Lecture Notes

Fundamentals of Control Systems

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Chapter 2

Mathematical Models of Continuous Control Systems

Content

- The concept of mathematical model
- Transfer function
- Block diagram algebra
- Signal flow diagram
- State space equation
- Linearized models of nonlinear systems
  - Nonlinear state equation
  - Linearized equation of state

The concept of mathematical models

Question

- If you design a control system, what do you need to know about the plant or the process to be controlled?

- What are the advantages of mathematical models?

- Practical control systems are diverse and different in nature.
- It is necessary to have a common method for analysis and design of different type of control systems ⇒ Mathematics
- The relationship between input and output of a LTI system of can be described by linear constant coefficient equations:

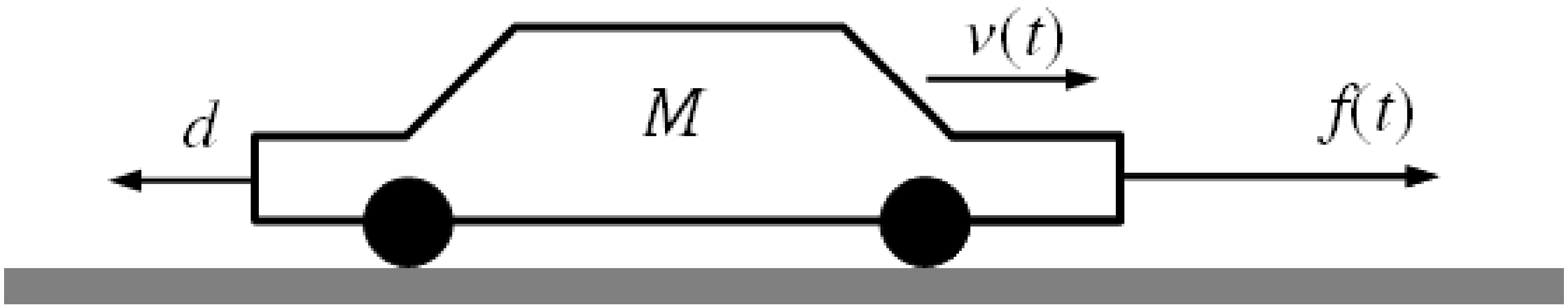
$$a_0\frac{d^n y(t)}{dt^n}+a_1\frac{d^{n-1}y(t)}{dt^{n-1}}+\cdots+a_{n-1}\frac{dy(t)}{dt}+a_n y(t)=$$

$$b_0\frac{d^m u(t)}{dt^m}+b_1\frac{d^{m-1}u(t)}{dt^{m-1}}+\cdots+b_{m-1}\frac{du(t)}{dt}+b_m u(t)$$

n: system order, for proper systems: $n \geq m$.

a_i, b_i: parameter of the system

Example: Car dynamics



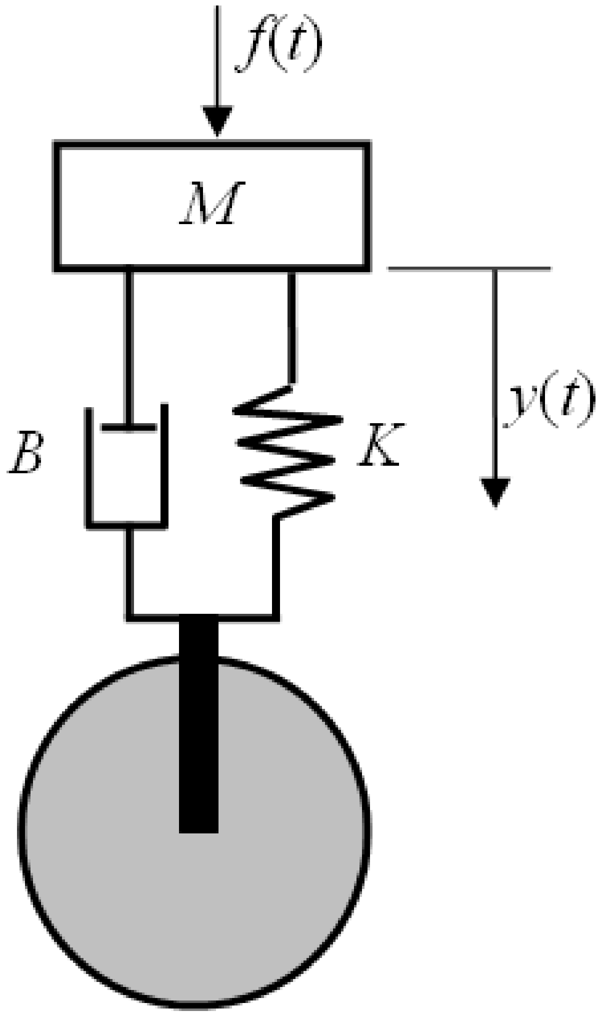
$$M\frac{dv(t)}{dt} + Bv(t) = f(t)$$

M: mass of the car, *B* friction coefficient: system parameters

f(*t*): engine driving force: input

v(*t*): car speed: output

Example: Car suspension



$$M\frac{d^2y(t)}{dt^2}+B\frac{dy(t)}{dt}+Ky(t)=f(t)$$

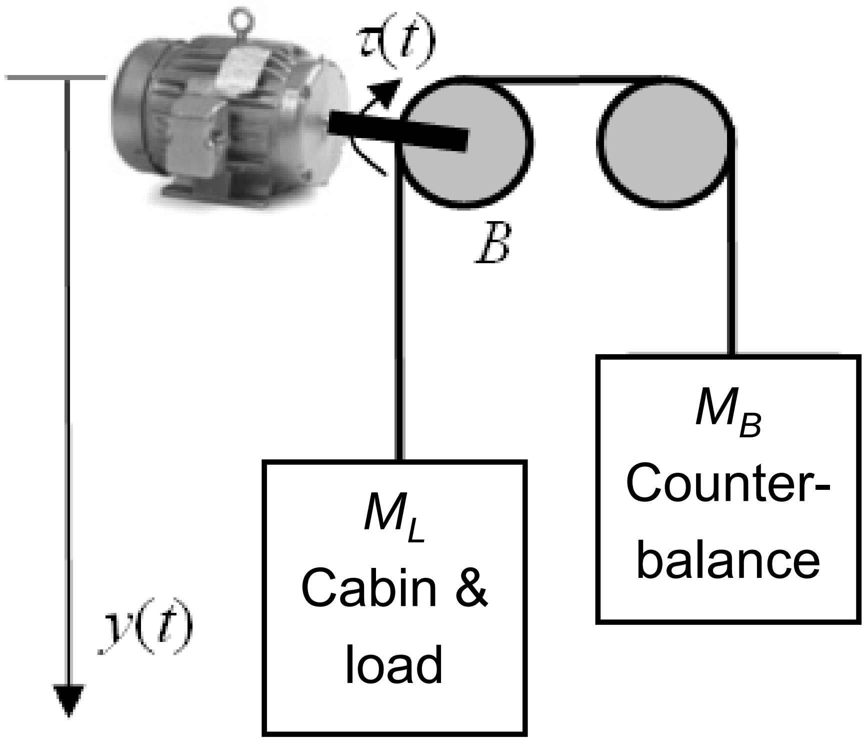
M: equivalent mass

B friction constant, *K* spring stiffness

f(*t*): external force: input

y(*t*): travel of the car body: output

Example: Elevator



M_L: mass of cabin and load,
M_B: counterbalance
B friction constant,
K gear box constant
$\tau(t)$: driving moment of the motor
$y(t)$: position of the cabin

$$M_L \frac{d^2y(t)}{dt^2} + B\frac{dy(t)}{dt} + M_T g = K\tau(t) + M_B g$$

Disadvantages of differential equation model

- Difficult to solve differential equation order *n* (*n*>2)

$$a_0 \frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t) = b_0 \frac{d^m u(t)}{dt^m} + b_1 \frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_{m-1} \frac{du(t)}{dt} + b_m u(t)$$

- System analysis based on differential equation model is difficult.
- System design based on differential equations is almost impossible in general cases.
- It is necessary to have another mathematical model that makes the analysis and design of control systems easier:
  - transfer function
  - state space equation

Transfer functions

Definition of Laplace transform

The Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, defined by:

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^{+\infty} f(t).e^{-st}dt$$

where:

– s : complex variable (Laplace variable)

– $\mathscr{L}$: Laplace operator

– $F(s)$ Laplace transform of $f(t)$.

The Laplace transform exists if the integral of $f(t)$ in the interval $[0,+\infty)$ is convergence.

Properties of Laplace transform

Given the functions *f*(*t*) and *g*(*t*), and their respective Laplace transforms *F*(*s*) and *G*(*s*):

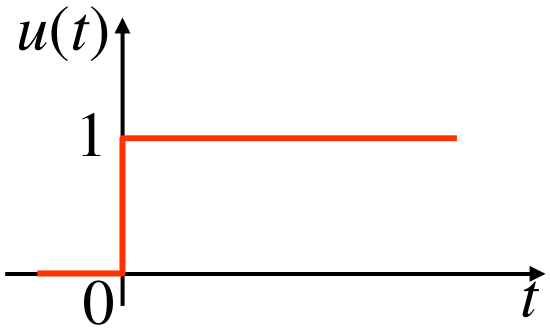
$$\mathscr{L}\{f(t)\} = F(s) \qquad \mathscr{L}\{g(t)\} = G(s)$$

- **Linearity** $$\mathscr{L}\{a.f(t) + b.g(t)\} = a.F(s) + b.G(s)$$
- **Time shifting** $$\mathscr{L}\{f(t-T)\} = e^{-Ts}.F(s)$$
- **Differentiation** $$\mathscr{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^+)$$
- **Integration** $$\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$
- **Final value theorem** $$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Laplace transform of basic functions

- Unit step function:

$$u(t)=\begin{cases}1 & \text{if } t\geq 0\\ 0 & \text{if } t<0\end{cases}$$

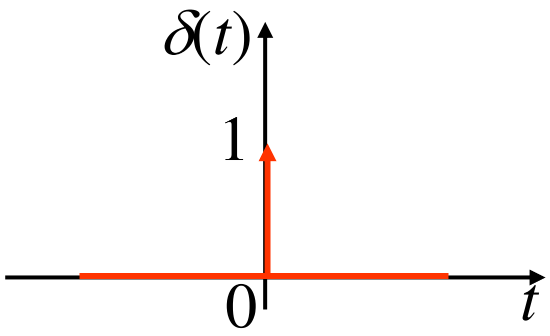


$$\mathscr{L}\{u(t)\}=\frac{1}{s}$$

- Dirac function:

$$\delta(t)=\begin{cases}0 & \text{if } t\neq 0\\ \infty & \text{if } t=0\end{cases}$$

$$\int_{-\infty}^{+\infty}\delta(t)dt=1$$

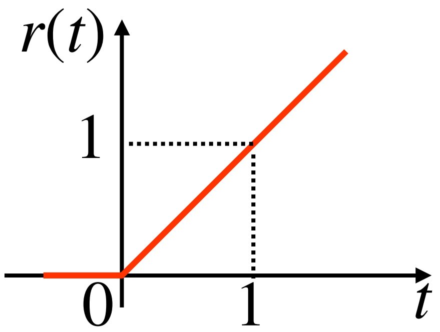


$$\mathscr{L}\{\delta(t)\}=1$$

Laplace transform of basic functions (cont')

- Ramp function:

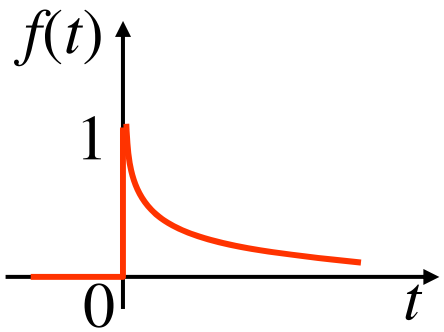
$$r(t) = tu(t) = \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$



$$\mathscr{L}\{t.u(t)\} = \frac{1}{s^2}$$

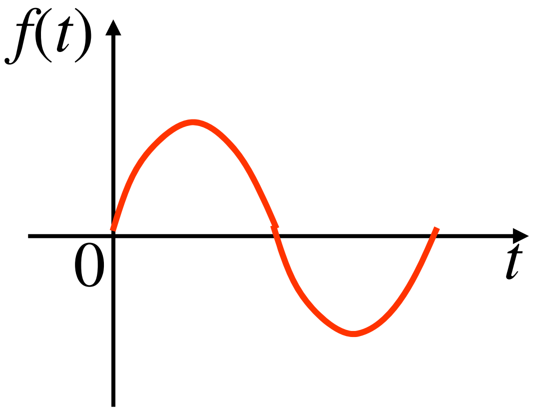
- Exponential function

$$f(t) = e^{-at}.u(t) = \begin{cases} e^{-at} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$



$$\mathscr{L}\{e^{-at}.u(t)\} = \frac{1}{s+a}$$

* Sinusoidal function



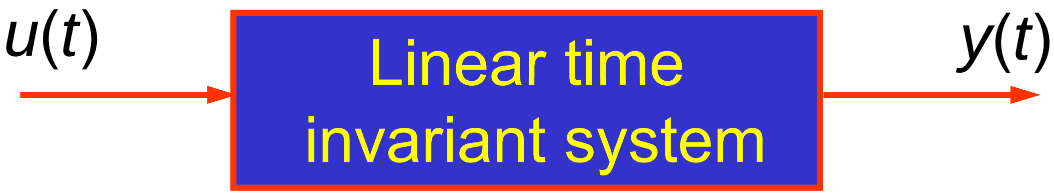
$$f(t) = (\sin \omega t).u(t) = \begin{cases} \sin \omega t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$$\mathscr{L}\{(\sin \omega t)u(t)\} = \frac{\omega}{s^2 + \omega^2}$$

* ***Table of Laplace transform*:** Appendix A, *Feedback control of dynamic systems*, Franklin *et. al*.

Definition of transfer function

✶ Consider a system described by the differential equation:



$$a_0\frac{d^n y(t)}{dt^n}+a_1\frac{d^{n-1}y(t)}{dt^{n-1}}+\cdots+a_{n-1}\frac{dy(t)}{dt}+a_n y(t)=$$

$$b_0\frac{d^m u(t)}{dt^m}+b_1\frac{d^{m-1}u(t)}{dt^{m-1}}+\cdots+b_{m-1}\frac{du(t)}{dt}+b_m u(t)$$

✶ Taking the Laplace transform the two sides of the above equation, using differentiation property and assuming that the initial condition are zeros, we have:

$$a_0 s^n Y(s)+a_1 s^{n-1}Y(s)+\cdots+a_{n-1}sY(s)+a_n Y(s)=$$

$$b_0 s^m U(s)+b_1 s^{m-1}U(s)+\cdots+b_{m-1}sU(s)+b_m U(s)$$

Definition of transfer function (cont')

- Transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

- **Definition:** Transfer function of a system is the ratio between the Laplace transform of the output signal and the Laplace transform of the input signal assuming that initial conditions are zeros.

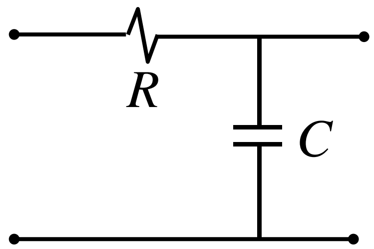
Transfer function of components

Procedure to find the transfer function of a component

- ***Step 1:*** Establish the differential equation describing the input-output relationship of the components by:
  - Applying Kirchhoff's law, current-voltage relationship of resistors, capacitors, inductors,... for the electrical components.
  - Applying Newton's law, the relationship between friction and velocity, the relationship between force and deformation of springs ... for the mechanical elements.
  - Apply heat transfer law, law of conservation of energy, for the thermal process.
  - ...
- ***Step 2:*** Taking the Laplace transform of the two sides of the differential equation established in step 1, we find the transfer function of the component.

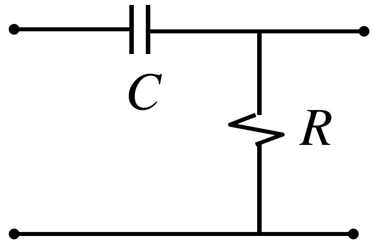
Passive compensators

★ First order integrator:



$$G(s) = \frac{1}{RCs+1}$$

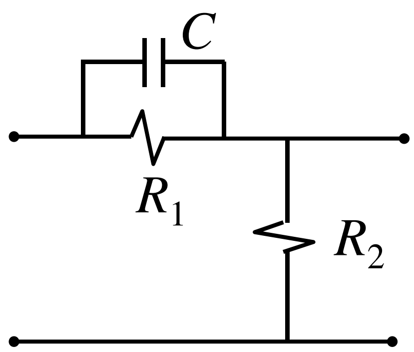
★ First order differentiator:



$$G(s) = \frac{RCs}{RCs+1}$$

Passive compensators

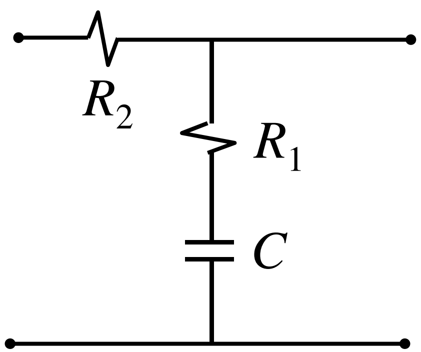
★ Phase lead compensator:



$$G(s) = K_C \frac{\alpha Ts + 1}{Ts + 1}$$

$$K_C = \frac{R_2}{R_1 + R_2} \qquad T = \frac{R_2 R_1 C}{R_1 + R_2} \qquad \alpha = \frac{R_1 + R_2}{R_2} > 1$$

★ Phase lag compensator :

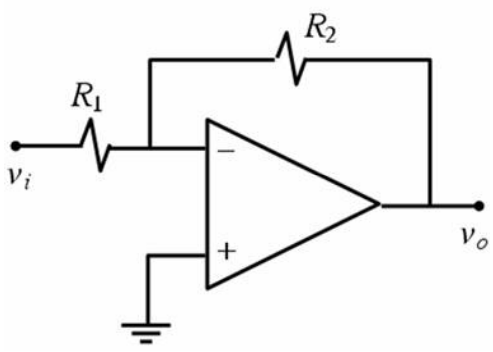


$$G(s) = K_C \frac{\alpha Ts + 1}{Ts + 1}$$

$$K_C = 1 \qquad T = (R_1 + R_2)C \qquad \alpha = \frac{R_1}{R_1 + R_2} < 1$$

Active controllers

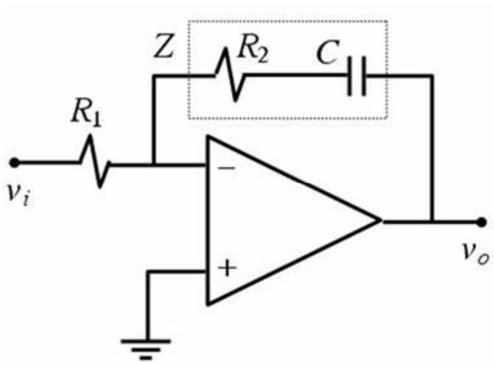
- Proportional Controller (P)



$$G(s) = K_P$$

$$K_P = -\frac{R_2}{R_1}$$

- Proportional Integral controller (PI)

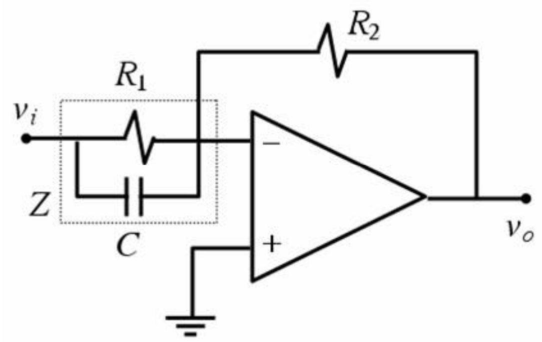


$$G(s) = K_P + \frac{K_I}{s}$$

$$K_P = -\frac{R_2}{R_1} \qquad K_I = -\frac{1}{R_1 C}$$

Active controllers

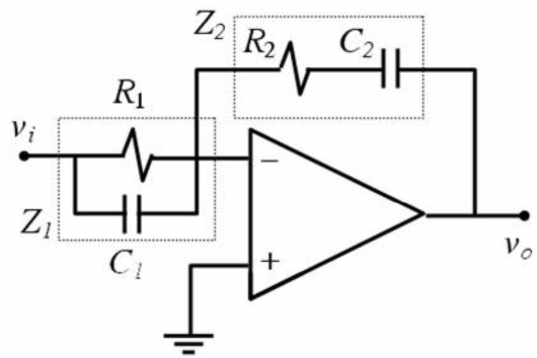
✶ Proportional Derivative controller (PD)



$$G(s) = K_P + K_D s$$

$$K_P = -\frac{R_2}{R_1} \qquad K_D = -R_2 C$$

✶ Proportional Integral Derivative controller (PID)



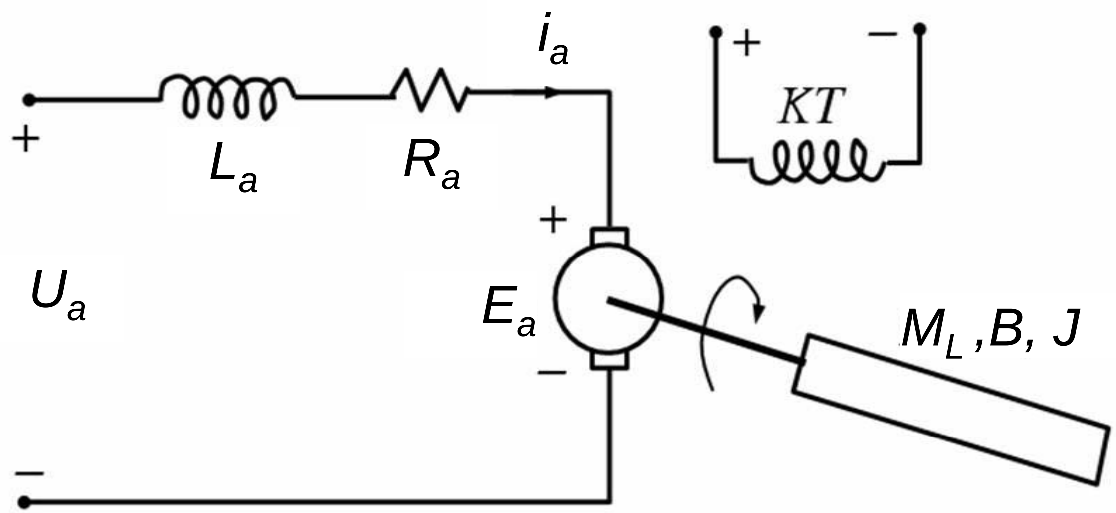
$$G(s) = K_P + \frac{K_I}{s} + K_D s$$

$$K_P = -\frac{R_1 C_1 + R_2 C_2}{R_1 C_2} \qquad K_I = -\frac{1}{R_1 C_2}$$

$$K_D = -R_2 C_1$$

Transfer function of DC motors

Equivalent diagram of a DC motor



- L_a : armature induction
- R_a : armature resistance
- U_a : armature voltage
- E_a : back electromotive force
- ω : motor speed
- M_L : load inertia
- B : friction constant
- J : moment of inertia of the rotor

Transfer function of DC motors

- Applying Kirchhoff's law for the armature circuit:

$$U_a(t) = i_a(t).R_a + L_a \frac{di_a(t)}{dt} + E_a(t) \qquad (1)$$

where: $$E_a(t) = K\Phi\,\omega(t) \qquad (2)$$

K : electromotive force constant

Φ : excitation magnetic flux

- Applying Newton's law for the rotating part of the motor:

$$M(t) = M_L(t) + B\omega(t) + J\frac{d\omega(t)}{dt} \qquad (3)$$

where: $$M(t) = K\Phi i_a(t) \qquad (4)$$

Transfer function of DC motors

✶ Taking the Laplace transform of (1), (2), (3), (4) leads to:

$$U_a(s) = I_a(s).R_a + L_a s I_a(s) + E_a(s) \tag{5}$$

$$E_a(s) = K\Phi\,\omega(s) \tag{6}$$

$$M(s) = M_t(s) + B\omega(s) + Js\omega(s) \tag{7}$$

$$M(s) = K\Phi i_a(s) \tag{8}$$

✶ Denote:

$$T_a = \frac{L_a}{R_a}$$ Electromagnetic time constant

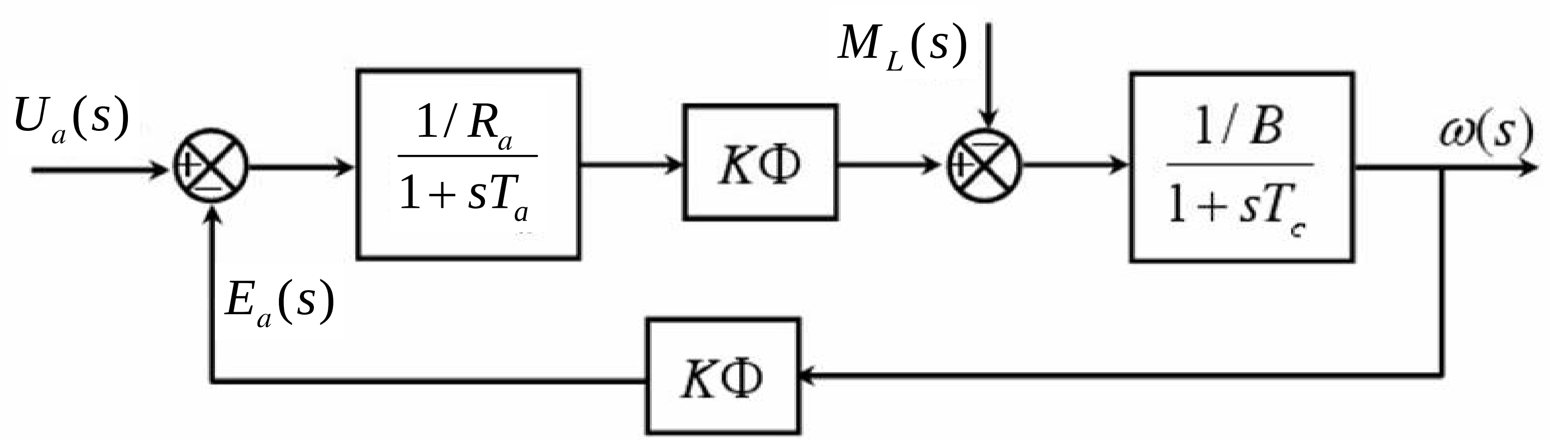
$$T_c = \frac{J}{B}$$ Mechanical time constant

* From (5) and (7), we have:

$$I_a(s) = \frac{U_a(s) - E_a(s)}{R_a(1 + T_a s)} \qquad (5')$$

$$\omega(s) = \frac{M(s) - M_L(s)}{B(1 + T_c s)}$$

* From (5'), (6), (7') and (8) we can develop the block diagram of the DC motor as follow:



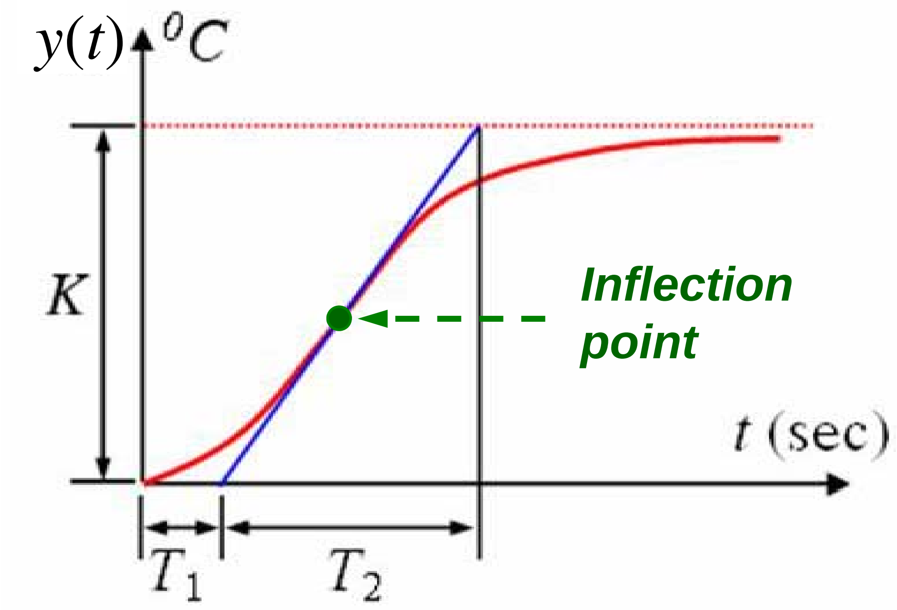
Transfer function of a thermal process

$u(t)$

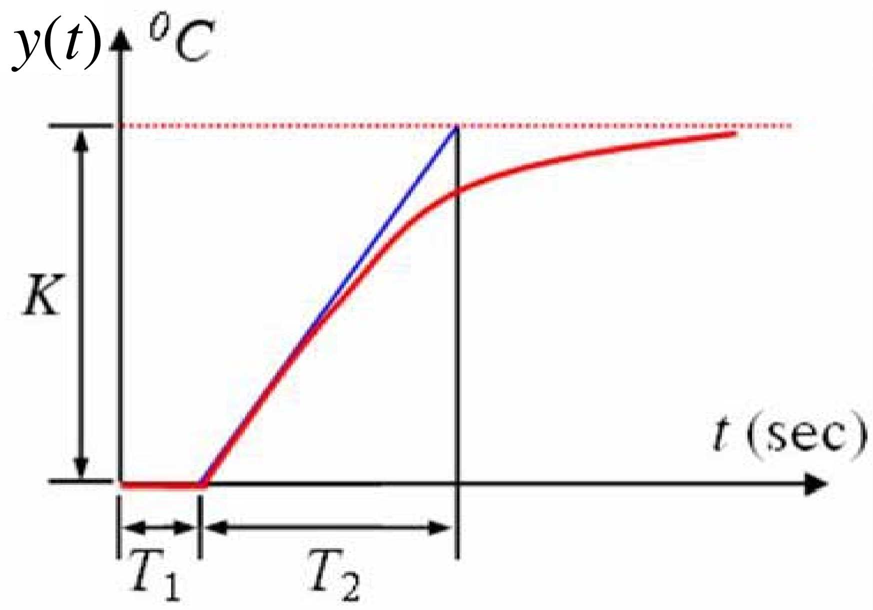
Electric power supplying to the oven 100%

$y(t)$

Temperature of the oven



"Exact" characteristic of the oven



Approximate characteristic of the oven

Transfer function of a thermal process (cont')

- The approximate transfer function of the thermal process can be calculated by using the equation: $G(s) = \dfrac{Y(s)}{U(s)}$
- The input is the unit step signal, then $U(s) = \dfrac{1}{s}$
- The approximate output is: $y(t) = f(t - T_1)$

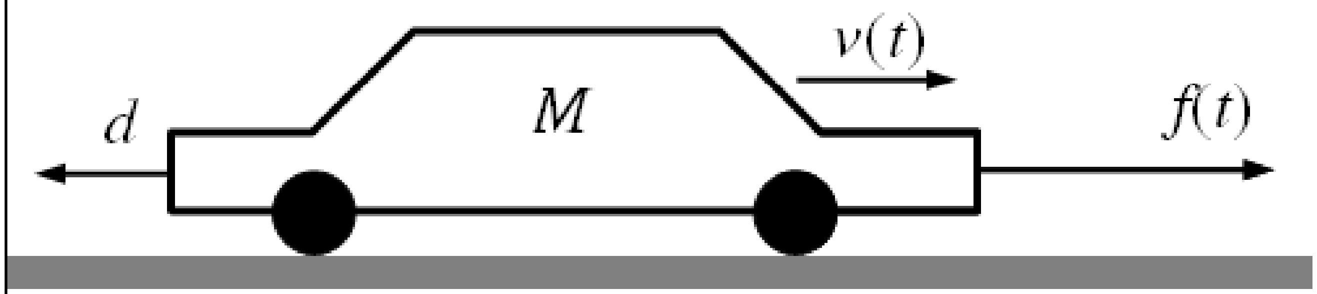
  where: $f(t) = K(1 - e^{-t/T_2})$

  The Laplace transform of $f(t)$ is: $F(s) = \dfrac{K}{s(1+T_2 s)}$

  Applying the time delay theorem: $Y(s) = \dfrac{Ke^{-T_1 s}}{s(1+T_2 s)}$

$$\Rightarrow \boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{Ke^{-T_1 s}}{T_2 s + 1}}$$

Transfer function of a car



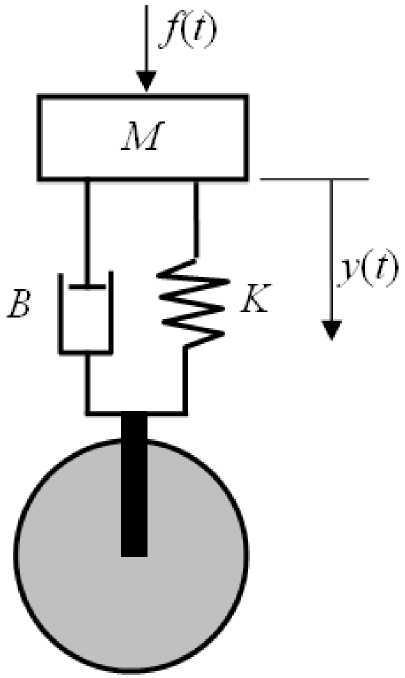
M: car mass
B: friction constant
f(*t*): driving force
v(*t*): car speed

- Differential equation: $$M\frac{dv(t)}{dt}+Bv(t)=f(t)$$

- Transfer function $$G(s)=\frac{V(s)}{F(s)}=\frac{1}{Ms+B} \Leftrightarrow G(s)=\frac{K}{Ts+1}$$

where $$K=\frac{1}{B} \qquad T=\frac{M}{B}$$

Transfer function of an suspension system



M: equivalent car mass
B: friction constant
K: spring stiffness
f(*t*): external force
y(*t*): travel of car body

- ★ Differential equation: $$M\frac{d^2y(t)}{dt^2} + B\frac{dy(t)}{dt} + Ky(t) = f(t)$$

- ★ Transfer function: $$G(s) = \frac{Y(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

Transfer functions of sensors

$$y(t) \longrightarrow \boxed{\text{Sensor}} \longrightarrow y_{fb}(t)$$

- Feedback signal $y_{fb}(t)$ is proportional to $y(t)$, so transfer functions of sensors are usually constant:

$$H(s) = K_{fb}$$

- Ex: Suppose that temperature of a furnace changing in the range $y(t) = 0 \div 500^0$C, if a sensor converts the temperature to a voltage in the range $y_{fb}(t)$ $0 \div 5$V, then the transfer function of the sensor is:

$$H(s) = K_{fb} = 5(V)/500(^0C) = 0.01(V/^0C)$$

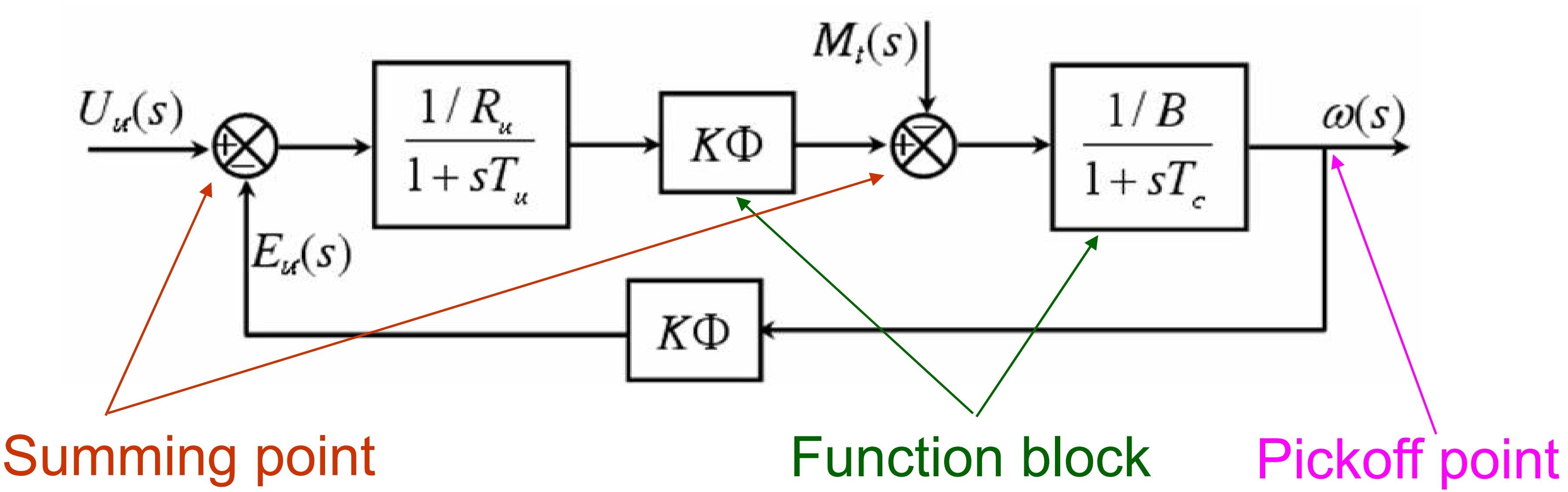
- If the sensor has a delay time, then the transfer function of the sensor is:

$$H(s) = \frac{K_{fb}}{1 + T_{fb}s}$$

Transfer functions of control systems

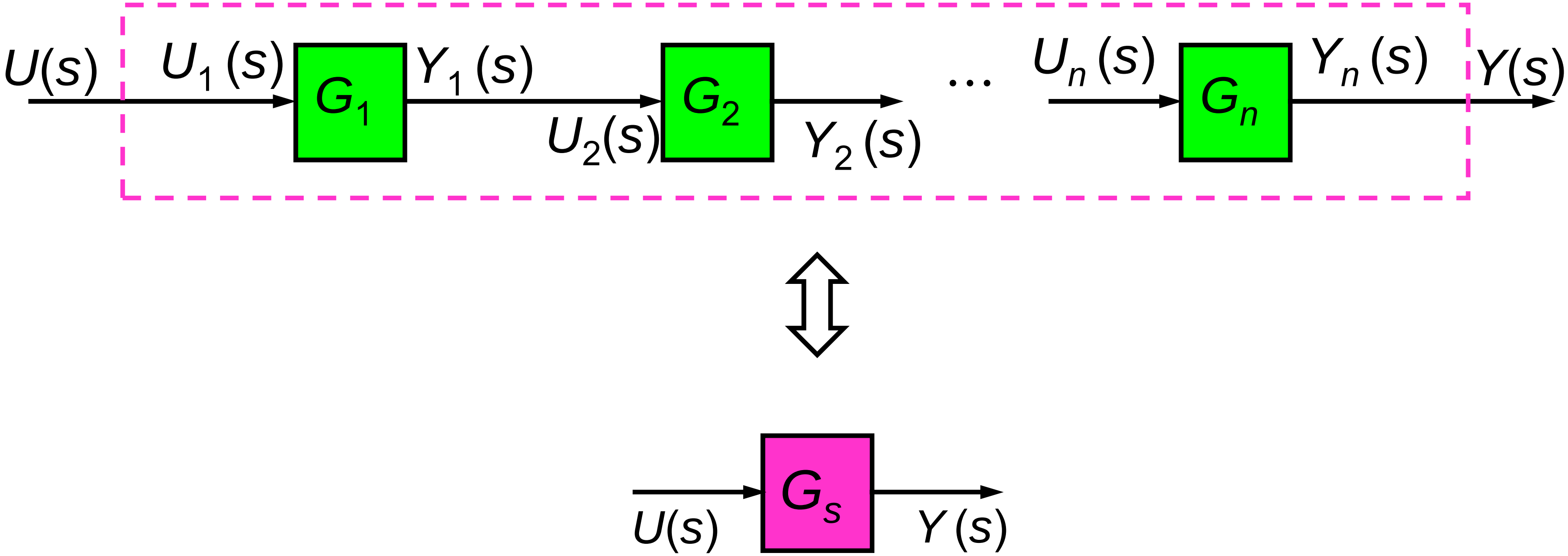
Block diagram

- Block diagram is a diagram of a system, in which the principal parts or functions are represented by blocks connected by lines, that show the relationships of the blocks.
- A block diagram composes of 3 components:
  - Function block
  - Summing point
  - Pickoff point



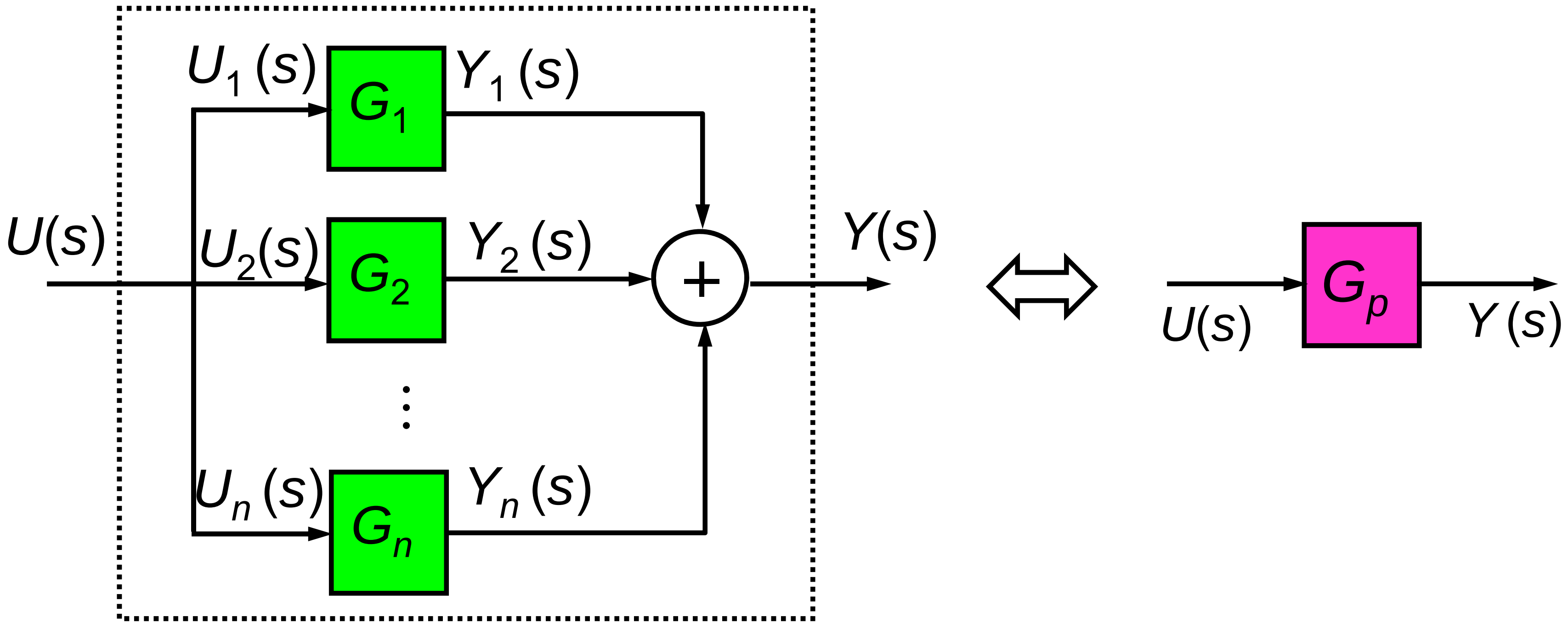
Block diagram algebra

Transfer function of systems in series



$$G_s(s) = \prod_{i=1}^{n} G_i(s)$$

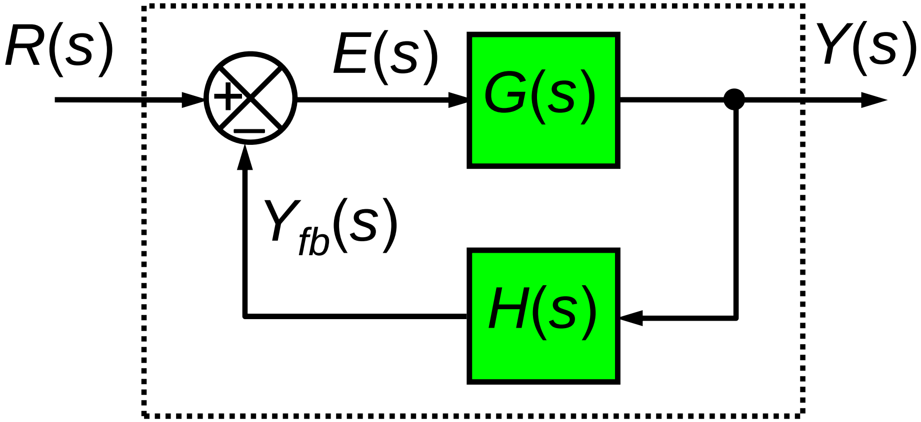
Transfer function of systems in parallel



$$G_p(s) = \sum_{i=1}^{n} G_i(s)$$

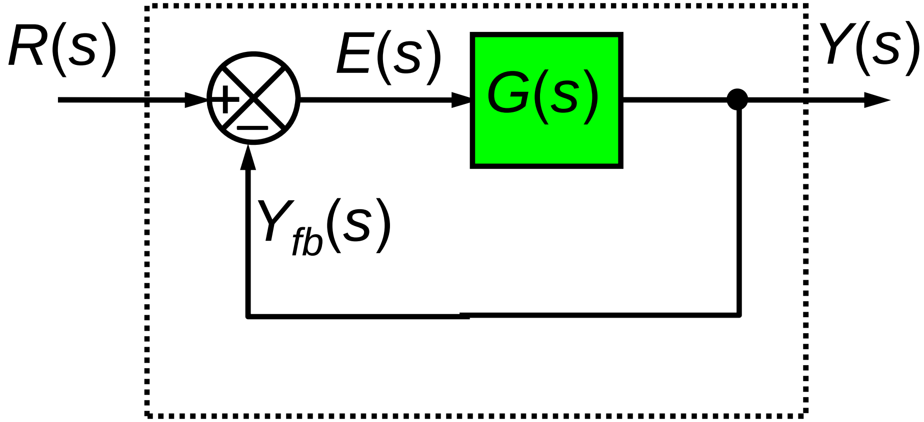
Transfer function of feedback systems

- Negative feedback



$$G_{cl}(s) = \frac{G(s)}{1+G(s).H(s)}$$

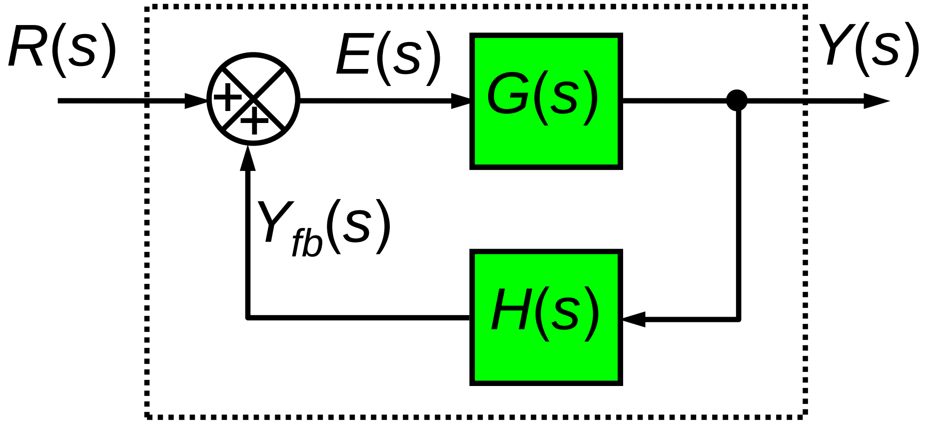
- Unity negative feedback



$$G_{cl}(s) = \frac{G(s)}{1+G(s)}$$

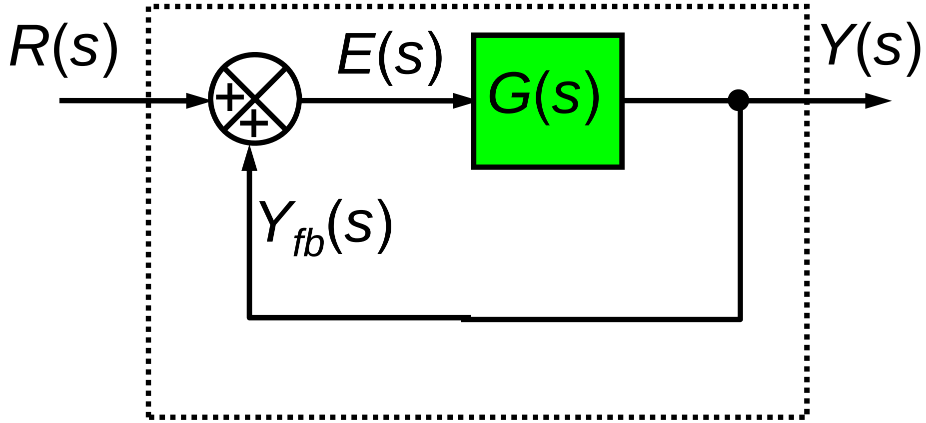
Transfer function of feedback systems

★ Positive feedback



$$G_{cl}(s)=\frac{G(s)}{1-G(s).H(s)}$$

★ Unity positive feedback



$$G_{cl}(s)=\frac{G(s)}{1-G(s)}$$

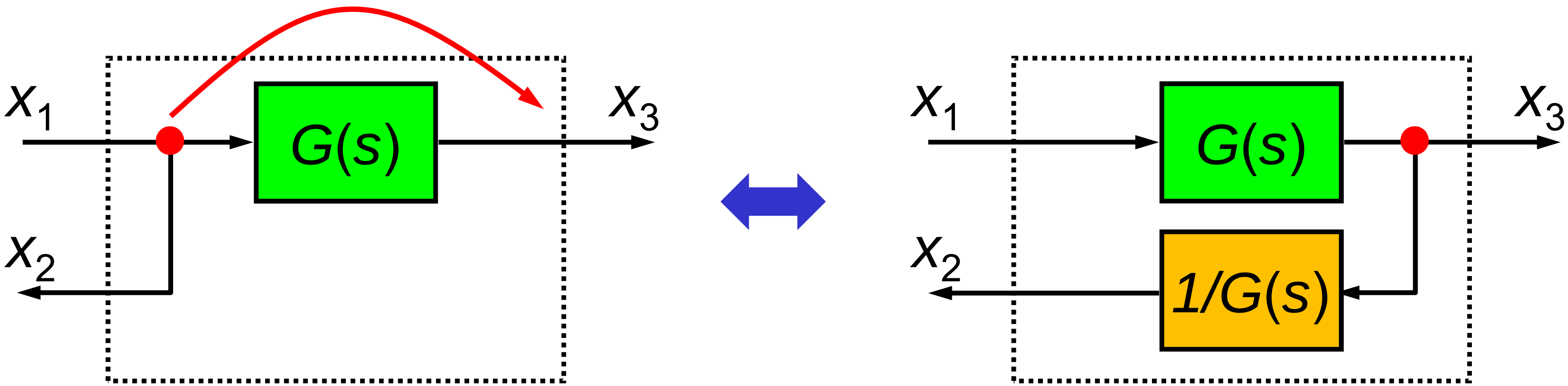
Transfer function of multi-loop systems

- For a complex system consisting of multi feedback loops, we perform equivalent block diagram transformation so that simple connecting blocks appears, and then we simplify the block diagram from the inner loops to the outer loops.

- ***Two block diagrams are equivalent if their input-output relationship are the same.***

Block diagram algebra (cont’)

Moving a pickoff point behind a block



$$x_2 = x_1$$

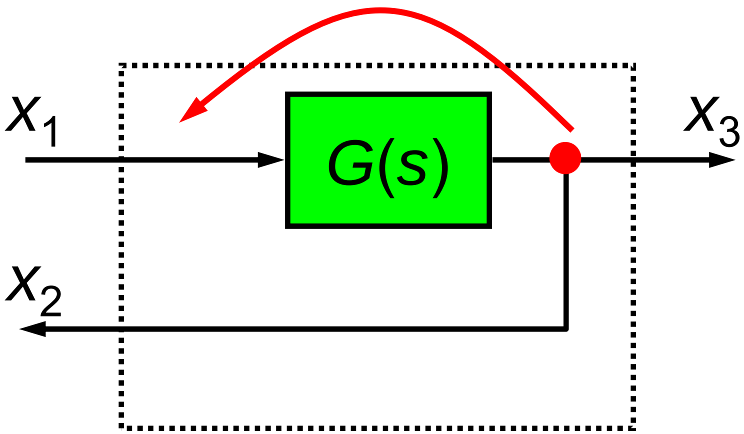
$$x_3 = x_1 G$$

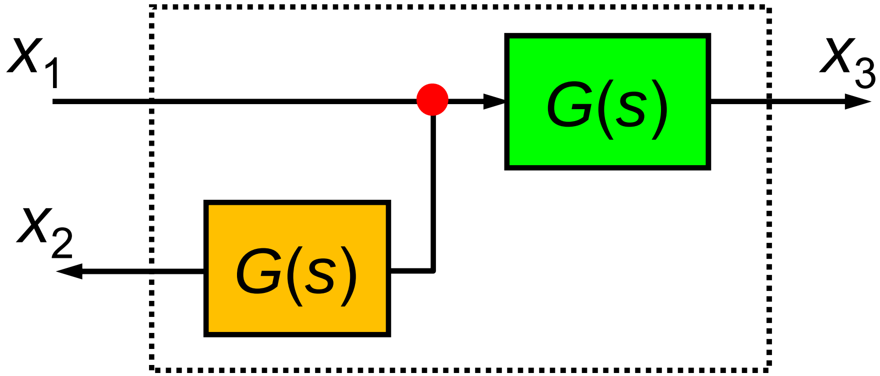
$$x_3 = x_1 G$$

$$x_2 = x_3 / G = x_1$$

Block diagram algebra (cont')

Moving a pickoff point ahead a block





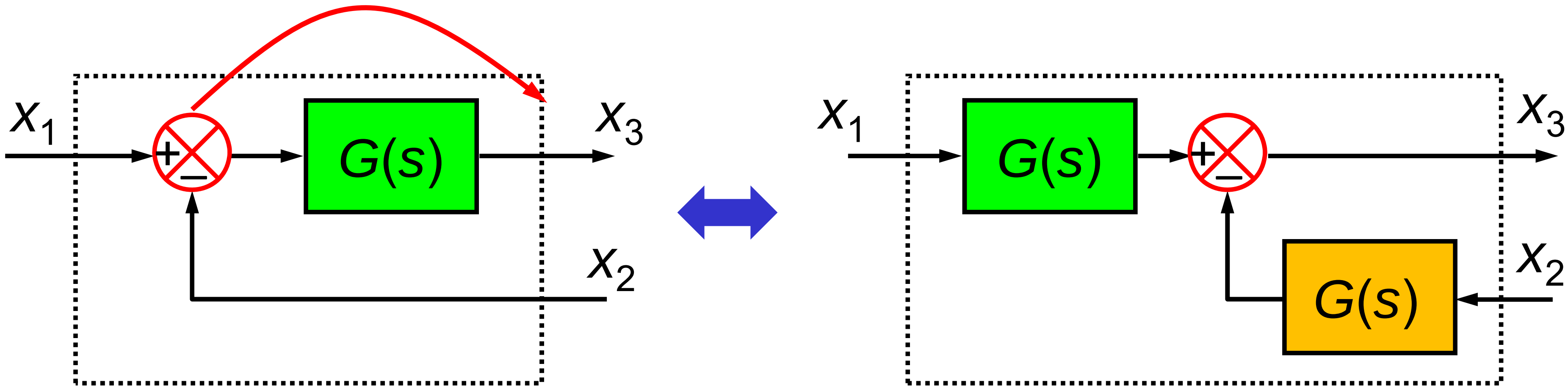
$$x_2 = x_3 = x_1 G$$

$$x_2 = x_1 G$$

$$x_3 = x_1 G$$

Block diagram algebra (cont')

Moving a summing point behide a block

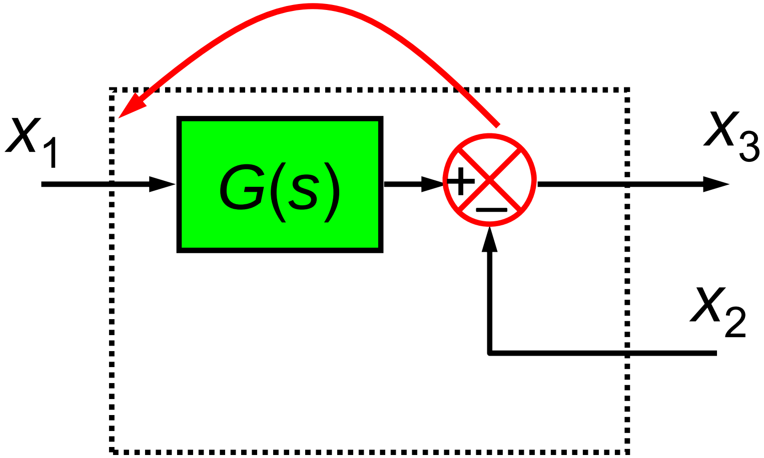


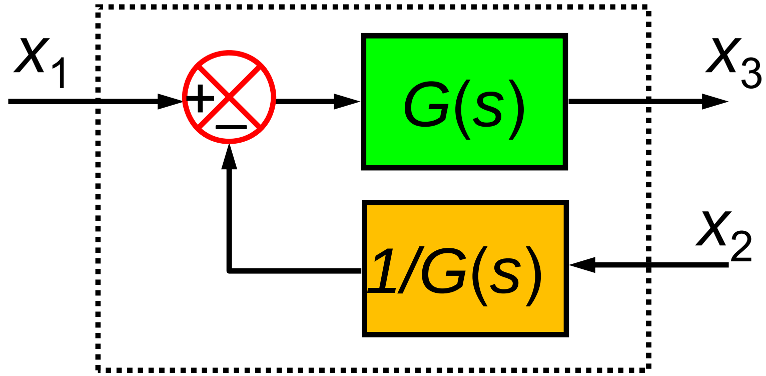
$$x_3 = (x_1 - x_2)G$$

$$x_3 = x_1 G - x_2 G = (x_1 - x_2)G$$

Block diagram algebra (cont')

Moving a summing point ahead a block



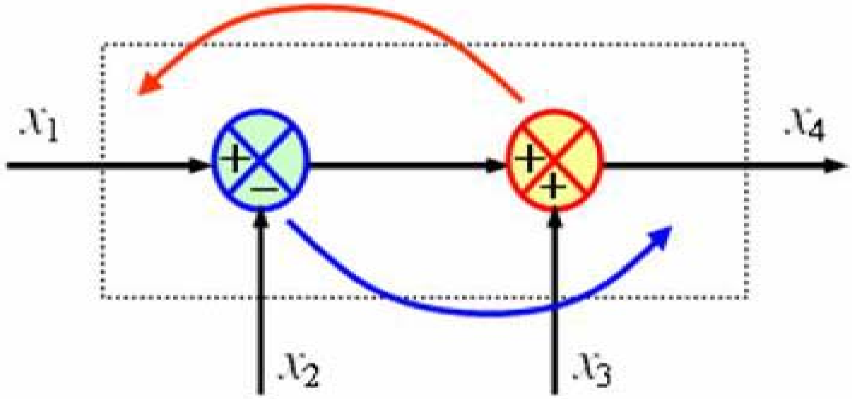


$$x_3 = x_1 G - x_2$$

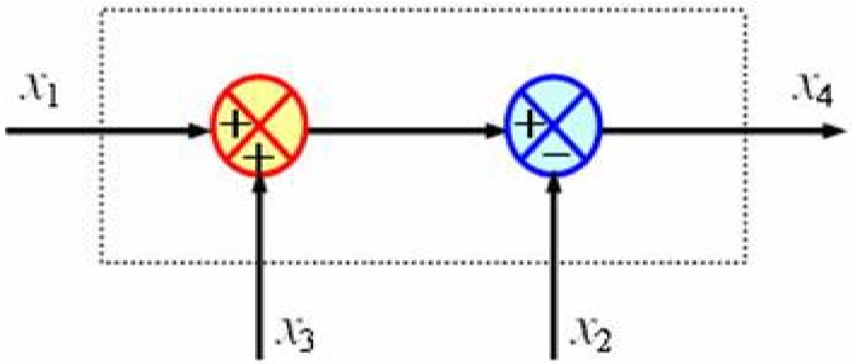
$$x_3 = (x_1 - x_2 / G)G = x_1 G - x_2$$

Block diagram algebra (cont')

Interchanging the positions of the two consecutive summing points



$\Leftrightarrow$

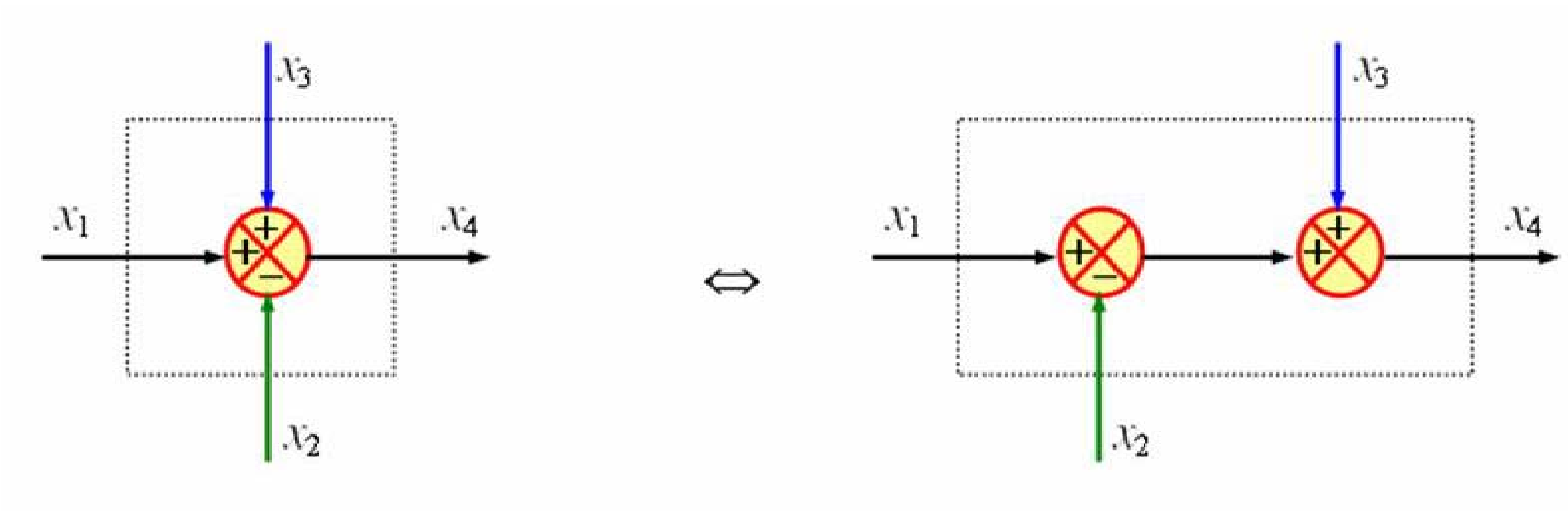


$$x_4 = (x_1 - x_2) + x_3$$

$$x_4 = (x_1 + x_3) - x_2$$

Block diagram algebra (cont')

Splitting a summing point

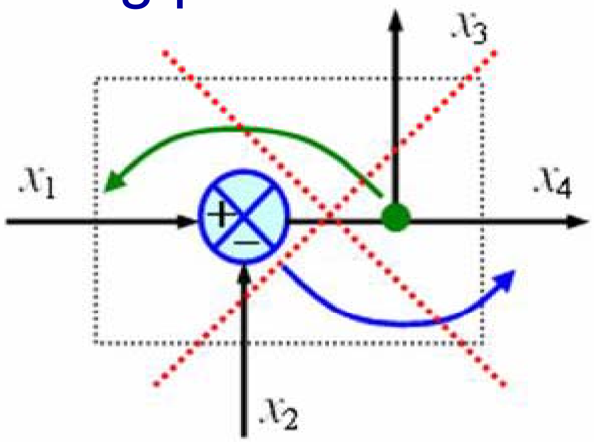


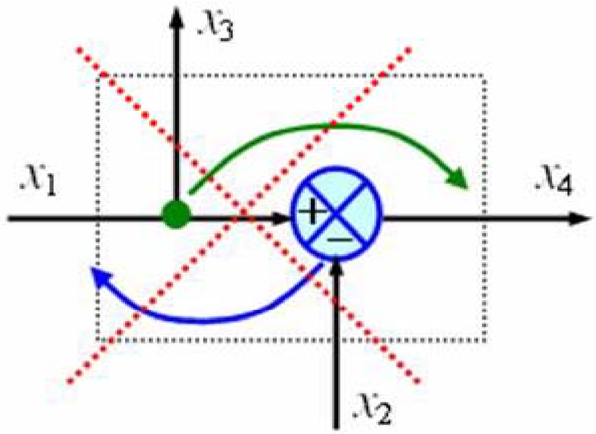
$$x_4 = x_1 - x_2 + x_3$$

$$x_4 = (x_1 - x_2) + x_3$$

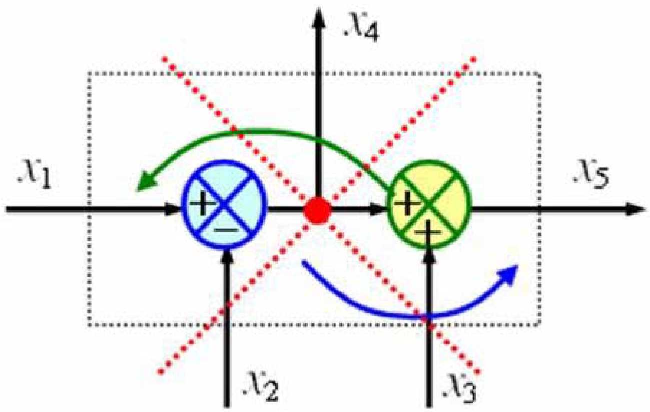
Note

- ✶ Do not interchange the positions of a pickoff point and a summing point :



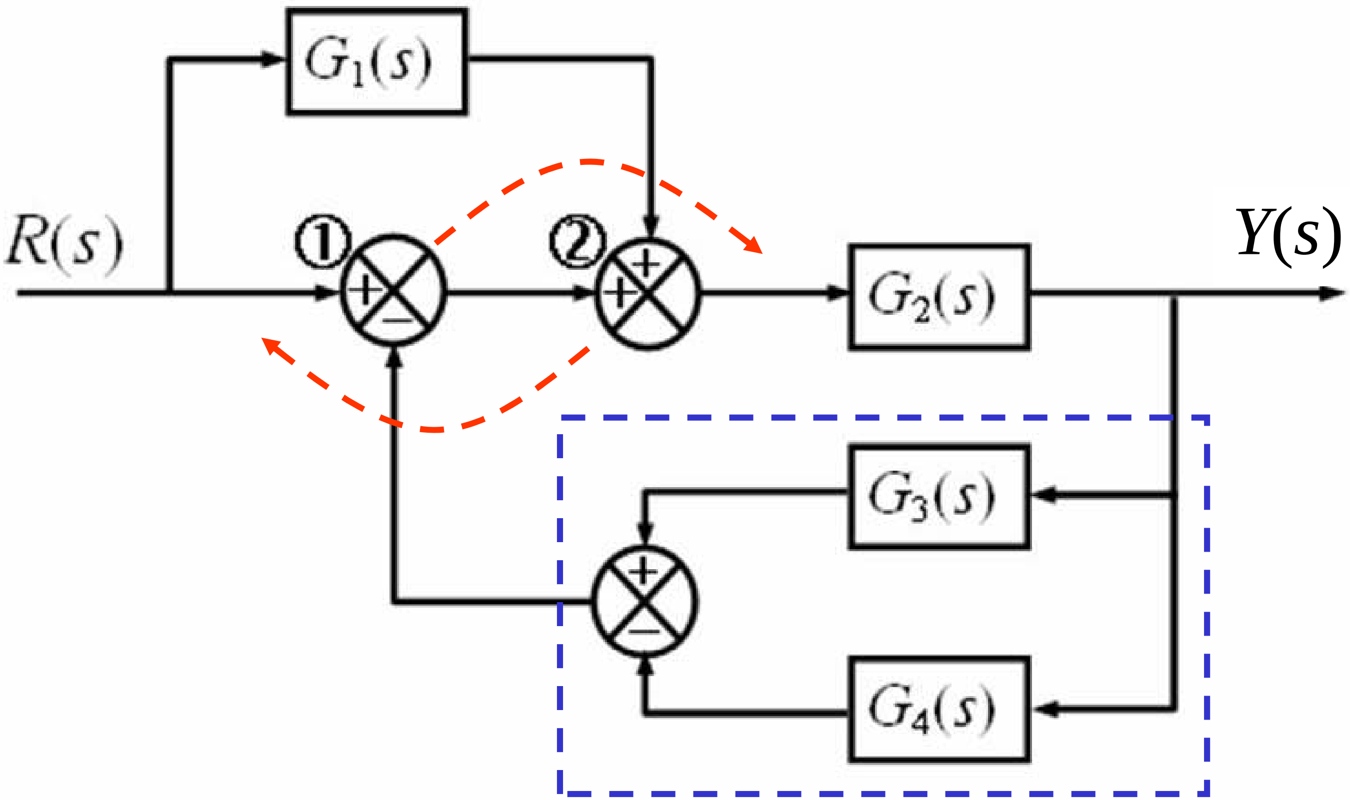


- ✶ Do not interchange the positions of two summing points if there exists a pickoff point between them:



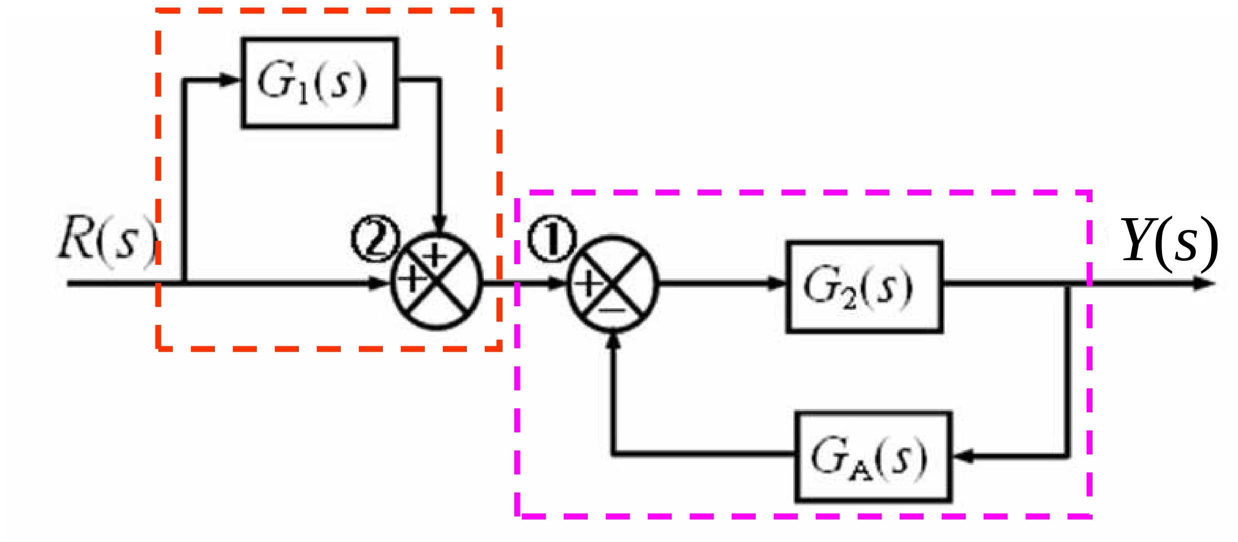
Block diagram algebra – Example 1

✶ Find the equivalent transfer function of the following system:



Block diagram algebra – Example 1 (cont')

- Interchanging the summing points ① and ②, Eliminating $G_A(s)$=[$G_3(s)$//$G_4(s)$]



$$G_A(s) = G_3(s) - G_4(s)$$

Block diagram algebra – Example 1 (cont')

- $G_B(s)$=[$G_1(s)$ // unity block] ,
  $G_C(s)$= feedback loop[$G_2(s)$,$G_A(s)$]:

$$G_B(s) = 1 + G_1(s)$$

$$G_C(s) = \frac{G_2(s)}{1 + G_2(s).G_A(s)} = \frac{G_2(s)}{1 + G_2(s).[G_3(s) - G_4(s)]}$$

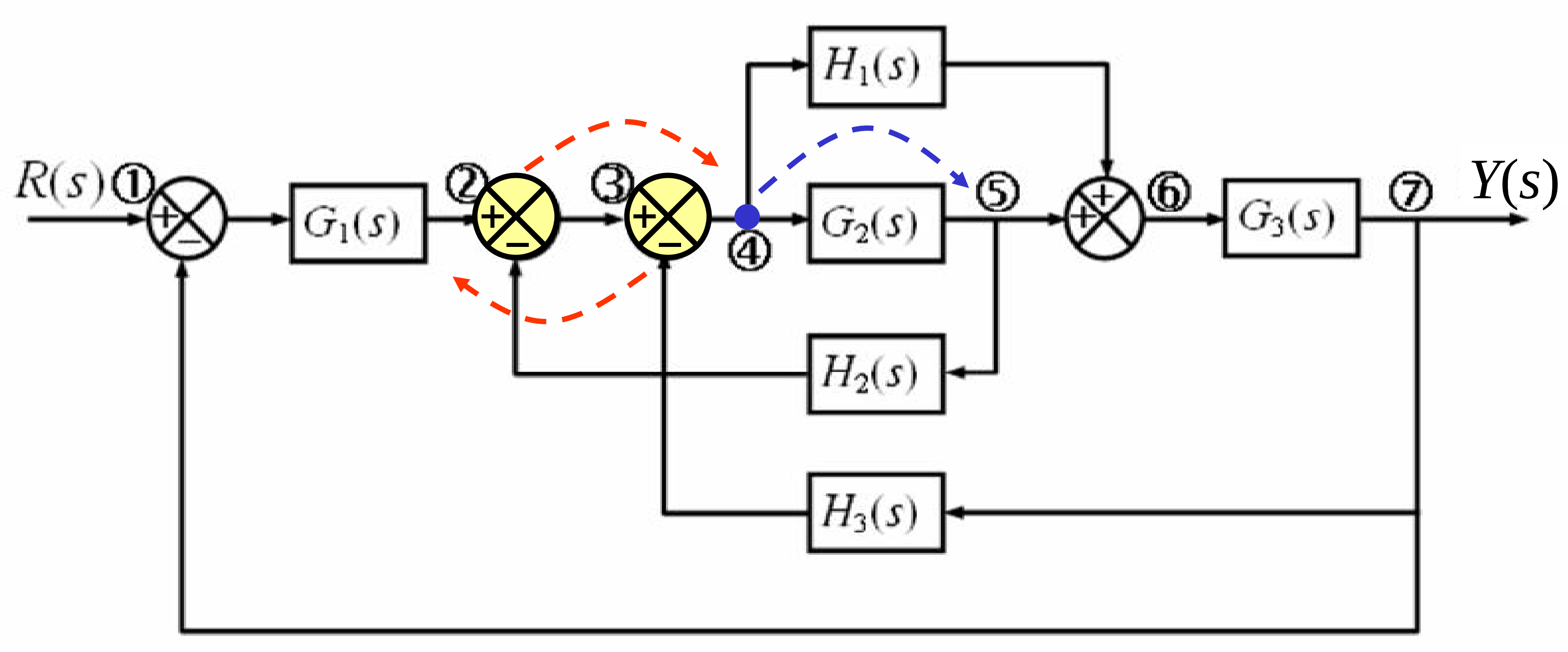
- Equivalent transfer function of the system:

$$G_{eq}(s) = G_B(s).G_C(s)$$

$$G_{eq}(s) = \frac{[1 + G_1(s)].G_2(s)}{1 + G_2(s).[G_3(s) - G_4(s)]}$$

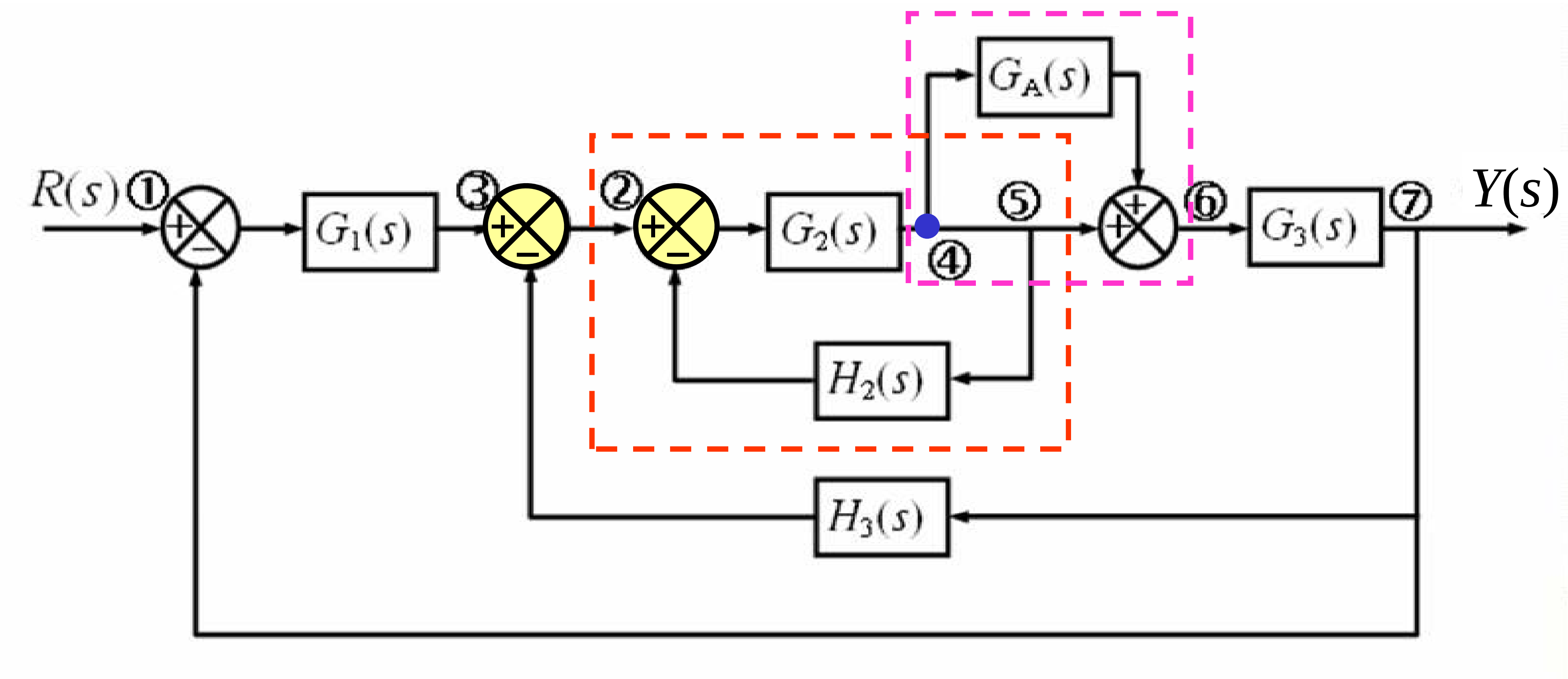
Block diagram algebra – Example 2

- Find the equivalent transfer function of the following system:



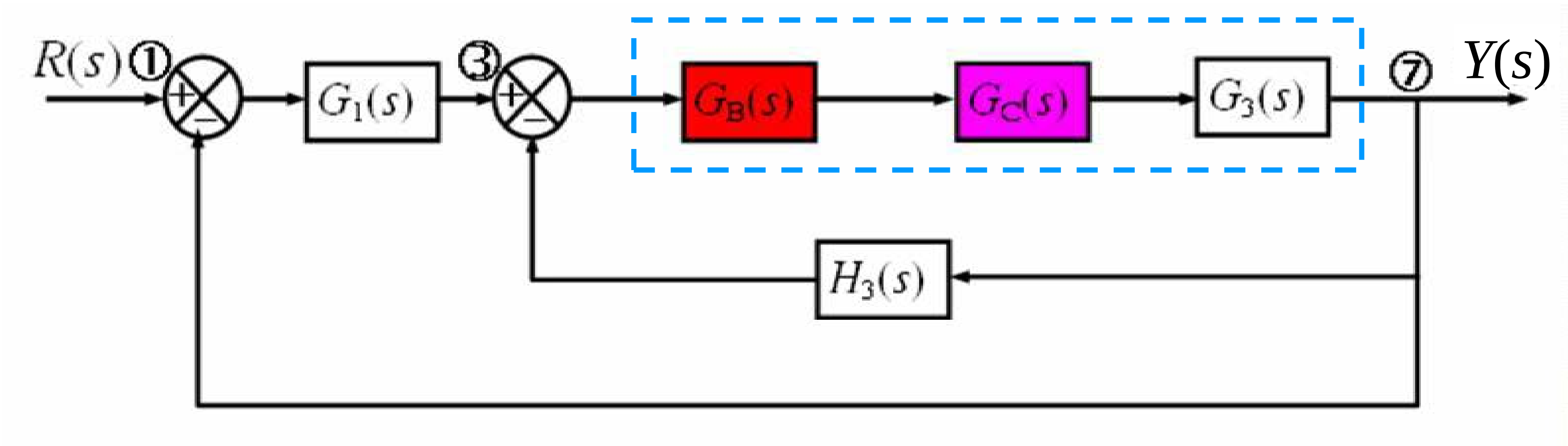
Block diagram algebra – Example 2 (cont')

- Interchanging the positions of the summing points ② and ③
  Moving the pickoff point ④ behind the block $G_2(s)$



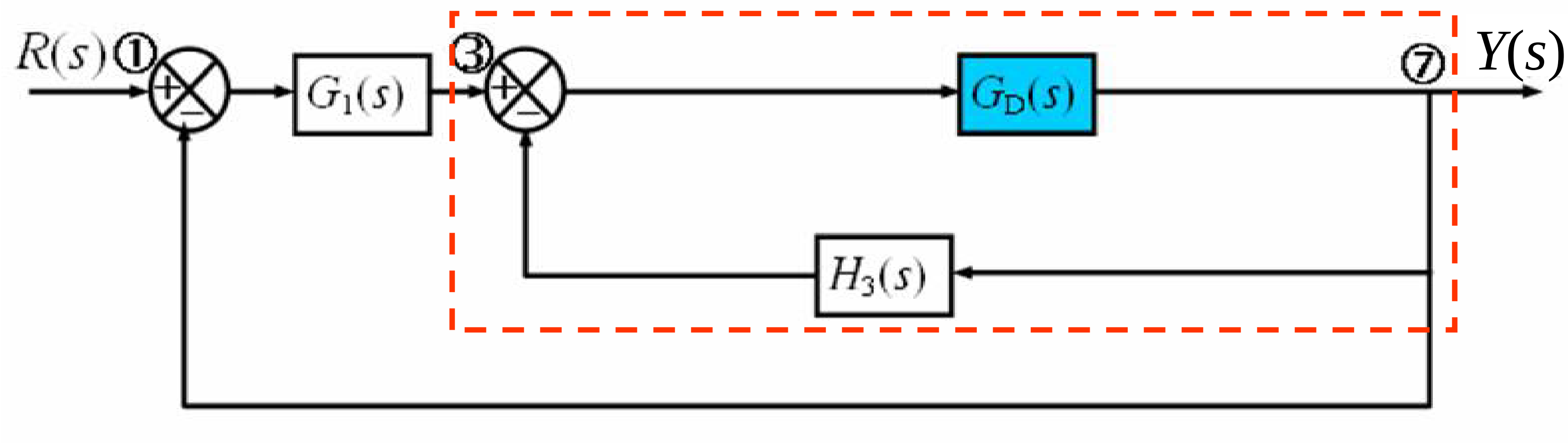
Block diagram algebra – Example 2 (cont')

- $G_B(s)$ = feedback loop [$G_2(s)$, $H_2(s)$]
  $G_C(s)$ = [$G_A(s)$// unity block]

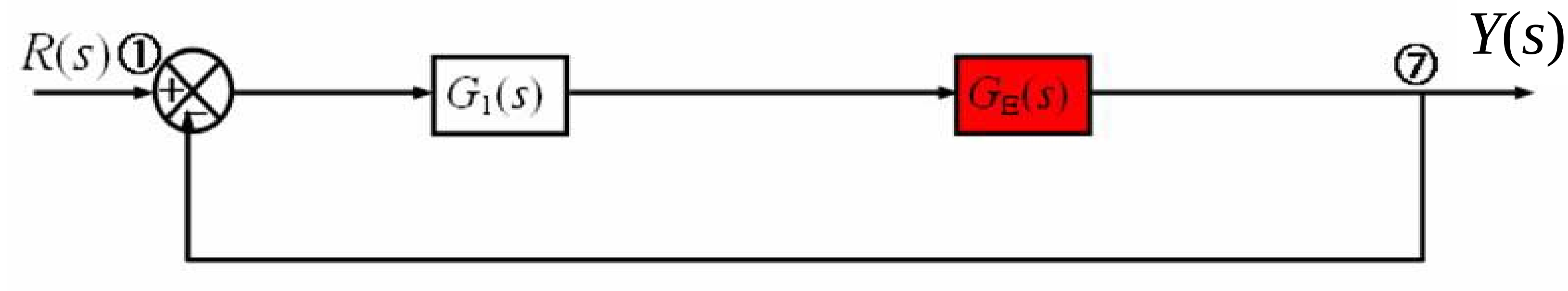


Block diagram algebra – Example 2 (cont')

- $G_D(s)$ = cascade(G_B (s), G_C(s), G_3(s))



- $G_E(s)$ = feedback loop($G_D(s)$, $H_3(s)$)



Block diagram algebra – Example 2 (cont')

- Detailed calculation:

$$* \quad G_A = \frac{H_1}{G_2}$$

$$* \quad G_B = \frac{G_2}{1 + G_2 H_2}$$

$$* \quad G_C = 1 + G_A = 1 + \frac{H_1}{G_2} = \frac{G_2 + H_1}{G_2}$$

$$* \quad G_D = G_B . G_C . G_3 = \left(\frac{G_2}{1 + G_2 H_2} \right) \left(\frac{G_2 + H_1}{G_2} \right) G_3 = \frac{G_2 G_3 + G_3 H_1}{1 + G_2 H_2}$$

Block diagram algebra – Example 2 (cont')

$$* \quad G_E = \frac{G_D}{1+G_D H_3} = \frac{\dfrac{G_2G_3+G_3H_1}{1+G_2H_2}}{1+\dfrac{G_2G_3+G_3H_1}{1+G_2H_2}H_3} = \frac{G_2G_3+G_3H_1}{1+G_2H_2+G_2G_3H_3+G_3H_1H_3}$$

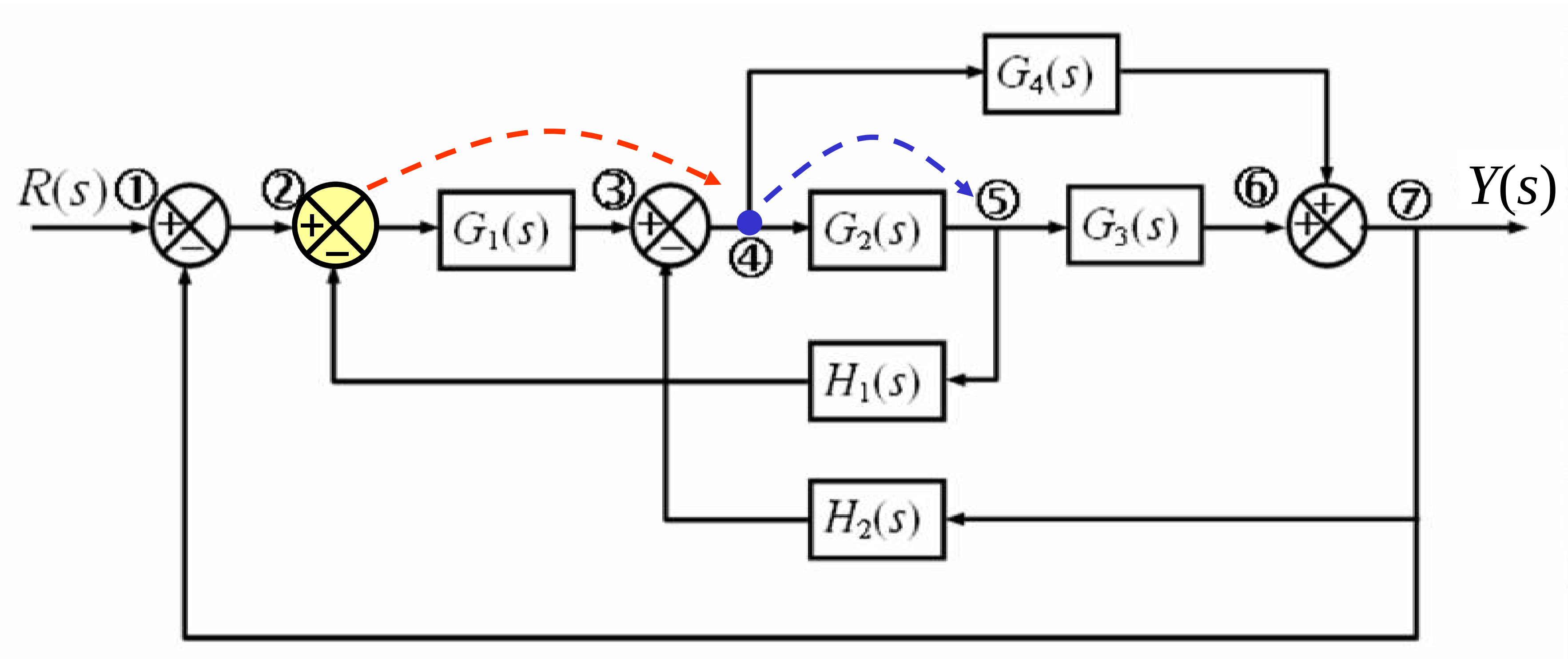
✶ Equivalent transfer function of the system:

$$* \quad G_{eq} = \frac{G_1G_E}{1+G_1G_E} = \frac{G_1.\dfrac{G_2G_3+G_3H_1}{1+G_2H_2+G_2G_3H_3+G_3H_1H_3}}{1+G_1.\dfrac{G_2G_3+G_3H_1}{1+G_2H_2+G_2G_3H_3+G_3H_1H_3}}$$

$$\Rightarrow \quad G_{eq} = \frac{G_1G_2G_3+G_1G_3H_1}{1+G_2H_2+G_2G_3H_3+G_3H_1H_3+G_1G_2G_3+G_1G_3H_1}$$

Block diagram algebra – Example 3

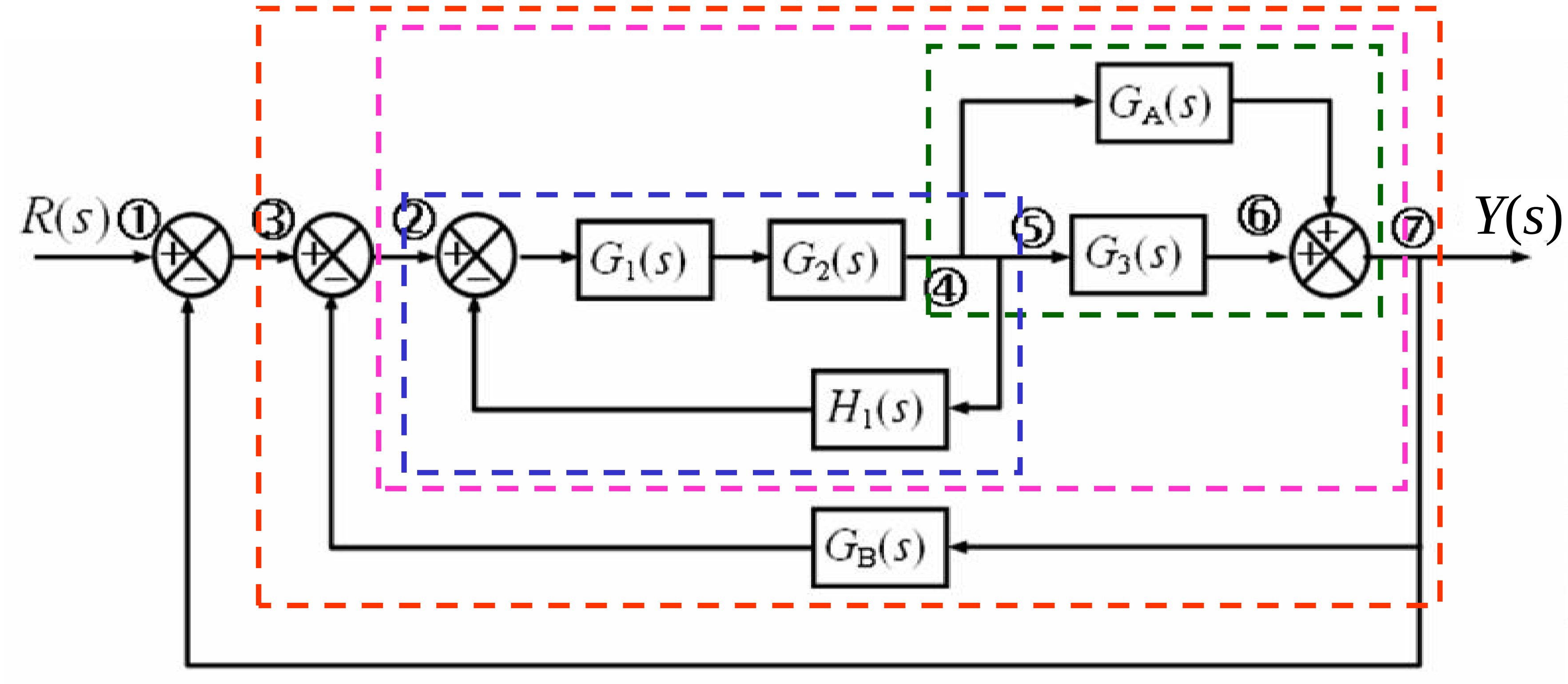
- Find the equivalent transfer function of the following system:



Block diagram algebra – Example 3 (cont')

Hint to solve example 3

✶ Move the summing point ③ ahead the block $G_1(s)$, then interchange the position of the summing points ② and ③
Move the pickoff point ④ behind the block $G_2(s)$



Block diagram algebra – Example 3 (cont')

Solution to Example 3

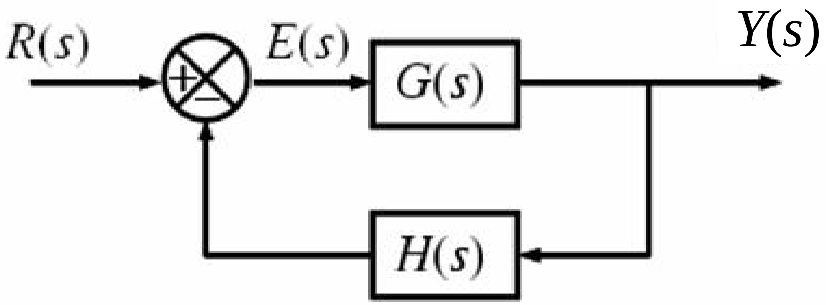
- Students calculate the equivalent transfer function themselves using the hints in the previous slide.

Remarks on block diagram algebra

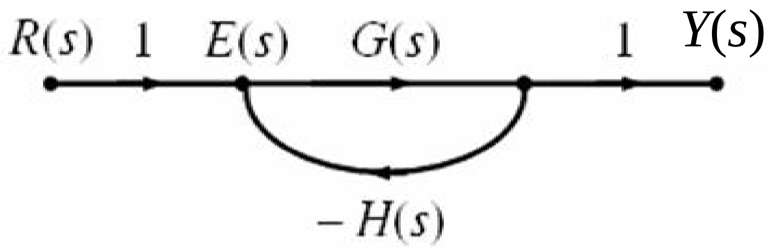
- ✶ Block diagram algebra is a relatively simple method to calculate the equivalent transfer function of a control system.
- ✶ The main disadvantage of block diagram algebra is its lack of systematic procedure to perform the block diagram transformation; each particular block diagram can be transformed by different heuristic ways.
- ✶ When calculating the equivalent transfer function, it is necessary to manipulate many calculations on algebraic fractions. This could be a potential source of error if the system is complex enough.

⇒ Block diagram algebra is only appropriate for finding equivalent transfer function of simple systems.

To find equivalent transfer function of complex systems, signal flow graph method (to be discussed later) is more effective.



Block diagram

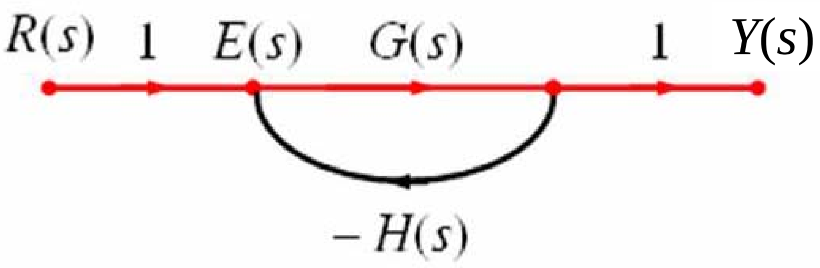


Signal flow graph

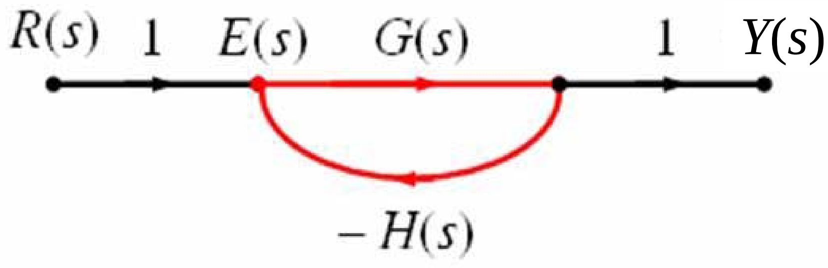
- ***Signal flow graph:*** a networks consisting of nodes and branches.
- ***Node:*** a point representing a signal or a variable in the system.
- ***Branch:*** a line directly connecting two nodes, each branch has an arrow showing the signal direction and a transfer function representing the relationship between the signal at the two nodes of the branch
- ***Source node:*** a node from which there are only out-going branches.
- ***Sink node:*** a node to which there are only in-going branches.
- ***Hybrid node:*** a node which both has in-going branches and out-going branches.

Definition of signal flow graph (cont')

- ***Forward path:*** is a path consisting of continuous sequence of branches that goes in the same direction from a source node to a sink node without passing any single node more than once.
  Path gain is the product of all transfer functions of the branches belonged to the path.
- ***Loop:*** is a closed path consisting of continuous sequence of branches that goes in the same direction without passing any single node more than once.
  Loop gain is the product of all transfer functions of the branches belonged to the loop.



Forward path



Loop

Mason's formula

- The equivalent transfer function from a source node to a sink node of a system can be found by using the Mason's formula:

$$G = \frac{1}{\Delta}\sum_k \Delta_k P_k$$

  - P_k: is the gain of k^{th} forward path from the considered source node to the considered sink node.
  - Δ: is the determinant of the signal flow graph.

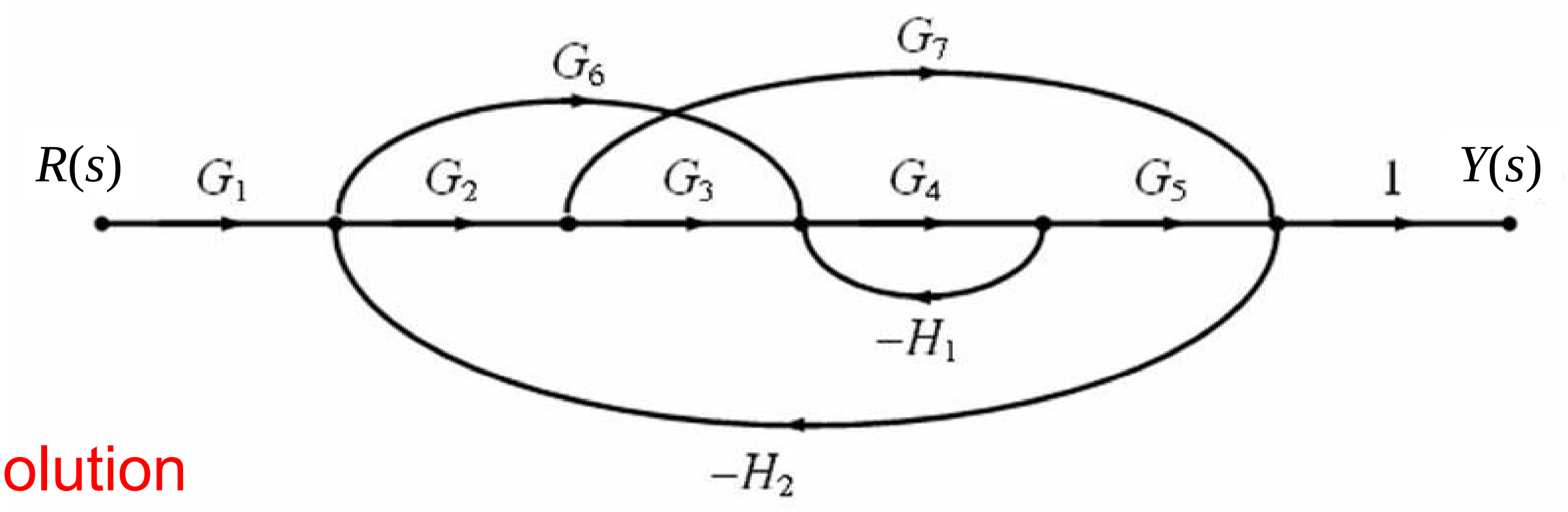
$$\Delta = 1 - \sum_i L_i + \sum_{\substack{i,j \\ nontouching}} L_i L_j - \sum_{\substack{i,j,m \\ nontouching}} L_i L_j L_m + \ldots$$

    L_i : is the gain of the i^{th} loop
  - Δ_k: is the cofactor of the k^{th} path .
    Δ_k is inferred from Δ by removing all the gain(s) of the loop(s) touching the forward path P_k

- ***Note***: Nontouching loops do not have any common nodes. A loop and a path touch together if they have at least one common node.

Signal flow graph – Example 1

- Find the equivalent transfer function of the system described by the following signal flow graph:



- Solution

  - Forward paths:

$$P_1 = G_1G_2G_3G_4G_5$$

$$P_2 = G_1G_6G_4G_5$$

$$P_3 = G_1G_2G_7$$

  - Loop:

$$L_1 = -G_4H_1$$

$$L_2 = -G_2G_7H_2$$

$$L_3 = -G_6G_4G_5H_2$$

$$L_4 = -G_2G_3G_4G_5H_2$$

Signal flow graph – Example 1 (cont')

- The determinant of the SFG:

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + L_1 L_2$$

- The cofactors of the paths

$$\Delta_1 = 1$$

$$\Delta_2 = 1$$

$$\Delta_3 = 1 - L_1$$

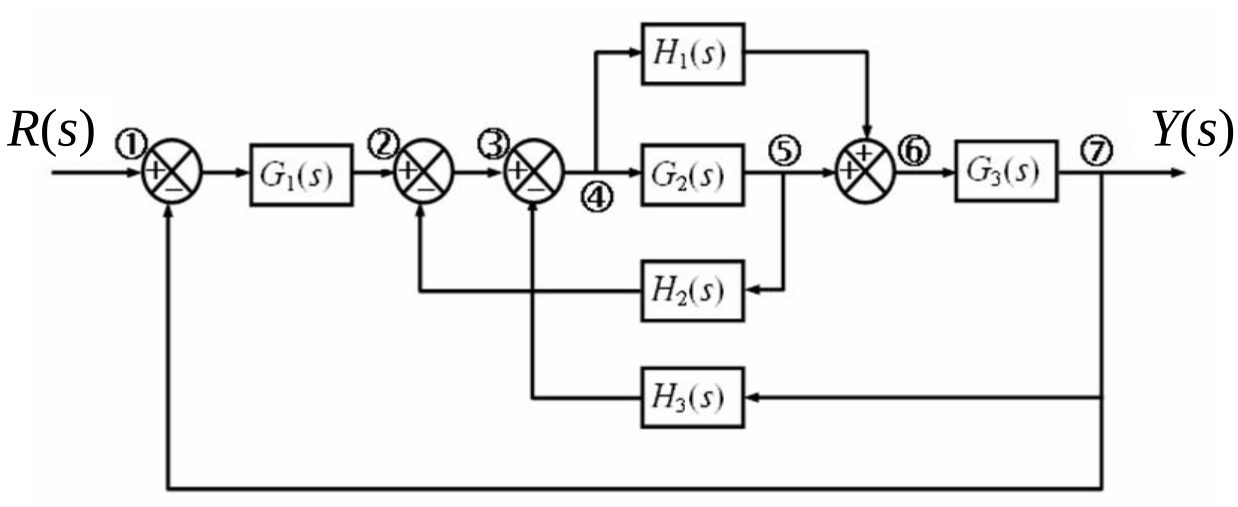
- The equivalent transfer function of the system:

$$G_{eq} = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3)$$

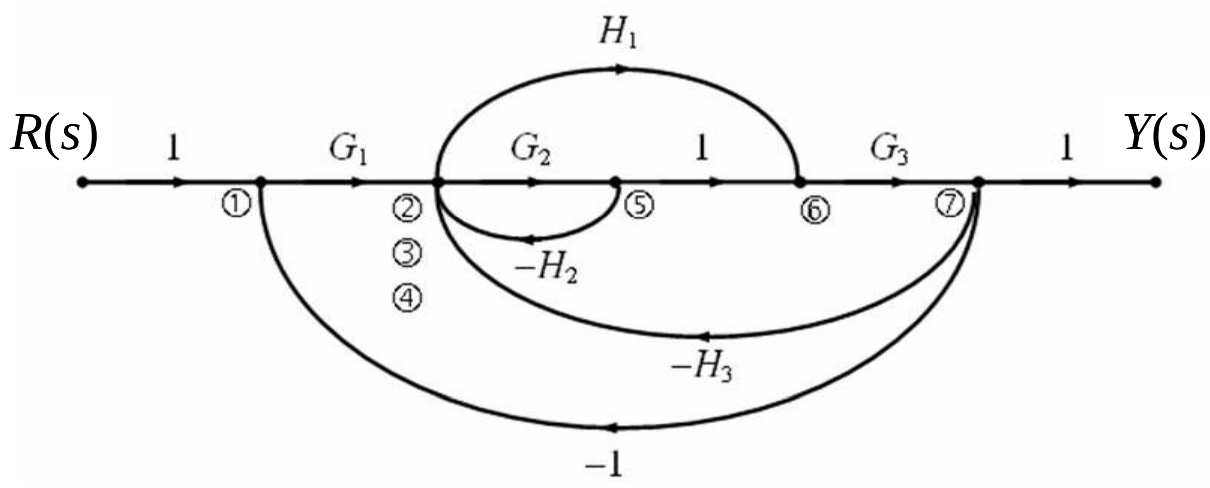
$$G_{eq} = \frac{G_1G_2G_3G_4G_5 + G_1G_6G_4G_5 + G_1G_2G_7(1 + G_4H_1)}{1 + G_4H_1 + G_2G_7H_2 + G_6G_4G_5H_2 + G_2G_3G_4G_5H_2 + G_4H_1G_2G_7H_2}$$

Signal flow graph – Example 2

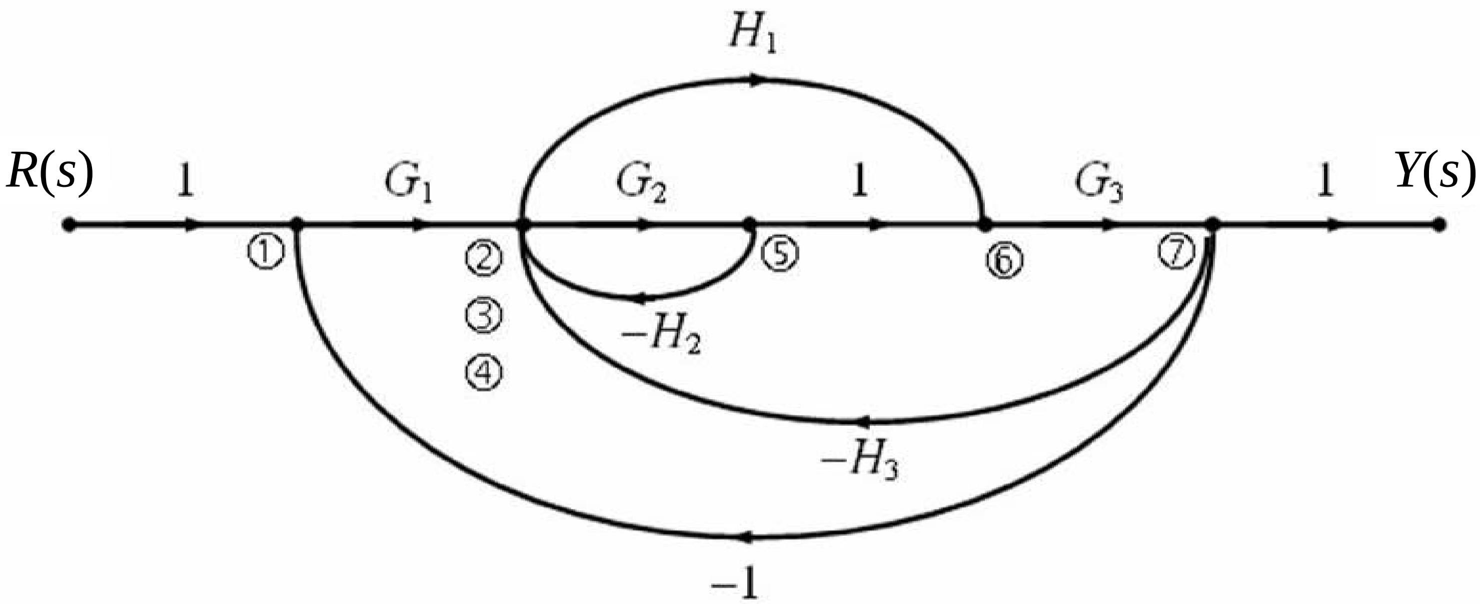
- Find the equivalent transfer function of the system described by the following block diagram:



- Solution:



Signal flow graph – Example 2 (cont')



- Forward paths:

$$P_1 = G_1G_2G_3$$

$$P_2 = G_1H_1G_3$$

- Loop

$$L_1 = -G_2H_2$$

$$L_2 = -G_2G_3H_3$$

$$L_3 = -G_1G_2G_3$$

$$L_4 = -G_3H_1H_3$$

$$L_5 = -G_1G_3H_1$$

Signal flow graph – Example 2 (cont')

- The determinant of the SFG:

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5)$$

- The cofactors of the paths

$$\Delta_1 = 1$$

$$\Delta_2 = 1$$

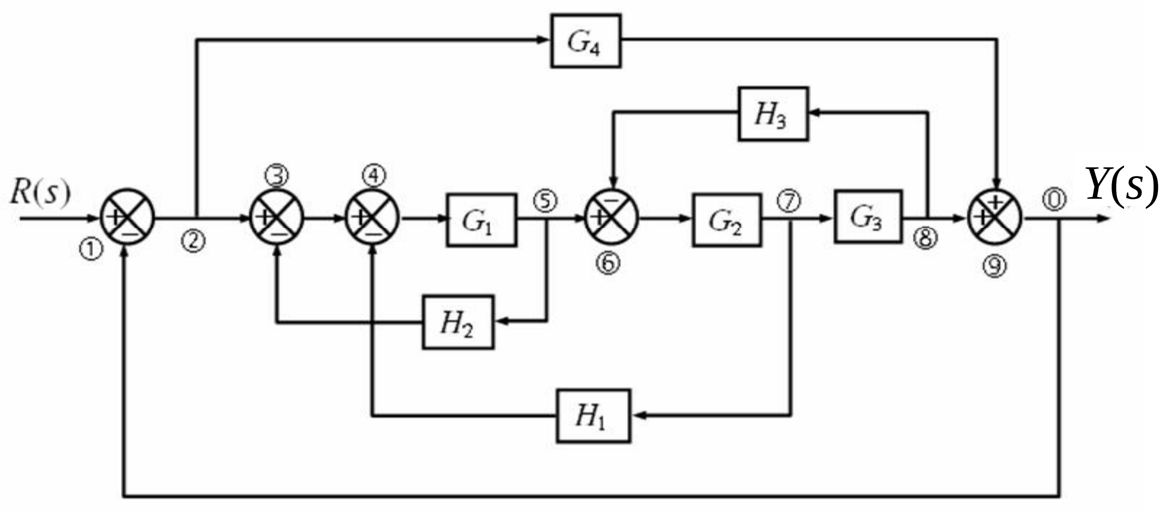
- The equivalent transfer function of the system:

$$G_{eq} = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2)$$

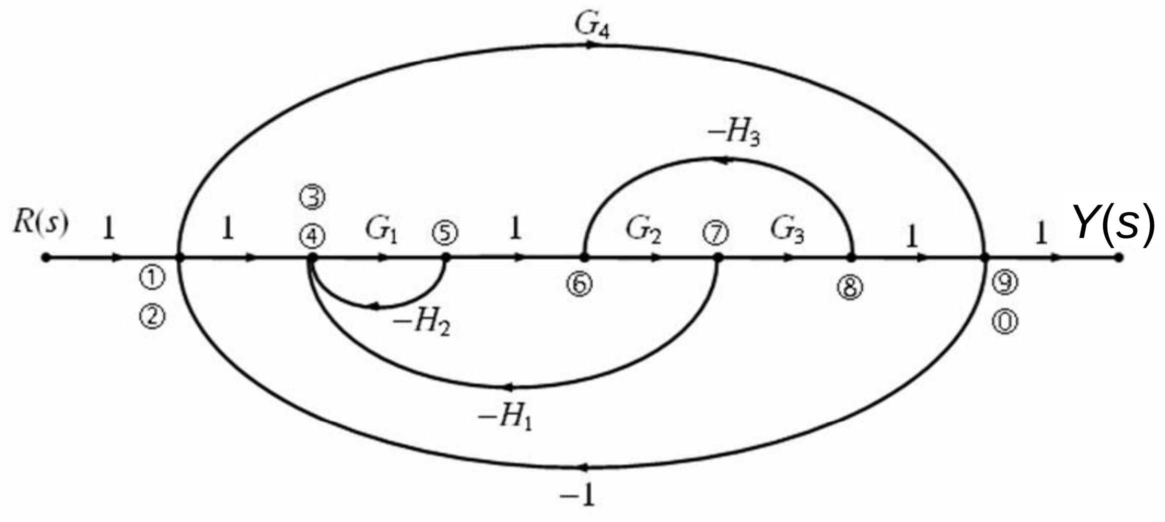
$$G_{eq} = \frac{G_1G_2G_3 + G_1G_3H_1}{1 + G_2H_2 + G_2G_3H_3 + G_1G_2G_3 + G_3H_1H_3 + G_1G_3H_1}$$

Signal flow graph – Example 3

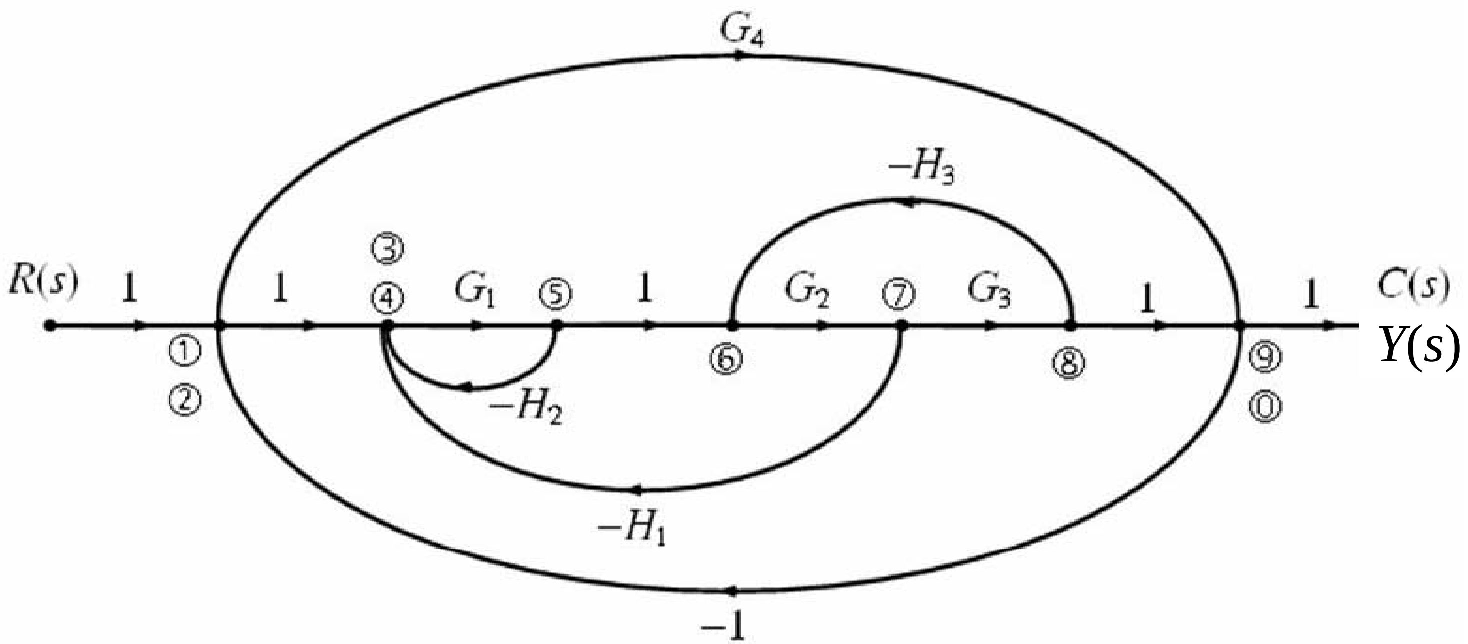
- Find the equivalent transfer function of the system described by the following block diagram:



- Solution:



Signal flow graph – Example 3 (cont')



- Forward path

$$P_1 = G_1G_2G_3$$

$$P_2 = G_4$$

- Loop

$$L_1 = -G_1H_2$$

$$L_2 = -G_1G_2H_1$$

$$L_3 = -G_1G_2G_3$$

$$L_4 = -G_2G_3H_3$$

$$L_5 = -G_4$$

Signal flow graph – Example 3 (cont')

- Determinant of the SFG:

$$\Delta = 1-(L_1+L_2+L_3+L_4+L_5)+(L_1L_4+L_1L_5+L_2L_5+L_4L_5)-L_1L_4L_5$$

- The cofactor:

$$\Delta_1 = 1$$
$$\Delta_2 = 1-(L_1+L_2+L_4)+(L_1L_4)$$

- The equivalent transer function of the system:

$$G = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2)$$

$$Num = G_1G_2G_3 + G_4(1+G_1H_2+G_1G_2H_1+G_2G_3H_3+G_1H_2G_2G_3H_3)$$
$$Den = 1+G_1H_2+G_1G_2H_1+G_1G_2G_3+G_2G_3H_3+G_4+G_1G_2G_3H_2H_3$$
$$+G_1G_4H_2+G_1G_2G_4H_1+G_2G_3G_4H_3+G_1G_2G_3G_4H_2H_3$$

State space equations

State of a system

- ***State:*** The state of a system is a set of variables whose values, together with the equations described the system dynamics, will provide future state and output of the system.

  A n^{th} order system has n state variables. The state variables can be physical variables, but not necessary.

- ***State vector:*** *n* state variables form a column vector called the state vector.

$$\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$$

State equations

- By using state variables, we can transform the n-order differential equation describing the system dynamics into a set of *n* first order differential equations (called state equations) of the form:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

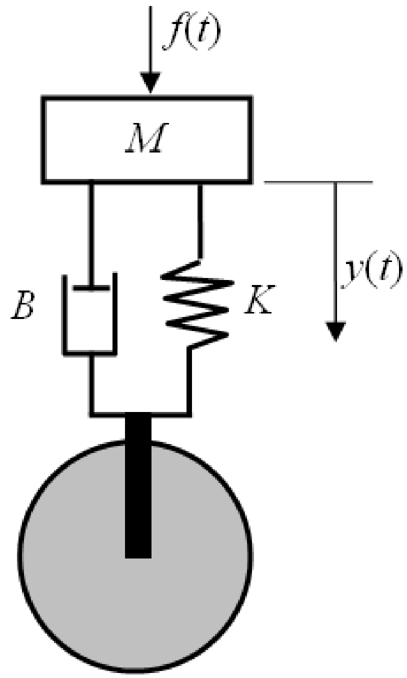
where

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}$$

- ***Note:*** Depending on how we chose the state variables, a system can be described by many different state equations.

State equations – Example 1

A suspension system



★ Differential equation:

$$M\frac{d^2y(t)}{dt^2}+B\frac{dy(t)}{dt}+Ky(t)=f(t)$$

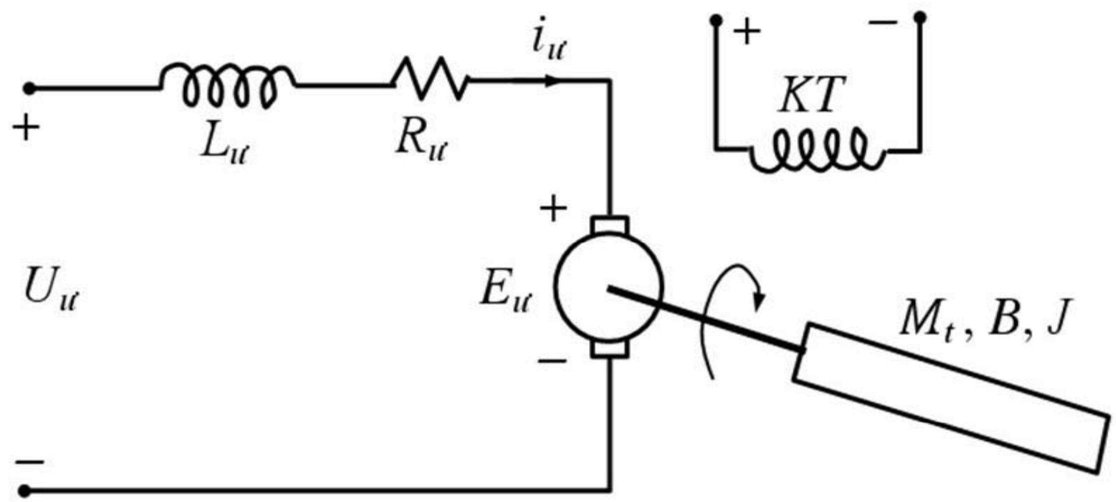
★ Denote:

$$\begin{cases} x_1(t)=y(t) \\ x_2(t)=\dot{y}(t) \end{cases} \Rightarrow \begin{cases} \dot{x}_1(t)=x_2(t) \\ \dot{x}_2(t)=-\dfrac{K}{M}x_1(t)-\dfrac{B}{M}x_2(t)+\dfrac{1}{M}f(t) \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{K}{M} & -\dfrac{B}{M} \end{bmatrix} . \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix} f(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

$$\Leftrightarrow \begin{cases} \dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{B}f(t) \\ y(t)=\boldsymbol{C}\boldsymbol{x}(t) \end{cases} \qquad \boldsymbol{A}=\begin{bmatrix} 0 & 1 \\ -\dfrac{K}{M} & -\dfrac{B}{M} \end{bmatrix} \quad \boldsymbol{B}=\begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix} \quad \boldsymbol{C}=\begin{bmatrix} 1 & 0 \end{bmatrix}$$

DC motor



- L_a : armature induction
- R_a : armature resistance
- U_a : armature voltage
- E_a : back electromotive force
- ω : motor speed
- M_t : load inertia
- B : friction constant
- J : moment of inertia of the rotor

State equations – Example 2 (cont')

- Applying Kirchhoff's law for the armature circuit:

$$U_a(t) = i_a(t).R_a + L_a \frac{di_a(t)}{dt} + E_a(t) \qquad (1)$$

where: $$E_a(t) = K\Phi\,\omega(t) \qquad (2)$$

K : electromotive force constant

Φ : excitation magnetic flux

- Applying Newton's law for the rotating part of the motor: (for simplicity, assuming that load torque is zero)

$$M(t) = B\omega(t) + J\frac{d\omega(t)}{dt} \qquad (3)$$

where: $$M(t) = K\Phi i_a(t) \qquad (4)$$

State equations – Example 2 (cont')

- (1) & (2) $\Rightarrow$ $$\frac{di_{ư}(t)}{dt} = -\frac{R_{ư}}{L_{ư}} i_{ư}(t) - \frac{K\Phi}{L_{ư}}\omega(t) + \frac{1}{L_{ư}}U_{ư}(t) \qquad (5)$$

- (3) & (4) $\Rightarrow$ $$\frac{d\omega(t)}{dt} = \frac{K\Phi}{J} i_{ư}(t) - \frac{B}{J}\omega(t) \qquad (6)$$

- Denote: $$\begin{cases} x_1(t) = i_{ư}(t) \\ x_2(t) = \omega(t) \end{cases}$$

- (5) & (6) $\Rightarrow$ $$\begin{cases} \dot{x}_1(t) = -\dfrac{R_{ư}}{L_{ư}} x_1(t) - \dfrac{K\Phi}{L_{ư}} x_2(t) + \dfrac{1}{L_{ư}} U_{ư}(t) \\ \dot{x}_2(t) = \dfrac{K\Phi}{J} x_1(t) - \dfrac{B}{J} x_2(t) \end{cases}$$

State equations – Example 2 (cont')

$$\Leftrightarrow \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{R_ư}{L_ư} & -\dfrac{K\Phi}{L_ư} \\ \dfrac{K\Phi}{J} & -\dfrac{B}{J} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L_ư} \\ 0 \end{bmatrix} U_ư(t) \\ \omega(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

$$\Leftrightarrow \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}U_u(t) \\ \omega(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

where: $\boldsymbol{A} = \begin{bmatrix} -\dfrac{R_ư}{L_ư} & -\dfrac{K\Phi}{L_ư} \\ \dfrac{K\Phi}{J} & -\dfrac{B}{J} \end{bmatrix}$ $\boldsymbol{B} = \begin{bmatrix} \dfrac{1}{L_ư} \\ 0 \end{bmatrix}$ $\boldsymbol{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$

Case #1: The differential equation does not involve the input derivatives

- The differential equation describing the system dynamics is:

$$a_0 \frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t) = b_0 u(t)$$

- Define the state variables as follow:
  - The first state is the system output:
  - The i th state (i=2..n) is chosen to be the first derivative of the (i–1)th state :

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{x}_1(t)$$
$$x_3(t) = \dot{x}_2(t)$$
$$\vdots$$
$$x_n(t) = \dot{x}_{n-1}(t)$$

Case #1 (cont')

- State equation: $$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

where:

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} \qquad \boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -\dfrac{a_n}{a_0} & -\dfrac{a_{n-1}}{a_0} & -\dfrac{a_{n-2}}{a_0} & \ldots & -\dfrac{a_1}{a_0} \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \dfrac{b_0}{a_0} \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

Case #1: Example

- Write the state equations describing the following system:

$$2\dddot{y}(t) + 5\ddot{y}(t) + 6\dot{y}(t) + 10y(t) = u(t)$$

- Define the state variables as: $\begin{cases} x_1(t) = y(t) \\ x_2(t) = \dot{x}_1(t) \\ x_3(t) = \dot{x}_2(t) \end{cases}$

- State equation: $\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}r(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\dfrac{a_3}{a_0} & -\dfrac{a_2}{a_0} & -\dfrac{a_1}{a_0} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -3 & -2.5 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{b_0}{a_0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Case #2: The differential equation involve the input derivatives

- Consider a system described by the differential equation:

$$a_0\frac{d^n y(t)}{dt^n}+a_1\frac{d^{n-1}y(t)}{dt^{n-1}}+\cdots+a_{n-1}\frac{dy(t)}{dt}+a_n y(t)=$$

$$b_0\frac{d^{n-1}u(t)}{dt^{n-1}}+b_1\frac{d^{n-2}u(t)}{dt^{n-1}}+\cdots+b_{n-2}\frac{du(t)}{dt}+b_{n-1}u(t)$$

- Define the state variables as follow:
  - The first state is the system output:
  - The i th state (i=2..n) is equal to the first derivative of the (i–1)th state minus a quantity proportional to the input:

$$x_1(t)=y(t)$$
$$x_2(t)=\dot{x}_1(t)-\beta_1 u(t)$$
$$x_3(t)=\dot{x}_2(t)-\beta_2 u(t)$$
$$\vdots$$
$$x_n(t)=\dot{x}_{n-1}(t)-\beta_{n-1} u(t)$$

Case #2 (cont')

✶ State equation:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

where:

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} \qquad \boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\dfrac{a_n}{a_0} & -\dfrac{a_{n-1}}{a_0} & -\dfrac{a_{n-2}}{a_0} & \dots & -\dfrac{a_1}{a_0} \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Case #2 (cont')

The coefficients β in the vector $\boldsymbol{B}$ are calculated as follow:

$$\beta_1 = \frac{b_0}{a_0}$$

$$\beta_2 = \frac{b_1 - a_1\beta_1}{a_0}$$

$$\beta_3 = \frac{b_2 - a_1\beta_2 - a_2\beta_1}{a_0}$$

$$\vdots$$

$$\beta_n = \frac{b_{n-1} - a_1\beta_{n-1} - a_2\beta_{n-2} - \ldots - a_{n-1}\beta_1}{a_0}$$

Case #2: Example

- Write the state equations describing the following system:

$$2\dddot{y}(t)+5\ddot{y}(t)+6\dot{y}(t)+10y(t)=10\dot{u}(t)+20u(t)$$

- Define the state variables:
$$\begin{cases} x_1(t)=y(t) \\ x_2(t)=\dot{x}_1(t)-\beta_1 u(t) \\ x_3(t)=\dot{x}_2(t)-\beta_2 u(t) \end{cases}$$

- The state equation:
$$\begin{cases} \dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{B}u(t) \\ y(t)=\boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

$$\boldsymbol{A}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\dfrac{a_3}{a_0} & -\dfrac{a_2}{a_0} & -\dfrac{a_1}{a_0} \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -3 & -2.5 \end{bmatrix} \qquad \boldsymbol{B}=\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

$$\boldsymbol{C}=\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Case #2: Example (cont'))

★ The elements of vector ***B*** are calculated as follow:

$$\begin{cases} \beta_1 = \dfrac{b_0}{a_0} = \dfrac{0}{2} = 0 \\ \beta_2 = \dfrac{b_1 - a_1\beta_1}{a_0} = \dfrac{10 - 5\times 0}{2} = 5 \\ \beta_3 = \dfrac{b_2 - a_1\beta_2 - a_2\beta_1}{a_0} = \dfrac{20 - 5\times 5 - 6\times 0}{2} = -\dfrac{5}{2} \end{cases}$$

$$\Rightarrow \qquad \boldsymbol{B} = \begin{bmatrix} 0 \\ 5 \\ -\dfrac{5}{2} \end{bmatrix}$$

State-space equations in controllable canonical form

- Consider a system described by the differential equation:

$$a_0\frac{d^n y(t)}{dt^n}+a_1\frac{d^{n-1}y(t)}{dt^{n-1}}+\cdots+a_{n-1}\frac{dy(t)}{dt}+a_n y(t)=$$
$$b_0\frac{d^m u(t)}{dt^m}+b_1\frac{d^{m-1}u(t)}{dt^{m-1}}+\cdots+b_{m-1}\frac{du(t)}{dt}+b_m u(t)$$

or equivalently by the transfer function:

$$G(s)=\frac{b_0s^m+b_1s^{m-1}+\ldots+b_{m-1}s+b_m}{a_0s^n+a_1s^{n-1}+\ldots+a_{n-1}s+a_n}$$

- The controllable canonical state equations of the system is presented in the next slide.

State-space equations in controllable canonical form (cont')

★ State equations:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}r(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

Where:

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad \boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\dfrac{a_n}{a_0} & -\dfrac{a_{n-1}}{a_0} & -\dfrac{a_{n-2}}{a_0} & \dots & -\dfrac{a_1}{a_0} \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} \dfrac{b_m}{a_0} & \dfrac{b_{m-1}}{a_0} & \dots & \dfrac{b_0}{a_0} & 0 & \dots & 0 \end{bmatrix}$$

- Write the controllable canonical state equations of the following system:

$$2\dddot{y}(t)+\ddot{y}(t)+5\dot{y}(t)+4y(t)=\ddot{u}(t)+3u(t)$$

- Solution:

$$\begin{cases}\boldsymbol{x}(t)=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{B}r(t)\\ y(t)=\boldsymbol{C}\boldsymbol{x}(t)\end{cases}$$

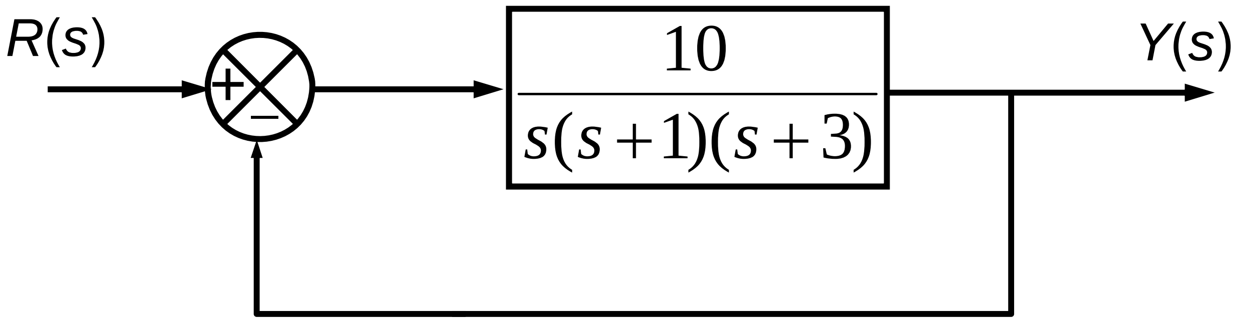
where:

$$\boldsymbol{A}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\dfrac{a_3}{a_0} & -\dfrac{a_2}{a_0} & -\dfrac{a_1}{a_0}\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2 & -2.5 & -0.5\end{bmatrix}\qquad \boldsymbol{B}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

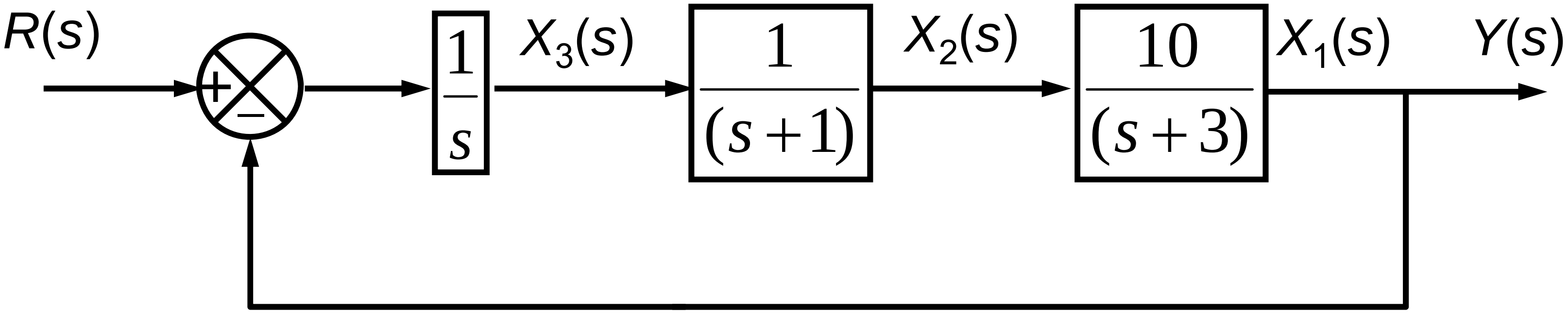
$$\boldsymbol{C}=\begin{bmatrix}\dfrac{b_2}{a_0} & \dfrac{b_1}{a_0} & \dfrac{b_0}{a_0}\end{bmatrix}=\begin{bmatrix}1.5 & 0 & 0.5\end{bmatrix}$$

Example

★ Establish the state equations describing the system below:



★ Define the state variables as in the block diagram:



Example (cont')

✶ From the block diagram, we have:

- $X_1(s) = \dfrac{10}{s+3} X_2(s) \quad \Rightarrow \quad sX_1(s) + 3X_1(s) = 10X_2(s)$

  $\Rightarrow \quad \dot{x}_1(t) = -3x_1(t) + 10x_2(t) \qquad (1)$

- $X_2(s) = \dfrac{1}{s+1} X_3(s) \quad \Rightarrow \quad sX_2(s) + X_2(s) = X_3(s)$

  $\Rightarrow \quad \dot{x}_2(t) = -x_2(t) + x_3(t) \qquad (2)$

- $X_3(s) = \dfrac{1}{s}\left(R(s) - Y(s)\right) \quad \Rightarrow \quad sX_3(s) = R(s) - X_1(s)$

  $\Rightarrow \quad \dot{x}_3(t) = -x_1(t) + r(t) \qquad (3)$

Establishing state equations from block diagrams

Example (cont')

- Combining (1), (2), and (3) leads to the state equations:

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}}_{\dot{\boldsymbol{x}}(t)} = \underbrace{\begin{bmatrix} -3 & 10 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}}_{\boldsymbol{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{B}} r(t)$$

- Output equation:

$$y(t) = x_1(t) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{\boldsymbol{C}} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

State equation to transfer function

- Given a system described by the state equations:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

- Then the transfer function of the system is:

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \boldsymbol{C}\left(s\boldsymbol{I} - \boldsymbol{A}\right)^{-1}\boldsymbol{B}}$$

State equation to transfer function – Example

- Calculate the transfer function of the system described by the state equation:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- Solution: The transfer function of the system is:

$$G(s) = \frac{Y(s)}{U(s)} = \boldsymbol{C}\left(s\boldsymbol{I} - \boldsymbol{A}\right)^{-1}\boldsymbol{B}$$

Example (cont')

$$(s\boldsymbol{I}-\boldsymbol{A}) = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(s\boldsymbol{I}-\boldsymbol{A})^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \frac{1}{s(s+3)-2.(-1)}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1} = \frac{1}{s^2+3s+2}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \frac{1}{s^2+3s+2}\begin{bmatrix} s+3 & 1 \end{bmatrix}$$

$$\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B} = \frac{1}{s^2+3s+2}\begin{bmatrix} s+3 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{3(s+3)+1}{s^2+3s+2}$$

$$\Rightarrow \quad \boxed{G(s) = \frac{3s+10}{s^2+3s+2}}$$

Solution to state equations

- Solution to the state equation $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t)$?

$$\boldsymbol{x}(t) = \Phi(t)\boldsymbol{x}(0^+) + \int_0^t \Phi(t-\tau)\boldsymbol{B}u(\tau)d\tau$$

where $\Phi(t) = \mathscr{L}^{-1}[\Phi(s)]$ transient matrix

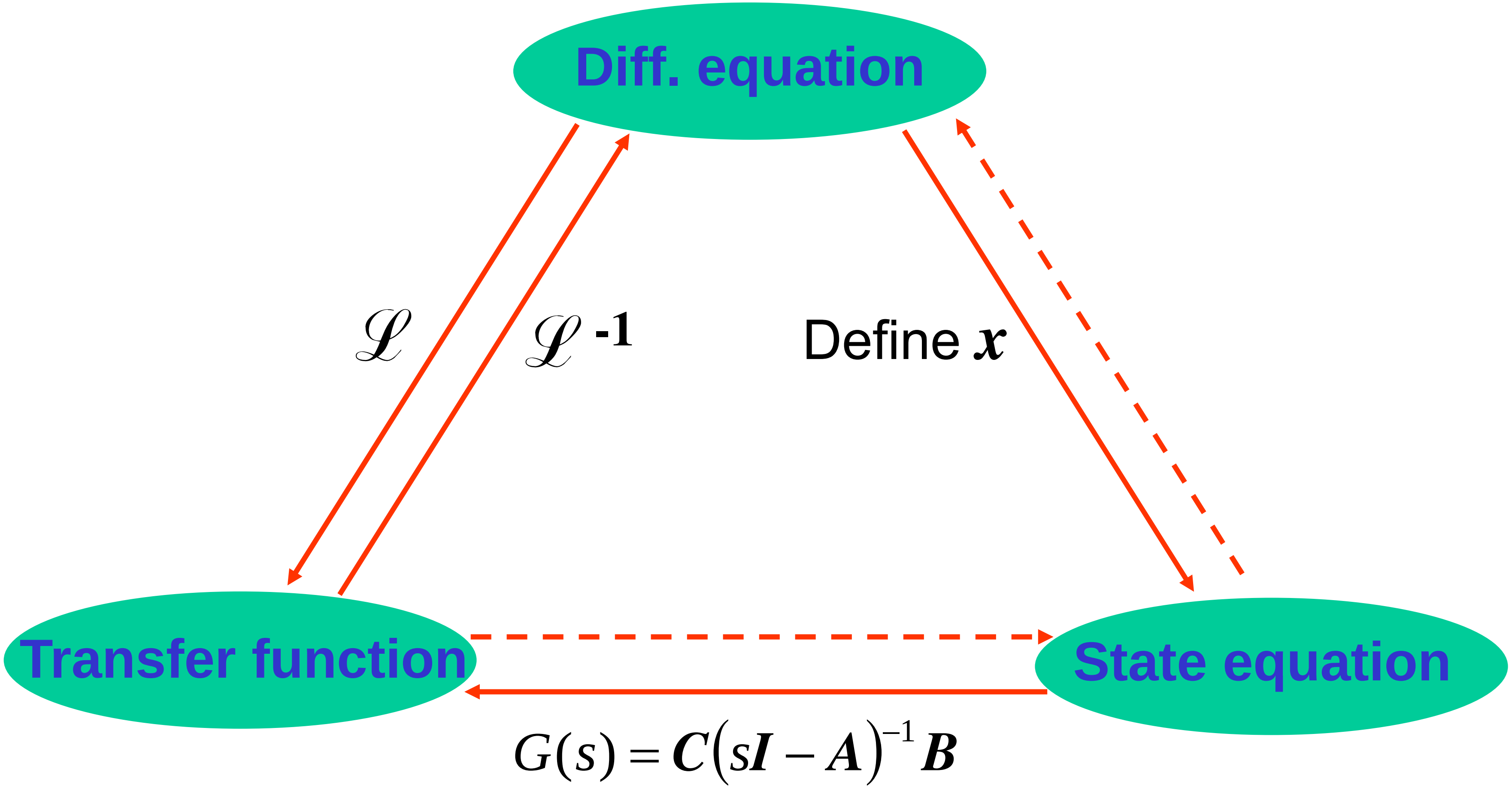
$$\Phi(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}$$

- System response?

$$y(t) = \boldsymbol{C}\boldsymbol{x}(t)$$

- Example:

Relationship between the mathematical models



Linearized models of nonlinear systems

Nonlinear systems

- Nonlinear systems do not satisfy the superposition principle and cannot be described by a linear differential equation.
- Most of the practical systems are nonlinear:
  - Fluid system (Ex: liquid tank,…)
  - Thermal system (Ex: furnace,…)
  - Mechanical system (Ex: robot arm,….)
  - Electro-magnetic system (TD: motor,…)
  - Hybrid system ,…

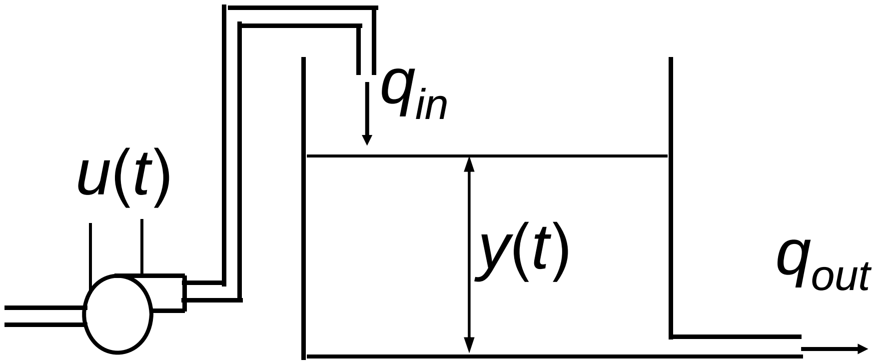
Mathematical model of nonlinear systems

- Input – output relationship of a continuous nonlinear system can be expressed in the form of a nonlinear differential equations.

$$\frac{d^n y(t)}{dt^n} = g\left(\frac{d^{n-1} y(t)}{dt^{n-1}}, \cdots, \frac{dy(t)}{dt}, y(t), \frac{d^m u(t)}{dt^m}, \cdots, \frac{du(t)}{dt}, u(t)\right)$$

where: $u(t)$: input signal,
$y(t)$: output signal,
$g(.)$: nonlinear function

Nonlinear system – Example 1



a: cross area of the dischage valve
A: cross area of the tank
g: gravity acceleration
k: constant
C_D: discharge constant

- Balance equation: $$A\dot{y}(t) = q_{in}(t) - q_{out}(t)$$

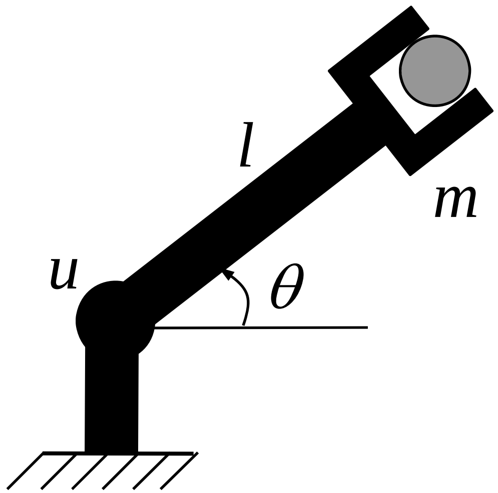
where: $$q_{in}(t) = ku(t)$$

$$q_{out}(t) = aC_D\sqrt{2gy(t)}$$

$$\Rightarrow \boxed{\dot{y}(t) = \frac{1}{A}\left(ku(t) - aC_D\sqrt{2gy(t)}\right)}$$

(first order nonlinear system)

Nonlinear system – Example 2



J: moment inertia of the robot arm
M: mass of the robot arm
m: object mass
l: length of robot arm
l_C : distance from center of gravity to rotary axis
B: friction constant
g: gravitational acceleration
u(*t*): input torque
$\theta(t)$: robot arm angle

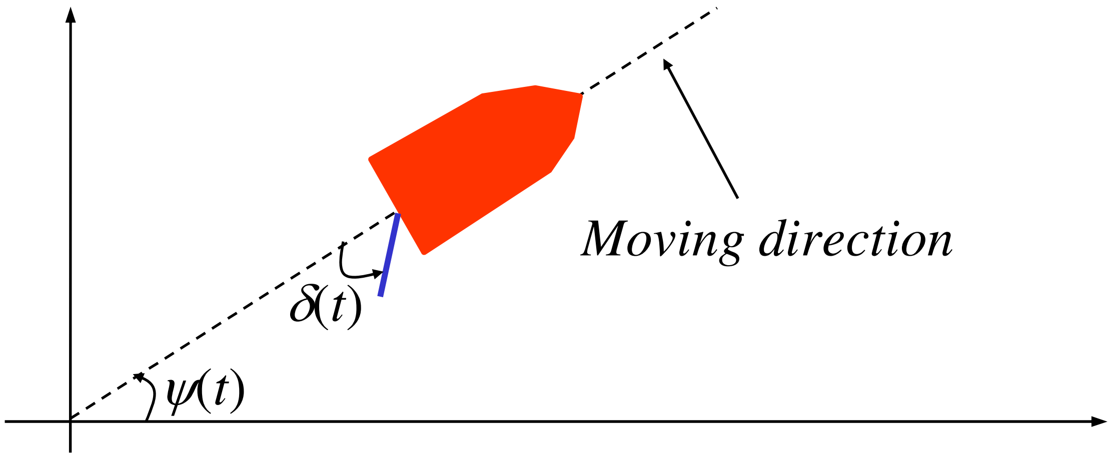
- According to Newton's Law

$$(J+ml^2)\ddot{\theta}(t)+B\dot{\theta}(t)+(ml+Ml_C)g\cos\theta=u(t)$$

$$\Rightarrow \quad \ddot{\theta}(t)=-\frac{B}{(J+ml^2)}\dot{\theta}(t)-\frac{(ml+Ml_C)}{(J+ml^2)}g\cos\theta+\frac{1}{(J+ml^2)}u(t)$$

(second order nonlinear system)

Nonlinear system – Example 3



δ: steering angle
ψ: ship angle
k: constant
τ_i: constant

- The differential equation describing the steering dynamic of a ship:

$$\dddot{\psi}(t) = -\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\ddot{\psi}(t) - \left(\frac{1}{\tau_1\tau_2}\right)\left(\dot{\psi}^3(t) + \dot{\psi}(t)\right) + \left(\frac{k}{\tau_1\tau_2}\right)\left(\tau_3\dot{\delta}(t) + \delta(t)\right)$$

(third order nonlinear system)

Describing nonlinear systems by state equations

- A continuous nonlinear system can be described by the state equation:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), u(t)) \\ y(t) = h(\boldsymbol{x}(t), u(t)) \end{cases}$$

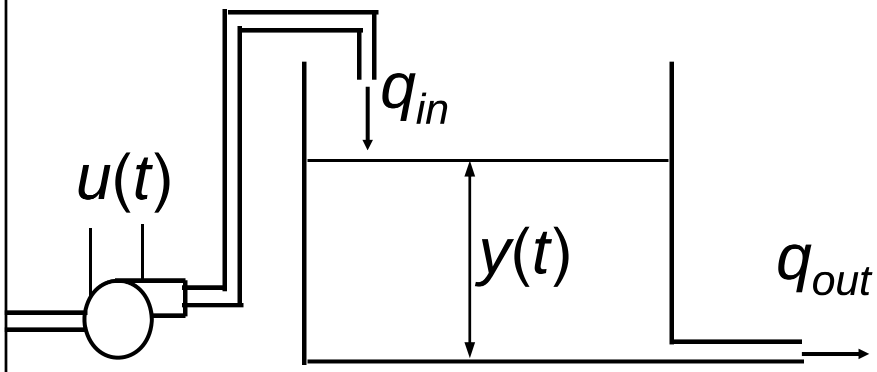
where: $u(t)$: input,

$y(t)$: output,

$\boldsymbol{x}(t)$: state vector,

$$\boldsymbol{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$$

$\boldsymbol{f}(.)$, $h(.)$: nonlinear functions



- Differential equation:

$$\dot{y}(t) = \frac{1}{A}\left(ku(t) - aC_D\sqrt{2gy(t)}\right)$$

- Define the state variable:

$$x_1(t) = y(t)$$

- State equation:

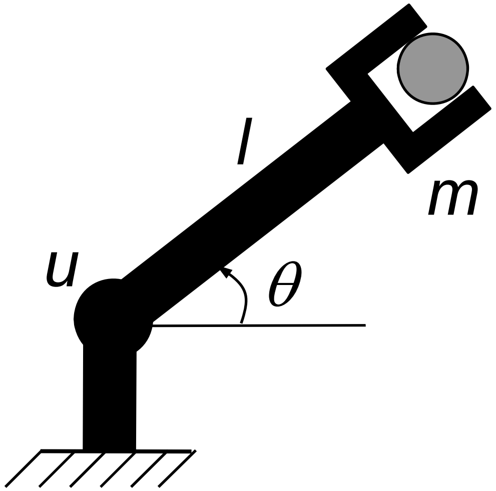
$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), u(t)) \\ y(t) = h(\boldsymbol{x}(t), u(t)) \end{cases}$$

where

$$\boldsymbol{f}(\boldsymbol{x}, u) = -\frac{aC_D\sqrt{2gx_1(t)}}{A} + \frac{k}{A}u(t)$$

$$h(\boldsymbol{x}(t), u(t)) = x_1(t)$$

State-space model of nonlinear system – Example 2



★ Differential equation:

$$\ddot{\theta}(t) = -\frac{B}{(J+ml^2)}\dot{\theta}(t) - \frac{(ml+Ml_C)}{(J+ml^2)} g\cos\theta + \frac{1}{(J+ml^2)}u(t)$$

★ Define the state variable: $\begin{cases} x_1(t) = \theta(t) \\ x_2(t) = \dot{\theta}(t) \end{cases}$

★ State equation:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), u(t)) \\ y(t) = h(\boldsymbol{x}(t), u(t)) \end{cases}$$

where

$$\boldsymbol{f}(\boldsymbol{x}, u) = \begin{bmatrix} x_2(t) \\ -\dfrac{(ml+Ml_C)g}{(J+ml^2)}\cos x_1(t) - \dfrac{B}{(J+ml^2)}x_2(t) + \dfrac{1}{(J+ml^2)}u(t) \end{bmatrix}$$

$$h(\boldsymbol{x}(t), u(t)) = x_1(t)$$

Equilibrium points of a nonlinear system

- Consider a nonlinear system described by the diff. equation:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), u(t)) \\ y(t) = h(\boldsymbol{x}(t), u(t)) \end{cases}$$

- The state $\bar{\boldsymbol{x}}$ is called the equilibrium point of the nonlinear system if the system is at the state $\bar{\boldsymbol{x}}$ and the control signal is fixed at $\bar{u}$ then the system will stay at state $\bar{\boldsymbol{x}}$ forever.

- If $(\bar{\boldsymbol{x}}, \bar{u})$ is equilibrium point of the nonlinear system then:

$$\boldsymbol{f}(\boldsymbol{x}(t), u(t))\Big|_{\boldsymbol{x}=\bar{\boldsymbol{x}}, u=\bar{u}} = 0$$

- The equilibrium point is also called the stationary point of the nonlinear system.

Equilibrium point of nonlinear system – Example 1

- Consider a nonlinear system described by the state equation:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t).x_2(t) + u \\ x_1(t) + 2x_2(t) \end{bmatrix}$$

Find the equilibrium point when $u(t) = \bar{u} = 1$

- Solution:

The equilibrium point(s) are the solution to the equation:

$$\boldsymbol{f}(\boldsymbol{x}(t), u(t))\Big|_{\boldsymbol{x}=\bar{\boldsymbol{x}}, u=\bar{u}} = 0$$

$$\Leftrightarrow \quad \begin{cases} \bar{x}_1.\bar{x}_2 + 1 = 0 \\ \bar{x}_1 + 2\bar{x}_2 = 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \bar{x}_1 = \sqrt{2} \\ \bar{x}_2 = -\dfrac{\sqrt{2}}{2} \end{cases} \quad \text{or} \quad \begin{cases} \bar{x}_1 = -\sqrt{2} \\ \bar{x}_2 = +\dfrac{\sqrt{2}}{2} \end{cases}$$

Equilibrium point of nonlinear system – Example 2

- Consider a nonlinear system described by the state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 + x_2^2 + x_3^2 + u \\ x_3 + \sin(x_1 - x_3) \\ x_3^2 + u \end{bmatrix}$$

$$y = x_1$$

Find the equilibrium point when $u(t) = \bar{u} = 0$

- Consider a nonlinear system described by the diff. equation:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), u(t)) \\ y(t) = h(\boldsymbol{x}(t), u(t)) \end{cases} \qquad (1)$$

- Expanding Taylor series for $\boldsymbol{f}(\boldsymbol{x},u)$ and $h(\boldsymbol{x},u)$ around the equilibrium point $(\bar{\boldsymbol{x}}, \bar{u})$, we can approximate the nonlinear system (1) by the following linearized state equation:

$$\begin{cases} \dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{A}\tilde{\boldsymbol{x}}(t) + \boldsymbol{B}\tilde{u}(t) \\ \tilde{y}(t) = \boldsymbol{C}\tilde{\boldsymbol{x}}(t) + \boldsymbol{D}\tilde{u}(t) \end{cases} \qquad (2)$$

where:

$$\tilde{\boldsymbol{x}}(t) = \boldsymbol{x}(t) - \bar{\boldsymbol{x}}$$

$$\tilde{u}(t) = u(t) - \bar{u}$$

$$\tilde{y}(t) = y(t) - \bar{y} \qquad (\bar{y} = h(\bar{\boldsymbol{x}}, \bar{u}))$$

★ The matrix of the linearized state equation are calculated as follow:

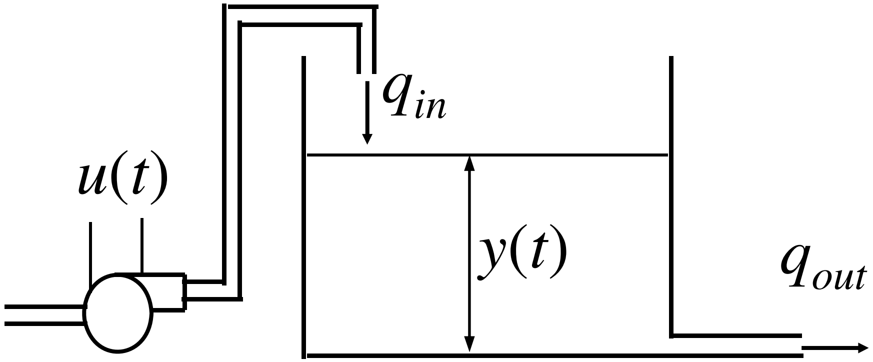
$$\boldsymbol{A} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}_{(\bar{\boldsymbol{x}},\bar{u})}$$

$$\boldsymbol{B} = \begin{bmatrix} \dfrac{\partial f_1}{\partial u} \\ \dfrac{\partial f_2}{\partial u} \\ \vdots \\ \dfrac{\partial f_n}{\partial u} \end{bmatrix}_{(\bar{\boldsymbol{x}},\bar{u})}$$

$$\boldsymbol{C} = \begin{bmatrix} \dfrac{\partial h}{\partial x_1} & \dfrac{\partial h}{\partial x_2} & \cdots & \dfrac{\partial h}{\partial x_n} \end{bmatrix}_{(\bar{\boldsymbol{x}},\bar{u})}$$

$$\boldsymbol{D} = \begin{bmatrix} \dfrac{\partial h}{\partial u} \end{bmatrix}_{(\bar{\boldsymbol{x}},\bar{u})}$$

Linearized state-space model – Example 1



The parameter of the tank:

$$a = 1cm^2, \quad A = 100cm^2$$

$$k = 150cm^3/\sec.V, \quad C_D = 0.8$$

$$g = 981cm/\sec^2$$

- Nonlinear state equation:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), u(t)) \\ y(t) = h(\boldsymbol{x}(t), u(t)) \end{cases}$$

where

$$\boldsymbol{f}(\boldsymbol{x}, u) = -\frac{aC_D\sqrt{2gx_1(t)}}{A} + \frac{k}{A}u(t) = -0.3544\sqrt{x_1(t)} + 0.9465u(t)$$

$$h(\boldsymbol{x}(t), u(t)) = x_1(t)$$

Linearized state-space model – Example 1 (cont')

Linearize the system around $y = 20\text{cm}$:

- The equilibrium point:

$$\bar{x}_1 = 20$$

$$\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{u}) = -0.3544\sqrt{\bar{x}_1} + 1.5\bar{u} = 0 \quad \Rightarrow \quad \bar{u} = 0.9465$$

Linearized state-space model – Example 1 (cont')

- The matrix of the linearized state-space model:

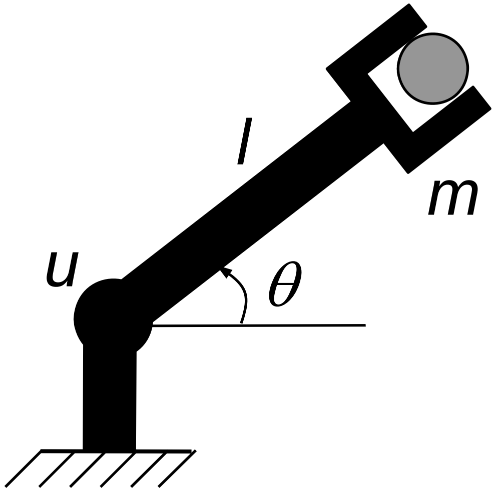
$$\boldsymbol{A} = \left.\frac{\partial f_1}{\partial x_1}\right|_{(\bar{\boldsymbol{x}},\bar{u})} = -\left.\frac{aC_D\sqrt{2g}}{2A\sqrt{x_1}}\right|_{(\bar{\boldsymbol{x}},\bar{u})} = -0.0396 \qquad \boldsymbol{C} = \left.\frac{\partial h}{\partial x_1}\right|_{(\bar{\boldsymbol{x}},\bar{u})} = 1$$

$$\boldsymbol{B} = \left.\frac{\partial f_1}{\partial u}\right|_{(\bar{\boldsymbol{x}},\bar{u})} = \left.\frac{k}{A}\right|_{(\bar{\boldsymbol{x}},\bar{u})} = 1.5 \qquad \boldsymbol{D} = \left.\frac{\partial h}{\partial u}\right|_{(\bar{\boldsymbol{x}},\bar{u})} = 0$$

- The linearized state equation describing the system around the equilibrium point y=20cm is:

$$\begin{cases} \dot{\tilde{\boldsymbol{x}}}(t) = -0.0396\tilde{\boldsymbol{x}}(t) + 1.5\tilde{u}(t) \\ \tilde{y}(t) = \tilde{\boldsymbol{x}}(t) \end{cases}$$

Linearized state-space model – Example 2



The parameters of the robot:

$$l = 0.5m, \quad l_C = 0.2m, m = 0.1kg$$

$$M = 0.5kg, J = 0.02kg.m^2$$

$$B = 0.005, \quad g = 9.81m/\sec^2$$

- Nonlinear state equation :

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), u(t)) \\ y(t) = h(\boldsymbol{x}(t), u(t)) \end{cases}$$

where:

$$\boldsymbol{f}(\boldsymbol{x}, u) = \begin{bmatrix} x_2(t) \\ -\dfrac{(ml + Ml_C)g}{(J + ml^2)} \cos x_1(t) - \dfrac{B}{(J + ml^2)} x_2(t) + \dfrac{1}{(J + ml^2)} u(t) \end{bmatrix}$$

$$h(\boldsymbol{x}(t), u(t)) = x_1(t)$$

Linearized state-space model – Example 2 (cont')

Linearize the system around the equilibrium point $y = \pi/6$ (rad):

- Calculating the equilibrium point:

$$\bar{x}_1 = \pi/6$$

$$\boldsymbol{f}(\bar{\boldsymbol{x}},\bar{u}) = \begin{bmatrix} \bar{x}_2 \\ -\dfrac{(ml+Ml_C)g}{(J+ml^2)}\cos\bar{x}_1 - \dfrac{B}{(J+ml^2)}\bar{x}_2 + \dfrac{1}{(J+ml^2)}\bar{u} \end{bmatrix} = 0$$

$$\Rightarrow \quad \begin{cases} \bar{x}_2 = 0 \\ \bar{u} = 1.2744 \end{cases}$$

Then the equilibrium point is:

$$\bar{\boldsymbol{x}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \pi/6 \\ 0 \end{bmatrix}$$

$$\bar{u} = 1.2744$$

Linearized state-space model – Example 2 (cont')

★ The system matrix around the equilibrium point:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$a_{11} = \left.\frac{\partial f_1}{\partial x_1}\right|_{(\bar{\boldsymbol{x}},\bar{u})} = 0 \qquad a_{12} = \left.\frac{\partial f_1}{\partial x_2}\right|_{(\bar{\boldsymbol{x}},\bar{u})} = 1$$

$$a_{21} = \left.\frac{\partial f_2}{\partial x_1}\right|_{(\bar{\boldsymbol{x}},\bar{u})} = \left.\frac{(ml + Ml_C)}{(J + ml^2)}\sin x_1(t)\right|_{(\bar{\boldsymbol{x}},\bar{u})}$$

$$a_{22} = \left.\frac{\partial f_2}{\partial x_2}\right|_{(\bar{\boldsymbol{x}},\bar{u})} = -\left.\frac{B}{(J + ml^2)}\right|_{(\bar{\boldsymbol{x}},\bar{u})}$$

Linearized state-space model – Example 2 (cont')

- The input matrix around the equilibrium point:

$$\boldsymbol{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$b_1 = \left.\frac{\partial f_1}{\partial u}\right|_{(\overline{\boldsymbol{x}},\overline{u})} = 0$$

$$b_2 = \left.\frac{\partial f_2}{\partial u}\right|_{(\overline{\boldsymbol{x}},\overline{u})} = \frac{1}{J + ml^2}$$

Linearized state-space model – Example 2 (cont')

- The output matrix around the equilibrium point:

$$\boldsymbol{C} = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \qquad c_1 = \left.\frac{\partial h}{\partial x_1}\right|_{(\bar{\boldsymbol{x}},\bar{u})} = 1 \qquad c_2 = \left.\frac{\partial h}{\partial x_2}\right|_{(\bar{\boldsymbol{x}},\bar{u})} = 0$$

$$\boldsymbol{D} = d_1 \qquad d_1 = \left.\frac{\partial h}{\partial u}\right|_{(\bar{\boldsymbol{x}},\bar{u})} = 0$$

- Then the linearized state equation is: $\begin{cases} \dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{A}\tilde{\boldsymbol{x}}(t) + \boldsymbol{B}\tilde{u}(t) \\ \tilde{y}(t) = \boldsymbol{C}\tilde{\boldsymbol{x}}(t) + \boldsymbol{D}\tilde{u}(t) \end{cases}$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ a_{21} & a_{22} \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \boldsymbol{D} = 0$$

Regulating nonlinear system around equilibrium point

- Drive the nonlinear system to the neighbor of the equilibrium point (the simplest way is to use an ON-OFF controller)
- Around the equilibrium point, use a linear controller to maintain the system around the equilibrium point.