Lecture Notes

Fundamentals of Control Systems

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Chapter 8

ANALYSIS OF DISCRETE CONTROL SYSTEMS

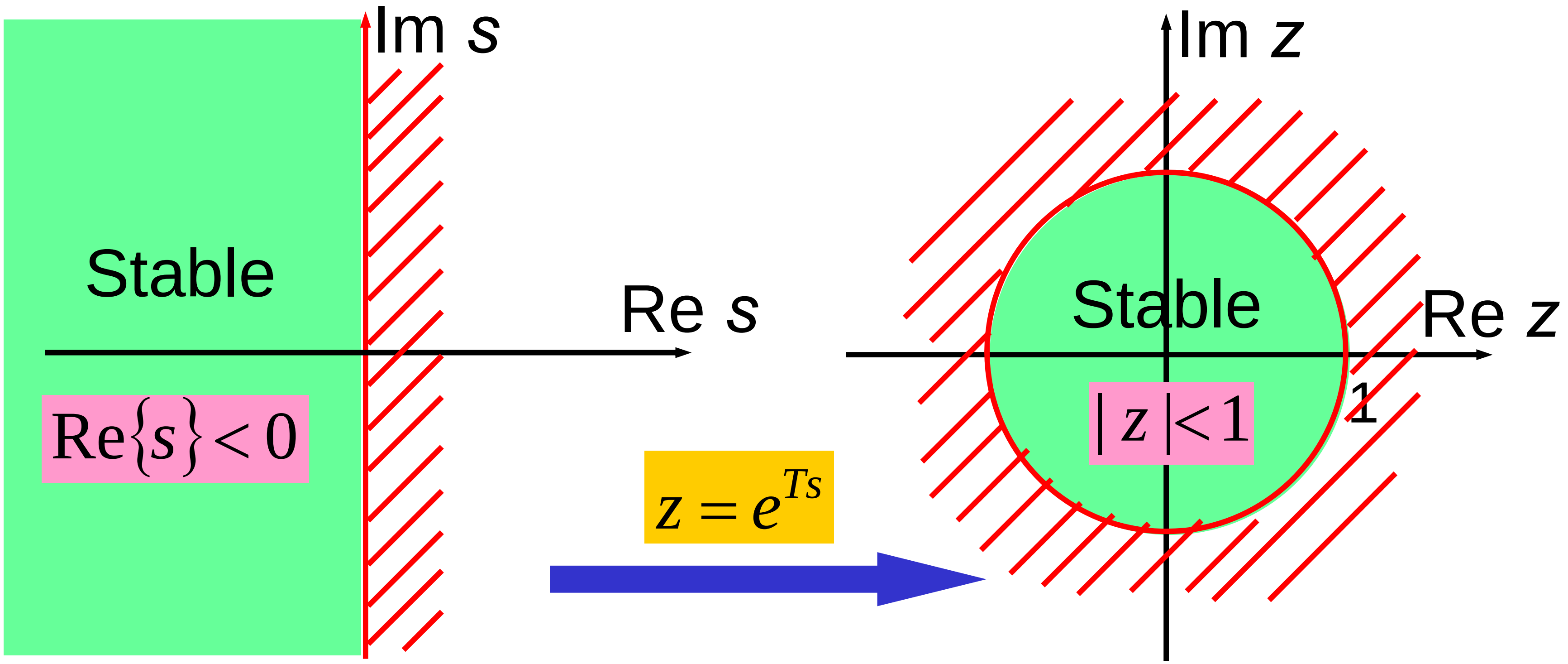
Content

- Stability conditions for discrete systems
- Extension of Routh-Hurwitz criteria
- Jury criterion
- Root locus
- Steady state error
- Performance of discrete systems

Stability conditions for discrete systems

Stability conditions for discrete systems

✶ A system is defined to be BIBO stable if every bounded input to the system results in a bounded output.

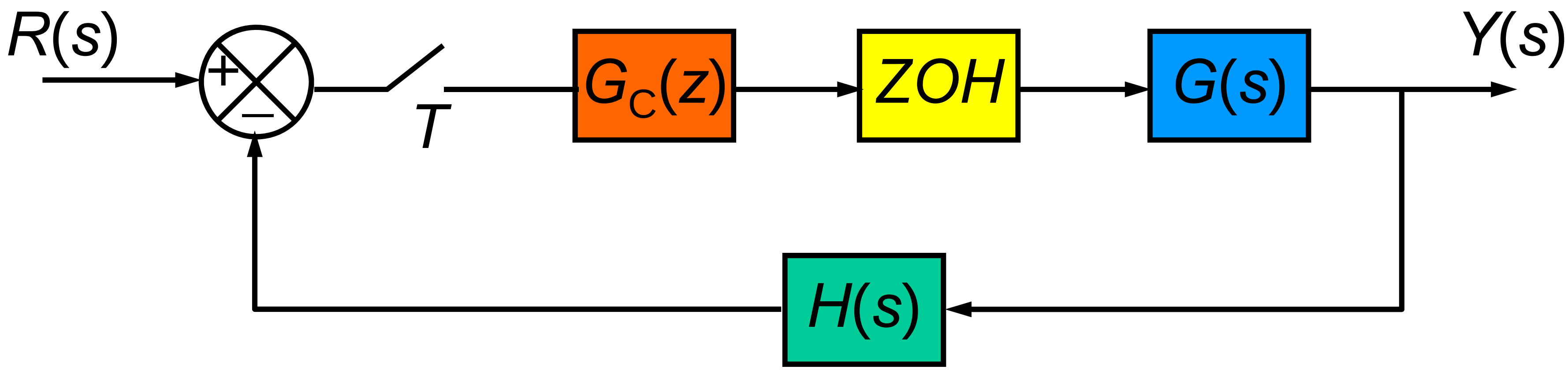


The region of stability for a continuous system is the left-half s-plane

The region of stability for a discrete system is the interior of the unit circle

Characteristic equation of discrete systems

✶ Discrete systems described by block diagram:



⇒ Characteristic equation: $1 + G_C(z)GH(z) = 0$

✶ Discrete systems described by the state equation

$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d \boldsymbol{x}(k) + \boldsymbol{B}_d r(k) \\ y(k) = \boldsymbol{C}_d \boldsymbol{x}(k) \end{cases}$$

⇒ Characteristic equation: $\det(z\boldsymbol{I} - \boldsymbol{A}_d) = 0$

Methods for analysis the stability of discrete systems

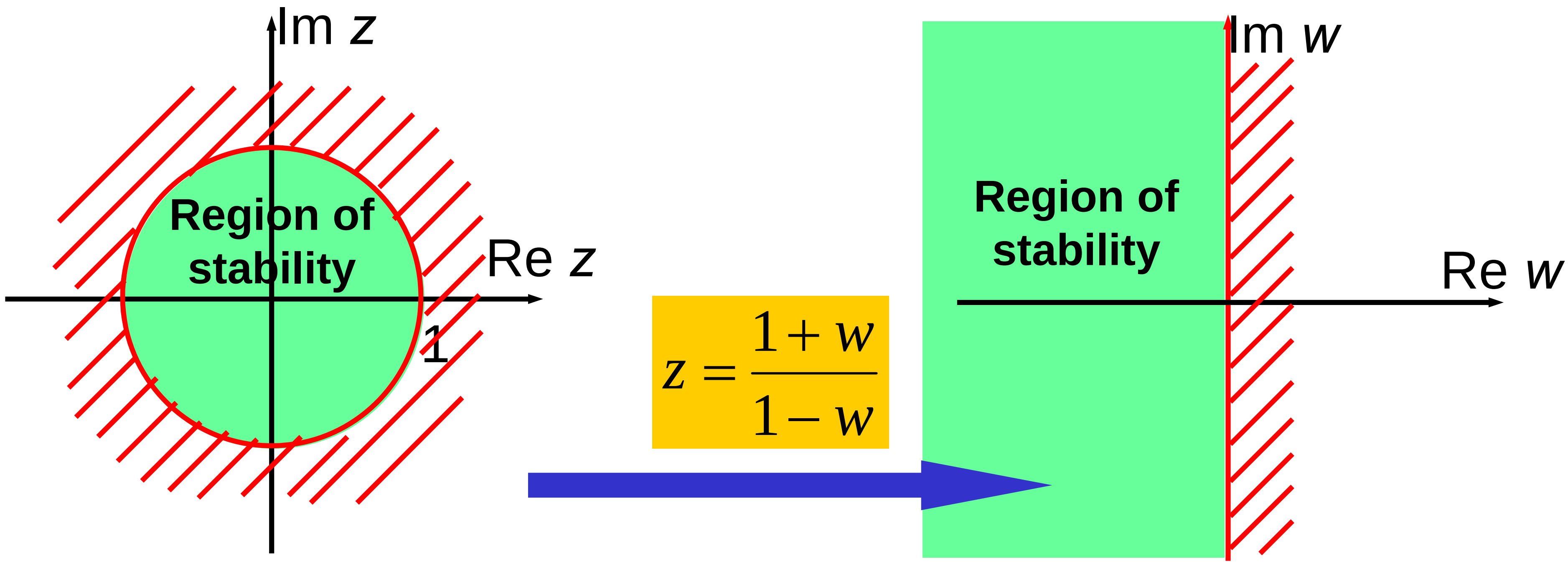
- Algebraic stability criteria
  - The extension of the Routh-Hurwitz criteria
  - Jury's stability criterion
- The root locus method

The extension of the Routh-Hurwitz criteria

The extension of the Routh-Hurwitz criteria

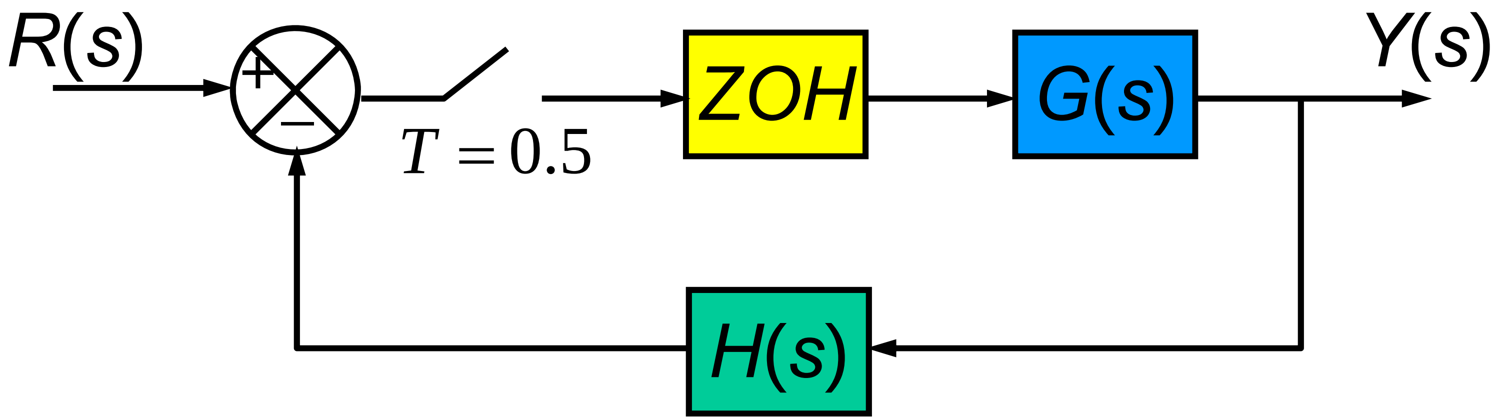
★ Characteristic equation of discrete systems:

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$$



★ The extension of the Routh-Hurwitz criteria: transform z→w, and then apply the Routh – Hurwitz criteria to the characteristic equation of the variable w.

★ Analyze the stability of the following system:



Given that: $G(s) = \dfrac{3e^{-s}}{s+3}$ $\quad H(s) = \dfrac{1}{s+1}$

★ **Solution:**

The characteristic equation of the system:

$$1 + GH(z) = 0$$

- $$GH(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)H(s)}{s}\right\}$$

$$= (1 - z^{-1})\mathcal{Z}\left\{\frac{3e^{-s}}{s(s+3)(s+1)}\right\}$$

$$= 3(1 - z^{-1})z^{-2}\frac{z(Az + B)}{(z-1)(z - e^{-3\times 0.5})(z - e^{-1\times 0.5})}$$

$$A = \frac{(1 - e^{-3\times 0.5}) - 3(1 - e^{-0.5})}{3(1-3)} = 0.0673$$

$$B = \frac{3e^{-3\times 0.5}(1 - e^{-0.5}) - e^{-0.5}(1 - e^{-3\times 0.5})}{3(1-3)} = 0.0346$$

$$\Rightarrow \quad GH(z) = \frac{0.202z + 0.104}{z^2(z - 0.223)(z - 0.607)}$$

The extension of the Routh-Hurwitz criteria – Example (cont')

⇒ The characteristic equation:

$$1 + GH(z) = 0$$

$$\Rightarrow \quad 1 + \frac{0.202z + 0.104}{z^2(z - 0.223)(z - 0.607)} = 0$$

$$\Rightarrow \quad z^4 - 0.83z^3 + 0.135z^2 + 0.202z + 0.104 = 0$$

★ Perform the transformation: $z = \dfrac{1+w}{1-w}$

$$\Rightarrow \left(\frac{1+w}{1-w}\right)^4 - 0.83\left(\frac{1+w}{1-w}\right)^3 + 0.135\left(\frac{1+w}{1-w}\right)^2 + 0.202\left(\frac{1+w}{1-w}\right) + 0.104 = 0$$

$$\Rightarrow 1.867w^4 + 5.648w^3 + 6.354w^2 + 1.52w + 0.611 = 0$$

The extension of the Routh-Hurwitz criteria – Example (cont')

★ The Routh table

w^4	1.867	6.354	0.611
w^3	5.648	1.52	0
w^2	$6.354-\dfrac{1.867}{5.648}\times 1.52=5.852$	0.611	0
w^1	$1.52-\dfrac{5.648}{5.852}\times 0.611=0.93$	0	
w^0	0.611		

★ Conclusion: The system is stable because all the terms in the first column of the Routh table are positive.

Jury stability criterion

Jury stability criterion

- Analyze the stability of the discrete system which has the characteristic equation:

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$

- Jury table: consist of $(2n+1)$ rows.
  - *The first row* consists of the coefficients of the characteristic polynomial in the increasing index order.
  - *The even row* (any) consists of the coefficients of the previous row in the reverse order.
  - The odd row $i = 2k+1$ ($k \geq 1$) consists $(n-k+1)$ terms, the term at the row i column j defined by:

$$c_{ij} = \frac{1}{c_{i-2,1}} \begin{vmatrix} c_{i-2,1} & c_{i-2,n-j-k+3} \\ c_{i-1,1} & c_{i-1,n-j-k+3} \end{vmatrix}$$

Jury stability criterion (cont')

- **Jury criterion statement:** The *necessary and sufficient condition* for the discrete system to be stable is that all the first terms of the odd rows of the Jury table are positive.

Jury stability criterion – Example

- Analyze the stability of the system which has the characteristic equation: $$5z^3 + 2z^2 + 3z + 1 = 0$$

- Solution: Jury table

Row 1	5	2	3	1
Row 2	1	3	2	5
Row 3	$\frac{1}{5}\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} = 4.8$	$\frac{1}{5}\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 1.4$	$\frac{1}{5}\begin{vmatrix} 5 & 2 \\ 1 & 3 \end{vmatrix} = 2.6$	
Row 4	2.6	1.4	4.8	
Row 5	$\frac{1}{4.8}\begin{vmatrix} 4.8 & 2.6 \\ 2.6 & 4.8 \end{vmatrix} = 3.39$	$\frac{1}{4.8}\begin{vmatrix} 4.8 & 1.4 \\ 2.6 & 1.4 \end{vmatrix} = 0.61$		
Row 6	0.61	3.39		
Row 7	$\frac{1}{3.39}\begin{vmatrix} 3.39 & 0.61 \\ 0.61 & 3.39 \end{vmatrix} = 3.28$			

- Since all the first terms of the odd rows are positive, the system is stable.

The root locus of discrete systems

The root locus (RL) method

- RL is a set of all the roots of the characteristic equation of a system when a real parameter changing from $0 \to +\infty$.

- Consider a discrete system which has the characteristic equation:

$$1 + K\frac{N(z)}{D(z)} = 0$$

Denote: $$G_0(z) = K\frac{N(z)}{D(z)}$$

Assume that $G_0(z)$ has n poles and m zeros.

- The rules for construction of the RL of continuous system can be applied to discrete systems, except for the step 8.

Rules for construction of the RL of discrete systems

- ***Rule 1:*** The number of branches of a RL = the order of the characteristic equation = number of poles of $G_0(z) = n$.

- ***Rule 2:***
  - For $K = 0$: the RL begin at the poles of $G_0(z)$.
  - As K goes to $+\infty$: m branches of the RL end at m zeros of $G_0(z)$, the $n-m$ remaining branches goes to ∞ approaching the asymptote defined by the rule 5 and rule 6.

- ***Rule 3:*** The RL is symmetric with respect to the real axis.

- ***Rule 4:*** A point on the real axis belongs to the RL if the total number of poles and zeros of $G_0(z)$ to its right is odd.

Rules for construction of the RL of discrete system (cont')

- ***Rule 5:*** The angles between the asymptotes and the real axis are given by:

$$\alpha = \frac{(2l+1)\pi}{n-m} \qquad (l = 0, \pm 1, \pm 2, \ldots)$$

- ***Rule 6:*** The intersection between the asymptotes and the real axis is a point *A* defined by:

$$OA = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m}$$

(p_i and z_i are poles and zeros of $G_0(z)$)

- ***Rule 7:*** *:* Breakaway / break-in points (or break points for short), if any, are located in the real axis and are satisfied the equation:

$$\frac{dK}{dz} = 0$$

★ ***Rule 8:*** The intersections of the RL with the unit circle can be determined by using the extension of the Routh-Hurwitz criteria or by substituting $z=a+jb$ ($a^2+b^2=1$) into the characteristic equation.

★ ***Rule 9:*** The departure angle of the RL from a pole p_j (of multiplicity 1) is given by:

$$\theta_j = 180^0 + \sum_{i=1}^{m} \arg(p_j - z_i) - \sum_{i=1, i\neq j}^{n} \arg(p_j - p_i)$$

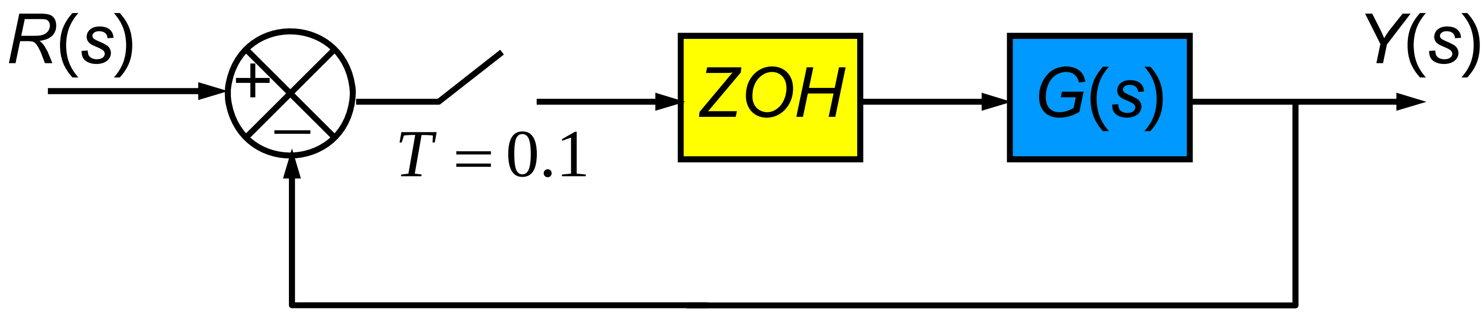
The geometric form of the above formula is

$\theta_j = 180^0 + (\sum$angle from z_i (i=1..m) to p_j)

– ($\sum$angle p_i (i=1..m, $i \neq j$) to p_j)

The root locus of discrete systems – Example

- Consider a discrete system described by a block diagram:



$$G(s) = \frac{5K}{s(s+5)}$$

- Sketch the RL of the system when K=0→+∞. Determine the critical gain K_{cr}

- **Solution:** The characteristic equation of the system:

$$1 + G(z) = 0$$

The root locus of discrete systems – Example (cont')

- $G(z) = (1 - z^{-1})\mathcal{Z}\left\{\dfrac{G(s)}{s}\right\}$

$$= (1 - z^{-1})\mathcal{Z}\left\{\frac{5K}{s^2(s+5)}\right\}$$

$$= K(1 - z^{-1})\left(\frac{z[(0.5 - 1 + e^{-0.5})z + (1 - e^{-0.5} - 0.5e^{-0.5})]}{5(z-1)^2(z - e^{-0.5})}\right)$$

$$\Rightarrow \quad G(z) = K\frac{0.021z + 0.018}{(z-1)(z-0.607)}$$

✶ The characteristic equation : $\boxed{1 + K\dfrac{0.021z + 0.018}{(z-1)(z-0.607)} = 0}$ (*)

✶ Poles: $p_1 = 1 \quad p_2 = 0.607$

✶ Zeros: $z_1 = -0.857$

The root locus of discrete systems – Example (cont')

- The asymptotes:

$$\alpha = \frac{(2l+1)\pi}{n-m} = \frac{(2l+1)\pi}{2-1} \quad \Rightarrow \quad \alpha = \pi$$

$$OA = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} = \frac{[1+0.607]-(-0.857)}{2-1} \Rightarrow OA = 2.464$$

- The breakaway/break-in points:

(*) $\Leftrightarrow \quad K = -\dfrac{(z-1)(z-0.607)}{0.021z+0.018} = -\dfrac{z^2-1.607z+0.607}{0.021z+0.018}$

$$\Rightarrow \quad \frac{dK}{dz} = -\frac{0.021z^2+0.036z-0.042}{(0.021z+0.018)^2}$$

Then $\dfrac{dK}{dz} = 0 \quad \Leftrightarrow \quad \begin{cases} z_1 = -2.506 \\ z_2 = 0.792 \end{cases}$

The root locus of discrete systems – Example (cont')

- The intersection of the root locus with the unit circle:

(*) $\Leftrightarrow \quad (z-1)(z-0.607)+K(0.021z+0.018)=0$

$$\Leftrightarrow \quad z^2+(0.021K-1.607)z+(0.018K+0.607)=0 \qquad (**)$$

Method 1: Apply the extension of Routh – Hurwitz criteria:

Perform the transformation $z=\dfrac{w+1}{w-1}$, (**) becomes:

$$\left(\frac{w+1}{w-1}\right)^2+(0.021K-1.607)\left(\frac{w+1}{w-1}\right)+(0.018K+0.607)=0$$

$$\Leftrightarrow 0.039Kw^2+(0.786-0.036K)w+(3.214-0.003K)=0$$

The root locus of discrete systems – Example (cont')

According to the corollary of the Hurwitz criterion, the stability conditions are:

$$\begin{cases} K > 0 \\ 0.786 - 0.036K > 0 \\ 3.214 - 0.003K > 0 \end{cases} \Leftrightarrow \begin{cases} K > 0 \\ K < 21.83 \\ K < 1071 \end{cases} \Rightarrow K_{cr} = 21.83$$

Substitute $K_{cr} = 21.83$ into (**), we have:

$$z^2 - 1.1485z + 1 = 0 \quad \Rightarrow \quad z = 0.5742 \pm j0.8187$$

Then the intersection of the RL with the unit circle are:

$$z = 0.5742 \pm j0.8187$$

The root locus of discrete systems – Example (cont')

Method 2: Substitute $z = a + jb$ into (**) :

$$(a + jb)^2 + (0.021K - 1.607)(a + jb) + (0.018K + 0.607) = 0$$

$$\Rightarrow a^2 + j2ab - b^2 + (0.021K - 1.607)a + j(0.021K - 1.607)b + (0.018K + 0.607) = 0$$

$$\Rightarrow \begin{cases} a^2 - b^2 + (0.021K - 1.607)a + (0.018K + 0.607) = 0 \\ j2ab + j(0.021K - 1.607)b = 0 \end{cases}$$

The root locus of discrete systems – Example (cont')

- Combine with $a^2 + b^2 = 1$, we have the set of equations:

$$\begin{cases} a^2 - b^2 + (0.021K - 1.607)a + (0.018K + 0.607) = 0 \\ j2ab + j(0.021K - 1.607)b = 0 \\ a^2 + b^2 = 1 \end{cases}$$

- Solve the above set of equation, we obtain 4 intersection:

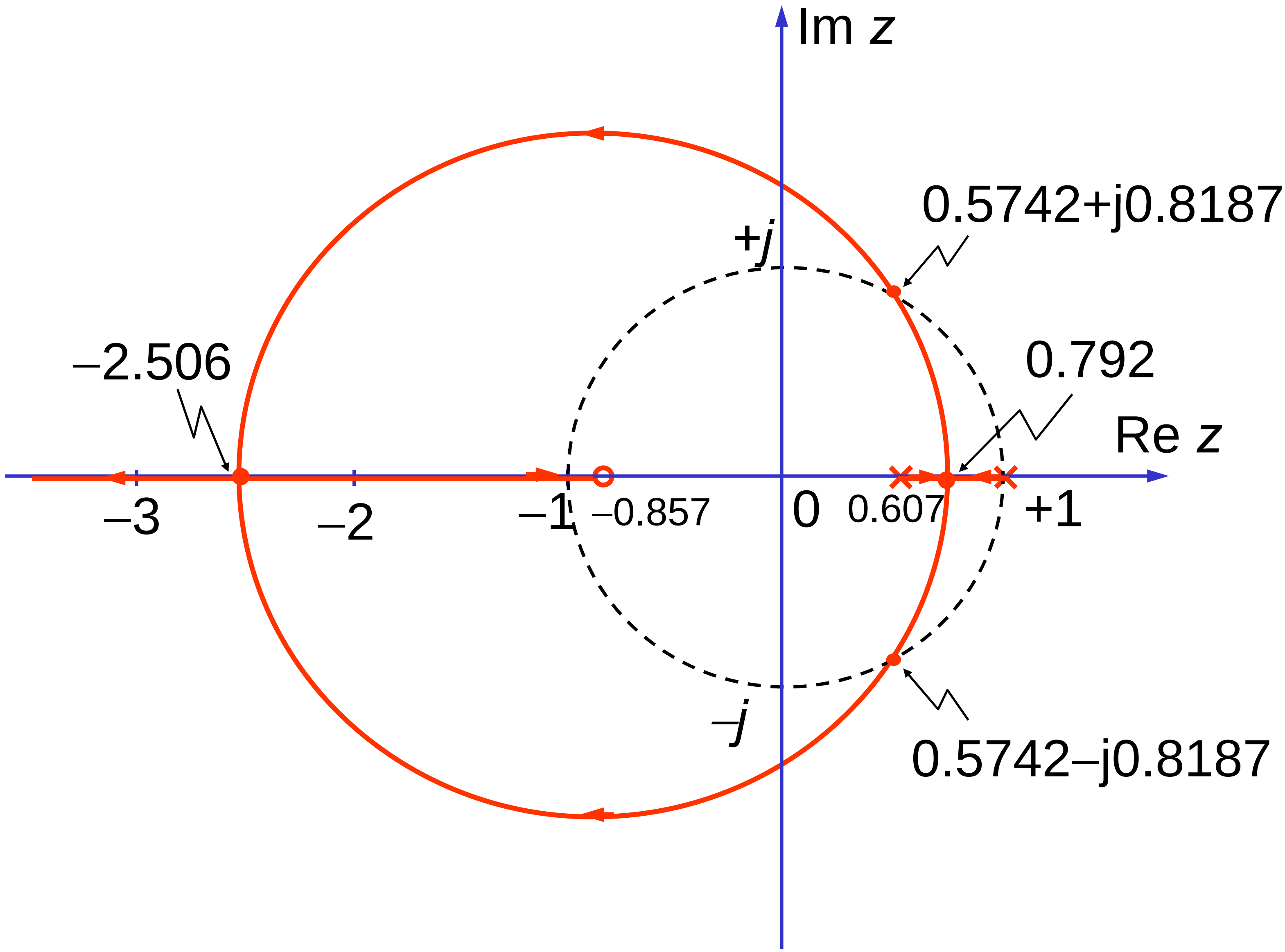
$z = 1$ when $K = 0$

$z = -1$ when $K = 1071$

$z = 0.5742 \pm j0.8187$ when $K = 21.83$

$$\Rightarrow \quad \boxed{K_{cr} = 21.83}$$

The root locus of discrete systems – Example (cont')



Frequency response of discrete systems

Frequency response of discrete systems

- Exact frequency response: substitute $z = e^{j\omega T}$ into the transfer function $G(z)$ $\Rightarrow$ $G(e^{j\omega T})$

- Ex: Transfer function: $G(z) = \dfrac{10}{z(z-0.6)}$

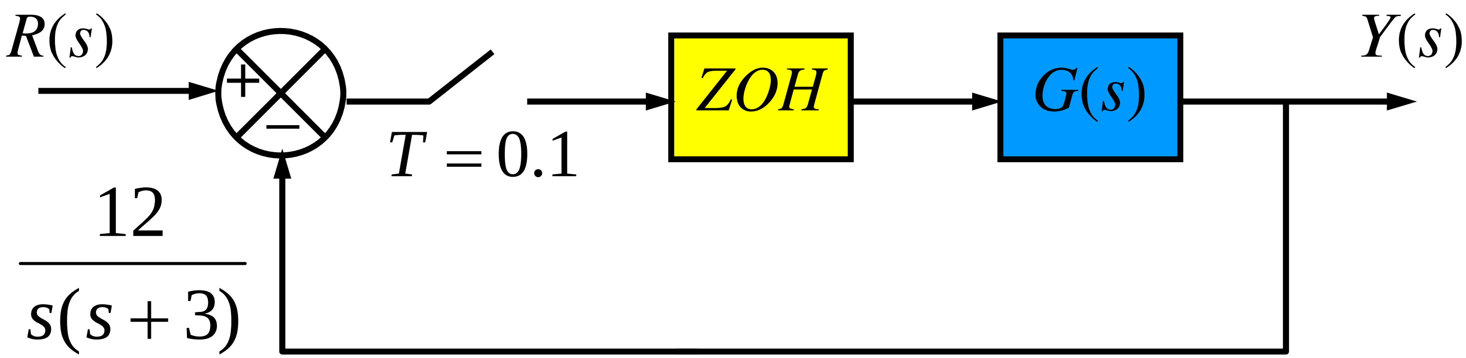
$\Rightarrow$ Frequency response: $G(e^{j\omega T}) = \dfrac{10}{e^{j\omega T}(e^{j\omega T} - 0.6)}$

- Draw the exact Bode diagram of discrete systems:
  - Difficult
  - Cannot apply the addition property of the Bode plot

- Note: Sampling theorem: $f \le \dfrac{f_s}{2} \quad \Rightarrow \quad \omega \le \dfrac{\pi}{T}$

Frequency response of discrete systems – Example

- Consider the discrete system:



$$G(s)=\frac{12}{s(s+3)}$$

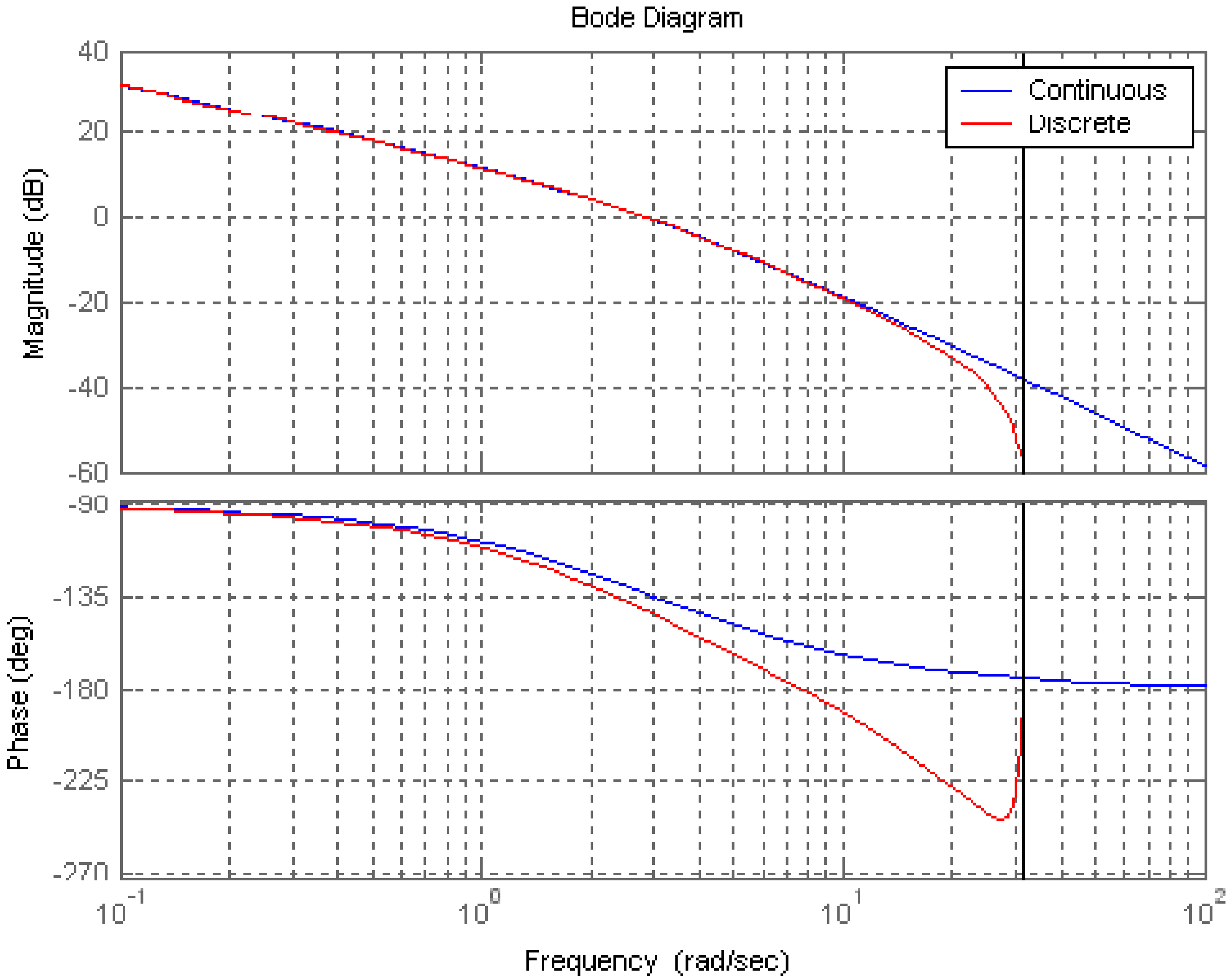
- Plot the frequency response of the open-loop system
- ***Solution:*** The transfer function of the open-loop system:

$$G(z)=(1-z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\} \Rightarrow G(z)=\frac{0.0544z+0.0493}{z^2-1.741z+0.741}$$

- Frequency response: $G(e^{j\omega T})=\dfrac{0.0544e^{j\omega T}+0.0493}{(e^{j\omega T})^2-1.741e^{j\omega T}+0.741}$

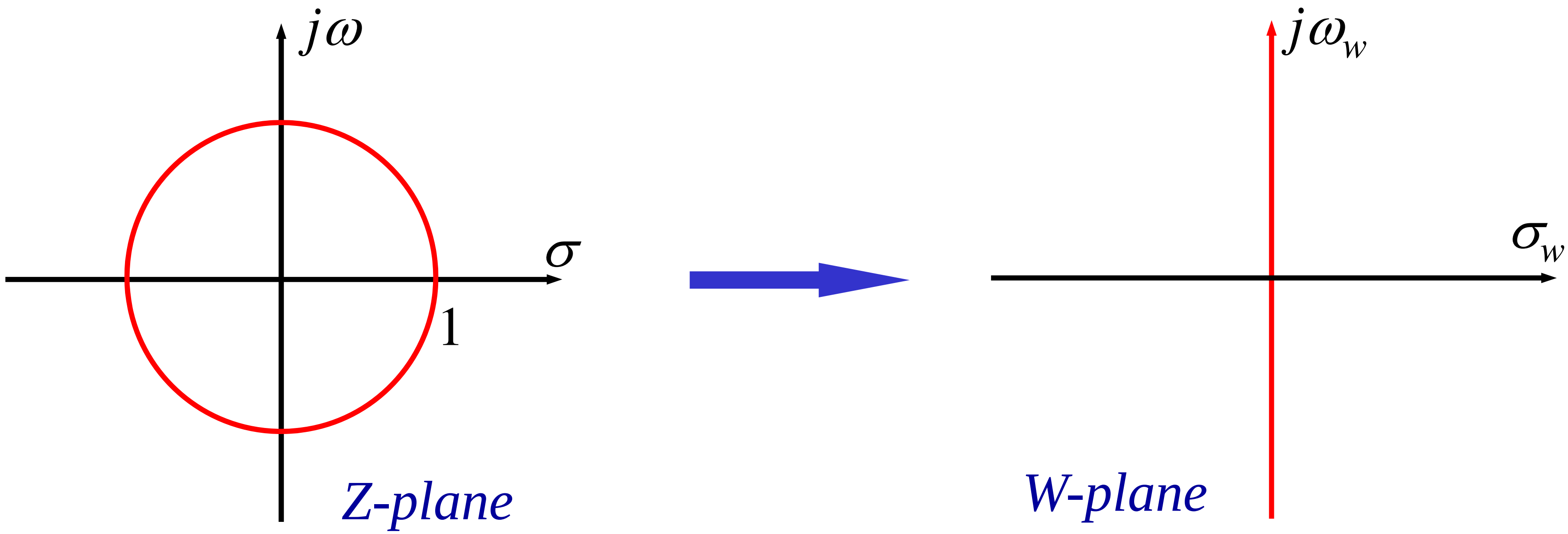
Frequency response of discrete systems – Example (cont')

Exact Bode diagram (Matlab)



The bilinear transformation

★ Bilinear transformation: $z = \dfrac{1 + Tw/2}{1 - Tw/2} \Leftrightarrow w = \dfrac{2}{T}\left[\dfrac{z-1}{z+1}\right]$



★ Frequency response of discrete system after bilinear transformation: $G(z)\big|_{z=e^{j\omega}} \Rightarrow G(w)\big|_{w=j\omega_w}$

Relationship between frequency in w-plane and continuous frequency

- In the imaginary axis of the w-plane: $w = j\omega_w$
- In the unit circle of the z-plane:

$$\frac{2}{T}\left[\frac{z-1}{z+1}\right]\Bigg|_{z=e^{j\omega T}} = \frac{2}{T}\left[\frac{e^{j\omega T}-1}{e^{j\omega T}+1}\right] = j\frac{2}{T}\tan\left(\frac{\omega T}{2}\right)$$

- Due to the bilinear transformation: $w = \dfrac{2}{T}\left[\dfrac{z-1}{z+1}\right]$

$$\Rightarrow \quad \boxed{j\omega_w = j\frac{2}{T}\tan\left(\frac{\omega T}{2}\right)}$$

- At low frequency: $\omega T/2 \approx 0 \;\Rightarrow\; \tan\left(\dfrac{\omega T}{2}\right) \approx \dfrac{\omega T}{2}$, then:

$$\boxed{j\omega_w \approx j\omega}$$

Procedure for drawing approximate Bode diagram

- ***Step 1:*** Perform the bilinear transformation:

$$z = \frac{1 + Tw/2}{1 - Tw/2}$$

- ***Step 2:*** Substitute $w = j\omega_w$, then apply the procedure for drawing the Bode diagram presented in chapter 4.

Stability analysis of discrete systems using frequency response

- When determine the gain crossover frequency, phase crossover frequency, remember the relationship:

$$j\omega_w = j\frac{2}{T}\tan\left(\frac{\omega T}{2}\right)$$

- Gain margin and phase margin are determined in a similar way as continuous systems.

⇒ The stability of discrete systems can be analyzed by using Bode diagrams as continuous systems.

Performance of discrete systems

Time response of discrete systems

- Time response of a discrete system can be calculated by using one of the two methods below:
  - *Method 1:* if the discrete system described by a transfer function, first we calculate $Y(z)$, and then apply the inverse z-transform to find $y(k)$.
  - *Method 2:* if the discrete system described by state equations, first we find the solution $\boldsymbol{x}(k)$ to the state equations, then calculate $y(k)$.

- Dominant poles of a discrete system are the poles lying closest to the unit circle.

Transient performances

Method 1: Analyzing the transient performance based on the time response $y(k)$ of discrete systems.

- Percentage of overshoot: $$POT = \frac{y_{\max} - y_{ss}}{y_{ss}} 100\%$$

y_{max} and y_{ss} are the maximum and steady-state values of $y(k)$

- Settling time: $$t_s = k_s T$$

where k_s satisfying the condition:

$$\left| y(k) - y_{ss} \right| \le \frac{\varepsilon . y_{ss}}{100}, \qquad \forall k \ge k_s$$

$$\Leftrightarrow \quad \left(1 - \frac{\varepsilon}{100}\right) y_{ss} \le y(k) \le \left(1 + \frac{\varepsilon}{100}\right) y_{ss}, \qquad \forall k \ge k_s$$

Transient performances

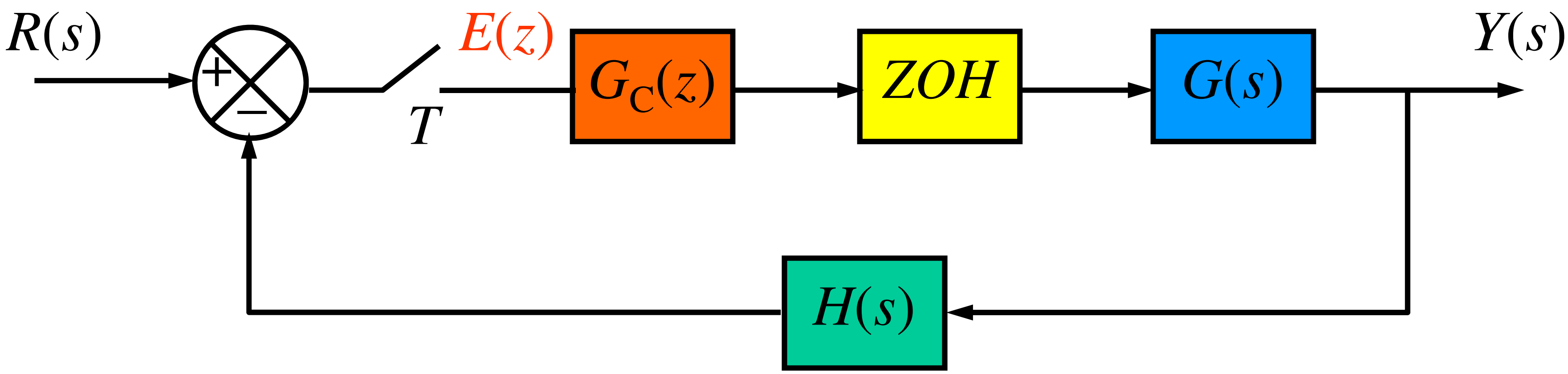
Method 2: Analyzing the transient performances based on the dominant poles.

- The dominant poles: $z_{1,2}^{*} = re^{j\varphi}$ $\Rightarrow$ $$\begin{cases} \xi = \dfrac{-\ln r}{\sqrt{(\ln r)^2 + \varphi^2}} \\ \omega_n = \dfrac{1}{T}\sqrt{(\ln r)^2 + \varphi^2} \end{cases}$$

- Percentage of overshoot: $$POT = \exp\left(-\frac{\xi\pi}{\sqrt{1-\xi^2}}\right) \times 100\%$$

- Settling time: $t_s = \dfrac{3}{\xi\omega_n}$ (according to 5% criterion)

Steady state error



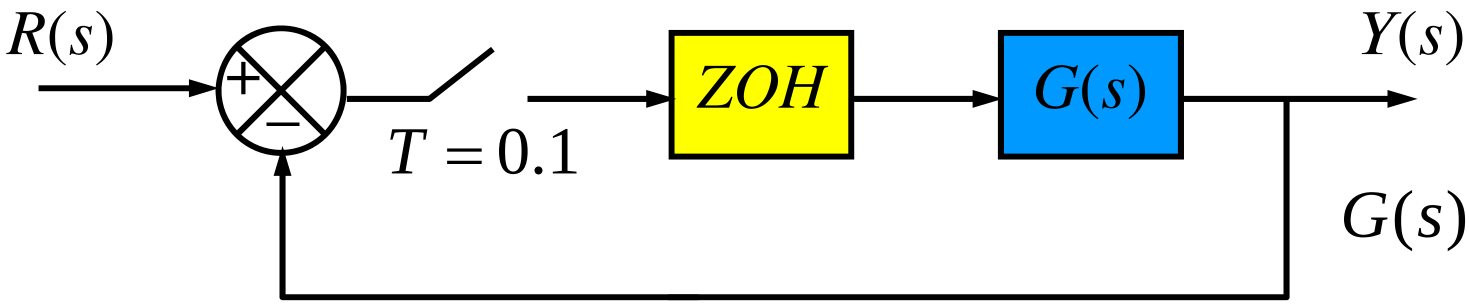
- Error expression:

$$E(z) = \frac{R(z)}{1 + G_C(z)GH(z)}$$

- Steady state error:

$$e_{ss} = \lim_{k\to\infty} e(k) = \lim_{z\to 1}(1 - z^{-1})E(z)$$

Performances of discrete system – Example 1



$$G(s) = \frac{10}{(s+2)(s+3)}$$

1. Calculate the closed-loop transfer function of the system.
2. Calculate the time response of the system to step input.
3. Evaluate the performance of the system: POT, settling time, steady-state error.

★ Solution:

1. The closed-loop TF of the system: $G_{cl}(z) = \dfrac{G(z)}{1+G(z)}$

Performance of discrete system – Example 1 (cont')

- $$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

$$= (1 - z^{-1})\mathcal{Z}\left\{\frac{10}{s(s+2)(s+3)}\right\}$$

$$= 10(1 - z^{-1})\frac{z(Az+B)}{(z-1)(z-e^{-2\times 0.1})(z-e^{-3\times 0.1})}$$

$$\Rightarrow \quad G(z) = \frac{0.042z + 0.036}{(z-0.819)(z-0.741)}$$

Performance of discrete system – Example 1 (cont')

- $$G_{cl}(z) = \frac{G(z)}{1+G(z)}$$

$$= \frac{\dfrac{0.042z + 0.036}{(z-0.819)(z-0.741)}}{1 + \dfrac{0.042z + 0.036}{(z-0.819)(z-0.741)}}$$

$$\Rightarrow \quad \boxed{G_{cl}(z) = \frac{0.042z + 0.036}{z^2 - 1.518z + 0.643}}$$

Performance of discrete system – Example 1 (cont’)

2. The time response of the system to step input.

$$Y(z) = G_k(z)R(z)$$

$$= \frac{0.042z + 0.036}{z^2 - 1.518z + 0.643} R(z)$$

$$= \frac{0.042z^{-1} + 0.036z^{-2}}{1 - 1.518z^{-1} + 0.643z^{-2}} R(z)$$

$$\Rightarrow \quad (1 - 1.518z^{-1} + 0.643z^{-2})Y(z) = (0.042z^{-1} + 0.036z^{-2})R(z)$$

$$\Rightarrow \quad y(k) - 1.518y(k-1) + 0.643y(k-2) = 0.042r(k-1) + 0.036r(k-2)$$

$$\Rightarrow \quad y(k) = 1.518y(k-1) - 0.643y(k-2) + 0.042r(k-1) + 0.036r(k-2)$$

Performance of discrete system – Example 1 (cont')

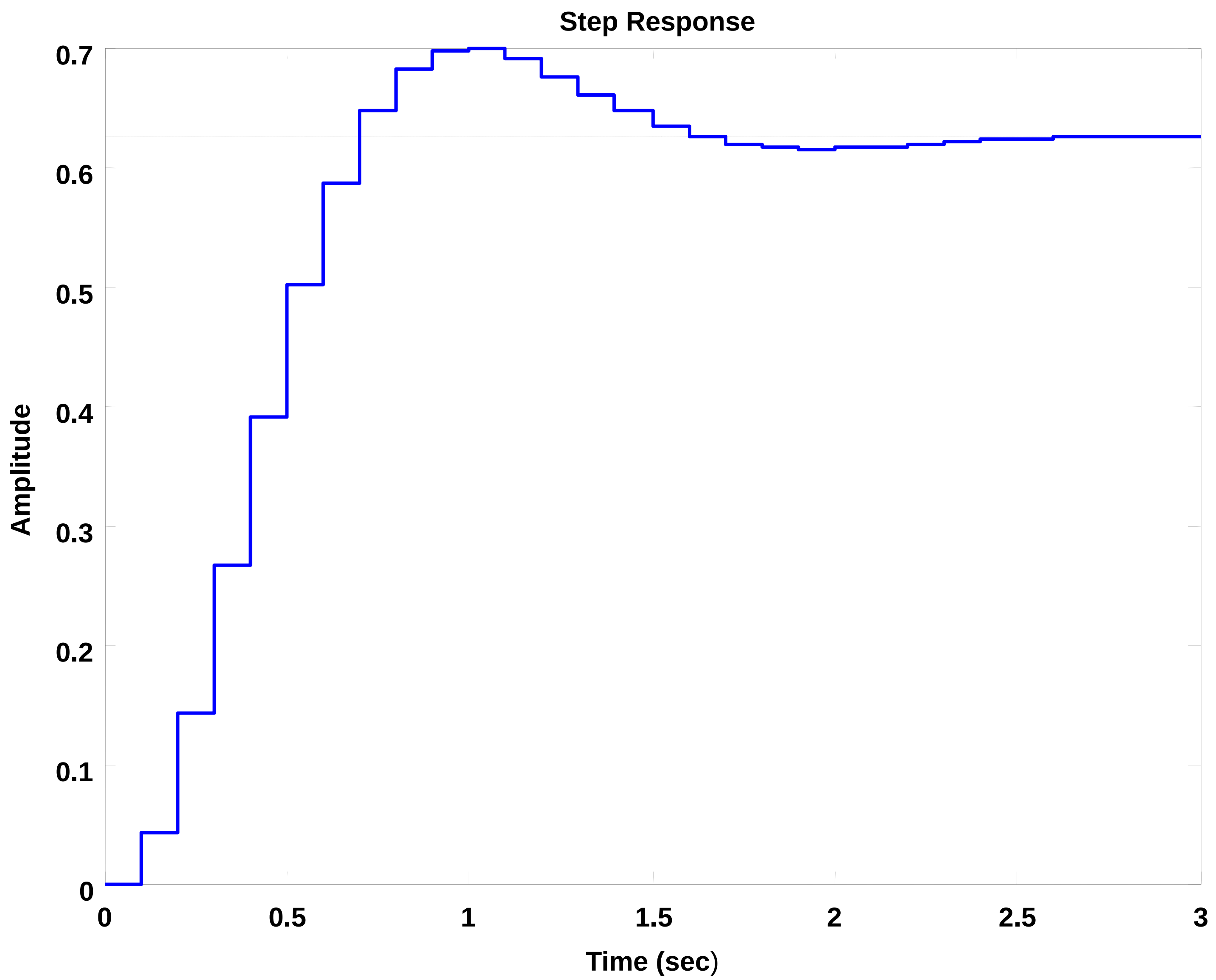
Unit step input: $r(k) = 1, \forall k \geq 0$

Initial condition: $y(-1) = y(-2) = 0$

Substitute the initial condition to the recursive equation of $y(k)$, we have:

$$
\begin{aligned}
y(k) = \{&0; 0.0420; 0.1418; 0.2662; 0.3909; 0.5003;\ldots \\
&0.5860;\ 0.6459;\ 0.6817; 0.6975;\ 0.6985;\ 0.6898;\ldots \\
&0.6760;\ 0.6606;\ 0.6461;\ 0.6341;\ 0.6251;\ 0.6191;\ldots\}
\end{aligned}
$$

Performance of discrete system – Example 1 (cont')



Performance of discrete system – Example 1 (cont')

3. Transient performances:

The steady state response:

$$y_{ss} = \lim_{z\to 1}(1-z^{-1})Y(z)$$

$$= \lim_{z\to 1}(1-z^{-1})G_k(z)R(z)$$

$$= \lim_{z\to 1}(1-z^{-1})\left(\frac{0.042z+0.036}{z^2-1.518z+0.643}\right)\left(\frac{1}{1-z^{-1}}\right)$$

$$\Rightarrow \quad y_{ss} = 0.624$$

The maximum value: $y_{\max} = 0.6985$

★ Percentage of overshoot:

$$POT = \frac{y_{\max}-y_{ss}}{y_{ss}}100\% = \frac{0.6985-0.624}{0.624}100\% = 11.94\%$$

Performance of discrete system – Example 1 (cont')

- Settling time (5% criterion):

  First, we need to find k_s satisfying:

$$(1-\varepsilon)y_{ss} \le y(k) \le (1+\varepsilon)y_{ss}, \forall k \ge k_s$$

$$\Leftrightarrow \quad 0.593 \le y(k) \le 0.655, \forall k \ge k_s$$

  From the time response calculated before $\Rightarrow$ $k_s = 14$

$$t_s = k_s T = 14 \times 0.1$$

$$\Rightarrow \quad t_s = 1.4\sec$$

- Steady state error:

  Since the system is unity negative feedback, we have:

$$e_{ss} = r_{ss} - y_{ss} = 1 - 0.624 \quad \Rightarrow \quad e_{ss} = 0.376$$

Performance of discrete system – Example 1 (cont')

- Note: It is possible to calculate *POT* and t_s based on the dominant poles

The poles of the closed-loop system are the roots of the equation:

$$z^2 - 1.518z + 0.643 = 0$$

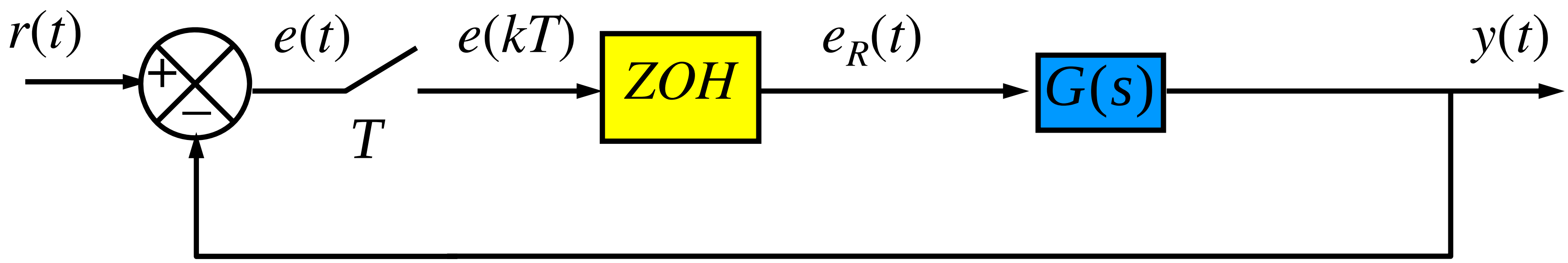
$$\Rightarrow \quad z^*_{1,2} = 0.7590 \pm j0.2587 = 0.8019\angle 0.3285$$

$$\Rightarrow \begin{cases} \xi = \dfrac{-\ln r}{\sqrt{(\ln r)^2 + \varphi^2}} = \dfrac{-\ln 0.8019}{\sqrt{(\ln 0.8019)^2 + 0.3285^2}} = 0.5579 \\[2ex] \omega_n = \dfrac{1}{T}\sqrt{(\ln r)^2 + \varphi^2} = \dfrac{1}{0.1}\sqrt{(\ln 0.8019)^2 + 0.3285^2} = 0.3958 \end{cases}$$

$$POT = \exp\left(-\frac{\xi\pi}{\sqrt{1-\xi^2}}\right).100\% = \exp\left(-\frac{0.5579\times 3.14}{\sqrt{1-0.5579^2}}\right).100\% = 12.11\%$$

$$t_{qđ} = \frac{3}{\xi\omega_n} = \frac{3}{0.5579\times 0.3958} = 1.36\,\text{sec}$$

Performance of discrete system – Example 2



with $T = 0.1$ $$G(s) = \frac{2(s+5)}{(s+2)(s+3)}$$

1. Formulate the state equations describing the system
2. Calculate the response of the system to unit step input (assuming the initial conditions are zeros) using the state equation formulated above.
3. Calculate POT, settling time, steady state error

Performance of discrete system – Example 2 (cont')

★ **Solution:**

1. Formulate the state equation:

$$G(s)=\frac{Y(s)}{E_R(s)}=\frac{2(s+5)}{(s+2)(s+3)}=\frac{2s+10}{s^2+5s+6}$$

★ The state equation of the continuous plant:

$$\Rightarrow \begin{cases} \begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\underbrace{\begin{bmatrix}0 & 1\\ -6 & -5\end{bmatrix}}_{\boldsymbol{A}}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{\boldsymbol{B}}e_R(t) \\ y(t)=\underbrace{\begin{bmatrix}10 & 2\end{bmatrix}}_{\boldsymbol{C}}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} \end{cases}$$

Performance of discrete system – Example 2 (cont')

★ The transient matrix:

$$\Phi(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1} = \left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \right)^{-1} = \left(\begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix} \right)^{-1}$$

$$= \frac{1}{s(s+5)-6}\begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix} = \begin{bmatrix} \dfrac{s+5}{(s+2)(s+3)} & \dfrac{1}{(s+2)(s+3)} \\ \dfrac{-6}{(s+2)(s+3)} & \dfrac{s}{(s+2)(s+3)} \end{bmatrix}$$

$$\Phi(t) = \mathscr{L}^{-1}[\Phi(s)] == \begin{bmatrix} \mathscr{L}^{-1}\left\{ \dfrac{3}{s+2} - \dfrac{2}{s+3} \right\} & \mathscr{L}^{-1}\left\{ \dfrac{1}{s+2} - \dfrac{1}{s+3} \right\} \\ \mathscr{L}^{-1}\left\{ -\dfrac{6}{s+2} + \dfrac{6}{s+3} \right\} & \mathscr{L}^{-1}\left\{ -\dfrac{2}{s+2} + \dfrac{3}{s+3} \right\} \end{bmatrix}$$

$$\Rightarrow \quad \Phi(t) = \begin{bmatrix} (3e^{-2t} - 2e^{-3t}) & (e^{-2t} - e^{-3t}) \\ (-6e^{-2t} + 6e^{-3t}) & (-2e^{-2t} + 3e^{-3t}) \end{bmatrix}$$

Performance of discrete system – Example 2 (cont')

- The state equation of the discrete open-loop system:

$$\begin{cases} \boldsymbol{x}[(k+1)T] = \boldsymbol{A}_d \boldsymbol{x}(kT) + \boldsymbol{B}_d e_R(kT) \\ y(kT) = \boldsymbol{C}_d \boldsymbol{x}(kT) \end{cases}$$

$$\boldsymbol{A}_d = \Phi(T) = \begin{bmatrix} (3e^{-2T} - 2e^{-3T}) & (e^{-2T} - e^{-3T}) \\ (-6e^{-2T} + 6e^{-3T}) & (-2e^{-2T} + 3e^{-3T}) \end{bmatrix}_{T=0.1} = \begin{bmatrix} 0.9746 & 0.0779 \\ -0.4675 & 0.5850 \end{bmatrix}$$

$$\boldsymbol{B}_d = \int_0^T \Phi(\tau)\boldsymbol{B}d\tau = \int_0^T \left\{ \begin{bmatrix} (3e^{-2\tau} - 2e^{-3\tau}) & (e^{-2\tau} - e^{-3\tau}) \\ (-6e^{-2\tau} + 6e^{-3\tau}) & (-2e^{-2\tau} + 3e^{-3\tau}) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau \right\}$$

$$= \int_0^T \left\{ \begin{bmatrix} (e^{-2\tau} - e^{-3\tau}) \\ (-2e^{-2\tau} + 3e^{-3\tau}) \end{bmatrix} d\tau \right\} = \begin{bmatrix} (-\dfrac{e^{-2\tau}}{2} + \dfrac{e^{-3\tau}}{3}) \\ (e^{-2\tau} - e^{-3\tau}) \end{bmatrix}_0^{0.1} = \begin{bmatrix} 0.0042 \\ 0.0779 \end{bmatrix}$$

$$\boldsymbol{C}_d = \boldsymbol{C} = \begin{bmatrix} 10 & 2 \end{bmatrix}$$

Performance of discrete system – Example 2 (cont')

★ The state equation of the discrete closed-loop system:

$$\begin{cases} \boldsymbol{x}[(k+1)T] = [\boldsymbol{A}_d - \boldsymbol{B}_d\boldsymbol{C}_d]\boldsymbol{x}(kT) + \boldsymbol{B}_d r(kT) \\ y(kT) = \boldsymbol{C}_d\boldsymbol{x}(kT) \end{cases}$$

with

$$[\boldsymbol{A}_d - \boldsymbol{B}_d\boldsymbol{C}_d] = \begin{bmatrix} 0.9746 & 0.0779 \\ -0.4675 & 0.5850 \end{bmatrix} - \begin{bmatrix} 0.0042 \\ 0.0779 \end{bmatrix}[10 \quad 2] = \begin{bmatrix} 0.9326 & 0.0695 \\ -1.2465 & 0.4292 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.9326 & 0.0695 \\ -1.2465 & 0.4292 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.0042 \\ 0.0779 \end{bmatrix} r(kT)$$

$$y(k) = [10 \quad 2].\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Performance of discrete system – Example 2 (cont')

2. Time response of the system:

From the closed-loop state equations, we have:

$$\begin{cases} x_1(k+1) = 0.9326x_1(k) + 0.0695x_2(k) + 0.0042r(k) \\ x_2(k+1) = -1.2465x_1(k) + 0.4292x_2(k) + 0.0779r(t) \end{cases}$$

With initial condition $x_1(-1)=x_2(-1)=0$, unit step input, we can calculate the solution to the state equation:

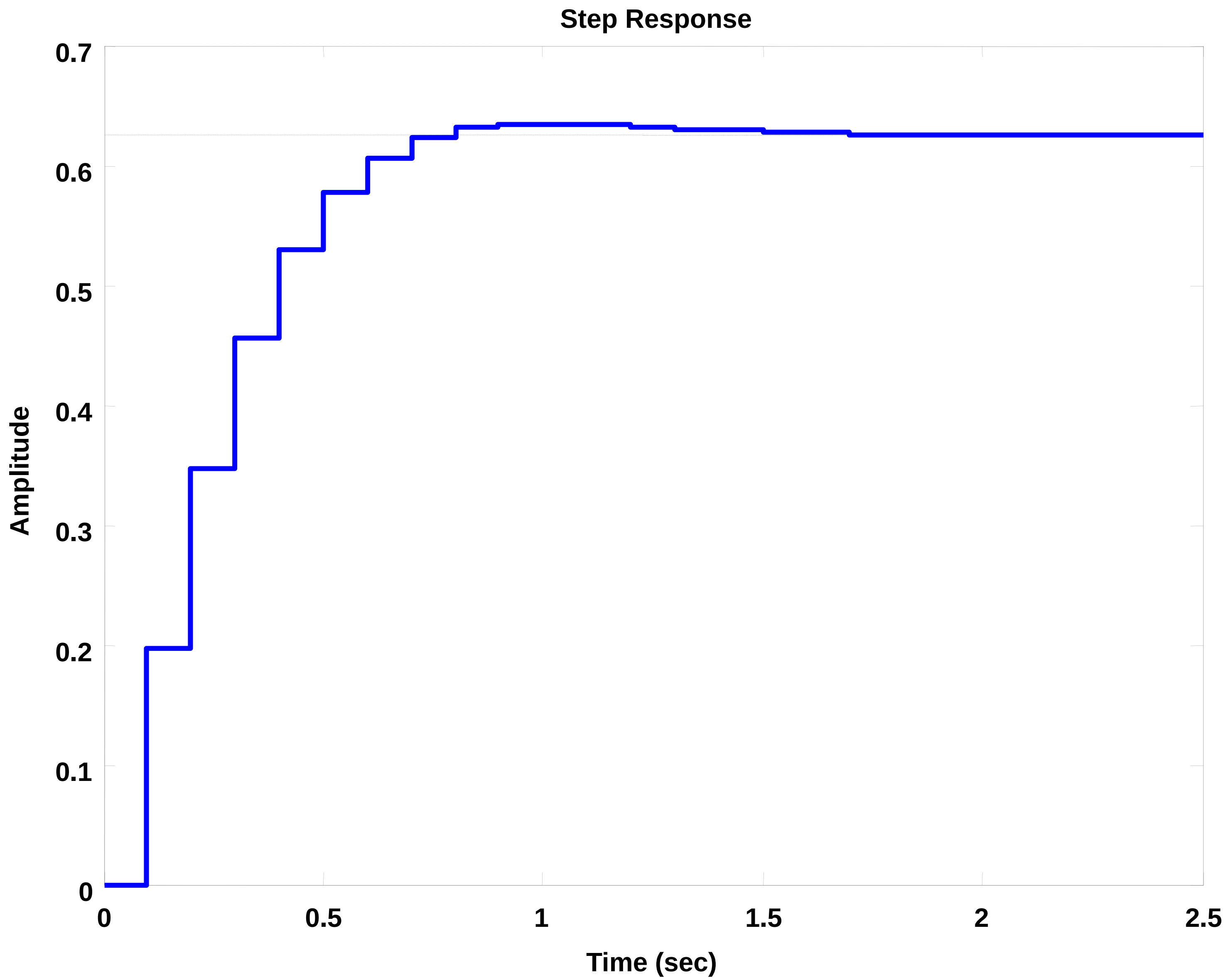
$$x_1(k) = 10^{-3} \times \{0;\, 4.2;\, 13.5;\, 24.2;\, 34.2;\, 42.6;\, 49.1;\, 54.0;\, 57.4;\, 59.7;\ldots$$
$$61.2;\, 62.0;\, 62.5;\, 62.7;\, 62.8;\, 62.8;\, 62.7;\, 62.7;\, 62.6;\, 62.6\ldots\}$$

$$x_2(k) = 10^{-3} \times \{0;\, 77.9;\, 106.1;\, 106.6;\, 93.5;\, 75.4;\, 57.2;\, 41.2;\, 28.3;\, 18.5;\ldots$$
$$11.4;\, 6.5;\, 3.4;\, 1.4;\, 0.3;\, -0.3;\, -0.5;\, -0.5;\, -0.5;\, -0.4\ldots\}$$

The closed-loop system response: $y(k) = 10x_1(k) + 2x_2(k)$

$$y(k) = \{0;\, 0.198;\, 0.348;\, 0.455;\, 0.529;\, 0.577;\, 0.606;\, 0.622;\, 0.631;\, 0.634;\ldots$$
$$0.635;\, 0.634;\, 0.632;\, 0.630;\, 0.629;\, 0.627;\, 0.627;\, 0.626;\, 0.625;\, 0.625\ldots\}$$

Performance of discrete system – Example 2 (cont')



Performance of discrete system – Example 2 (cont')

3. Performances of the system:

✶ Percentage of overshoot:

$$y_{\max} = 0.635 \qquad y_{ss} = 0.625 \quad \Rightarrow \quad POT = \frac{y_{\max} - y_{ss}}{y_{ss}} 100\% = 1.6\%$$

✶ The settling time:

$$(1-0.05)y \;\; \le y(k) \le (1+0.05)y \;\; , \forall k \ge k_s$$

According to the response of the system:

$$0.594 \le y(k) \le 0.656, \qquad \forall k \ge 6$$

$$\Rightarrow \quad k_s = 6 \quad \Rightarrow \quad t_s = k_s T = 0.6\sec$$

✶ Steady state error: $e_{ss} = r_{ss} - y_{ss} = 1 - 0.625 = 0.375$