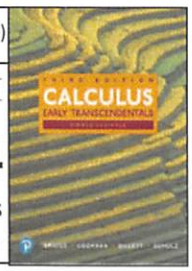
MTH211 (Cameron)

Calculus II

Unit 4

Limits, Series & Sequences



Unit 4: Limits, Sequences, Series

4.7 L'Hôpital's Rule

CO 4.1 Use L'Hopital's Rule to evaluate limits involving indeterminate forms of the type $0/0$ and ∞/∞.

CO 4.2 Apply algebraic techniques to reduce limits involving indeterminate forms of type $0 \cdot \infty$, and $\infty - \infty$ to forms for which L'Hopital's Rule can be applied.

CO 4.3 Use logarithmic techniques to reduce limits involving indeterminate forms of type 0^0, ∞^0, and 1^∞ to forms for which L'Hopital's Rule can be applied.

L'Hôpital's Rule

If f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a), and

$$\left\{\begin{array}{c}\lim\limits_{x\to a} f(x) = 0 \\ \text{and} \\ \lim\limits_{x\to a} g(x) = 0\end{array}\right\} \text{ or } \left\{\begin{array}{c}\lim\limits_{x\to a} f(x) = \pm\infty \\ \text{and} \\ \lim\limits_{x\to a} g(x) = \pm\infty\end{array}\right\} \text{ i.e. we have an indeterminate form of type } \frac{0}{0} \text{ or } \pm\frac{\infty}{\infty}$$

Then $\boxed{\lim\limits_{x\to a}\left(\dfrac{f(x)}{g(x)}\right) = \lim\limits_{x\to a}\left(\dfrac{f'(x)}{g'(x)}\right)}$

The rule also applies if $x \to a$ is replaced with $x \to \infty$, $x \to -\infty$, $x \to a^-$ or $x \to a^+$

How to use L'Hôpital's Rule

Step 1: Identify the form of a limit by substitution

Step 2: If the form of the limit is not of type $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$ rewrite the function so the limit **is** in one of these forms

Step 3: Use L'Hôpital's Rule – but <u>only</u> if the limit is an indeterminant form of type $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$

L'Hôpital's Rule for the Form 0/0

EXAMPLE 1. **0/0 form** Evaluate the following limits using L'Hôpital's Rule

a) $\lim\limits_{x\to 1}\left(\dfrac{\ln x}{4x - x^2 - 3}\right) \overset{\text{① SUB}}{=} \dfrac{\ln(1)}{4(1)-(1)^2-3} = \boxed{\text{form } \frac{0}{0}}$ indeterminate form value unknown BUT L'hopital CAN be used

no limit when substituting

L'H

$= \lim\limits_{x\to 1} \dfrac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(4x - x^2 - 3)} = \lim\limits_{x\to 1} \dfrac{\frac{1}{x}}{4-2x} \overset{\text{SUB}}{=} \dfrac{\frac{1}{1}}{4-2(1)} = \boxed{\frac{1}{2}}$

limit REQUIRED w/ L'Hopital

b) $\lim\limits_{x\to 0}\left(\dfrac{e^{2x}-1}{x}\right) \overset{\text{SUB}}{=} \dfrac{e^{2(0)}-1}{0} = \dfrac{e^0-1}{0} = \dfrac{1-1}{0} = \boxed{\text{form } \frac{0}{0}}$ indeterminate form value unknown But L'Hopital can be used!

L'H

$= \lim\limits_{x\to 0} \dfrac{\frac{d}{dx}(e^{2x}-1)}{\frac{d}{dx}(x)} = \lim\limits_{x\to 0} \dfrac{e^{2x}[2]}{1} \overset{\text{SUB}}{=} 2e^{2(0)} = 2e^0 = \boxed{2}$

Chain Rule!

c)#25/35 $\lim_{x\to\pi}\left(\frac{\cos x+1}{(x-\pi)^2}\right) \overset{SUB}{=} \frac{\cos\pi+1}{(\pi-\pi)^2} = \frac{-1+1}{0^2} = \boxed{\text{form } \frac{0}{0}}$ indeterminate (value is unknown) L'Hopital CAN BE used with this form to try & find its value

$\overset{L'H}{=} \lim_{x\to\pi} \frac{\frac{d}{dx}(\cos x+1)}{\frac{d}{dx}(x-\pi)^2} = \lim_{x\to\pi} \frac{-\sin x}{2(x-\pi)} \overset{SUB}{=} \frac{-\sin\pi}{2(\pi-\pi)} = \boxed{\text{form } \frac{0}{0}}$ use L'H again

$\overset{L'H}{=} \lim_{x\to\pi} \frac{-\cos x}{2} \overset{SUB}{=} \frac{-\cos\pi}{2} = \frac{-(-1)}{2} = \boxed{\frac{1}{2}}$

d)#33/47 $\lim_{x\to 3}\left(\frac{x-1-\sqrt{x^2-5}}{x-3}\right) \overset{SUB}{=} \frac{3-1-\sqrt{3^2-5}}{3-3} = \frac{2-\sqrt{4}}{0} = \boxed{\text{form } \frac{0}{0}}$ try using L'Hopital to find the value

$\overset{L'H}{=} \lim_{x\to 3} \frac{1-\frac{1}{\cancel{2}}(x^2-5)^{-1/2}[\cancel{2}x]}{1} \overset{SUB}{=} 1-\frac{3}{\sqrt{3^2-5}}$

$= 1-\frac{3}{\sqrt{4}} = 1-\frac{3}{2} = \boxed{\frac{-1}{2}}$

e)S5x $\lim_{x\to\pi^-}\left(\frac{\sin x}{1-\cos x}\right) \overset{SUB}{=} \frac{\sin\pi}{1-\cos\pi} = \frac{0}{1-(-1)} = \frac{0}{2} = \boxed{0}$

NOTE: using L'Hopital without substituting first can lead to an $\boxed{\text{incorrect result}}$

Ex $\overset{L'H}{=} \lim_{x\to\pi^-} \frac{\cos x}{+\sin x} \overset{SUB}{=} \frac{\cos\pi^-}{\sin\pi^-} = \frac{-1}{0^+} = \boxed{-\infty}$ ← wrong result!

Indeterminate Form ∞/∞

EXAMPLE 2. ∞/∞ **form** Evaluate the following limits using L'Hôpital's Rule

a) $\lim_{x\to\infty}\left(\frac{\ln\left(\frac{1}{x}\right)}{2x+3}\right) \overset{SUB}{=} \frac{\ln\left(\frac{1}{\infty}\right)}{2(\infty)+3} = \frac{\ln(0)}{\infty} = \boxed{\text{form } \frac{-\infty}{\infty}}$ indeterminate form (value can't be determined) L'Hopital can be used with this form.

Chain Rule! Simplify Complex Fractions

$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1/x}\frac{d}{dx}(1/x)}{2} = \lim_{x\to\infty} \frac{1}{2}\left(\frac{1}{x^{-1}}[-x^{-2}]\right) = \lim_{x\to\infty} \frac{-1}{2}\frac{\frac{1}{x^2}}{\frac{1}{x}}\left[\frac{x^2}{x^2}\right]$

$= \lim_{x\to\infty} \frac{-1}{2}\frac{1}{x} \overset{SUB}{=} \frac{-1}{2(\infty)} = \boxed{0}$

b) $\lim_{x\to\infty}\left(\frac{e^x}{x^2}\right) \overset{SUB}{=} \frac{e^\infty}{(\infty)^2} = \boxed{\text{form } \frac{\infty}{\infty}}$ indeterminate form. L'Hopital can be used with this type of form.

$\overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{SUB}{=} \frac{e^\infty}{2(\infty)} = \boxed{\text{form } \frac{\infty}{\infty}}$ indeterminate L'H CAN be used with this form

$\overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{2} \overset{SUB}{=} \frac{e^\infty}{2} = \frac{1}{2}(\infty) = \boxed{\infty}$ or more generally, $\boxed{\text{dne}}$

c) $\lim_{x\to\infty}\left(\frac{e^x}{x^{-2}}\right) \overset{SUB}{=} \frac{e^\infty}{(\infty)^{-2}} = e^\infty\cdot(\infty)^2 = \boxed{\infty}$

Start D1 #1

Indeterminate Form $0 \bullet \infty$

To evaluate $\lim\limits_{x\to a}\left(f(x)\cdot g(x)\right)$ which has indeterminate form $0 \bullet \infty$,

first write $f(x)\cdot g(x)$ as either $\dfrac{f(x)}{1/g(x)}$ or $\dfrac{g(x)}{1/f(x)}$ to get either $\dfrac{0}{0}$ or $\pm\dfrac{\infty}{\infty}$

Given an indeterminate product, the choice of what to use for a denominator in the rewritten form can make a difference!

EXAMPLE 3. **($0 \bullet \infty$ form)** Evaluate the following limits using L'Hôpital's Rule

a) $\lim\limits_{x\to 0^+}(x\cot x) \overset{\text{SUB}}{=} 0^+ \cot 0^+ = \boxed{0\cdot\infty}$ indeterminate form (value is unknown)
L'H can't be used so Rewrite 1st.

Two Different Ways to Rewrite

$$\lim_{x\to 0^+}\frac{x}{\frac{1}{\cot x}} = \lim_{x\to 0^+}\frac{x}{\tan x} \overset{\text{SUB: } \boxed{\frac{0}{0}}\ \text{L'H}}{=} \lim_{x\to 0^+}\frac{1}{\sec^2 x} \overset{\text{SUB}}{=} \frac{1}{\sec^2 0} = \frac{1}{1} = \boxed{1}$$

$$\lim_{x\to 0^+}\frac{\cot x}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\cot x}{x^{-1}} \overset{\text{SUB: } \boxed{\frac{\infty}{\infty}}\ \text{L'H}}{=} \lim_{x\to 0^+}\frac{-\csc^2 x}{-x^{-2}} \overset{\text{SUB: } \boxed{\frac{\infty}{\infty}}\ \text{L'H}}{=} \lim_{x\to 0^+}\frac{2\csc x[-\csc x\cot x)}{2x^{-3}}$$

getting more & more complicated!

b) $\lim\limits_{x\to 0^+}(x^2\csc^2(x)) \overset{\text{SUB}}{=} 0^2 \csc^2(0) = \boxed{0\cdot\infty}$ indeterminate form
can't use L'H so Rewrite 1st

Two different ways to rewrite.

$$= \lim_{x\to 0^+}\frac{x^2}{\frac{1}{\csc^2 x}} = \lim_{x\to 0^+}\frac{x^2}{\sin^2 x} \overset{\boxed{\frac{0}{0}}\ \text{L'H}}{=} \lim_{x\to 0}\frac{2x}{2\sin x\cos x} \overset{\text{trig ident.}}{=} \lim_{x\to 0}\frac{2x}{\sin(2x)} \overset{\boxed{\frac{0}{0}}\ \text{L'H}}{=} \lim_{x\to 0}\frac{2}{2\cos(2x)}$$

$$\overset{\text{SUB}}{=} \frac{2}{2\cos(0)} = \frac{2}{2} = \boxed{1}$$

$$= \lim_{x\to 0^+}\frac{\csc^2 x}{\frac{1}{x^2}} \overset{\boxed{\frac{\infty}{\infty}}\ \text{L'H}}{=} \lim_{x\to 0^+}\frac{2\csc x(-\csc x\cot x)}{-2x^{-3}} = \ldots$$

(getting more complicated!)

Finish D1 #2

ch solns

Indeterminate Form $\infty - \infty$

To evaluate $\lim_{x\to a}\left(f(x)-g(x)\right) = \infty - \infty$, $f(x)-g(x)$ must be written as a single term. When the trigonometric functions are involved, switching to all sines and cosines may help to get either $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$

Given an indeterminate difference, rewrite the difference as a single fraction

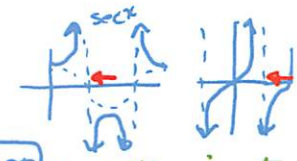
indeterminate form
L'Hopital can NOT be used
so rewrite in a form L'H can be used

EXAMPLE 4. ($\infty - \infty$ **form**) Evaluate the following limits using L'Hôpital's Rule

a) $\lim_{x\to 0^+}\left(\frac{1}{\ln(x^2+1)} - \frac{1}{x}\right) \overset{SUB}{=} \frac{1}{\ln(0^+ + 1)} - \frac{1}{0^+} = \frac{1}{\ln(1^+)} - \frac{1}{0^+} = \boxed{\infty - \infty}$

$= \lim_{x\to 0^+}\left(\frac{x - \ln(x^2+1)}{x\ln(x^2+1)}\right) \overset{SUB}{=} \frac{0^+ - \ln(0^+ + 1)}{0^+\ln(0^+ + 1)} = \frac{0-0}{0(0)} = \boxed{\text{form } \frac{0}{0}}$ indeterminate But L'H can be used to try and get a value

$\overset{L'H}{=} \lim_{x\to 0^+}\frac{1 + \frac{1}{x^2+1}[2x]}{\ln(x^2+1) + \frac{x}{x^2+1}[2x]} \overset{SUB}{=} \frac{1 + \frac{0^+}{1}}{\ln(0^+ + 1) + \frac{0^+}{1}} = \frac{1+0}{0+0} = \frac{1}{0^+} = \boxed{\infty}$ or dne



b) #62 $\lim_{x\to(\pi/2)^+}(\sec x - \tan x) \overset{SUB}{=} \sec\frac{\pi}{2}^+ - \tan\frac{\pi}{2}^+ = (-\infty) - (-\infty) = \boxed{-\infty + \infty}$ form: indeterminate

L'H CAN'T be used on this form so REWRITE!

$= \lim_{x\to\frac{\pi}{2}^+}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x\to\frac{\pi}{2}^+}\left(\frac{1-\sin x}{\cos x}\right) \overset{SUB}{=} \frac{1-\sin\frac{\pi}{2}^+}{\cos\pi/2^+} = \frac{1-1^-}{0^-} = \boxed{\text{form } \frac{0}{0}}$

indeterminate BUT L'H CAN be use on this form

$\overset{L'H}{=} \lim_{x\to\frac{\pi}{2}^+}\frac{-\cos x}{-\sin x} \overset{SUB}{=} \frac{\cos\frac{\pi}{2}^+}{\sin\frac{\pi}{2}^+} = \frac{0^-}{1} = \boxed{0}$

c) $\lim_{y\to\infty}\left(y - \sqrt{y^2-8}\right) \overset{SUB}{=} \infty - \sqrt{\infty^2 - 8} = \boxed{\text{form } \infty - \infty}$ indeterminate L'H can't be used with this form so rewrite

"Rationalize numerator"

$= \lim_{y\to\infty}\left((y - \sqrt{y^2-8})\left[\frac{y+\sqrt{y^2-8}}{y+\sqrt{y^2-8}}\right]\right) = \lim_{y\to\infty}\frac{\cancel{y^2} - (\cancel{y^2}-8)}{y+\sqrt{y^2-8}} \overset{SUB}{=} \frac{8}{\infty + \sqrt{\infty^2-8}} = \frac{8}{\infty} = \boxed{0}$

d) $\lim_{x\to\infty}\left(\frac{x^2-2x}{3x^2+5}\right) \overset{SUB}{=} \frac{\boxed{\text{form } \infty - \infty}}{\infty}$ ← can't use L'H so rewrite

$= \lim_{x\to\infty}\left(\frac{x(x-2)}{3x^2+5}\right) \overset{SUB}{=} \frac{\infty(\infty)}{\infty^2} = \boxed{\text{form } \frac{\infty}{\infty}}$ indeterminate L'H can be used with this form.

$\overset{L'H}{=} \lim_{x\to\infty}\frac{x[1] + [1](x-2)}{6x} = \lim_{x\to\infty}\frac{x + x - 2}{6x} = \lim_{x\to\infty}\frac{2x}{6x} - \frac{2}{6x} = \lim_{x\to\infty}\frac{1}{3} - \frac{1}{3x}$

$\overset{SUB}{=} \frac{1}{3} - \frac{1}{\infty} = \frac{1}{3} - 0 = \boxed{\frac{1}{3}}$

Start D2 #3

Indeterminate Forms 1^∞, 0^0 and ∞^0

For limits of the type $\lim_{x\to a}\left(f(x)^{g(x)}\right)$ indeterminate forms of type 1^∞, 0^0 or ∞^0 are possible. In these cases,

Step 1. Use the inverse exponential rule, $u = e^{\ln(u)}$.

$$\lim_{x\to a}\left(f(x)^{g(x)}\right) = \lim_{x\to a} e^{\ln\left(f(x)^{g(x)}\right)}$$

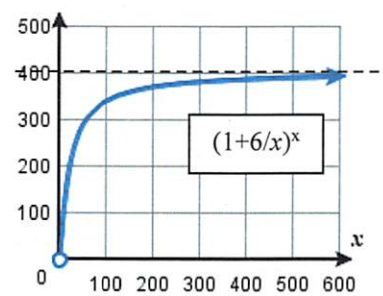
Step 2. Use the Limit of a Composition Rule: if f is continuous, $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$

$$= \lim_{x\to a} e^{\ln\left(f(x)^{g(x)}\right)} = e^{\lim_{x\to a}\ln\left(f(x)^{g(x)}\right)} = e^{\lim_{x\to a}\left(g(x)\cdot\ln(f(x))\right)}$$

Step 3. If $\lim_{x\to a}\left(g(x)\cdot\ln f(x)\right) = L$ a finite real number, then $\lim_{x\to a}\left(f(x)^{g(x)}\right) = e^L$

EXAMPLE 5. **1^∞, 0^0 and ∞^0 forms** Evaluate the following limits using L'Hôpital's Rule

a)#60 $\lim_{x\to\infty}\left(1+\frac{6}{x}\right)^x = 1^\infty$ (Rewrite)



$A = e^{\ln(A)}$ (inverse log property)

$$= \lim_{x\to\infty} e^{\ln(1+6/x)^x}$$

(limit of a composition) $\lim_{u\to a} e^u = e^{\lim_{u\to a} u}$ b/c e^x is continuous

$$= e^{\boxed{\lim_{x\to\infty}\ln(1+6/x)^x}}$$

$$\lim_{x\to\infty}\ln(1+6/x)^x$$

$$= \lim_{x\to\infty} x\ln(1+6/x) = \infty\cdot\ln(1) = \infty\cdot 0 \text{ (Rewrite)}$$

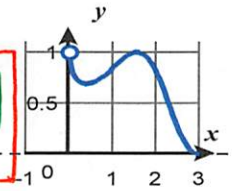
$$= \lim_{x\to\infty}\frac{\ln(1+6/x)}{1/x} = \frac{\ln(1)}{0} = \frac{0}{0} \text{ form } \checkmark$$

$$\overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{1+6x^{-1}}\left[\cancel{-6x^{-2}}\right]}{\left[\cancel{-x^{-2}}\right]} = \lim_{x\to\infty}\frac{6}{1+6/x} = \frac{6}{1+0} = \boxed{6}$$

$$= \boxed{e^6}$$

b) $\lim_{x\to 0^+} (\sin x)^x = \boxed{0^0}$ Rewrite

$$= \lim_{x\to 0^+} e^{\ln[(\sin x)^x]} = e^{\boxed{\lim_{x\to 0^+}[x\ln(\sin x)]}} = e^{\boxed{}}$$



$$\lim_{x\to 0^+}(x\ln(\sin x)) = 0\cdot\ln(0) = \boxed{0\cdot\infty} \text{ Rewrite}$$

$$= \lim_{x\to 0^+}\left(\frac{\ln(\sin x)}{1/x}\right) = \frac{\ln(0)}{1/0} = \boxed{\frac{\infty}{\infty}} \text{ use L'H}$$

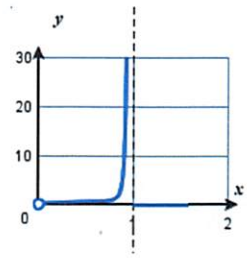
$$\overset{L'H}{=} \lim_{x\to 0^+}\frac{[1/\sin x][\cos x]}{-x^{-2}} \overset{\text{simplify}}{=} -\lim_{x\to 0^+}\frac{\cot x}{x^{-2}} = -\lim_{x\to 0^+}\frac{x^2}{\tan x} = \boxed{\frac{0}{0}} \text{ use L'H}$$

$$\overset{L'H}{=} -\lim_{x\to 0^+}\frac{2x}{\sec^2 x} = -\frac{2\cdot 0}{1^2} = \boxed{0}$$

$$= e^0 = \boxed{1}$$

c) $\lim_{x\to 0^+}(\cot x)^{1/\ln(x)} = \infty^{1/\infty} = \boxed{\infty^0} \to$ Rewrite

$$= \lim_{x\to 0^+} e^{\ln(\cot x)^{1/\ln x}} = e^{\boxed{\lim_{x\to 0^+}\left(\frac{1}{\ln x}\ln(\cot x)\right)}} = e^{\boxed{}}$$



$$\lim_{x\to 0^+}\frac{\ln(\cot x)}{\ln(x)} = \frac{\ln(\infty)}{\ln(0)} = \boxed{\frac{\infty}{-\infty}} \text{ use L'H}$$

$$\overset{L'H}{=} \lim_{x\to 0^+}\frac{\frac{1}{\cot x}[-\csc^2 x]}{1/x} \overset{\text{simplify}}{=} \lim_{x\to 0^+}\left(\frac{-x}{\sin^2 x}\left(\frac{\sin x}{\cos x}\right)\right)$$

$$= \lim_{x\to 0^+}\left(\frac{-x}{\frac{1}{2}\sin(2x)}\right) = \boxed{\frac{0}{0}} \text{ use L'H}$$

$$\overset{L'H}{=} \lim_{x\to 0^+}\frac{-2}{2\cos(2x)} \overset{\text{sub}}{=} \frac{-1}{\cos(0)} = \frac{-1}{1} = \boxed{-1}$$

$$= e^{-1} = \boxed{\frac{1}{e}}$$

Finish D2 #4

Pitfalls in using L'Hôpital's Rule

In this section, seven indeterminate forms have been identified. There are similar forms that are "determinate". Included in these "determinate" forms are: $\infty + \infty \to \infty$, $-\infty - \infty \to -\infty$, and $0^\infty = 0$. A form is indeterminate if the expression is sensitive to small errors i.e. $1^\infty = 1$ and $(1^+)^\infty = \infty$ so 1^∞ is an indeterminate form, whereas $0^\infty = 0$ and $(0^+)^\infty = 0$ so 0^∞ is a determinate form.

EXAMPLE 6. Evaluate the following limits

Always substitute first

a) $\lim_{x\to 0}\left(\frac{e^{2x}-1}{e^x}\right) \overset{SUB}{=} \frac{e^0-1}{e^0} = \frac{1-1}{1} = \frac{0}{1} = \boxed{0}$

WRONG: $\overset{L'H}{=} \lim_{x\to 0} \frac{2e^{2x}}{e^x} = \lim_{x\to 0} 2 = \boxed{2}$ X

b) $\lim_{x\to\infty}\left(x\cos\left(\frac{1}{x}\right)\right) \overset{SUB}{=} \infty \cos\left(\frac{1}{\infty}\right) = \infty \cdot \cos 0 = \infty \to$ $\boxed{\text{limit diverges}}$

$= \lim_{x\to\infty} \frac{\cos(1/x)}{1/x}$ WRONG: $\overset{L'H}{=} \lim_{x\to\infty} \frac{-\sin(1/x)\,[\cancel{-x^{-2}}]}{[\cancel{-x^{-2}}]} \overset{SUB}{=} -\sin\frac{1}{\infty}$

$= -\sin(0) = \boxed{0}$ X

$\overset{SUB}{=} \frac{\cos 0}{0} = \frac{1}{0} = \boxed{\infty}$

Some exponential forms are not indeterminate

c) $\lim_{x\to\infty}\left(\frac{4}{x}\right)^{\frac{x}{2}} \overset{SUB}{=} \left(\frac{4}{\infty}\right)^{\frac{\infty}{2}} = 0^\infty = \boxed{0}$

b/c $\begin{cases} (0^+)^\infty = 0 \\ (0^-)^\infty = 0 \\ (\text{exactly } 0)^\infty = 0 \end{cases}$

d) $\lim_{x\to\infty}(1)^x = (\text{exactly } 1)^\infty = \boxed{1}$

$$\lim_{x\to\infty}\underbrace{\left(1+\frac{1}{x}\right)}_{\text{almost } 1}^{x} = e \qquad \lim_{x\to\infty}\underbrace{\left(1+\frac{1}{x}\right)}_{\text{almost } 1}^{x^2} = \infty \qquad \lim_{x\to\infty}\underbrace{\left(1+\frac{1}{x}\right)}_{\text{almost } 1}^{\sqrt{x}} = 1$$

L'hopital doesn't work on dne forms

e) $\lim_{x\to\infty}\left(\frac{\sin x}{x}\right) \overset{SUB}{=} \frac{\sin(\infty)}{\infty} = \frac{dne}{\infty}$ ← NOT $\frac{\infty}{\infty}$

Instead, use the Squeeze Theorem

① $-1 \le \sin x \le 1$

② $\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

③ $\lim_{x\to\infty} \frac{-1}{x} \le \lim_{x\to\infty} \frac{\sin x}{x} \le \lim_{x\to\infty} \frac{1}{x}$

④ $0 \le \lim_{x\to\infty} \frac{\sin x}{x} \le 0$

① Value of $\sin x$ always $\in [-1, 1]$

② make center match argument of limit

③ take limit of each part
middle part matches limit to be evaluated

④ Evaluate limits on each end

Start D3 #5

Conclusion: Therefore, $\lim_{x\to\infty} \frac{\sin x}{x} = 0$

8.9 Improper Integrals

CO 5.1 Evaluate improper integrals with infinite limits of integration.
CO 5.2 Evaluate improper integrals with discontinuous integrands.

Improper integrals have two different forms: those where the limits of integration include ∞ or -∞ ("Type I") and those where the integrand grows infinitely large somewhere between the limits of integration ("Type II").

Improper Integrals over Infinite Intervals

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx, \qquad f \text{ is continuous on } [a, \infty)$$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx, \qquad f \text{ is continuous on } (-\infty, b]$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx, \qquad f \text{ is continuous on } (-\infty,\infty),\ c\in\mathbb{R}$$

If the limit exists, the improper integral **converges**; otherwise it **diverges**

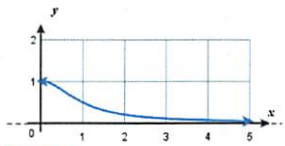
value of limit is a ℝeal # — limit = ±∞ or dne

EXAMPLE 7. **Infinite integrals of integration** Evaluate the following integrals or state that they diverge.

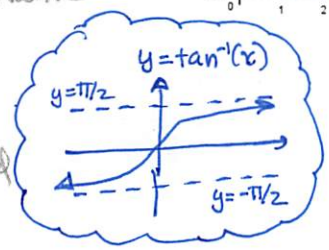
- State the limit definition of the improper integral.
- Find the corresponding antiderivative to be evaluated.
- Evaluate the integral.
- Evaluate the limit.

a) $\displaystyle\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{a\to\infty}\int_1^a \frac{1}{x^2+1}\,dx$ — Limit Def^n

$\displaystyle = \lim_{a\to\infty}\Big[\tan^{-1}(x)\Big]_1^a$ — Anti Derivative



$\displaystyle = \lim_{a\to\infty}\Big[\tan^{-1}(a) - \tan^{-1}(1)\Big]$ — Eval Integral



$\displaystyle = \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{4}}$ — Eval Limit

~~Integral Diverges~~
Integral Converges to $\pi/4$

b) $\displaystyle\int_1^{\infty} \frac{1}{2x+7}\,dx = \lim_{a\to\infty}\int_1^a \frac{1}{2x+7}\,dx$

$u = 2x+7$
$du = 2dx \rightarrow \frac{1}{2}du = dx$
$x = a \to \infty \Rightarrow u = a \to \infty$
$x = 1 \Rightarrow u = 9$

$\displaystyle = \lim_{a\to\infty}\int_9^a \frac{1}{u}\cdot\frac{1}{2}du$

$\displaystyle = \frac{1}{2}\lim_{a\to\infty}\Big[\ln u\Big]_9^a$

$\displaystyle = \frac{1}{2}\lim_{a\to\infty}\Big[\ln a - \ln 9\Big]$ ($\ln a \to \infty$)

$= \infty$

← alternatively (make it very clear the limits are for x not u): $\displaystyle\lim_{a\to\infty}\int_{x=1}^{x=a}\frac{1}{u}\cdot\frac{1}{2}du$

$\displaystyle = \frac{1}{2}\lim_{a\to\infty}\Big[\ln|u|\Big]_{x=1}^{x=a}$

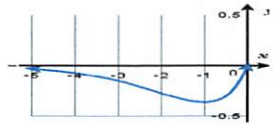
$\displaystyle = \frac{1}{2}\lim_{a\to\infty}\Big[\ln|2x+7|\Big]_{x=1}^{x=a}$

$\displaystyle = \frac{1}{2}\lim_{a\to\infty}\Big[\ln|2a+7| - \ln|9|\Big] = \infty$ ($\ln|2a+7| \to \infty$)

Integral Diverges
Integral Converges to _____

c) $\int_{-\infty}^{0} xe^x\,dx$

Integration By Parts



$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$$= \lim_{a\to-\infty} \int_a^0 x e^x\,dx$$

$$= \lim_{a\to-\infty} \left[xe^x\Big|_a^0 - \int_a^0 e^x\,dx \right] = \lim_{a\to-\infty} \left[xe^x - e^x \right]_a^0$$

$$= \lim_{a\to-\infty} \left\{ \left[0e^0 - e^0 \right] - \left[ae^a - e^a \right] \right\}$$

$$= 0 - 1 - \lim_{a\to-\infty} \left[ae^a \right] + 0 = -1 - \left[-\infty \cdot 0 \right] \quad \text{(Rewrite)}$$

$$= -1 - \lim_{a\to-\infty} \left[\frac{a}{e^{-a}} \right] \Rightarrow \frac{-\infty}{\infty} \text{ form}$$

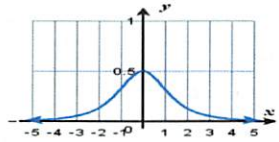
$$\overset{L'H}{=} -1 - \lim_{a\to-\infty} \left[\frac{1}{-e^{-a}} \right] = -1 - \frac{1}{-e^{\infty}}$$

$$= -1 + \frac{1}{e^{\infty}} = -1 + 0$$

$$= \boxed{-1}$$

~~Integral Diverges~~
Integral Converges to -1

d) $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\,dx + \int_0^{\infty} \frac{e^x}{1+e^{2x}}\,dx$



$u = e^x$

$du = e^x dx$

$x = a \to -\infty \Rightarrow u = a \to 0$

$x = 0 \Rightarrow u = 1$

$x = b \to \infty \Rightarrow u = a \to \infty$

def^n
$$= \lim_{a\to-\infty} \int_a^0 \frac{e^x}{1+e^{2x}}\,dx + \lim_{b\to\infty} \int_0^b \frac{e^x}{1+e^{2x}}\,dx$$

$$= \lim_{a\to 0} \int_a^1 \frac{1}{1+u^2}\,du + \lim_{b\to\infty} \int_1^b \frac{1}{1+u^2}\,du$$

anti deriv
$$= \lim_{a\to 0} \left[\tan^{-1}|u| \right]_a^1 + \lim_{b\to\infty} \left[\tan^{-1}|u| \right]_1^b$$

Eval integral
$$= \lim_{a\to 0} \left[\cancel{\tan^{-1}(1)} - \tan^{-1}(a) \right] + \lim_{b\to\infty} \left[\tan^{-1}(b) - \cancel{\tan^{-1}(1)} \right]$$

Eval limit
$$= -0 + \frac{\pi}{2} = \frac{\pi}{2}$$

~~Integral Diverges~~
Integral Converges to $\pi/2$

Finish D3 #6

Unbounded Integrals

Improper Integrals with an Unbounded Integrand

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx, \qquad f \text{ is continuous on } (a,\,b] \text{ with } \lim_{x\to a^+} f(x) = \pm\infty$$

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx, \qquad f \text{ is continuous on } [a,\,b) \text{ with } \lim_{x\to b^-} f(x) = \pm\infty$$

$$\int_a^b f(x)\,dx = \lim_{c\to p^-}\int_a^c f(x)\,dx + \lim_{d\to p^+}\int_d^b f(x)\,dx, \quad f \text{ is continuous on } [a,\,b] \text{ except at } p \text{ where } f \text{ is unbounded}$$

If the limit exists, the improper integral **converges**; otherwise it **diverges**

EXAMPLE 8. **Integrals with unbounded integrands** Evaluate the following integrals or state that they diverge.

a) $\displaystyle\int_{-1/3}^{2} \frac{1}{\sqrt{3x+1}}\,dx$

discontinuous when $3x+1=0 \to x=-1/3$

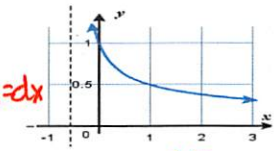
$u = 3x+1$

$du = 3dx \to \frac{1}{3}du = dx$

$x=2 \to u=7$

$x=a \to -\frac{1}{3}^+ \Rightarrow u=a \Rightarrow 3\left(\frac{-1}{3}\right)^+ + 1$

$u=a \to 0^+$



$$= \lim_{a\to -1/3^+}\int_a^2 \frac{1}{\sqrt{3x+1}}\,dx$$

$$= \lim_{a\to 0^+}\int_a^7 \frac{1}{\sqrt{u}}\cdot\frac{1}{3}\,du$$

$$= \lim_{a\to 0^+} \frac{1}{3}\,\frac{u^{1/2}}{1/2}\Big|_a^7 = \frac{2}{3}\lim_{a\to 0^+}\sqrt{u}\Big|_a^7$$

$$= \frac{2}{3}\lim_{a\to 0^+}\left[\sqrt{7}-\sqrt{a}\right]$$

$$= \frac{2}{3}\left[\sqrt{7}-0\right]$$

$$= \boxed{\frac{2\sqrt{7}}{3}}$$

~~Integral Diverges~~
Integral Converges to $\frac{2\sqrt{7}}{3}$

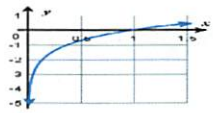
b) /53 $\displaystyle\int_0^1 \ln(x)\,dx$

grows w/o bound as $x\to 0$

Integration by Parts

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$



$$= \lim_{a\to 0^+}\int_a^1 \ln(x)\,dx$$

$$= \lim_{a\to 0^+}\left[x\ln x\Big|_a^1 - \int_a^1 dx\right] = \lim_{a\to 0^+}\left[x\ln x - x\right]_a^1$$

$$= \lim_{a\to 0^+}\left\{\left[\underbrace{\ln 1}_{0} - 1\right] - \left[\underbrace{a\ln a}_{0\cdot(-\infty)} - \underbrace{a}_{0}\right]\right\}$$

$$= -1 - \lim_{a\to 0^+}\frac{\ln(a)}{1/a} \leftarrow \frac{-\infty}{\infty} \text{ form}$$

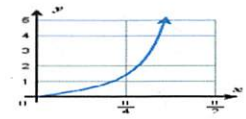
$$\overset{L'H}{=} -1 - \lim_{a\to 0^+}\frac{\frac{1}{a}}{-a^{-2}} = -1 + \lim_{a\to 0^+}\frac{a^2}{a}$$

$$= -1 + \lim_{a\to 0^+}(a) = -1+0 = \boxed{-1}$$

~~Integral Diverges~~
Integral Converges to -1

Start D4 #7

c) #39/41 $\int_0^{\pi/2} \sec x \tan x \, dx = \lim_{a \to \pi/2^-} \int_0^a \sec x \tan x \, dx$



$\sec \frac{\pi}{2}$ dne

$\tan \frac{\pi}{2}$ dne

$$= \lim_{a \to \pi/2^-} \left[\sec x \right]_0^a$$

$$= \lim_{a \to \pi/2^-} \left[\sec(a) - \sec(0) \right]$$

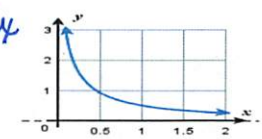
$$= \infty - 1$$

$$= \boxed{\infty}$$

Integral Diverges

~~Integral Converges to~~ _____

d) $\int_0^\infty \frac{1}{\sqrt{x}(1+x)} dx = \lim_{a \to 0^+} \int_a^1 \frac{1}{x^{1/2}(1+x)} dx + \lim_{b \to \infty} \int_1^b \frac{1}{x^{1/2}(1+x)} dx$



discontinuous at $x=0$

$u = x^{1/2} \rightarrow u^2 = x$

$du = \frac{1}{2} x^{-1/2} dx \rightarrow 2du = \frac{1}{\sqrt{x}} dx$

$x = a \to 0^+ \rightarrow u = a \to 0^+$

$x = 1 \rightarrow u = 1$

$x = b \to \infty \rightarrow u = b \to \infty$

$$= \lim_{a \to 0^+} \int_a^1 \frac{1}{1+u^2} 2du + \lim_{b \to \infty} \int_1^b \frac{1}{1+u^2} 2du$$

$$= \lim_{a \to 0^+} 2 \left[\tan^{-1}(u) \right]_a^1 + \lim_{b \to \infty} 2 \left[\tan^{-1}(u) \right]_1^b$$

$$= 2 \lim_{a \to 0^+} \left\{ \cancel{\tan^{-1}(1)} - \underbrace{\cancel{\tan^{-1}(a)}}_{0} \right\} + 2 \lim_{b \to \infty} \left\{ \tan^{-1}(b) - \cancel{\tan^{-1}(1)} \right\}$$

~~Integral Diverges~~

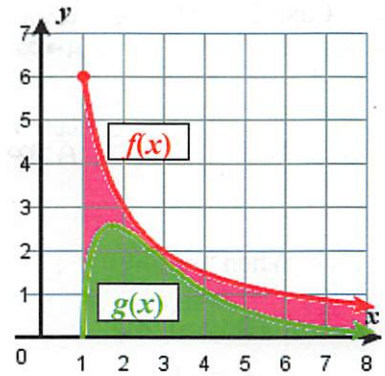
Integral Converges to π

$$= 2 \left[\frac{\pi}{2} \right] = \pi$$

A Comparison Test for Improper Integrals[1]

Sometime it is possible to determine whether an improper integral converges or not without having to find its exact value (which may be impossible!). It is useful information to know an improper integral exists before doing the work of approximating it!

EXAMPLE 9. Use the graphs to the right with $f(x) \geq g(x) \geq 0$ and $\boxed{k > 0}$ to help determine the answers to the following questions.



If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges

If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ is unknown (may or may not converge)

If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ is unknown

If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges

If $\int_a^\infty f(x)\,dx$ converges, then $\int_{a+k}^\infty f(x)\,dx$ converges

If $\int_a^\infty f(x)\,dx$ diverges, then $\int_{a+k}^\infty f(x)\,dx$ diverges

Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$

- If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent
- If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent

A similar theorem is true for integrals with unbounded integrands

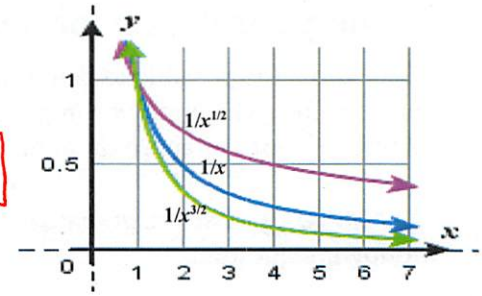
EXAMPLE 10. Determine the convergence or non-convergence of $\int_1^4 \frac{1+e^{-x}}{x}\,dx$. Explain.

$\frac{1+e^{-x}}{x}$
- is continuous on $[1, 4]$
- is defined on $[1, 4]$

$\Rightarrow$ The integral converges (to a finite real # value)

$\rightarrow$ no need to use the Comparison Theorem to determine if it converges.

EXAMPLE 11. For what values of p is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?



Case 1. $p=1$ $= \lim_{A\to\infty} \int_1^A \frac{1}{x}\,dx = \lim_{A\to\infty} \left[\ln|x|\right]_1^A$

$= \lim_{A\to\infty} \ln|A| - \ln|1| = \boxed{\infty}$ Divergent when $p=1$

Case 2. $p \neq 1$ $= \lim_{A\to\infty} \int_1^A \frac{1}{x^p}\,dx = \lim_{A\to\infty} \int_1^A x^{-p}\,dx$

$= \lim_{A\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^A = \frac{1}{1-p}\left[\lim_{A\to\infty} x^{-p+1}\right]_1^A = \boxed{\frac{1}{1-p}\lim_{A\to\infty}\left(A^{1-p}\right) - \frac{1}{1-p}}$

When $1-p<0$

$\Rightarrow p-1>0$

$\to p>1$

$= \frac{1}{1-p}\lim_{A\to\infty}\frac{1}{A^{p-1}} + \frac{1}{p-1}$

$= \frac{1}{1-p}\left[\lim_{A\to\infty}\frac{1}{A}\right]^{p-1} + \frac{1}{p-1} = \frac{1}{1-p}(0)^{p-1} + \frac{1}{p-1}$

$= 0 + \frac{1}{p-1} = \boxed{\frac{1}{p-1}}$ Converges when $p>1$

When $1-p>0$

$= \frac{1}{1-p}\left(\lim_{A\to\infty} A\right)^{1-p} + \frac{1}{p-1}$

$= \frac{1}{1-p}(\infty)^{1-p} + \frac{1}{p-1} = \boxed{\infty}$ Divergent when $p<1$

Conclusion: $\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$ is convergent when $p>1$. $\int_1^\infty \frac{1}{x^p}\,dx$ is divergent when $p \le 1$

This result is useful when trying to determine if an improper integral is convergent. For example, determine the convergence or divergence of the integrals below:

a) $\int_4^\infty x^{-5}\,dx$ $= \int_4^\infty \frac{1}{x^5}\,dx < \int_1^\infty \frac{1}{x^5}\,dx$ $\boxed{\text{converges}}$ b/c $p=5>1$

b) $\int_2^\infty \frac{1}{(2x-3)^4}\,dx$ $= \int_1^\infty \frac{1}{2u^4}\,du < \int_1^\infty \frac{1}{u^4}\,du$ $\boxed{\text{converges}}$ b/c $p=4>1$

$u = 2x-3$ $du = 2dx$

$x=2 \to u=1$ $x=\infty \to u=\infty$

c) $\int_1^\infty \frac{1}{x}\,dx$ $\boxed{\text{diverges}}$ b/c $p=1$ (so $p \not> 1$)

d) $\int_8^\infty \frac{1}{x^{2/3}}\,dx$ $\boxed{\text{diverges}}$ b/c $\int_1^\infty \frac{1}{x^{2/3}}\,dx$ diverges b/c $p = 2/3 \not> 1$

and if $\int_A^\infty f(x)\,dx$ diverges, so does $\int_{A+B}^\infty f(x)\,dx$, $B>0$

How to use the Comparison Test

Step 1: Confirm the integrand is positive and continuous on the interval described by the limits of integration
Step 2: Compare the integrand with the integrand (like $1/x^p$) of a known convergent or divergent integral
Step 3: State the convergence or divergence of the known integral
Step 4: State the conclusion that can be reached about the unknown (given) integral.

EXAMPLE 13. Comparison Test for Improper Integrals Use the Comparison test to show the convergence

a) $\int_3^\infty \frac{1+e^{-x}}{x}\,dx$

① $\frac{1+e^{-x}}{x}$ for $x \geq 3$ is $\begin{cases} C & \text{continuous} \checkmark \\ P & \text{positive} \checkmark \end{cases}$

② $\frac{1+e^{-x}}{x} = \frac{1}{x} + \frac{e^{-x}}{x} > \frac{1}{x}$

③ $\int_1^\infty \frac{1}{x}\,dx$ diverges b/c $p=1 \not> 1$; ($\int_3^\infty \frac{1}{x}\,dx$ also diverges)

④ $\int_3^\infty \frac{1+e^{-x}}{x}\,dx > \int_3^\infty \frac{1}{x}\,dx$, a known divergent integral
Therefore the given integral, which is larger, also $\boxed{\text{diverges}}$

b) $\int_1^\infty \frac{x^2}{x^5+3}\,dx$

① $\frac{x^2}{x^5+3}$ for $x \geq 1$ is $\begin{cases} \text{Continuous} \checkmark \\ \text{Positive} \checkmark \end{cases}$

② $\frac{x^2}{x^5+3} < \frac{x^2}{x^5} = \frac{1}{x^3}$
Equal numerators, larger denom. makes a smaller fraction

③ $\int_1^\infty \frac{1}{x^3}\,dx$ converges b/c $p=3>1$

④ $0 \leq \int_1^\infty \frac{x^2}{x^5+3}\,dx \leq \int_1^\infty \frac{1}{x^3}\,dx$, a known convergent integral. Therefore the given integral, which is smaller, also $\boxed{\text{converges}}$

c) $\int_1^\infty e^{-x^2}\,dx$

① e^{-x^2} for $x \geq 1$ is $\begin{cases} \text{Continuous} \checkmark \\ \text{Positive} \checkmark \end{cases}$

② $e^{-x^2} = \frac{1}{e^{x^2}} < \frac{1}{x}$ b/c $e^{x^2} > x$ and a larger denom. makes a smaller fraction (numerators the same)

③ $\int_1^\infty \frac{1}{x}\,dx$ diverges b/c $p=1 \not> 1$

④ $0 \leq \int_1^\infty e^{-x^2}\,dx \leq \int_1^\infty \frac{1}{x}\,dx$
The given integral is smaller than a known divergent integral, but $\boxed{\text{no conclusion}}$ can be made about it.

TRY again

② $e^{-x^2} = \frac{1}{e^{x^2}} < \frac{1}{x^2}$ b/c $e^{x^2} > x^2$

③ $\int_1^\infty \frac{1}{x^2}\,dx$ converges b/c $p=2>1$

④ $0 \leq \int_1^\infty e^{-x^2}\,dx \leq \int_1^\infty \frac{1}{x^2}\,dx$, a known convergent integral
so the given integral, which is smaller, also $\boxed{\text{converges}}$

Finish D4 #8

[1]Stewart James. Calculus: Concepts and Contexts – Single Variable. Pacific Grove, CA: Brooks/Cole Publishing Company, 1997.

10.1 An Overview

CO 6.1 Determine if a sequence converges or diverges, and if it converges, find its limit. [8.1]
CO 6.2 Given a series, calculate a sequence of its partial sums and construct an expression for the nth partial sum when possible [8.1]

Sequence an **ordered list** of numbers which is formed according to some rule or law

A sequence is denoted $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ or $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Each number is a **term**

The subscript is an **index** indicating the position of the term in the sequence

The first term in a sequence is typically 0 or 1

Explicit formula a way of defining a sequence by defining a formula for each term as a function of its position in the sequence: $a_n = f(n)$

EXAMPLE 14. Explicit formulas Write the first four terms of the following sequences

a) $\{5n\}_{n=0}^{\infty}$

$n=0$: $5(0)$, $n=1$: $5(1)$, $n=2$: $5(2)$, $n=3$: $5(3)$

$\boxed{\{0, 5, 10, 15, \ldots\}}$

b) #12 $\{2+(-1)^n\}_{n=1}^{\infty}$

$n=1$: $2+(-1)^1$, $n=2$: $2+(-1)^2$, $n=3$: $2+(-1)^3$, $n=4$: $2+(-1)^4$

$2-1$, $2+1$, $2-1$, $2+1$

$\boxed{\{1, 3, 1, 3, \ldots\}}$

c) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$

$n=1$: $\frac{1}{1+1}$, $n=2$: $\frac{2}{2+1}$, $n=3$: $\frac{3}{3+1}$, $n=4$: $\frac{4}{4+1}$

$\boxed{\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}}$

Recurrence relation a way of defining a sequence by defining a starting value and a formula for the next term as a function of previous terms in the sequence: $a_n = f(a_{n-1})$.
(The position of the term in the sequence cannot be used in the recurrence formula!)

EXAMPLE 15. Recurrence relations Write the first four terms of the following sequences

a) $a_1 = 0$, $a_{n+1} = a_n + 5$

$a_1 = 0$
$a_2 = 0 + 5 = 5$
$a_3 = 5 + 5 = 10$
$a_4 = 10 + 5 = 15$

$\boxed{\{0, 5, 10, 15, \ldots\}}$

b) #22 Fibonacci Sequence: $a_1 = 1$, $a_2 = 1$, $a_{n+1} = a_n + a_{n-1}$, $n \geq 3$

$a_1 = 1$
$a_2 = 1$
$a_3 = 1 + 1 = 2$
$a_4 = 2 + 1 = 3$
$a_5 = 3 + 2 = 5$
$a_6 = 5 + 3 = 8$

$\boxed{\{1, 1, 2, 3, 5, 8, \ldots\}}$

Start D5 #9

EXAMPLE 16. **Recurrence relations** Write the next two terms of each sequence and find a recurrence relation and an explicit formula for the sequence.

a) $\left\{4, -2, 1, -\frac{1}{2}, \cdots\right\}$

next 2 terms : $\boxed{+\frac{1}{4}}$, $\boxed{\frac{-1}{8}}$

Recurrance Relation (a function of previous terms)

$$\boxed{\begin{cases} a_1 = 4 \\ a_{n+1} = -a_n\left(\frac{1}{2}\right) \end{cases}}$$

Explicit formula (a function of n)

n	a_n
1	4
2	$[4](-\frac{1}{2})$
3	$[4(-\frac{1}{2})](-\frac{1}{2})$
4	$[4(-\frac{1}{2})(-\frac{1}{2})](-\frac{1}{2})$ $= 4\frac{(-1)^3}{2^3}$
5	$[4(\frac{-1}{2})(\frac{-1}{2})(\frac{-1}{2})](\frac{-1}{2})$ $= 4\frac{(-1)^4}{2^4}$
n	$4\frac{(-1)^{n-1}}{2^{n-1}}$

$$\boxed{\left\{4\left(\frac{-1}{2}\right)^{n-1}\right\}_{n=1}^{\infty}}$$

b) {-1, 3, -5, 7, -9, …}

next 2 terms: $\boxed{11}$, $\boxed{-13}$

Recurrance Relation (in terms of a_n)

$$\boxed{\begin{cases} a_1 = -1 \\ a_{n+1} = \frac{|a_n|}{-a_n}\left[|a_n| + 2\right] \end{cases}}$$

alternating sign

$-\frac{|-1|}{-1} = +1$; $-\frac{|3|}{3} = -1$

Explicit formula (in terms of n)

n	a_n
1	-1
2	$(-1)^2\ [3]$
3	$(-1)^3\ [5]$
4	$(-1)^4\ [7]$
n	$(-1)^n(1+2^{n-1})$

$3 = 1+2 = 1 + 2(1)$

$5 = 1+2+2 = 1 + 2(2)$

$7 = 1+2+2+2 = 1 + 2(3)$

$\boxed{\{(-1)^n(1+2(n-1))\}_1^{\infty}}$ or $\boxed{\{(-1)^n(2n-1)\}_1^{\infty}}$

Now do D5 #10

Limit of a Sequence

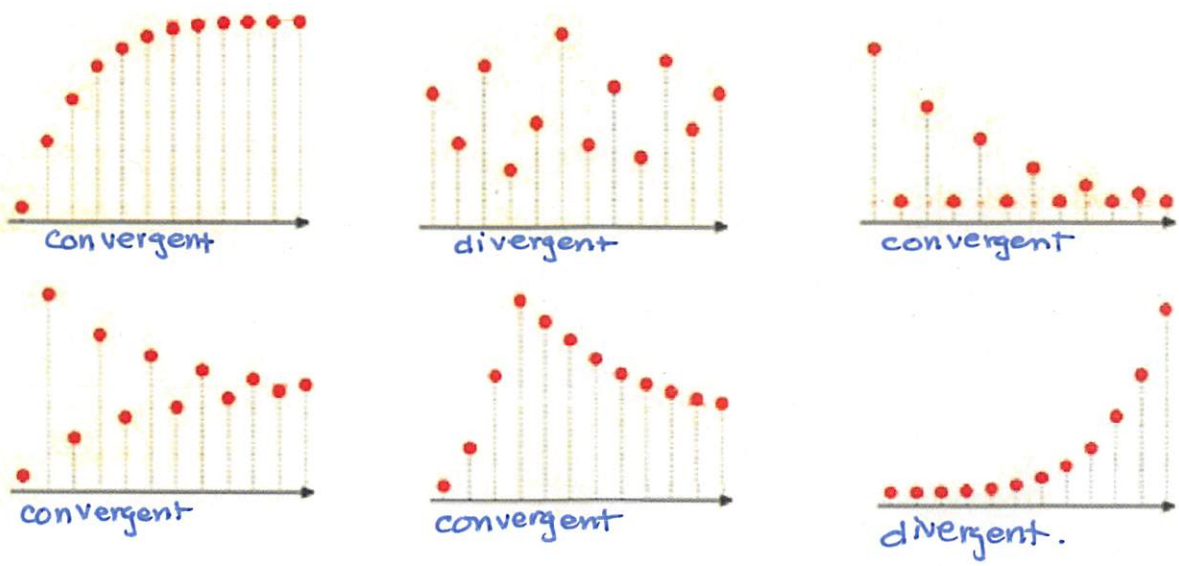
The primary focus of this section and the next concerns whether or not a sequence has terms that approach a limiting finite real number value. If the terms in the sequence converge to a single finite real number value, the sequence is said to **converge**.

Convergent sequence — A sequence whose terms eventually approach a single finite real number value. The terms of a sequence approach a unique real number L as n increases.

Divergent sequence — A sequence whose terms do NOT eventually approach a single finite real number. Instead, the terms in the sequence may oscillate forever, or their magnitude may grow without bound

Limit of a sequence — the long-term behavior of the terms, a_n, in a sequence
When $\lim_{n\to\infty} a_n$ does not exist, the terms don't approach a single value and the **sequence diverges**. When $\lim_{n\to\infty} a_n$ does exist, then $\lim_{n\to\infty} a_n = L$ which means that the terms approach the value L, and the **sequence converges to L**

EXAMPLE 17. The terms of some sequences are graphed below. Determine which sequences are convergent.



10.2 Sequences

Limit of a Sequence and Limit Laws

Limits of sequences from Limits of Functions

Given $a_n = f(n)$ for function f and all positive integers n,
If $\lim_{n\to\infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L.

If a function has a limit, then its corresponding sequence has the same limit. (Note however that the converse is not true – for example, $\{\cos(2\pi n)\}$ converges, but its corresponding function $f(x)=\cos(2\pi x)$ does not.)

Algebraic Properties of Convergent Sequences

Given convergent sequences $\{a_n\}$ and $\{b_n\}$.

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(ca_n) = c \lim_{n\to\infty} a_n$
3. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
4. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, provided $\lim_{n\to\infty} b_n \neq 0$

In general, nothing can be said about the convergence properties of a sequence $\{a_n b_n\}$ if one of the sequences $\{a_n\}$ or $\{b_n\}$ diverges. For example,

- If $a_n = (-1)^n$ and $b_n=(-1)^{n+1}$, then both $\{a_n\}$ and $\{b_n\}$ diverge, but the sequence $\{a_n b_n\}=\{-1\}$ converges.
- If $a_n = n$ and $b_n = \frac{1}{n^2}$, then $\{a_n\}$ diverges, $\{b_n\}$ converges to 0, but the sequence $\{a_n b_n\}=\{\frac{1}{n}\}$ converges.
- If $a_n = n^2$ and $b_n = \frac{1}{n}$, then $\{a_n\}$ diverges, $\{b_n\}$ converges to 0, but the sequence $\{a_n b_n\}=\{n\}$ diverges.

EXAMPLE 18. Limits of sequences Find the limit as n approaches infinity of the following sequences or determine that the limit does not exist.

Technique: Simplify

a) $\left\{\frac{3^{n+1}+3}{3^n}\right\}_{n=0}^{\infty}$

$$\lim_{n\to\infty}\left(\frac{3^{n+1}+3}{3^n}\right) = \lim_{n\to\infty}\left(\frac{3^{n+1}}{3^n}+\frac{3}{3^n}\right) = \lim_{n\to\infty}\left(3+\frac{1}{3^{n-1}}\right)$$

$$= \lim_{n\to\infty} 3 + \lim_{n\to\infty}\left(\tfrac{1}{3}\right)^{n-1} = 3+0 = 3$$

(the terms in the sequence approach 3)

Limit of sequence 3

~~the limit does not exist~~

Technique: Use L'Hopital's Rule

c) $\left\{\sqrt{\left(\frac{3n}{3n+1}\right)^n}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty}\sqrt{\left(\frac{3n}{3n+1}\right)^n} = \lim_{n\to\infty}\left(\frac{3n}{3n+1}\right)^{n/2} = \lim_{n\to\infty}\left(\frac{3n+1}{3n}\right)^{-\frac{n}{2}} = \lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^{-n/2} = 1^{-\infty}$$

$$= e^{\boxed{\lim_{n\to\infty}\ln\left(1+\frac{1}{3n}\right)^{-n/2}}} \to \lim_{n\to\infty} -\frac{n}{2}\ln\left(1+\frac{1}{3n}\right) = -\left(\frac{1}{2}\right)\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{3n}\right)}{1/n} = \boxed{\frac{0}{0}}$$

$$\overset{L'H}{=} -\frac{1}{2}\lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{3n}}\left[\frac{1}{3}\cancel{(-n^{-2})}\right]}{\cancel{(-n^{-2})}} = -\frac{1}{2}\left(\frac{1}{3}\right)$$

$$= e^{\boxed{-1/6}}$$

Limit of sequence $e^{-1/6}$

~~the limit does not exist~~

Technique: Multiply by a "convenient 1" - the reciprocal of the highest power in the denominator

b) $\left\{\frac{3n^5+n^2+1}{n^6+4n^3+2n}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty}\frac{3n^5+n^2+1}{n^6+4n^3+2n} = \lim_{n\to\infty}\frac{3n^5+n^2+1}{n^6+4n^3+2n}\cdot\left[\frac{1/n^6}{1/n^6}\right]$$

Could use L'H but this method is faster.

$$= \lim_{n\to\infty}\frac{\frac{3}{n}+\frac{1}{n^4}+\frac{1}{n^6}}{1+\frac{4}{n^3}+\frac{2}{n^5}} = \frac{0+0+0}{1+0+0} = \boxed{0}$$

Limit of sequence 0

~~the limit does not exist~~

Finish D5 #11

e) $\left\{\dfrac{2-\sqrt{n^2+1}}{n+3}\right\}_{n=0}^{\infty}$

$$\lim_{n\to\infty}\frac{2-\sqrt{n^2+1}}{n+3}\overset{\boxed{\frac{-\infty}{\infty}}}{\underset{L'H}{=}}\lim_{n\to\infty}\frac{-\frac{1}{\cancel{2}}(n^2+1)^{-1/2}[\cancel{2}n]}{1}=\boxed{\lim_{n\to\infty}\frac{n}{(n^2+1)^{1/2}}}\overset{\boxed{\frac{\infty}{\infty}}}{\underset{L'H}{=}}$$

$$=\lim_{n\to\infty}\frac{1}{\frac{1}{\cancel{2}}(n^2+1)^{-1/2}[\cancel{2}n]}=\lim_{n\to\infty}\frac{(n^2+1)^{1/2}}{n}\overset{\frac{\infty}{\infty}}{\underset{L'H}{=}}\lim_{n\to\infty}\frac{\frac{1}{\cancel{2}}(n^2+1)^{-1/2}[\cancel{2}n]}{1}=\boxed{\lim_{n\to\infty}\frac{n}{(n^2+1)^{1/2}}}$$

circular !!

L'Hopital does not always work! multiply by a convenient 1 instead:

$$\lim_{n\to\infty}\frac{2-\sqrt{n^2+1}}{n+3}\boxed{\frac{1/n}{1/n}}=\lim_{n\to\infty}\frac{\frac{2}{n}-\frac{\sqrt{n^2+1}}{\sqrt{n^2}}}{\frac{n}{n}+\frac{3}{n}}=\lim_{n\to\infty}\frac{\frac{2}{n}-\sqrt{1+\frac{1}{n^2}}}{1+\frac{3}{n}}=\frac{0-\sqrt{1+0}}{1+0}$$

$$=\boxed{-1}$$

Limit of sequence $\boxed{-1}$

~~the limit does not exist~~

Technique for Radical expressions: Multiply by the conjugate

f) $\left\{n-\sqrt{n^2+5n}\right\}_{n=1}^{\infty}$

$\lim_{n\to\infty} n-\sqrt{n^2+5n} = \infty-\infty$ indeterminate. Must Rewrite first!

$$=\lim_{n\to\infty} n-\sqrt{n^2+5n}\left[\frac{n+\sqrt{n^2+5n}}{n+\sqrt{n^2+5n}}\right]=\lim_{n\to\infty}\frac{n^2-(n^2+5n)}{n+\sqrt{n^2+5n}}=\lim_{n\to\infty}\frac{-5n}{n+\sqrt{n^2+5n}}$$

$$=\lim_{n\to\infty}\frac{-5n}{n+\sqrt{n^2+5n}}\left[\frac{1/n}{1/n}\right]=\lim_{n\to\infty}\frac{-5}{1+\sqrt{\frac{n^2+5n}{\sqrt{n^2}}}}=\lim_{n\to\infty}\frac{-5}{1+\sqrt{1+\frac{5}{n}}}=\frac{-5}{1+1}$$

Limit of sequence $\boxed{-\frac{5}{2}}$

~~the limit does not exist~~

Technique: Use limits of a composition and trig limits

g) $\left\{\ln\left(2\sin\left(\frac{1}{n}\right)\right)+\ln(n)\right\}_{n=1}^{\infty}$

$\lim_{n\to\infty}\ln\left(2\sin\frac{1}{n}\right)+\ln(n)=\ln(2\sin(0))+\ln(\infty)=\ln(0)+\ln(\infty)=-\infty+\infty$ indeterminate must Rewrite

$=\lim_{n\to\infty}\ln\left(\left(2\sin\frac{1}{n}\right)\cdot\ln\right)=\ln(0\cdot\infty)$ indeterminate. must Rewrite

limit of a composition when outside function is continuous.

$$=\lim_{n\to\infty}\ln\left(\frac{2\sin\frac{1}{n}}{\frac{1}{n}}\right)=\ln\lim_{n\to\infty}\frac{2\sin 1/n}{1/n}\overset{\boxed{\frac{0}{0}}}{\underset{L'H}{=}}$$

Start D6 #12

$$=\ln\lim_{n\to\infty}\frac{2\cos(1/n)\cancel{[-n^{-2}]}}{\cancel{[-n^{-2}]}}=\ln\left(2\cos\left(\frac{1}{\infty}\right)\right)$$

$$=\ln(2\cos(0))=\ln(2(1))=\ln 2$$

Limit of sequence $\ln 2$

~~the limit does not exist~~

Caution

g) $\left\{\dfrac{\sin x}{n}\right\}_{n=1}^{\infty}$

x is not a variable from the perspective of n

$$\lim_{n\to\infty}\frac{\sin x}{n}=\sin x\lim_{n\to\infty}\frac{1}{n}=$$

$$=\sin x(0)=0$$

Limit of sequence 0

~~the limit does not exist~~

Geometric sequences

Monotonic sequence — terms are only **non-increasing** ($a_{n+1} \le a_n$) or only **non-decreasing** ($a_{n+1} \ge a_n$)

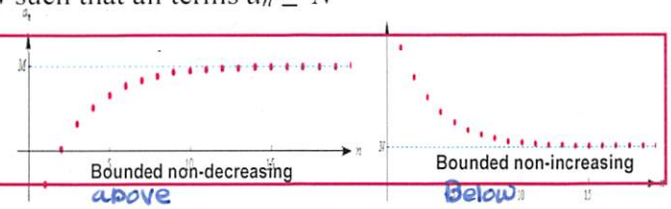
Sequence bounded above — there exists a number M such that all terms $a_n \le M$

Sequence bounded below — there exists a number N such that all terms $a_n \ge N$

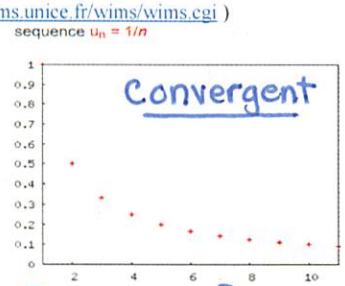
Bounded Monotonic Sequences

A bounded monotonic sequence **converges**.

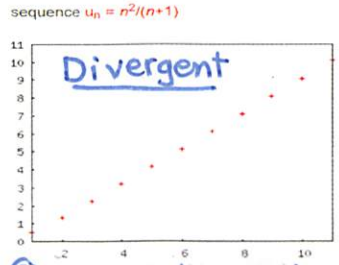
① ②



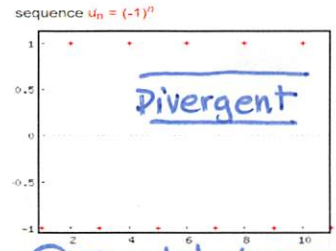
EXAMPLE 19. Determine whether the sequence is bounded, monotonic and/or convergent (http://wims.unice.fr/wims/wims.cgi)



① Bounded Below
② Non-increasing



② non-decreasing BUT
× NOT Bounded above.



② Bounded above and below
× But NOT non decreasing
nor non increasing

We are already familiar with limits of the form x^n where the base x is increasing without bound and the exponent n is a positive constant.

when n is an even integer: $\lim_{x \to \pm\infty} x^n = \infty$

when n is an odd integer: $\lim_{x \to \infty} x^n = \infty$ and $\lim_{x \to -\infty} x^n = -\infty$

The limit of n^x is examined below. Although similar, this time the base is a positive constant and it is the exponent that is increasing without bound.

the variable is in the exponent

Geometric Sequences, $\{ar^n\}$ — each term is obtained by multiplying the previous term by a fixed constant r, called the **ratio**

$$\lim_{n \to \infty}(ar^n) = a \lim_{n \to \infty}(r^n)$$

$$\lim_{n \to \infty}(r^n) = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ dne & \text{if } r \le -1 \text{ or } r > 1 \end{cases}$$

The sequence is **monotonic** if $r \ge 0$ The sequence **oscillates** if $r < 0$

Proof:

If $r = 0$ $\quad \lim_{n \to \infty} 0^n = 0$

If $0 < |r| < 1$ $\quad$ Let $|r| = c$. then for some $a < 0$, $c = \frac{1}{1-a}$

$|r|^n = |\frac{1}{1-a}|^n = \frac{1}{1-an+\ldots}$ (by the Binomial Theorem) $< \frac{1}{1-an}$

$0 < |r|^n < \frac{1}{1-an}$. Taking the limit of all three parts: $\lim_{n \to \infty} 0 < \lim_{n \to \infty} |r|^n < \lim_{n \to \infty} \frac{1}{1-an}$.

Because $\lim_{n \to \infty} 0 = 0$ and $\lim_{n \to \infty} \frac{1}{1-an} = 0$ then $\lim_{n \to \infty} |r|^n = 0$ by the Squeeze Theorem

If $r = 1$ $\quad \lim_{n \to \infty} 1^n = 1$

If $r > 1$ $\quad$ then $0 < \frac{1}{r} < 1$. $\lim_{n \to \infty} r^n = \lim_{n \to \infty} \frac{1}{\left(\frac{1}{r}\right)^n} = \frac{1}{\lim_{n \to \infty} \left(\frac{1}{r}\right)^n} = \frac{1}{0} = \infty$

If $r = -1$ $\quad \lim_{n \to \infty} (-1)^n = dne$

or Have the form $\{ar^n\}_{n=0}^{\infty}$

EXAMPLE 20. **Geometric sequences** Determine whether the following sequences converge or diverge. State whether they are monotonic or they oscillate. Find the limit when the sequence converges.

a) $\left\{\left(\frac{8}{9}\right)^n\right\}_{n=0}^{\infty}$

☑ Geometric Sequence: $a =$ 1, $r =$ $\frac{8}{9}$

{Converges to 0 / ~~Diverges~~} because $|r|<1$
Choose one

{is Monotonic / ~~Oscillates~~} because $r>0$
Choose one

b) $\left\{\frac{1}{2}\left(\frac{8}{3}\right)^n\right\}_{n=0}^{\infty}$

☑ Geometric Sequence: $a =$ $\frac{1}{2}$, $r =$ $\frac{8}{3}$

{~~Converges to~~ ____ / Diverges} because $|r| \not< 1$
Choose one

{is Monotonic / ~~Oscillates~~} because $r>0$
Choose one

c) $\left\{(-.2)^n\right\}_{n=0}^{\infty}$

☑ Geometric Sequence: $a =$ 1, $r =$ $-.2$

{Converges to 0 / ~~Diverges~~} because $|r|<1$
Choose one

{~~is Monotonic~~ / Oscillates} because $r<0$
Choose one

d) $\left\{e^{2+n}\left(\ln\frac{1}{3}\right)^{-n}\right\}_{n=0}^{\infty}$ $= \left\{\frac{e^2 e^n}{(\ln\frac{1}{3})^n}\right\}_0^{\infty}$

$= \left\{e^2\left(\frac{e}{\ln 1/3}\right)^n\right\}_0^{\infty}$

☑ Geometric Sequence: $a =$ e^2, $r =$ $\frac{e}{\ln 1/3} \approx -2.47$

{~~Converges to~~ ____ / Diverges} because $|r| \not< 1$
Choose one

{~~is Monotonic~~ / Oscillates} because $r<0$
Choose one

Finish D6 #13

10.3 Infinite Series

CO 6.3 Determine the convergence or divergence of a telescoping series by evaluating the limit of the nth partial sum, and give its sum if it converges.
CO 6.4 Recognize geometric series, determine their convergence or divergence, and find their sum if convergent.

Know the rules of exponents!

	Product Rule	Quotient Rule	P/Q extensions ('Like power' rules)		Power Rule
Rules:	$a^{x+y} = a^x a^y$	$a^{x-y} = \frac{a^x}{a^y}$	$(ab)^x = a^x b^x$	$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$	$(a^x)^y = a^{xy}$
Pitfalls:	$a^{x+y} \neq a^x + a^y$	$a^{x-y} \neq a^x - a^y$	$(ab)^x \neq a^x + b^x$	$\left(\frac{a}{b}\right)^x \neq a^x - b^x$	$(a^x)^y \neq a^{x^y}$

A **sequence** is an ordered list of numbers; an **infinite series** is an infinite sum of numbers

EXAMPLE 21. Evaluate each partial sum.

a) $\sum_{k=0}^{4} 3^k = 3^0 + 3^1 + 3^2 + 3^3 + 3^4 = 1+3+9+27+81 = \boxed{121}$

b) $\sum_{k=0}^{4} (-3)^k = (-3)^0 + (-3)^1 + (-3)^2 + (-3)^3 + (-3)^4 = 1 - 3 + 9 - 27 + 81 = \boxed{61}$

c) $\sum_{k=3}^{6} 3^k = 3^3 + 3^4 + 3^5 + 3^6 = \underbrace{3^3(1+3+3^2+3^3)}_{} = 27(1+3+9+27)$
$= 27(40) = \boxed{1080}$

Rewrite this partial sum in the form $a\sum_{k=0}^{n} r^k$

$= 3^3 \sum_{k=0}^{6-3=3} 3^k$

Infinite series, $\sum_{k=1}^{\infty} a_k$ The sum of the terms of a sequence $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$

Sequence of partial sums, $\{S_n\}$ where $S_n = \sum_{k=1}^{n} a_k$ for $n = 1, 2, 3, \ldots$

If the sequence of partial sums has a limit, L $\lim_{n\to\infty} S_n = L$

then the value of the infinite series is also L $\lim_{n\to\infty} \sum_{k=1}^{n} a_k = L$

If the sequence of partial sums doesn't have a limit, the infinite series diverges

$\{S_n\} = \{S_1, S_2, S_3, S_4, \ldots, S_n, \ldots\}$ limit of sequence $= \lim_{n\to\infty} S_n$

$= \{a_1, a_1+a_2, a_1+a_2+a_3, a_1+a_2+a_3+a_4, \ldots, \sum_{k=1}^{n} a_k, \ldots\}$ $= \lim_{n\to\infty}\left(\sum_{1}^{n} a_k\right)$

If a formula for the sequence of partial sums can be found,
then the value of the infinite series is the limit of that sequence of partial sums.

Geometric Sums and Series

Geometric Series A series of the form $\sum_{k=0}^{\infty} ar^k = a\sum_{k=0}^{\infty} r^k$.

Show that the *nth* partial geometric sum S_n is $S_n = a\dfrac{1-r^n}{1-r}$

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$
$$r\,S_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n$$
$$S_n - rS_n = a - ar^n \rightarrow \frac{S_n\cancel{(1-r)}}{\cancel{1-r}} = \frac{a(1-r^n)}{1-r} \rightarrow S_n = \frac{a(1-r^n)}{1-r}$$

The value of the geometric series $\sum_{k=0}^{\infty} ar^k$ is the limit of its sequence of partial sums. Symbolically

$$\sum_{k=0}^{\infty} ar^k = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(a\frac{1-r^n}{1-r}\right) = a\,\frac{1-\lim_{n\to\infty}(r^n)}{1-r}$$

Find this limit for three cases: $|r| < 1$, $|r| > 1$, and $|r| = 1$,

Case 1. $|r| < 1$ $\Rightarrow \lim_{n\to\infty}(r^n) = 0 \Rightarrow \sum_{k=0}^{\infty} ar^k = \boxed{a\left(\frac{1}{1-r}\right)}$

Case 2. $|r| > 1$ $\Rightarrow \lim_{n\to\infty}(r^n) = \text{dne} \Rightarrow \sum_{k=0}^{\infty} ar^k = \text{dne}$

Case 3. $|r| = 1$ $\rightarrow r = 1 \rightarrow \lim_{n\to\infty} 1^n = 1 \Rightarrow \sum_{k=0}^{\infty} a = \infty$

$r = -1 \rightarrow \lim_{n\to\infty}(-1)^n \text{ dne} \Rightarrow \sum_{k=0}^{\infty} a(-1)^k = \text{dne}$

A **geometric sequence** nas the form $\{ar^n\}$

A **geometric series** with ratio r has the form $\sum_{k=0}^{\infty} ar^k$ $\begin{cases} \text{If } |r| \not< 1 \text{ the series diverges} \\ \text{If } |r| < 1, \text{ then the series converges to the sum } \dfrac{a}{1-r} \end{cases}$

To be able to use this formula, you must be in the correct form:
k starts at 0 and the exponent on r must be k

EXAMPLE 22. Geometric series Evaluate each geometric series or state that it diverges.

a) #21/25 $\sum_{k=0}^{\infty} 0.9^k$

☑ Geometric Series $a =$ 1, $r =$ .9

Converges because: $|r| < 1$

Diverges

Value of Series:

$$\frac{1}{1-r} = \frac{1}{1-.9} = \frac{1}{.1} = \boxed{10}$$

b) #27/31~ $\sum_{k=0}^{\infty} 2^{-3k}$ $= \sum_{0}^{\infty} (2^{-3})^k$

$= \sum_{0}^{\infty} \left(\frac{1}{8}\right)^k$

☑ Geometric Series $a = 1$, $r = \frac{1}{8}$

Converges because: $|r| < 1$

~~Diverges~~

Value of Series:

$\frac{1}{1-r} = \frac{1}{1-\frac{1}{8}} = \boxed{\frac{8}{7}}$

c) #35/23 $\sum_{k=0}^{\infty} \left(-\frac{9}{10}\right)^k$

☑ Geometric Series $a = 1$, $r = \frac{-9}{10}$

Converges because: $|r| < 1$

~~Diverges~~

Value of Series:

$= \frac{1}{1-r} = \frac{1}{1-\frac{-9}{10}} = \frac{10}{10+9} = \boxed{\frac{10}{19}}$

d) $1 + \ln\left(\frac{1}{4}\right) + \left(\ln\left(\frac{1}{4}\right)\right)^2 + \left(\ln\left(\frac{1}{4}\right)\right)^3 + \left(\ln\left(\frac{1}{4}\right)\right)^4 + \cdots$

$= \sum_{k=0}^{\infty} \left(\ln \frac{1}{4}\right)^k$

☑ Geometric Series $a = 1$, $r = \ln \frac{1}{4} \approx -1.386$

~~Converges~~

Diverges because: $|r| \not< 1$

Value of Series: n/a $\boxed{\text{dne}}$

$6\left(\frac{3}{4}\right) = \frac{18}{4}$ $\times\left(\frac{3}{4}\right)$ $\times\left(\frac{3}{4}\right)$

e) $6 + \frac{9}{2} + \frac{27}{8} + \frac{81}{32} + \frac{243}{128} + \cdots$

$= 6 + 6\left(\frac{3}{4}\right) + 6\left(\frac{3}{4}\right)^2 + 6\left(\frac{3}{4}\right)^3 + \cdots$

$= \sum_{k=0}^{\infty} 6\left(\frac{3}{4}\right)^k$

$= 6 \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k$

☑ Geometric Series $a = 6$, $r = \frac{3}{4}$

Converges because: $|r| < 1$

~~Diverges~~

Value of Series:

$a\left(\frac{1}{1-r}\right) = 6\left(\frac{1}{1-\frac{3}{4}}\right)$

$= 6\left(\frac{4}{4-3}\right) = \boxed{24}$

f) #33/75 $\sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k 5^{3-k}$

$$= \sum_{1}^{\infty} \frac{5^3\, 5^{-k}}{4^k}$$

$$= \sum_{1}^{\infty} 5^3 \left(\frac{1}{4^k}\right)\left(\frac{1}{5^k}\right)$$

$$= 5^3 \sum_{k=1}^{\infty} \left(\frac{1}{4(5)}\right)^k = 5^3 \sum_{1}^{\infty} \left(\frac{1}{20}\right)^k$$

$$= 5^3 \left(\frac{1}{20}\right) \sum_{k=0}^{\infty} \left(\frac{1}{20}\right)^k = \boxed{\frac{25}{4} \sum_{0}^{\infty} \left(\frac{1}{20}\right)^k}$$

☐ Geometric Series $a = \frac{25}{4}$, $r = \frac{1}{20}$

Converges because: $|r| < 1$

~~Diverges~~

Value of Series: $\frac{25}{4}\left(\frac{1}{1-\frac{1}{20}}\right)$

$$= \frac{25}{4}\left(\frac{20}{19}\right)$$

$$= \boxed{\frac{125}{19}}$$

g) st#16 $\sum_{k=2}^{\infty} \frac{4^{k+1}}{5^k}$

$$= \sum_{k=2}^{\infty} \frac{4 \cdot 4^k}{5^k}$$

$$= \sum_{k=2}^{\infty} 4 \left(\frac{4}{5}\right)^k$$

$$= 4\left(\frac{4}{5}\right)^2 \sum_{k=0}^{\infty} \left(\frac{4}{5}\right)^k$$

$$= \boxed{\frac{64}{25} \sum_{k=0}^{\infty} \left(\frac{4}{5}\right)^k}$$

☑ Geometric Series $a = \frac{64}{25}$, $r = \frac{4}{5}$

Converges because: $|r| < 1$

~~Diverges~~

Value of Series:

$$\frac{64}{25}\left(\frac{1}{1-\frac{4}{5}}\right) = \frac{64}{25}\left(\frac{5}{5-4}\right)$$

$$= \boxed{\frac{64}{5}}$$

h) $\sum_{k=3}^{\infty} 2^{2k-1} 7^{2-k}$

$$= \sum_{k=3}^{\infty} 2^{2k}\, 2^{-1}\, 7^{2}\, 7^{-k}$$

$$= \sum_{k=3}^{\infty} \frac{2^{2k}(7^2)}{7^k(2)}$$

$$= \frac{49}{2} \sum_{k=3}^{\infty} \frac{(2^2)^k}{7^k}$$

$$= \frac{49}{2} \sum_{k=3}^{\infty} \left(\frac{4}{7}\right)^k$$

$$= \frac{49}{2}\left(\frac{4}{7}\right)^3 \sum_{k=0}^{\infty} \left(\frac{4}{7}\right)^k$$

$$= \boxed{\frac{32}{7} \sum_{k=0}^{\infty} \left(\frac{4}{7}\right)^k}$$

☑ Geometric Series $a = \frac{32}{7}$, $r = \frac{4}{7}$

Converges because: $|r| < 1$

~~Diverges~~

Value of Series:

Finish D7 #14

$$= \frac{32}{7}\left(\frac{1}{1-\frac{4}{7}}\right)$$

$$= \frac{32}{7}\left(\frac{7}{3}\right) = \boxed{\frac{32}{3}}$$

Properties of Convergent Series

Properties of Convergent Series

Given real numbers A, B and c:

- If $\sum a_k = A$ and c is a real number then $\sum (c \cdot a_k) = c \cdot \sum (a_k) = cA$
- If $\sum a_k =$ diverges then $\sum (c \cdot a_k)$ also diverges (for any real number $c \neq 0$)
- If $\sum a_k = A$ and $\sum b_k = B$ then $\sum (a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$
- If $\sum a_k =$ diverges and $\sum b_k = B$ then $\sum (a_k \pm b_k)$ also diverges
- If M is a positive integer, then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=M}^{\infty} a_k$ either both converge or both diverge

Adding or removing a finite number of terms

- Does not affect the convergence of a series
- Does affect the value of a convergent series

EXAMPLE 23. Properties of series Use the properties of infinite series to evaluate the following series

a) $\sum_{k=1}^{\infty} 3^{k+1} \pi^{-2k}$

$$= \sum_{1}^{\infty} \frac{3^k\, 3}{\pi^{2k}}$$

$$= 3 \sum_{1}^{\infty} \frac{3^k}{(\pi^2)^k} = 3 \sum_{k=1}^{\infty} \left(\frac{3}{\pi^2}\right)^k$$

$$= 3 \left(\frac{3}{\pi^2}\right) \sum_{k=0}^{\infty} \left(\frac{3}{\pi^2}\right)^k = \frac{9}{\pi^2} \sum_{k=0}^{\infty} \left(\frac{3}{\pi^2}\right)^k$$

$$= \frac{9}{\pi^2} \left(\frac{1}{1 - \frac{3}{\pi^2}}\right) = \frac{9}{\pi^2} \left(\frac{\pi^2}{\pi^2 - 3}\right) = \boxed{\frac{9}{\pi^2 - 3}}$$

b) #47/83 $\sum_{k=1}^{\infty} \left(\frac{1}{3}\left(\frac{5}{6}\right)^k + \frac{3}{5}\left(\frac{7}{9}\right)^k \right)$

$$= \sum_{k=1}^{\infty} \frac{1}{3}\left(\frac{5}{6}\right)^k + \sum_{k=1}^{\infty} \frac{3}{5}\left(\frac{7}{9}\right)^k$$

$$= \frac{1}{3}\left(\frac{5}{6}\right) \sum_{k=0}^{\infty} \left(\frac{5}{6}\right)^k + \frac{3}{5}\left(\frac{7}{9}\right) \sum_{k=0}^{\infty} \left(\frac{7}{9}\right)^k$$

$$= \frac{5}{18}\left(\frac{1}{1 - \frac{5}{6}}\right) + \frac{7}{15}\left(\frac{1}{1 - \frac{7}{9}}\right)$$

$$= \frac{5}{\cancel{18}_{3}}\left(\frac{\cancel{6}}{1}\right) + \frac{7}{\cancel{15}_{5}}\left(\frac{\cancel{9}^{3}}{2}\right)$$

$$= \frac{5}{3} + \frac{21}{10} = \frac{50 + 63}{30} = \boxed{\frac{113}{30}}$$

Now do D8 #15

Telescoping Series

Telescoping series a series whose partial sums have a fixed number of terms after cancellation

EXAMPLE 24. Telescoping series Find a formula for the nth term of the sequence of partial sums $\{S_n\}$. Then evaluate $\lim_{n\to\infty} S_n$ to obtain the value of the series or state that the series diverges.

a) $\sum_{k=1}^{\infty}\left(\frac{1}{k+2}-\frac{1}{k+3}\right)$

☑ Telescoping Series $S_n = \frac{1}{3} - \frac{1}{n+3}$

Formula for n^{th} partial sum

Converges ~~Diverges~~ because: $\lim_{n\to\infty} S_n$ exists

Value of Series: $\boxed{\frac{1}{3}}$

$$= \lim_{n\to\infty} \boxed{\sum_{k=1}^{n}\left(\frac{1}{k+2}-\frac{1}{k+3}\right)}$$

$$= \lim_{n\to\infty} \boxed{S_n}$$

$$S_n = \sum_{1}^{n}\left(\frac{1}{k+2}-\frac{1}{k+3}\right)$$

$k=1$ $k=2$ $k=n$

$$S_n = \left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots+\left(\frac{1}{n+2}-\frac{1}{n+3}\right)$$

$$\boxed{S_n = \frac{1}{3}-\frac{1}{n+3}}$$

$$= \lim_{n\to\infty}\boxed{\left(\frac{1}{3}-\frac{1}{n+3}\right)} = \frac{1}{3}-\frac{1}{\infty} = \frac{1}{3}-0 = \boxed{\frac{1}{3}}$$

b) $\sum_{k=1}^{\infty}\left(\frac{3}{k^2+3k}\right) = \sum_{k=1}^{\infty}\left(\frac{A}{k}+\frac{B}{k+3}\right)$

☑ Telescoping Series $S_n = 1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}$

Formula for n^{th} partial sum

Converges ~~Diverges~~ because: $\lim_{n\to\infty} S_n$ exists

Value of Series: $\boxed{\frac{11}{6}}$

Partial Fractions - identity equation

$$3 = A(k+3) + Bk$$

$k=0 \rightarrow 3 = A(3) \rightarrow A=1$

$k=-3 \rightarrow 3 = B(-3) \rightarrow B=-1$

$$= \sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+3}\right)$$

$$= \lim_{n\to\infty}\boxed{\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+3}\right)}$$

$$= \lim_{n\to\infty}\boxed{S_n}$$

$$S_n = \left(\frac{1}{1}-\frac{1}{4}\right)+\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{3}-\frac{1}{6}\right)$$
$$+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{5}-\frac{1}{8}\right)+\left(\frac{1}{6}-\frac{1}{9}\right)$$
$$+\cdots \quad +\cdots \quad +\cdots$$
$$+\cdots \quad +\cdots \quad +\cdots$$
$$+\left(\frac{1}{n-2}-\frac{1}{n+1}\right)+\left(\frac{1}{n-1}-\frac{1}{n+2}\right)+\left(\frac{1}{n}-\frac{1}{n+3}\right)$$

$k=n-2$ $k=n-1$ $k=n$

$$\boxed{S_n = 1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}}$$

$$= \lim_{n\to\infty}\boxed{1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}}$$

$$= 1+\frac{1}{2}+\frac{1}{3}-0-0-0$$

$$= \frac{6+3+2}{6} = \boxed{\frac{11}{6}}$$

c) $\sum_{k=1}^{\infty}\left(\ln\frac{k+2}{k}\right) = \sum_{1}^{\infty}\left(\ln(k+2) - \ln(k)\right)$

$= \sum_{1}^{\infty}\left(-\ln(k) + \ln(k+2)\right)$ more convenient to put lesser expression 1st

$= \lim_{n\to\infty} \boxed{\sum_{1}^{n}\left(-\ln(k) + \ln(k+2)\right)}$

$= \lim_{n\to\infty} \boxed{S_n}$

$S_n = \sum_{1}^{n}\left(-\ln(k) + \ln(k+2)\right)$

$= (-\ln 1 + \ln 3) + (-\ln 2 + \ln 4)$
$+ (-\ln 3 + \ln 5) + (-\ln 4 + \ln 6)$
$+ \cdots + \cdots$
$+ \cdots + \cdots$
$+ (-\ln(n-1) + \ln(n+1)) + (-\ln(n) + \ln(n+2))$

$S_n = -\ln 1 - \ln 2 + \ln(n+1) + \ln(n+2)$

$= \lim_{n\to\infty} \boxed{\left(-\ln 2 + \ln(n+1) + \ln(n+2)\right)} = -\ln 2 + \infty + \infty = \boxed{\infty}$

☐ Telescoping Series $S_n =$ $-\ln 2 + \ln(n+1) + \ln(n+2)$
Formula for n^{th} partial sum

~~Converges~~ Diverges because: $\lim_{n\to\infty} S_n$ dne

Value of Series: —

d) $\sum_{k=0}^{\infty}\left(\sin\left(\frac{(k+1)\pi}{2k+1}\right) - \sin\left(\frac{k\pi}{2k-1}\right)\right)$

$= \lim_{n\to\infty} \boxed{\sum_{1}^{n}\left(-\sin\frac{k\pi}{2k-1} + \sin\frac{(k+1)\pi}{2k+1}\right)}$

$= \lim_{n\to\infty} \boxed{S_n}$

$S_n = \sum_{1}^{n}\left(-\sin\frac{k\pi}{2k-1} + \sin\frac{(k+1)\pi}{2k+1}\right)$

$= \left(-\sin\frac{\pi}{1} + \sin\frac{2\pi}{3}\right)$
$+ \left(-\sin\frac{2\pi}{3} + \sin\frac{3\pi}{5}\right)$
$+ \left(-\sin\frac{3\pi}{5} + \sin\frac{4\pi}{9}\right)$
$+ (\cdots\cdots \quad \cdots\cdots)$
$+ (\cdots\cdots \quad \cdots\cdots)$
$+ \left(-\sin\frac{n\pi}{2n-1} + \sin\frac{(n+1)\pi}{2n-1}\right)$

$S_n = -\sin\pi + \sin\left(\frac{n+1}{2n-1}\pi\right)$

$= \lim_{n\to\infty} \boxed{\left(-\sin\pi + \sin\left(\frac{n+1}{2n-1}\pi\right)\right)}$

$= -\sin\pi + \lim_{n\to\infty} \sin\frac{(n+1)}{(2n-1)}\left(\frac{\frac{1}{n}}{\frac{1}{n}}\right)\pi$

$= 0 + \lim_{n\to\infty} \sin\left(\left(\frac{1+1/n}{2-1/n}\right)\pi\right) = \sin\left(\frac{1}{2}\pi\right) = \boxed{1}$

☑ Telescoping Series $S_n =$ $\sin\left(\frac{(n+1)\pi}{2n-1}\right)$
Formula for n^{th} partial sum

Converges ~~Diverges~~ because: $\lim_{n\to\infty} S_n$ exists

Value of Series: $\boxed{1}$

Why not???

$$0 = \sum_{n=1}^{\infty} 0 = \sum_{n=1}^{\infty}(1-1) = 1 + \sum_{n=2}^{\infty}(-1+1) = 1$$

Finish D8 #16

Growth Rates of Functions

Growth rates compare function values as $x \to \infty$

Function *f* grows faster than function *g*, $g<<f$	$\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right)=\infty$
Function *f* grows slower than function *g*, $f<<g$	$\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right)=0$
***f* and *g* have comparable growth rates**	$\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right)=M$, M is a finite, and non-zero

EXAMPLE 25. Use limit methods to determine which of the two given functions grows faster or state that they have comparable growth rates.

a) $\ln(x^{20})$; $\ln(x)$

$$\lim_{x\to\infty}\frac{(\ln x)^{20}}{\ln(x)} = \lim_{x\to\infty}\frac{20\cancel{\ln x}}{\cancel{\ln x}} = \lim_{x\to\infty} 20 = \boxed{20}$$

The limit is a constant, so the functions have comparable growth rates.

b) x^x; $(x/2)^x$

$$\lim_{x\to\infty}\frac{x^x}{\left(\frac{x}{2}\right)^x} = \lim_{x\to\infty}\left(\frac{x}{x/2}\right)^x = \lim_{x\to\infty} 2^x = \infty$$

The limit is ∞ so the numerator (x^x) grows faster than the denominator $\left(\frac{x}{2}\right)^x$.

c) x^7; 7^x

$$\lim_{x\to\infty}\frac{x^7}{7^x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{7x^6}{7^x\ln 7} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{7\cdot 6x^5}{7^x(\ln 7)^2} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{7\cdot 6\cdot 5x^4}{7^x(\ln 7)^3}$$

$$= \cdots = \lim_{x\to\infty}\frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2x}{7^x(\ln 7)^6} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{7^x(\ln 7)^7} = 0$$

The limit is 0 so the numerator (x^7) grows slower than the denominator 7^x.

Start D9 #17

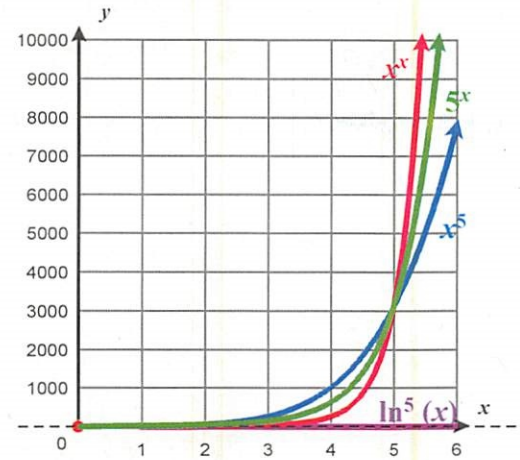
P.S. $7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 7!$ "Seven factorial"

Comparison of Growth Rates for common functions

For positive real numbers p and b, with $b > 1$

$$\ln x \ll x^p \ll b^x \ll x! \ll x^x$$

Logarithms << power functions << exponential functions << factorials << exponential powers



EXAMPLE 26. Use growth rates to find the following limits.

a) $\lim_{n\to\infty} \frac{\ln^3(n)}{n}$ $= 0$ b/c $\ln^3(n) \ll n$

b) $\lim_{n\to\infty} \frac{n^3}{2^n}$ $= 0$ b/c $n^3 \ll 2^n$

c) $\lim_{n\to\infty} \frac{3^n}{n!}$ $= 0$ b/c $3^n \ll n!$

d) $\lim_{n\to\infty} \frac{n^{10}}{\ln^{20}(n)}$ $= \infty$ b/c $n^{10} \gg \ln(n)$

e) $\lim_{n\to\infty} \frac{e^{n/10}}{2^n}$ $= \lim_{n\to\infty} \left(\frac{e^{\frac{1}{10}}}{2}\right)^n$

$= 0$

This is a term in a geometric sequence w/ $r = \frac{1}{2}e^{1/10}$. $|r| < 1$ so the limit $\lim_{n\to\infty} r^n = 0$

Now do D9 #18

10.4 The Divergence and Integral Tests

CO 6.5 Use the nth Term Test for Divergence. [10.4]

With geometric series and telescoping series, the sequence of partial sums can be found and its limit can be evaluated (when it exists). More typically finding a formula for the sequence of partial sums is difficult or impossible and consequently, getting an exact value of most convergent series is not possible.
In light of this, the focus is shifted to simply whether or not a series converges (not what it converges to). If it does converge, it may be possible to get an estimate of its value.

Divergence Test (also known as the nth Term Test)

If $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$ — If the series converges then the terms go to 0

Ex. If you live in Rochester, then you live in NY

Equivalently

If $\lim_{k\to\infty} a_k \neq 0$, then $\sum a_k$ diverges — If the terms don't go to 0 then the series diverges

Ex. If you don't live in NY, then you don't live in Rochester

The divergence test CANNOT be used to determine if a series converges:

If $\lim_{k\to\infty} a_k = 0$ the Divergence Test is <u>inconclusive</u>. **CANNOT** say: If the terms go to 0 then the series converges

If you live in NY, then you live in Rochester

EXAMPLE 27. Divergence Test Use the Divergence Test to determine whether the following series diverge or state that the test is inconclusive.

a) $\sum_{k=0}^{\infty} \frac{k}{2k+1}$

Divergence Test

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{2k+1} \overset{\boxed{\frac{\infty}{\infty}}}{\overset{L'H}{=}} \lim_{k\to\infty} \frac{1}{2} = \boxed{\frac{1}{2}}$$

Divergence Test
Terms in the series approach: $\frac{1}{2}$
☐ The series converges
☑ The series diverges
☐ The test is inconclusive
because the terms approach a non-zero value.

b) $\sum_{k=0}^{\infty} \frac{k}{k^2+1}$

Divergence Test

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{k^2+1} \overset{\boxed{\frac{\infty}{\infty}}}{\overset{L'H}{=}} \lim_{k\to\infty} \frac{1}{2k}$$

$$= \frac{1}{\infty} = 0$$

Divergence Test
Terms in the series approach: 0
☐ The series converges
☐ The series diverges
☑ The test is inconclusive
because the terms approach 0

c) $\sum_{k=2}^{\infty} \frac{\sqrt{k}}{\ln^{10}(k)}$

Divergence Test

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^{1/2}}{\ln^{10}(k)} = \infty \quad \text{b/c } k^{1/2} >> \ln^{10}(k)$$

Divergence Test
Terms in the series approach: ∞
☐ The series converges
☑ The series diverges
☐ The test is inconclusive
because the terms grow without bound

d) $\displaystyle\sum_{k=0}^{\infty} \frac{k}{\sqrt{k^2+1}}$

Divergence Test

$$\lim_{k\to\infty} \frac{k}{\sqrt{k^2+1}} \cdot \boxed{\frac{1/k}{1/k}} = \lim_{k\to\infty} \frac{1}{\sqrt{\frac{k^2+1}{k^2}}} = \lim_{k\to\infty} \frac{1}{\sqrt{1+\frac{1}{k^2}}}$$

$$= \frac{1}{1} = \boxed{1}$$

Divergence Test

Terms in the series approach: 1

- ☐ The series converges
- ☑ The series diverges
- ☐ The test is inconclusive

because the terms approach a non-zero value.

Now do D9 #19

Be aware of subtle differences here. The Divergence Test only examines the convergence of the terms that make up a series. It does not say anything about the convergence of the series. For example:

Determine the convergence of the terms of the series $\displaystyle\sum_{k=1}^{\infty} 2^{-k}$

$$\lim_{k\to\infty} 2^{-k} = \lim_{k\to\infty} \frac{1}{2^k} = \lim_{k\to\infty} \left(\frac{1}{2}\right)^k = 0$$

The terms of the series converge to 0

The convergence of the series, according to the Divergence Test, is unknown (the Divergence Test is inconclusive!)

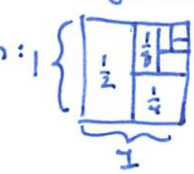
Determine the convergence of the series $\displaystyle\sum_{k=1}^{\infty} 2^{-k}$ (the infinite sum of the terms in this series)

$$\sum_{k=1}^{\infty} 2^{-k} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{2}\left(\frac{1}{1-1/2}\right) = \frac{1}{2}\left(\frac{2}{2-1}\right) = \boxed{1}$$

geometric Series

The series converges to 1

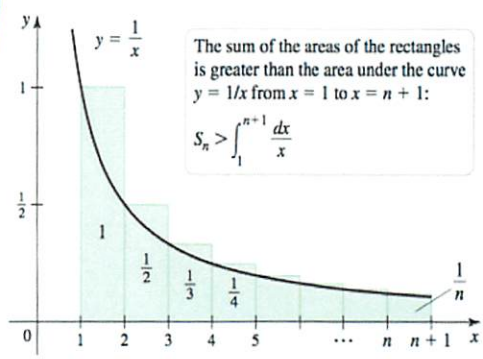
Geometric Explanation:



Harmonic Series

CO 6.10 Recognize the Harmonic Series and know that it diverges.

Harmonic Series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$



The Divergence Test is inconclusive because $\displaystyle\lim_{k\to\infty} a_k = 0$

The Convergence of the terms $= \displaystyle\lim_{k\to\infty} \frac{1}{k} = 0$
("the terms go to...")

The Harmonic Series diverges – even though the terms of the series approach zero

Rational $\displaystyle\sum_{1}^{\infty} \frac{1}{k} = 1$

$+\frac{1}{2}$

$+\frac{1}{3} + \frac{1}{4}$ $\leftarrow > \frac{1}{2}$

$+\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$ $\leftarrow > \frac{1}{2}$

$+\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}$ $\leftarrow > \frac{1}{2}$ etc

$\Rightarrow > 1 + \frac{1}{2} + \underbrace{\sum_{1}^{\infty} \frac{1}{2}}_{\text{terms} > 1/2}$

(not 0) $\Rightarrow$ series diverges.

Finish D9 #20

The Integral Test

CO 6.6. Interpret the terms of a series as areas of rectangles of width one and height equal to the value of the term. Interpret the Integral Test as a comparison of an improper integral to its upper- or lower-sum.

CO 6.7. Use the Integral Test to determine the convergence or divergence of a series when applicable. Use an appropriate integral to give an upper bound for the error in using a partial sum to approximate the sum of the series.

Integral Test

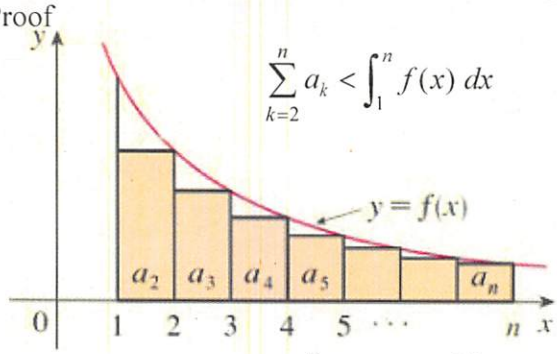
Suppose f is a continuous, positive, decreasing function for $x \geq 1$, and $a_k = f(k)$ for $k = 1, 2, 3, \ldots$

Then $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\ dx$ either both converge or both diverge.

In the case of convergence, the value of the integral is not equal to the value of the series.

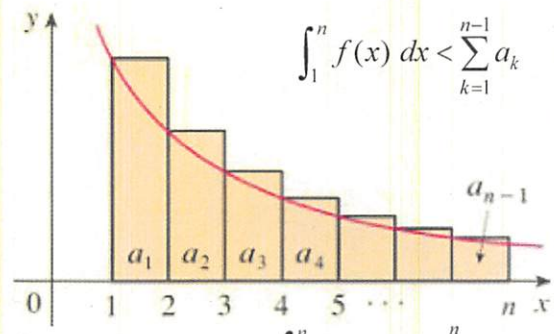
Instead, the integral provides an upper bound on the value for the series: $\sum_{k=1}^{\infty} a_k < a_1 + \int_1^{\infty} f(x)\ dx$

Proof



$\rightarrow \quad a_1 + \sum_{k=2}^{n} a_k < a_1 + \int_1^n f(x)\ dx$

$\rightarrow \quad \boxed{\sum_{k=1}^{\infty} a_k < a_1 + \int_1^{\infty} f(x)\ dx}$



$\rightarrow \quad \int_1^n f(x)\ dx < \sum_{k=1}^{n} a_k$

$\rightarrow \quad \int_1^{\infty} f(x)\ dx < \sum_{k=1}^{\infty} a_k$

If $\int_1^{\infty} f(x)\ dx$ converges then $\sum_{k=1}^{\infty} a_k$ converges.

If $\sum_{k=1}^{\infty} a_k$ diverges then $\int_1^{\infty} f(x)\ dx$ diverges.

If $\sum_{k=1}^{\infty} a_k$ converges then $\int_1^{\infty} f(x)\ dx$ converges.

If $\int_1^{\infty} f(x)\ dx$ diverges then $\sum_{k=1}^{\infty} a_k$ diverges.

How to determine a series converges using the Integral Test

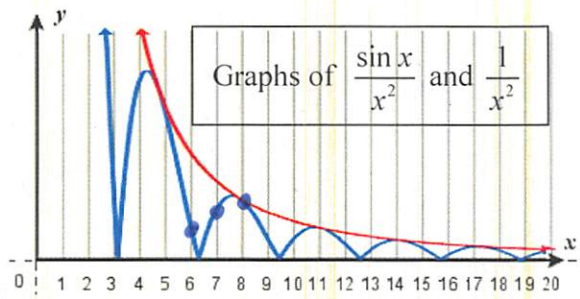
1. Check conditions: State the associated function $f(x)$ and interval I of interest
   - Continuous — Show $f(x)$ is continuous for all x in I
   - Positive — Show $f(x)$ is positive for all x in I
   - Decreasing — Show $f(x)$ is <u>eventually</u> decreasing. For unobvious cases, show $f'(x) < 0$ or $f(x+1) \leq f(x)$
2. Evaluate the integral of the associated function
3. State the conclusion: "The integral converges/diverges, so therefore the series behaves likewise"

EXAMPLE 28. Use the Integral Test to determine the convergence or divergence of the following series, or state that the test does not apply. If convergent, state an upper bound for the value of the series.

a) $\sum_{k=1}^{\infty} \frac{|\sin(k)|}{k^2}$

<u>Integral Test</u> $f(x) = \frac{|\sin x|}{x^2}$, $x \geq 1$

- f is continuous for $x \geq 1$
- f is positive for $x \geq 1$
- f is <u>NOT</u> decreasing: $f(x+1) \leq f(x)$ not always true



b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k+10}}$

Integral Test

① $f(x) = \frac{1}{(x+10)^{1/3}}$ for $x \geq 1$ $\begin{cases} C & \text{continuous for } x \geq 1 \\ P & \text{positive for } x \geq 1 \\ D & \text{decreasing b/c } \frac{1}{(x+1+10)^{1/3}} \leq \frac{1}{(x+10)^{1/3}} \end{cases}$

② $\int_1^{\infty} \frac{1}{(x+10)^{1/3}}\,dx = \lim_{A\to\infty} \int_1^A \frac{1}{(x+10)^{1/3}}\,dx$

USUB $u = x+10$ $du = dx$ $x = A \to \infty \Rightarrow u = B = \infty + 10 \to \infty$, $x = 1 \Rightarrow u = 11$

$= \lim_{B\to\infty} \int_{11}^{B} u^{-1/3}\,du = \lim_{B\to\infty} \frac{3}{2} u^{2/3} \Big|_{11}^{B}$

$= \lim_{B\to\infty} \frac{3}{2} B^{2/3} - \frac{3}{2} 11^{2/3} = \boxed{\infty}$

③ the integral diverges so the $\boxed{\text{series also diverges}}$

$\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k+10}}$ $\begin{cases} = _____ \\ \leq _____ \\ \text{does not exist} \\ \text{unknown} \end{cases}$

c) $\sum_{k=1}^{\infty} \frac{k}{e^k}$

Integral Test

① $f(x) = \frac{x}{e^x}$, $x \geq 1$ $\begin{cases} C & \text{continuous} \\ P & \text{positive for } x \geq 1 \\ D & \text{decreasing - show } f' < 0 \text{ eventually} \end{cases}$

$f'(x) = \frac{e^x - xe^x}{e^{2x}} = \frac{e^x(1-x)}{e^{2x}} = \frac{1-x}{e^x} < 0$ when $1 - x < 0 \Rightarrow 1 < x$

② $\int_1^{\infty} \frac{x}{e^x}\,dx = \lim_{A\to\infty} \int_1^A x e^{-x}\,dx$ PARTS $u = x$, $du = dx$, $dv = e^{-x}dx$, $v = -e^{-x}$

$= \lim_{A\to\infty} \left[-xe^{-x} \Big|_1^A - \int_1^A -e^{-x}\,dx \right]$

$= \lim_{A\to\infty} \left[-xe^{-x} - e^{-x} \right] \Big|_1^A$

$= \lim_{A\to\infty} \left[\underbrace{-Ae^{-A}}_{\substack{\infty \cdot 0 \\ \text{indet.}}} - \underbrace{e^{-A}}_{\to 0} \right] - \left[-1e^{-1} - e^{-1} \right]$

$= \lim_{A\to\infty} \underbrace{\frac{-A}{e^A}}_{\substack{\frac{-\infty}{\infty} \\ \overset{L'H}{=} \lim_{A\to\infty} \frac{-1}{e^A} = \frac{-1}{\infty} = \boxed{0}}} + 2e^{-1} = 0 + \frac{2}{e} = \frac{2}{e}$

③ the integral converges so the $\boxed{\text{series also converges.}}$

$\sum_{k=1}^{\infty} \frac{k}{e^k}$ $\begin{cases} = _____ \\ \leq \frac{1}{e^1} + \int_1^{\infty} \frac{x}{e^x}\,dx \\ \text{does not exist} \\ \text{unknown} \end{cases}$

NOT equal to $\frac{2}{e}$!

d) $\sum_{k=3}^{\infty} \frac{1}{k\ln(k)\ln(\ln k)}$

Integral Test

$\ln 1 = 0$ ↯

① $f(x) = \frac{1}{x(\ln x)\ln(\ln x)}$, $x \geq 3$

C ✓ continuous for $x > e$
P ✓ positive for $x > e$
D ✓ decreasing b/c:

$$\frac{1}{(x+1)\ln(x+1)\ln(\ln(x+1))} \leq \frac{1}{x\ln x\ln(\ln x)}$$

② $\int_3^{\infty} \frac{1}{x\ln x\ \ln(\ln x)}\,dx$

$= \lim_{A\to\infty} \int_3^A \frac{1}{x\ln x\ \ln(\ln x)}\,dx$

USUB $u = \ln(\ln x)$ $\quad x = A \to \infty \Rightarrow u = B \to \ln(\ln\infty) = \infty$

$du = \frac{1}{\ln x}\left[\frac{1}{x}\right]dx$ $\quad x = 3 \Rightarrow u = \ln(\ln 3)$

$= \lim_{B\to\infty} \int_{\ln(\ln 3)}^{B} \frac{1}{u}\,du = \lim_{B\to\infty} \ln|u|\Big|_{\ln(\ln 3)}^{B}$

$= \lim_{B\to\infty} \ln|B| - \ln|\ln(\ln 3)|$

$= \infty$

③ The integral diverges so $\boxed{\text{the series also diverges}}$

$\sum_{k=3}^{\infty} \frac{1}{k\ln(k)\ln(\ln k)}$ = _____ ≤ _____ does not exist unknown

Start D10 #21,22

d) $\sum_{k=3}^{\infty} \frac{k}{k^4+25}$

Integral Test

① $f(x) = \frac{x}{x^4+25}$, $x \geq 3$

C continuous for $x \geq 3$
P positive for $x \geq 3$
D decreasing – show $f'(x) < 0$ eventually.

$$f'(x) = \frac{(x^4+25) - x[4x^3]}{(x^4+25)^2} = \frac{x^4+25-4x^4}{(x^4+25)^2} = \frac{25-3x^4}{(x^4+25)^2} < 0?$$

$\Rightarrow 25 - 3x^4 < 0 \to 25 < 3x^4 \to x^4 > \frac{25}{3}$

true for $x \geq 2$ ✓

② $\int_3^{\infty} \frac{x}{x^4+25}\,dx = \lim_{A\to\infty} \int_3^A \frac{x\,dx}{(x^2)^2+5^2}$

USUB $u = x^2$ $\quad x = A \to \infty \Rightarrow u = B \to \infty$

$du = 2x\,dx$ $\quad x = 3 \to u = 9$

$\frac{1}{2}du = x\,dx$

$= \lim_{B\to\infty} \int_9^B \frac{\frac{1}{2}du}{u^2+5^2} = \lim_{B\to\infty} \frac{1}{2}\frac{\tan^{-1}\left(\frac{u}{5}\right)}{5}\Big|_9^B$

$= \lim_{B\to\infty} \frac{\tan^{-1}\left(\frac{B}{5}\right)}{10} - \frac{1}{10}\tan^{-1}\left(\frac{9}{5}\right)$

$= \frac{1}{10}\left(\frac{\pi}{2}\right) - \frac{1}{10}\tan^{-1}\left(\frac{9}{5}\right)$ ← this is the value of the integral. It is NOT the value of the series.

③ the integral converges so $\boxed{\text{the series also converges}}$

$\sum_{k=3}^{\infty} \frac{k}{k^4+25}$ = _____ ≤ $\frac{3}{3^4+25} + \int_3^{\infty} \frac{x}{x^4+25}\,dx$ does not exist unknown

1st term in series + integral starting at beginning k value

Estimating the Value of an Infinite Series

CO 6.7 Use the Integral Test to determine the convergence or divergence of a series when applicable. Use an appropriate integral to give an upper bound for the error in using a partial sum to approximate the sum of the series.

Estimating Series with Positive Terms

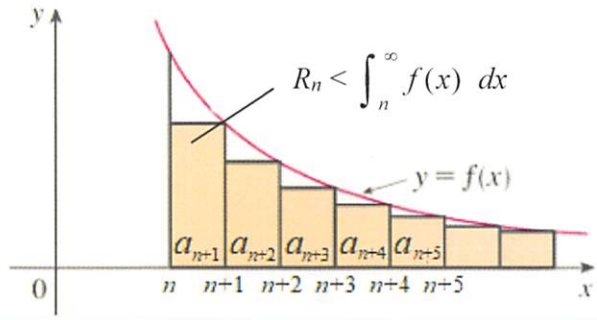
Given • f, a continuous, positive, decreasing function for $x \geq 1$, and $a_k = f(k)$ for $k = 1, 2, 3, \ldots$

• $S = \sum_{k=1}^{\infty} a_k$ is a convergent series

$S_n = \sum_{k=1}^{n} a_k$ — S_n is the sum of the first n terms of the series

$R_n = \sum_{k=n+1}^{\infty} a_k$ — R_n is the remainder or error that occurs when only the first n terms are used to approximate the series

$R_n < \int_n^{\infty} f(x)\ dx$



EXAMPLE 29. Given the series $\sum_{k=1}^{\infty} ke^{-k^2}$

a) Show that the associated function $f(x)$ meets the requirements for estimating the series. (These are the same requirements that are used in the Integral Test.)

$f(x) = xe^{-x^2}$, for $x \geq 1$

✓ C Continuous

✓ P Positive for $x \geq 1$

✓ D Decreasing

$f'(x) = x(-2x)e^{-x^2} + e^{-x^2}$

$= e^{-x^2}(-2x^2+1) < 0$?

yes, if $-2x^2+1 < 0 \rightarrow 2x^2 > 1$

$x^2 > 1/2$ ✓

b) Find an upper bound for the remainder if n terms are used to estimate the series.

$|R_n| < \int_n^{\infty} xe^{-x^2}\,dx$

USUB

$u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$x \to \infty \Rightarrow u \to -\infty$

$x = n \Rightarrow u = -n^2$

$\lim_{A\to-\infty} \int_{-n^2}^{A} -\frac{1}{2}e^u\,du = \lim_{A\to-\infty} \frac{-1}{2}e^u \Big|_{-n^2}^{A}$

$\lim_{A\to-\infty} \frac{-1}{2}e^u - \frac{-1}{2}e^{-n^2} = 0 + \frac{1}{2}e^{-n^2} = \boxed{\frac{1}{2e^{n^2}}}$

$\Rightarrow \boxed{|R_n| < \frac{1}{2e^{n^2}}}$

c) What is the minimum number of terms needed to ensure that the remainder is less than 10^{-2}.

$\frac{1}{2e^{n^2}} < 10^{-2} \rightarrow 10^2 < 2e^{n^2} \rightarrow 50 < e^{n^2}$

$\rightarrow \ln 50 < \ln(e^{n^2}) \rightarrow n^2 > \ln 50 = 3.91$

$\Rightarrow n > \sqrt{3.91} \Rightarrow \boxed{\text{use } n = 2}$

d) What is the sum of the first two terms in the series?

$S_2 = 1e^{-1^2} + 2e^{-2^2} \approx \boxed{.40451\ldots}$

e) What is the upper bound on the error if $n = 2$ terms are used to approximate the value of the series?

$|R_2| < \frac{1}{2e^{2^2}} = \boxed{.009157\ldots}$

HW Finish D10 #23

The *p*-Series

CO 6.8 Apply the p-Series Test to determine the convergence or divergence of a p-series.

Convergence of the *p*-Series

The *p*-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$; diverges for $p \not> 1$

Integral Test (see p. 12) $\int \frac{1}{x^p}\,dx$ $\begin{cases} \text{converges if } p>1 \\ \text{diverges if } p \not> 1 \end{cases}$

EXAMPLE 30. p-series Determine the convergence or divergence of the following series

a) $\sum_{k=1}^{\infty} k^{-10} = \sum_{1}^{\infty} \frac{1}{k^{10}}$

☑ *p*-series $p =$ 10

Converges because $p>1$

Diverges

b) $\sum_{k=1}^{\infty} \sqrt[5]{k^3} = \sum_{1} k^{\frac{3}{5}} = \sum_{1} \frac{1}{k^{-3/5}}$

☑ *p*-series $p =$ $-3/5$

Converges because $p \not> 1$

Diverges

c) $\sum_{k=1}^{\infty} \frac{1}{7(k+4)^3} = \frac{1}{7}\sum_{k=1}^{\infty} \frac{1}{(k+4)^3} = \frac{1}{7}\sum_{n=5}^{\infty} \frac{1}{n^3}$

use $n = k+4$

☑ *p*-series $p =$ 3

Converges because $p>1$

Diverges

Compare with the following Geometric Series: $\sum_{k=0}^{\infty} r^k$ converges if $|r|<1$, diverges if $|r| \not< 1$

Start D11 #24

$\sum_{k=1}^{\infty} (-10)^k$

$r = -10$

$|r| = 10 \not< 1$

series diverges

$\sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^k$

$r = \frac{3}{5}$

$|r| < 1$ so converges

$= \frac{3}{5}\sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^k$

$= \frac{3}{5}\left(\frac{1}{1-\frac{3}{5}}\right)$

$= \frac{3}{5}\left(\frac{5}{2}\right) = \boxed{\frac{3}{2}}$

$\sum_{k=1}^{\infty} \frac{1}{3^{k+4}} = \sum_{1}^{\infty} \frac{1}{3^k 3^4}$

$= \frac{1}{81}\sum_{1} \frac{1}{3^k} = \frac{1}{81}\sum_{1}\left(\frac{1}{3}\right)^k$

$r = \frac{1}{3}$, $|r|<1 \rightarrow$ conv.

$= \frac{1}{81}\left(\frac{1}{3}\right)\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k$

$= \frac{1}{81(3)}\left(\frac{1}{1-\frac{1}{3}}\right)$

$= \frac{1}{81(3)}\,\frac{3}{2} = \boxed{\frac{1}{162}}$

Choose a test

EXAMPLE 31. Determine whether the following series converge or diverge. Determine the exact value of the series or an upper bound for the series if possible.

a) $\sum_{k=1}^{\infty} \frac{8}{(k+1)(k+2)}$

Partial Fractions $\frac{A}{k+1} + \frac{B}{k+2}$

$A(k+2) + B(k+1) = 8$

$k=-1: \quad A = 8$

$k=-2: \quad B(-1) = 8 \rightarrow B = -8$

$= \sum_{1}^{\infty} \left(\frac{8}{k+1} - \frac{8}{k+2}\right)$

$S_n = \sum_{1}^{n} a_k = \left(\frac{8}{2} - \frac{8}{3}\right) + \left(\frac{8}{3} - \frac{8}{4}\right) + \left(\frac{8}{4} - \frac{8}{5}\right) + \cdots \left(\frac{8}{n+1} - \frac{8}{n+2}\right)$

$S_n = 4 - \frac{8}{n+2}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(4 - \frac{8}{n+2}\right) = \boxed{4}$

- ☐ Geometric Series
- ☑ Telescoping Series
- ☐ p-series
- ☐ Harmonic Series
- ☐ Divergence Test
- ☐ Integral Test

Convergent/~~Divergent/Unknown~~

because: $\lim_{n\to\infty} S_n = 4$

Value of Series:

$\sum_{k=1}^{\infty} \frac{8}{(k+1)(k+2)} = 4$

b) $\sum_{k=1}^{\infty} \frac{k}{\sqrt{16+k^2}}$

Divergence Test

$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{\sqrt{16+k^2}} \left[\frac{1/k}{1/k}\right]$

$= \lim_{k\to\infty} \frac{1}{\sqrt{16+k^2}} \frac{1}{\sqrt{\frac{1}{k^2}}}$

$= \lim_{k\to\infty} \frac{1}{\sqrt{\frac{16}{k^2}+1}} = \frac{1}{\sqrt{0+1}}$

$= \boxed{1}$

- ☐ Geometric Series
- ☐ Telescoping Series
- ☐ p-series
- ☐ Harmonic Series
- ☑ Divergence Test
- ☐ Integral Test

~~Convergent~~/Divergent/~~Unknown~~

because: terms approach a non-zero value

Value of Series:

$\sum_{k=1}^{\infty} \frac{k}{\sqrt{16+k^2}}$

D9 Assn

c) $\sum_{k=2}^{\infty} \frac{1}{k\left((\ln(k))^2+1\right)}$

Integral Test

① $f(x) = \frac{1}{x((\ln x)^2+1)}$ for $x \geq 2$

C Continuous
P Positive
D Decreasing

$u = \ln x$

$du = \frac{1}{x}dx$

USUB

$x = A \to \infty \Rightarrow$

$u = B \to \infty$

$x = 2 \to u = \ln 2$

② $\int_2^{\infty} \frac{dx}{x(\ln^2 x+1)} = \lim_{A\to\infty} \int_2^{A} \frac{dx}{x(\ln^2 x+1)}$

$= \lim_{B\to\infty} \int_{\ln 2}^{B} \frac{du}{u^2+1} = \lim_{B\to\infty} \tan^{-1}(u)\Big|_{\ln 2}^{B}$

$= \lim_{B\to\infty} \tan^{-1}(B) - \tan^{-1}(\ln 2) = \frac{\pi}{2} - \tan^{-1}(\ln 2)$

③ The integral converges so the series also converges

- ☐ Geometric Series
- ☐ Telescoping Series
- ☐ p-series
- ☐ Harmonic Series
- ☐ Divergence Test
- ☑ Integral Test

Convergent/~~Divergent/Unknown~~

because: the integral converges

Value of Series:

$\sum_{k=2}^{\infty} \frac{1}{k\left((\ln(k))^2+1\right)}$

Finish D11 #25

Section 10.4 Choose your test….

Use Geometric Series, p-series, Telescoping series, the Divergence Test, or the Integral Test to determine the convergence of the following series. The convergence of two series in this list cannot be determined using only these techniques.

9. $\sum_{k=0}^{\infty} \frac{k}{2k+1}$

10. $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$

11. $\sum_{k=0}^{\infty} \frac{1}{1000+k}$

12. $\sum_{k=1}^{\infty} \frac{k^3}{k^3+1}$

13. $\sum_{k=2}^{\infty} \frac{k}{\ln k}$

15. $\sum_{k=2}^{\infty} \frac{\sqrt{k}}{\ln^{10} k}$

17. $\sum_{k=0}^{\infty} \frac{1}{\sqrt{k+8}}$

18. $\sum_{k=1}^{\infty} \frac{1}{(2k+4)^2}$

19. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{5k+3}}$

20. $\sum_{k=1}^{\infty} \frac{e^k}{1+e^{2k}}$

21. $\sum_{k=1}^{\infty} ke^{-2k^2}$

22. $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$

23. $\sum_{k=1}^{\infty} k^{-1/5}$

24. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3}}$

25. $\sum_{k=1}^{\infty} \frac{k^3}{e^{k^4}}$

26. $\sum_{k=1}^{\infty} \frac{k^2}{k^3+5}$

27. $\sum_{k=1}^{\infty} k^{1/k}$

28. $\sum_{k=1}^{\infty} \frac{\sqrt{k^2+1}}{k}$

29. $\sum_{k=1}^{\infty} \frac{1}{k^{10}}$

30. $\sum_{k=2}^{\infty} \frac{k^e}{k^\pi}$

31. $\sum_{k=3}^{\infty} \frac{1}{(k-2)^4}$

32. $\sum_{k=1}^{\infty} 2k^{-3/2}$

33. $\sum_{k=1}^{\infty} \frac{k}{e^k}$

34. $\sum_{k=3}^{\infty} \frac{1}{k(\ln k)\ln\ln k}$

35. $\sum_{k=1}^{\infty} \left(\frac{k}{k+10}\right)^k$

36. $\sum_{k=1}^{\infty} \frac{k}{(k^2+1)^3}$

37. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}}$

38. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{27k^2}}$

39. $\sum_{k=1}^{\infty} \frac{1}{k^7}; n = 2$

40. $\sum_{k=1}^{\infty} \frac{1}{k^3}; n = 5$

41. $\sum_{k=1}^{\infty} \frac{1}{k^6}$

42. $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$

43. $\sum_{k=1}^{\infty} \frac{1}{3^k}$

44. $\sum_{k=1}^{\infty} ke^{-k^2}$

48. $\sum_{k=1}^{\infty} \ln\left(\frac{2k^6}{1+k^6}\right)$

49. $\sum_{k=2}^{\infty} \frac{1}{e^k}$

50. $\sum_{k=1}^{\infty} \sqrt{\frac{k+1}{k}}$

51. $\sum_{k=1}^{\infty} \frac{1}{(3k+1)(3k+4)}$

52. $\sum_{k=0}^{\infty} \frac{10}{k^2+9}$

53. $\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2+1}}$

54. $\sum_{k=1}^{\infty} \frac{2^k+3^k}{4^k}$

55. $\sum_{k=3}^{\infty} \frac{4}{k\sqrt{\ln k}}$

56. $\sum_{k=1}^{\infty} \frac{1}{k^2+7k+12}$

57. $\sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{-k}$

58. $\sum_{k=1}^{\infty} \left(\frac{k+6}{k}\right)^k$

59. $\sum_{k=4}^{\infty} \frac{3}{(k-3)^4}$

60. $\sum_{k=1}^{\infty} \frac{3^k}{k^2+1}$

61. $\frac{2}{4^2} + \frac{2}{5^2} + \frac{2}{6^2} + \cdots$

62. $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$

63. $\sum_{k=1}^{\infty} \frac{3^{k+2}}{5^k}$

Summary of Log and Exponent Properties

Properties of Exponents

$a > 0$, $b > 0$, $a \neq 1$, $b \neq 1$, n, m are real numbers

Rules of 1	$a^0 = 1$	$a^1 = a$	
Negative Exponent Rule	$a^{-n} = \dfrac{1}{a^n}$	and	$\dfrac{1}{a^{-n}} = a^n$
Product Rule (for like bases)	$a^n \cdot a^m = a^{n+m}$		
Quotient Rule (for like bases)	$\dfrac{a^n}{a^m} = a^{n-m}$	and	$\dfrac{a^n}{a^m} = \dfrac{1}{a^{m-n}}$
Power Rule	$(a^n)^m = a^{nm}$		
Product Power Rule	$(ab)^n = a^n b^n$		
Quotient Power Rule	$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	and	$\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n = \dfrac{b^n}{a^n}$
Product Rule (for like exponents)	$a^n \cdot b^n = (ab)^n$		
Quotient Rule (for like exponents)	$\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$		

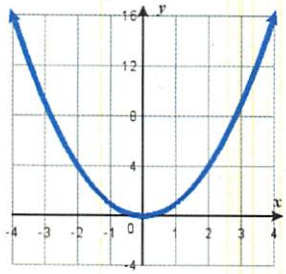
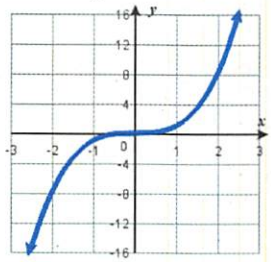
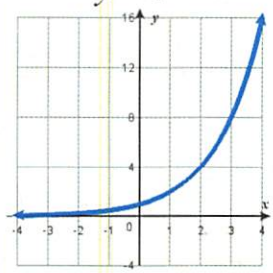
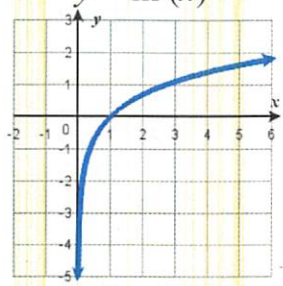
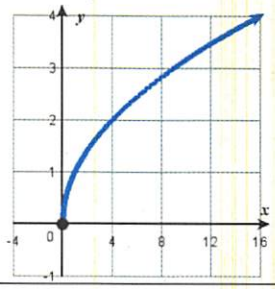
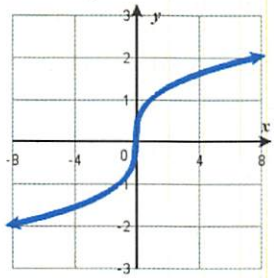
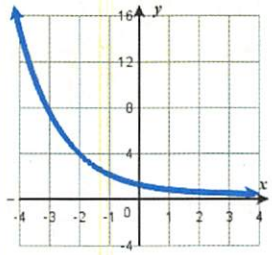
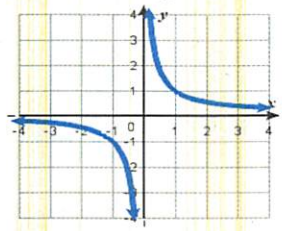
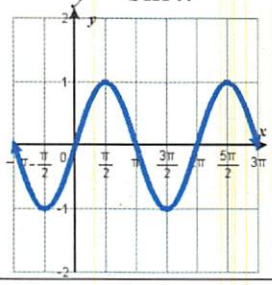
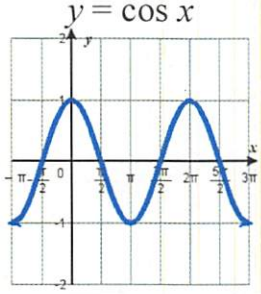
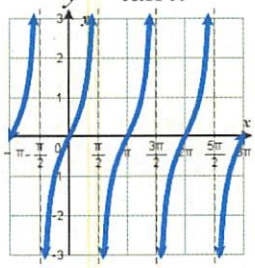
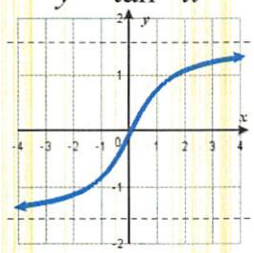
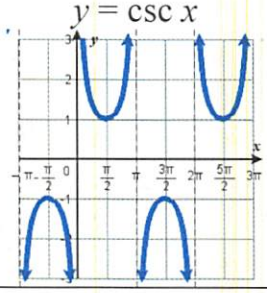
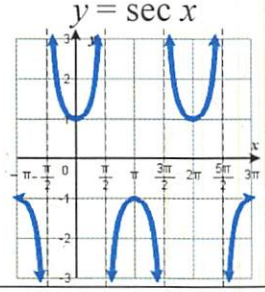
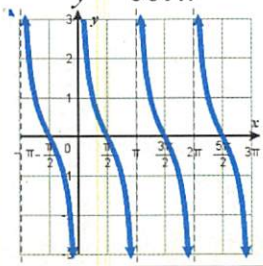
$y = \log_a x$ if and only if $x = a^y$ where $a > 0$, $a \neq 1$

Properties of Logarithms

$a > 0$, $a \neq 1$, $M > 0$, $N > 0$, and a, M, N, r are all real numbers:

Properties of 1	$\log_a(1) = 0$	$\log_a(a) = 1$	$\ln(1) = 0$	$\ln e = 1$
Inverse Properties	$\log_a(a)^r = r$	$a^{\log_a(M)} = M$	$\ln(e)^r = r$	$e^{\ln(M)} = M$
Product Rule	$\log_a(MN) = \log_a(M) + \log_a(N)$		$\ln(MN) = \ln(M) + \ln(N)$	
Quotient Rule	$\log_a\left(\dfrac{M}{N}\right) = \log_a(M) - \log_a(N)$		$\ln\left(\dfrac{M}{N}\right) = \ln(M) - \ln(N)$	
P/Q Extension	$\log_a\left(\dfrac{AB}{CD}\right) = \log_a A + \log_a B - \log_a C - \log_a D$		$\ln\left(\dfrac{AB}{CD}\right) = \ln A + \ln B - \ln C - \ln D$	
Power Rule	$\log_a(M)^r = r\log_a(M)$	$a^x = e^{x\ln(a)}$	$\ln(M)^r = r\ln(M)$	

Graphs of Some Common Functions

$y = x^2$ 	$y = x^3$ 	$y = 2^x$ 	$y = \ln(x)$ 
$y = \sqrt{x}$ 	$y = \sqrt[3]{x}$ 	$y = 2^{-x}$ 	$y = \frac{1}{x}$ 
$y = \sin x$ 	$y = \cos x$ 	$y = \tan x$ 	$y = \tan^{-1} x$ 
$y = \csc x$ 	$y = \sec x$ 	$y = \cot x$ 	

Some Trigonometric Properties

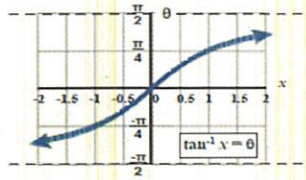
θ	$\sin\theta$	$\csc\theta$
$0, \pi, 2\pi$	0	u
$\frac{\pi}{6}, \frac{5\pi}{6}$	$\frac{1}{2}$	2
$\frac{\pi}{4}, \frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
$\frac{\pi}{3}, \frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{2\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	1
$\frac{7\pi}{6}, \frac{11\pi}{6}$	$-\frac{1}{2}$	-2
$\frac{5\pi}{4}, \frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$
$\frac{4\pi}{3}, \frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{2\sqrt{3}}{3}$
$\frac{3\pi}{2}$	-1	-1

θ	$\cos\theta$	$\csc\theta$
$0, 2\pi$	1	1
π	-1	-1
$\frac{\pi}{6}, \frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{2\sqrt{3}}{3}$
$\frac{\pi}{4}, \frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
$\frac{\pi}{3}, \frac{5\pi}{3}$	$\frac{1}{2}$	2
$\frac{\pi}{2}$	0	u
$\frac{5\pi}{6}, \frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-\frac{2\sqrt{3}}{3}$
$\frac{3\pi}{4}, \frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$
$\frac{2\pi}{3}, \frac{4\pi}{3}$	$-\frac{1}{2}$	-2
$\frac{3\pi}{2}$	0	u

θ	$\tan\theta$	$\cot\theta$
$0, \pi, 2\pi$	0	u
$\frac{\pi}{6}, \frac{7\pi}{6}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}, \frac{5\pi}{4}$	1	1
$\frac{\pi}{3}, \frac{4\pi}{3}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	u	0
$\frac{5\pi}{6}, \frac{11\pi}{6}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$
$\frac{3\pi}{4}, \frac{7\pi}{4}$	-1	-1
$\frac{2\pi}{3}, \frac{5\pi}{3}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$
$\frac{3\pi}{2}$	u	0

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

$$\cos(2\alpha) = \begin{cases} \cos^2\alpha - \sin^2\alpha \\ 2\cos^2\alpha - 1 \\ 1 - 2\sin^2\alpha \end{cases}$$



General Derivative Formulas

Constant Rule $\frac{d}{dx}c = 0$

Power Rule $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$

Product Rule $\frac{d}{dx}[f(x)g(x)] = f(x)\left[\frac{d}{dx}g(x)\right] + g(x)\left[\frac{d}{dx}f(x)\right]$ $\quad (uv)' = uv' + vu'$

Quotient Rule $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)[f'(x)] - f(x)[g'(x)]}{(g(x))^2}$ $\quad \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$

Chain Rule $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$ $\quad u(v(x)) = u'(v) \cdot v'(x)$

Trigonometric Functions

$\frac{d}{dx}(\sin x) = \cos x$ $\quad \frac{d}{dx}(\tan x) = \sec^2 x$ $\quad \frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\cos x) = -\sin x$ $\quad \frac{d}{dx}(\cot x) = -\csc^2 x$ $\quad \frac{d}{dx}(\csc x) = -\csc x \cot x$

Exponential and Logarithmic Functions

$\frac{d}{dx}\left(e^x\right) = e^x$ $\quad \frac{d}{dx}\left(b^x\right) = b^x \ln b$

$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ $\quad \frac{d}{dx}(\log_b|x|) = \frac{1}{x \ln b}$

Inverse Trigonometric Functions

$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$ $\quad \frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$ $\quad$ for $-1 < x < 1$

$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$ $\quad \frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$ $\quad$ for $-\infty < x < \infty$

$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$ $\quad \frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$ $\quad$ for $|x| > 1$

FORMULAS

Growth Rates for $p, q, r, s > 0$ and $b > 1$ $\{\ln^q n\} << \{n^p\} << \{n^p \ln^r n\} << \{n^{p+s}\} << \{b^n\} << \{n!\} << \{n^n\}$	if $\{a_n\} << \{b_n\}$ then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$
$\lim_{n\to\infty}(r^n) = \begin{cases} 0 & \text{if } \lvert r \rvert < 1 \\ 1 & \text{if } r = 1 \\ dne & \text{if } r \le -1 \text{ or } r > 1 \end{cases}$	

Geometric Series	$\sum a_k = \sum_{k=0}^{\infty} ar^k$	$\sum a_k$ converges to $a\,\frac{1}{1-r}$ when $\lvert r \rvert < 1$	$\sum a_k$ diverges when $\lvert r \rvert \not< 1$
***p*-series**	$\sum a_k = \sum_{k=1}^{\infty} \frac{1}{k^p}$	$\sum a_k$ converges when $p > 1$	$\sum a_k$ diverges when $p \not> 1$
Harmonic Series	$\sum a_k = \sum_{k=1}^{\infty} \frac{1}{k}$	$\sum a_k$ Does NOT converge	$\sum a_k$ diverges
Telescoping series	$\sum a_k = \sum_{k=1}^{\infty} (a_n - a_{n+1})$	$\sum a_k$ converges to $a_1 - \lim_{k\to\infty} a_k$	

	Conditions	$\sum a_k$ converges	$\sum a_k$ diverges	inconclusive
Divergence Test	$\lim_{k\to\infty} a_k$		$\sum a_k$ diverges if $\lim_{k\to\infty} a_k \neq 0$	Inconclusive if $\lim_{k\to\infty} a_k = 0$
Integral Test	$a_k = f(k)$ f is (eventually) • continuous, • positive, • decreasing	$\sum a_k$ converges if $\int_1^{\infty} f(x)\,dx$ converges	$\sum a_k$ diverges if $\int_1^{\infty} f(x)\,dx$ diverges	

Remainder for a convergent Series	$\sum_{k=1}^{\infty} a_k$ converges $a_k = f(k)$ for $k = 1, 2, 3, \ldots$ f is (eventually) • continuous, • positive, • decreasing	Remainder due to using just the sum of the first n terms: $S - S_n = R_n < \int_n^{\infty} f(x)\,dx$ Bounds: $\sum_{k=1}^{n} a_k + \int_{n+1}^{\infty} f(x)\,dx < \sum_{k=1}^{\infty} a_k < \sum_{k=1}^{n} a_k + \int_n^{\infty} f(x)\,dx$

Section 10.4 Exercises Geometric series (exponent has k in it: 43, 49, 54, 57, 63), p-series (base of exponential expression has a k in it: 11, 18, 23, 24, 29, 30, 31, 32, 37, 38, 39, 40, 41, 59, 61), Divergence Test (terms do not approach zero: 9, 12, 13, 15, 27, 28, 35, 48, 50, 53, 58, 60), Integral Test (expression can be integrated: 10, 17, 19, 20, 21, 22, 25, 26, 33, 34, 36, 42, 44, 52, 55, 62), Telescoping Series (expression can be written as a difference of related expressions: 51, 56)

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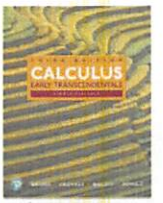
Briggs, Cochran, Gillett, and Schulz. Single Variable Calculus (Early Transcendentals). 3rd ed. Boston: Pearson Education, 2019



Unit 4: Tentative Schedule

Assignments that are a day late only get partial credit.
Assignments handed in after solutions have been posted on Blackboard are NOT accepted.

Week	Mon					
10		Notes pp 1-3 D1 4.7 L'Hôpital's Rule (0/0, ∞/∞ and $0\cdot\infty$forms) Notes pp 4-6 D2 4.7 L'Hôpital's Rule (∞-∞,1^∞, 0^0 and ^{0} forms) Notes pp 7-9 D3 4.7 L'H pitfalls, Squeeze Theorem 8.9 Improper Integrals (∞ limits of integration)				
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12		Notes pp 26-28 D8 10.4 series properties, Divergence Test 10.3 Infinite Series: Telescoping Series Notes pp 29-32 D9 Growth Rates, 10.4 Divergence Test 10.4 Harmonic series Notes pp 33-36 D10 10.4 Integral Test, errors Notes pp 37-38 D11 10.4 p-series, Choosing a test				
13		**Exam #4 Limits, Sequences Series**				