Lecture Notes

Fundamentals of Control Systems

Instructor: Assoc. Prof. Dr. Huynh Thai Hoang
Department of Automatic Control
Faculty of Electrical & Electronics Engineering
Ho Chi Minh City University of Technology
Email: hthoang@hcmut.edu.vn
huynhthaihoang@yahoo.com
Homepage: www4.hcmut.edu.vn/~hthoang/

Chapter 7

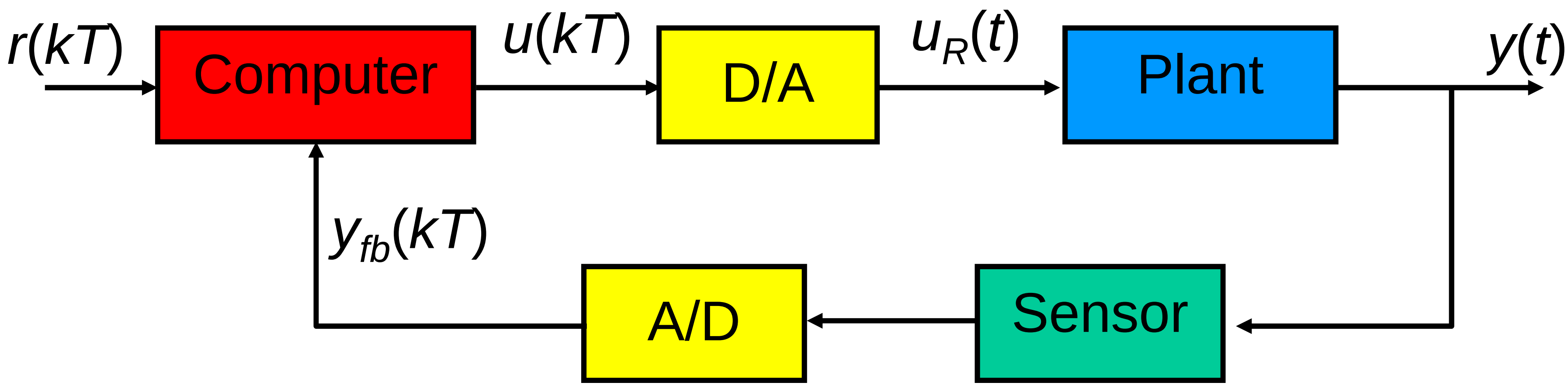
MATHEMATICAL MODEL OF DISCRETE TIME CONTROL SYSTEMS

Content

- ✶ Introduction to discrete-time system
- ✶ The Z-transform
- ✶ Transfer function of discrete-time system
- ✶ State-space equation of discrete-time system

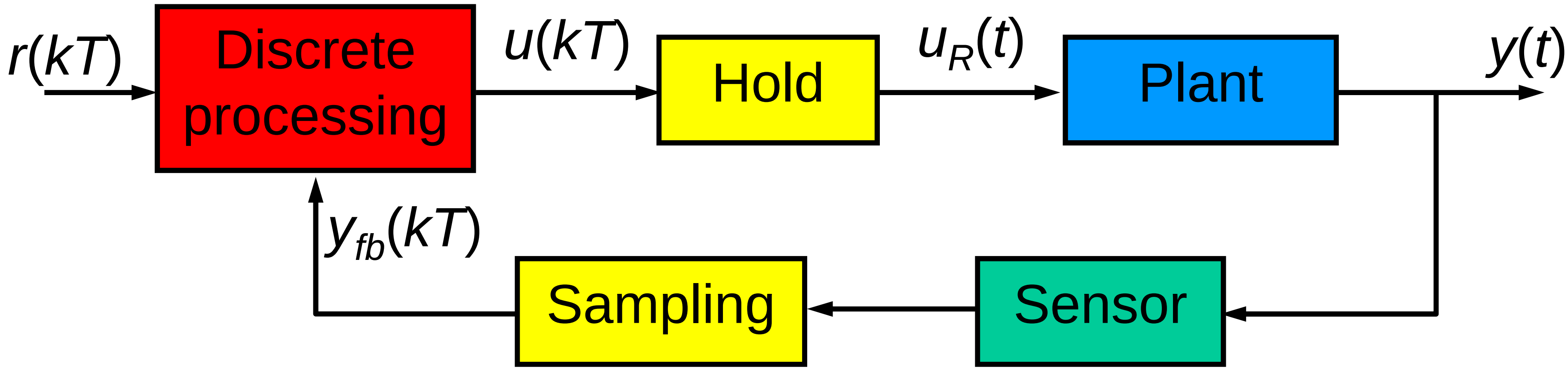
Introduction to discrete-time systems

Digital control systems



- "Computer" = computational equipments based on microprocessor technology (microprocessor, microcontroller, PC, DSP,…).
- Advantages of digital control system:
  - Flexibility
  - Easy to implement complex control algorithms
  - Computer can control many plants at the same time.

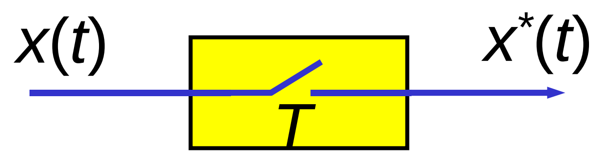
Discrete control systems



- Discrete control systems are control systems which have signals at several points being discrete signal.

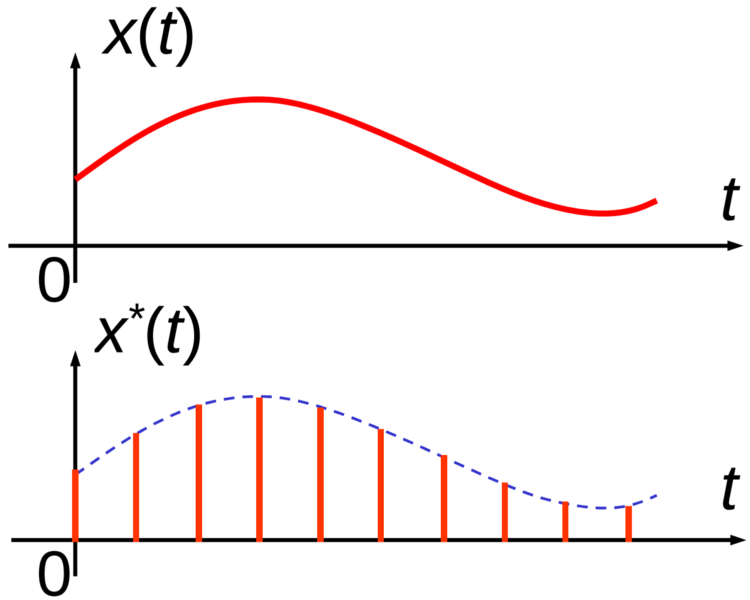
Sampling

- Sampling is the reduction of a continuous signal to a discrete signal.



- Mathematical expression describing the sampling process:

$$X^*(s) = \sum_{k=0}^{+\infty} x(kT)e^{-kTs}$$



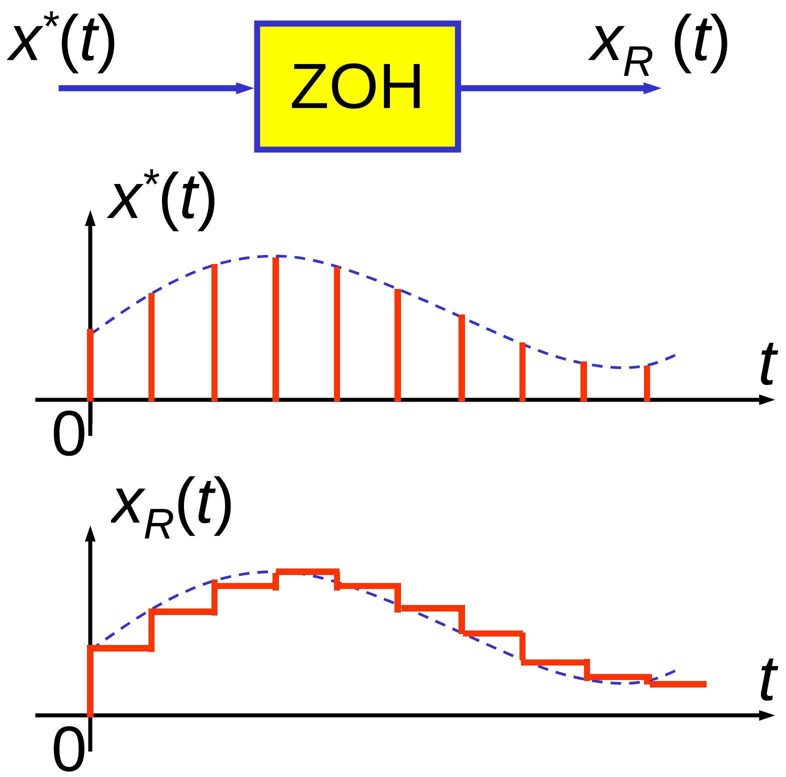
- Shannon's Theorem:

$$f = \frac{1}{T} \geq 2f_c$$

- If quantization error is negligible, then A/D converters are approximate the ideal samplers.

Sampled-data hold

- Sampled-data hold is the reconstruction of discrete signal to a continuous signal.



- Zero-order hold (ZOH): keep signal unchanged between two consecutive sampling instants.

- Transfer function of the ZOH.

$$G_{ZOH}(s) = \frac{1 - e^{-Ts}}{s}$$

- If quantization error is negligible, then D/A converters are approximate the zero-order hold.

The Z-transform

Definition of the Z-transform

- Consider $x(k)$, k=0,1,2,… being a discrete signal. The Z-transform of $x(k)$ is defined as:

$$X(z) = \mathcal{Z}\{x(k)\} = \sum_{k=-\infty}^{+\infty} x(k) z^{-k}$$

where:

– $z = e^{Ts}$ (s is the Laplace variable, T is the sampling period)

– $X(z)$: Z-transform of $x(k)$.

Notation: $x(k) \xleftrightarrow{\mathcal{Z}} X(z)$

- If $x(k) = 0$, $\forall\, k < 0$ then $$X(z) = \mathcal{Z}\{x(k)\} = \sum_{k=0}^{+\infty} x(k) z^{-k}$$

- Region Of Convergence (ROC): set of z such that $X(z)$ is finite.

An interpretation of the Z-transform

- Suppose $x(t)$ being a continuous signal, sample $x(t)$ at the sampling periode T, we have a discrete signal $x(k) = x(kT)$.
- The mathematic model of the process of sampling $x(t)$

$$X^*(s) = \sum_{k=0}^{+\infty} x(kT)e^{-kTs} \qquad (1)$$

- The Z-transform of the sequence $x(k) = x(kT)$.

$$X(z) = \sum_{k=0}^{+\infty} x(k)z^{-k} \qquad (2)$$

- Due to $z = e^{Ts}$, the right hand-side of the expression (1) and (2) are identical. So performing Z-transform of a signal is equivalent to discretizing this signal.

Properties of the Z-transform

Given $x(k)$ and $y(k)$ being two sequences which have the Z-transforms:

$$\mathcal{Z}\{x(k)\} = X(z) \qquad \mathcal{Z}\{y(k)\} = Y(z)$$

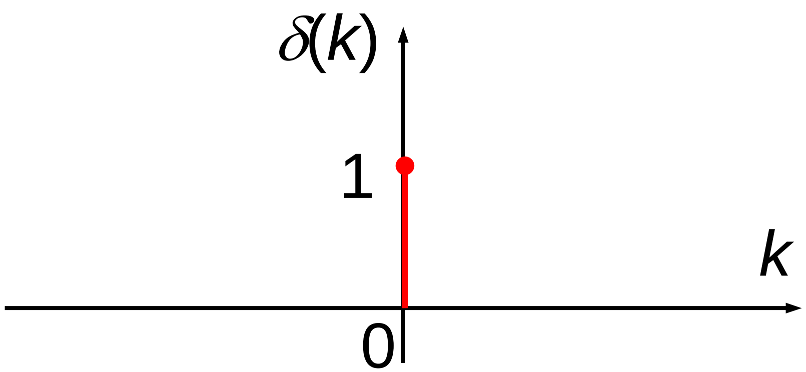
- Linearity: $\mathcal{Z}\{ax(k) + by(k)\} = aX(z) + bY(z)$
- Time shifting: $\mathcal{Z}\{x(k - k_0)\} = z^{-k_0} X(z)$
- Scale in Z-domain: $\mathcal{Z}\{a^k x(k)\} = X(a^{-1} z)$
- Derivative in Z-domain: $\mathcal{Z}\{kx(k)\} = -z \dfrac{dX(z)}{dz}$
- Initial-value theorem: $x(0) = \lim\limits_{z \to \infty} X(z)$
- Final-value theorem: $x(\infty) = \lim\limits_{z \to 1} (1 - z^{-1}) X(z)$

The Z-transform of basic discrete signals

- ✶ Dirac impulse:

$$\delta(k)=\begin{cases}1 & \text{if } k=0\\ 0 & \text{if } k\neq 0\end{cases}$$

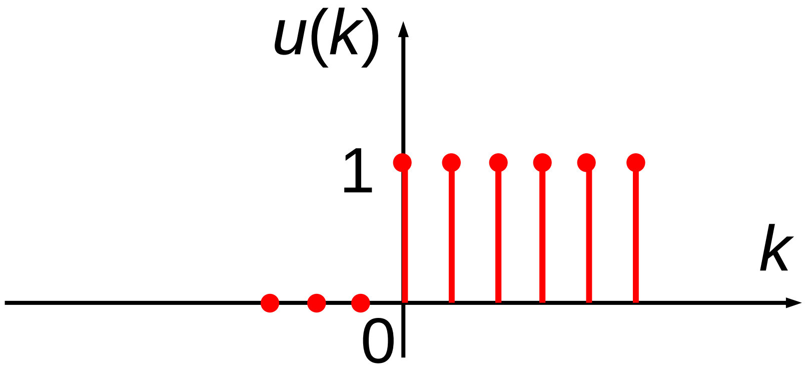
$$\mathcal{Z}\{\delta(k)\}=1$$



- ✶ Step function:

$$u(k)=\begin{cases}1 & \text{if } k\geq 0\\ 0 & \text{if } k<0\end{cases}$$

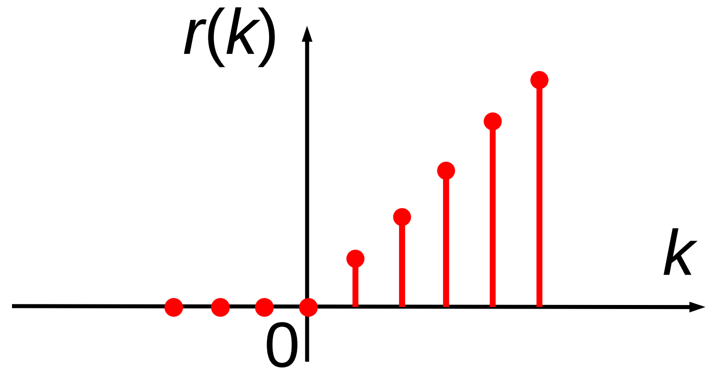
$$\mathcal{Z}\{u(k)\}=\frac{z}{z-1}$$



The Z-transform of basic discrete signals (cont')

★ Ramp function:

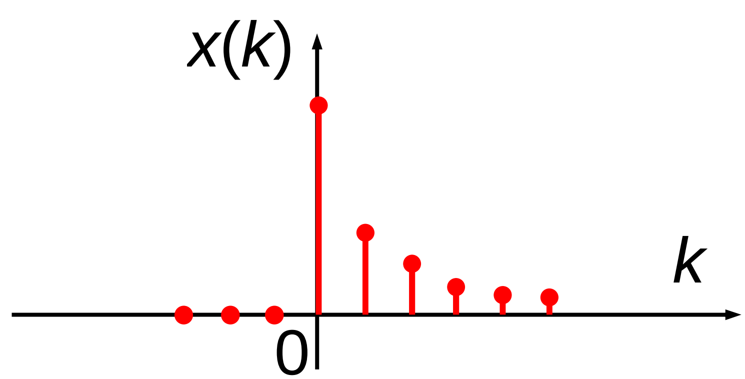
$$r(k)=\begin{cases}kT & \text{if } k\ge 0\\ 0 & \text{if } k<0\end{cases}$$



$$\mathcal{Z}\{u(k)\}=\frac{Tz}{(z-1)^2}$$

★ Exponential function:

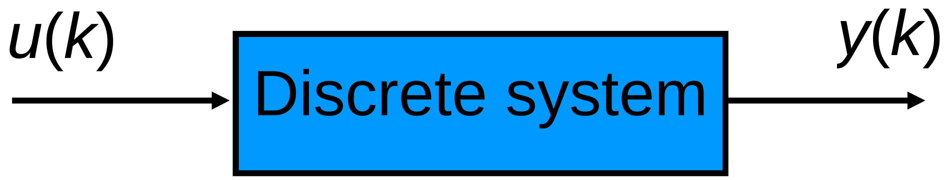
$$x(k)=\begin{cases}e^{-akT} & \text{if } k\ge 0\\ 0 & \text{if } k<0\end{cases}$$



$$\mathcal{Z}\{x(k)\}=\frac{z}{z-e^{-aT}}$$

Discrete transfer function

Derive transfer function (TF) from difference equation



- The input-output relation ship of a discrete system can be described by the difference equation:

$$a_0 y(k+n) + a_1 y(k+n-1) + \ldots + a_{n-1} y(k+1) + a_n y(k) =$$
$$b_0 u(k+m) + b_1 u(k+m-1) + \ldots + b_{m-1} u(k+1) + b_m u(k)$$

where *n*>*m*, *n* is the order of the system.

- Taking the Z-transform the two sides of the above equation:

$$a_0 z^n Y(z) + a_1 z^{n-1} Y(z) + \ldots + a_{n-1} z Y(z) + a_n Y(z) =$$
$$b_0 z^m U(z) + b_1 z^{m-1} U(z) + \ldots + b_{m-1} z U(z) + b_m U(z)$$

Derive TF from difference equation (con't)

- Taking the ratio $Y(z)/U(z)$ to obtain the transfer function:

$$G(z)=\frac{Y(z)}{U(z)}=\frac{b_0z^m+b_1z^{m-1}+\ldots+b_{m-1}z+b_m}{a_0z^n+a_1z^{n-1}+\ldots+a_{n-1}z+a_n}$$

- The above transfer function can be transformed into the equivalent form:

$$G(z)=\frac{Y(z)}{U(z)}=\frac{z^{-(n-m)}[b_0+b_1z^{-1}+\ldots+b_{m-1}z^{-m+1}+b_mz^{-m}]}{a_0+a_1z^{-1}+\ldots+a_{n-1}z^{-n+1}+a_nz^{-n}}$$

Derive TF from difference equation _ Example

- Consider a system described by the difference equation. Derive its transfer function:

$$y(k+3)+2y(k+2)-5y(k+1)+3y(k)=2u(k+2)+u(k)$$

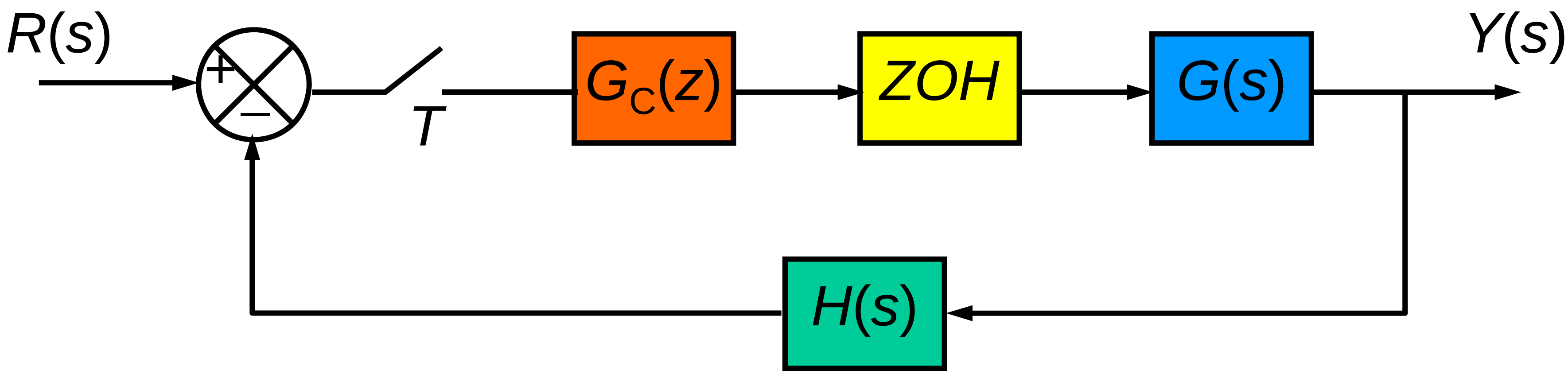
- Solution: Taking the Z-transform the difference equation:

$$z^3Y(z)+2z^2Y(z)-5zY(z)+3Y(z)=2z^2U(z)+U(z)$$

$$\Rightarrow \quad G(z)=\frac{Y(z)}{U(z)}=\frac{2z^2+1}{z^3+2z^2-5z+3}$$

$$\Leftrightarrow \quad G(z)=\frac{Y(z)}{U(z)}=\frac{z^{-1}(2+z^{-2})}{1+2z^{-1}-5z^{-2}+3z^{-3}}$$

Calculate transfer function from block diagram



★ The closed-loop TF:

$$G_k(z) = \frac{Y(z)}{R(z)} = \frac{G_C(z)G(z)}{1 + G_C(z)GH(z)}$$

where

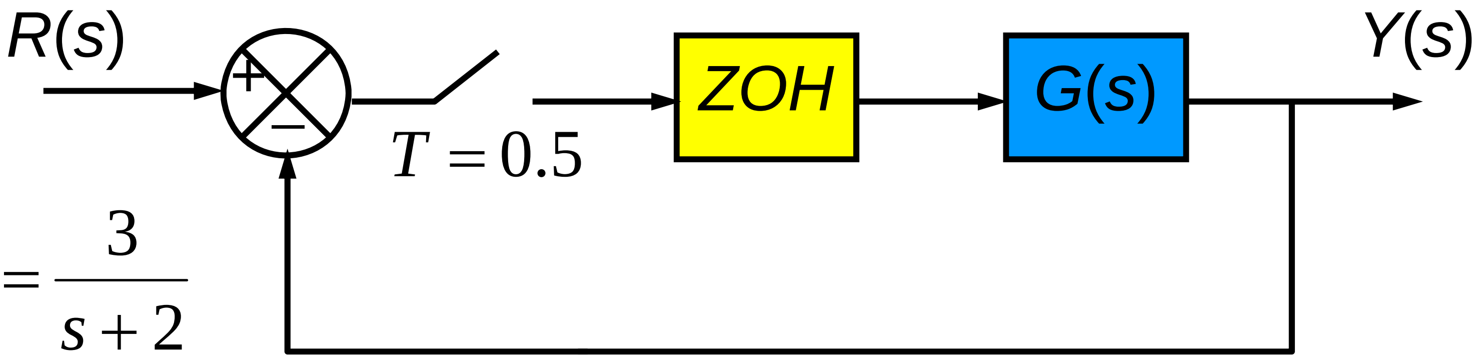
$G_C(z)$: TF of the controller, derive from difference equation

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

$$GH(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)H(s)}{s}\right\}$$

Calculate TF from block diagram – Example 1

★ Find the closed-loop transfer function of the system:



$$G(s)=\frac{3}{s+2}$$

Solution: $G(z)=(1-z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}=(1-z^{-1})\mathcal{Z}\left\{\frac{3}{s(s+2)}\right\}$

$$=(1-z^{-1})\frac{3}{2}\frac{z(1-e^{-2\times 0.5})}{(z-1)(z-e^{-2\times 0.5})}$$

$$\Rightarrow \quad \boxed{G(z)=\frac{0.948}{z-0.368}}$$

Calculate transfer function from block diagram – Example 1 (cont')

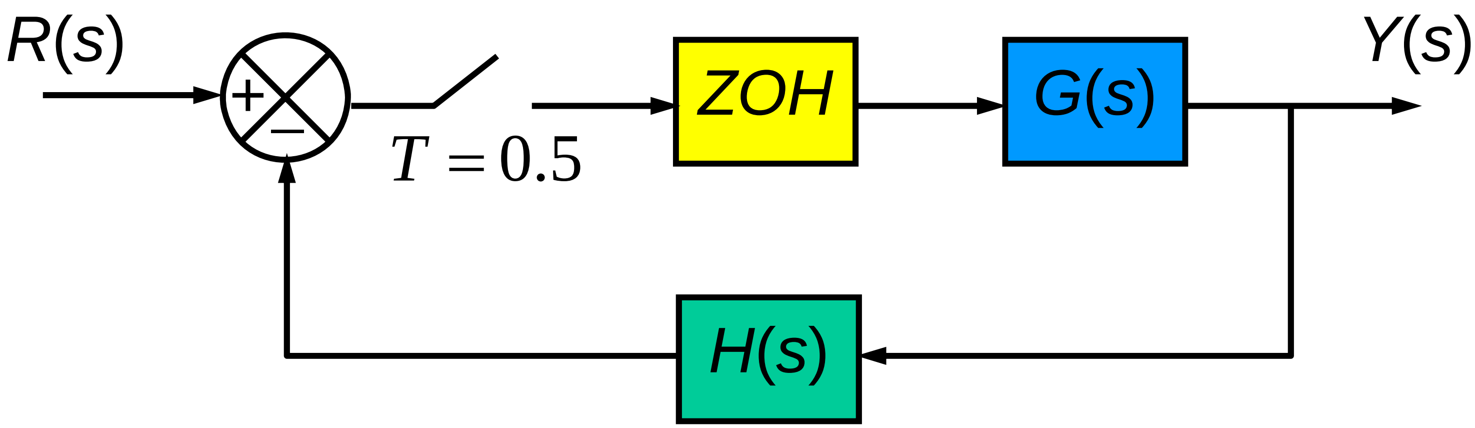
- The closed-loop transfer function:

$$G_k(z) = \frac{G(z)}{1+G(z)} = \frac{\dfrac{0.948}{z-0.368}}{1+\dfrac{0.948}{z-0.368}}$$

$$\Rightarrow \quad \boxed{G_k(z) = \frac{0.948}{z+0.580}}$$

Calculate TF from block diagram – Example 2

- Calculate the transfer function of the system:



Given that $G(s)=\dfrac{3e^{-s}}{s+3}$ $\quad H(s)=\dfrac{1}{s+1}$

- Solution:

The closed-loop transfer function:

$$G_k(z)=\frac{G(z)}{1+GH(z)}$$

Calculate TF from block diagram – Example 2 (cont')

- $$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

$$= (1 - z^{-1})\mathcal{Z}\left\{\frac{3e^{-s}}{s(s+3)}\right\}$$

$$= (1 - z^{-1})z^{-2}\frac{z(1 - e^{-3\times 0.5})}{(z-1)(z - e^{-3\times 0.5})}$$

$$\Rightarrow \quad \boxed{G(z) = \frac{0.777}{z^2(z - 0.223)}}$$

Calculate TF from block diagram – Example 2 (cont')

- $$GH(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)H(s)}{s}\right\}$$

$$= (1 - z^{-1})\mathcal{Z}\left\{\frac{3e^{-s}}{s(s+3)(s+1)}\right\}$$

$$= 3(1 - z^{-1})z^{-2}\frac{z(Az+B)}{(z-1)(z-e^{-3\times 0.5})(z-e^{-1\times 0.5})}$$

$$A = \frac{(1-e^{-3\times 0.5}) - 3(1-e^{-0.5})}{3(1-3)} = 0.0673$$

$$B = \frac{3e^{-3\times 0.5}(1-e^{-0.5}) - e^{-0.5}(1-e^{-3\times 0.5})}{3(1-3)} = 0.0346$$

$$\Rightarrow \quad \boxed{GH(z) = \frac{0.202z + 0.104}{z^2(z-0.223)(z-0.607)}}$$

Calculate TF from block diagram – Example 2 (cont')

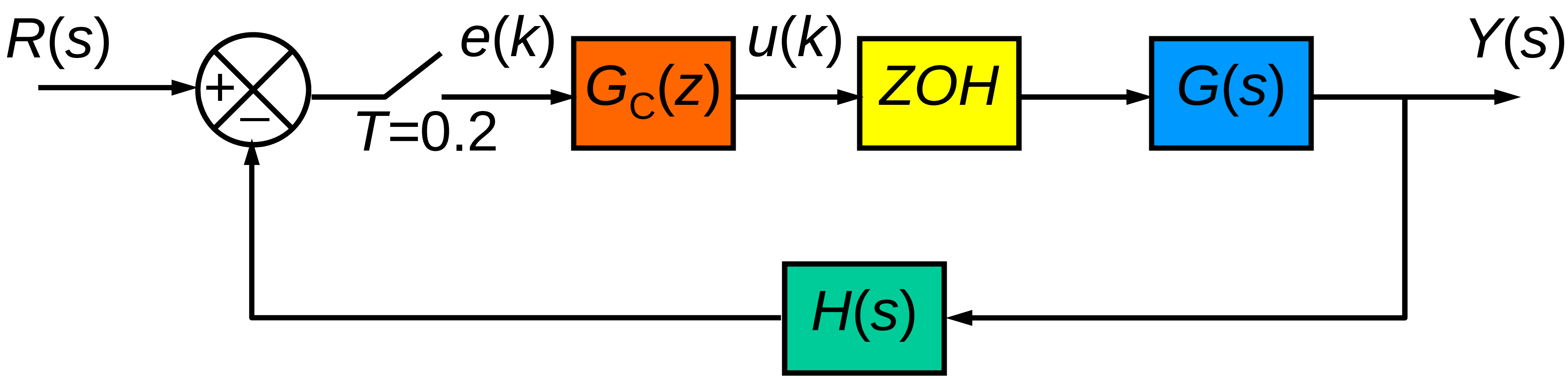
★ The closed-loop transfer function:

$$G_k(z) = \frac{G(z)}{1+GH(z)} = \frac{\dfrac{0.777}{z^2(z-0.223)}}{1+\dfrac{0.202z+0.104}{z^2(z-0.223)(z-0.607)}}$$

$$\Rightarrow \quad \boxed{G_k(z) = \frac{0.777(z-0.607)}{z^4-0.83z^3+0.135z^2+0.202z+0.104}}$$

Calculate TF from block diagram – Example 3

- Calculate the closed-loop transfer function of the system:



Given that: $G(s)=\dfrac{5e^{-0.2s}}{s^2}$ $\qquad H(s)=0.1$

The controller is described by the difference equation:

$$u(k)=10e(k)-2e(k-1)$$

Calculate TF from block diagram – Example 3 (cont')

- **Solution:**

  The closed-loop transfer function:

$$G_k(z) = \frac{G_C(z)G(z)}{1 + G_C(z)GH(z)}$$

- The TF of the controller is calculated from the difference equation:

$$u(k) = 10e(k) - 2e(k-1)$$

$$\Rightarrow \quad U(z) = 10E(z) - 2z^{-1}E(z)$$

$$\Rightarrow \quad G_C(z) = \frac{U(z)}{E(z)} = 10 - 2z^{-1}$$

Calculate TF from block diagram – Example 3 (cont')

- $$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

$$= (1 - z^{-1})\mathcal{Z}\left\{\frac{5e^{-0.2s}}{s^3}\right\} = 5(1 - z^{-1})z^{-1}\frac{(0.2)^2 z(z+1)}{2(z-1)^3}$$

$\Rightarrow$ $$\boxed{G(z) = \frac{0.1(z+1)}{z(z-1)^2}}$$

- $$GH(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)H(s)}{s}\right\}$$

$$= 0.1(1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

$\Rightarrow$ $$\boxed{GH(z) = \frac{0.01(z+1)}{z(z-1)^2}}$$

Calculate TF from block diagram – Example 3 (cont')

- The closed-loop transfer function:

$$G_k(z) = \frac{G_C(z)G(z)}{1+G_C(z)GH(z)} = \frac{\left[\dfrac{10z-2}{z}\right].\left[\dfrac{0.1(z+1)}{z(z-1)^2}\right]}{1+\left[\dfrac{10z-2}{z}\right].\left[\dfrac{0.01(z+1)}{z(z-1)^2}\right]}$$

$$\Rightarrow \quad \boxed{G_k(z) = \frac{z^2+0.8z-0.2}{z^4-2z^3+1.1z^2+0.08z-0.02}}$$

State-space model of discrete system

The discrete state space (SS) equation

- The state-space model of a discrete system is a set of first-order difference equations of the form:

$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d \boldsymbol{x}(k) + \boldsymbol{B}_d r(k) \\ y(k) = \boldsymbol{C}_d \boldsymbol{x}(k) \end{cases}$$

where:

$$\boldsymbol{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} \qquad \boldsymbol{A}_d = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \qquad \boldsymbol{B}_d = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\boldsymbol{C}_d = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}$$

Derive SS equation from difference equation

- ***Case 1:*** The right-hand side of the difference equation does not involve the differences of the input:

$$a_0 y(k+n) + a_1 y(k+n-1) + \ldots + a_{n-1} y(k+1) + a_n y(k) = b_0 u(k)$$

- Define the state variables:
  - The first state variable is the output of the system;
  - The i^{th} state variable (i=2..n) is set to be one sample time-advanced of the (i–1)th state variable.

$$x_1(k) = y(k)$$
$$x_2(k) = x_1(k+1)$$
$$x_3(k) = x_2(k+1)$$
$$\vdots$$
$$x_n(k) = x_{n-1}(k+1)$$

Derive SS equation from difference equation

Case 1 (cont')

- The state equations: $\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d\boldsymbol{x}(k) + \boldsymbol{B}_d u(k) \\ y(k) = \boldsymbol{C}_d\boldsymbol{x}(k) \end{cases}$

where:

$$\boldsymbol{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} \quad \boldsymbol{A}_d = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\dfrac{a_n}{a_0} & -\dfrac{a_{n-1}}{a_0} & -\dfrac{a_{n-2}}{a_0} & \dots & -\dfrac{a_1}{a_0} \end{bmatrix} \quad \boldsymbol{B}_d = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \dfrac{b_0}{a_0} \end{bmatrix}$$

$$\boldsymbol{C}_d = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Derive SS equation from difference equation – Case 1 example

- Write the state equations of the system described by:

$$2y(k+3)+y(k+2)+5y(k+1)+4y(k)=3u(k)$$

- Define the state variables: $\begin{cases} x_1(k)=y(k) \\ x_2(k)=x_1(k+1) \\ x_3(k)=x_2(k+1) \end{cases}$

- The state equations: $\begin{cases} \boldsymbol{x}(k+1)=\boldsymbol{A}_d\boldsymbol{x}(k)+\boldsymbol{B}_d r(k) \\ y(k)=\boldsymbol{C}_d\boldsymbol{x}(k) \end{cases}$

where:

$$\boldsymbol{A}_d=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\dfrac{a_3}{a_0} & -\dfrac{a_2}{a_0} & -\dfrac{a_1}{a_0} \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -2.5 & -0.5 \end{bmatrix}$$

$$\boldsymbol{B}_d=\begin{bmatrix} 0 \\ 0 \\ \dfrac{b_0}{a_0} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 1.5 \end{bmatrix}$$

$$\boldsymbol{C}_d=\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Derive SS equation from difference equation

- ***Case 2:*** The right-hand side of the difference equation involve the differences of the input:

$$a_0 y(k+n) + a_1 y(k+n-1) + \ldots + a_{n-1} y(k+1) + a_n y(k) =$$
$$b_0 u(k+n-1) + b_1 u(k+n-2) + \ldots + b_{n-2} u(k+1) + b_{n-1} u(k)$$

- Define the state variable:
  - The first state variable is the output of the system;
  - The i^{th} state variable (i=2..n) is set to be one sample time-advanced of the (i–1)th state variable minus a quantity proportional to the input

$$x_1(k) = y(k)$$
$$x_2(k) = x_1(k+1) - \beta_1 u(k)$$
$$x_3(k) = x_2(k+1) - \beta_2 u(k)$$
$$\vdots$$
$$x_n(k) = x_{n-1}(k+1) - \beta_{n-1} u(k)$$

Derive SS equation from difference equation

Case 2 (cont')

★ The state equation: $\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d \boldsymbol{x}(k) + \boldsymbol{B}_d u(k) \\ y(k) = \boldsymbol{C}_d \boldsymbol{x}(k) \end{cases}$

where:

$$\boldsymbol{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} \quad \boldsymbol{A}_d = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\dfrac{a_n}{a_0} & -\dfrac{a_{n-1}}{a_0} & -\dfrac{a_{n-2}}{a_0} & \dots & -\dfrac{a_1}{a_0} \end{bmatrix} \quad \boldsymbol{B}_d = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix}$$

$$\boldsymbol{C}_d = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Derive SS equation from difference equation

Case 2 (cont')

The coefficient β_i in the vector $\boldsymbol{B}_d$ are defined as:

$$\beta_1 = \frac{b_0}{a_0}$$

$$\beta_2 = \frac{b_1 - a_1\beta_1}{a_0}$$

$$\beta_3 = \frac{b_2 - a_1\beta_2 - a_2\beta_1}{a_0}$$

$$\vdots$$

$$\beta_n = \frac{b_{n-1} - a_1\beta_{n-1} - a_2\beta_{n-2} - \ldots - a_{n-1}\beta_1}{a_0}$$

Derive SS equation from difference equation – Case 2 example

- Write the state equations of the system described by:

$$2y(k+3)+y(k+2)+5y(k+1)+4y(k)=u(k+2)+3u(k)$$

- Define the state variables: $\begin{cases} x_1(k)=y(k) \\ x_2(k)=x_1(k+1)-\beta_1 r(k) \\ x_3(k)=x_2(k+1)-\beta_2 r(k) \end{cases}$

- The state equations: $\begin{cases} \boldsymbol{x}(k+1)=\boldsymbol{A}_d\boldsymbol{x}(k)+\boldsymbol{B}_d u(k) \\ y(k)=\boldsymbol{C}_d\boldsymbol{x}(k) \end{cases}$

where:

$$\boldsymbol{A}_d=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\dfrac{a_3}{a_0} & -\dfrac{a_2}{a_0} & -\dfrac{a_1}{a_0} \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -2.5 & -0.5 \end{bmatrix} \qquad \boldsymbol{B}_d=\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

$$\boldsymbol{C}_d=\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

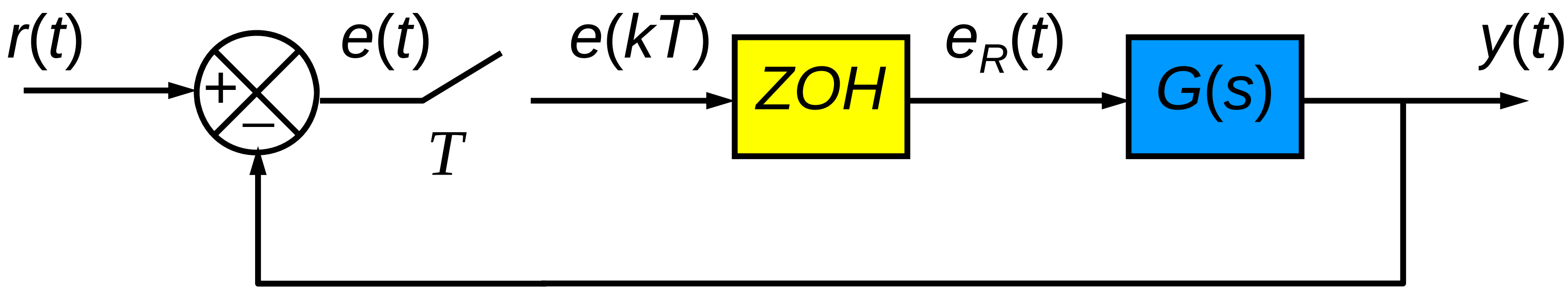
Derive SS from difference equation – Case 2 example (cont')

★ The coefficient β_i in the vector $\boldsymbol{B}_d$ are calculated as:

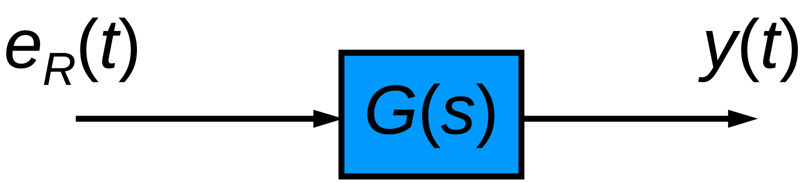
$$\begin{cases} \beta_1 = \dfrac{b_0}{a_0} = \dfrac{1}{2} = 0.5 \\ \beta_2 = \dfrac{b_1 - a_1\beta_1}{a_0} = \dfrac{0 - 1\times 0.5}{2} = -0.25 \\ \beta_3 = \dfrac{b_2 - a_1\beta_2 - a_2\beta_1}{a_0} = \dfrac{3 - 1\times(-0.25) - 5\times 0.5}{2} = 0.375 \end{cases}$$

$$\Rightarrow \quad \boldsymbol{B}_d = \begin{bmatrix} 0.5 \\ -0.25 \\ 0.375 \end{bmatrix}$$

Formulation of SS from block diagram



- ***Step 1:*** Write the state space equations of the open-loop continuous system:



$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}e_R(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

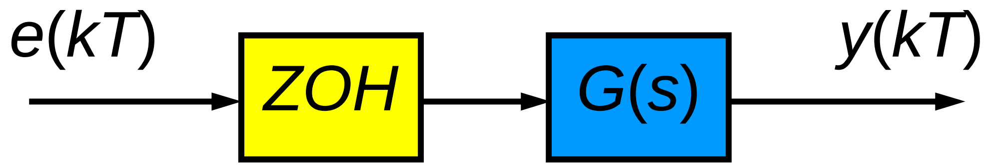
- ***Step 2:*** Calculate the transient matrix:

$$\Phi(t) = \mathscr{L}^{-1}[\Phi(s)]$$

where $$\Phi(s) = \left(s\boldsymbol{I} - \boldsymbol{A}\right)^{-1}$$

Formulation of SS equations from block diagram (cont')

- ***Step 3:*** Discretizing the open-loop continuous SS equation:



$$\begin{cases} \boldsymbol{x}[(k+1)T] = \boldsymbol{A}_d \boldsymbol{x}(kT) + \boldsymbol{B}_d e_R(kT) \\ y(kT) = \boldsymbol{C}_d \boldsymbol{x}(kT) \end{cases}$$

with

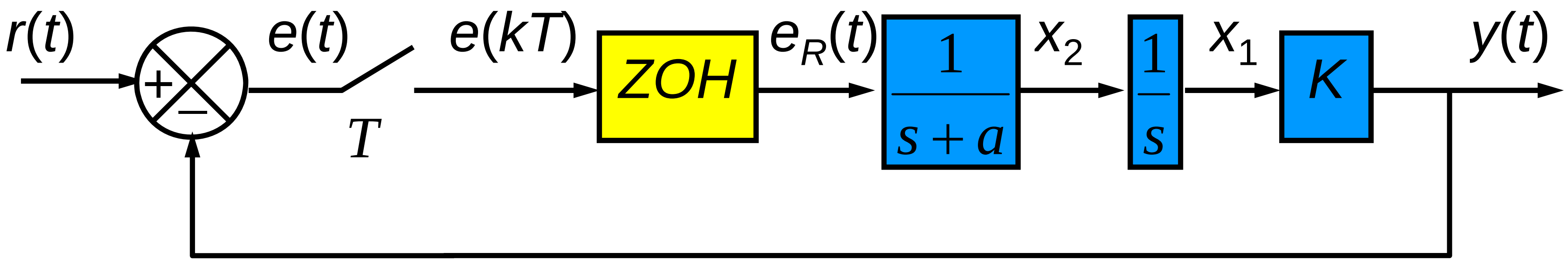
$$\begin{cases} \boldsymbol{A}_d = \Phi(T) \\ \boldsymbol{B}_d = \int_0^T \Phi(\tau) B d\tau \\ \boldsymbol{C}_d = \boldsymbol{C} \end{cases}$$

- ***Step 4:*** Write the closed-loop discrete state equations (which has input signal $r(kT)$)

$$\begin{cases} \boldsymbol{x}[(k+1)T] = \left[\boldsymbol{A}_d - \boldsymbol{B}_d \boldsymbol{C}_d\right] \boldsymbol{x}(kT) + \boldsymbol{B}_d r(kT) \\ y(kT) = \boldsymbol{C}_d \boldsymbol{x}(kT) \end{cases}$$

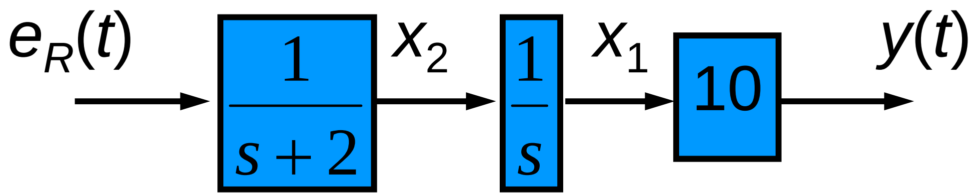
Formulation of SS equations from block diagram – Example

- Formulate the SS equations describing the system:



where $a = 2$, $T = 0.5$, $K = 10$

- Solution:
- Step 1:



$$X_1(s)=\frac{X_2(s)}{s} \quad \Rightarrow \quad sX_1(s)=X_2(s) \quad \Rightarrow \quad \dot{x}_1(t)=x_2(t)$$

$$X_2(s)=\frac{E_R(s)}{s+2} \quad \Rightarrow \quad (s+2)X_2(s)=E_R(s) \quad \Rightarrow \quad \dot{x}_2(t)=-2x_2(t)+e_R(t)$$

$$\Rightarrow \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\boldsymbol{B}} e_R(t) \\ y(t)=10x_1(t)=\underbrace{\begin{bmatrix} 10 & 0 \end{bmatrix}}_{\boldsymbol{C}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

★ Step 2: Calculate the transient matrix

$$\Phi(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1} = \left(s\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1\\0 & -2\end{bmatrix}\right)^{-1} = \left(\begin{bmatrix}s & -1\\0 & s+2\end{bmatrix}\right)^{-1}$$

$$= \frac{1}{s(s+2)}\begin{bmatrix}s+2 & 1\\0 & s\end{bmatrix} = \begin{bmatrix}\dfrac{1}{s} & \dfrac{1}{s(s+2)}\\0 & \dfrac{1}{s+2}\end{bmatrix}$$

$$\Phi(t) = \mathcal{L}^{-1}[\Phi(s)] = \mathcal{L}^{-1}\left\{\begin{bmatrix}\dfrac{1}{s} & \dfrac{1}{s(s+2)}\\0 & \dfrac{1}{s+2}\end{bmatrix}\right\} = \begin{bmatrix}\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} & \mathcal{L}^{-1}\left\{\dfrac{1}{s(s+2)}\right\}\\0 & \mathcal{L}^{-1}\left\{\dfrac{1}{s+2}\right\}\end{bmatrix}$$

$$\Rightarrow \quad \Phi(t) = \begin{bmatrix}1 & \dfrac{1}{2}(1-e^{-2t})\\0 & e^{-2t}\end{bmatrix}$$

- Step 3: Discretizing the open-loop continuous state equations:

$$\begin{cases} \boldsymbol{x}[(k+1)T] = \boldsymbol{A}_d \boldsymbol{x}(kT) + \boldsymbol{B}_d e_R(kT) \\ y(kT) = \boldsymbol{C}_d \boldsymbol{x}(kT) \end{cases}$$

$$\boldsymbol{A}_d = \Phi(T) = \left.\begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}\right|_{t=T} = \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2\times 0.5}) \\ 0 & e^{-2\times 0.5} \end{bmatrix} = \begin{bmatrix} 1 & 0.316 \\ 0 & 0.368 \end{bmatrix}$$

$$\boldsymbol{B}_d = \int_0^T \Phi(\tau)\boldsymbol{B}\,d\tau = \int_0^T \left\{ \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2\tau}) \\ 0 & e^{-2\tau} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau \right\} = \int_0^T \left\{ \begin{bmatrix} \frac{1}{2}(1-e^{-2\tau}) \\ e^{-2\tau} \end{bmatrix} d\tau \right\}$$

$$= \left.\begin{bmatrix} \left(\frac{\tau}{2} + \frac{e^{-2\tau}}{2^2}\right) \\ -\frac{e^{-2\tau}}{2} \end{bmatrix}\right|_0^T = \begin{bmatrix} \left(\frac{0.5}{2} + \frac{e^{-2\times 0.5}}{2^2} - \frac{1}{2^2}\right) \\ -\frac{e^{-2\times 0.5}}{2} + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0.092 \\ 0.316 \end{bmatrix}$$

$$\boldsymbol{C}_d = \boldsymbol{C} = \begin{bmatrix} 10 & 0 \end{bmatrix}$$

Formulation of state equations from block diagram – Example (cont')

- Step 4: The closed-loop discrete state equations:

$$\begin{cases} \boldsymbol{x}[(k+1)T] = [\boldsymbol{A}_d - \boldsymbol{B}_d\boldsymbol{C}_d]\boldsymbol{x}(kT) + \boldsymbol{B}_d r(kT) \\ y(kT) = \boldsymbol{C}_d\boldsymbol{x}(kT) \end{cases}$$

where $[\boldsymbol{A}_d - \boldsymbol{B}_d\boldsymbol{C}_d] = \begin{bmatrix} 1 & 0.316 \\ 0 & 0.368 \end{bmatrix} - \begin{bmatrix} 0.092 \\ 0.316 \end{bmatrix}[10 \quad 0] = \begin{bmatrix} 0.080 & 0.316 \\ -3.160 & 0.368 \end{bmatrix}$

- Conclusion: The closed-loop state equation is:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.080 & 0.316 \\ -3.160 & 0.368 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.092 \\ 0.316 \end{bmatrix} r(k)$$

$$y(k) = [10 \quad 0].\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Calculate transfer function from state equation

- Given the state equation

$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d \boldsymbol{x}(k) + \boldsymbol{B}_d u(k) \\ y(k) = \boldsymbol{C}_d \boldsymbol{x}(k) \end{cases}$$

- The corresponding transfer function is:

$$G(z) = \frac{Y(z)}{U(z)} = \boldsymbol{C}_d (z\boldsymbol{I} - \boldsymbol{A}_d)^{-1} \boldsymbol{B}_d$$

Calculate transfer function from state equation - Example

- Calculate the TF of the system described by the SS equation:

$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d \boldsymbol{x}(k) + \boldsymbol{B}_d u(k) \\ y(k) = \boldsymbol{C}_d \boldsymbol{x}(k) \end{cases}$$

$$\boldsymbol{A}_d = \begin{bmatrix} 0 & 1 \\ -0.7 & -0.1 \end{bmatrix} \qquad \boldsymbol{B}_d = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \boldsymbol{C}_d = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- Solution: The transfer function is:

$$G(z) = \boldsymbol{C}_d (z\boldsymbol{I} - \boldsymbol{A}_d)^{-1} \boldsymbol{B}_d$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.7 & -0.1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \boxed{G(z) = \frac{2}{z^2 + 0.1z + 0.7}}$$