Lecture Notes

Fundamentals of Control Systems

Course coordinator: Assoc. Prof. Dr. Huynh Thai Hoang
Department of Automatic Control
Faculty of Electrical & Electronics Engineering
Ho Chi Minh City University of Technology
Email: hthoang@hcmut.edu.vn

Instructor: Lecturers of Department of Automatic Control

Chapter 4

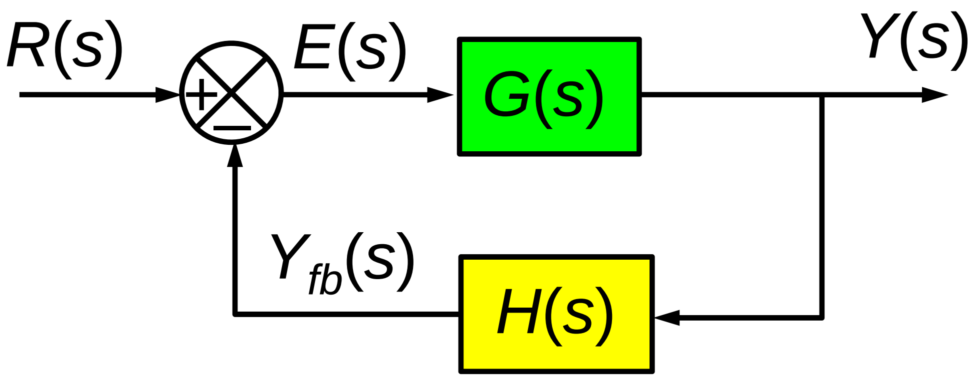
ANALYSIS OF CONTROL SYSTEM PERFORMANCE

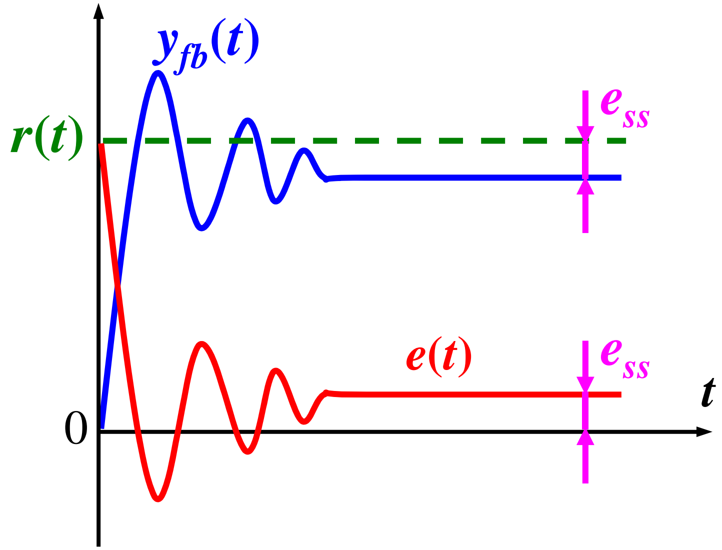
Content

- Performance criteria
- Steady state error
- Transient response
- The optimal performance index
- Relationship between frequency domain performances and time domain performances.

Performance criteria

Performance criteria: Steady state error





- Error: is the difference between the set-point (input) and the feedback signal.

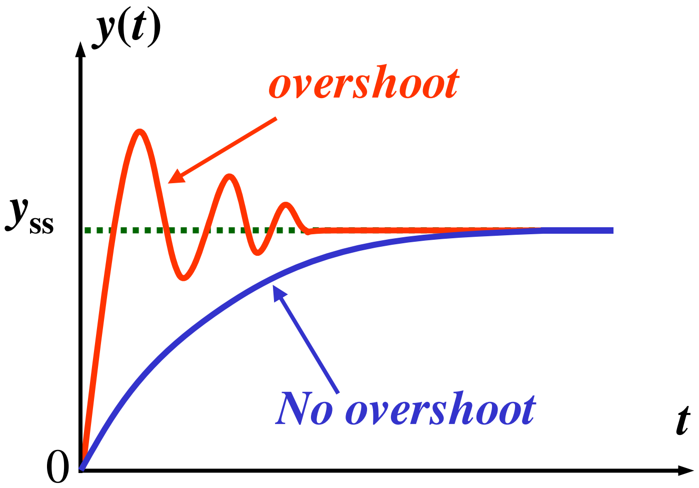
$$e(t) = r(t) - y_{fb}(t) \quad \Leftrightarrow \quad E(s) = R(s) - Y_{fb}(s)$$

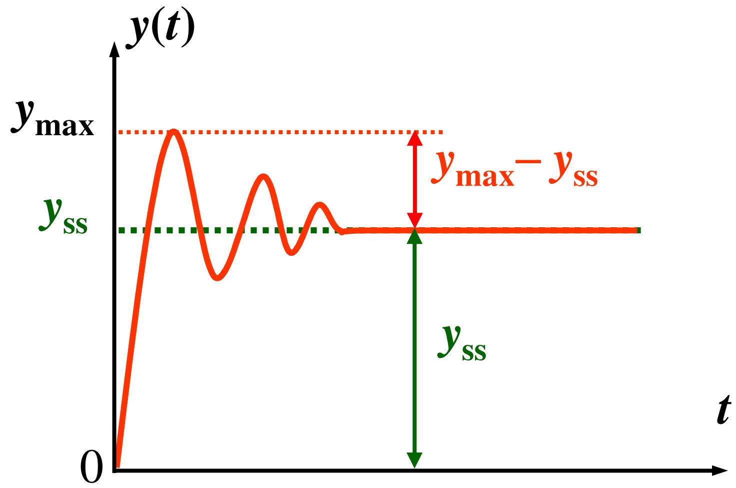
- Steady-state error: is the error when time approaching infinity.

$$e_{ss} = \lim_{t\to\infty} e(t) \quad \Leftrightarrow \quad e_{ss} = \lim_{s\to 0} sE(s)$$

Performance criteria – Percent of Overshoot (POT)

- Overshoot: refers to an output exceeding its steady-state value.



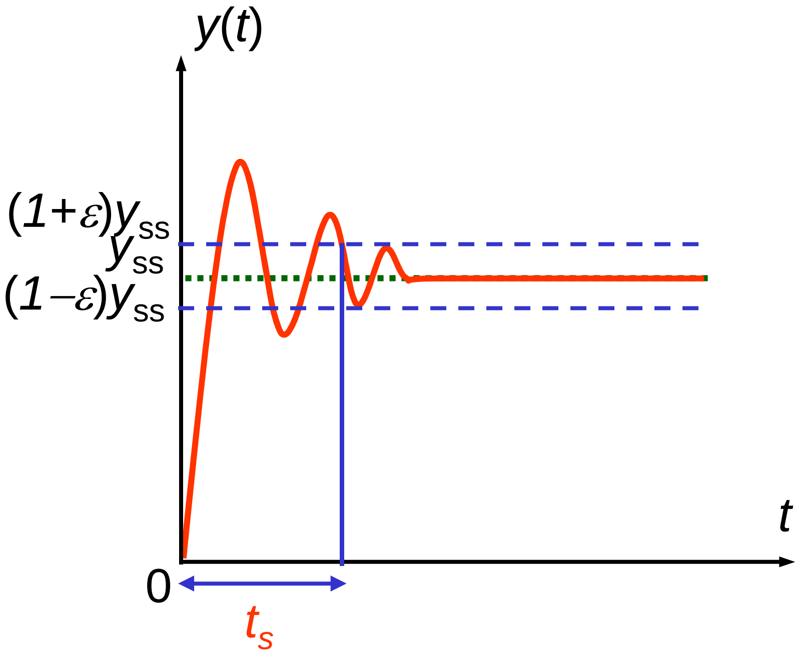


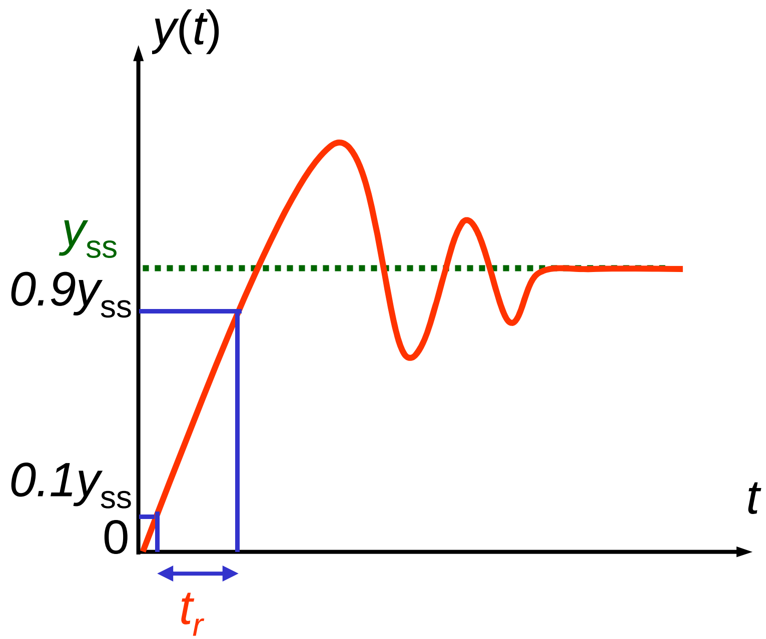
- Percentage of Overshoot (POT) is an index to quantify the overshoot of a system, POT is calculated as:

$$POT = \frac{y_{\max} - y_{ss}}{y_{ss}} \times 100\%$$

Performance criteria – Settling time and rise time

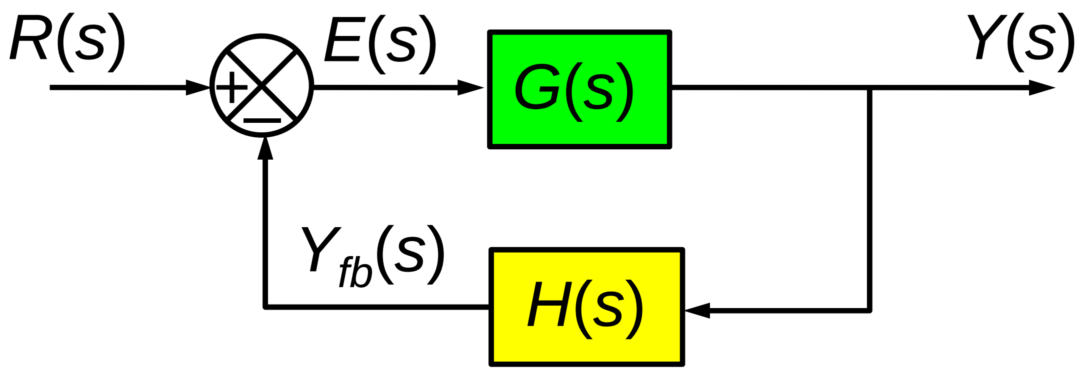
- Settling time (t_s): is the time required for the response of a system to reach and stay within a range about the steady-state value of size specified by absolute percentage of the steady-state value (usually 2% or 5%)
- Rise time (t_r): is the time required for the response of a system to rise from 10% to 90% of its steady-state value.





Steady-state error

Steady-state error



- Error expression: $$E(s) = \frac{R(s)}{1+G(s)H(s)}$$

- Steady-state error: $$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}$$

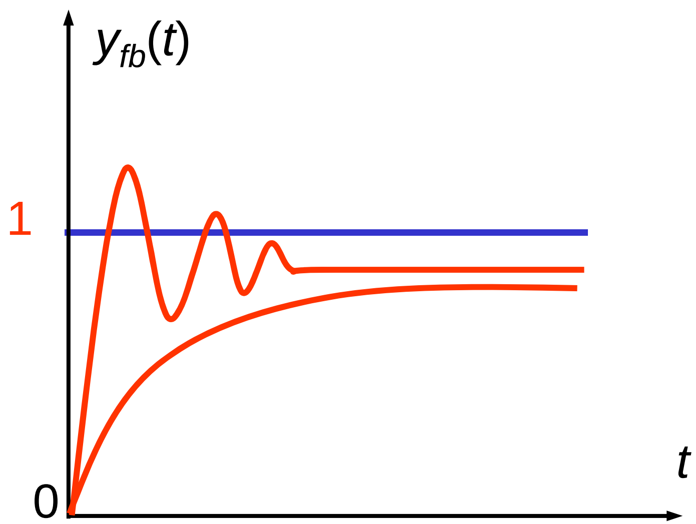
- Remark: Steady-state error not only depends on the structure and parameters of the system but also depends on the input signal.

Steady-state error to step input

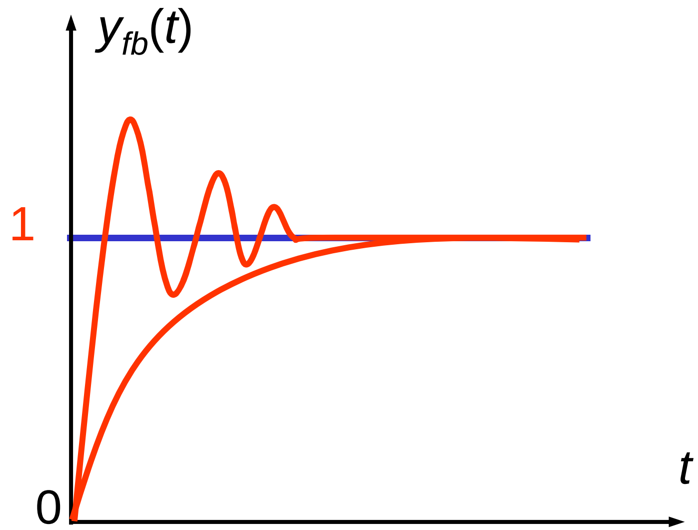
✶ Step input: $R(s) = 1/s$

⇒ Steady-state error: $e_{ss} = \dfrac{1}{1+K_p}$

with $K_p = \lim_{s \to 0} G(s)H(s)$ (position constant)



G(*s*)*H*(*s*) does not have any deal integral factor

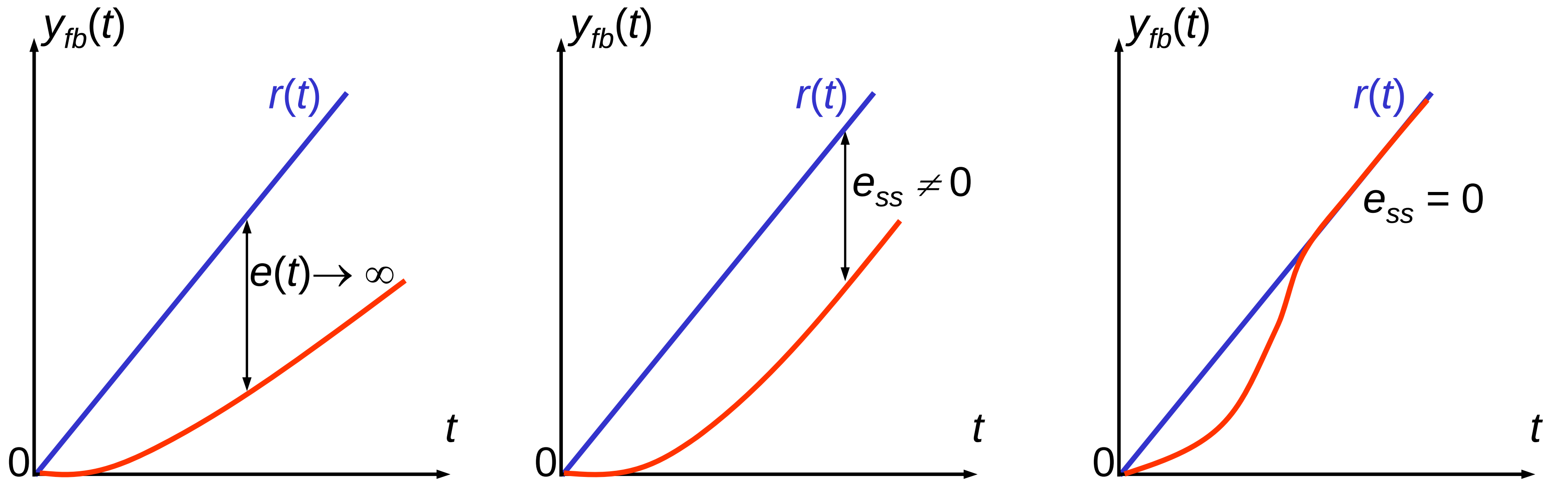


G(*s*)*H*(*s*) has at least 1 ideal integral factor

Steady-state error to ramp input

- Ramp input: $R(s) = 1/s^2$

$$\Rightarrow \quad e_{ss} = \frac{1}{K_v} \quad \text{with} \quad K_v = \lim_{s \to 0} sG(s)H(s) \quad \text{(velocity constant)}$$



G(*s*)*H*(*s*) does not have deal integral factor

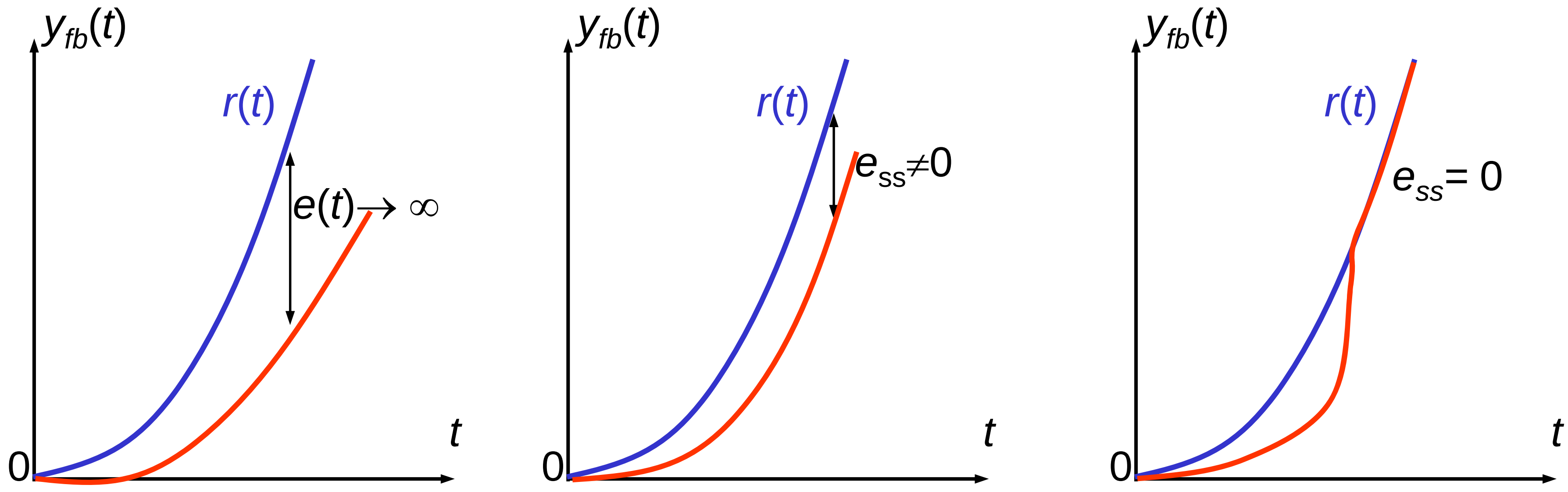
G(*s*)*H*(*s*) has 1 ideal integral factor

G(*s*)*H*(*s*) has at least 2 ideal integral factors

Steady-state error to parabolic input

★ Parabolic input: $R(s) = 1/s^3$

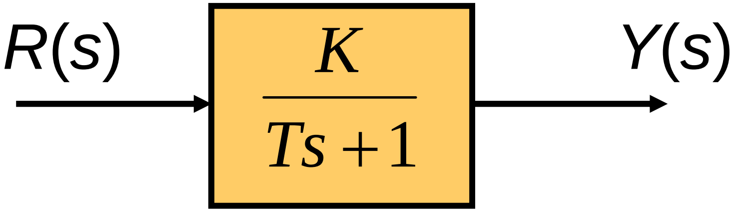
$$\Rightarrow \quad e_{ss} = \frac{1}{K_a} \quad \text{with} \quad K_a = \lim_{s \to 0} s^2 G(s)H(s)$$ (acceleration constant)



$G(s)H(s)$ has less than 2 ideal integral factors	$G(s)H(s)$ has 2 ideal integral factors	$G(s)H(s)$ has more than 2 ideal integral factors

Transient response

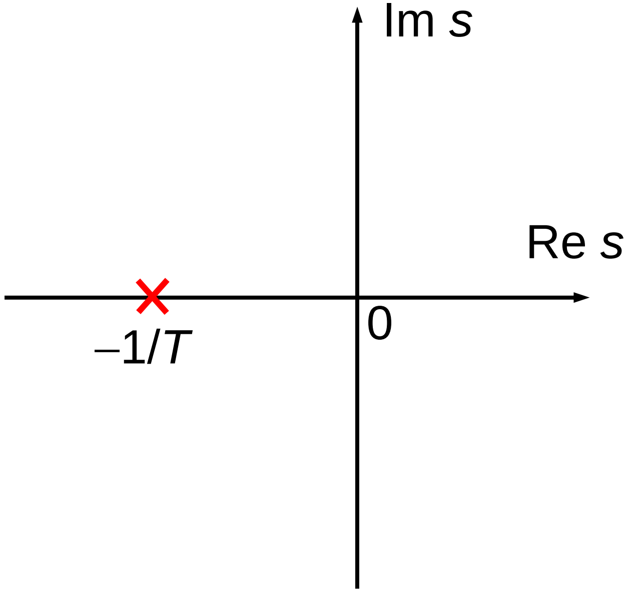
First-order system



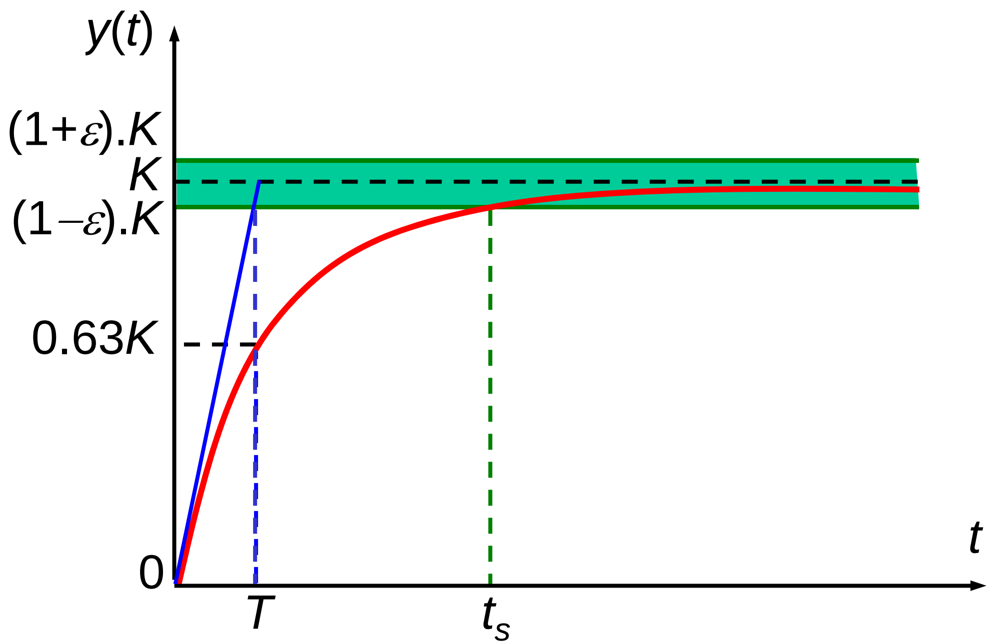
- Transfer function: $$G(s) = \frac{K}{Ts+1}$$
- First order system has 1 real pole: $$p_1 = -\frac{1}{T}$$
- Transient response: $$Y(s) = R(s)G(s) = \frac{1}{s}.\frac{K}{Ts+1}$$

$$\Rightarrow \quad y(t) = K(1 - e^{-t/T})$$

First-order system (cont')



Pole – zero plot
of a first order system



Transient response
of the first order

$$y(t) = K(1 - e^{-t/T})$$

First-order system – Remarks

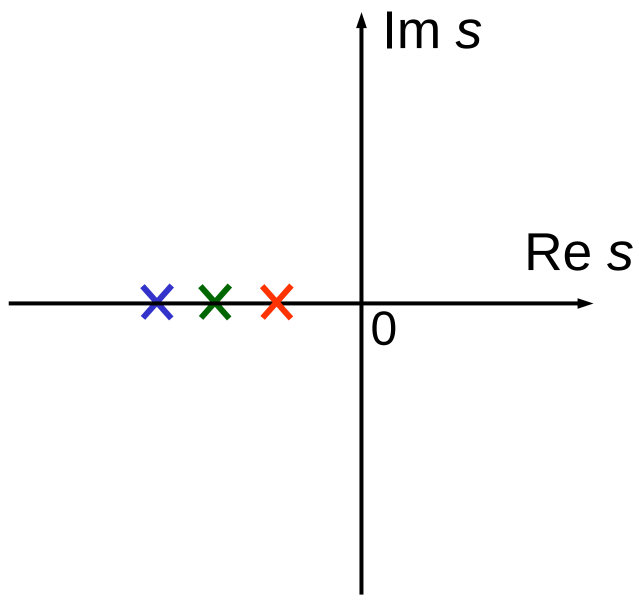
- First order system has only one real pole at $(-1/T)$, its transient response doesn't have overshoot.
- Time constant T: is the time required for the step response of the system to reach 63% its steady-state value.
- The further the pole $(-1/T)$ of the system is from the imaginary axis, the smaller the time constant and the faster the time response of the system.
- Settling time of the first order system is:

$$t_s = T\ln\left(\frac{1}{\varepsilon}\right)$$

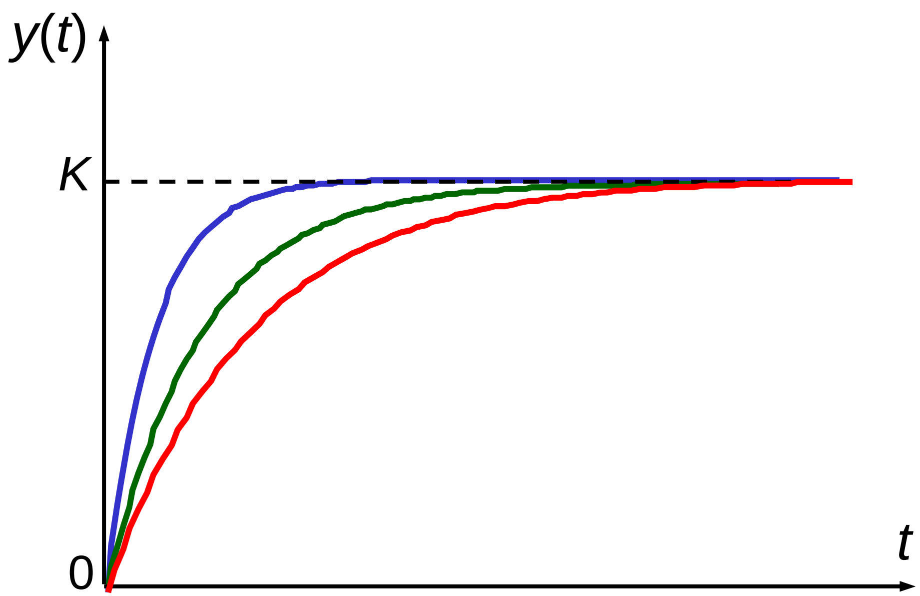
where $\varepsilon = 0.02$ (2% criterion) or $\varepsilon = 0.05$ (5% criterion)

The relationship between the pole and the time response

- The further the pole of the system is from the imaginary axis, the smaller the time constant and the faster the time response of the system.

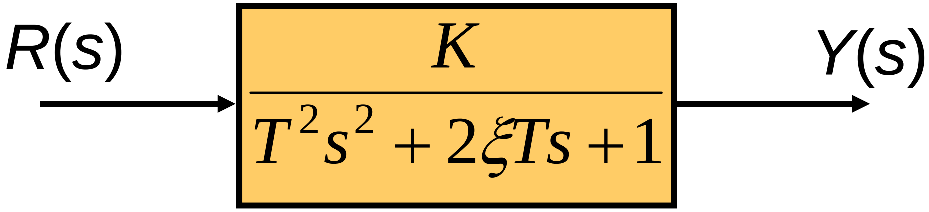


Pole – zero plot of a first order system



Transient response of the first order

Second-order oscillating system



- The transfer function of the second-order oscillating system:

$$G(s)=\frac{K}{T^2s^2+2\xi Ts+1}=\frac{K\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \qquad \left(\omega_n=\frac{1}{T},\ 0<\xi<1\right)$$

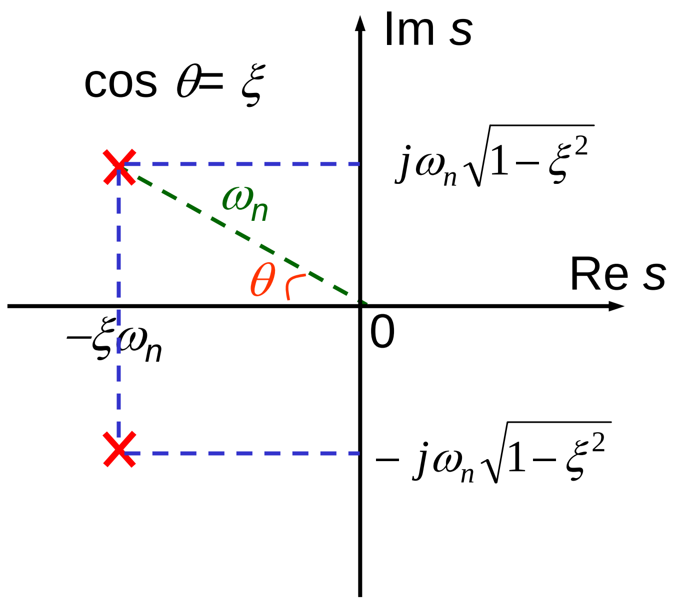
- The system has two complex conjugate poles:

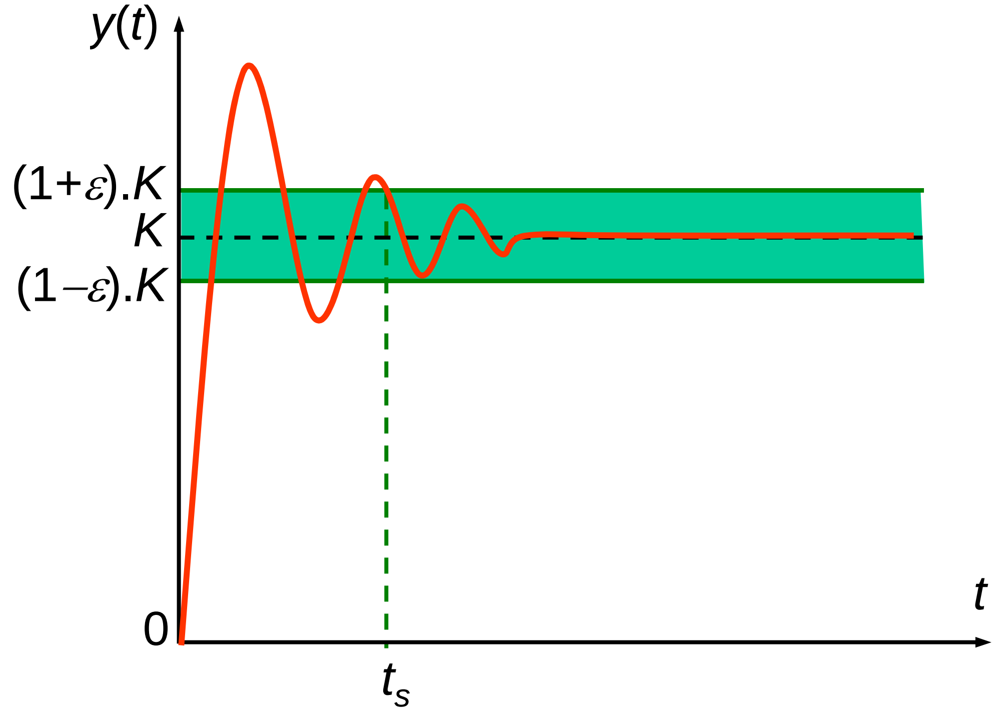
$$p_{1,2}=-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

- Transient response: $Y(s)=R(s)G(s)=\dfrac{1}{s}.\dfrac{K\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$$\Rightarrow \qquad y(t)=K\left\{1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t+\theta\right]\right\} \qquad (\cos\theta=\xi)$$

Second-order oscillating system (cont')





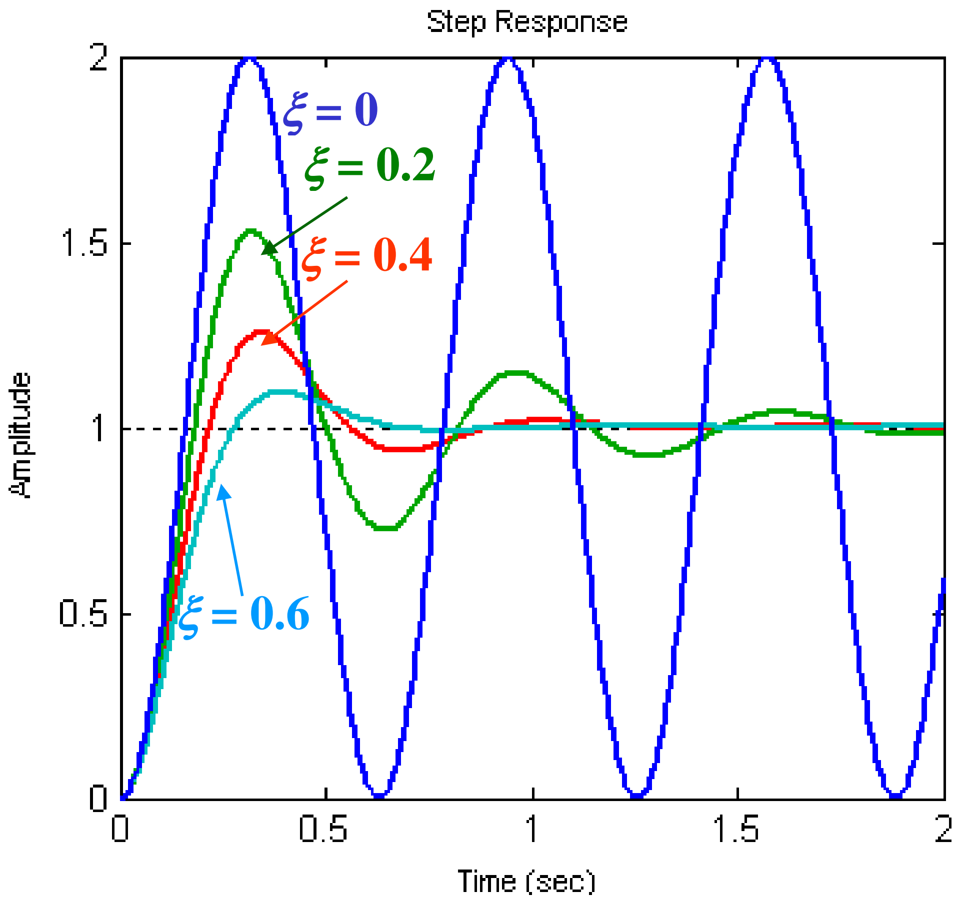
Pole – zero plot of a second order oscillating system

Transient response of a second order oscillating system

Second-order oscillating system – Remark

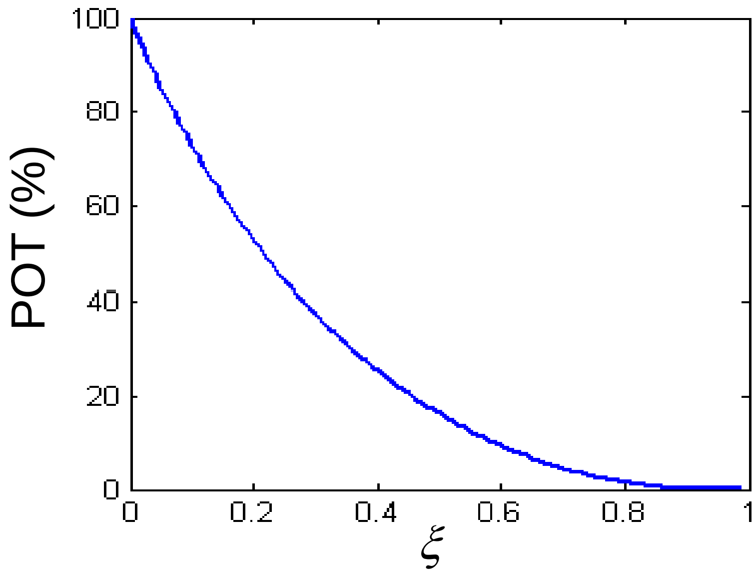
- A second order oscillation system has two conjugated complex poles, its transient response is a oscillation signal.

  - If $\xi = 0$, transient response is a stable oscillation signal at the frequency $\omega_n \Rightarrow \omega_n$ is called natural oscillation frequency.
  - If $0<\xi<1$, transient response is a decaying oscillation signal $\Rightarrow \xi$ is called damping constant, the larger the value ξ, (the closer the poles are to the real axis) the faster the response decays.



- Transient response of the second order oscillating system has overshoot.

The percentage of overshoot: $$POT = \exp\left(-\frac{\xi\pi}{\sqrt{1-\xi^2}}\right).100\%$$



The relationship between *POT* and ξ

- The larger the value ξ, (the closer the poles are to the real axis) the smaller the POT.
- The smaller the value ξ, (the closer the poles are to the imaginary axis) the larger the POT

Second-order oscillating system – Settling time

- Settling time:

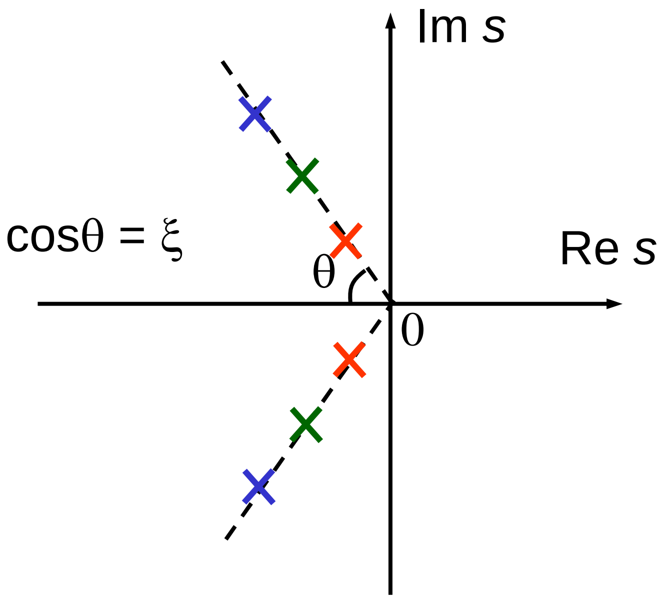
  5% criterion: $$t_s = \frac{3}{\xi\omega_n}$$

  2% criterion: $$t_s = \frac{4}{\xi\omega_n}$$

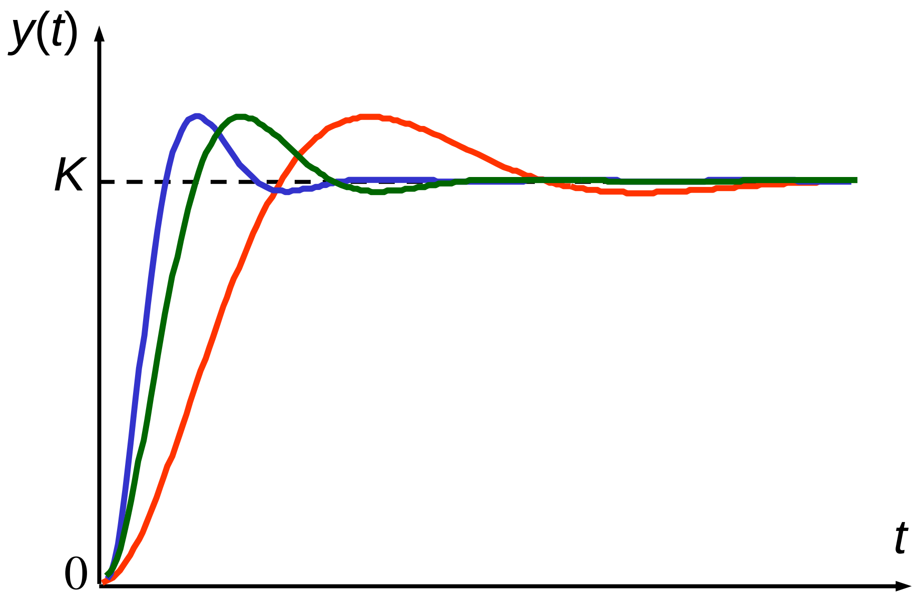
Second-order oscillating system

Relationship between pole location and transient response

- The 2nd order systems that have the poles located in the same rays starting from the origin have the same damping constant, then the percentage of overshoots are the same. The further the poles from the origin, the shorter the settling time.



Pole – zero plot of a second order oscillating system

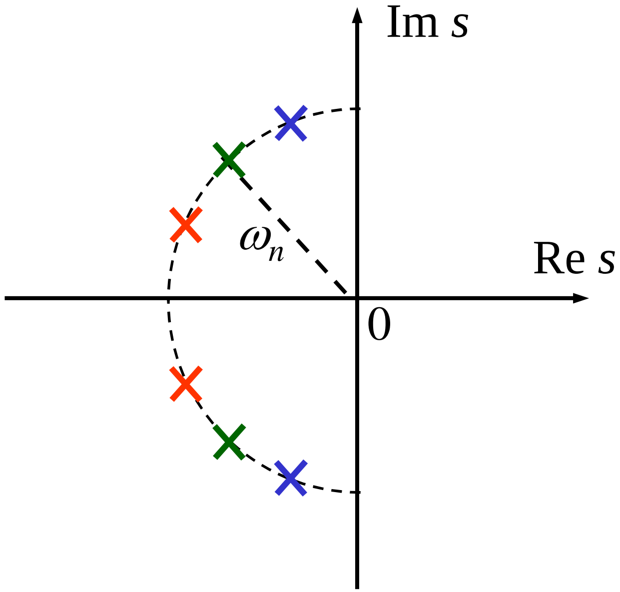


Transient response of a second order oscillating system

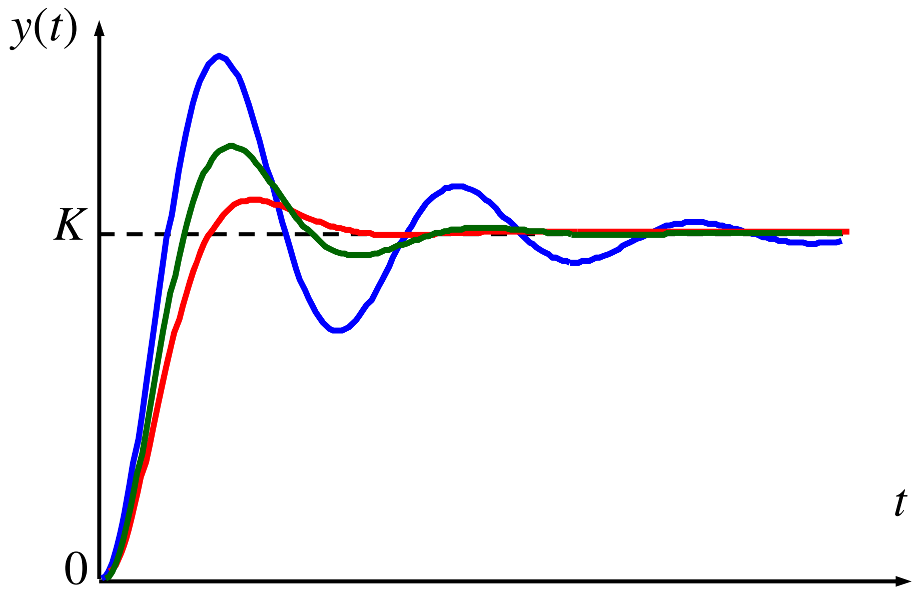
Second-order oscillating system

Relationship between pole location and transient response (cont')

- The 2nd order systems that have the poles located in the same distance from the origin have the same natural oscillation frequency. The closer the poles to the imaginary axis, the smaller the damping constant, then the higher the POT.



Pole – zero plot of a second order oscillating system

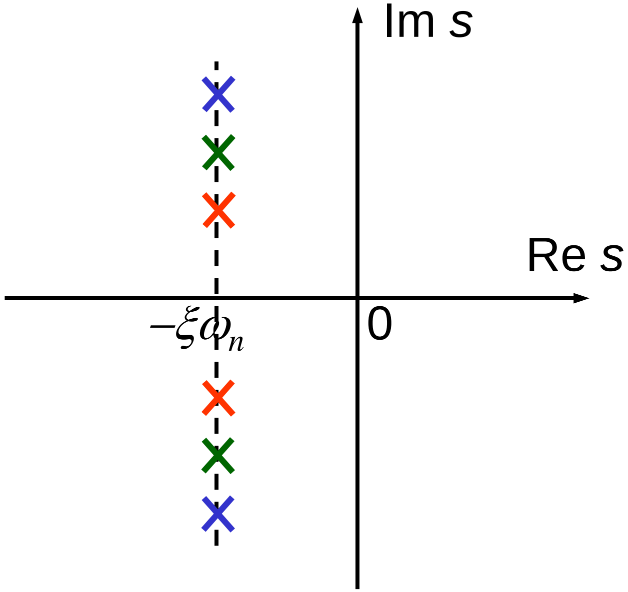


Transient response of a second order oscillating system

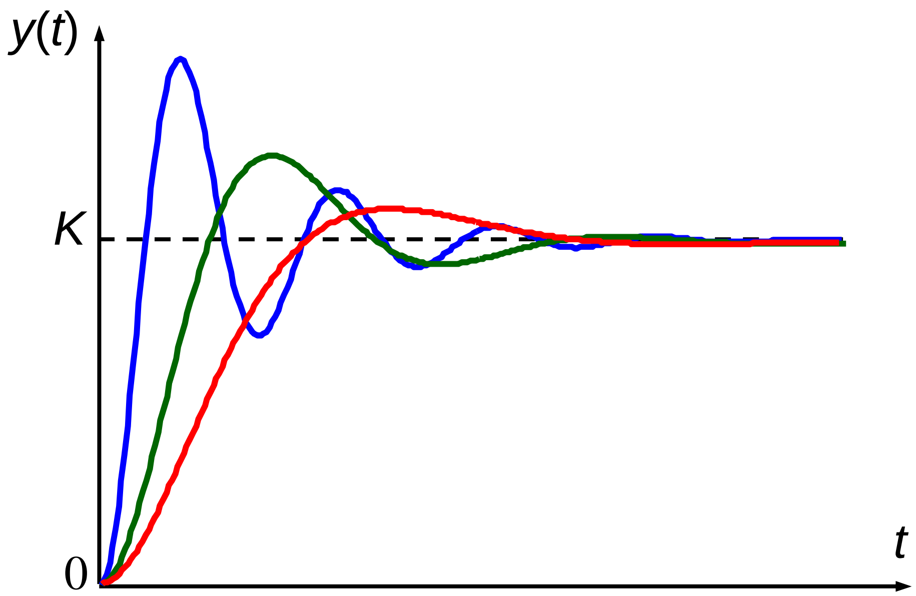
Second-order oscillating system

Relationship between pole location and transient response (cont')

- The 2nd order systems that have the poles located in the same distance from the imaginary axis have the same $\xi\omega_n$, then the settling time are the same. The further the poles from the real axis, the smaller the damping constant, then the higher the POT



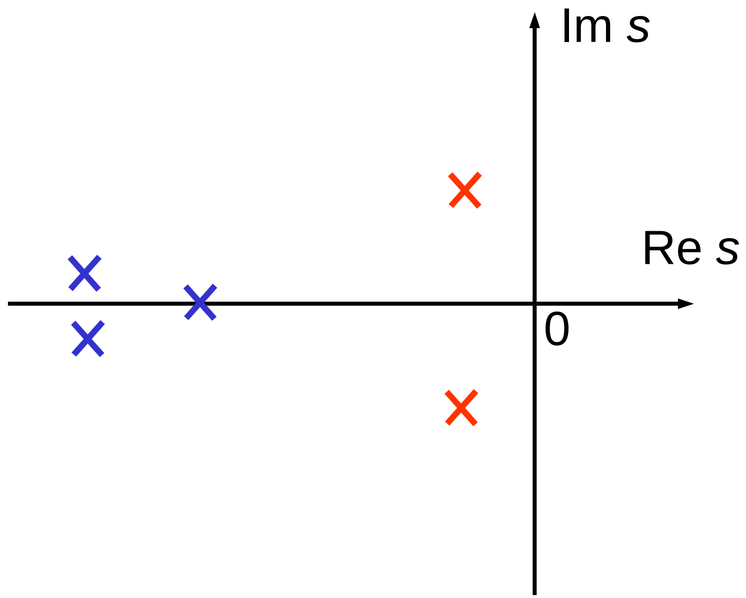
Pole – zero plot of a second order oscillating system



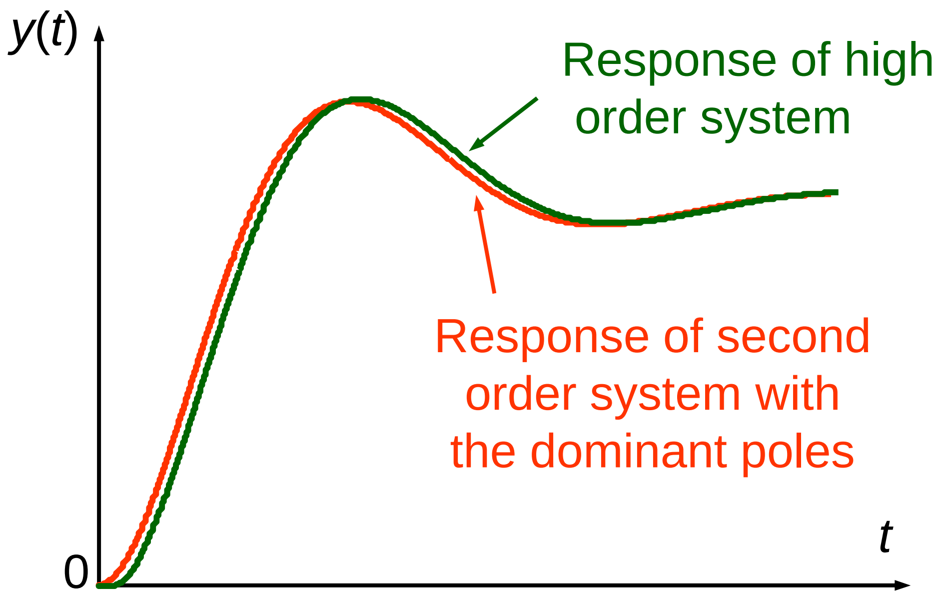
Transient response of a second order oscillating system

Transient response of high order system

- High-order systems are the system that have more than 2 poles
- If a high order system have a pair of poles located closer to the imaginary axis than the others then the high order system can be approximated to a second order system. The pair of poles nearest to the imaginary axis are called the dominant poles.



High order systems have more than 2 poles



A high order system can be approximated by a 2nd order system

Performance indices

Integral performance indices

* **IAE criterion**

  (Integral of the Absolute Magnitude of the Error **)**

$$J_{IAE} = \int_0^{+\infty} |e(t)| dt$$

* **ISE criterion**

  (Integral of the Square of the Error)

$$J_{ISE} = \int_0^{+\infty} e^2(t) dt$$

* **ITAE criterion**

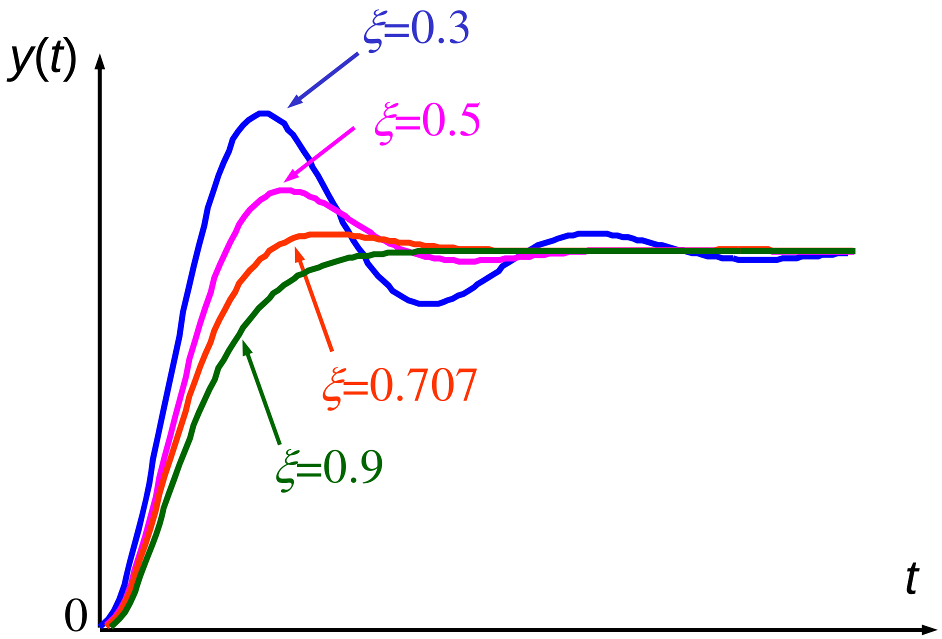
  (Integral of Time multiplied by the Absolute Value of the Error)

$$J_{ITAE} = \int_0^{+\infty} t|e(t)| dt$$

Optimal systems

- A control system is optimal when the selected performance index is minimized
- Second order system:

$$J_{IAE} \to \min \quad \text{when} \quad \xi \to 0.707$$
$$J_{ISE} \to \min \quad \text{when} \quad \xi \to 0.5$$
$$J_{ITAE} \to \min \quad \text{when} \quad \xi \to 0.707$$



Transient response of second order systems

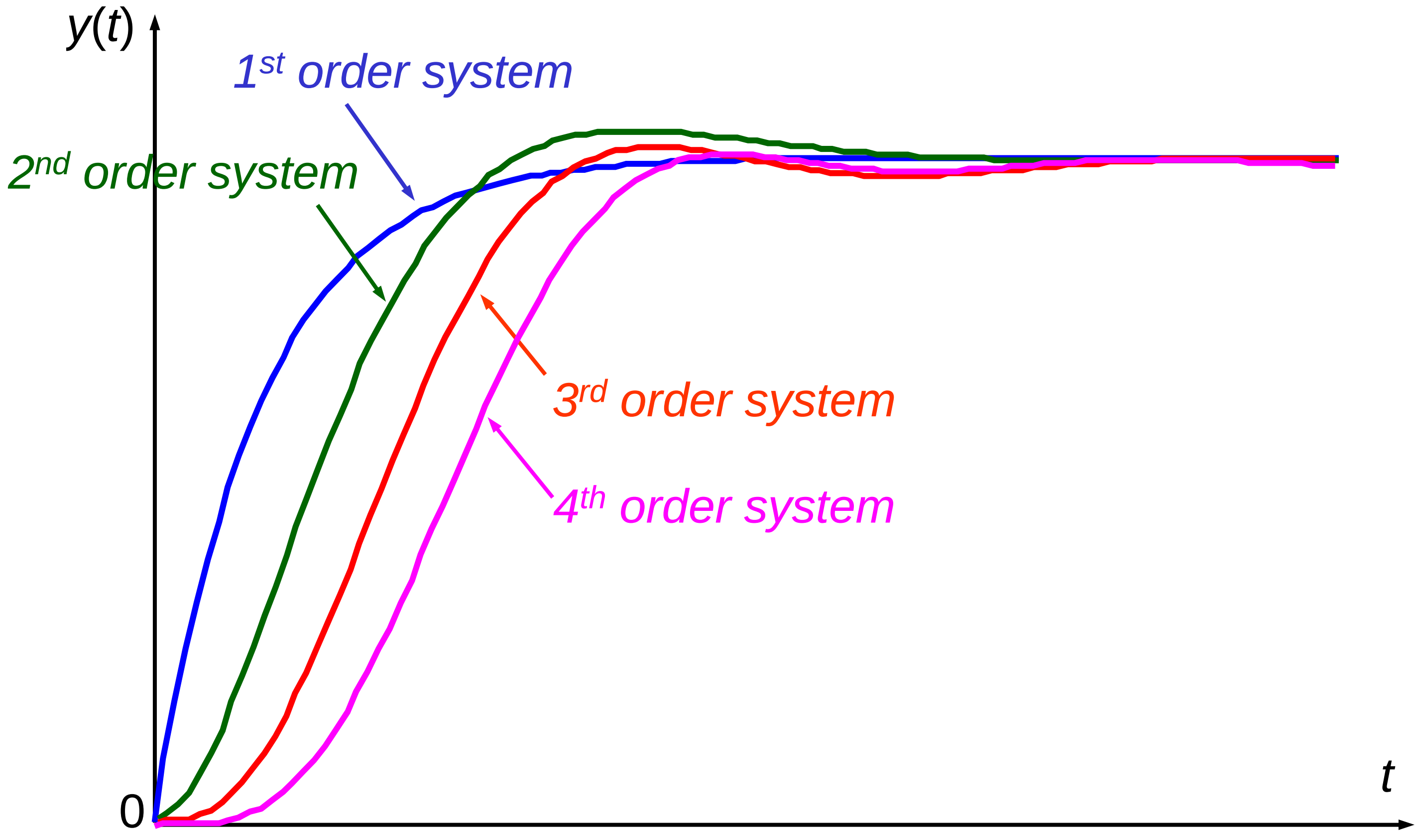
ITAE optimal control

- ITAE is usually used in design of control system
- An n-order system is optimal according to ITAE criterion if the denominator of its transfer function has the form:

Order	Denominator of transfer function
1	$s+\omega_n$
2	$s^2+1{,}414\omega_n s+\omega_n^2$
3	$s^3+1{,}75\omega_n s^2+2{,}15\omega_n^2 s+\omega_n^3$
4	$s^4+2{,}1\omega_n s^3+3{,}4\omega_n^2 s^2+2{,}7\omega_n^3 s+\omega_n^4$

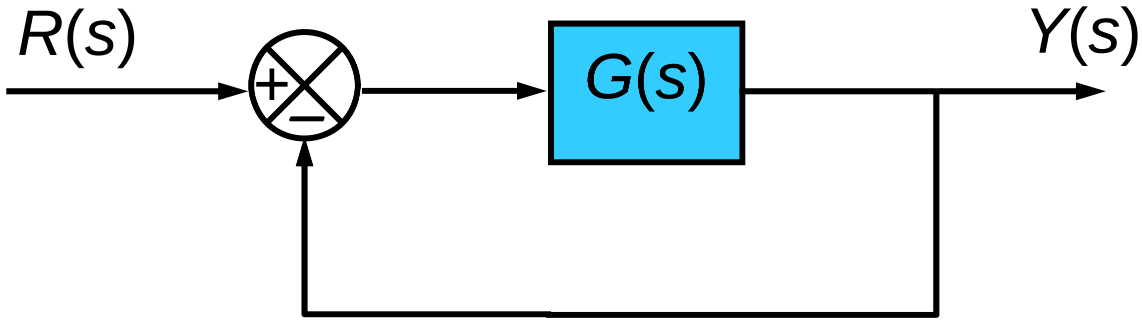
ITAE optimal control (cont')

- Optimal response according to ITAE criterion



Relationship between frequency domain performances & time domain performances

Relationship between frequency response & steady state error

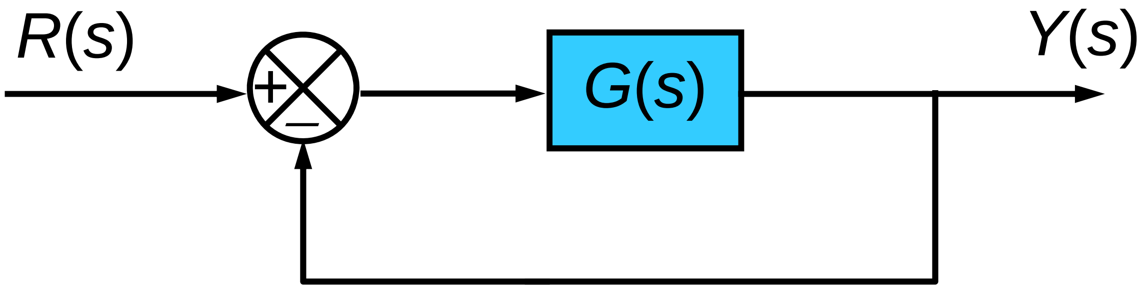


$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{\omega\to 0} G(j\omega)H(j\omega)$$

$$K_v = \lim_{s\to 0} s\,G(s)H(s) = \lim_{\omega\to 0} j\omega G(j\omega)H(j\omega)$$

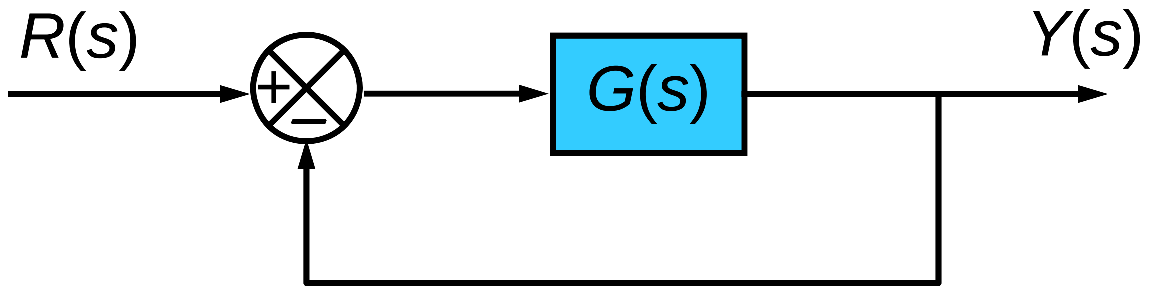
$$K_a = \lim_{s\to 0} s^2\,G(s)H(s) = \lim_{\omega\to 0} (j\omega)^2 G(j\omega)H(j\omega)$$

Relationship between frequency response & steady state error



- Steady state error of the closed-loop system depends on the magnitude response of the open-loop system at low frequencies but not at high frequencies.
- The higher the magnitude response of the open-loop system at low frequencies, the smaller the steady-state error of the closed-loop system.
- In particular, if the magnitude response of the open-loop system is infinity as frequency approaching zero, then the steady-state error of the closed-loop system to step input is zero.

Relationship between frequency response & transient response



✶ In the frequency range $\omega < \omega_c$, because $|G(j\omega)| > 1$ then:

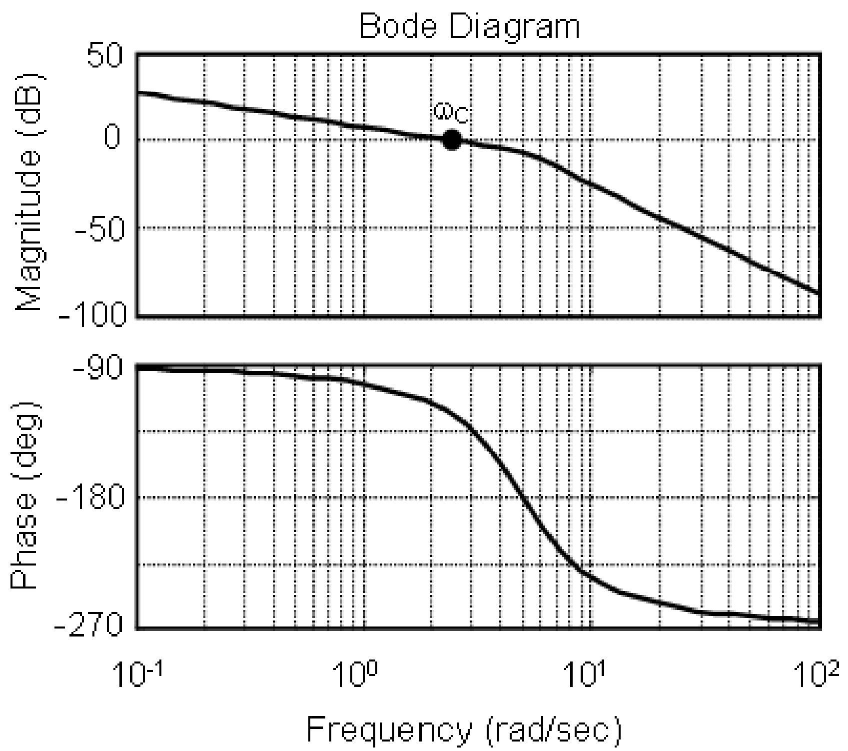
$$|G_{cl}(j\omega)| = \frac{|G(j\omega)|}{|1+G(j\omega)|} \approx \frac{|G(j\omega)|}{|G(j\omega)|} = 1$$

✶ In the frequency range $\omega > \omega_c$, because $|G(j\omega)| < 1$ then:

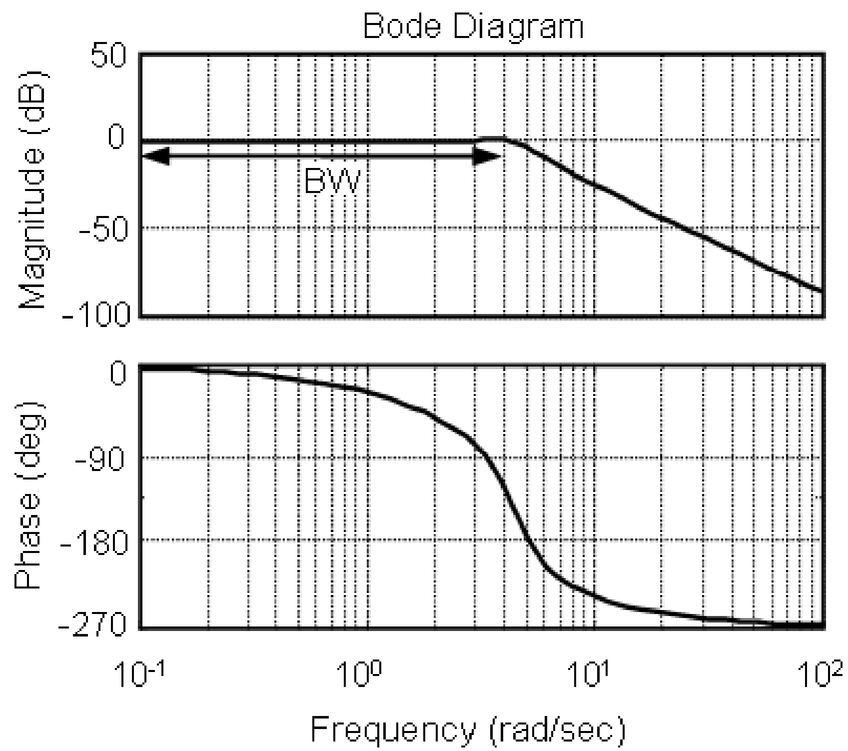
$$|G_{cl}(j\omega)| = \frac{|G(j\omega)|}{|1+G(j\omega)|} \approx \frac{|G(j\omega)|}{1} = |G(j\omega)|$$

$\Rightarrow$ Bandwidth of the closed-loop system is approximate the gain crossover frequency of the open-loop system.

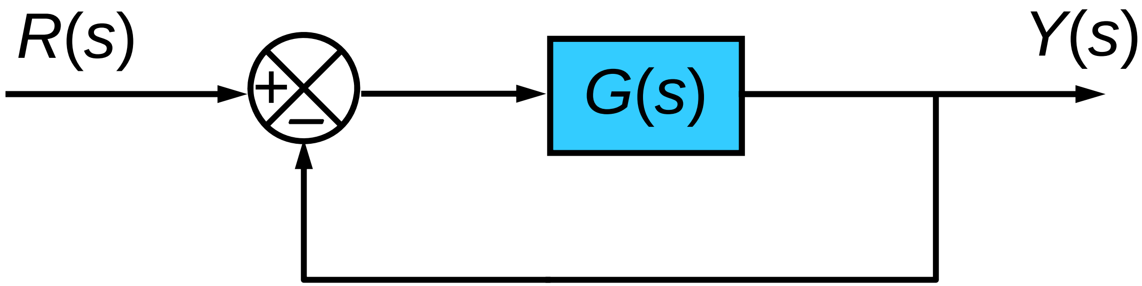
Relationship between frequency response & transient response



Bode plot of a open-loop system



Bode plot of the corresponding closed-loop system

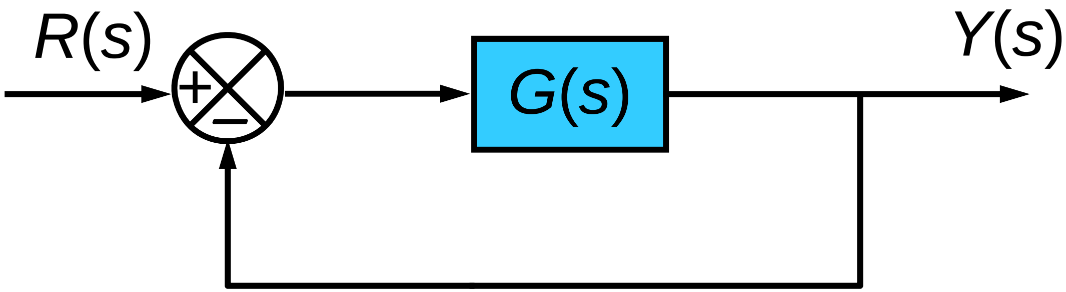


- The higher the gain crossover frequency of open-loop system, the wider the bandwidth of closed-loop system ⇒ the faster the response of close-loop system, the shorter the settling time.

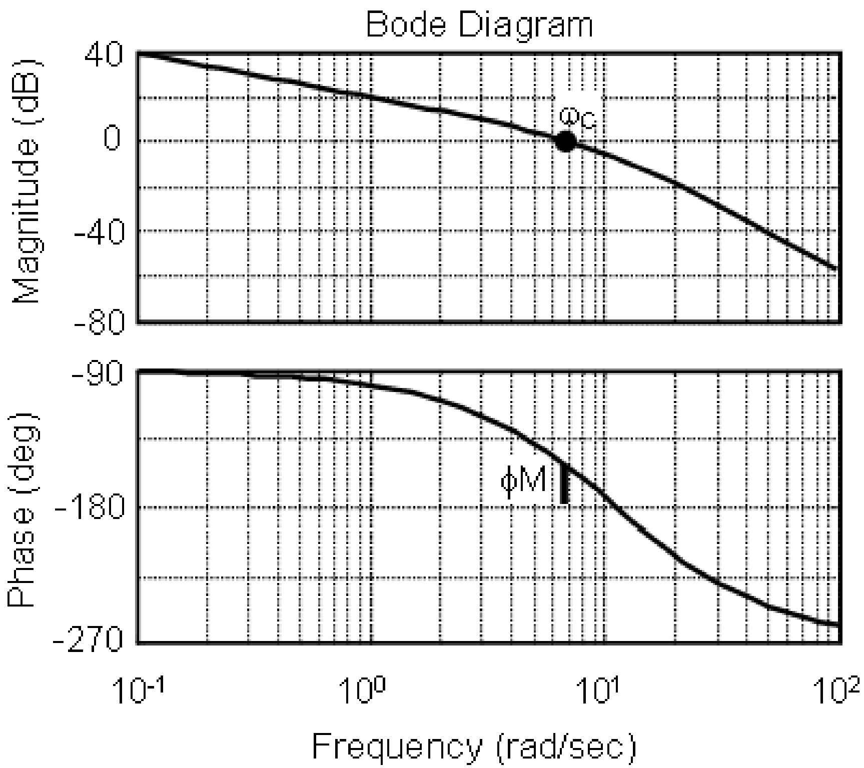
$$\frac{\pi}{\omega_c} < t_{qd} < \frac{4\pi}{\omega_c}$$

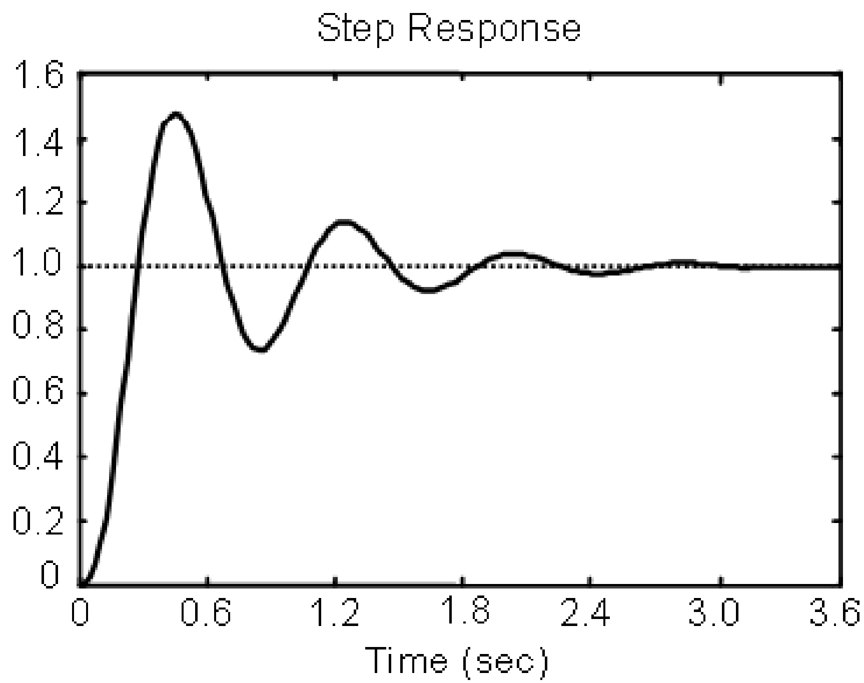
- The higher the phase margin of the open-loop system, the smaller the POT of closed-loop system. In most of the cases, if the phase margin of the open-loop system is larger than 60^0 then the POT of the closed-loop system is smaller than 10%.

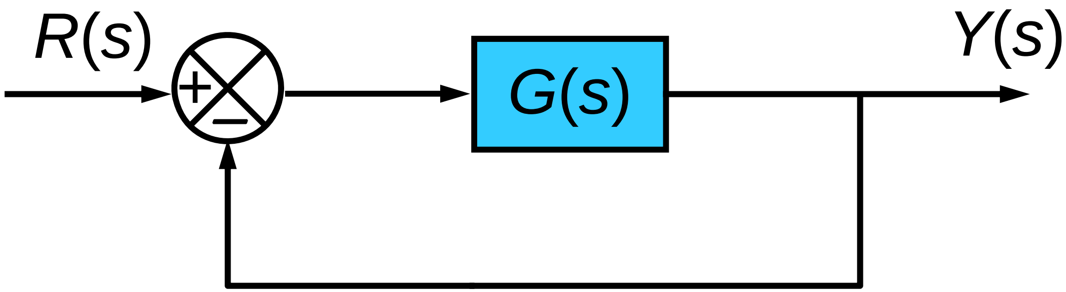
Ex: relationship between gain crossover frequency & settling time



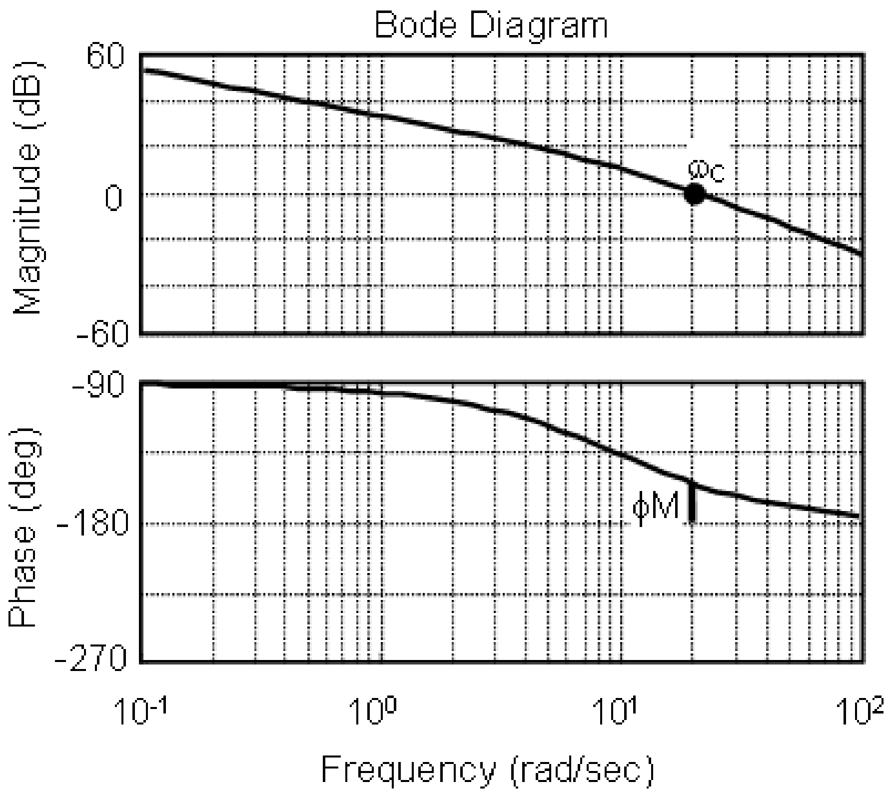
$$G(s)=\frac{10}{s(0.1s+1)(0.08s+1)}$$

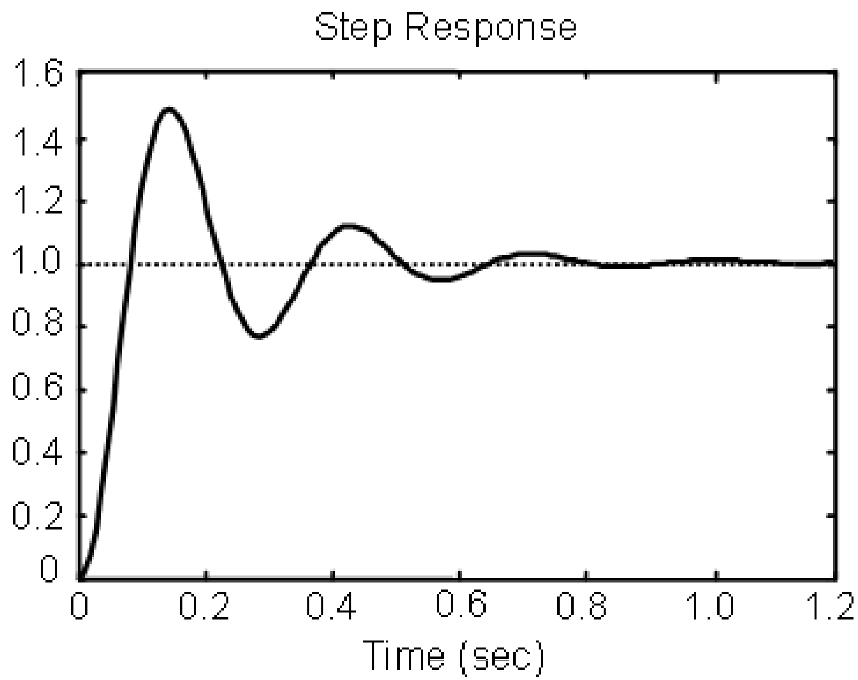




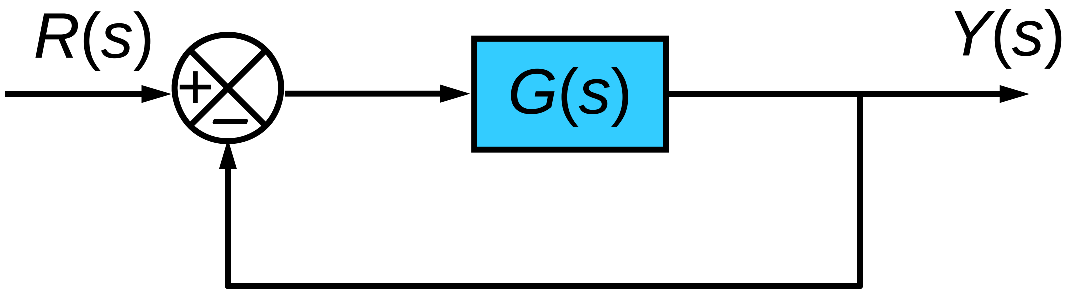


$$G(s) = \frac{50}{s(0.1s+1)}$$

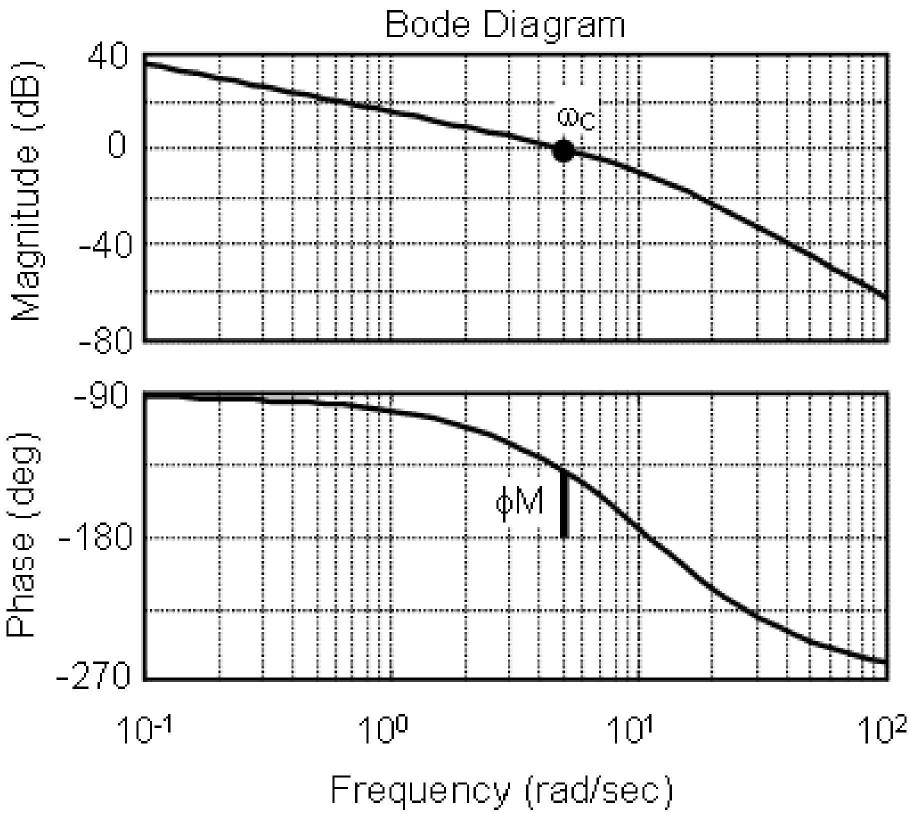


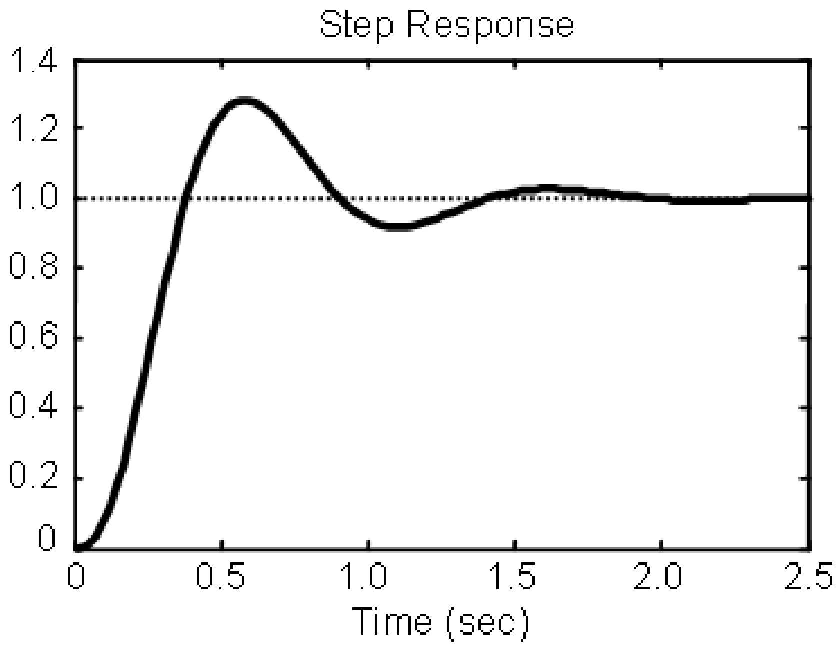


Example of relationship between phase margin and POT

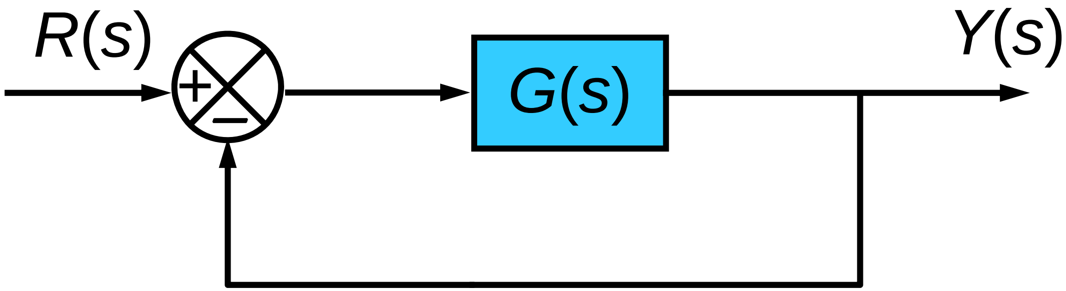


$$G(s) = \frac{6}{s(0.1s+1)(0.08s+1)}$$

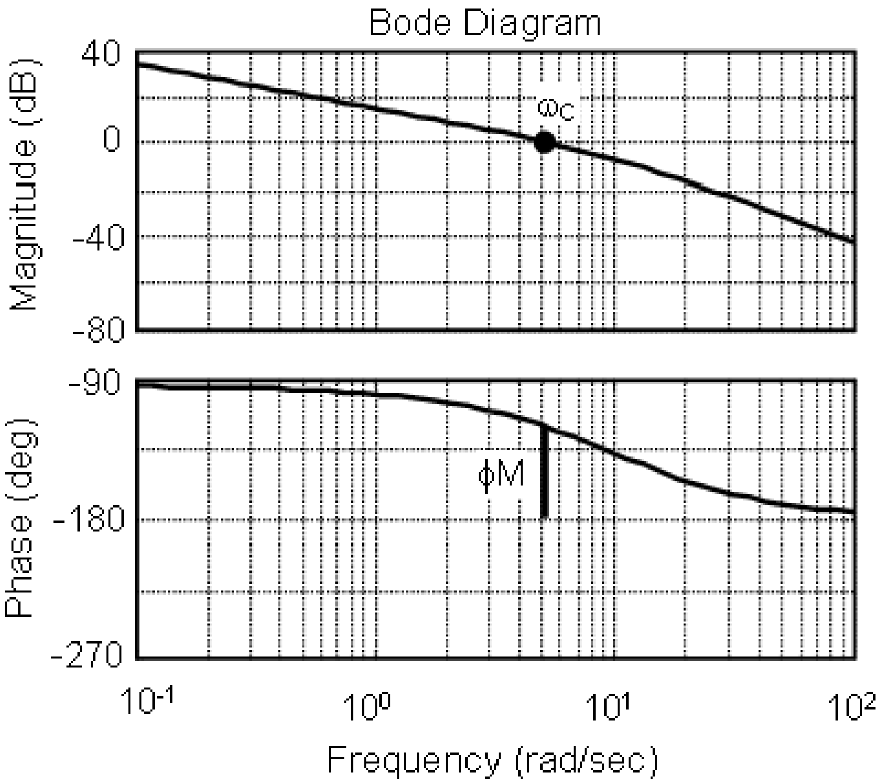


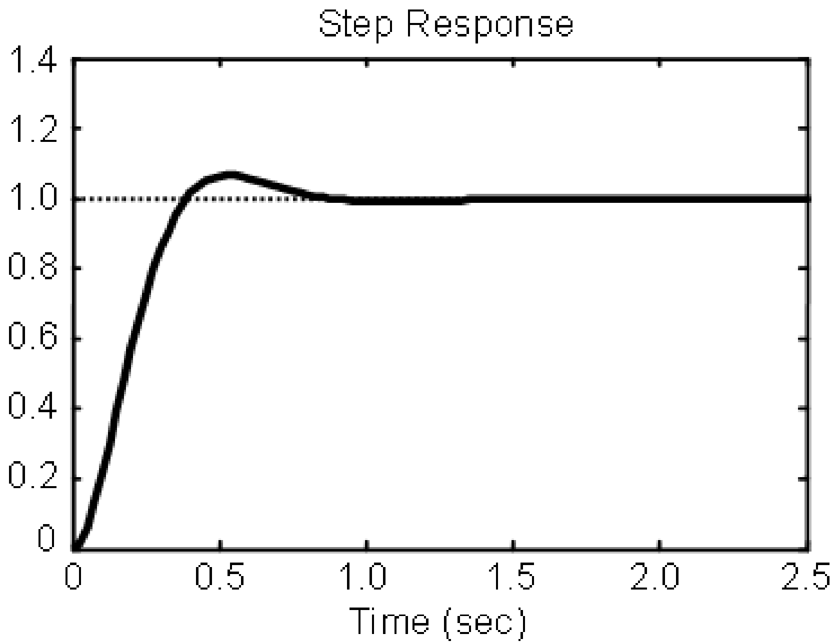


Ex of relationship between phase margin and POT (cont')



$$G(s) = \frac{6}{s(0.1s+1)}$$





End of Chapter 4