Lecture Notes

Introduction of Control Systems

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Chapter 3

SYSTEM STABILITY ANALYSIS

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- Stability concept
- Algebraic stability criteria
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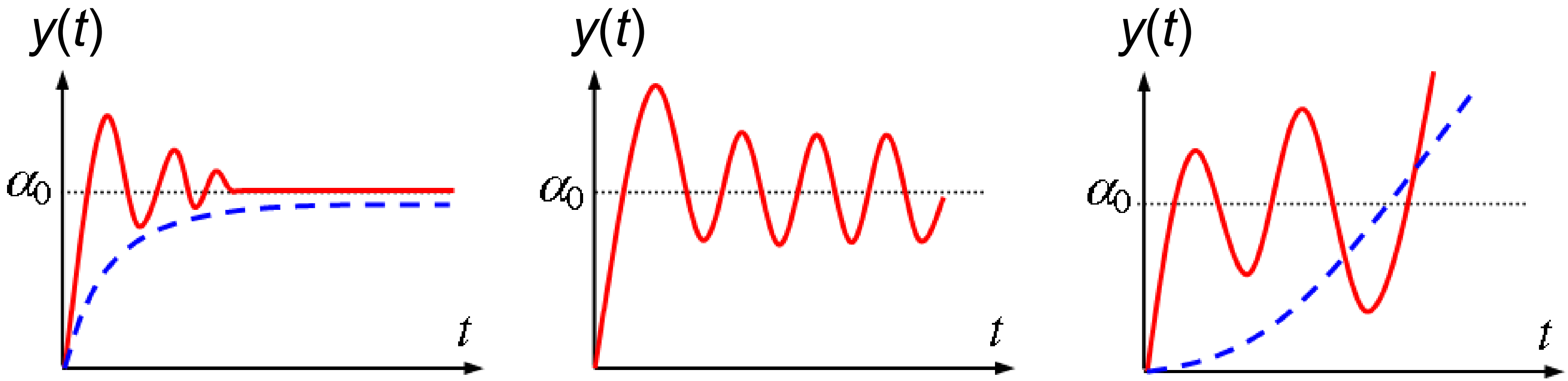
Stability concept

BIBO stability

- A system is defined to be BIBO stable if every bounded input to the system results in a bounded output over the time interval $[t_0, +\infty)$ for all initial times t_0.



Stable system

System at stability boundary

Unstable system

Poles and zeros

- Consider a system described by the transfer function (TF):

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_0s^m+b_1s^{m-1}+\ldots+b_{m-1}s+b_m}{a_0s^n+a_1s^{n-1}+\ldots+a_{n-1}s+a_n}$$

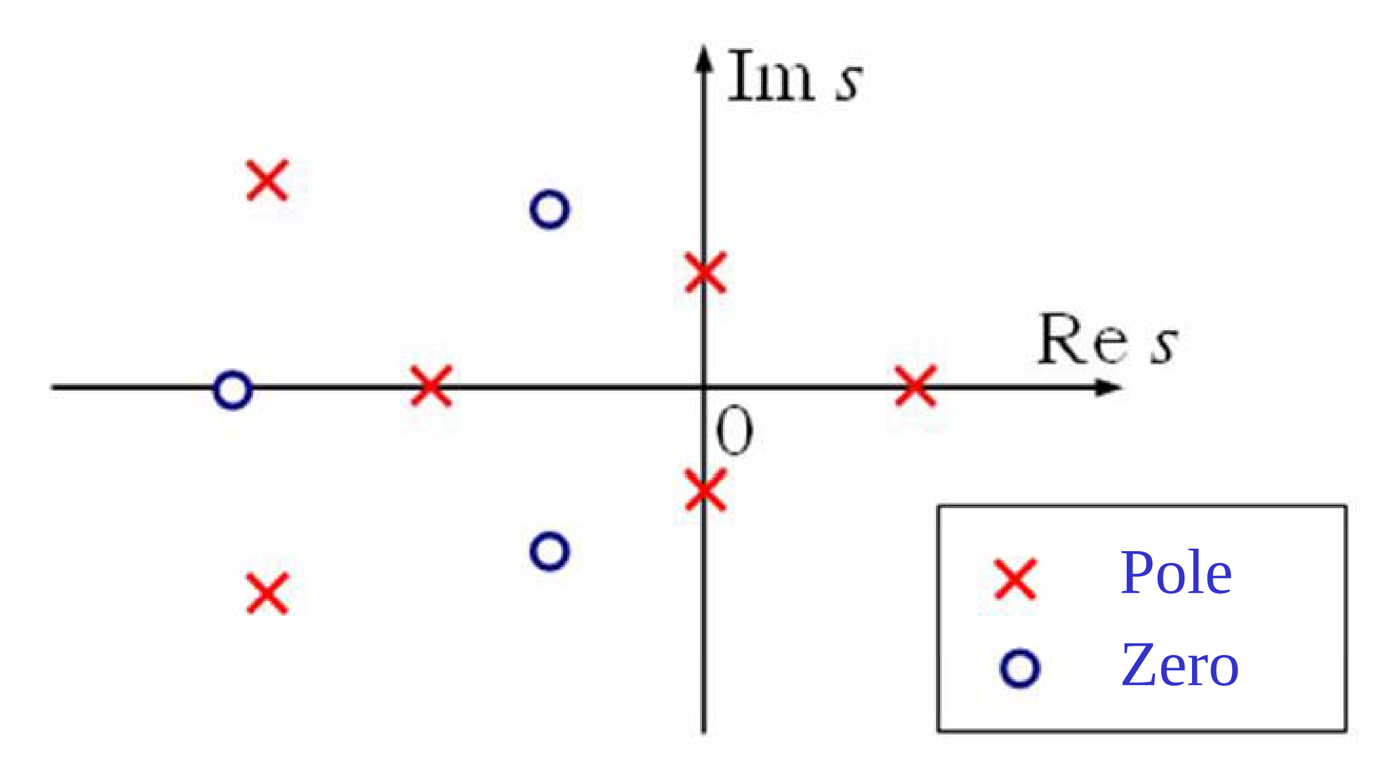
- Denote: $A(s)=a_0s^n+a_1s^{n-1}+\ldots+a_{n-1}s+a_n$ (TF's denominator)

  $B(s)=b_0s^m+b_1s^{m-1}+\ldots+b_{m-1}s+b_m$ (TF' numerator)

- ***Poles***: are the roots of the denominator of the transfer function, i.e. the roots of the equation $A(s) = 0$. Since $A(s)$ is of order n, the system has n poles denoted as p_i , i =1,2,…n.
- ***Zeros***: are the roots of the numerator of the transfer function, i.e. the roots of the equation $B(s) = 0$. Since $B(s)$ is of order m, the system has m zeros denoted as z_i, i =1,2,…m.

Pole – zero plot

- Pole – zero plot is a graph which represents the position of poles and zeros in the complex s-plane.



Stability analysis in the complex plane

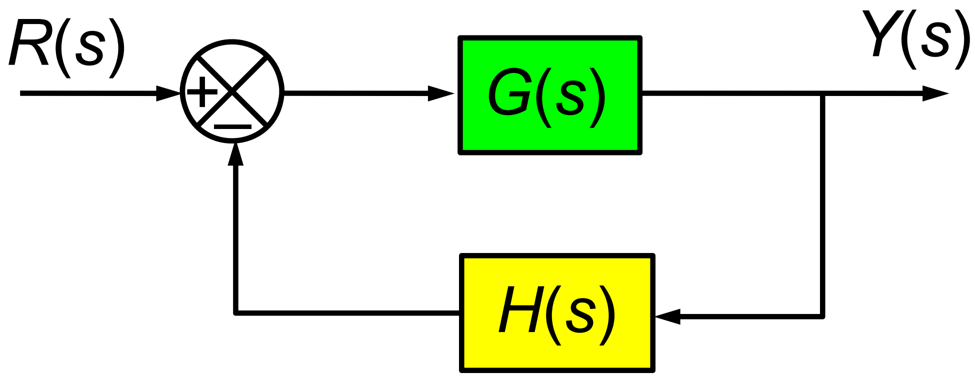
- The stability of a system depends on the location of its poles.
- If all the poles of the system lie in the left-half s-plane then the system is stable.
- If any of the poles of the system lie in the right-half s-plane then the system is unstable.
- If some of the poles of the system lie in the imaginary axis and the others lie in the left-half s-plane then the system is at the stability boundary.

Characteristic equation

- Characteristic equation: is the equation $A(s) = 0$
- Characteristic polynomial: is the denominator $A(s)$

- ***Note:***

Feedback systems



Characteristic equation

$$1+G(s)H(s)=0$$

Systems described by state equations

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

Characteristic equation

$$\det\left(s\boldsymbol{I} - \boldsymbol{A}\right) = 0$$

Algebraic stability criteria

Necessary condition

- The necessary condition for a linear system to be stable is that all the coefficients of the characteristic equation of the system must be positive.
- Example: Consider the systems which have the characteristic equations:

$s^3 + 3s^2 - 2s + 1 = 0$	Unstable
$s^4 + 2s^2 + 5s + 3 = 0$	Unstable
$s^4 + 4s^3 + 5s^2 + 2s + 1 = 0$	Cannot conclude about the stability

Routh's stability criterion

Rules for forming the Routh table

- Consider a linear system whose characteristic function is:

$$a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0$$

- To analyze the system stability using Routh's criterion, it is necessary to form the Routh table according to the rules:
  - The Routh table has n+1 rows.
  - The 1st row consists of the even-indexed coefficients.
  - The 2nd row consists of the odd-indexed coefficients.
  - The element at row i^{th} column j^{th} ($i \geq 3$) is calculated as:

$$c_{ij} = c_{i-2,j+1} - \alpha_i . c_{i-1,j+1}$$

with

$$\alpha_i = \frac{c_{i-2,1}}{c_{i-1,1}}$$

Routh's stability criterion

Routh table

	s^n	$c_{11} = a_0$	$c_{12} = a_2$	$c_{13} = a_4$	$c_{14} = a_6$	...
	s^{n-1}	$c_{21} = a_1$	$c_{22} = a_3$	$c_{23} = a_5$	$c_{24} = a_7$	...
$\alpha_3 = \dfrac{c_{11}}{c_{21}}$	s^{n-2}	$c_{31} = c_{12} - \alpha_3 c_{22}$	$c_{32} = c_{13} - \alpha_3 c_{23}$	$c_{33} = c_{14} - \alpha_3 c_{24}$	$c_{34} = c_{15} - \alpha_3 c_{25}$	...
$\alpha_4 = \dfrac{c_{21}}{c_{31}}$	s^{n-3}	$c_{41} = c_{22} - \alpha_4 c_{32}$	$c_{42} = c_{23} - \alpha_4 c_{33}$	$c_{43} = c_{24} - \alpha_4 c_{34}$	$c_{44} = c_{25} - \alpha_4 c_{35}$	...
...	...	...	...	...	...	...
$\alpha_n = \dfrac{c_{n-2,1}}{c_{n-1,1}}$	s^0	$c_{n1} = c_{n-2,2} - \alpha_n c_{n-1,2}$				

$$c_{ij} = c_{i-2,j+1} - \alpha_i . c_{i-1,j+1}$$

$$\alpha_i = \frac{c_{i-2,1}}{c_{i-1,1}}$$

Routh's stability criterion

Routh's criterion statement

- The *necessary and sufficient condition* for a system to be stable is that all the coefficients of the characteristic equation are positive and all terms in the first column of the Routh table have positive signs.
- The number of sign changes in the first column of the Routh table is equal the number of roots lying in the right-half s-plane.

Routh's stability criterion – Example 1

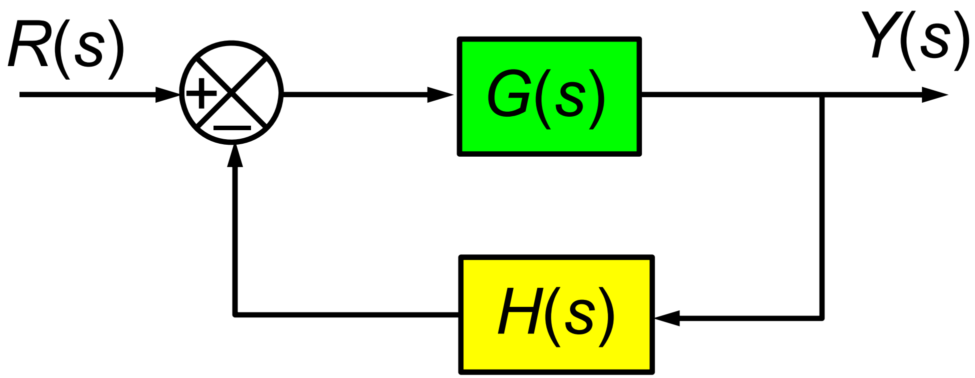
- Analyze the stability of the system which have the following characteristic equation: $s^4 + 4s^3 + 5s^2 + 2s + 1 = 0$
- Solution: Routh table

	s^4	1	5	1
	s^3	4	2	0
$\alpha_3 = \frac{1}{4}$	s^2	$5 - \frac{1}{4}.2 = \frac{9}{2}$	1	
$\alpha_4 = \frac{8}{9}$	s^1	$2 - \frac{8}{9}.1 = \frac{10}{9}$	0	
$\alpha_5 = \frac{81}{20}$	s^0	1		

- Conclusion: The system is stable because all the terms in the first column are positive.

Routh's stability criterion – Example 2

★ Analyze the system described by the following block diagram:



$$G(s) = \frac{50}{s(s+3)(s^2+s+5)}$$

$$H(s) = \frac{1}{s+2}$$

★ Solution: The characteristic equation of the system:

$$1 + G(s).H(s) = 0$$

$$\Leftrightarrow \quad 1 + \frac{50}{s(s+3)(s^2+s+5)} \cdot \frac{1}{(s+2)} = 0$$

$$\Leftrightarrow \quad s(s+3)(s^2+s+5)(s+2) + 50 = 0$$

$$\Leftrightarrow \quad s^5 + 6s^4 + 16s^3 + 31s^2 + 30s + 50 = 0$$

Routh's stability criterion – Example 2 (cont')

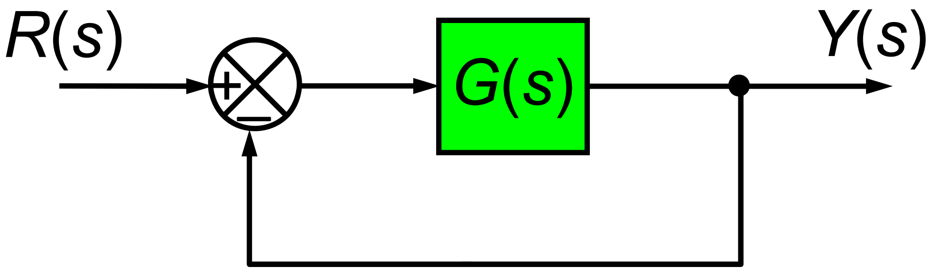
✶ Routh table

	s^5	1	16	30
	s^4	6	31	50
$\alpha_3 = \dfrac{1}{6}$	s^3	$16 - \dfrac{1}{6}.31 = 10.83$	$30 - \dfrac{1}{6}.50 = 21.67$	0
$\alpha_4 = \dfrac{6}{10.83}$	s^2	$31 - \dfrac{6}{10.83} \times 21.67 = 18.99$	50	
$\alpha_5 = \dfrac{10.83}{18.99}$	s^1	$21.67 - \dfrac{10.83}{18.99} \times 50 = -6.84$	0	
	s^0	50		

✶ Conclusion: The system is unstable because the terms in the first column change their signs two times. The characteristic equation has two roots with positive real parts.

Routh's stability criterion – Example 3

★ Find the condition of K for the following system to be stable.



$$G(s) = \frac{K}{s(s^2+s+1)(s+2)}$$

★ Solution: The characteristic equation of the system is:

$$1 + G(s) = 0$$

$$\Leftrightarrow \quad 1 + \frac{K}{s(s^2+s+1)(s+2)} = 0$$

$$\Leftrightarrow \quad s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

Routh's stability criterion – Example 3 (cont')

★ Routh table

	s^4	1	3	K
	s^3	3	2	0
$\alpha_3 = \frac{1}{3}$	s^2	$3 - \frac{1}{3}.2 = \frac{7}{3}$	K	
$\alpha_4 = \frac{9}{7}$	s^1	$2 - \frac{9}{7}.K$	0	
	s^0	K		

★ The necessary & sufficient condition for the system to be stable:

$$\begin{cases} 2 - \dfrac{9}{7}K > 0 \\ K > 0 \end{cases} \quad \Leftrightarrow \quad 0 < K < \frac{14}{9}$$

Routh’s stability criterion – Special case #1

- If a first-column term in any row is zero, but the remaining terms in that row are not zero or there is no remaining term, then the zero term is replaced by a very small positive number ε and the rest rows of the Routh table is calculated as the normal case.

Routh's stability criterion – Example 4

- Analyze the stability of the system whose characteristic equation is:

$$s^4 + 2s^3 + 4s^2 + 8s + 3 = 0$$

- Solution: Routh table

	s^4	1	4	3
	s^3	2	8	0
$\alpha_3 = \frac{1}{2}$	s^2	$4 - \frac{1}{2}.8 = 0$	3	
$\Rightarrow$	s^2	$\varepsilon > 0$	3	
$\alpha_4 = \frac{2}{\varepsilon}$	s^1	$8 - \frac{2}{\varepsilon}.3 < 0$	0	
	s^0	3		

- Conclusion: Because the terms in the first column change their signs two times, the system is unstable and it has two poles lying in the right-half complex plane.

Routh's stability criterion – Special case #2

- If all the coefficients in any row are zero:
  - Forming an auxiliary polynomial with coefficients of the last row above the "all-zero-term row", denote the auxiliary polynomial as $A_0(s)$.
  - Replace the "all-zero-term row" by another row whose elements are the coefficients of the derivative $dA_0(s)/ds$.
  - Then continue to calculate the Routh table as the normal case.

- ***Note:*** The roots of $A_0(s)$ are also the roots of characteristic equation.

Routh's stability criterion – Example 5

- Analyze the stability of the system whose characteristic equation is:

$$s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$$

- Solution: Routh table

	s^5	1	8	7
	s^4	4	8	4
$\alpha_3 = \frac{1}{4}$	s^3	$8 - \frac{1}{4} \times 8 = 6$	$7 - \frac{1}{4} \times 4 = 6$	0
$\alpha_4 = \frac{4}{6}$	s^2	$8 - \frac{4}{6} \times 6 = 4$	4	
$\alpha_5 = \frac{6}{4}$	s^1	$6 - \frac{6}{4} \times 4 = 0$	0	
$\Rightarrow$	s^1	8	0	
$\alpha_6 = \frac{4}{8}$	s^0	$4 - \frac{4}{8} \times 0 = 4$		

Routh's stability criterion – Example 5 (cont')

- The auxiliary polynomial:

$$A_0(s) = 4s^2 + 4 \qquad \Rightarrow \qquad \frac{dA_0(s)}{ds} = 8s + 0$$

- The roots of the auxiliary polynomial (are also the roots the characteristic equation):

$$A_0(s) = 4s^2 + 4 = 0 \quad \Leftrightarrow \quad s = \pm j$$

- Conclusion:
  - All the terms in the first column are positive ⇒ characteristic equation has no root lying in the right-half s-plane.
  - The characteristic equation has two roots lying in the imaginary axis.
  - The number of roots lying in the left-half s-plane is 5 – 2 = 3.

The system is at the stability boundary.

Hurwitz's stability criterion

Rules for forming the Hurwitz matrix

- Given a system whose characteristic equation is:

$$a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0$$

- To analyze the system stability using Hurwitz's criterion, it is necessary to form the Hurwitz matrix according to the rules:
  - The Hurwitz matrix is a square matrix of order $n \times n$.
  - ***The diagonal*** consists of the coefficients a_1 to a_n.
  - ***The odd row*** of the Hurwitz matrix consists of the odd-indexed coefficients of the characteristic polynomial; the indexes increase on the right and decrease on the left of the diagonal.
  - ***The even row*** of the Hurwitz matrix consists of the even-indexed coefficients of the characteristic polynomial; the indexes increase on the right and decrease on the left of the diagonal .

Hurwitz's stability criterion

Hurwitz matrix

$$\begin{bmatrix} a_1 & a_3 & a_5 & a_7 & \dots & 0 \\ a_0 & a_2 & a_4 & a_6 & \dots & 0 \\ 0 & a_1 & a_3 & a_5 & \dots & 0 \\ 0 & a_0 & a_2 & a_4 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & \dots & a_n \end{bmatrix}$$

Hurwitz's criterion statement

- The *necessary and sufficient condition* for the system to be stable is that all the determinants of the principal sub-matrices of the Hurwitz matrix are positive.

Hurwitz's stability criterion – Example 1

Analyze the stability of the system whose characteristic equation is:

$$s^3 + 4s^2 + 3s + 2 = 0$$

✶ Solution:

Hurwitz matrix:
$$\begin{bmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 2 \end{bmatrix}$$

The determinants: $\Delta_1 = a_1 = 4$

$$\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = 4 \times 3 - 1 \times 2 = 10$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{vmatrix} = a_3 \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = 2 \times \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = 2 \times 10 = 20$$

✶ Conclusion: The system is stable because all the determinants are positive.

Hurwitz's stability criterion – Some corollaries

- ***A 2nd order system*** is stable if the coefficients of the characteristic polynomial satisfy the conditions:

$$a_i > 0, \quad i = \overline{0,2}$$

- ***A 3rd order system*** is stable if the coefficients of the characteristic polynomial satisfy the conditions:

$$\begin{cases} a_i > 0, & i = \overline{0,3} \\ a_1 a_2 - a_0 a_3 > 0 \end{cases}$$

- ***A 4th order system*** is stable if the coefficients of the characteristic polynomial satisfy the conditions:

$$\begin{cases} a_i > 0, & i = \overline{0,4} \\ a_1 a_2 - a_0 a_3 > 0 \\ a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4 > 0 \end{cases}$$

The root locus method

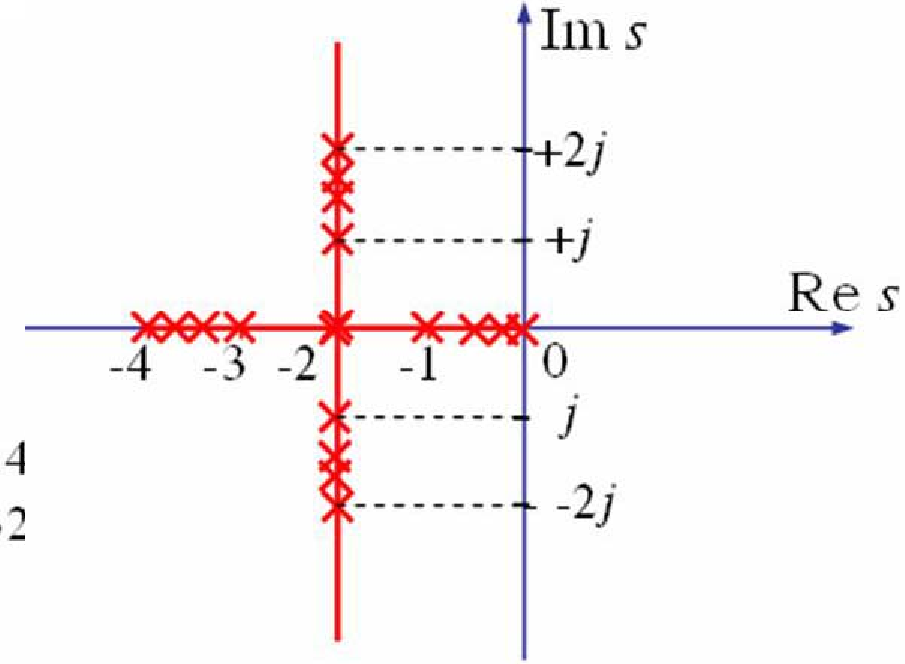
The concept of root locus (RL)

- Example: Plot of all the roots of the following characteristic equation when K changes from $0 \rightarrow +\infty$.

$$s^2 + 4s + K = 0$$

$K = 0$:	$s_1 = 0$,	$s_2 = -4$
$K = 1$:	$s_1 = -0.268$,	$s_2 = -3.732$
$K = 2$:	$s_1 = -0.586$,	$s_2 = -3.414$
$K = 3$:	$s_1 = -1$,	$s_2 = -3$
$K = 4$:	$s_1 = -2$,	$s_2 = -2$
$K = 5$:	$s_1 = -2 + j$,	$s_2 = -2 - j$
$K = 6$:	$s_1 = -2 + j1.414$,	$s_2 = -2 - j1.414$
$K = 7$:	$s_1 = -2 + j1.732$,	$s_2 = -2 - j1.732$
$K = 8$:	$s_1 = -2 + j2$,	$s_2 = -2 - j2$

...



- Definition: Root locus is the set of all the roots of the characteristic equation of a system when a real parameter changing from $0 \rightarrow +\infty$.

Magnitude and phase condition of the root locus

- In order to apply the rules for construction of the root locus, first we have to equivalently transform the characteristic equation to standard form:

$$1 + K\frac{N(s)}{D(s)} = 0 \qquad (1)$$

where *K* is the changing parameter.

Denote: $$G_0(s) = K\frac{N(s)}{D(s)}$$

Assume that $G_0(s)$ has n poles p_i and m zeros z_i.

(1) $\Leftrightarrow$ $$1 + G_0(s) = 0$$

$$\Leftrightarrow \quad \begin{cases} |G_0(s)| = 1 & \text{magnitude condition} \\ \angle G_0(s) = (2l+1)\pi & \text{phase condition} \end{cases}$$

Rules for construction of the root locus

- ***Rule 1:*** The number of branches of a root locus = the order of the characteristic equation = number of poles of $G_0(s) = n$.

- ***Rule 2:***
  - For $K = 0$: the root locus begins at the poles of $G_0(s)$.
  - As K goes to $+\infty$: m branches of the root locus end at m zeros of $G_0(s)$, the $n-m$ remaining branches go to infinity approaching the asymptote defined by the rule 5 & rule 6.

- ***Rule 3:*** The root locus is symmetric with respect to the real axis.

- ***Rule 4:*** A point on the real axis belongs to the root locus if the total number of poles and zeros of $G_0(s)$ to its right is odd.

Rules for construction of the root locus (cont')

- ***Rule 5:*** The angles between the asymptotes and the real axis are calculated by:

$$\alpha = \frac{(2l+1)\pi}{n-m} \qquad (l = 0, \pm 1, \pm 2, \ldots)$$

- ***Rule 6:*** The intersection between the asymptotes and the real axis is a point *A* defined by:

$$OA = \frac{\sum \text{pole} - \sum \text{zero}}{n-m} = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m}$$

(p_i & z_i are poles and zeros of $G_0(s)$)

- ***Rule 7:*** Breakaway / break-in points (or break points for short), if any, are located in the real axis and are satisfied the equation:

$$\frac{dK}{ds} = 0$$

Rules for construction of the root locus (cont')

- ***Rule 8:*** The intersections of the root locus with the imaginary axis can be determined by using the Routh-Hurwitz criteria or by substituting $s=j\omega$ into the characteristic equation.

- ***Rule 9:*** The departure angle of the root locus from a pole p_j (of multiplicity 1) is given by:

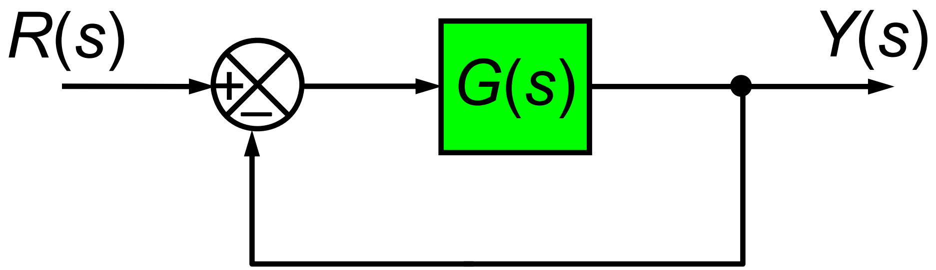
$$\theta_j = 180^0 + \sum_{i=1}^{m} \arg(p_j - z_i) - \sum_{\substack{i=1 \\ i \neq j}}^{n} \arg(p_j - p_i)$$

The geometric form of the above formula is:

θ_j = 180^0 + (Σangle from the zero z_i (i=1..m) to the pole p_j)
– (Σangle from the poles p_i (i=1..m, $i \neq j$) to the pole p_j)

The root locus method – Example 1

- Sketch the root locus of the following system when K=0→+∞.



$$G(s) = \frac{K}{s(s+2)(s+3)}$$

- Solution:
- The characteristic equation of the system:

$$1 + G(s) = 0 \quad \Leftrightarrow \quad 1 + \frac{K}{s(s+2)(s+3)} = 0 \qquad (1)$$

- Poles: $p_1 = 0 \quad p_2 = -2 \quad p_3 = -3$
- Zeros: none

The root locus method – Example 1 (cont')

- The asymptotes:

$$\alpha = \frac{(2l+1)\pi}{n-m} = \frac{(2l+1)\pi}{3-0} \Rightarrow \begin{cases} \alpha_1 = \dfrac{\pi}{3} & (l=0) \\ \alpha_2 = -\dfrac{\pi}{3} & (l=\text{-}1) \\ \alpha_3 = \pi & (l=1) \end{cases}$$

$$OA = \frac{\sum \text{pole} - \sum \text{zero}}{n-m} = \frac{[0+(-2)+(-3)]-0}{3-0} = -\frac{5}{3}$$

- The break points:

(1) $\Leftrightarrow \quad K = -s(s+2)(s+3) = -(s^3+5s^2+6s)$

$$\Rightarrow \quad \frac{dK}{ds} = -(3s^2+10s+6)$$

Then $\dfrac{dK}{ds} = 0 \quad \Leftrightarrow \quad \begin{cases} s_1 = -2.549 & \text{(rejected)} \\ s_2 = -0.785 \end{cases}$

The root locus method – Example 1 (cont')

- The intersections of the root locus with the imaginary axis:

  Method 1: Using the Hurwitz's criterion

  (1) $\Leftrightarrow \quad s^3 + 5s^2 + 6s + K = 0$ (2)

  Stability condition:

$$\begin{cases} K > 0 \\ a_1 a_2 - a_0 a_3 > 0 \end{cases} \Leftrightarrow \begin{cases} K > 0 \\ 5 \times 6 - 1 \times K > 0 \end{cases} \Leftrightarrow 0 < K < 30 \Rightarrow \boxed{K_{cr} = 30}$$

Substitute K_{cr} = 30 into the equation (2) and solve the equation, we have the intersections of the root locus with the imaginary axis.

$$s^3 + 5s^2 + 6s + 30 = 0 \Leftrightarrow \begin{cases} s_1 = -5 \\ s_2 = j\sqrt{6} \\ s_3 = -j\sqrt{6} \end{cases}$$

The root locus method – Example 1 (cont')

- The intersections of the root locus with the imaginary axis:

  Method 2:

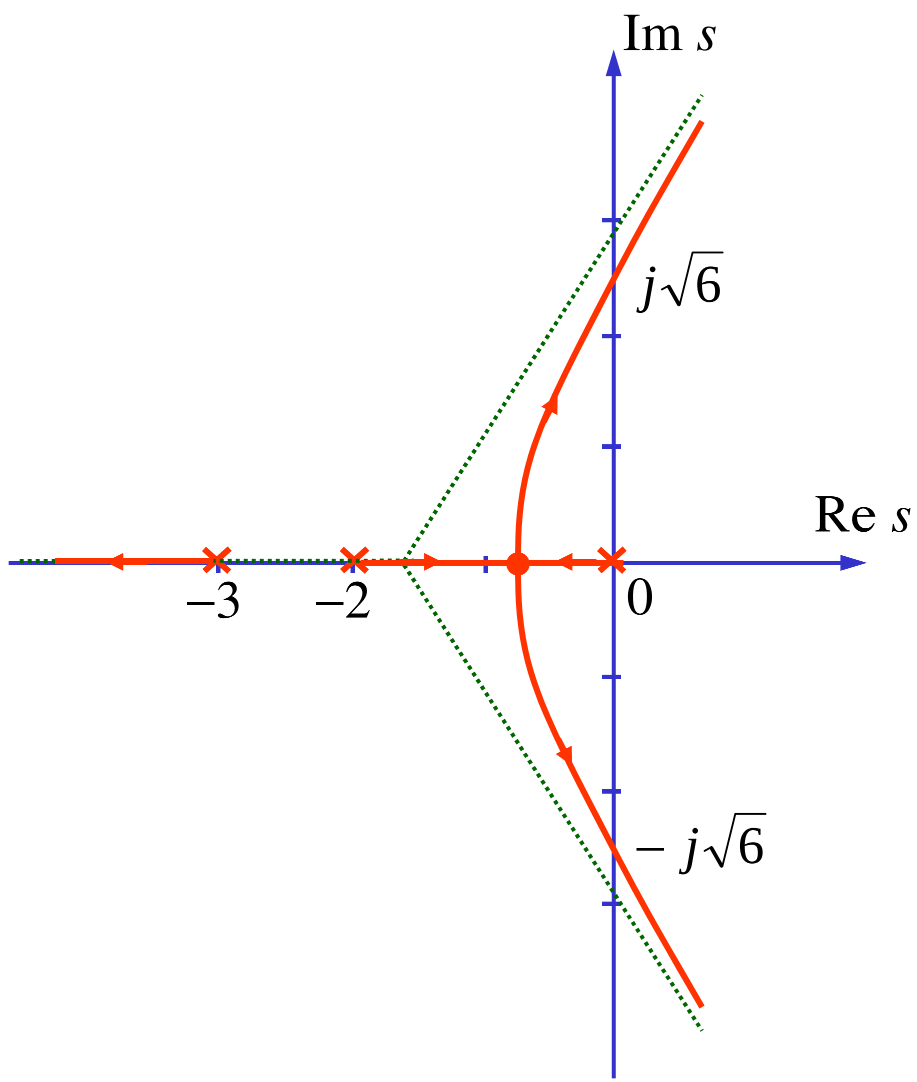
  (1) $\Leftrightarrow \quad s^3 + 5s^2 + 6s + K = 0$ (2)

  Substitute $s=j\omega$ into the equation (2):

$$(j\omega)^3 + 5(j\omega)^2 + 6(j\omega) + K = 0 \quad \Leftrightarrow \quad -j\omega^3 - 5\omega^2 + 6j\omega + K = 0$$

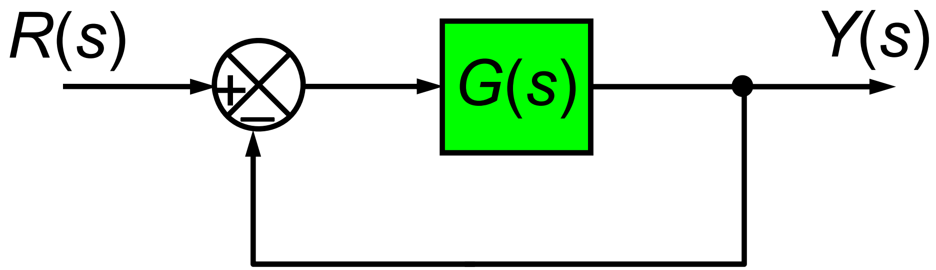
$$\Leftrightarrow \begin{cases} -j\omega^3 + 6j\omega = 0 \\ -5\omega^2 + K = 0 \end{cases} \quad \Leftrightarrow \quad \left[\begin{array}{l} \begin{cases} \omega = 0 \\ K = 0 \end{cases} \\ \begin{cases} \omega = \pm\sqrt{6} \\ K = 30 \end{cases} \end{array} \right.$$

The root locus method – Example 1 (cont’)



The root locus method – Example 2

- Sketch the root locus of the system below when K=0→+∞.



$$G(s) = \frac{K}{s(s^2 + 8s + 20)}$$

- Solution:
- The characteristic equation of the system:

$$1 + G(s) = 0 \quad \Leftrightarrow \quad 1 + \frac{K}{s(s^2 + 8s + 20)} = 0 \qquad (1)$$

- Poles: $p_1 = 0 \qquad p_{2,3} = -4 \pm j2$
- Zeros: none

The root locus method – Example 2 (cont')

- The asymptotes:

$$\alpha = \frac{(2l+1)\pi}{n-m} = \frac{(2l+1)\pi}{3-0} \Rightarrow \begin{cases} \alpha_1 = \dfrac{\pi}{3} & (l=0) \\ \alpha_2 = -\dfrac{\pi}{3} & (l=-1) \\ \alpha_3 = \pi & (l=1) \end{cases}$$

$$OA = \frac{\sum \text{pole} - \sum \text{zero}}{n-m} = \frac{[0+(-4+j2)+(-4-j2)]-(0)}{3-0} = -\frac{8}{3}$$

- The break points:

(1) $\Leftrightarrow \quad K = -(s^3 + 8s^2 + 20s)$

$$\Rightarrow \quad \frac{dK}{ds} = -(3s^2 + 16s + 20)$$

Then $\dfrac{dK}{ds} = 0 \quad \Leftrightarrow \quad \begin{cases} s_1 = -3.33 \\ s_2 = -2.00 \end{cases}$ (2 break points accepted)

The root locus method – Example 2 (cont')

- The intersections of the root locus with the imaginary axis:

(1) $\Leftrightarrow \quad s^3 + 8s^2 + 20s + K = 0$ (2)

Substitute $s = j\omega$ into the equation (2):

$$(j\omega)^3 + 8(j\omega)^2 + 20(j\omega) + K = 0$$

$$\Leftrightarrow \quad -j\omega^3 - 8\omega^2 + 20j\omega + K = 0$$

$$\Leftrightarrow \begin{cases} -8\omega^2 + K = 0 \\ -\omega^3 + 20\omega = 0 \end{cases} \quad \Leftrightarrow \left[\begin{array}{l} \begin{cases} \omega = 0 \\ K = 0 \end{cases} \\ \begin{cases} \omega = \pm\sqrt{20} \\ K = 160 \end{cases} \end{array} \right.$$

The root locus method – Example 2 (cont')

✶ The departure angle of the root locus from the pole p_2

$$\theta_2 = 180^0 - [\arg(p_2 - p_1) + \arg(p_2 - p_3)]$$

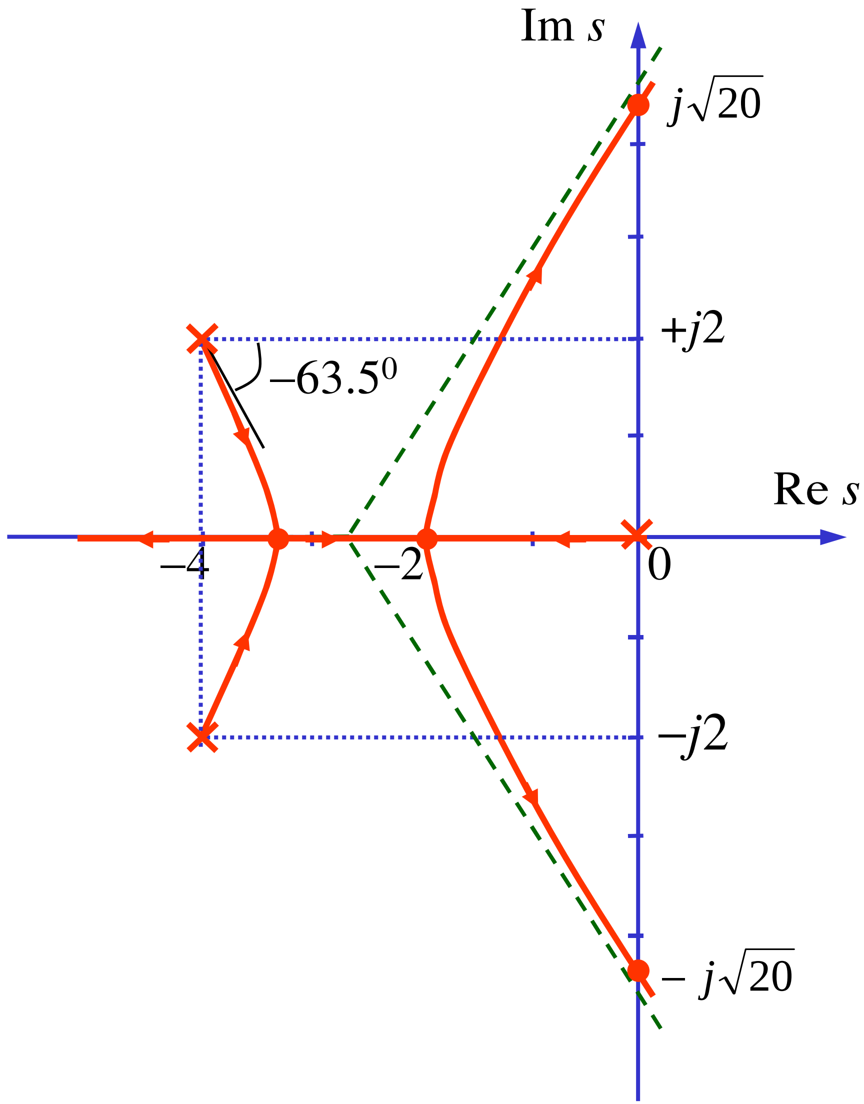
$$= 180^0 - \{\arg[(-4 + j2) - 0] + \arg[(-4 + j2) - (-4 - j2)]\}$$

$$= 180^0 - \left\{ tg^{-1}\left(\frac{2}{-4}\right) + 90 \right\}$$

$$= 180^0 - \{153.5 + 90\}$$

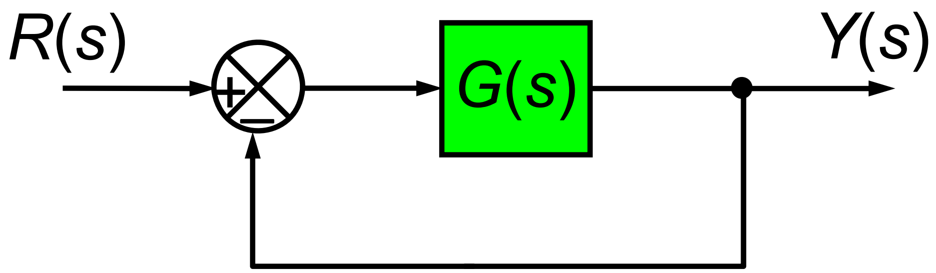
$$\theta_2 = -63.5^0$$

The root locus method – Example 2 (cont')



The root locus method – Example 3

- Sketch the root locus of the system below when K=0→+∞.



$$G(s)=\frac{K(s+1)}{s(s+3)(s^2+8s+20)}$$

- Solution:
- The characteristic equation of the system:

$$1+G(s)=0 \quad \Leftrightarrow \quad 1+\frac{K(s+1)}{s(s+3)(s^2+8s+20)}=0 \qquad (1)$$

- Poles: $p_1=0 \quad p_2=-3 \quad p_{3,4}=-4\pm j2$
- Zeros: $z_1=-1$

The root locus method – Example 3 (cont')

- The asymptotes:

$$\alpha = \frac{(2l+1)\pi}{n-m} = \frac{(2l+1)\pi}{4-1} \Rightarrow \begin{cases} \alpha_1 = \dfrac{\pi}{3} & (l=0) \\ \alpha_2 = -\dfrac{\pi}{3} & (l=\text{-}1) \\ \alpha_3 = \pi & (l=1) \end{cases}$$

$$OA = \frac{\sum \text{pole} - \sum \text{zero}}{n-m} = \frac{[0+(-3)+(-4+j2)+(-4-j2)]-(-1)}{4-1} = -\frac{10}{3}$$

- The break points:

(1) $\Leftrightarrow K = -\dfrac{s(s+3)(s^2+8s+20)}{(s+1)} \Rightarrow \dfrac{dK}{ds} = -\dfrac{3s^4+26s^3+77s^2+88s+60}{(s+1)^2}$

Then $\dfrac{dK}{ds} = 0 \quad \Leftrightarrow \quad \begin{cases} s_{1,2} = -3{,}67 \pm j1{,}05 & \text{(rejected)} \\ s_{3,4} = -0{,}66 \pm j0.97 & \text{(rejected)} \end{cases}$

The root locus method – Example 3 (cont')

✶ The intersections of the root locus with the imaginary axis:

(1) $\Leftrightarrow$ $s^4 + 11s^3 + 44s^2 + (60+K)s + K = 0$ (2)

Substitute *s=jω* into the equation (2):

$$\omega^4 - 11j\omega^3 - 44\omega^2 + (60+K)j\omega + K = 0$$

$$\Leftrightarrow \begin{cases} \omega^4 - 44\omega^2 + K = 0 \\ -11\omega^3 + (60+K)\omega = 0 \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} \omega = 0 \\ K = 0 \end{cases} \\ \begin{cases} \omega = \pm 5{,}893 \\ K = 322 \end{cases} \\ \begin{cases} \omega = \pm j1{,}314 \text{ (rejected)} \\ K = -61{,}7 \end{cases} \end{array} \right.$$

$\Rightarrow$ the intersections are: $s = \pm j5{,}893$ Critical gain: $K_{cr} = 322$

The root locus method – Example 3 (cont')

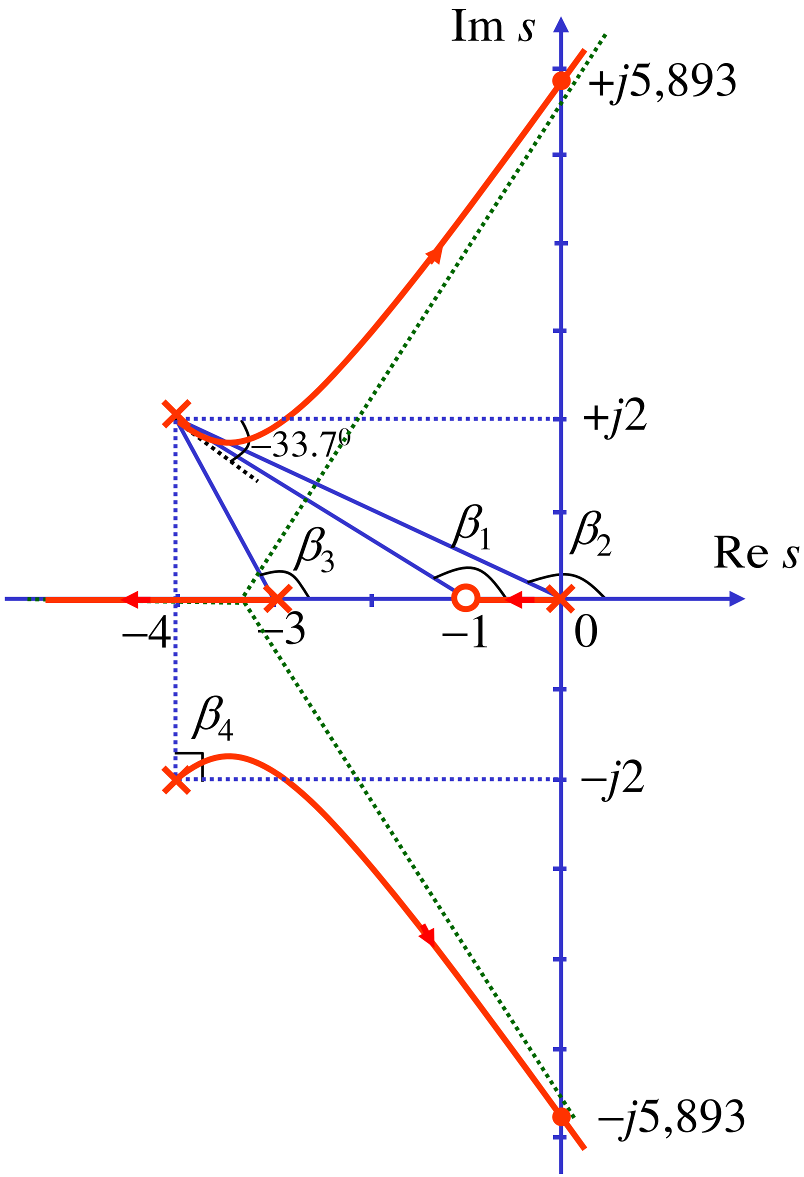
- The departure angle of the root locus from the pole p_3

$$\theta_3 = 180 + \beta_1 - (\beta_2 + \beta_3 + \beta_4)$$

$$= 180 + 146,3 - (153,4 + 116,6 + 90)$$

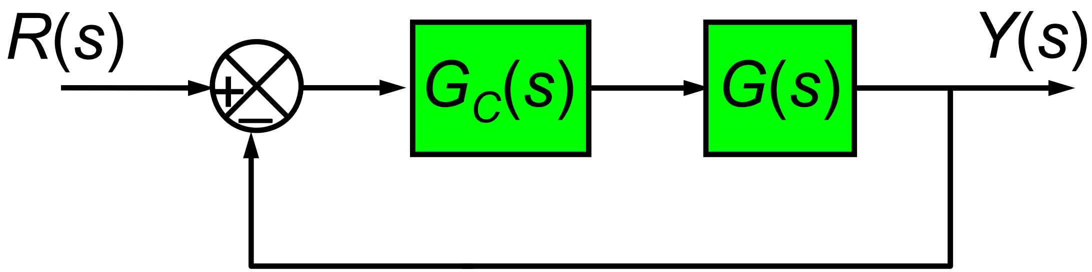
$$\theta_3 = -33.7^0$$

The root locus method – Example 3 (cont')



The root locus method – Example 4

* Given the system below:



$$G(s)=\frac{10}{(s^2+9s+3)}$$

$$G_C(s)=K_P+\frac{K_I}{s}$$

* For $K_I = 2.7$, sketch the root locus of the system when $K_I=0\rightarrow+\infty$, note that $dK_P/ds=0$ has 3 roots at –3, – 3, 1.5.

* For $K_P=270$, $K_I = 2.7$, the system is stable or not?

The root locus method – Example 4 (cont')

- Solution:
- The characteristic equation of the system:

$$1+G_C(s)G(s)=0$$

$$\Leftrightarrow \quad 1+\left(K_P+\frac{2.7}{s}\right)\left(\frac{10}{s^2+9s+3}\right)=0$$

$$\Leftrightarrow \quad 1+\frac{10K_P s}{(s+9)(s^2+3)}=0 \qquad (1)$$

- Poles: $p_1=-9 \qquad p_2=+j\sqrt{3} \qquad p_3=-j\sqrt{3}$
- Zeros: $z_1=0$

The root locus method – Example 4 (cont')

- The asymptotes:

$$\alpha = \frac{(2l+1)\pi}{n-m} = \frac{(2l+1)\pi}{3-1} \Rightarrow \begin{cases} \pi/2 & (l=0) \\ -\pi/2 & (l=-1) \end{cases}$$

$$OA = \frac{\sum \text{pole} - \sum \text{zero}}{n-m} = \frac{[-9+(j\sqrt{3})+(-j\sqrt{3})]-(0)}{3-1} = -\frac{9}{2}$$

- The break points:

$$\frac{dK_P}{ds} = 0 \quad \Leftrightarrow \quad \begin{cases} s_1 = -3 \\ s_2 = -3 \\ s_3 = 1.5 \quad \text{(rejected)} \end{cases}$$

The root locus has two break points at the same location –3

The root locus method – Example 4 (cont')

- The departure angle of the root locus from the pole p_2

$$\theta_2 = 180^0 + \arg(p_2 - z_1) - [\arg(p_2 - p_1) + \arg(p_2 - p_3)]$$

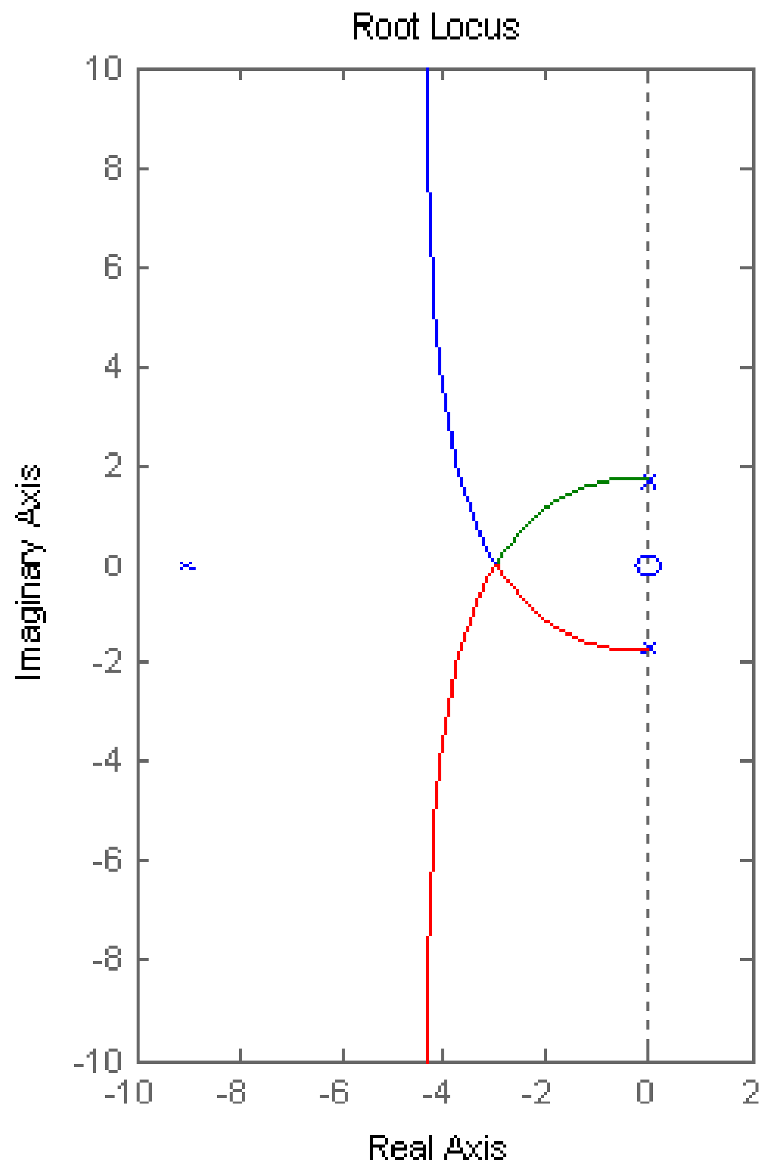
$$= 180^0 + \arg(j\sqrt{3} - 0) - [\arg(j\sqrt{3} - (-9)) + \arg(j\sqrt{3} - (-j\sqrt{3}))]$$

$$= 180^0 + 90 - \left\{ tg^{-1}\left(\frac{\sqrt{3}}{-9}\right) + 90 \right\}$$

$$\theta_2 = -169^0$$

The root locus method – Example 4 (cont')

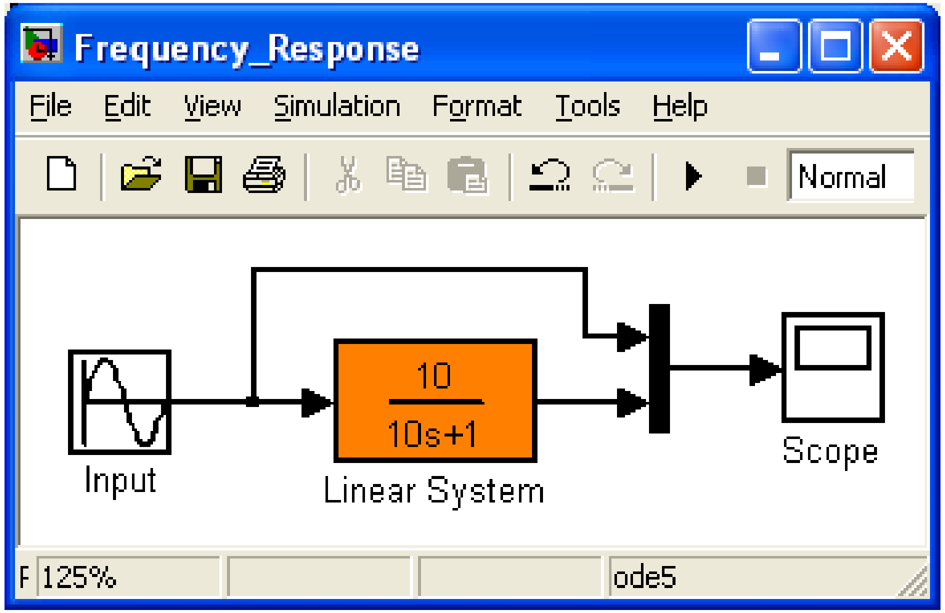
- For K_I =2.7 the root locus is located completely in the left-half s-plane when K_P =0→+∞, so the system is stable when K_I =2.7, K_P =270.

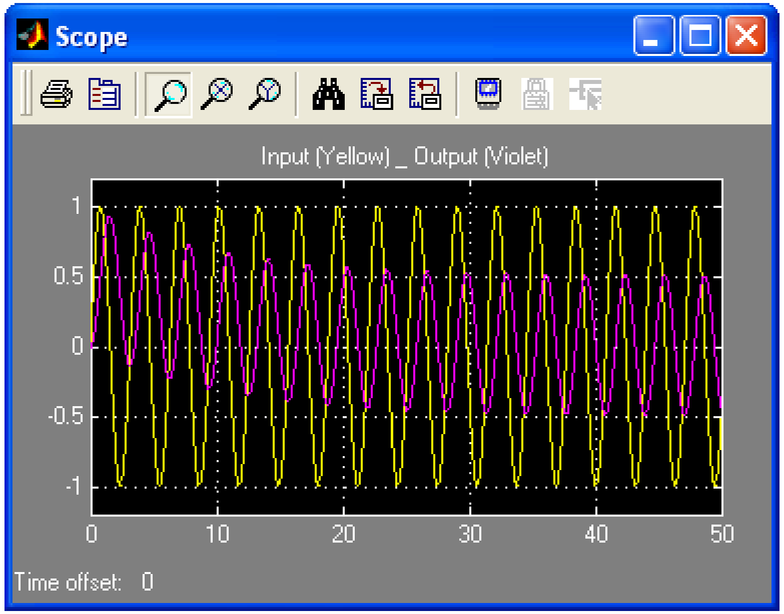


Frequency domain analysis

Frequency response

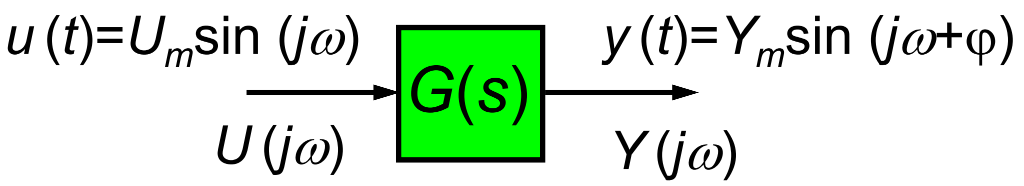
✶ Observe the response of a linear system at steady state when the input is a sinusoidal signal.





Frequency response definition

- It can be observed that, for linear system, if the input is a sinusoidal signal then the output signal at steady-state is also a sinusoidal signal with the same frequency as the input, but different amplitude and phase.



- Definition: Frequency response of a system is the ratio between the steady-state output and the sinusoidal input.

$$\text{Frequency response} = \frac{Y(j\omega)}{U(j\omega)}$$

It is proven that: $\text{Frequency response} = G(s)\big|_{s=j\omega} = G(j\omega)$

Magnitude response and phase response

- In general, $G(j\omega)$ is a complex function and it can be represented in algebraic form or polar form.

$$G(j\omega) = P(\omega) + jQ(\omega) = M(\omega).e^{j\varphi(\omega)}$$

where:

$$M(\omega) = |G(j\omega)| = \sqrt{P^2(\omega) + Q^2(\omega)}$$ Magnitude response

$$\varphi(\omega) = \angle G(j\omega) = tg^{-1}\left[\frac{Q(\omega)}{P(\omega)}\right]$$ Phase response

- Physical meaning of frequency response:
  - The magnitude response provides information about the gain of the system with respect to frequency .
  - The phase response provides information about the phase shift between the output & the input with respect to frequency

Graphical representation of frequency response

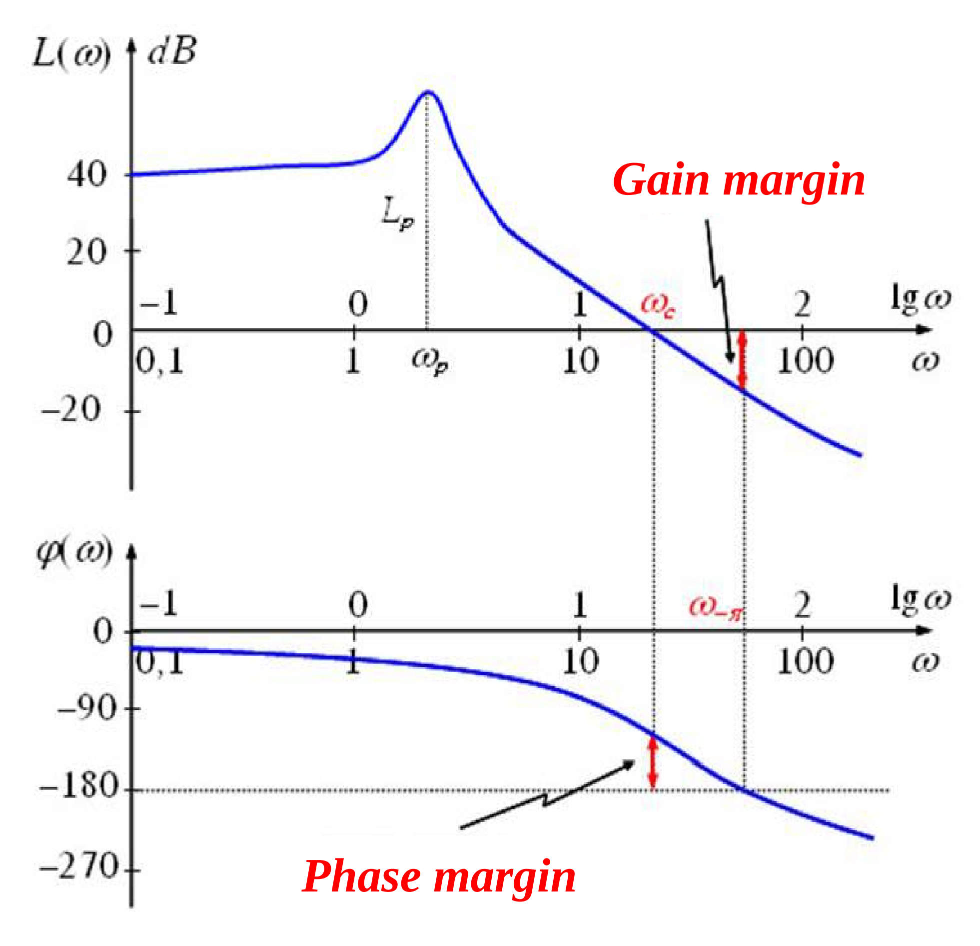
- ***Bode diagram:*** is a graph of the frequency response of a linear system versus frequency plotted with a log-frequency axis. Bode diagram consists of two plots:
  - ***Bode magnitude plot*** expresses the magnitude response gain $L(\omega)$ versus frequency ω .

$$L(\omega) = 20 \lg M(\omega) \quad \text{[dB]}$$

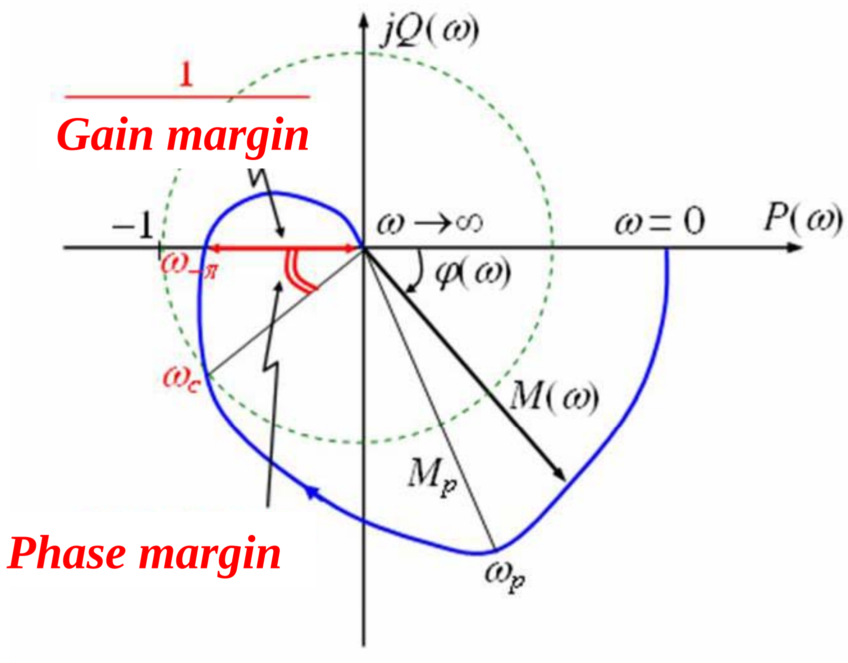
  - ***Bode phase plot*** expresses the phase response $\varphi(\omega)$ versus frequency ω.

- ***Nyquist plot:*** is a graph in polar coordinates in which the gain and phase of a frequency response $G(j\omega)$ are plotted when ω changing from $0 \rightarrow +\infty$.

Bode diagram



Nyquist plot



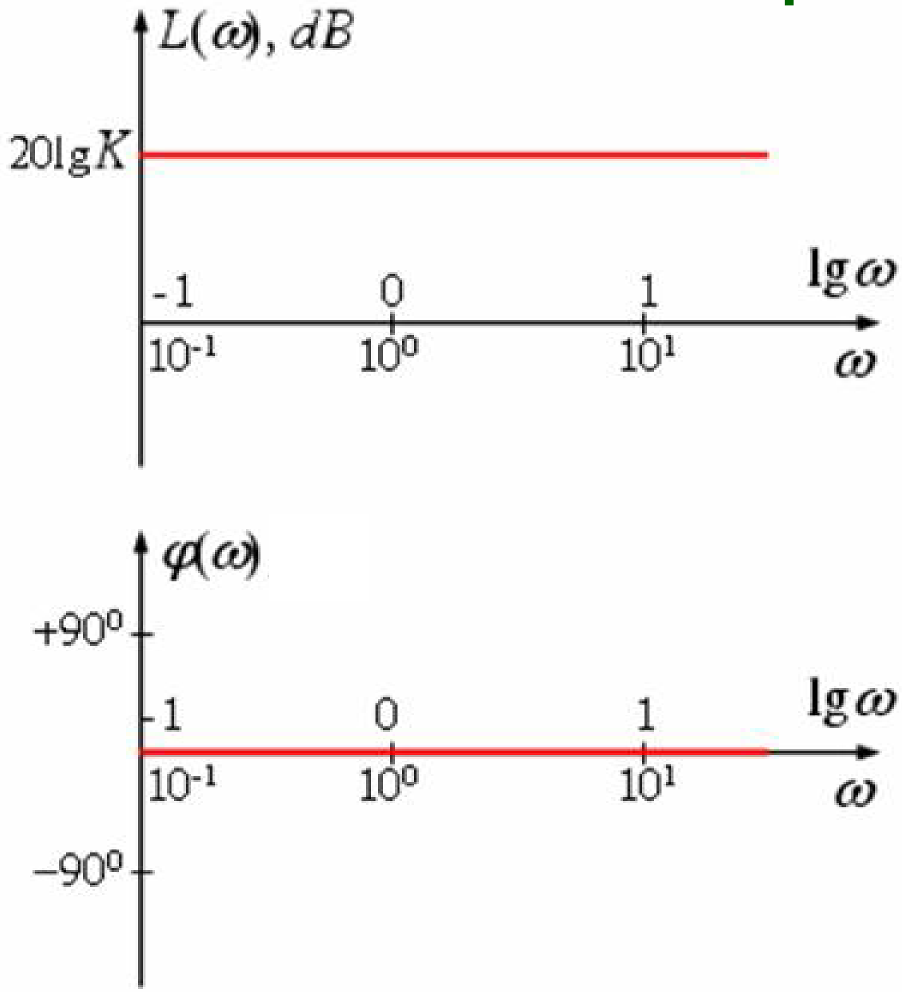
Frequency response of basic factor

Proportional gain

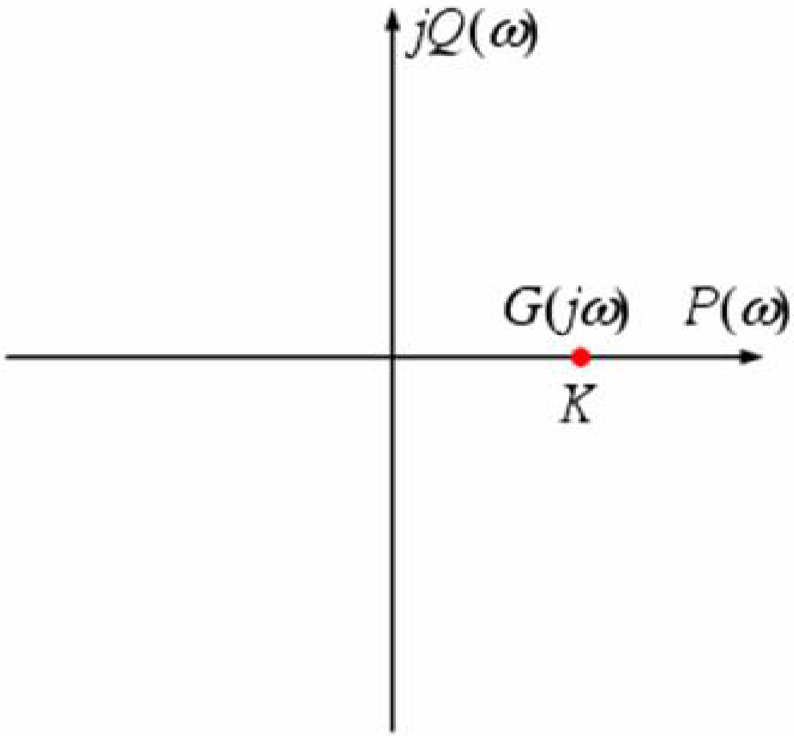
- Transfer function: $G(s) = K$
- Frequency response: $G(j\omega) = K$
  - Magnitude response: $M(\omega) = K \quad \Rightarrow \quad L(\omega) = 20\lg K$
  - Phase response: $\varphi(\omega) = 0$

Frequency response of basic factor (cont.)

Proportional gain



Bode diagram



Nyquist plot

Frequency response of basic factor (cont.)

Integral factor

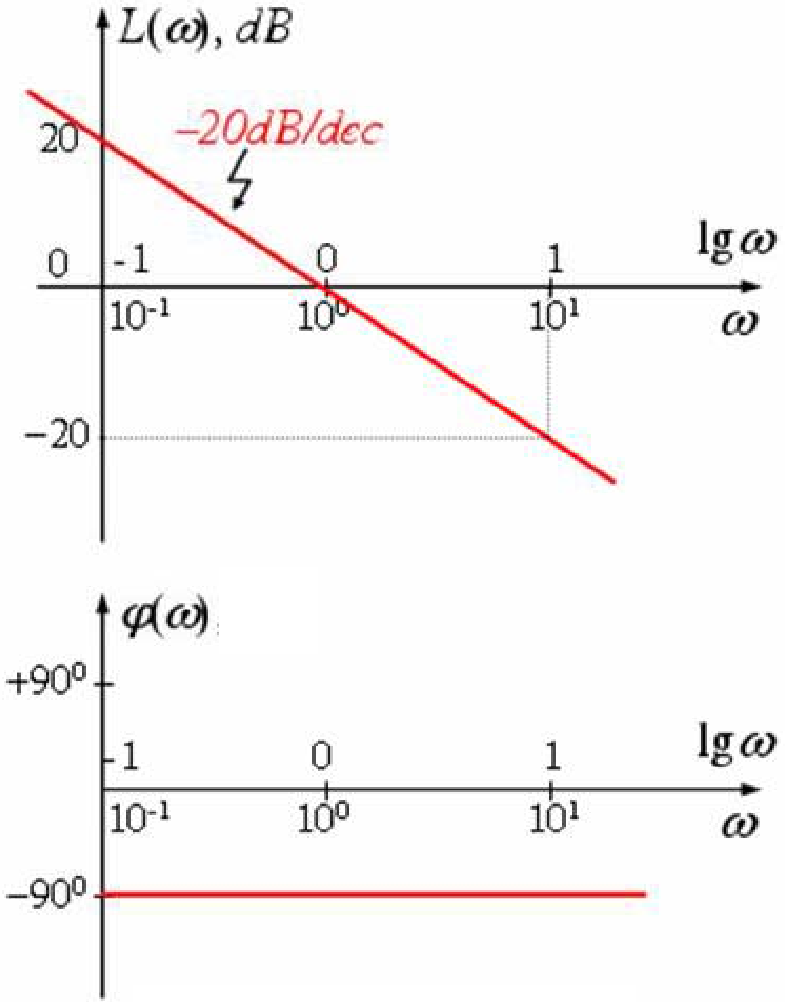
- Transfer function: $G(s) = \dfrac{1}{s}$

- Frequency response: $G(j\omega) = \dfrac{1}{j\omega} = -j\dfrac{1}{\omega}$

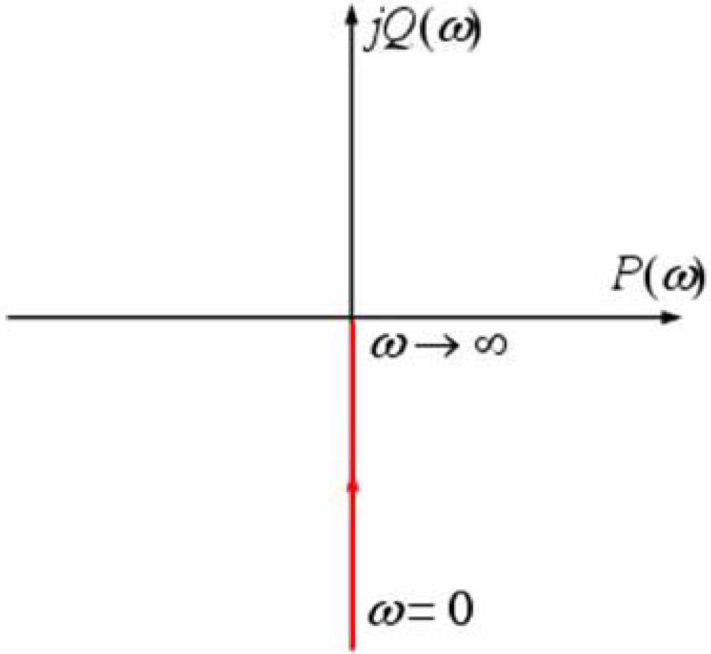
  - Magnitude response: $M(\omega) = \dfrac{1}{\omega} \quad \Rightarrow \quad L(\omega) = -20\lg\omega$

  - Phase response: $\varphi(\omega) = -90^0$

Integral factor



Bode diagram



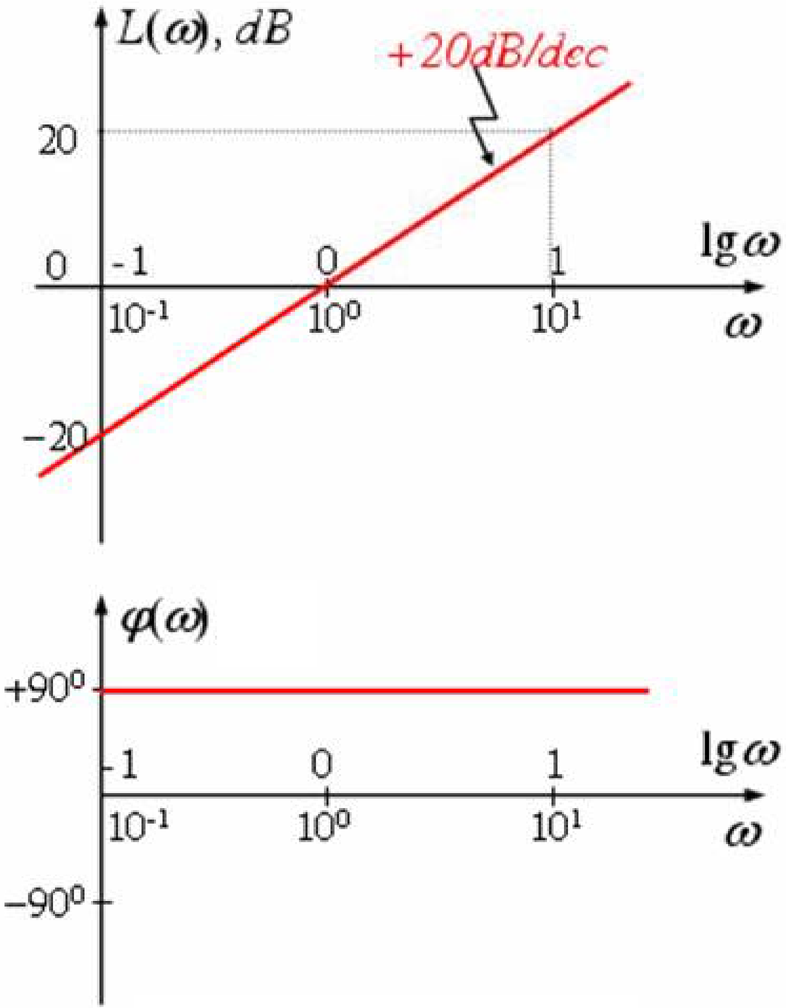
Nyquist plot

Frequency response of basic factor (cont.)

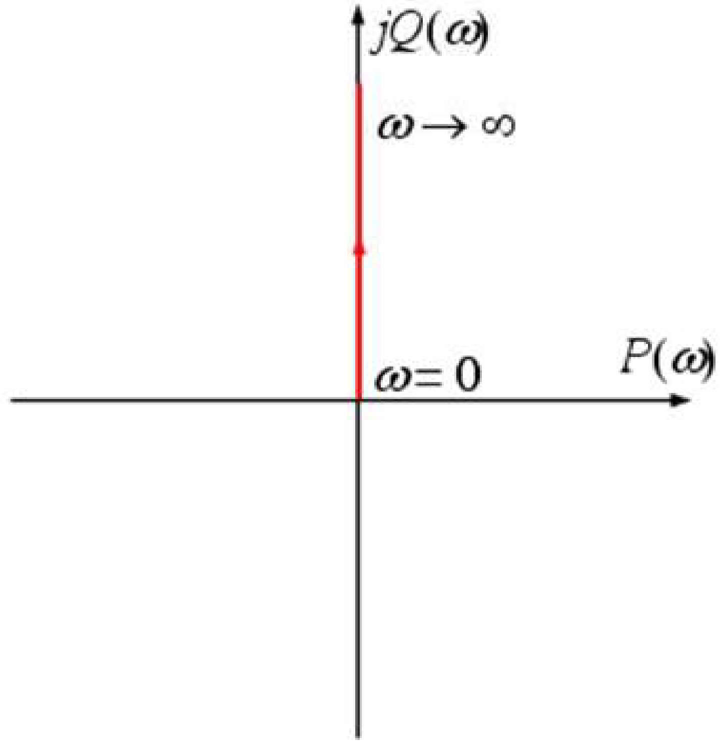
Derivative factor

- Transfer function: $G(s) = s$
- Frequency response: $G(j\omega) = j\omega$
  - Magnitude response: $M(\omega) = \omega \quad \Rightarrow \quad L(\omega) = 20\lg\omega$
  - Phase response: $\varphi(\omega) = 90^0$

Derivative factor



Bode diagram



Nyquist plot

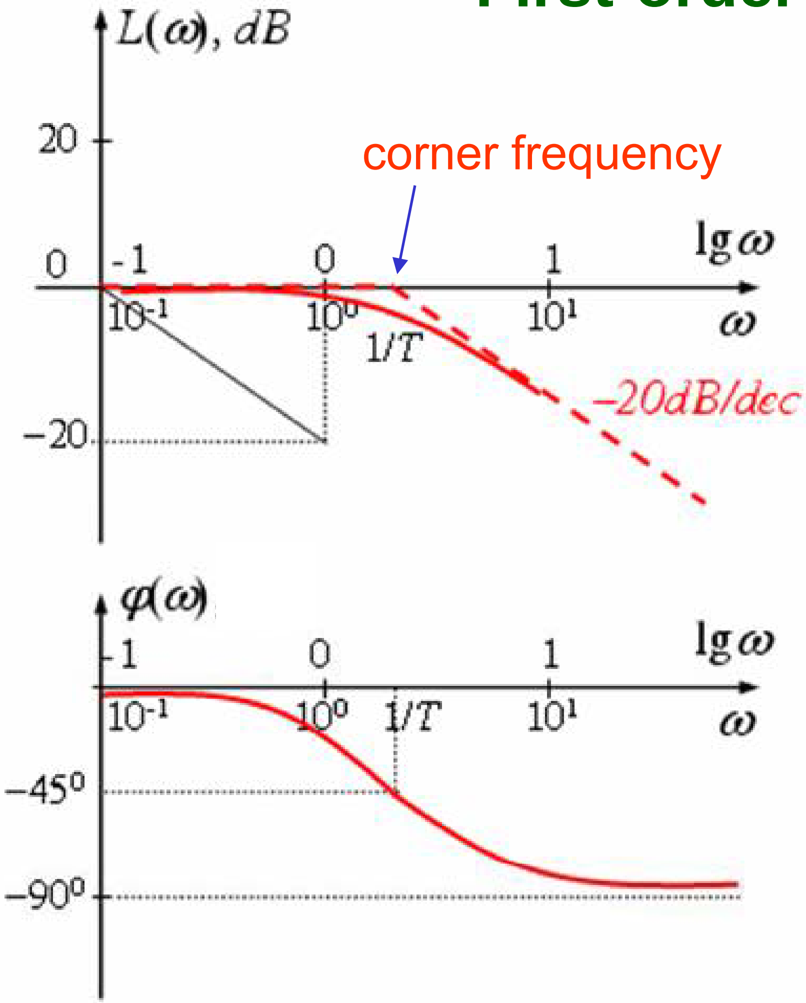
Frequency response of basic factor (cont.)

First-order lag factor

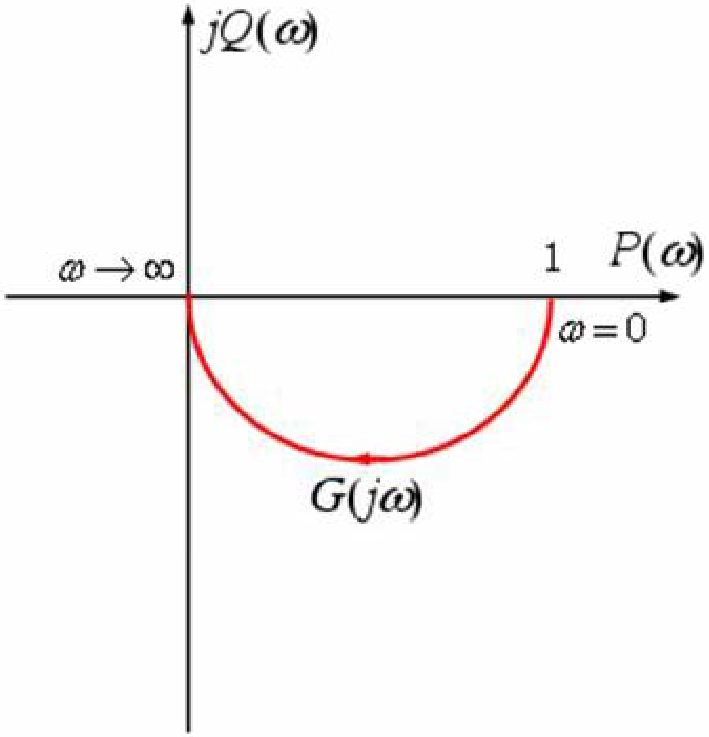
- Transfer function: $$G(s) = \frac{1}{Ts+1}$$
- Frequency response: $$G(j\omega) = \frac{1}{Tj\omega+1}$$
  - Magnitude response: $$M(\omega) = \frac{1}{\sqrt{1+T^2\omega^2}}$$

    $\Rightarrow$ $$L(\omega) = -20\lg\sqrt{1+T^2\omega^2}$$
  - Phase response: $$\varphi(\omega) = -tg^{-1}(T\omega)$$
- Approximation of the Bode diagram by asymptotes:
  - $\omega < 1/T$: the asymptote lies on the horizontal axis
  - $\omega > 1/T$: the asymptote has the slope of –20dB/dec

First-order lag factor



Bode diagram



Nyquist plot

Frequency response of basic factor (cont.)

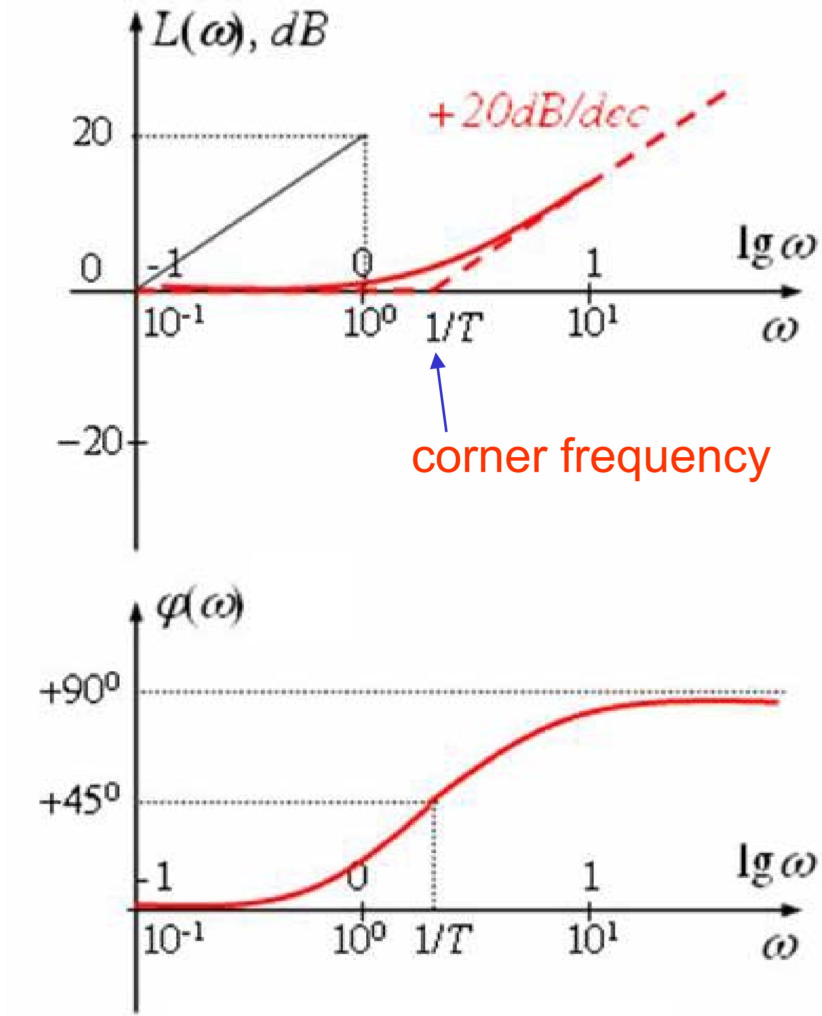
First-order lead factor

- Transfer function: $G(s) = Ts + 1$
- Frequency response: $G(j\omega) = Tj\omega + 1$
  - Magnitude response: $M(\omega) = \sqrt{1 + T^2\omega^2}$

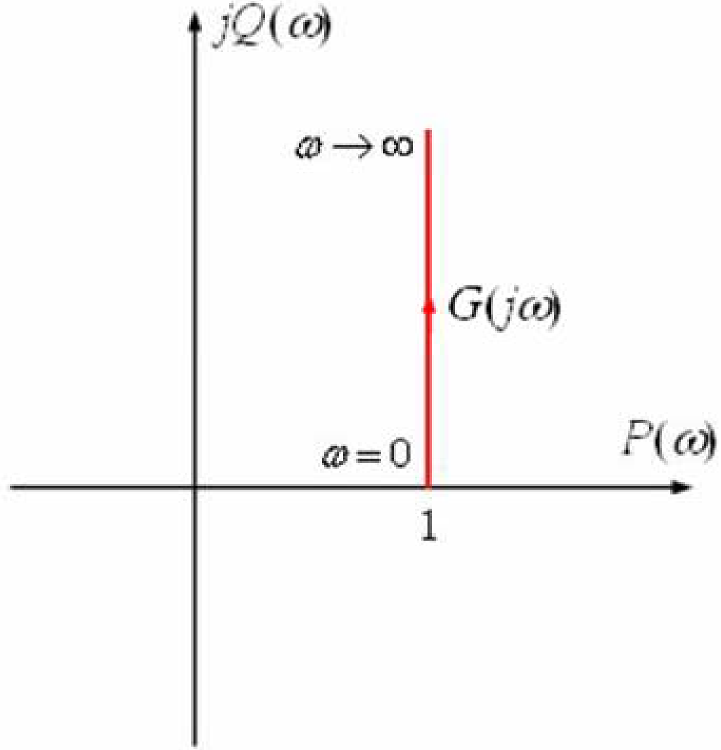
    $\Rightarrow \quad L(\omega) = 20\lg\sqrt{1 + T^2\omega^2}$
  - Phase response: $\varphi(\omega) = tg^{-1}(T\omega)$
- Approximation of the Bode diagram by asymptotes:
  - $\omega < 1/T$: the asymptote lies on the horizontal axis
  - $\omega > 1/T$: the asymptote has the slope of +20dB/dec

Frequency response of basic factor (cont.)

First-order lead factor



Bode diagram



Nyquist plot

Frequency response of basic factor (cont.)

Second-order oscilating factor

- Transfer function: $$G(s)=\frac{1}{T^2s^2+2\xi Ts+1}\quad(0<\xi<1)$$

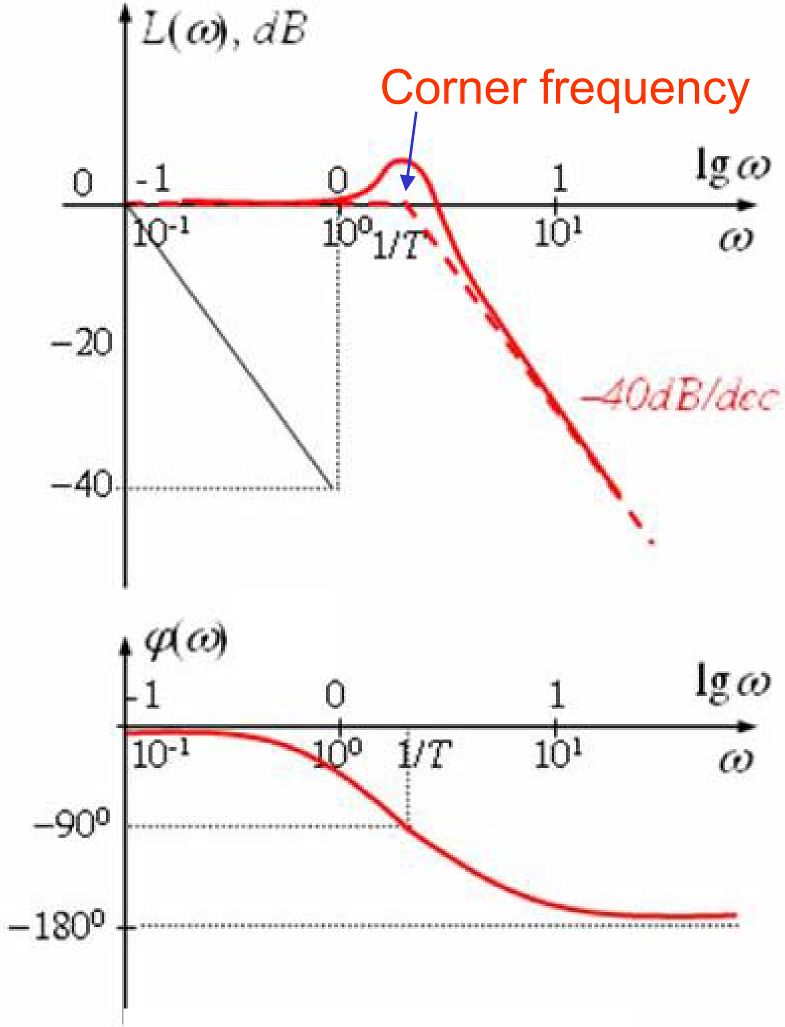
Frequency response of basic factor (cont.)

Second-order oscillating factor

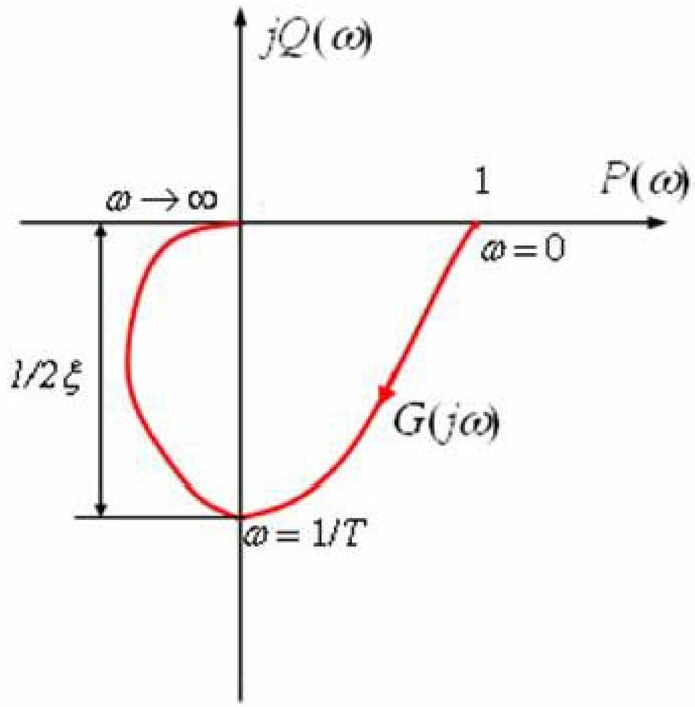
- Frequency response: $$G(j\omega)=\frac{1}{-T^2\omega^2+2\xi Tj\omega+1}$$
  - Magnitude response: $$M(\omega)=\frac{1}{\sqrt{(1-T^2\omega^2)^2+4\xi^2T^2\omega^2}}$$
    $$\Rightarrow \quad L(\omega)=-20\lg\sqrt{(1-T^2\omega^2)^2+4\xi^2T^2\omega^2}$$
  - Phase response: $$\varphi(\omega)=-tg^{-1}\left(\frac{2\xi T\omega}{1-T^2\omega^2}\right)$$
- Approximation of the Bode diagram by asymptotes:
  - $\omega<1/T$: the asymptote lies on the horizontal axis
  - $\omega>1/T$: the asymptote has the slope of –40dB/dec

Frequency response of basic factor (cont.)

Second-order oscilating factor



Bode diagram



Nyquist plot

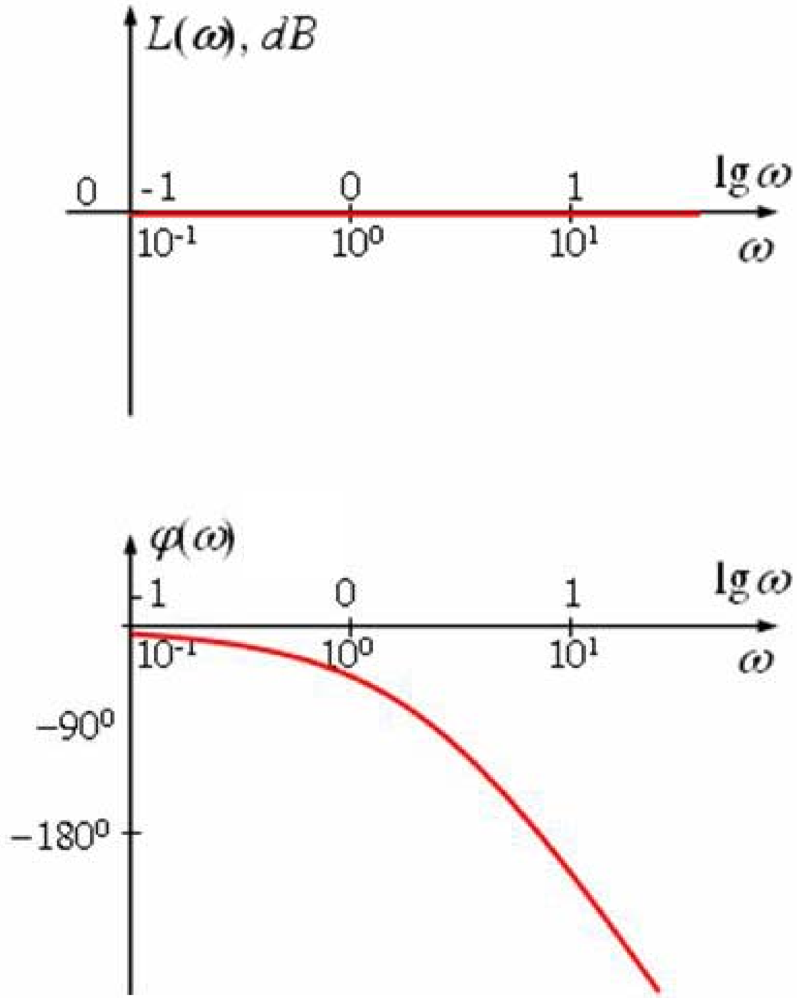
Frequency response of basic factor (cont.)

Time delay factor

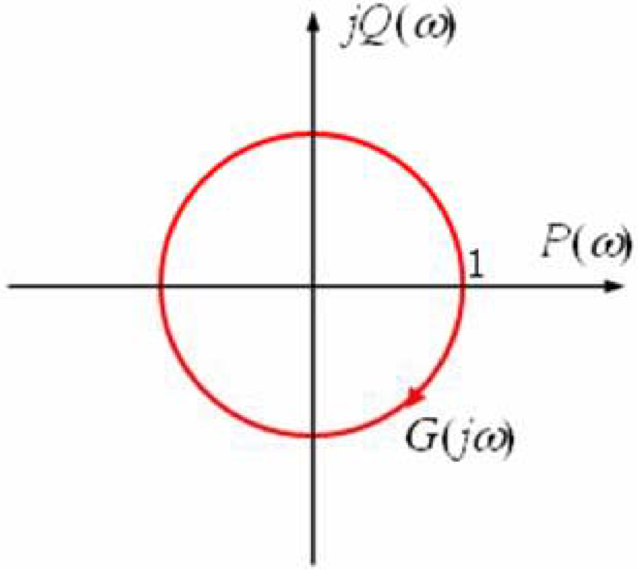
- Transfer function: $G(s) = e^{-Ts}$
- Frequency response: $G(j\omega) = e^{-Tj\omega}$
  - Magnitude response: $M(\omega) = 1 \quad \Rightarrow \quad L(\omega) = 0$
  - Phase response: $\varphi(\omega) = -T\omega$

Frequency response of basic factor (cont.)

Time delay factor



Bode diagram



Nyquist plot

Frequency response of control systems

- Consider a control system which has the transfer function $G(s)$. Suppose that $G(s)$ consists of basis factors in series:

$$G(s)=\prod_{i=1}^{l} G_i(s)$$

- Frequency response: $$G(j\omega)=\prod_{i=1}^{l} G_i(j\omega)$$
  - Magnitude response: $M(\omega)=\prod_{i=1}^{l} M_i(\omega) \Rightarrow L(\omega)=\sum_{i=1}^{l} L_i(\omega)$
  - Phase response: $$\varphi(\omega)=\sum_{i=1}^{l} \varphi_i(\omega)$$

$\Rightarrow$ The Bode diagram of a system consisting of basic factors in series equals to the sum of the Bode diagram of the basic factors.

Approximation of Bode diagram

- Suppose that the TF of the system is of the form:

$$G(s) = Ks^{\alpha} G_1(s) G_2(s) G_3(s) \ldots$$

(α>0: the system has ideal derivative factor(s)
α<0: the system has ideal integral factor(s))

- Step 1: Determine all the corner frequencies $\omega_i = 1/T_i$, and sort them in ascending order $\omega_1 < \omega_2 < \omega_3 \ldots$

- Step 2: The approximated Bode diagram passes through the point *A* having the coordinates:

$$\begin{cases} \omega = \omega_0 \\ L(\omega) = 20\lg K + \alpha \times 20\lg \omega_0 \end{cases}$$

where ω_0 is a frequency satisfying $\omega_0 < \omega_1$. If $\omega_1 > 1$ then it is possible to chose $\omega_0 = 1$.

Approximation of Bode diagram (cont')

- Step 3: Through point A, draw an asymptote with the slope:
  - ($-$ 20 *dB/dec* $\times\ \alpha$) if $G(s)$ has α ideal integral factors
  - (+ 20 *dB/dec* $\times\ \alpha$) if $G(s)$ has α ideal derivative factors

  The asymptote extends to the next corner frequency.

- Step 4: At the corner frequency $\omega_i = 1/T_i$, the slope of the asymptote is added with:
  - ($-$20*dB/dec* $\times\ \beta_i$) if $G_i(s)$ is a first-order lag factor (multiple β_i)
  - (+20*dB/dec* $\times\ \beta_i$) if $G_i(s)$ is a first-order lead factor (multiple β_i)
  - ($-$40*dB/dec* $\times\ \beta_i$) if $G_i(s)$ is a 2nd order oscillating factor (multiple β_i)
  - (+40*dB/dec* $\times\ \beta_i$) if $G_i(s)$ is a 2nd order lead factor (multiple β_i)

  The asymptote extends to the next corner frequency.

- Step 5: Repeat the step 4 until the asymptote at the last corner frequency is plotted.

Approximation of Bode diagram – Example 1

- Plot the Bode diagram using asymptotes:

$$G(s) = \frac{100(0{,}1s+1)}{s(0{,}01s+1)}$$

Based on the Bode diagram, determine the gain cross frequency of the system.

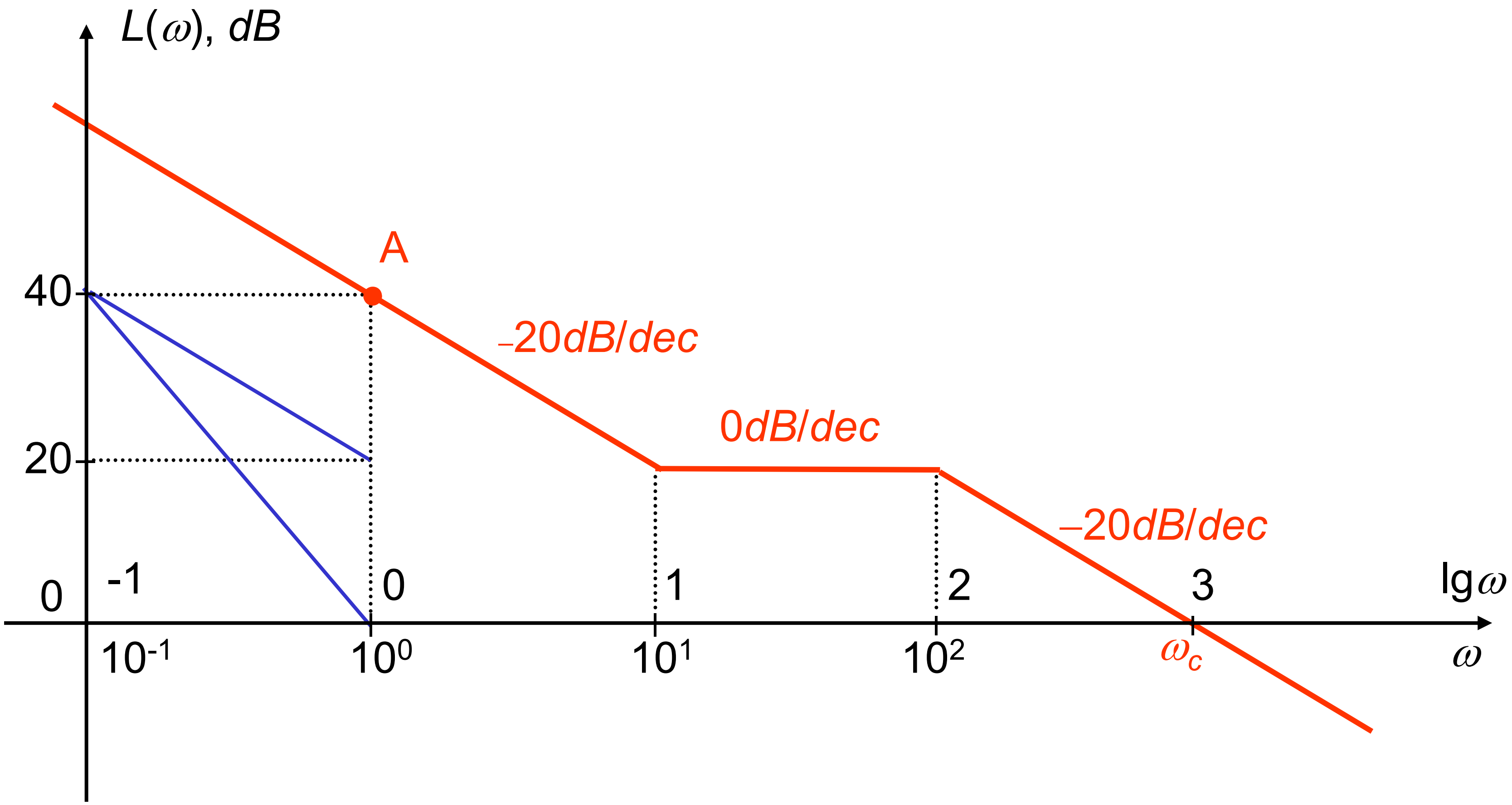
- Solution:
- Corner frequencies:

$$\omega_1 = \frac{1}{T_1} = \frac{1}{0{,}1} = 10\ \text{(rad/sec)} \qquad \omega_2 = \frac{1}{T_2} = \frac{1}{0{,}01} = 100\ \text{(rad/sec)}$$

- The Bode diagram pass the point A at the coordinate:

$$\begin{cases} \omega = 1 \\ L(\omega) = 20\lg K = 20\lg 100 = 40 \end{cases}$$

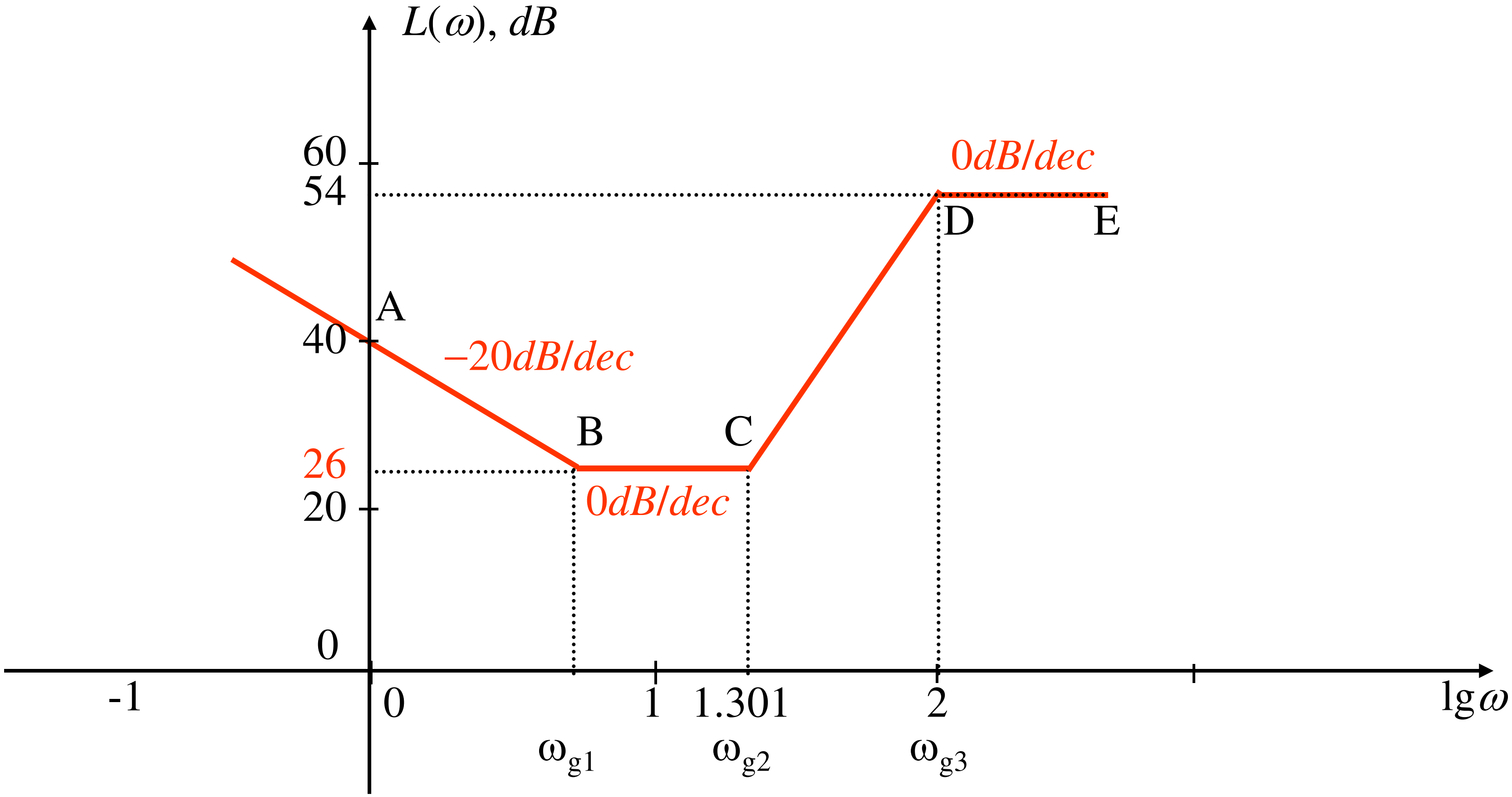
Approximation of Bode diagram – Example 1 (cont')



★ In the Bode diagram, the gain crossover frequency is 10^3 rad/sec.

Example 2 – Bode diagram to transfer function

* Determine the transfer function of the system which has the approximation Bode diagram as below:



Example 2 – Bode diagram to transfer function (cont')

- The slope of segment CD: $\dfrac{54-26}{2-1.301} = +40\,(\text{dB/dec})$
- The corner frequencies:

$$\lg\omega_{g1} = 0 + \frac{40-26}{20} = 0.7 \quad \Rightarrow \quad \omega_{g1} = 10^{0.7} = 5\,(\text{rad/sec})$$

$$\lg\omega_{g2} = 1.301 \quad \Rightarrow \quad \omega_{g2} = 10^{1.301} = 20\,(\text{rad/sec})$$

$$\lg\omega_{g3} = 2 \quad \Rightarrow \quad \omega_{g3} = 10^{2} = 100\,(\text{rad/sec})$$

- The transfer function has the form: $G(s) = \dfrac{K(T_1 s+1)(T_2 s+1)^2}{s(T_3 s+1)^2}$

$$20\lg K = 40 \quad \Rightarrow \quad K = 100$$

$$T_1 = \frac{1}{\omega_{g1}} = \frac{1}{5} = 0.2 \qquad T_2 = \frac{1}{\omega_{g2}} = \frac{1}{20} = 0.05 \qquad T_3 = \frac{1}{\omega_{g3}} = \frac{1}{100} = 0.01$$

Crossover frequency

- Gain crossover frequency(ω_c): is the frequency where the amplitude of the frequency response is 1 (or 0 dB).

$$M(\omega_c) = 1 \quad \Leftrightarrow \quad L(\omega_c) = 0$$

- Phase crossover frequency ($\omega_{-\pi}$): is the frequency where phase shift of the frequency response is equal to -180^0 (or equal to $-\pi$ radian).

$$\varphi(\omega_{-\pi}) = -180^0 \quad \Leftrightarrow \quad \varphi(\omega_{-\pi}) = -\pi \text{ rad}$$

Stability margin

- **Gain margin** (*GM*):

$$GM = \frac{1}{M(\omega_{-\pi})} \quad \Leftrightarrow \quad GM = -L(\omega_{-\pi}) \quad \text{[dB]}$$

Physical meaning: The gain margin is the amount of positive gain at the phase crossover frequency required to bring the system to the stability boundary.

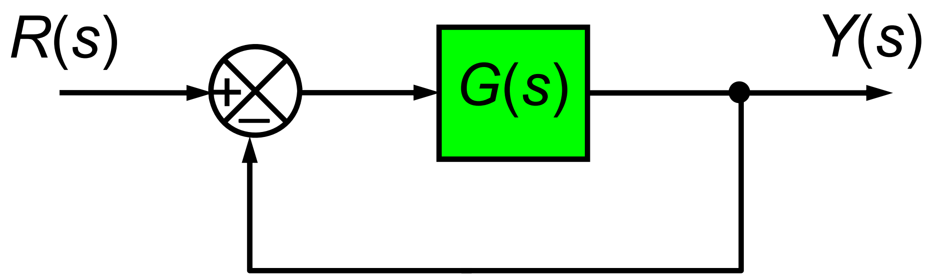
- **Phase margin** (*ΦM*)

$$\Phi M = 180^0 + \varphi(\omega_c)$$

Physical meaning: The phase margin is the amount of additional phase lag at the gain crossover frequency required to bring the system to the stability boundary.

Nyquist stability criterion

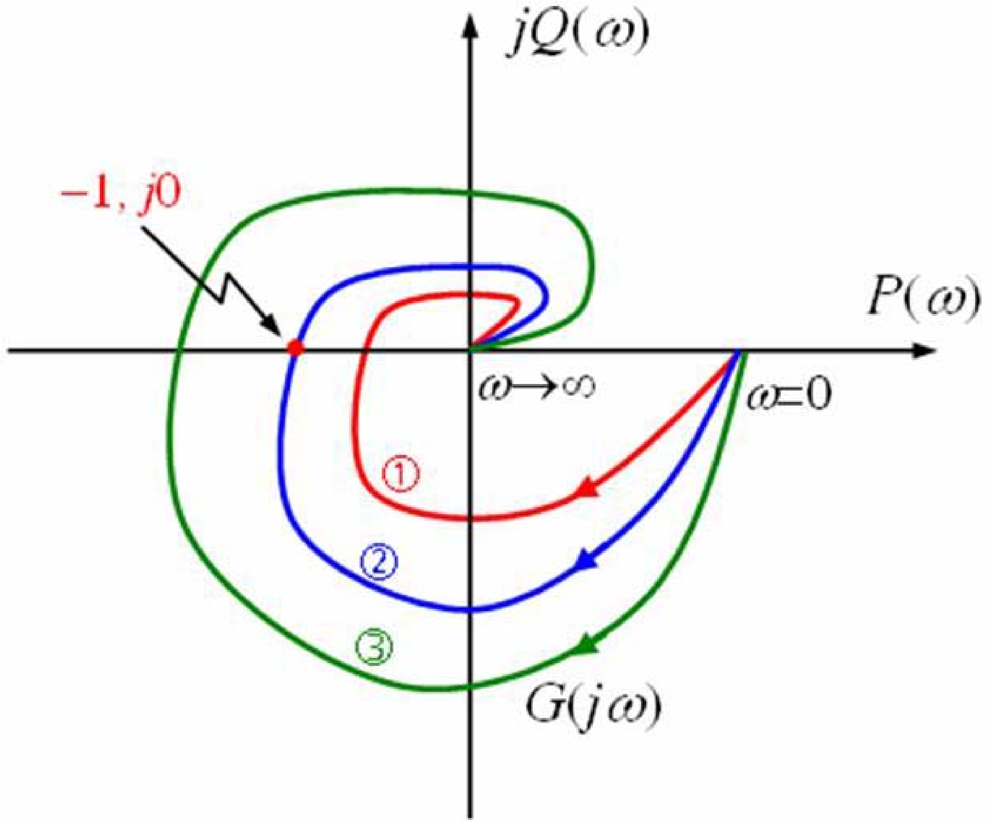
- Consider a unity feedback system shown below, suppose that we know the Nyquist plot of the open loop system $G(s)$, the problem is to determine the stability of the closed-loop system $G_{cl}(s)$.



- **Nyquist criterion:** The closed-loop system $G_{cl}(s)$ is stable if and only if the Nyquist plot of the open-loop system $G(s)$ encircles the critical point $(-1, j0)$ $l/2$ times in the counterclockwise direction when ω changes from 0 to $+\infty$ (l is the number of poles of $G(s)$ lying in the right-half s-plane).

Nyquist stability criterion – Example 1

- Consider an unity negative feedback system, whose open-loop system $G(s)$ is stable and has the Nyquist plots below (three cases). Analyze the stability of the closed-loop system.



Nyquist stability criterion – Example 1

- Solution

  The number of poles of $G(s)$ lying in the right-half s-plane is 0 because $G(s)$ is stable. Then according to the Nyquist criterion, the closed-loop system is stable if the Nyquist plot $G(j\omega)$ does not encircle the critical point $(-1, j0)$

- Case ①: $G(j\omega)$ does not encircle $(-1, j0)$

  $\Rightarrow$ the close-loop system is stable.

- Case ②: $G(j\omega)$ pass $(-1, j0)$

  $\Rightarrow$ the close-loop system is at the stability boundary;

- Case ③: $G(j\omega)$ encircles $(-1, j0)$

  $\Rightarrow$ the close-loop system is unstable.

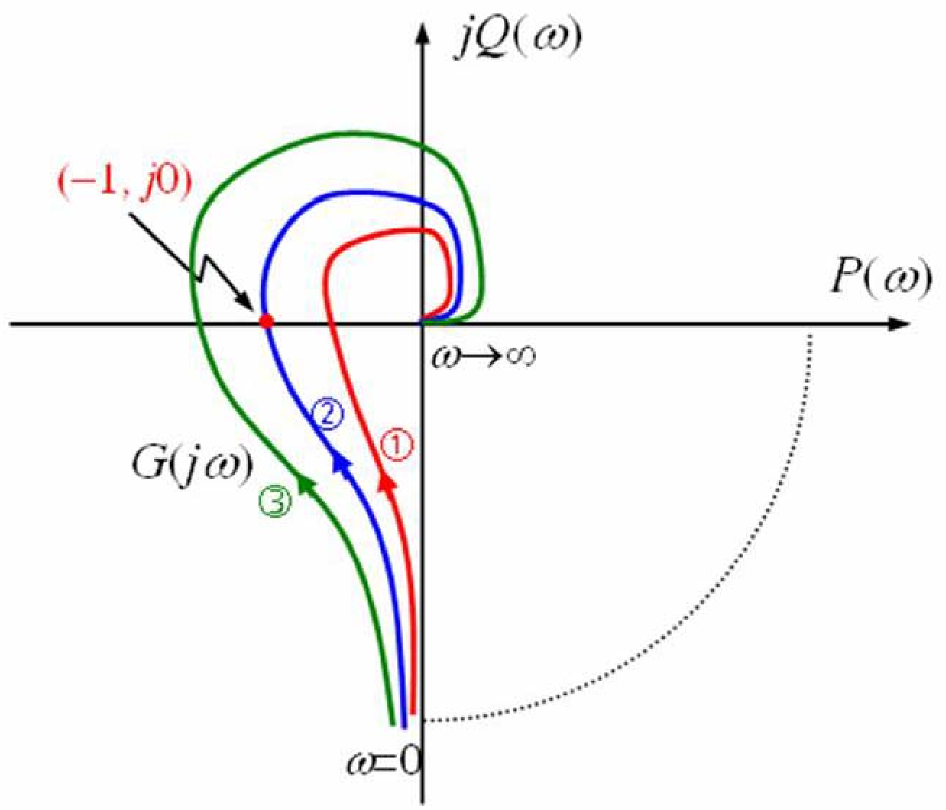
Nyquist stability criterion – Example 2

- Analyze the stability of a unity negative feedback system whose open loop transfer function is:

$$G(s) = \frac{K}{s(T_1 s + 1)(T_2 s + 1)(T_3 s + 1)}$$

- Solution:

- Nyquist plot: Depending on the values of T_1, T_2, T_4 and K, the Nyquist plot of G(s) could be one of the three curves 1, 2 or 3.



Nyquist stability criterion – Example 2 (cont')

The number of poles of $G(s)$ lying in the right-half s-plane is 0 because $G(s)$ is stable. Then according to the Nyquist criterion, the closed-loop system is stable if the Nyquist plot $G(j\omega)$ does not encircle the critical point $(-1, j0)$

- Case ①: $G(j\omega)$ does not encircle $(-1, j0)$

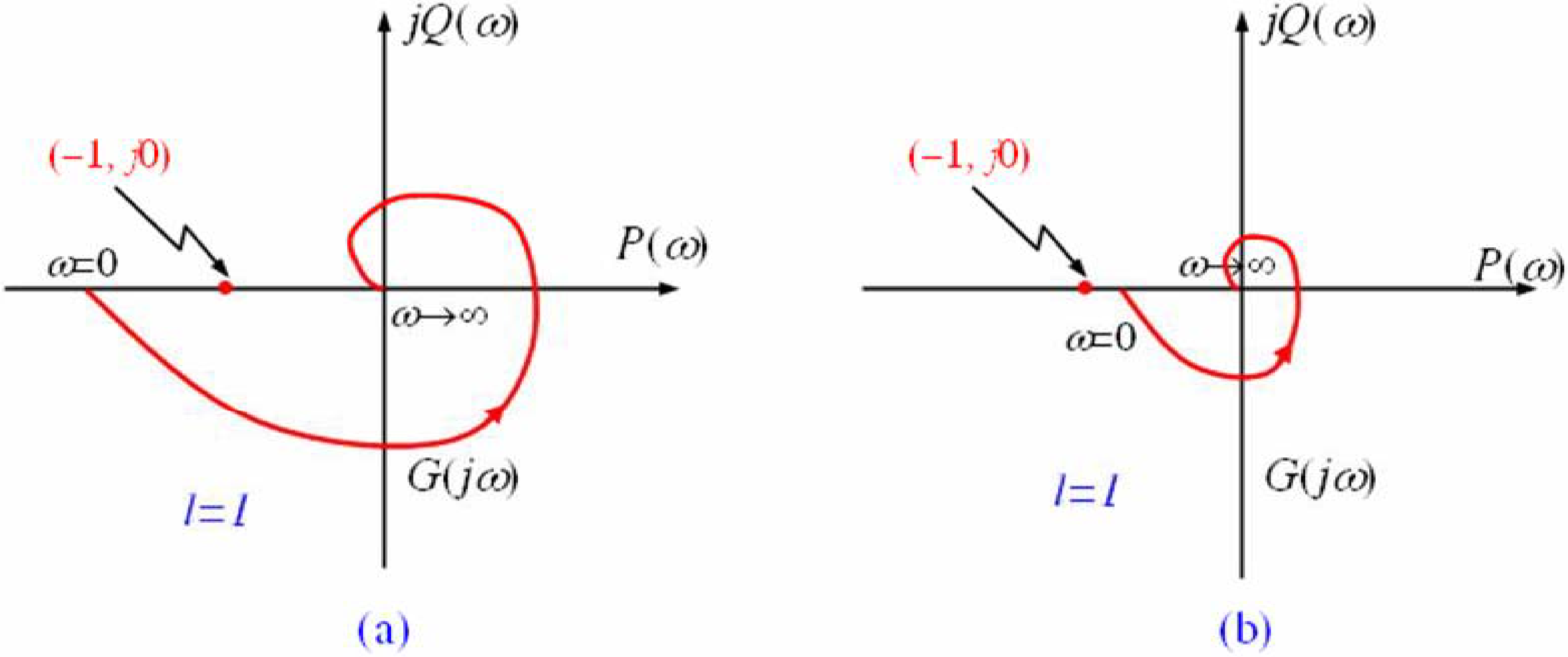
  $\Rightarrow$ the close-loop system is stable.
- Case ②: $G(j\omega)$ pass $(-1, j0)$

  $\Rightarrow$ the close-loop system is at the stability boundary;
- Case ③: $G(j\omega)$ encircles $(-1, j0)$

  $\Rightarrow$ the close-loop system is unstable.

Nyquist stability criterion – Example 3

Given an unstable open-loop systems which have the Nyquist plot as below. In which cases the closed-loop system is stable?

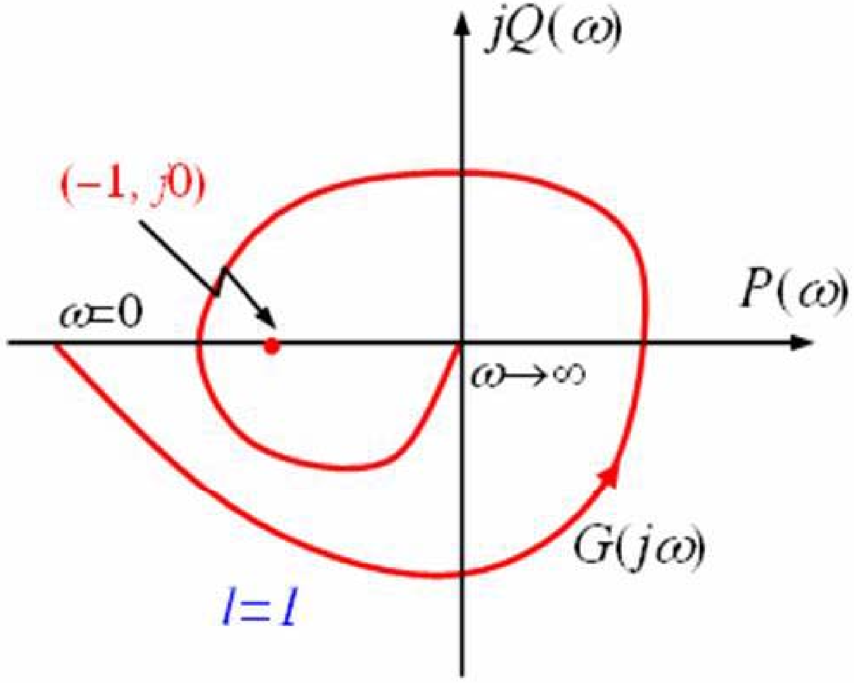


Stable

Unstable

Nyquist stability criterion – Example 3 (cont')

Given an unstable open-loop systems which have the Nyquist plot as below. In which cases the closed-loop system is stable?

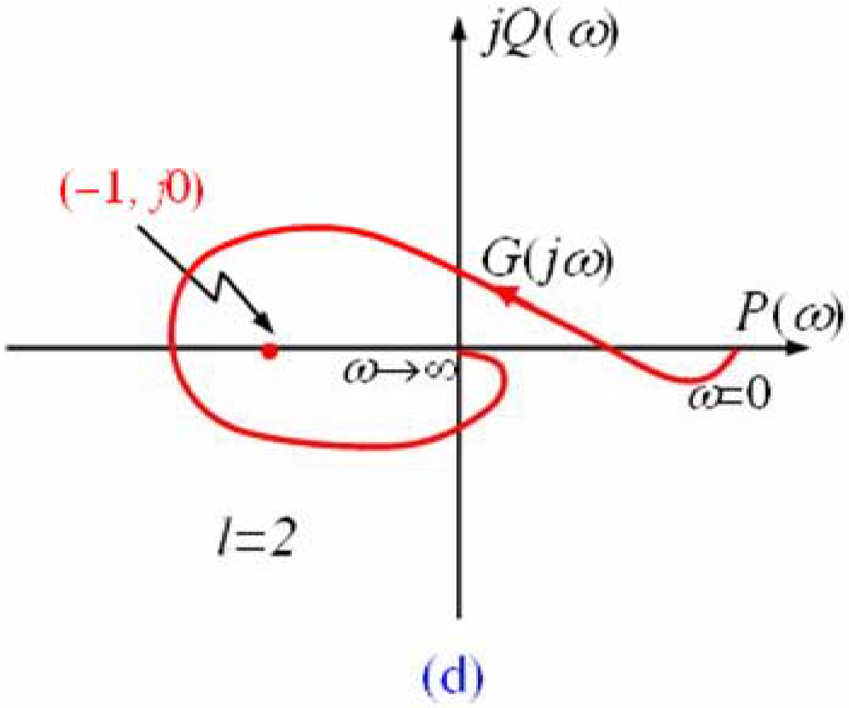


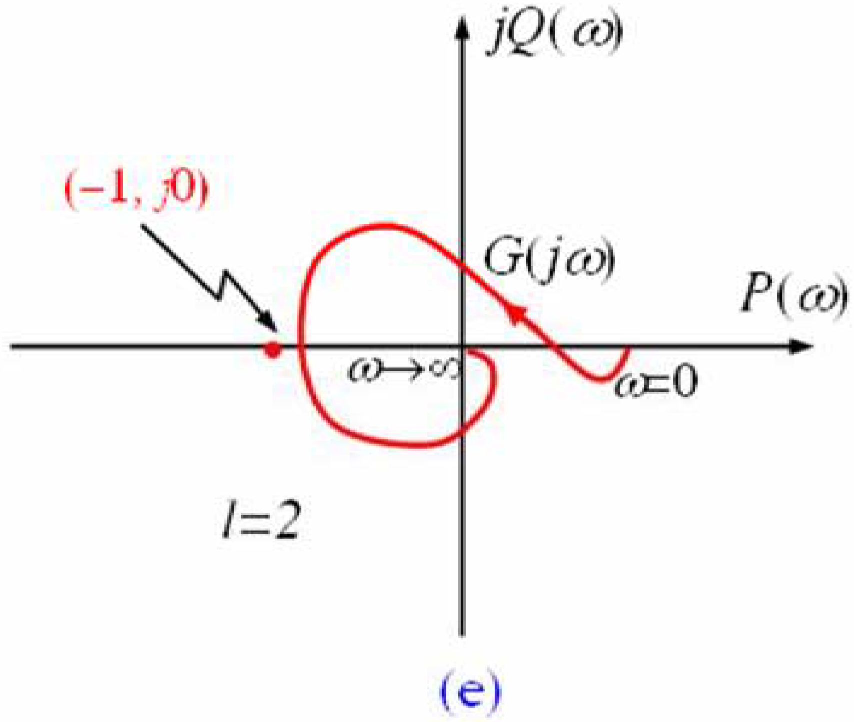
(c)

Unstable

Nyquist stability criterion – Example 3 (cont')

Given an unstable open-loop systems which have the Nyquist plot as below. In which cases the closed-loop system is stable?





Stable

Unstable

Nyquist stability criterion – Example 4

- Given a open-loop system which has the transfer function:

$$G(s) = \frac{K}{(Ts+1)^n} \qquad (K>0,\ T>0,\ n>2)$$

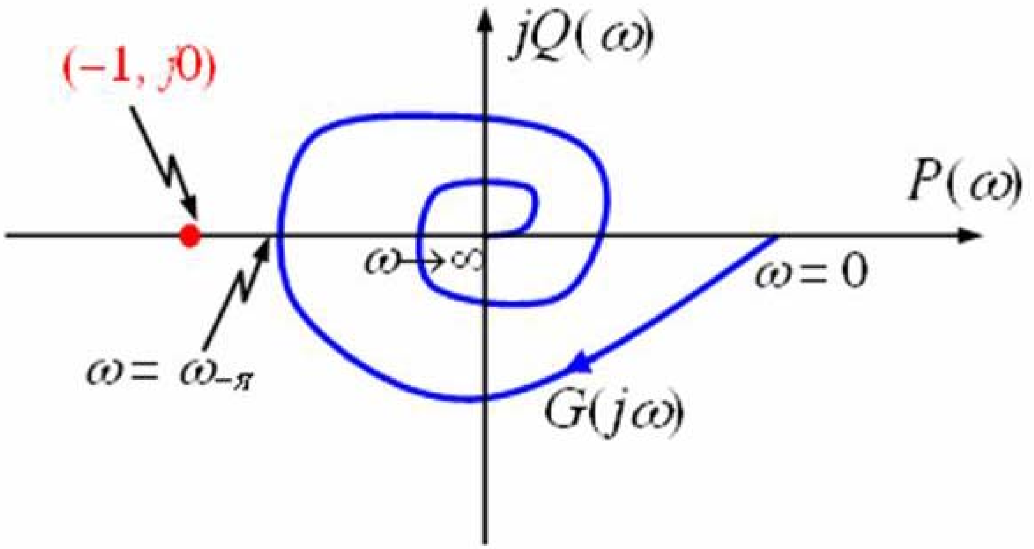
  Find the condition of *K* and *T* for the unity negative feedback closed-loop system to be stable.

- Solution:
- Frequency response of the open-loop system:

$$G(j\omega) = \frac{K}{(Tj\omega+1)^n}$$

  - Magnitude: $M(\omega) = \dfrac{K}{\left(\sqrt{T^2\omega^2+1}\right)^n}$
  - Phase: $\varphi(\omega) = -n\,tg^{-1}(T\omega)$

- Nyquist plot:



- Stability condition: the Nyquist plot of $G(j\omega)$ does not encircle the critical point (–1,j0). According to the Nyquist plot, this requires:

$$M(\omega_{-\pi}) < 1$$

Nyquist stability criterion – Example 4 (cont')

- We have: $\varphi(\omega_{-\pi}) = -n\,tg^{-1}(T\omega_{-\pi}) = -\pi$

$$\Rightarrow \quad tg^{-1}(T\omega_{-\pi}) = \frac{\pi}{n} \quad \Rightarrow \quad (T\omega_{-\pi}) = tg\left(\frac{\pi}{n}\right)$$

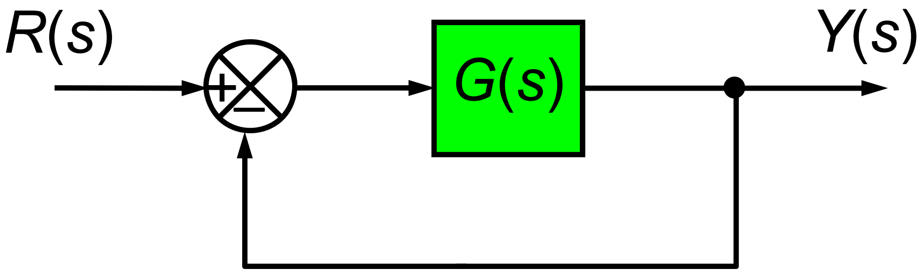
$$\Rightarrow \quad \boxed{\omega_{-\pi} = \frac{1}{T}tg\left(\frac{\pi}{n}\right)}$$

- Then: $M(\omega_{-\pi}) < 1 \quad \Leftrightarrow \quad \dfrac{K}{\left(\sqrt{T^2\left[\dfrac{1}{T}tg\left(\dfrac{\pi}{n}\right)\right]^2 + 1}\right)^n} < 1$

$$\Leftrightarrow \quad \boxed{K < \left(\sqrt{tg^2\left(\frac{\pi}{n}\right) + 1}\right)^n}$$

Bode criterion

- Consider a unity feedback system, suppose that we know the Nyquist plot of the open loop system $G(s)$, the problem is to determine the stability of the closed-loop system $G_{cl}(s)$.

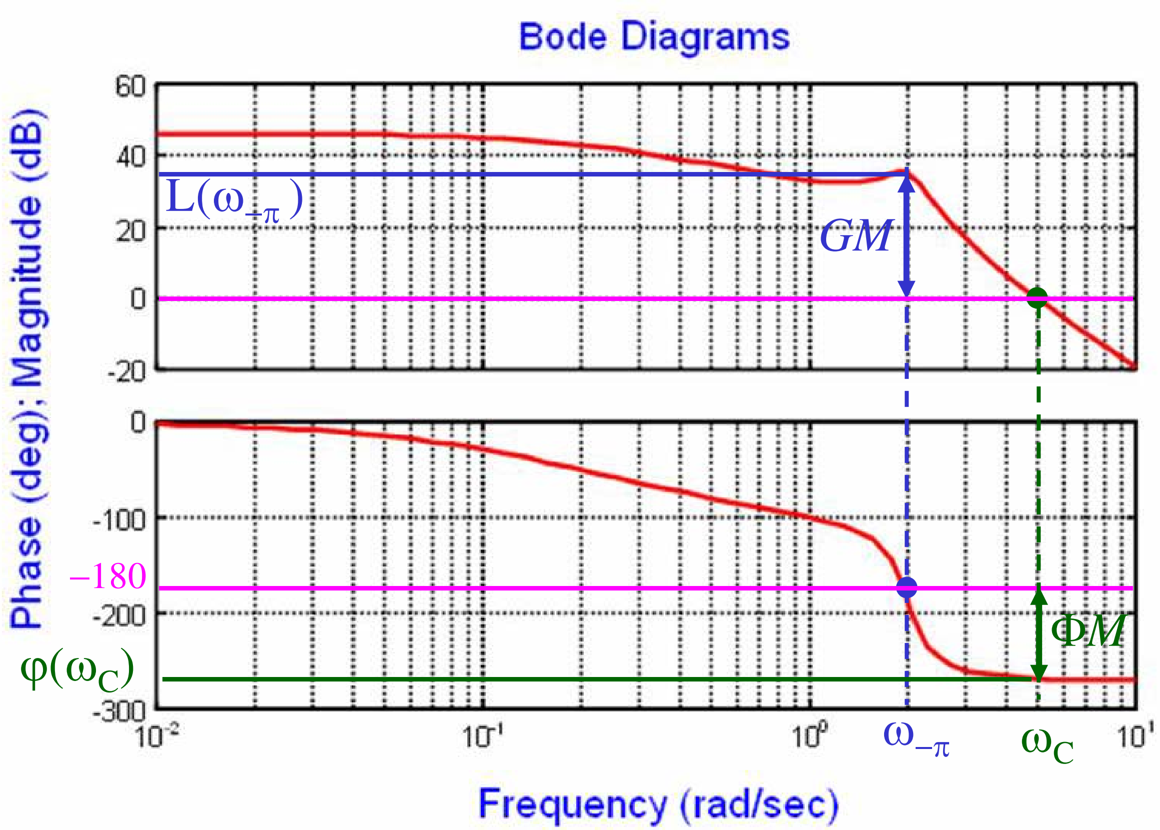


- **Bode criterion:** The closed-loop system $G_{cl}(s)$ is stable if the gain margin and phase margin of open-loop system $G(s)$ are positive.

$$\begin{cases} GM > 0 \\ \Phi M > 0 \end{cases} \Leftrightarrow \quad \text{The closed-loop system is stable}$$

Bode criterion – Example

- Consider a unity negative feedback system whose open-loop system has the Bode diagram as below. Determine the gain margin, phase margin of the open-loop system. Is the closed-loop system stable or not?



Bode diagram:

$\omega_c = 5$

$\omega_{-\pi} = 2$

$L(\omega_{-\pi}) = 35dB$

$\varphi(\omega_c) = -270^0$

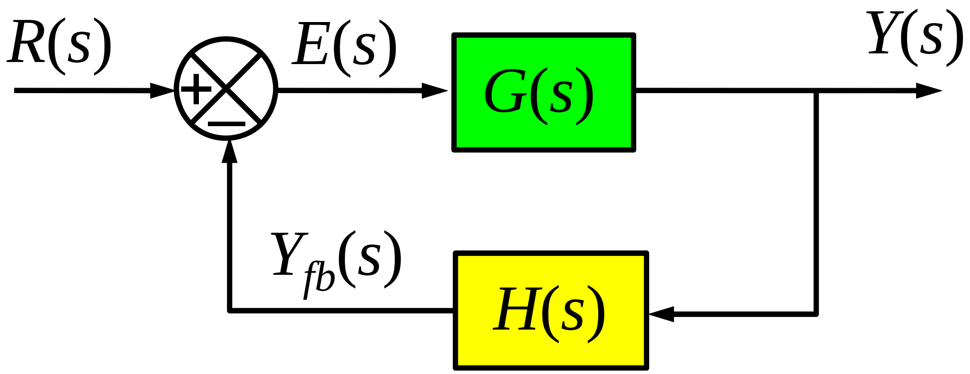
$GM = -35dB$

$\Phi M = 180^0 + (-270^0) = -90^0$

Because GM<0 and ΦM<0, the closed-loop system is unstable.

Remark on the frequency domain analysis

- If the closed-loop system as below, the Nyquist and Bode criteria can also be applied and in this case the open-loop system is $G(s)H(s)$.



End of chapter 3