

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

MSc and EEE/ISE PART IV: MEng and ACGI

IDENTIFICATION AND ADAPTIVE CONTROL

Corrected Copy

Wednesday, 29 April 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

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Information for candidates:

The probability density $N(m, Q)$ of an n -vector, normal random variable with mean m and covariance matrix Q ($Q > 0$) is

$$N(m, Q)(x) = \frac{1}{(\sqrt{2\pi})^{\frac{n}{2}} (\det Q)^{\frac{1}{2}}} \exp -\frac{1}{2} \left((x - m)^T Q^{-1} (x - m) \right) .$$

In the case that $n = 1$, m is a scalar and $Q = \sigma^2$ ($\sigma^2 > 0$),

$$N(m, \sigma^2)(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left(-\frac{(x - m)^2}{2\sigma^2} \right)$$

and, if X is a scalar random variable with probability density $N(m, \sigma^2)$,

$$\text{Prob}\{m - 2\sigma \leq X \leq m + 2\sigma\} \approx 0.95 .$$

1. Two coupled, stationary, chemical processes y_t and z_t are governed by the equations

$$\begin{aligned}y_t &= \alpha y_{t-1} + e_t \\z_t &= \alpha z_{t-1} + \gamma y_{t-1},\end{aligned}$$

in which e_t is a white noise process with variance $\sigma^2 > 0$. The constant α ($|\alpha| < 1$) is the reaction rate parameter for both processes. The constant γ is the coupling coefficient.

Let $x_t = (y_t, z_t)^T$. Develop a state space model for x_t , of the form

$$x_t = Fx_{t-1} + ge_t. \quad [2]$$

Show that $R = E[x_t x_t^T]$ satisfies the equation

$$R = FRF^T + bb^T \sigma^2$$

and derive formulae for the entries of R in terms of σ^2 , α and γ . [4]

Now assume that the value of the reaction rate parameter is [6]

$$\alpha = 0.5.$$

Assume also that, by means of an identification experiment, it has been possible to establish the following relation between the variances $r_z = E[z_t^2]$ and $r_y = E[y_t^2]$:

$$\frac{r_z}{r_y} = 1.25.$$

Determine the value of the coupling coefficient γ . [8]

2. An N -dimensional measurement vector is assumed to be modelled by the equation

$$y = x\theta + e$$

in which x is a known, non-zero, deterministic N -vector and e is a normal N -vector random variable zero mean and covariance matrix Q ($Q > 0$). θ is an unknown scalar parameter.

Consider the linear estimate $\hat{\theta}$ of θ given y :

$$\hat{\theta} = (x^T Q^{-1} x)^{-1} x^T Q^{-1} y .$$

Show that the estimate $\hat{\theta}$ is unbiased. [3]

Determine the variance of the estimate $\hat{\theta}$. [3]

Show that the estimate $\hat{\theta}$ minimizes the mean square error

$$E \left[|\hat{\theta} - \theta|^2 \right]$$

over all unbiased linear estimates $\hat{\theta}$ of θ given y . [10]

Hint: Use the fact that an arbitrary linear, unbiased estimate $\hat{\theta}$ can be expressed as

$$\hat{\theta} = (x^T Q^{-1} x)^{-1} x^T Q^{-1} y + b^T y ,$$

where b is an N -vector satisfying the condition:

$$b^T x = 0 .$$

Now assume that, for some integer $N \geq 2$, x is the N -vector $[1, \dots, 1]^T$ and

$$Q^{-1} = \begin{bmatrix} 1 & 0.5 & 0 & 0 & . & 0 \\ 0.5 & 1 & 0.5 & 0 & . & 0 \\ . & . & . & . & . & . \\ 0 & . & 0 & 0.5 & 1 & 0.5 \\ 0 & . & 0 & 0 & 0.5 & 1 \end{bmatrix} .$$

Determine a 0.95 confidence interval for θ , given the estimate $\hat{\theta}$. [4]

3. Let y_t be a stationary, ergodic, scalar process satisfying

$$y_t + ay_{t-1} = e_t ,$$

in which e_t is a Gaussian, white noise process, with variance $\sigma^2 > 0$. The number a is an unknown, scalar parameter satisfying $|a| < 1$. Write

$$\hat{R}^N = \frac{1}{N} \sum_{t=0}^{N-1} y_t^2 \quad (\text{the sample covariance function of } y_t) .$$

What is the linear least squares estimate \hat{a}_N of the parameter a , based on observations $y_t, t = 0, 1, \dots, N$? [2]

Show that, as $N \rightarrow \infty$,

$$\hat{a}_N \rightarrow a . \quad [8]$$

Show further that

$$\hat{R}^N(\hat{a}_N - a) = - \sum_{i=1}^N y_{i-1} e_i . \quad [5]$$

Hence, or otherwise, show that

$$\text{var}\{\hat{R}^N(\hat{a}_N - a)\} \leq \frac{1}{N} \times \frac{\sigma^2}{(1 - a^2)} ,$$

for all $N > 1$. [5]

Hint: To evaluate $\text{var}\{\hat{R}^N(\hat{a}_N - a)\}$, use the fact that

$$E[y_{t-1} e_t y_{t'-1} e_{t'}] = 0 \quad \text{for } t' \neq t .$$

- 4 A control system relating the scalar control signal u_t and the scalar output signal y_t is modelled by the equations

$$y_t - ay_{t-1} = bu_t + e_t$$

in which e_t is a white noise sequence with variance σ^2 . Here, a and b are scalar parameters. The true value of the parameter a is $a = 0$. b is non-zero.

In an identification experiment, the input signal is chosen to be samples of a process modelled as

$$u_t = v_t + 0.5v_{t-1}$$

in which v_t is a white noise sequence with unit variance, uncorrelated with e_t . It can be assumed that the joint process (y_t, u_t) is stationary and ergodic.

Let (\hat{a}, \hat{b}) be linear least squares estimate of (a, b) , given $\{y_0, \dots, y_N\}$ and $\{u_1, \dots, u_N\}$, based on the assumption that both a and b are unknown parameters.

Let $\hat{\hat{b}}$ be the linear least squares estimate of b , based on the assumption that $a = 0$.

- (a): Calculate $R_u(0) = E[u_t^2]$, $R_y(0) = E[y_t^2]$ and $R_{uy}(1) = E[u_t y_{t-1}]$. (In performing this calculation you should assume that $a = 0$ and b is an arbitrary non-zero number). [5]

- (b): Obtain formulae for \hat{b} and $\hat{\hat{b}}$, expressed in terms of sample covariances and cross-covariances of y_y and u_t , and the constant b . [7]

- (c): Show that the conditional covariances $\hat{\gamma}$ and $\hat{\hat{\gamma}}$ of \hat{b} and $\hat{\hat{b}}$ respectively, given $\{y_0, \dots, y_N\}$ and $\{u_1, \dots, u_N\}$, are, for N large, approximately

$$\hat{\gamma} = \frac{1}{N}(b^2 + \frac{4}{5})\sigma^2 \quad \text{and} \quad \hat{\hat{\gamma}} = \frac{1}{N} \times \frac{4}{5}\sigma^2.$$

Comment on the relative magnitudes of the $\hat{\gamma}$ and $\hat{\hat{\gamma}}$. [7]

In (c) you should assume that sample covariances/cross-covariances can be replaced by covariances/cross-covariances. [1]

5. (a): Measurements y_1 and y_2 are taken at times $t = 1$ and $t = 2$ of a process governed by the ARMA model equations

$$y_t = e_t + c e_{t-1} ,$$

in which $e_0 = 0$, and e_1 and e_2 are independent, zero mean, normal random variables, each with variance σ^2 . c and σ^2 are unknown parameters.

Show that the 2-vector random variables $y = (y_1, y_2)^T$ and $e = (e_1, e_2)^T$ are related by

$$y = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} e .$$

Calculate the log likelihood function $L(c, \sigma^2; y)$ of c and σ^2 :

$$L(c, \sigma^2; y) = \log_e p(y|c, \sigma^2)$$

in which $p(y|c, \sigma^2)$ denotes the probability density of y , given c and σ^2 . [3]

Calculate the maximum likelihood estimates \hat{c} and $\hat{\sigma}^2$ of c and σ^2 ; that is, the values of c and σ^2 maximizing $L(c, \sigma^2)$. [7]

Show that $\hat{\sigma}^2$ is an biased estimate of σ^2 . [2]

Hint: Maximize the likelihood function first over c (for fixed σ^2) and then over σ^2 .

(b): Describe the Generalized Least Squares Algorithm for estimating the parameters a_1, \dots, a_n and d_1, \dots, d_n in the model

$$\begin{cases} A(z)y_t = B(z)u_t + n_t \\ D(z)n_t = e_t \end{cases}$$

given measurements $y_1, \dots, y_N, u_1, \dots, u_N$ (and appropriate starting values). Here

$$A(z) = 1 + a_1 z^{-1} + \dots + a_n z^{-n}, \quad B(z) = b_0 + b_1 z^{-1} + \dots + b_n z^{-n}, \quad D(z) = 1 + d_1 z^{-1} + \dots + d_n z^{-n},$$

and $\{e_t\}$ is a white noise sequence with random variables, each with variance σ^2 . [8]

Model Answer. Identification + Adaptive Control, 2009

1. Set $x_t = \begin{pmatrix} y_t \\ z_t \end{pmatrix}$. Then $x_t^{(1)} = \alpha x_{t-1}^{(1)} + e_t$, $x_t^{(2)} = \gamma x_{t-1}^{(2)} + \alpha x_{t-1}^{(1)}$.

It follows

$$x_t = \begin{bmatrix} \alpha & 0 \\ \gamma & \alpha \end{bmatrix} x_{t-1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e_t \quad (E[e_t] = 0, \text{var}\{e_t\} = \sigma^2)$$

Write $x_t = F x_{t-1} + b e_t$, with

$$F = \begin{bmatrix} \alpha & 0 \\ \gamma & \alpha \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Write $R = E[x_t x_t^T]$. We have

$$E[x_t x_t^T] = E[F x_{t-1} x_{t-1}^T F^T] + b b^T \sigma^2$$

or

$$R = F R F^T + b b^T \sigma^2$$

Expand as

$$\begin{aligned} \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} &= \begin{bmatrix} \alpha & 0 \\ \gamma & \alpha \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} \alpha & \gamma \\ 0 & \alpha \end{bmatrix} + \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \alpha & 0 \\ \gamma & \alpha \end{bmatrix} \begin{bmatrix} \alpha r_{11} & \gamma r_{11} + \alpha r_{12} \\ \alpha r_{12} & \gamma r_{12} + \alpha r_{22} \end{bmatrix} + \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \alpha^2 r_{11} + \sigma^2 & \alpha \gamma r_{11} + \alpha^2 r_{12} \\ \alpha \gamma r_{11} + \alpha^2 r_{12} & \gamma^2 r_{11} + \alpha \gamma r_{12} + \alpha^2 r_{22} \end{bmatrix} \end{aligned}$$

Equating matrix entries gives

$$r_{11} = \alpha^2 r_{11} + \sigma^2, \quad r_{12} = \alpha \gamma r_{11} + \alpha^2 r_{12}$$

$$\text{and } r_{22} = \gamma^2 r_{11} + 2\alpha \gamma r_{12} + \alpha^2 r_{22}$$

Hence

$$r_{11} = \frac{\sigma^2}{1-\alpha^2}, \quad r_{12} = \frac{\alpha \gamma}{(1-\alpha^2)} \times \frac{\sigma^2}{1-\alpha^2} = \frac{\alpha \gamma \sigma^2}{(1-\alpha^2)^2}$$

$$r_{22} = \frac{1}{(1-\alpha^2)} \left(\frac{\gamma^2 \sigma^2}{1-\alpha^2} + 2 \frac{\alpha^2 \gamma^2}{1-\alpha^2} \times \frac{\sigma^2}{1-\alpha^2} \right)$$

$$= \frac{1}{(1-\alpha^2)^3} \left(\gamma^2 \sigma^2 (1-\alpha^2) + 2 \alpha^2 \gamma^2 \sigma^2 \right) = \frac{\sigma^2 (1+\alpha^2) \gamma^2}{(1-\alpha^2)^3}$$

If $\alpha = 1/2$

$$r_{11} = \frac{4}{3} \sigma^2, \quad r_{22} = \left(\frac{4}{3} \right)^3 \times 1.25 \gamma^2 \sigma^2$$

and

$$r_{22}/r_{11} = \frac{4^2}{3^2} \times 1.25 \gamma^2 = 1.25$$

So $\gamma = 3/4$

2. Take $\hat{\theta} = (X^T Q^{-1} X)^{-1} X^T Q^{-1} y$, $y = X\theta + e$, $e \sim N(0, Q)$
 Then $E[\hat{\theta}] = E[(X^T Q^{-1} X)^{-1} X^T Q^{-1} (X\theta + e)] = \theta + E[\underbrace{(X^T Q^{-1} X)^{-1} X^T Q^{-1} e}_{=0}]$
 We have shown $\hat{\theta}$ is unbiased.

$$\hat{\theta} - \theta = (X^T Q^{-1} X)^{-1} X^T Q^{-1} e \quad \text{so}$$

$$E|\hat{\theta} - \theta|^2 = E[(X^T Q^{-1} X)^{-2} X^T Q^{-1} e e^T Q^{-1} X] = (X^T Q^{-1} X)^{-2} \underbrace{[X^T Q^{-1} X]}_{=I} = (X^T Q^{-1} X)^{-1}$$

An arbitrary linear, unbiased estimator can be written

$$\hat{\theta} = (X^T Q^{-1} X)^{-1} X^T Q^{-1} y + b^T y$$

for some N -vector b . Because $\hat{\theta}$ is unbiased

$$E[\hat{\theta}] = \theta + (X^T Q^{-1} X)^{-1} X^T Q^{-1} E[e] + b^T X \theta + b^T E[e]$$

Since $E[\hat{\theta}] = \theta$, for all θ , we must have

$$b^T X = 0 \quad \text{--- (1)}$$

Now examine mean square error of estimate θ (by (1))

$$\begin{aligned} E[|\hat{\theta} - \theta|^2] &= E[|(X^T Q^{-1} X)^{-1} X^T Q^{-1} e + b^T X \theta + b^T e|^2] \\ &= E\left[\left[(X^T Q^{-1} X)^{-1} X^T Q^{-1} + b^T\right] e e^T \left[Q^{-1} X (X^T Q^{-1} X)^{-1} + b\right]\right] \\ &= \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix} Q \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix} \\ &= (X^T Q^{-1} X)^{-1} + 0 + 0 + b^T Q b \end{aligned}$$

(we have used (1))

Since $b^T Q b \geq 0$ we have shown

$$E[|\hat{\theta} - \theta|^2] \geq (X^T Q^{-1} X)^{-1} = E[|\hat{\theta} - \theta|^2]$$

This establishes that $\hat{\theta}$ is BLUE ("best linear unbiased estimator.")

We know that $\hat{\theta}$ has mean θ and variance $(X^T Q^{-1} X)^{-1}$. Furthermore, it is an affine function of e , so it is normal: $\hat{\theta} \sim N(\theta, (X^T Q^{-1} X)^{-1})$

It is now supposed that

$$X = (\underbrace{1, 1, \dots, 1}_{N \text{ terms}}, 1)^T \quad \text{and} \quad Q^{-1} = \begin{bmatrix} 0.5 & 0 & & \\ & 0.5 & & \\ 0.5 & & 1 & \\ & & & 0.5 \end{bmatrix}$$

$$\begin{aligned} \text{We see } X^T Q^{-1} X &= [1, \dots, 1] \begin{bmatrix} 1.5 \\ 2 \\ 2 \\ 1.5 \end{bmatrix} = 2 \times 1.5 + (N-2) \times 2 \\ &= N-1 \quad (N > 2) \end{aligned}$$

Hence $\hat{\theta} \sim N(\theta, \frac{1}{N-1})$. So $|\hat{\theta} - \theta| \leq 2/\sqrt{N-1}$ w.p. 0.95

We conclude

$$\hat{\theta} - \frac{2}{\sqrt{N-1}} \leq \theta \leq \hat{\theta} + \frac{2}{\sqrt{N-1}} \quad \text{w.p. } 0.95$$

3. $y_t = -a y_{t-1} + e_t \quad (1)$

implies $\begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} -y_0 \\ \vdots \\ -y_{N-1} \end{pmatrix} a + \begin{pmatrix} e_1 \\ \vdots \\ e_N \end{pmatrix}$; write as $\underline{y} = \underline{X} \underline{a} + \underline{e}$

The linear l.s. estimate $\hat{a} = (X^T X)^{-1} X^T y = - \frac{\sum_{t=1}^N y_t y_{t-1}}{\sum_{t=1}^N y_{t-1}^2}$

But $N^{-1} \sum_{t=1}^N y_t y_{t-1} \rightarrow R_y(1)$, $N^{-1} \sum_{t=1}^N y_{t-1}^2 \rightarrow R_y(0)$ as $N \rightarrow \infty$.

It follows that $\hat{a} \rightarrow -R_y(1)/R_y(0) \neq 0$

But, from (1), $E\{y_t y_{t-1}\} = -a E\{y_{t-1}^2\} + E\{e_t y_{t-1}\}$,

whence $R_y(1) = -a R_y(0)$

We deduce that

$\hat{a} \rightarrow a$ as $N \rightarrow \infty$.

We have

$\left(\frac{1}{N} \sum_{t=1}^N y_{t-1}^2 \right) \hat{a} = - \frac{1}{N} \sum_{t=1}^N y_t y_{t-1}$

Multiply across 1 by y_{t-1} and summing gives

$\frac{1}{N} \sum_{t=1}^N y_t y_{t-1} = -a \frac{1}{N} \sum_{t=1}^N y_{t-1}^2 + \frac{1}{N} \sum_{t=0}^N y_{t-1} e_t$

It follows

$\left(\frac{1}{N} \sum_{t=0}^{N-1} y_t^2 \right) (\hat{a} - a) = - \frac{1}{N} \sum_{t=1}^N y_t y_{t-1} + \frac{1}{N} \sum_{t=1}^N y_t y_{t-1} - \frac{1}{N} \sum_{t=0}^N y_{t-1} e_t$

Hence

$\hat{R}_y(0) \times (\hat{a} - a) = - \frac{1}{N} \sum_{t=0}^N y_{t-1} e_t$

We see

$E[\hat{R}_y(0) \times (\hat{a} - a)] = -E\left[\frac{1}{N} \dots\right] = 0$

Hence

$\text{var}(\hat{R}_y(0) \times (\hat{a} - a)) = \frac{1}{N^2} E\left[\left(\sum_{t=1}^N y_{t-1} e_t\right) \times \left(\sum_{t=1}^N y_{t-1} e_t\right)\right]$
 $= \frac{1}{N^2} E\left[\sum_{t=1}^N y_{t-1}^2\right] \sigma^2 + 0 + \dots + 0$
 $= \frac{\sigma^2}{N} R_y(0)$

But, from (1) $R_y(0) = E[y_t^2] = E[(-a y_{t-1} + e_t)^2] = a^2 R_y(0) + \sigma^2$

Hence $R_y(0) = \sigma^2 / (1 - a^2)$

But then

$\text{var}(\hat{R}_y(0) \times (\hat{a} - a)) = \frac{1}{N} \times \frac{\sigma^4}{(1 - a^2)}$

4. (i) $R_u(0) = E[(v_t + \frac{1}{2}v_{t-1})(v_t + \frac{1}{2}v_{t-1})] = E[v_t^2] + 0 + 0 + \frac{1}{4}E[v_t^2] = \frac{5}{4}$
 $R_y(0) = E[(b(v_t + \frac{1}{2}v_{t-1}) + e_t)^2] = b^2(1 + \frac{1}{4}) + \sigma^2 = \frac{5}{4}b^2 + \sigma^2$
 $R_{uy}(L) = E[(v_t + \frac{1}{2}v_{t-1})(b(v_t + \frac{1}{2}v_{t-1}) + e_t)] = \frac{5}{4}b$

(ii) We write $\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} y_0 & u_1 \\ \vdots & \vdots \\ y_{N-1} & u_N \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix} \equiv y = X\theta + e$
 Then $\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = (X^T X)^{-1} X^T y = \begin{bmatrix} \frac{1}{N} \sum_{i=0}^{N-1} y_i^2 & \frac{1}{N} \sum_{i=1}^N u_i y_{i-1} \\ \frac{1}{N} \sum_{i=1}^N u_i y_{i-1} & \frac{1}{N} \sum_{i=1}^N u_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{N} \sum_{i=1}^N y_i y_{i-1} \\ \frac{1}{N} \sum_{i=1}^N u_i y_i \end{bmatrix}$
 $= \begin{bmatrix} \hat{R}_y(0) & \hat{R}_{uy}(L) \\ \hat{R}_{uy}(L) & \hat{R}_u(0) \end{bmatrix}^{-1} \begin{bmatrix} R_y(1) \\ R_{uy}(L) \end{bmatrix}$. So $\hat{b} = [0 \ 1] \begin{bmatrix} \hat{R}_y(0) & \hat{R}_{uy}(L) \\ \hat{R}_{uy}(L) & \hat{R}_u(0) \end{bmatrix}^{-1} \begin{bmatrix} R_y(1) \\ R_{uy}(L) \end{bmatrix}$

If however we assume $a=0$, $\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} b + \begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix} \equiv y = \bar{X}\bar{\theta} + e$

In this case, $\hat{b} = (\bar{X}^T \bar{X})^{-1} \bar{X}^T y = \left(\frac{1}{N} \sum_{i=1}^N u_i^2 \right)^{-1} \left(\frac{1}{N} \sum_{i=1}^N u_i y_i \right)$
 giving $\hat{b} = \hat{R}_u(0)^{-1} R_{uy}(0)$.

(iii) The conditional variance of \hat{b} (given $\{y_0, \dots, y_N\}$ and $\{u_1, \dots, u_N\}$):
 $\hat{\delta} = \sigma^2 [0 \ 1] (X^T X)^{-1} [0 \ 1]^T = \frac{\sigma^2}{N} [0 \ 1] \begin{bmatrix} \frac{1}{N} \sum_{i=0}^{N-1} y_i^2 & \frac{1}{N} \sum_{i=1}^N u_i y_{i-1} \\ \frac{1}{N} \sum_{i=1}^N u_i y_{i-1} & \frac{1}{N} \sum_{i=1}^N u_i^2 \end{bmatrix}^{-1} [0 \ 1]^T$
 $= \frac{\sigma^2}{N} \times \frac{1}{\left(\hat{R}_y(0) \hat{R}_u(0) - \frac{1}{N} R_{uy}(0)^2 \right)} \times \hat{R}_y(0)$

Also, the conditional variance of \hat{b} is $\hat{\delta} = \sigma^2 (\bar{X}^T \bar{X})^{-1} = \frac{\sigma^2}{N} \left(\frac{1}{N} \sum_{i=1}^N u_i^2 \right)^{-1} = \frac{\sigma^2}{N} \hat{R}_u(0)^{-1}$

Using the results from (i), and assuming $\hat{R}_y(0) = R_y(0)$, etc, we have
 $\hat{\delta} = \frac{\sigma^2}{N} \frac{(5/4 b^2 + \sigma^2)}{5/4 (5/4 b^2 + \sigma^2) - (\frac{5}{4})^2 b^2} = \frac{1}{N} (b^2 + 4/5 \sigma^2)$

$\hat{\delta} = \frac{1}{N} \times 4/5 \sigma^2$

We observe that $\hat{\delta} > \delta$, reflecting the fact that the model (with a unknown) is over-parameterized and therefore gives rise to estimates of non-zero parameters with increased variance.

5.(a) $y_1 = e_1 + 0$ and $y_2 = e_2 + ce_1$, so $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$

The prob. density of y is:

$$p(y|c, \sigma^2) = \frac{1}{2\pi \sigma^2 \det Q} \exp \left\{ -\frac{1}{2\sigma^2} y^T Q^{-1} y \right\}$$

in which $Q = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & c \\ c & 1+c^2 \end{bmatrix}$. Note, $\det Q = (1+c^2) = 1$.

So $Q^{-1} = \begin{bmatrix} 1+c^2 & -c \\ -c & 1 \end{bmatrix}$.

$$\begin{aligned} L(c, \sigma^2) &= \log_e p(y|c, \sigma^2) = -\log(2\pi) - \log(\sigma^2) - \frac{1}{2\sigma^2} (y_1, y_2) \begin{bmatrix} 1+c^2 & -c \\ -c & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= -\log(2\pi) - \log(\sigma^2) - \frac{1}{2\sigma^2} \left((1+c^2)y_1^2 - 2cy_1y_2 + y_2^2 \right) \end{aligned}$$

For fixed σ^2 , the maximizing $c = \hat{c}$ satisfies

$$\frac{\partial}{\partial c} [\hat{c}^2 y_1^2 - 2\hat{c} y_1 y_2] = 0 \Rightarrow \hat{c} = y_1 y_2 / y_1^2$$

Then

$$\begin{aligned} L(\hat{c}, \sigma^2) &= -\log(2\pi) - \log(\sigma^2) - \frac{1}{2\sigma^2} \left(y_1^2 + y_2^2 - \frac{(y_1 y_2)^2}{y_1^2} \right) \\ &= \text{const.} - \log(\sigma^2) - \frac{1}{2\sigma^2} y_1^2 \end{aligned}$$

The maximizing $\sigma^2 = \hat{\sigma}^2$ satisfies

$$\frac{\partial}{\partial \sigma^2} L(\hat{c}, \hat{\sigma}^2) = -\frac{1}{\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4} y_1^2 = 0$$

Hence $\hat{\sigma}^2 = \frac{1}{2} y_1^2$.

We have shown that the maximum likelihood estimates are

$$\hat{c} = y_1 / y_2 \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{2} y_1^2$$

We see $E[\hat{\sigma}^2] = \frac{1}{2} E[y_1^2] = \frac{1}{2} E[e_1^2] = \frac{1}{2} \sigma^2$, bias = $\frac{1}{2} \sigma^2$

(b) Generalized Least Squares Algorithm:

Choose $D(z)$ ($=1$, say). Compute $y'_t = D(z)y_t$, $u'_t = D(z)u_t$.
Obtain LS estimate $A'(z), B'(z)$ of $A(z), B(z)$ for model

$$A(z) y'_t = B(z) u'_t + \text{"noise"}$$

given y'_t, u'_t .

Calculate residuals $n'_t = A'(z)y'_t - B'(z)u'_t$.

Obtain LS estimate $\hat{D}(z)$ of $D(z)$ for model

$$\hat{D}(z) n'_t = 0 + \text{"noise"}$$

Repeat to obtain $(A^2(z), B^2(z)), (\hat{D}^3, A^3, B^3), \dots$