the binary representation of $F_i \in [0,1]$ to $l_i = \lceil -\log_2(p_i) \rceil$ bits.

i) Show that the average codeword length L_S satisfies the following bound

$$H(X) \leq L_S \leq H(X) + 1$$

where $H(X)$ is the entropy of the source.

ii) Construct the code for the probability distribution $\mathbf{p} = [0.5, 0.25, 0.125, 0.125]$.

iii) Show that Shannon's code is an instantaneous code. Your answer should be general, i.e., not limited to the example in ii).

[15]

3. Channel coding.

a) Converse of the channel coding theorem. Consider the channel model shown in Fig. 3.1. Justify each step of the following proof.

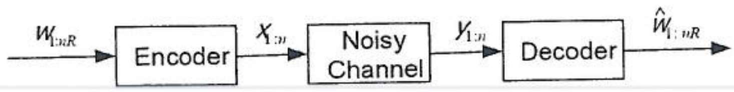


Fig. 3.1. Channel model.

$$
\begin{aligned}
nR &\overset{(1)}{=} H(\mathbf{w}) = H(\mathbf{w}\mid\mathbf{y}) + I(\mathbf{w};\mathbf{y}) \\
&\overset{(2)}{\le} H(\mathbf{w}\mid\mathbf{y}) + I(\mathbf{x}(\mathbf{w});\mathbf{y}) \\
&\overset{(3)}{\le} 1 + nRP_e^{(n)} + I(\mathbf{x};\mathbf{y}) \\
&\overset{(4)}{\le} 1 + nRP_e^{(n)} + nC \\
&\overset{(5)}{\Rightarrow} P_e^{(n)} \ge \frac{R - C - n^{-1}}{R} \underset{n\to\infty}{\to} 1 - \frac{C}{R} > 0 \quad \textit{if} \quad R > C
\end{aligned}
$$

$\overset{(6)}{\Rightarrow}$ For large (hence for all) n, $P_e^{(n)}$ has a lower bound of $(R–C)/R$ if w equiprobable.

[6]

b) Bandlimited channel. Consider a continuous channel bandlimited to bandwidth $[-W,W]$ and subject to white Gaussian noise with double-sided power spectral density $N_0/2$. Let T be the signal duration, in the sense that most energy of the signal falls into this time interval. Reproduce the Shannon capacity formula by following the steps given below:

i) Sample the signal at the Nyquist rate. Compute the energy constraint per sample E_s.

ii) Compute the noise power P_N associated with each sample.

iii) Compute the channel capacity C in bits/second.

[9]

c) Gaussian mutual information. Suppose X, Y, and Z are Gaussian random variables, each with zero mean and unit variance. Further, suppose that (X, Y, Z) are jointly Gaussian and that $X \to Y \to Z$ forms a Markov chain. Let X and Y have correlation coefficient ρ_1 and let Y and Z have correlation coefficient ρ_2.

i) Derive the correlation coefficient ρ_3 between X and Z.

ii) Find $I(X; Z)$.

Hint: Use the fact that $E[X\mid Y] = \rho_1 Y$ for jointly Gaussian random variables X, Y, each with zero mean and unit variance.

[10]

4. Network information theory.

a) Broadcast channel.

i) With the help of a diagram, explain what is a broadcast channel. Give an example of the broadcast channel.

ii) Consider the two-user scalar Gaussian broadcast channel. Write down the capacity region and describe the coding/decoding strategy.

[8]

b) Multi-access channel.

i) With the help of a diagram, explain what is a multi-access channel. Give an example.

ii) Write down the capacity region of the m-user multi-access channel with additive white Gaussian noise, where the users have equal powers P, and the noise power is N.

iii) Specialize to the two-user case. Draw its capacity region, and explain why CDMA is better than FDMA and TDMA.

[9]

c) Now a new user with power P_0 wishes to join the m-user multi-access channel.

i) At what rate R_0 can he send information without disturbing the other users?

ii) What should his power P_0 be so that the new user's rate is equal to the combined communication rate $C(mP/N)$ of all the other users?

[8]

B — Book work
A — Application
E — New example
T — New theory

ANSWERS

1\. a)

i) $H(p) = -\sum_{i=1}^{m} p(i) \log p(i)$ [3 B]

Entropy is the average Shannon information content, or the amount of uncertainty before we know the value.

ii) $\max H(p) = \log m$ iff all elements of p are equal.

$\min H(p) = 0$ iff only one element is nonzero. [2 A]

iii) $D(p\|q) = 0.35$

$D(q\|p) = 0.424$ [4 E]

iv) $D(p\|q) \neq D(q\|p)$ in general. Therefore, strictly speaking, $D(p\|q)$ is not a distance, although it can measure the similarity between p and q. [3 B]

b) $H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$ [2 A]

but $H(Y|X) = 0$ since Y is a function of X.

So $H(Y) = H(X) - H(X|Y) \leq H(X)$ with equality iff $H(X|Y) = 0$. This is true only if X is a function of Y. This means Y has to be a one-to-one function of X.

i) $H(Y) \leq H(X)$ since $Y = X^2$ is not one-to-one. [2 A]

ii) $H(Y) = H(X)$ since $Y = X^3$ is one-to-one. [2 A]

c) By data processing theorem, $I(X;Z) \leq I(X;Y)$ [2 A]

$= H(Y) - H(Y|X) \leq H(Y) \leq \log k$ (uniform bound). [2 A]

If $k=1$, $\log k = 0$. So $I(X;Z) = 0$, which means X and Z are independent. [3 E]

2. a) i)

(1) definition of $I(X;\hat{X})$. [1 each, B]

(2) Gaussian entropy; Shift doesn't change entropy.

(3) Conditioning reduces entropy.

(4) Given variance, Gaussian has max entropy.

(5) $Var(X-\hat{X}) \le D$.

(6) Algebra and $I(X;\hat{X}) \ge 0$

ii) (7) $\hat{X} + Z = X$

(8) $h(Z|\hat{X}) = h(Z) = \frac{1}{2}\log 2\pi e D$

(9) The lower bound is achievable $\Rightarrow$ $\geq$ becomes $=$.

(10) definition of rate-distortion.

b) i) Since $l_i = \lceil -\log p_i \rceil$, we have [5T]

$$-\log p_i \le l_i \le -\log p_i + 1. \quad (*)$$

Average length

$$-\sum p_i \log p_i \le L_S = \sum p_i l_i \le -\sum p_i \log p_i + 1$$

$$H(S) \le L_S \le H(S) + 1.$$

ii) [5T]

Symbol	p_i	F_i	binary	l_i	codeword
1	0.5	0	0.0	1	0
2	0.25	0.5	0.10	2	10
3	0.125	0.75	0.110	3	110
4	0.125	0.875	0.111	3	111

iii) [difficult]. From (*), we have [5T]

$$2^{-l_i} \le p_i < 2^{-(l_i - 1)}.$$

Thus, F_j $(j > i)$ must be larger than F_i by more than 2^{-l_i}, therefore F_j $(j>i)$ must differ from F_i by one bit in the first l_i bits, in binary expansion. Thus their codewords also differ by at least one bit. This means no codewords are a prefix of another.

3.

a) [1 each B]

(1) input is uniform.

(2) $W \to X \to Y$ form a Markov chain.

(3) Fano's inequality.

(4) $I(X;Y) \le nC$

(5) algebra

(6) (6) (5) is true for large n. It is also true for small n, since we can concatenate several short blocks to form a long block.

b) i) By sampling at Nyquist rate, we obtain 2WT samples.

~~Let~~

$$E_s \cdot 2WT \le PT$$ [3 B]

$$\Rightarrow E_s \le \frac{P}{2W}$$

ii) Noise power $P_N = \frac{N_0}{2} \cdot 2W \cdot \frac{T}{2WT} = \frac{N_0}{2}$ [3 B]

iii)

$$C = \frac{1}{2}\log\left(1 + \frac{P/2W}{N_0/2}\right)\frac{2WT}{T}$$ [3 B]

$$= W\log\left(1 + \frac{P}{WN_0}\right)$$

c) i) $$\rho_3 = \frac{E[XZ]}{\sigma_x \sigma_z} = \frac{E\{E[XZ|Y]\}}{\sigma_x \sigma_z}$$ [5 T]

Obvious, but $\sigma_x, \sigma_y, \rho_z$ not defined.

$$= \frac{E\{E[X|Y]\,E[Z|Y]\}}{\sigma_x \sigma_z}$$ Markov chain

$$= \frac{E\{\rho_1 Y \cdot \rho_2 Y\}}{\sigma_x \sigma_z}$$

$$= \frac{\rho_1 \rho_2 \sigma_y^2}{\sigma_x \sigma_z}$$ $\sigma_x = \sigma_y = \sigma_z = 1$

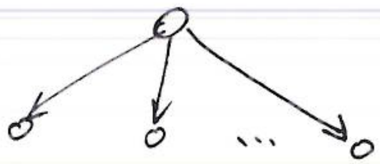
$$= \rho_1 \rho_2$$

ii) The covariance matrix

$$Cov(X, Z) = \begin{pmatrix} 1 & \rho_3 \\ \rho_3 & 1 \end{pmatrix}$$ [5 T]

$$\begin{aligned} I(X; Z) &= h(X) + h(Z) - h(X, Z) \\ &= \tfrac{1}{2}\log(2\pi e) + \tfrac{1}{2}\log(2\pi e) - \tfrac{1}{2}\log 2\pi e(1-\rho_3^2) \\ &= \tfrac{1}{2}\log\frac{2\pi e}{1-\rho_3^2}. \end{aligned}$$

4. a) i) Broadcast channel: [4 B]



One sender, many receivers. The sender can send different messages to different users.

Example: Mobile downlink.

ii) Capacity region: [4 B]

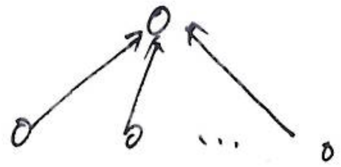
$$R_1 \le C\left(\frac{\alpha P}{N_1}\right) \qquad 0 \le \alpha \le 1$$

$$R_2 \le C\left(\frac{(1-\alpha)P}{\alpha P + N_2}\right)$$

Encoding: One codebook with power αP at rate R_1, another with power $(1-\alpha)P$ at rate R_2, send sum of two codewords.

Decoding: Bad receiver Y_2 treats Y_1 as noise; Good receiver Y_1 first decodes Y_2, subtracts it out, then decodes own message.

b) i) Multi-access channel



[3B]

One reciever, many senders. Example: CDMA uplink.

ii) Capacity region

$$R_i < c\left(\frac{P}{N}\right)$$

[3B]

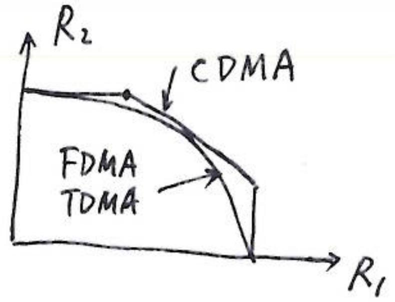
$$R_i + R_j < c\left(\frac{2P}{N}\right)$$

$$R_i + R_j + R_k < c\left(\frac{3P}{N}\right)$$

$$\vdots$$

$$\sum_{i=1}^{m} R_i \le c\left(\frac{mP}{N}\right)$$

iii) Two-user:



[3B]

c) i) If the new user can be decoded while treating other users as noise, then it is fine. Hence

$$R_0 < \frac{1}{2}\log\left(1+\frac{P_0}{mP+N}\right)$$

[4T]

ii)

$$\frac{1}{2}\log\left(1+\frac{P_0}{mP+N}\right) = \frac{1}{2}\log\left(1+\frac{mP}{N}\right)$$

[4T]

$$\frac{P_0}{mP+N} = \frac{mP}{N}$$

$$P_0 = (mP+N)\frac{mP}{N}$$