

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

MSc and EEE/ISE PART IV: MEng and ACGI

Corrected Copy

STABILITY AND CONTROL OF NON-LINEAR SYSTEMS

Thursday, 14 May 10:00 am

Time allowed: 3:00 hours

There are **SIX** questions on this paper.

Answer **FOUR** questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible

First Marker(s) : D. Angeli

Second Marker(s) : E.C. Kerrigan

*One corrector
made*

STABILITY AND CONTROL OF NONLINEAR SYSTEMS

1. Consider the second order nonlinear differential equation:

$$\ddot{y}(t) = \operatorname{atan}(y(t)) - \frac{\dot{y}(t)}{1+y^2(t)} - \frac{\pi}{4}y(t),$$

defined for all $y \in \mathbb{R}$.

- a) Choose a suitable state variable and write the corresponding state-space model. [4]
- b) Compute all equilibria of the system. [4]
- c) Linearize the system around each of the equilibria determined in part b) and classify the corresponding local phase-plane portrait (SADDLE, NODE, FOCUS, CENTER, STABLE, UNSTABLE). [6]
- d) Exploiting the local information obtained in part c), sketch a consistent global phase portrait for the system. [6]

2. Consider the three dimensional nonlinear system:

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1^3 - \frac{x_2}{x_1^2+1} - x_3 - x_2 - x_1, \\ \dot{x}_3 &= x_1^3 + \frac{x_2}{1+x_1^2} + x_3 + x_1.\end{aligned}$$

- a) Show that $y = x_1 + x_2 + x_3$ is constant along solutions. [4]
- b) Write the equations of the bidimensional system obtained for $x_1 + x_2 + x_3 = 0$. (Hint: use the coordinates x_1 and x_2) [4]
- c) Compute the unique equilibrium of the system determined in part b) and show, using a candidate Lyapunov function $V(x_1, x_2) = \alpha x_1^a + \beta x_2^b$, that this equilibrium is Globally Asymptotically Stable (choose the real parameters α, β and the integers a, b in a suitable way). [6]
- d) Can local stability properties of the system determined in part b) be assessed by Lyapunov's linearization method? Explain your answer. [6]

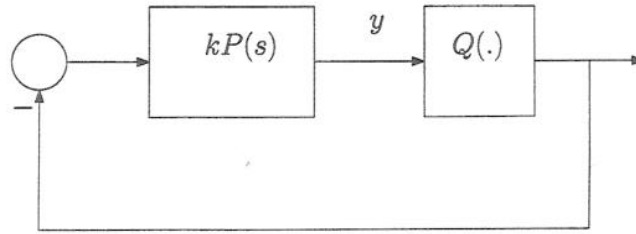


Figure 3.1 Closed loop system

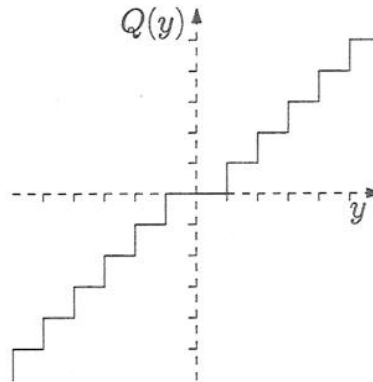


Figure 3.2 Quantization device: Input-Output map (assume equal units on both axis)

3. A SISO linear plant with transfer function $P(s)$ is controlled by means of a proportional controller k . Let $P(s) = \frac{1}{s^2 + s + 1}$. Due to the presence of a nonlinear static quantization device on the sensor, the overall feedback loop is as in Figure 3.1, where $Q(\cdot)$ is the nonlinear element with the characteristic given in Figure 3.2.
 - a) What is the smallest sector which comprises the quantization nonlinearity (assuming the same units on the two axis)? [4]
 - b) Draw the Nyquist plot of $P(s)$ and find out what is the maximum value of k which does not destabilize the system in the absence of quantization. [8]
 - c) What is the maximum value of k allowed by the circle criterion in order to preserve stability in the presence of quantization? [8]

4. Consider the time-invariant linear system:

$$\dot{x} = Ax$$

with $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ together with the following Statement:

Statement: If A is diagonalizable, there exists $P > 0$ so that

$$\frac{d}{dt}x'Px \leq 2\lambda_{\max}x'Px$$

with $\lambda_{\max} = \max\{\operatorname{Re}(\lambda) : \lambda \in \operatorname{sp}(A)\}$ and $\operatorname{sp}(A)$ denoting the spectrum of the matrix A .

- a) Show that the Statement is true. Hint: build first P for the simple systems:

$$\dot{x} = \lambda x \quad x \in \mathbb{R},$$

$$\dot{x} = \begin{bmatrix} \lambda & \omega \\ -\omega & \lambda \end{bmatrix} x \quad x \in \mathbb{R}^2.$$

[8]

- b) Show, by means of an example, that if A is not diagonalizable, there is no $P > 0$ such that an inequality as in the Statement holds. [5]

- c) Argue that $V(x) = x'Px$ as given in the Statement can be used to prove global exponential stability, provided A is Hurwitz. [7]

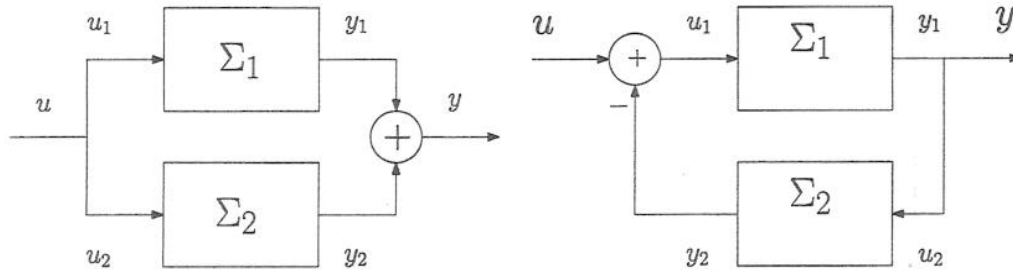


Figure 5.1 Interconnected systems

5. Consider a nonlinear time-invariant system

$$\dot{x} = f(x, u), \quad y = h(x).$$

- a) Recall the time-domain definition of passivity of a nonlinear system. [2]
- b) Consider the interconnected systems shown in the Figure 5.1. Show that each one of them is again a passive system, provided the individual subsystems are such. [2]
- c) Show, by means of an example, that the series of passive systems need not be passive. [4]
- d) Can you think of one input-output pair which violates the definition of passivity for a series of passive linear systems? [4]
- e) Consider the following nonlinear circuital components:

- i) Nonlinear resistor with characteristic equation:

$$V = R(I)$$

- ii) Nonlinear inductor with characteristic equation:

$$L(I)\dot{I} = V$$

- iii) Nonlinear capacitor with characteristic equation:

$$C(V)\dot{V} = I$$

Find conditions on the nonlinear functions $R(\cdot)$, $L(\cdot)$ and $C(\cdot)$ so that the resulting components are passive with respect to V and I as input and output variables. [4]

- f) Show that the network obtained by composing in series an inductor and a capacitor (in the sense of circuit theory) is lossless. [4]

6. Consider the parameter-dependent nonlinear system:

$$\begin{aligned}\dot{x}_1 &= -k \sin(x_1) + x_2, \\ \dot{x}_2 &= \operatorname{atan}(x_2) + x_3, \\ \dot{x}_3 &= \frac{u}{\operatorname{atan}(x_2) + x_3},\end{aligned}$$

with state $x = [x_1, x_2, x_3]$ taking values in \mathbb{R}^3 and control u taking values in \mathbb{R} .

- Show that the system with output $y = x_1$ has relative degree 3. [3]
- Is the system globally feedback linearizable? Why? [2]
- Build a global feedback stabilizer assuming k is known. [3]
- Let $y = x_2$. Compute the relative degree and discuss if the system can be globally stabilized by means of input-output feedback linearization. What are the zero-dynamics? Are they Input-to-State Stable? Design a local feedback stabilizer by means of Input-Output feedback linearization. How many equilibria has the closed-loop system? [6]
- Assume now k is only known to belong to the interval $[-\varepsilon, +\varepsilon]$. Design by means of backstepping a controller which robustly globally asymptotically stabilizes the origin irrespectively of the value of k . (Hint: find a robust virtual control for the x_1 equation.) [6]

STABILITY AND CONTROL OF NONLINEAR SYSTEMS

MODEL ANSWERS 2009

1. Exercise

- a) We may choose the following state variable $x(t) = [y(t), \dot{y}(t)]' \doteq [x_1(t), x_2(t)]'$.
With such choice the model of the system reads:

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= \operatorname{atan}(x_1) - \frac{x_2}{1+x_1^2} - \frac{\pi}{4}x_1.\end{aligned}$$

- b) The equilibria are obtained solving:

$$\begin{cases} x_2 = 0 \\ \operatorname{atan}(x_1) - \frac{x_2}{1+x_1^2} - \frac{\pi}{4}x_1 = 0 \end{cases}$$

Substituting $x_2 = 0$ in the second equation we get:

$$\operatorname{atan}(x_1) = \frac{\pi}{4}x_1$$

that is $x_1 = -1, 0, 1$. We have therefore 3 possible equilibria: $[-1, 0]', [0, 0]', [1, 0]'$.

- c) To compute the linearization $\delta \dot{x} = \left. \frac{\partial f}{\partial x} \right|_{x=x_e} \delta x$ around such points note that:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ \frac{1}{1+x_1^2} - \frac{\pi}{4} + \frac{2x_1x_2}{(1+x_1^2)^2} & -\frac{1}{1+x_1^2} \end{bmatrix}.$$

Evaluating the above expression at equilibria, yields

$$\left. \frac{\partial f}{\partial x} \right|_{x=[\pm 1, 0]'} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} - \frac{\pi}{4} & -\frac{1}{2} \end{bmatrix}$$

and

$$\left. \frac{\partial f}{\partial x} \right|_{x=[0, 0]'} = \begin{bmatrix} 0 & 1 \\ 1 - \frac{\pi}{4} & -1 \end{bmatrix}.$$

Computing the characteristic polynomial of the first matrix yields:

$$\chi(s) = s^2 + \frac{s}{2} + \frac{\pi-2}{4}$$

which admits two roots with negative real part; moreover the discriminant is given by: $\frac{1}{4} - (\pi-2) = 2.25 - \pi < 0$. This means the equilibria are stable foci. For $x_e = [0, 0]'$ we have:

$$\chi(s) = s^2 + s - \frac{4-\pi}{4}.$$

Hence, solutions have respectively positive and negative real parts, and are real. This is therefore a saddle point.

- d) Without further analysis we may conjecture a phase plot along the lines of the Figure 1.1.

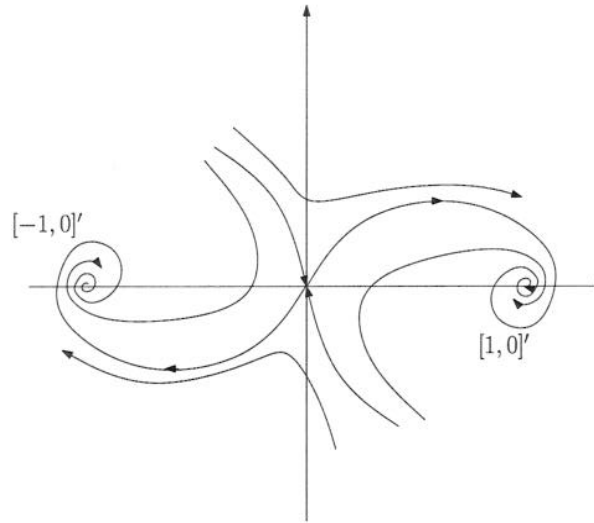


Figure 1.1 Qualitative phase portrait

2. Exercise

- a) In order to show that $y(t)$ is constant for all initial conditions, it is enough to show $\dot{y} = 0$. Hence we compute: $\dot{x}_1 + \dot{x}_2 + \dot{x}_3 = 0$.
- b) Next we substitute $x_3 = -x_1 - x_2$ in the systems equations yielding:

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1^3 - \frac{x_2}{1+x_1^2}.\end{aligned}$$

- c) The equilibrium is obtained solving:

$$\begin{cases} x_2 = 0 \\ -x_1^3 - \frac{x_2}{1+x_1^2} = 0 \end{cases}$$

Substituting $x_2 = 0$ in the second equation yields, $x_1^3 = 0$, that is $x_1 = 0$. Hence, there exists a unique equilibrium in $[0, 0]'$. Let us verify that $V(x_1, x_2)$ is a suitable candidate Lyapunov function to prove global asymptotic stability. We take $V(x_1, x_2) = x_1^4/4 + x_2^2/2$. Indeed V is differentiable, and positive definite:

$$x \neq 0 \Rightarrow x_1 \neq 0 \text{ or } x_2 \neq 0 \Rightarrow \begin{cases} \text{in the first case } V(x_1, x_2) \geq \frac{x_1^4}{4} > 0 \\ \text{otherwise } V(x_1, x_2) \geq \frac{x_2^2}{2} > 0 \end{cases}$$

So V is positive definite. It is straightforward to verify that V is radially unbounded. Next we compute \dot{V} .

$$\dot{V} = \dot{x}_1 x_1^3 + \dot{x}_2 x_2 = -\frac{x_2^2}{1+x_1^2} \leq 0$$

Hence, solutions are bounded and by Lasalle's principle converge to the largest invariant set contained in $K \doteq \{(x_1, x_2)' : x_2 = 0\}$. We claim that the only invariant set contained in K is actually the equilibrium itself. Indeed, asking for $\dot{x}_2 = 0$ simultaneously to $x_2 = 0$ yields $x_1 = 0$.

- d) We now proceed to linearizing the system. The Jacobian is given by:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -3x_1^2 + \frac{2x_1x_2}{(1+x_1^2)^2} & \frac{2x_2}{1+x_1^2} \end{bmatrix}$$

which evaluated at $[0,0]'$ yields

$$\left. \frac{\partial f}{\partial x} \right|_{x_1=0, x_2=0} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We do have a double eigenvalue at 0, that is on the imaginary axis. This is, henceforth, a critical case in which we cannot appeal to the Lyapunov theorem to claim local asymptotic stability of the origin.

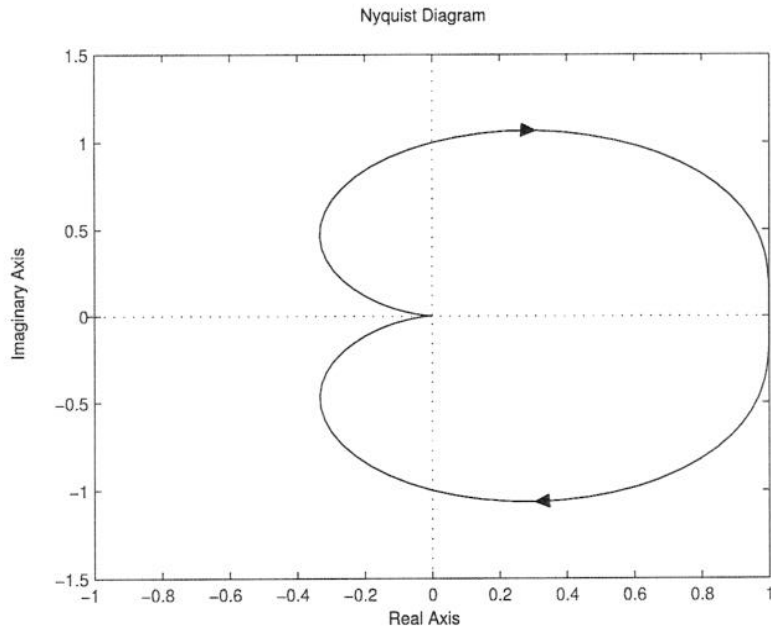


Figure 3.1 Nyquist plot of $P(s)$

3. Exercise

- The smallest sector containing the quantization nonlinearity is $[0, 1]$ (notice the local slope at the origin is 0 !).
- The Nyquist plot of $P(s)$ is as in Figure 3.1. Hence, for all $k > 0$ the resulting closed-loop system is asymptotically stable (the point $-1/k$ is never encircled by the diagram).
- Due to the effect of quantization, however, and applying circle criterion, we are only guaranteed GAS, for all ks such that the vertical line through $-\frac{1}{k}$ does not meet the Nyquist diagram. Hence we need to compute the minimum value of the real part of $P(j\omega)$ as $\omega \in \mathbb{R}$.

$$\operatorname{Re}[P(j\omega)] = \frac{1 - \omega^2}{(1 - \omega^2)^2 + \omega^2}$$

Zeroing the derivative with respect to ω^2 we have that the minimum is achieved for $\omega^2 = 2$. This in turn yields $\min \operatorname{Re}[P(j\omega)] = -\frac{1}{3}$. Hence, the maximal gain allowed is $k = 3$.

4. Exercise

- a) Up to a real change of coordinates A can be put in a block-diagonal form, with the diagonal blocks of type:

$$\dot{x} = \lambda x \quad x \in \mathbb{R}$$

for a real eigenvalue in λ , or:

$$\dot{x} = \begin{bmatrix} \lambda & \omega \\ -\omega & \lambda \end{bmatrix} x \quad x \in \mathbb{R}^2$$

for complex conjugate ones in $\lambda \pm j\omega$. In the first case, $V(x) = x^2$ provides the desired estimate; indeed:

$$\dot{V}(x) = 2x\dot{x} = 2\lambda x^2 \leq 2\lambda_{\max} V(x)$$

Similarly, in the case of complex conjugate eigenvalues we let $V(x) = x'x$:

$$\dot{V} = x'(A' + A)x = 2\lambda x'x = 2\lambda_{\max} V(x)$$

Hence, any linear combination of such functions works as a suitable Lyapunov function for the overall block-diagonal system. The inequality preserve their validity in original coordinates.

- b) Let A be given by:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Clearly $\lambda_{\max} = 0$. So, existence of P as requested yields for $V(x) = x'Px$

$$\dot{V} = x'(A'P + PA)x \leq 0$$

This in turn implies $V(x(t)) \leq V(x(0))$ and hence, boundedness of solutions. However, it is well known that the above system admits t as one of its modes, that is, admits unbounded solutions. This provides the sought contradiction.

- c) In fact, the differential inequality that we proved yields:

$$V(x(t)) \leq e^{2\lambda_{\max}t} V(x(0))$$

In particular then, if $\lambda_{\max} < 0$ this provides a proof of exponential stability, since:

$$\underline{\sigma}(P)\|x(t)\|^2 \leq V(x(t)) \leq e^{2\lambda_{\max}t} V(x(0)) \leq e^{2\lambda_{\max}t} \bar{\sigma}(P)\|x(0)\|^2$$

5. Exercise

- a) A system is said to be passive if, for all input output pairs y, u there exists some M so that it holds

$$\int_0^{+\infty} y(t)u(t)dt \geq M$$

(meaning that $\liminf_{T \rightarrow +\infty} \int_0^T y(t)u(t)dt \geq M$, as the above integral need not exist).

- b) For the parallel interconnection we have:

$$u = u_1 = u_2 \quad y = y_1 + y_2$$

Hence,

$$\begin{aligned} \int_0^{+\infty} y(t)u(t)dt &= \int_0^{+\infty} (y_1(t) + y_2(t))u(t)dt \\ &\geq \int_0^{+\infty} y_1(t)u_1(t)dt + \int_0^{+\infty} y_2(t)u_2(t)dt \geq M_1 + M_2. \end{aligned}$$

For the feedback interconnection we have:

$$u = u_1 + y_2 \quad y = y_1 = u_2$$

Hence:

$$\begin{aligned} \int_0^{+\infty} y(t)u(t)dt &= \int_0^{+\infty} y(t)(u_1(t) + y_2(t))dt \\ &\geq \int_0^{+\infty} y_1(t)u_1(t)dt + \int_0^{+\infty} y_2(t)u_2(t)dt \geq M_1 + M_2, \end{aligned}$$

which again shows passivity.

- c) Consider the series interconnection of two copies of the following elementary system:

$$\dot{x} = -x + u \quad y = x$$

The transfer function of the series is $G(s) = \frac{1}{(s+1)^2}$. For all $\omega > 1$, we have $\text{Arg}[G(j\omega)] < -\pi/2$. Hence the systems violates the frequency-domain characterization of passivity. Any sinusoidal input with frequency larger than 1, besides, violates the passivity definition. Indeed:

$$\int_0^{2\pi/\omega} \sin(\omega t) \sin(\omega t + \phi) dt = \frac{1}{2} \int_0^{2\pi/\omega} \cos(\phi) - \cos(2\omega t + \phi) dt = \frac{\pi}{\omega} \cos(\phi)$$

The latter is a negative quantity whenever the phase-lag introduced by the system is larger than $\pi/2$. Hence, for such ω s,

$$\liminf_{T \rightarrow +\infty} \int_0^T y(t)u(t)dt \leq \lim_{k \rightarrow +\infty} \int_0^{k2\pi/\omega} y(t)u(t)dt = k \frac{\pi}{\omega} \cos(\phi) = -\infty$$

which violates passivity definition.

- d) Consider next the nonlinear resistor:

$$VI = R(I)I$$

If $R(I)I \geq 0$ for all I , we have:

$$\int_0^t V(t)I(t)dt \geq 0$$

for all t For the nonlinear inductor:

$$VI = IL(I)\dot{I} = \frac{d}{dt} \int_0^I iL(i)di$$

Hence, if $L(i) \geq \varepsilon > 0$ for all i , the function $E(I) \doteq \int_0^I iL(i)di$ is positive semidefinite and:

$$\int_0^t V(t)I(t)dt = E(I(t)) - E(I(0)) \geq -E(I(0)) > -\infty$$

Similarly for the nonlinear capacitor:

$$VI = VC(V)\dot{V} = \frac{d}{dt} \int_0^V vC(v)dv$$

Hence, if $C(v) \geq \varepsilon > 0$ for all v the function $E(V) \doteq \int_0^V vC(v)dv$ is positive semidefinite and:

$$\int_0^t V(t)I(t)dt = E(V(t)) - E(V(0)) \geq -E(V(0))$$

- e) The series interconnection (in the sense of circuit theory) of an inductor and a capacitor is characterized by the following equations:

$$V = V_C + V_L \quad I = I_C = I_L$$

These are exactly the equations which characterize the feedback interconnection of two systems, taking, respectively $u = I_C$ and $y = V_C$ for the capacitor and $u = V_L$ and $y = I_L$ for the inductor. By the above considerations nonlinear capacitors and inductors correspond to lossless elements. The second order system (with input V and output I) arising from their feedback interconnection is therefore a lossless system.

6. Exercise

- a) It is assumed $y = x_1$. Hence deriving the output 3 times we obtain:

$$\begin{aligned} \dot{y} &= -k \sin(x_1) + x_2 \\ \ddot{y} &= -k \cos(x_1)[-k \sin(x_1) + x_2] + \text{atn}(x_2) + x_3 \\ &= \frac{1}{2}k^2 \sin(2x_1) - k \cos(x_1)x_2 + \text{atn}(x_2) + x_3 \\ y^{(3)} &= [k^2 \cos(2x_1) + k \sin(x_1)][-k \sin(x_1) + x_2] \\ &\quad + \left[\frac{1}{1+x_2^2} - k \cos(x_1) \right] \cdot [\text{atn}(x_2) + x_3] + u \end{aligned}$$

Since u only appears at the third derivative, the relative degree is 3.

- b) Moreover, the coefficient of u is constant (and different from 0), hence it is possible to globally feedback linearize the system.
c) A globally stabilizing control law is:

$$\begin{aligned} u = & -[k^2 \cos(2x_1) + k \sin(x_1)][-k \sin(x_1) + x_2] \\ & - \left[\frac{1}{1+x_2^2} - k \cos(x_1) \right] \cdot [\text{atn}(x_2) + x_3] - y - 3\dot{y} - 3\ddot{y} \end{aligned}$$

Under such feedback the equations read $y^{(3)} + 3\ddot{y} + 3\dot{y} + y = 0$, which is a linear system with 3 eigenvalues in -1 .

d) Let us fix $y = x_2$. This choice gives:

$$\begin{aligned}\dot{y} &= \text{atn}(x_2) + x_3 \\ \ddot{y} &= \frac{\text{atn}(x_2) + x_3}{1 + x_2^2} + u\end{aligned}$$

Hence, the relative degree is 2. The system is globally input-output feedback linearizable, however there are non-empty zero-dynamics. In particular the x_1 -equation is the zero-dynamics. For $x_2 = 0$ the zero dynamics have infinitely many equilibria at $x_1 = n\pi$ for all $n \in \mathbb{Z}$. Hence the zero-dynamics are not globally asymptotically stable (and not Input-to-State Stable). If $k > 0$ they are locally asymptotically stable at the origin (easy to see by linearization). Hence, a local feedback stabilizer can be obtained by letting:

$$u = -\frac{\text{atn}(x_2) + x_3}{1 + x_2^2} - y - 2\dot{y}.$$

Notice that the above feedback does not assume knowledge of k ; it is, however, only guaranteed to converge locally.

e) We now proceed to design a robust feedback stabilizer by means of backstepping. Consider the x_1 equation. This is ISS stabilized (with respect to actuators disturbances and regarding x_2 as an input), by applying the virtual control

$$x_2^v = -2\epsilon x_1$$

This is easily seen taking $x_1^2/2$ as a Lyapunov function and exploiting $k \in [-\epsilon, +\epsilon]$.

Next we consider the (x_1, x_2) subsystem and try to ISS stabilize it by means of the virtual input x_3 . To this end we pick the Lyapunov function:

$$V(x_1, x_2) = \frac{x_1^2 + \alpha(x_2 - x_2^v)^2}{2}$$

Taking derivatives yields:

$$\begin{aligned}\dot{V} &= -k \sin(x_1)x_1 + x_1x_2 + \alpha(x_2 + 2\epsilon x_1)[\text{atn}(x_2) + x_3 - 2\epsilon k \sin(x_1) + 2\epsilon x_2] \\ &= -k \sin(x_1)x_1 - 2\epsilon x_1^2 + (x_2 + 2\epsilon x_1)[x_1 + \alpha \text{atn}(x_2) + \alpha x_3 - 2\alpha \epsilon k \sin(x_1) + 2\alpha \epsilon x_2] \\ &\leq -\epsilon x_1^2 + x_1(1 + 2\alpha \epsilon^2)|x_2 + 2\epsilon x_1| + (x_2 + 2\epsilon x_1)[\alpha \text{atn}(x_2) + \alpha x_3 + 2\alpha \epsilon x_2]\end{aligned}$$

Hence, the (x_1, x_2) subsystem is ISS stabilized by picking

$$x_3^v = -\text{atn}(x_2) - 2\epsilon x_2 - \gamma(x_2 + 2\epsilon x_1)$$

provided γ is picked sufficiently large, for instance:

$$\gamma > \frac{(1/2 + \alpha \epsilon^2)^2}{\epsilon \alpha}.$$

The last step is to backstep x_3^v . We use the Lyapunov function:

$$W(x_1, x_2, x_3) = V(x_1, x_2) + \frac{\beta}{2}(x_3 - x_3^v)^2$$

Taking derivatives of W gives a term proportional to $k \sin(x_1)(x_3 - x_3^v)$. Since k is not known we need to dominate this by introducing a sufficiently large term: $-\delta(x_3 - x_3^v)$ in our control law.