Lecture Notes

Fundamentals of Control Systems

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Chapter 6

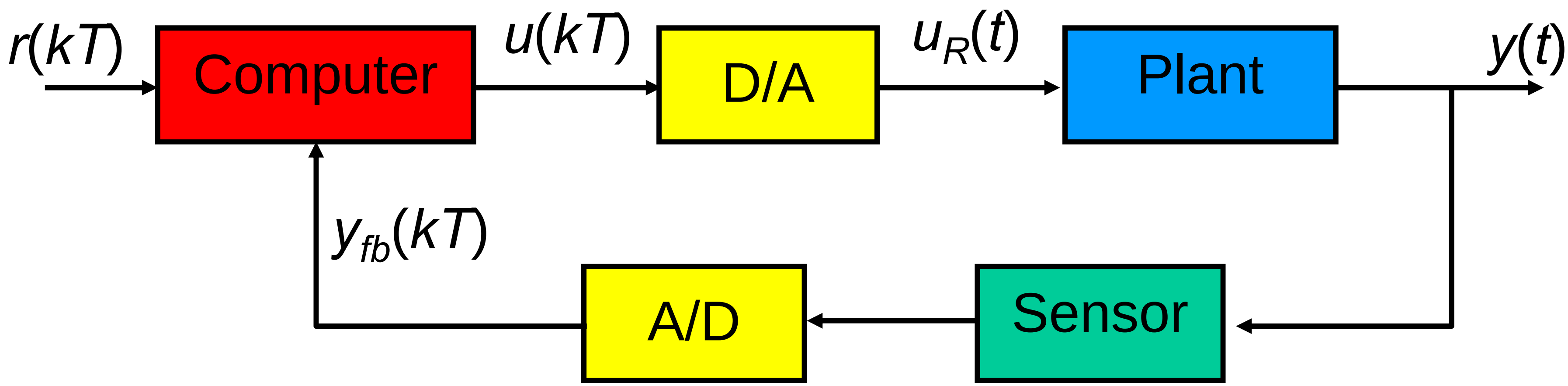
DISCRETE TIME CONTROL SYSTEMS

Content

- Introduction to discrete-time system
- Mathematical model of discrete time system
  - Transfer function
  - State-space equation
- Analyze the stability of discrete time system
  - Stability condition
  - Jury criterion
- Discrete PID controller
  - Transfer function of discrete PID controller
  - Manual tuning of PID controller
  - Implementation of dicrete PID controller

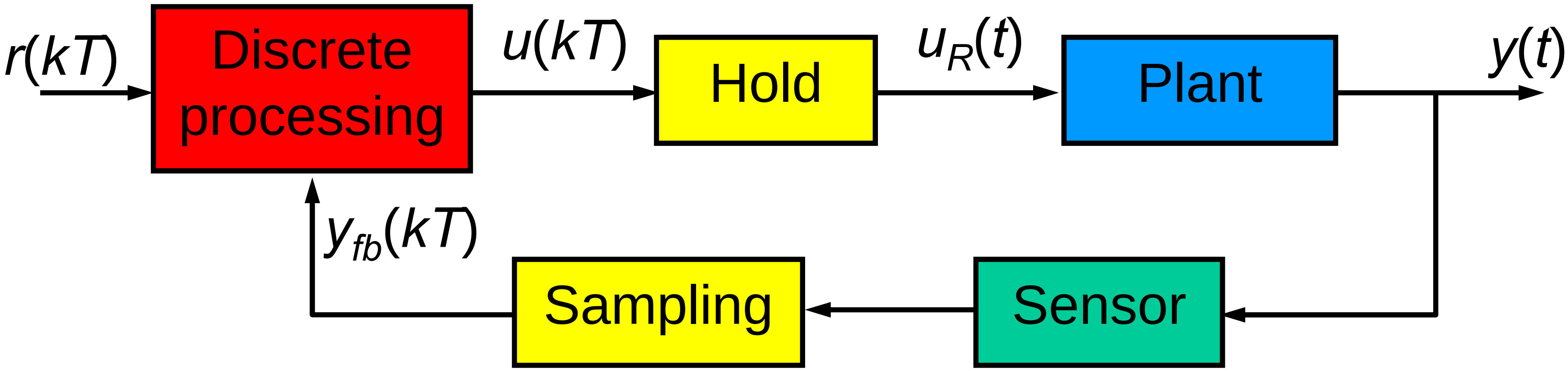
Introduction to discrete-time systems

Digital control systems



- "Computer" = computational equipments based on microprocessor technology (microprocessor, microcontroller, PC, DSP,…).
- Advantages of digital control system:
  - Flexibility
  - Easy to implement complex control algorithms
  - Computer can control many plants at the same time.

Discrete control systems



- ✶ Discrete control systems are control systems which have signals at several points being discrete signal.
- ✶ Note: Discrete time control systems are ideal model of real digital control systems.

Sampling

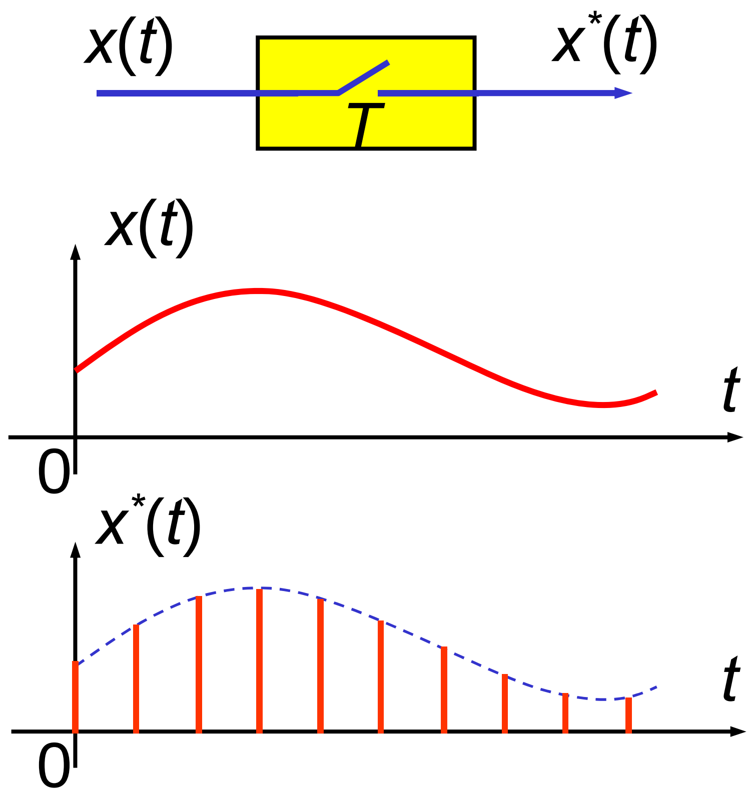
- Sampling is the reduction of a continuous signal to a discrete signal.

- Mathematical expression describing the sampling process:

$$X^*(s) = \sum_{k=0}^{+\infty} x(kT)e^{-kTs}$$

- Shannon's Theorem:

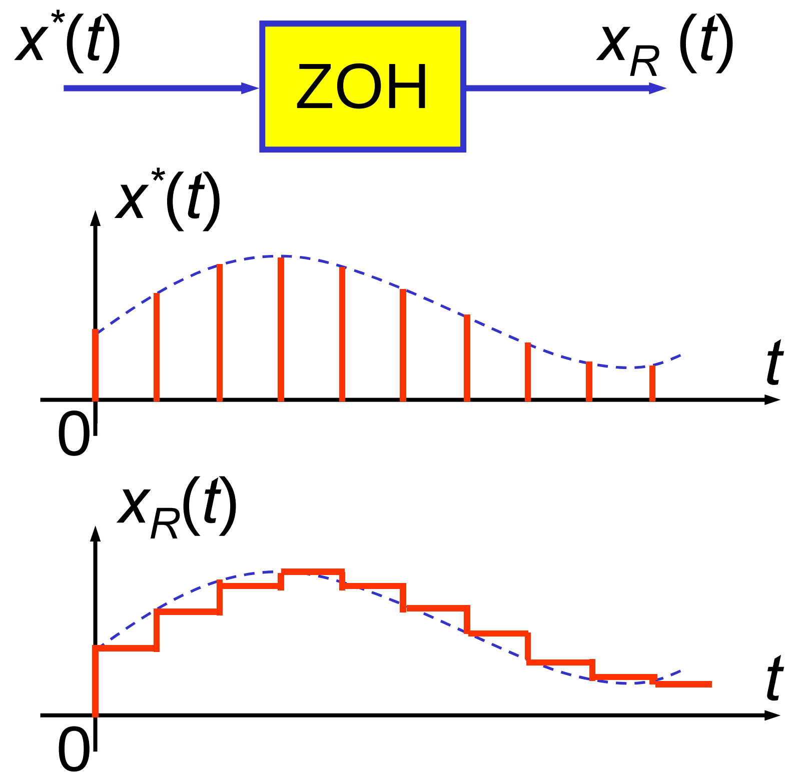
$$f = \frac{1}{T} \geq 2f_c$$



- If quantization error is negligible, then A/D converters are approximate the ideal samplers.

Sampled-data hold

- Sampled-data hold is the reconstruction of discrete signal to a continuous signal.



- Zero-order hold (ZOH): keep signal unchanged between two consecutive sampling instants.

- Transfer function of the ZOH.

$$G_{ZOH}(s) = \frac{1 - e^{-Ts}}{s}$$

- If quantization error is negligible, then D/A converters are approximate the zero-order hold.

Mathematical model
of discrete-time control systems

A brief review of the Z-transform

- Consider $x(k)$, k=0,1,2,… being a discrete signal. The Z-transform of $x(k)$ is defined as:

$$X(z) = \mathcal{Z}\{x(k)\} = \sum_{k=-\infty}^{+\infty} x(k) z^{-k}$$

where:

– $z = e^{Ts}$ (s is the Laplace variable, T is the sampling period)

– $X(z)$: Z-transform of $x(k)$.

Notation: $x(k) \xleftrightarrow{\mathcal{Z}} X(z)$

- If $x(k) = 0$, $\forall\ k < 0$ then

$$X(z) = \mathcal{Z}\{x(k)\} = \sum_{k=0}^{+\infty} x(k) z^{-k}$$

- Region Of Convergence (ROC): set of z such that $X(z)$ is finite.

An interpretation of the Z-transform

- Suppose $x(t)$ being a continuous signal, sample $x(t)$ at the sampling periode T, we have a discrete signal $x(k) = x(kT)$.
- The mathematic model of the process of sampling $x(t)$

$$X^*(s) = \sum_{k=0}^{+\infty} x(kT)e^{-kTs} \qquad (1)$$

- The Z-transform of the sequence $x(k) = x(kT)$.

$$X(z) = \sum_{k=0}^{+\infty} x(k)z^{-k} \qquad (2)$$

- Due to $z = e^{Ts}$, the right hand-side of the expression (1) and (2) are identical. So performing Z-transform of a signal is equivalent to discretizing this signal.

Properties of the Z-transform

Given $x(k)$ and $y(k)$ being two sequences which have the Z-transforms:

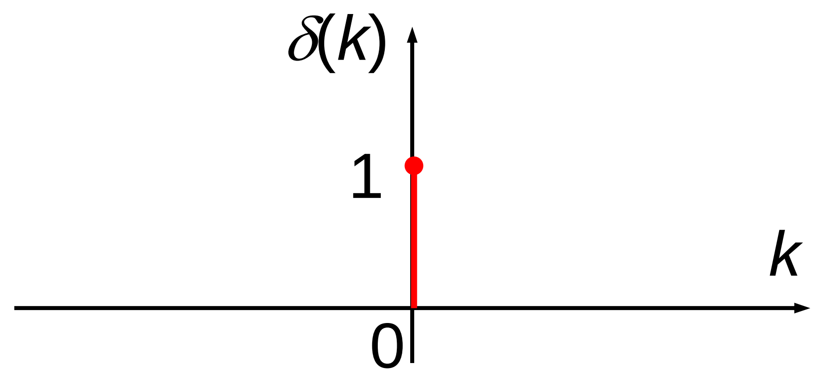
$$\mathcal{Z}\{x(k)\} = X(z) \qquad \mathcal{Z}\{y(k)\} = Y(z)$$

- Linearity: $$\mathcal{Z}\{ax(k) + by(k)\} = aX(z) + bY(z)$$
- Time shifting: $$\mathcal{Z}\{x(k - k_0)\} = z^{-k_0} X(z)$$
- Scale in Z-domain: $$\mathcal{Z}\{a^k x(k)\} = X(a^{-1} z)$$
- Derivative in Z-domain: $$\mathcal{Z}\{kx(k)\} = -z \frac{dX(z)}{dz}$$
- Initial-value theorem: $$x(0) = \lim_{z \to \infty} X(z)$$
- Final-value theorem: $$x(\infty) = \lim_{z \to 1} (1 - z^{-1}) X(z)$$

The Z-transform of basic discrete signals

✶ Dirac impulse:

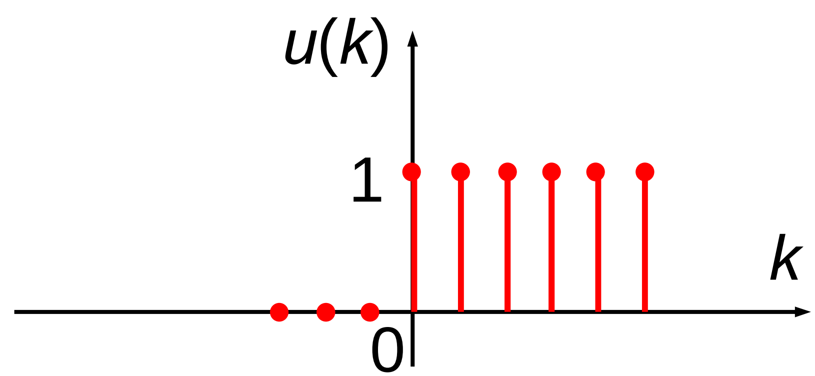
$$\delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$



$$\mathcal{Z}\{\delta(k)\} = 1$$

✶ Step function:

$$u(k) = \begin{cases} 1 & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$$



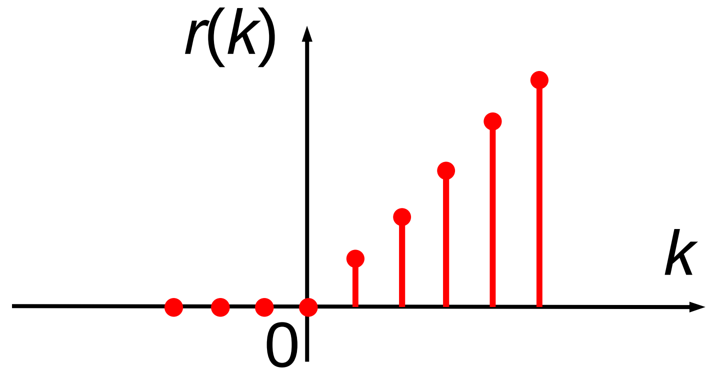
$$\mathcal{Z}\{u(k)\} = \frac{z}{z-1}$$

The Z-transform of basic discrete signals (cont’)

- ✶ Ramp function:

$$r(k)=\begin{cases} kT & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$$

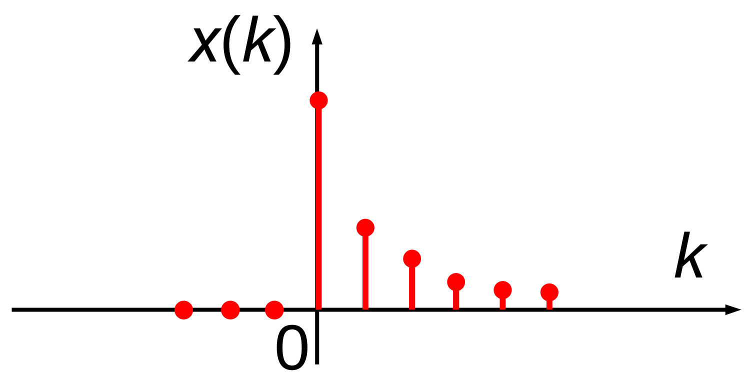
$$\mathcal{Z}\{u(k)\}=\frac{Tz}{(z-1)^2}$$



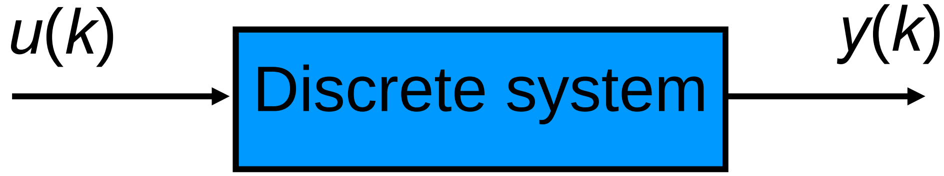
- ✶ Exponential function:

$$x(k)=\begin{cases} e^{-akT} & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$$

$$\mathcal{Z}\{x(k)\}=\frac{z}{z-e^{-aT}}$$



Transfer function of discrete time system



- The input-output relation ship of a discrete system can be described by the difference equation:

$$a_0 y(k+n) + a_1 y(k+n-1) + \ldots + a_{n-1} y(k+1) + a_n y(k) =$$
$$b_0 u(k+m) + b_1 u(k+m-1) + \ldots + b_{m-1} u(k+1) + b_m u(k)$$

where *n*>*m*, *n* is the order of the system.

- Taking the Z-transform the two sides of the above equation:

$$a_0 z^n Y(z) + a_1 z^{n-1} Y(z) + \ldots + a_{n-1} z Y(z) + a_n Y(z) =$$
$$b_0 z^m U(z) + b_1 z^{m-1} U(z) + \ldots + b_{m-1} z U(z) + b_m U(z)$$

Transfer function of discrete time system (cont.)

- Taking the ratio *Y*(*z*)/*U*(*z*) to obtain the transfer function:

$$G(z)=\frac{Y(z)}{U(z)}=\frac{b_0z^m+b_1z^{m-1}+\ldots+b_{m-1}z+b_m}{a_0z^n+a_1z^{n-1}+\ldots+a_{n-1}z+a_n}$$

- The above transfer function can be transformed into the equivalent form:

$$G(z)=\frac{Y(z)}{U(z)}=\frac{z^{-(n-m)}[b_0+b_1z^{-1}+\ldots+b_{m-1}z^{-m+1}+b_mz^{-m}]}{a_0+a_1z^{-1}+\ldots+a_{n-1}z^{-n+1}+a_nz^{-n}}$$

Transfer function of discrete system _ Example

- Consider a system described by the difference equation. Derive its transfer function:

$$y(k+3)+2y(k+2)-5y(k+1)+3y(k)=2u(k+2)+u(k)$$

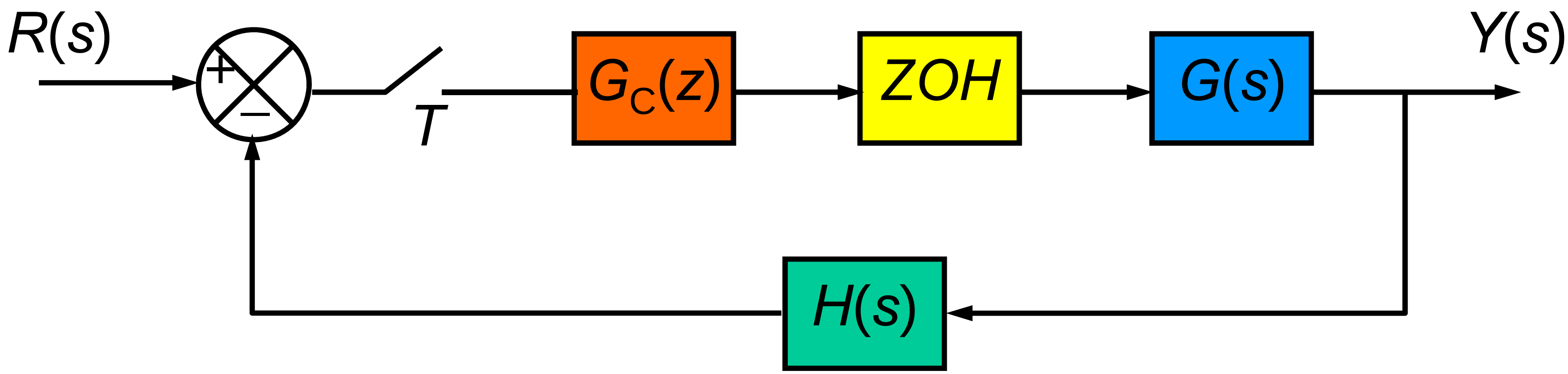
- Solution: Taking the Z-transform the difference equation:

$$z^3Y(z)+2z^2Y(z)-5zY(z)+3Y(z)=2z^2U(z)+U(z)$$

$$\Rightarrow \quad G(z)=\frac{Y(z)}{U(z)}=\frac{2z^2+1}{z^3+2z^2-5z+3}$$

$$\Leftrightarrow \quad G(z)=\frac{Y(z)}{U(z)}=\frac{z^{-1}(2+z^{-2})}{1+2z^{-1}-5z^{-2}+3z^{-3}}$$

Calculate transfer function from block diagram



✶ The closed-loop TF:

$$G_k(z) = \frac{Y(z)}{R(z)} = \frac{G_C(z)G(z)}{1 + G_C(z)GH(z)}$$

where

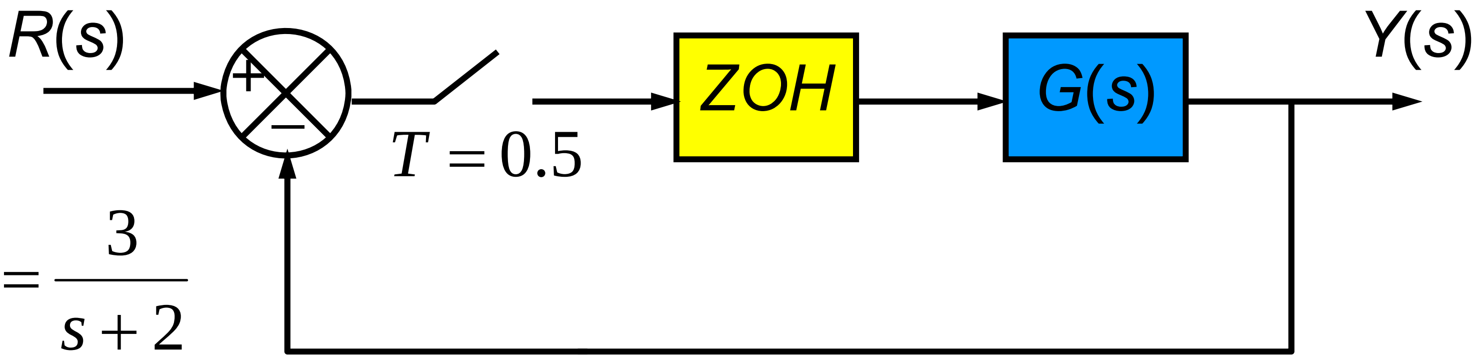
$G_C(z)$: TF of the controller, derive from difference equation

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

$$GH(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)H(s)}{s}\right\}$$

Calculate TF from block diagram – Example 1

- Find the closed-loop transfer function of the system:



$$G(s) = \frac{3}{s+2}$$

Solution: $G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = (1 - z^{-1})\mathcal{Z}\left\{\frac{3}{s(s+2)}\right\}$

$$= (1 - z^{-1})\frac{3}{2}\frac{z(1 - e^{-2\times 0.5})}{(z-1)(z - e^{-2\times 0.5})}$$

$$\Rightarrow \quad \boxed{G(z) = \frac{0.948}{z - 0.368}}$$

Calculate transfer function from block diagram – Example 1 (cont')

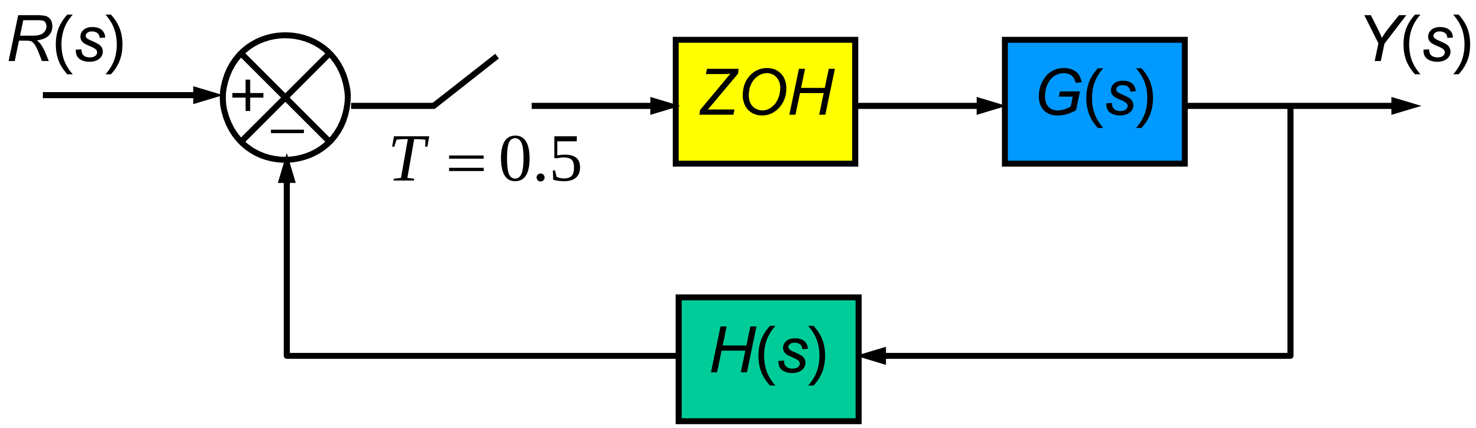
- The closed-loop transfer function:

$$G_k(z) = \frac{G(z)}{1+G(z)} = \frac{\dfrac{0.948}{z-0.368}}{1+\dfrac{0.948}{z-0.368}}$$

$$\Rightarrow \quad \boxed{G_k(z) = \frac{0.948}{z+0.580}}$$

Calculate TF from block diagram – Example 2

- Calculate the transfer function of the system:



Given that $G(s) = \dfrac{3e^{-s}}{s+3}$ $H(s) = \dfrac{1}{s+1}$

- Solution:

The closed-loop transfer function:

$$G_k(z) = \frac{G(z)}{1 + GH(z)}$$

Calculate TF from block diagram – Example 2 (cont')

- $$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

$$= (1 - z^{-1})\mathcal{Z}\left\{\frac{3e^{-s}}{s(s+3)}\right\}$$

$$= (1 - z^{-1})z^{-2}\frac{z(1 - e^{-3\times 0.5})}{(z-1)(z - e^{-3\times 0.5})}$$

$$\Rightarrow \quad \boxed{G(z) = \frac{0.777}{z^2(z - 0.223)}}$$

Calculate TF from block diagram – Example 2 (cont')

- $$GH(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)H(s)}{s}\right\}$$

$$= (1 - z^{-1})\mathcal{Z}\left\{\frac{3e^{-s}}{s(s+3)(s+1)}\right\}$$

$$= 3(1 - z^{-1})z^{-2}\frac{z(Az+B)}{(z-1)(z-e^{-3\times 0.5})(z-e^{-1\times 0.5})}$$

$$A = \frac{(1-e^{-3\times 0.5}) - 3(1-e^{-0.5})}{3(1-3)} = 0.0673$$

$$B = \frac{3e^{-3\times 0.5}(1-e^{-0.5}) - e^{-0.5}(1-e^{-3\times 0.5})}{3(1-3)} = 0.0346$$

$$\Rightarrow \quad GH(z) = \frac{0.202z + 0.104}{z^2(z-0.223)(z-0.607)}$$

Calculate TF from block diagram – Example 2 (cont')

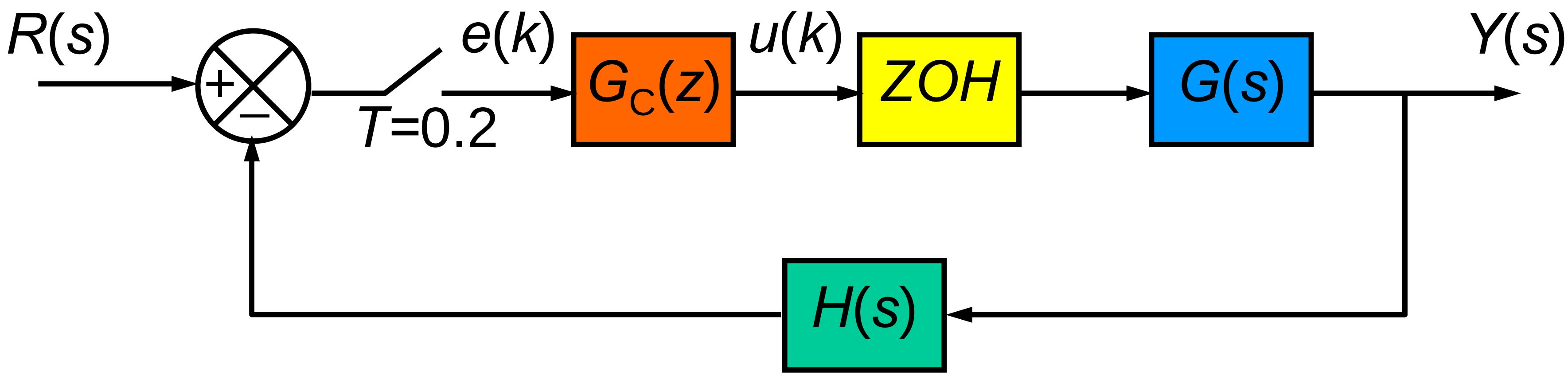
★ The closed-loop transfer function:

$$G_k(z) = \frac{G(z)}{1+GH(z)} = \frac{\dfrac{0.777}{z^2(z-0.223)}}{1+\dfrac{0.202z+0.104}{z^2(z-0.223)(z-0.607)}}$$

$$\Rightarrow \quad \boxed{G_k(z) = \frac{0.777(z-0.607)}{z^4-0.83z^3+0.135z^2+0.202z+0.104}}$$

Calculate TF from block diagram – Example 3

- Calculate the closed-loop transfer function of the system:



Given that: $$G(s)=\frac{5e^{-0.2s}}{s^2} \qquad H(s)=0.1$$

The controller is described by the difference equation:

$$u(k)=10e(k)-2e(k-1)$$

Calculate TF from block diagram – Example 3 (cont')

★ **Solution:**

The closed-loop transfer function:

$$G_k(z) = \frac{G_C(z)G(z)}{1 + G_C(z)GH(z)}$$

★ The TF of the controller is calculated from the difference equation:

$$u(k) = 10e(k) - 2e(k-1)$$

$$\Rightarrow \quad U(z) = 10E(z) - 2z^{-1}E(z)$$

$$\Rightarrow \quad G_C(z) = \frac{U(z)}{E(z)} = 10 - 2z^{-1}$$

Calculate TF from block diagram – Example 3 (cont')

- $$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

$$= (1 - z^{-1})\mathcal{Z}\left\{\frac{5e^{-0.2s}}{s^3}\right\} = 5(1 - z^{-1})z^{-1}\frac{(0.2)^2 z(z+1)}{2(z-1)^3}$$

$$\Rightarrow \quad G(z) = \frac{0.1(z+1)}{z(z-1)^2}$$

- $$GH(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)H(s)}{s}\right\}$$

$$= 0.1(1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

$$\Rightarrow \quad GH(z) = \frac{0.01(z+1)}{z(z-1)^2}$$

Calculate TF from block diagram – Example 3 (cont')

★ The closed-loop transfer function:

$$G_k(z)=\frac{G_C(z)G(z)}{1+G_C(z)GH(z)}=\frac{\left[\dfrac{10z-2}{z}\right].\left[\dfrac{0.1(z+1)}{z(z-1)^2}\right]}{1+\left[\dfrac{10z-2}{z}\right].\left[\dfrac{0.01(z+1)}{z(z-1)^2}\right]}$$

$$\Rightarrow \quad \boxed{G_k(z)=\frac{z^2+0.8z-0.2}{z^4-2z^3+1.1z^2+0.08z-0.02}}$$

State-space model of discrete system

State space equation of discrete system

- The state-space model of a discrete system is a set of first-order difference equations of the form:

$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d \boldsymbol{x}(k) + \boldsymbol{B}_d r(k) \\ y(k) = \boldsymbol{C}_d \boldsymbol{x}(k) \end{cases}$$

where:

$$\boldsymbol{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} \qquad \boldsymbol{A}_d = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \qquad \boldsymbol{B}_d = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\boldsymbol{C}_d = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}$$

Derive SS equation from difference equation

- ***Case 1:*** The right-hand side of the difference equation does not involve the differences of the input:

$$a_0 y(k+n) + a_1 y(k+n-1) + \ldots + a_{n-1} y(k+1) + a_n y(k) = b_0 u(k)$$

- Define the state variables:
  - The first state variable is the output of the system;
  - The *i*th state variable (i=2..n) is set to be one sample time-advanced of the (i–1)th state variable.

$$x_1(k) = y(k)$$
$$x_2(k) = x_1(k+1)$$
$$x_3(k) = x_2(k+1)$$
$$\vdots$$
$$x_n(k) = x_{n-1}(k+1)$$

Derive SS equation from difference equation

Case 1 (cont')

- The state equations: $$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d \boldsymbol{x}(k) + \boldsymbol{B}_d u(k) \\ y(k) = \boldsymbol{C}_d \boldsymbol{x}(k) \end{cases}$$

where:

$$\boldsymbol{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} \quad \boldsymbol{A}_d = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\dfrac{a_n}{a_0} & -\dfrac{a_{n-1}}{a_0} & -\dfrac{a_{n-2}}{a_0} & \dots & -\dfrac{a_1}{a_0} \end{bmatrix} \quad \boldsymbol{B}_d = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \dfrac{b_0}{a_0} \end{bmatrix}$$

$$\boldsymbol{C}_d = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

- Write the state equations of the system described by:

$$2y(k+3)+y(k+2)+5y(k+1)+4y(k)=3u(k)$$

- Define the state variables: $\begin{cases} x_1(k)=y(k) \\ x_2(k)=x_1(k+1) \\ x_3(k)=x_2(k+1) \end{cases}$

- The state equations: $\begin{cases} \boldsymbol{x}(k+1)=\boldsymbol{A}_d\boldsymbol{x}(k)+\boldsymbol{B}_d r(k) \\ y(k)=\boldsymbol{C}_d\boldsymbol{x}(k) \end{cases}$

where:

$$\boldsymbol{A}_d=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\dfrac{a_3}{a_0} & -\dfrac{a_2}{a_0} & -\dfrac{a_1}{a_0} \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -2.5 & -0.5 \end{bmatrix}$$

$$\boldsymbol{B}_d=\begin{bmatrix} 0 \\ 0 \\ \dfrac{b_0}{a_0} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 1.5 \end{bmatrix}$$

$$\boldsymbol{C}_d=\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Derive SS equation from difference equation

- ***Case 2:*** The right-hand side of the difference equation involve the differences of the input:

$$a_0 y(k+n) + a_1 y(k+n-1) + \ldots + a_{n-1} y(k+1) + a_n y(k) = b_0 u(k+n-1) + b_1 u(k+n-2) + \ldots + b_{n-2} u(k+1) + b_{n-1} u(k)$$

- Define the state variable:
  - The first state variable is the output of the system;
  - The i^{th} state variable (i=2..n) is set to be one sample time-advanced of the (i–1)th state variable minus a quantity proportional to the input

$$\begin{aligned}
x_1(k) &= y(k) \\
x_2(k) &= x_1(k+1) - \beta_1 u(k) \\
x_3(k) &= x_2(k+1) - \beta_2 u(k) \\
&\vdots \\
x_n(k) &= x_{n-1}(k+1) - \beta_{n-1} u(k)
\end{aligned}$$

Derive SS equation from difference equation

Case 2 (cont')

- The state equation: $\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d \boldsymbol{x}(k) + \boldsymbol{B}_d u(k) \\ y(k) = \boldsymbol{C}_d \boldsymbol{x}(k) \end{cases}$

where:

$$\boldsymbol{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} \quad \boldsymbol{A}_d = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\dfrac{a_n}{a_0} & -\dfrac{a_{n-1}}{a_0} & -\dfrac{a_{n-2}}{a_0} & \dots & -\dfrac{a_1}{a_0} \end{bmatrix} \quad \boldsymbol{B}_d = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix}$$

$$\boldsymbol{C}_d = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Derive SS equation from difference equation

Case 2 (cont')

The coefficient β_i in the vector $\boldsymbol{B}_d$ are defined as:

$$\beta_1 = \frac{b_0}{a_0}$$

$$\beta_2 = \frac{b_1 - a_1\beta_1}{a_0}$$

$$\beta_3 = \frac{b_2 - a_1\beta_2 - a_2\beta_1}{a_0}$$

$$\vdots$$

$$\beta_n = \frac{b_{n-1} - a_1\beta_{n-1} - a_2\beta_{n-2} - \ldots - a_{n-1}\beta_1}{a_0}$$

Derive SS equation from difference equation – Case 2 example

- Write the state equations of the system described by:

$$2y(k+3)+y(k+2)+5y(k+1)+4y(k)=u(k+2)+3u(k)$$

- Define the state variables: $\begin{cases} x_1(k)=y(k) \\ x_2(k)=x_1(k+1)-\beta_1 r(k) \\ x_3(k)=x_2(k+1)-\beta_2 r(k) \end{cases}$

- The state equations: $\begin{cases} \boldsymbol{x}(k+1)=\boldsymbol{A}_d\boldsymbol{x}(k)+\boldsymbol{B}_d u(k) \\ y(k)=\boldsymbol{C}_d\boldsymbol{x}(k) \end{cases}$

where:

$$\boldsymbol{A}_d=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\dfrac{a_3}{a_0} & -\dfrac{a_2}{a_0} & -\dfrac{a_1}{a_0} \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -2.5 & -0.5 \end{bmatrix} \qquad \boldsymbol{B}_d=\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

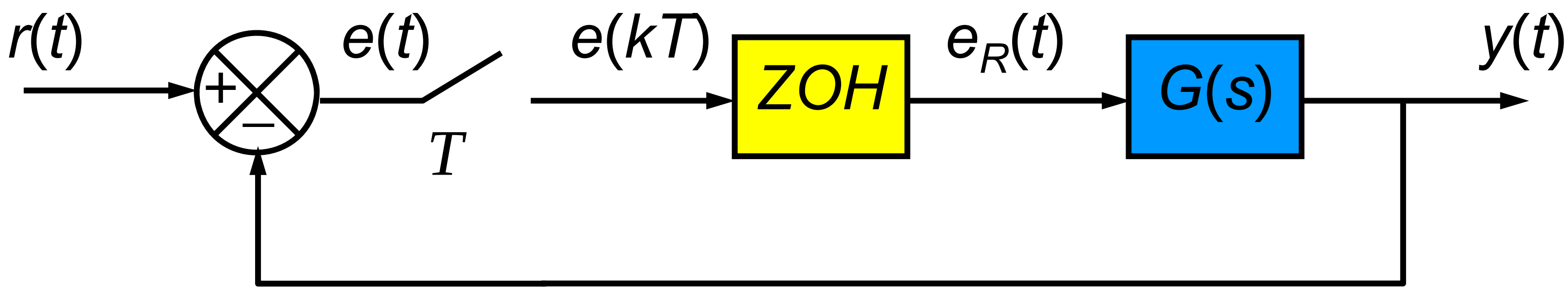
$$\boldsymbol{C}_d=\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

- The coefficient β_i in the vector $\boldsymbol{B}_d$ are calculated as:

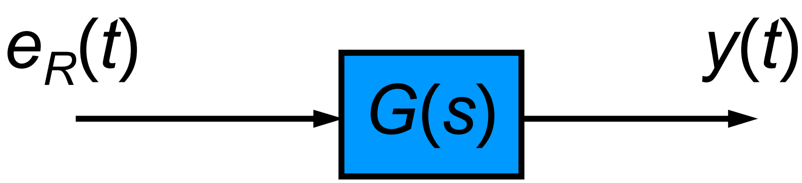
$$\begin{cases} \beta_1 = \dfrac{b_0}{a_0} = \dfrac{1}{2} = 0.5 \\ \beta_2 = \dfrac{b_1 - a_1\beta_1}{a_0} = \dfrac{0 - 1\times 0.5}{2} = -0.25 \\ \beta_3 = \dfrac{b_2 - a_1\beta_2 - a_2\beta_1}{a_0} = \dfrac{3 - 1\times(-0.25) - 5\times 0.5}{2} = 0.375 \end{cases}$$

$$\Rightarrow \quad \boldsymbol{B}_d = \begin{bmatrix} 0.5 \\ -0.25 \\ 0.375 \end{bmatrix}$$

Formulation of SS from block diagram



- ***Step 1:*** Write the state space equations of the open-loop continuous system:



$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}e_R(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

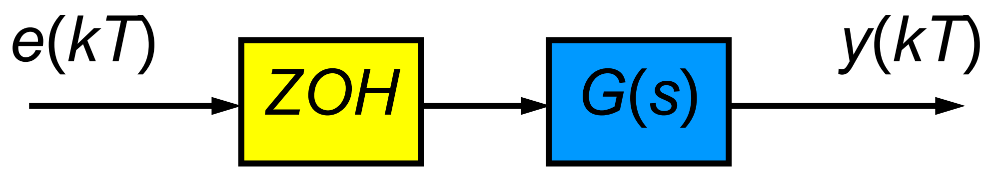
- ***Step 2:*** Calculate the transient matrix:

$$\Phi(t) = \mathscr{L}^{-1}[\Phi(s)]$$

where $$\Phi(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}$$

Formulation of SS equations from block diagram (cont')

- ***Step 3:*** Discretizing the open-loop continuous SS equation:



$$\begin{cases} \boldsymbol{x}[(k+1)T] = \boldsymbol{A}_d \boldsymbol{x}(kT) + \boldsymbol{B}_d e_R(kT) \\ y(kT) = \boldsymbol{C}_d \boldsymbol{x}(kT) \end{cases}$$

with

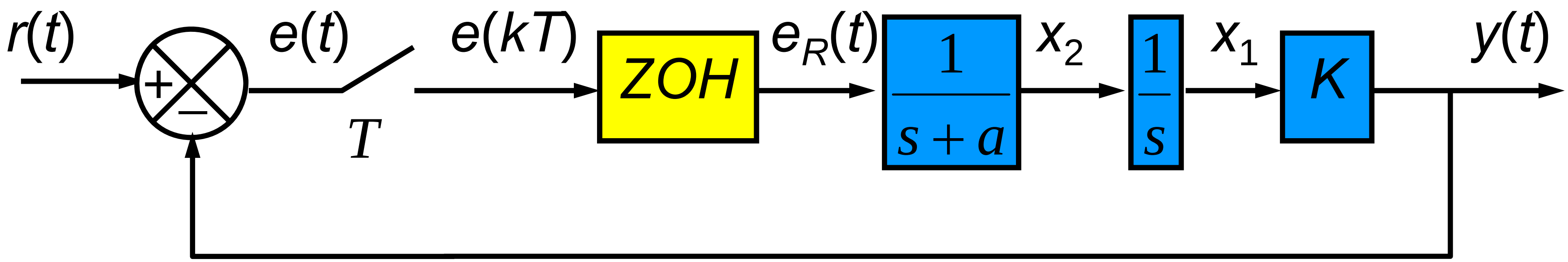
$$\begin{cases} \boldsymbol{A}_d = \Phi(T) \\ \boldsymbol{B}_d = \int_0^T \Phi(\tau) B d\tau \\ \boldsymbol{C}_d = \boldsymbol{C} \end{cases}$$

- ***Step 4:*** Write the closed-loop discrete state equations (which has input signal *r*(*kT*))

$$\begin{cases} \boldsymbol{x}[(k+1)T] = [\boldsymbol{A}_d - \boldsymbol{B}_d \boldsymbol{C}_d] \boldsymbol{x}(kT) + \boldsymbol{B}_d r(kT) \\ y(kT) = \boldsymbol{C}_d \boldsymbol{x}(kT) \end{cases}$$

Formulation of SS equations from block diagram – Example

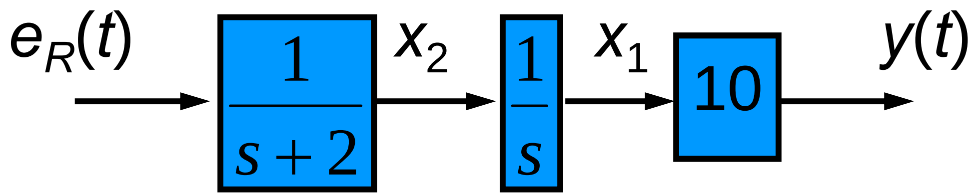
- Formulate the SS equations describing the system:



where $a = 2$, $T = 0.5$, $K = 10$

Formulation of state equations from block diagram – Example (cont')

- Solution:
- Step 1:



$$X_1(s) = \frac{X_2(s)}{s} \quad \Rightarrow \quad sX_1(s) = X_2(s) \quad \Rightarrow \quad \dot{x}_1(t) = x_2(t)$$

$$X_2(s) = \frac{E_R(s)}{s+2} \quad \Rightarrow \quad (s+2)X_2(s) = E_R(s) \quad \Rightarrow \quad \dot{x}_2(t) = -2x_2(t) + e_R(t)$$

$$\Rightarrow \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\boldsymbol{B}} e_R(t) \\ y(t) = 10x_1(t) = \underbrace{\begin{bmatrix} 10 & 0 \end{bmatrix}}_{\boldsymbol{C}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

★ Step 2: Calculate the transient matrix

$$\Phi(s) = (sI - A)^{-1} = \left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \right)^{-1} = \left(\begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix} \right)^{-1}$$

$$= \frac{1}{s(s+2)} \begin{bmatrix} s+2 & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$\Phi(t) = \mathscr{L}^{-1}[\Phi(s)] = \mathscr{L}^{-1}\left\{ \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} \right\} = \begin{bmatrix} \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} & \mathscr{L}^{-1}\left\{\frac{1}{s(s+2)}\right\} \\ 0 & \mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} \end{bmatrix}$$

$$\Rightarrow \quad \Phi(t) = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

Formulation of state equations from block diagram – Example (cont')

✶ Step 3: Discretizing the open-loop continuous state equations:

$$\begin{cases} \boldsymbol{x}[(k+1)T] = \boldsymbol{A}_d \boldsymbol{x}(kT) + \boldsymbol{B}_d e_R(kT) \\ y(kT) = \boldsymbol{C}_d \boldsymbol{x}(kT) \end{cases}$$

$$\boldsymbol{A}_d = \Phi(T) = \left.\begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}\right|_{t=T} = \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2\times 0.5}) \\ 0 & e^{-2\times 0.5} \end{bmatrix} = \begin{bmatrix} 1 & 0.316 \\ 0 & 0.368 \end{bmatrix}$$

$$\boldsymbol{B}_d = \int_0^T \Phi(\tau)\boldsymbol{B}d\tau = \int_0^T \left\{ \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2\tau}) \\ 0 & e^{-2\tau} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau \right\} = \int_0^T \left\{ \begin{bmatrix} \frac{1}{2}(1-e^{-2\tau}) \\ e^{-2\tau} \end{bmatrix} d\tau \right\}$$

$$= \left.\begin{bmatrix} \left(\frac{\tau}{2} + \frac{e^{-2\tau}}{2^2}\right) \\ -\frac{e^{-2\tau}}{2} \end{bmatrix}\right|_0^T = \begin{bmatrix} \left(\frac{0.5}{2} + \frac{e^{-2\times 0.5}}{2^2} - \frac{1}{2^2}\right) \\ -\frac{e^{-2\times 0.5}}{2} + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0.092 \\ 0.316 \end{bmatrix}$$

$$\boldsymbol{C}_d = \boldsymbol{C} = \begin{bmatrix} 10 & 0 \end{bmatrix}$$

Formulation of state equations from block diagram – Example (cont')

- Step 4: The closed-loop discrete state equations:

$$\begin{cases} \boldsymbol{x}[(k+1)T] = [\boldsymbol{A}_d - \boldsymbol{B}_d \boldsymbol{C}_d]\boldsymbol{x}(kT) + \boldsymbol{B}_d r(kT) \\ y(kT) = \boldsymbol{C}_d \boldsymbol{x}(kT) \end{cases}$$

where $[\boldsymbol{A}_d - \boldsymbol{B}_d \boldsymbol{C}_d] = \begin{bmatrix} 1 & 0.316 \\ 0 & 0.368 \end{bmatrix} - \begin{bmatrix} 0.092 \\ 0.316 \end{bmatrix} [10 \quad 0] = \begin{bmatrix} 0.080 & 0.316 \\ -3.160 & 0.368 \end{bmatrix}$

- Conclusion: The closed-loop state equation is:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.080 & 0.316 \\ -3.160 & 0.368 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.092 \\ 0.316 \end{bmatrix} r(k)$$

$$y(k) = [10 \quad 0] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Calculate transfer function from state equation

- Given the state equation

$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d \boldsymbol{x}(k) + \boldsymbol{B}_d u(k) \\ y(k) = \boldsymbol{C}_d \boldsymbol{x}(k) \end{cases}$$

- The corresponding transfer function is:

$$G(z) = \frac{Y(z)}{U(z)} = \boldsymbol{C}_d (z\boldsymbol{I} - \boldsymbol{A}_d)^{-1} \boldsymbol{B}_d$$

Calculate transfer function from state equation - Example

- Calculate the TF of the system described by the SS equation:

$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d \boldsymbol{x}(k) + \boldsymbol{B}_d u(k) \\ y(k) = \boldsymbol{C}_d \boldsymbol{x}(k) \end{cases}$$

$$\boldsymbol{A}_d = \begin{bmatrix} 0 & 1 \\ -0.7 & -0.1 \end{bmatrix} \qquad \boldsymbol{B}_d = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \boldsymbol{C}_d = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- Solution: The transfer function is:

$$G(z) = \boldsymbol{C}_d (z\boldsymbol{I} - \boldsymbol{A}_d)^{-1} \boldsymbol{B}_d$$

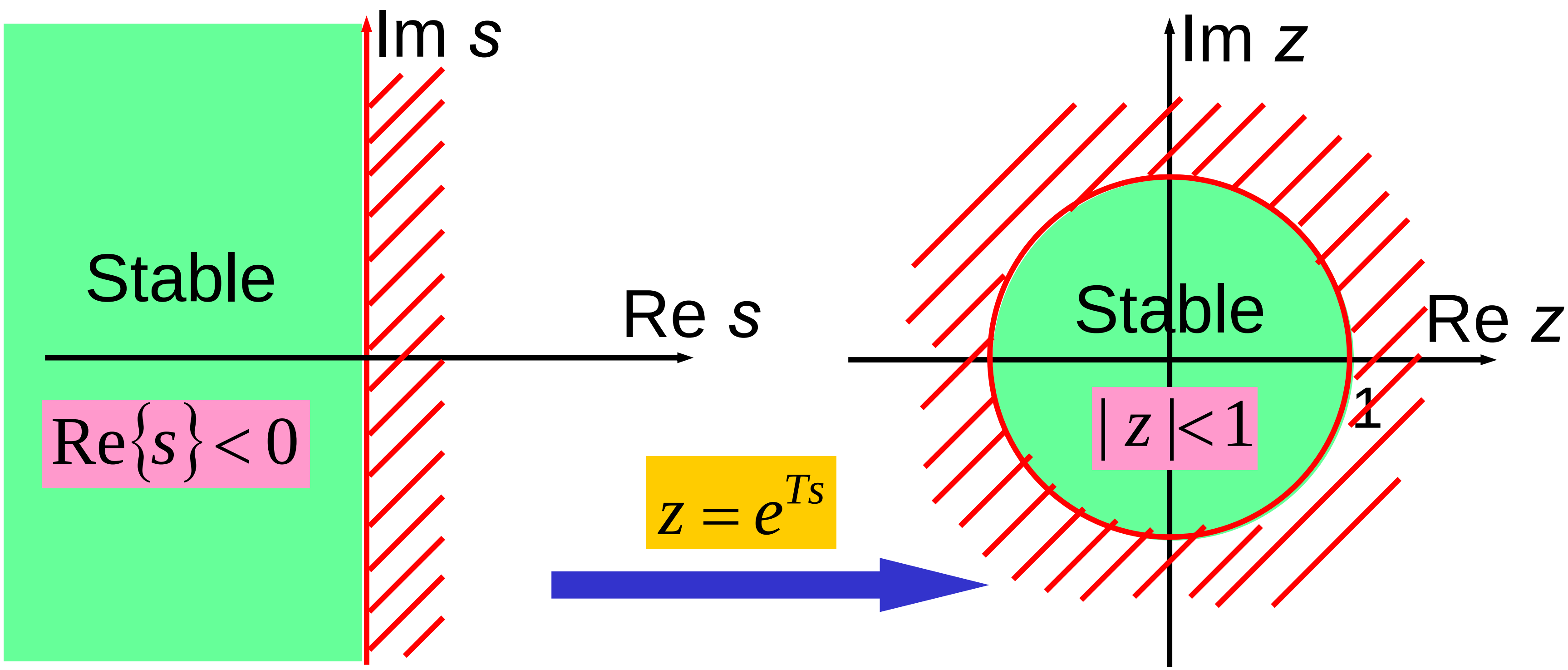
$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.7 & -0.1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \boxed{G(z) = \frac{2}{z^2 + 0.1z + 0.7}}$$

Analyze the stability of discrete control systems

Stability conditions for discrete systems

- A system is defined to be BIBO stable if every bounded input to the system results in a bounded output.

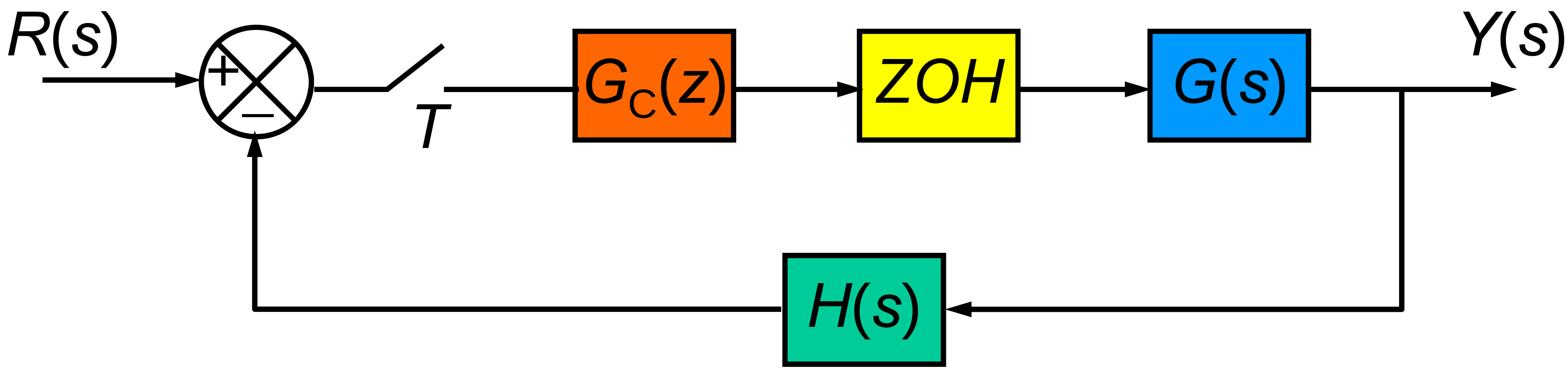


The region of stability for a continuous system is the left-half s-plane

The region of stability for a discrete system is the interior of the unit circle

Characteristic equation of discrete systems

✶ Discrete systems described by block diagram:



⇒ Characteristic equation: $1+G_C(z)GH(z)=0$

✶ Discrete systems described by the state equation

$$\begin{cases} \boldsymbol{x}(k+1)=\boldsymbol{A}_d\boldsymbol{x}(k)+\boldsymbol{B}_d r(k) \\ y(k)=\boldsymbol{C}_d\boldsymbol{x}(k) \end{cases}$$

⇒ Characteristic equation: $\det(z\boldsymbol{I}-\boldsymbol{A}_d)=0$

Methods for analyzing the stability of discrete systems

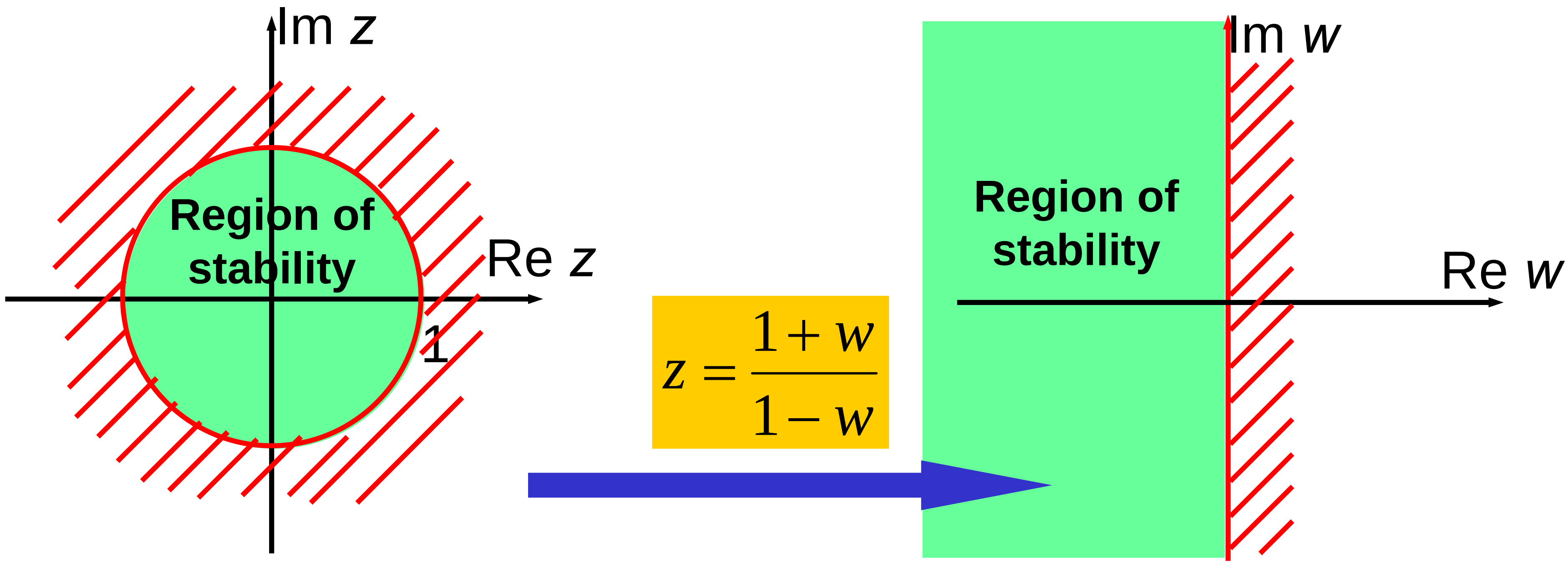
- Algebraic stability criteria
  - The extension of the Routh-Hurwitz criteria (student's further reading)
  - Jury's stability criterion

- The root locus method (student's further reading)

The extension of the Routh-Hurwitz criteria

- Characteristic equation of discrete systems:

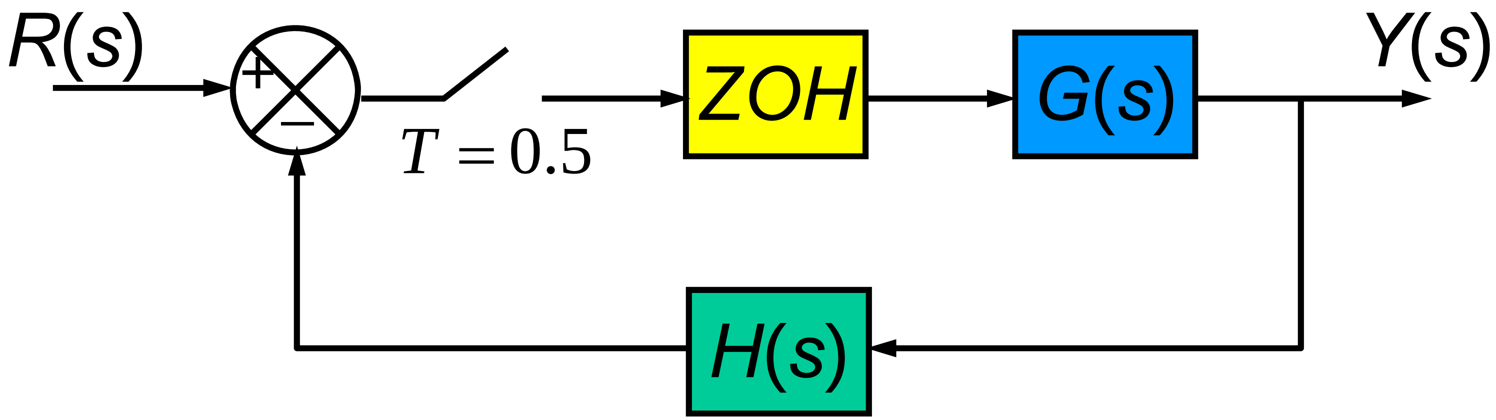
$$a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$$



$$z = \frac{1+w}{1-w}$$

- The extension of the Routh-Hurwitz criteria: transform z→w, and then apply the Routh – Hurwitz criteria to the characteristic equation of the variable w.

★ Analyze the stability of the following system:



Given that: $G(s)=\dfrac{3e^{-s}}{s+3}$ $\quad H(s)=\dfrac{1}{s+1}$

★ **Solution:**

The characteristic equation of the system:

$$1+GH(z)=0$$

The extension of the Routh-Hurwitz criteria – Example (cont')

- $$GH(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)H(s)}{s}\right\}$$

$$= (1 - z^{-1})\mathcal{Z}\left\{\frac{3e^{-s}}{s(s+3)(s+1)}\right\}$$

$$= 3(1 - z^{-1})z^{-2}\frac{z(Az + B)}{(z-1)(z - e^{-3\times 0.5})(z - e^{-1\times 0.5})}$$

$$A = \frac{(1 - e^{-3\times 0.5}) - 3(1 - e^{-0.5})}{3(1-3)} = 0.0673$$

$$B = \frac{3e^{-3\times 0.5}(1 - e^{-0.5}) - e^{-0.5}(1 - e^{-3\times 0.5})}{3(1-3)} = 0.0346$$

$$\Rightarrow \quad \boxed{GH(z) = \frac{0.202z + 0.104}{z^2(z - 0.223)(z - 0.607)}}$$

The extension of the Routh-Hurwitz criteria – Example (cont')

⇒ The characteristic equation:

$$1 + GH(z) = 0$$

$$\Rightarrow \quad 1 + \frac{0.202z + 0.104}{z^2(z-0.223)(z-0.607)} = 0$$

$$\Rightarrow \quad z^4 - 0.83z^3 + 0.135z^2 + 0.202z + 0.104 = 0$$

★ Perform the transformation: $z = \dfrac{1+w}{1-w}$

$$\Rightarrow \left(\frac{1+w}{1-w}\right)^4 - 0.83\left(\frac{1+w}{1-w}\right)^3 + 0.135\left(\frac{1+w}{1-w}\right)^2 + 0.202\left(\frac{1+w}{1-w}\right) + 0.104 = 0$$

$$\Rightarrow 1.867w^4 + 5.648w^3 + 6.354w^2 + 1.52w + 0.611 = 0$$

- The Routh table

w^4	1.867	6.354	0.611
w^3	5.648	1.52	0
w^2	$6.354 - \dfrac{1.867}{5.648} \times 1.52 = 5.852$	0.611	0
w^1	$1.52 - \dfrac{5.648}{5.852} \times 0.611 = 0.93$	0	
w^0	0.611		

- Conclusion: The system is stable because all the terms in the first column of the Routh table are positive.

Jury stability criterion

- Analyze the stability of the discrete system which has the characteristic equation:

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$

- Jury table: consist of (2*n*+1) rows.
  - *The first row* consists of the coefficients of the characteristic polynomial in the increasing index order.
  - *The even row* (any) consists of the coefficients of the previous row in the reverse order.
  - The odd row $i = 2k+1$ ($k \geq 1$) consists ($n-k+1$) terms, the term at the row *i* column *j* defined by:

$$c_{ij} = \frac{1}{c_{i-2,1}} \begin{vmatrix} c_{i-2,1} & c_{i-2,n-j-k+3} \\ c_{i-1,1} & c_{i-1,n-j-k+3} \end{vmatrix}$$

Jury stability criterion (cont')

- **Jury criterion statement:** The *necessary and sufficient condition* for the discrete system to be stable is that all the first terms of the odd rows of the Jury table are positive.

Jury stability criterion – Example

- Analyze the stability of the system which has the characteristic equation: $$5z^3 + 2z^2 + 3z + 1 = 0$$
- Solution: Jury table

Row 1	5	2	3	1
Row 2	1	3	2	5
Row 3	$\frac{1}{5}\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} = 4.8$	$\frac{1}{5}\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 1.4$	$\frac{1}{5}\begin{vmatrix} 5 & 2 \\ 1 & 3 \end{vmatrix} = 2.6$	
Row 4	2.6	1.4	4.8	
Row 5	$\frac{1}{4.8}\begin{vmatrix} 4.8 & 2.6 \\ 2.6 & 4.8 \end{vmatrix} = 3.39$	$\frac{1}{4.8}\begin{vmatrix} 4.8 & 1.4 \\ 2.6 & 1.4 \end{vmatrix} = 0.61$		
Row 6	0.61	3.39		
Row 7	$\frac{1}{3.39}\begin{vmatrix} 3.39 & 0.61 \\ 0.61 & 3.39 \end{vmatrix} = 3.28$			

- Since all the first terms of the odd rows are positive, the system is stable.

Root locus of discrete control system

- RL is a set of all the roots of the characteristic equation of a system when a real parameter changing from $0 \rightarrow +\infty$.

- Consider a discrete system which has the characteristic equation:

$$1 + K\frac{N(z)}{D(z)} = 0$$

Denote: $$G_0(z) = K\frac{N(z)}{D(z)}$$

Assume that $G_0(z)$ has n poles and m zeros.

- The rules for construction of the RL of continuous system can be applied to discrete systems, except for the step 8.

Rules for construction of the RL of discrete systems

- ***Rule 1:*** The number of branches of a RL = the order of the characteristic equation = number of poles of $G_0(z) = n$.

- ***Rule 2:***
  - For $K = 0$: the RL begin at the poles of $G_0(z)$.
  - As K goes to $+\infty$: m branches of the RL end at m zeros of $G_0(z)$, the $n-m$ remaining branches goes to ∞ approaching the asymptote defined by the rule 5 and rule 6.

- ***Rule 3:*** The RL is symmetric with respect to the real axis.

- ***Rule 4:*** A point on the real axis belongs to the RL if the total number of poles and zeros of $G_0(z)$ to its right is odd.

- ***Rule 5:*** The angles between the asymptotes and the real axis are given by:

$$\alpha = \frac{(2l+1)\pi}{n-m} \qquad (l = 0, \pm 1, \pm 2, \ldots)$$

- ***Rule 6:*** The intersection between the asymptotes and the real axis is a point *A* defined by:

$$OA = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m}$$

(p_i and z_i are poles and zeros of $G_0(z)$)

- ***Rule 7:*** **:** Breakaway / break-in points (or break points for short), if any, are located in the real axis and are satisfied the equation:

$$\frac{dK}{dz} = 0$$

- ***Rule 8:*** The intersections of the RL with the unit circle can be determined by using the extension of the Routh-Hurwitz criteria or by substituting $z=a+jb$ $(a^2+b^2=1)$ into the characteristic equation.

- ***Rule 9:*** The departure angle of the RL from a pole p_j (of multiplicity 1) is given by:

$$\theta_j = 180^0 + \sum_{i=1}^{m} \arg(p_j - z_i) - \sum_{i=1, i\neq j}^{n} \arg(p_j - p_i)$$

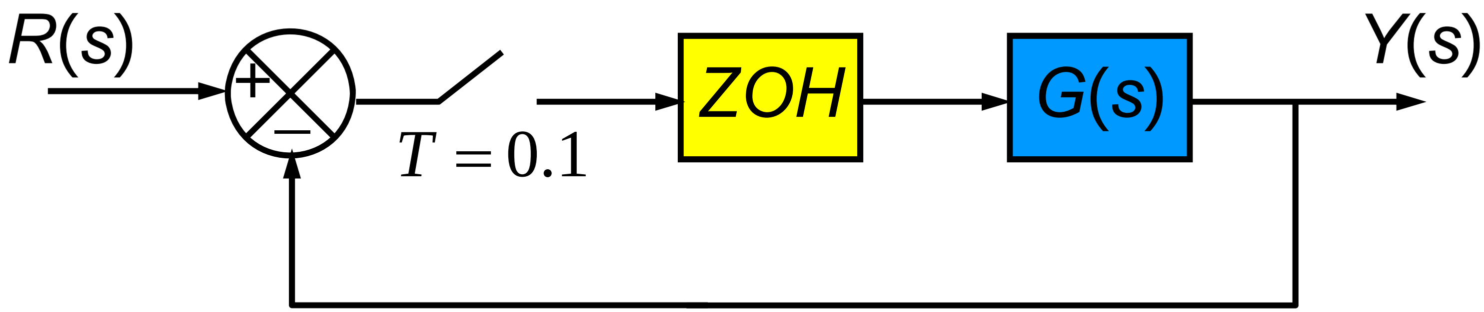
The geometric form of the above formula is

$\theta_j = 180^0 + (\sum$angle from z_i ($i=1..m$) to p_j)

– ($\sum$angle p_i ($i=1..m$, $i\neq j$) to p_j)

The root locus of discrete systems – Example

- Consider a discrete system described by a block diagram:



$$G(s)=\frac{5K}{s(s+5)}$$

- Sketch the RL of the system when K=0→+∞. Determine the critical gain K_{cr}

- **Solution:** The characteristic equation of the system:

$$1+G(z)=0$$

The root locus of discrete systems – Example (cont')

- $$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

$$= (1 - z^{-1})\mathcal{Z}\left\{\frac{5K}{s^2(s+5)}\right\}$$

$$= K(1 - z^{-1})\left(\frac{z[(0.5 - 1 + e^{-0.5})z + (1 - e^{-0.5} - 0.5e^{-0.5})]}{5(z-1)^2(z - e^{-0.5})}\right)$$

$$\Rightarrow \quad G(z) = K\frac{0.021z + 0.018}{(z-1)(z-0.607)}$$

★ The characteristic equation : $\boxed{1 + K\frac{0.021z + 0.018}{(z-1)(z-0.607)} = 0}$ (*)

★ Poles: $p_1 = 1 \quad p_2 = 0.607$

★ Zeros: $z_1 = -0.857$

The root locus of discrete systems – Example (cont')

- The asymptotes:

$$\alpha = \frac{(2l+1)\pi}{n-m} = \frac{(2l+1)\pi}{2-1} \quad \Rightarrow \quad \alpha = \pi$$

$$OA = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} = \frac{[1+0.607]-(-0.857)}{2-1} \Rightarrow OA = 2.464$$

- The breakaway/break-in points:

$$(*) \quad \Leftrightarrow \quad K = -\frac{(z-1)(z-0.607)}{0.021z+0.018} = -\frac{z^2-1.607z+0.607}{0.021z+0.018}$$

$$\Rightarrow \quad \frac{dK}{dz} = -\frac{0.021z^2+0.036z-0.042}{(0.021z+0.018)^2}$$

Then $\dfrac{dK}{dz} = 0 \quad \Leftrightarrow \quad \begin{cases} z_1 = -2.506 \\ z_2 = 0.792 \end{cases}$

The root locus of discrete systems – Example (cont')

- The intersection of the root locus with the unit circle:

$$(*) \Leftrightarrow \quad (z-1)(z-0.607)+K(0.021z+0.018)=0$$

$$\Leftrightarrow \quad z^2+(0.021K-1.607)z+(0.018K+0.607)=0 \qquad (**)$$

Method 1: Apply the extension of Routh – Hurwitz criteria:

Perform the transformation $z=\dfrac{w+1}{w-1}$, (**) becomes:

$$\left(\frac{w+1}{w-1}\right)^2+(0.021K-1.607)\left(\frac{w+1}{w-1}\right)+(0.018K+0.607)=0$$

$$\Leftrightarrow 0.039Kw^2+(0.786-0.036K)w+(3.214-0.003K)=0$$

The root locus of discrete systems – Example (cont')

According to the corollary of the Hurwitz criterion, the stability conditions are:

$$\begin{cases} K > 0 \\ 0.786 - 0.036K > 0 \\ 3.214 - 0.003K > 0 \end{cases} \Leftrightarrow \begin{cases} K > 0 \\ K < 21.83 \\ K < 1071 \end{cases} \Rightarrow K_{cr} = 21.83$$

Substitute $K_{cr} = 21.83$ into (**), we have:

$$z^2 - 1.1485z + 1 = 0 \quad \Rightarrow \quad z = 0.5742 \pm j0.8187$$

Then the intersection of the RL with the unit circle are:

$$z = 0.5742 \pm j0.8187$$

The root locus of discrete systems – Example (cont')

Method 2: Substitute $z = a + jb$ into (**) :

$$(a+jb)^2+(0.021K-1.607)(a+jb)+(0.018K+0.607)=0$$

$$\Rightarrow\ a^2+j2ab-b^2+(0.021K-1.607)a+j(0.021K-1.607)b+(0.018K+0.607)=0$$

$$\Rightarrow\ \begin{cases} a^2-b^2+(0.021K-1.607)a+(0.018K+0.607)=0 \\ j2ab+j(0.021K-1.607)b=0 \end{cases}$$

The root locus of discrete systems – Example (cont')

- Combine with $a^2 + b^2 = 1$, we have the set of equations:

$$\begin{cases} a^2 - b^2 + (0.021K - 1.607)a + (0.018K + 0.607) = 0 \\ j2ab + j(0.021K - 1.607)b = 0 \\ a^2 + b^2 = 1 \end{cases}$$

- Solve the above set of equation, we obtain 4 intersection:

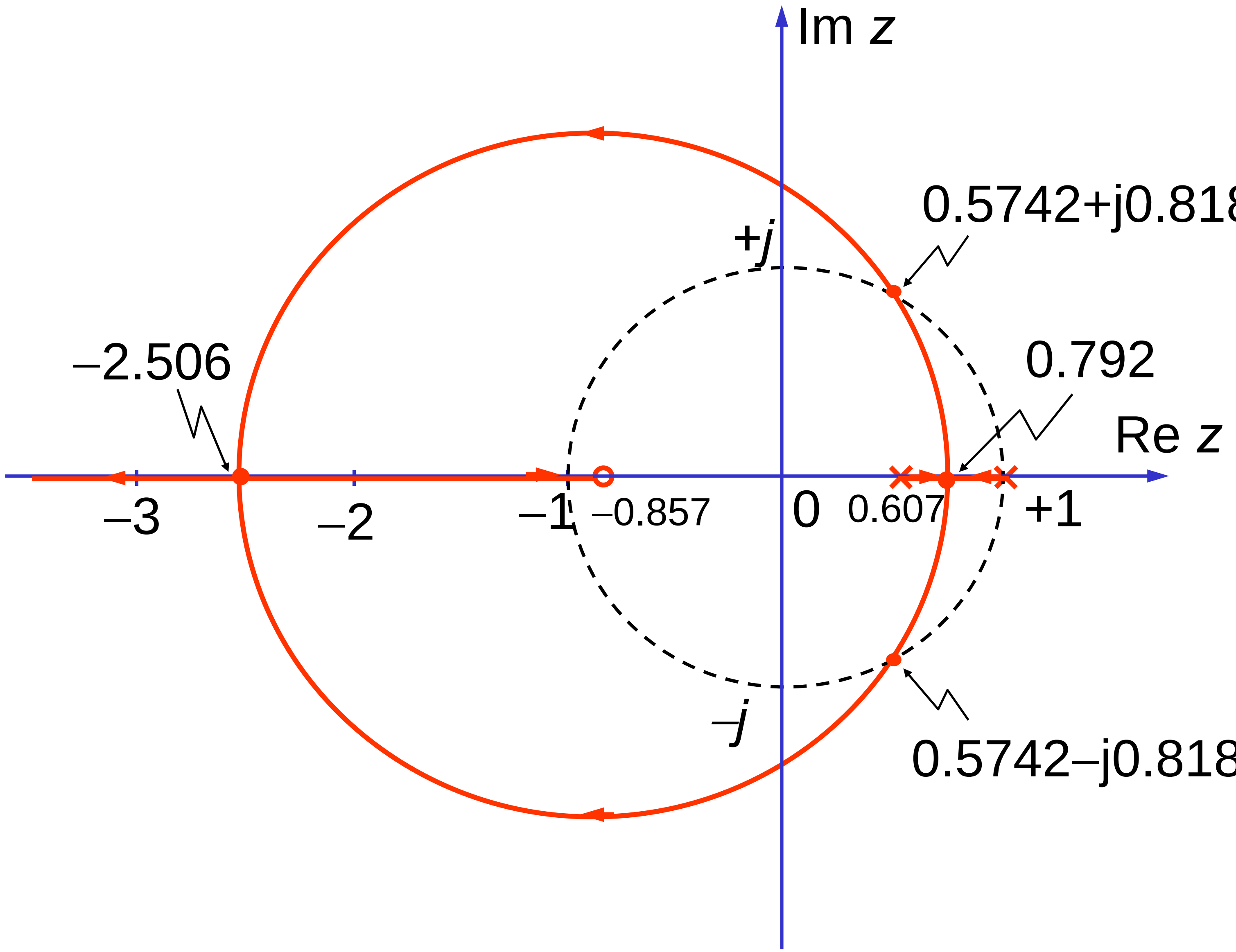
$z = 1$ when $K = 0$

$z = -1$ when $K = 1071$

$z = 0.5742 \pm j0.8187$ when $K = 21.83$

$$\Rightarrow \quad \boxed{K_{cr} = 21.83}$$

The root locus of discrete systems – Example (cont')



Analyze the performance of discrete systems

Time response of discrete systems

- Time response of a discrete system can be calculated by using one of the two methods below:
  - *Method 1:* if the discrete system described by a transfer function, first we calculate $Y(z)$, and then apply the inverse z-transform to find $y(k)$.
  - *Method 2:* if the discrete system described by state equations, first we find the solution $\boldsymbol{x}(k)$ to the state equations, then calculate $y(k)$.

- Dominant poles of a discrete system are the poles lying closest to the unit circle.

Transient performances

Method 1: Analyzing the transient performance based on the time response $y(k)$ of discrete systems.

- Percentage of overshoot: $$POT = \frac{y_{\max} - y_{ss}}{y_{ss}} 100\%$$

y_{max} and y_{ss} are the maximum and steady-state values of $y(k)$

- Settling time: $$t_s = k_s T$$

where k_s satisfying the condition:

$$\left| y(k) - y_{ss} \right| \le \frac{\varepsilon . y_{ss}}{100}, \qquad \forall k \ge k_s$$

$$\Leftrightarrow \quad \left(1 - \frac{\varepsilon}{100}\right) y_{ss} \le y(k) \le \left(1 + \frac{\varepsilon}{100}\right) y_{ss}, \qquad \forall k \ge k_s$$

Transient performances

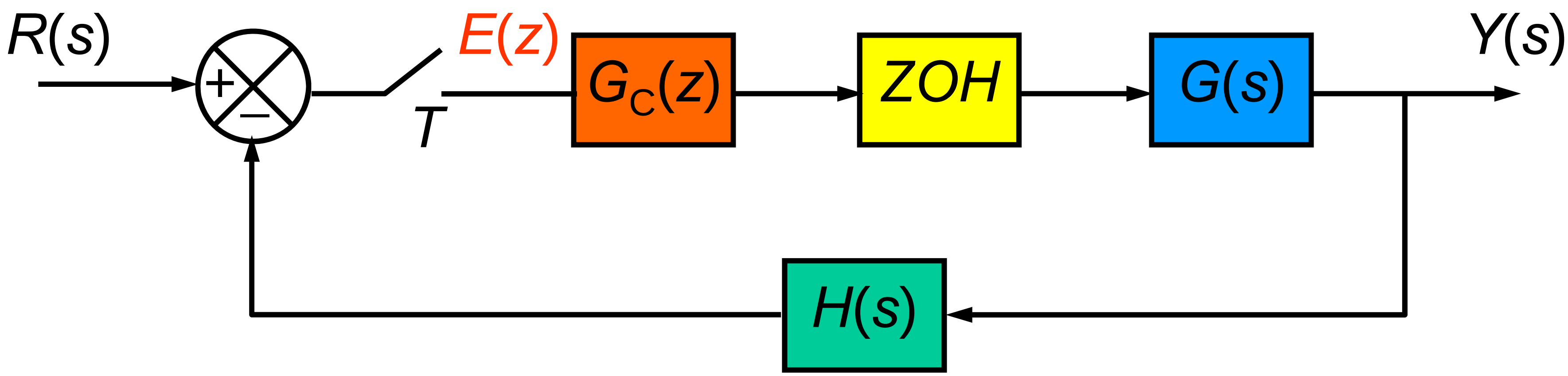
Method 2: Analyzing the transient performances based on the dominant poles.

- The dominant poles: $z_{1,2}^{*} = re^{j\varphi}$ $\Rightarrow$ $\begin{cases} \xi = \dfrac{-\ln r}{\sqrt{(\ln r)^2 + \varphi^2}} \\ \omega_n = \dfrac{1}{T}\sqrt{(\ln r)^2 + \varphi^2} \end{cases}$

- Percentage of overshoot: $POT = \exp\left(-\dfrac{\xi\pi}{\sqrt{1-\xi^2}}\right)\times 100\%$

- Settling time: $t_s = \dfrac{3}{\xi\omega_n}$ (according to 5% criterion)

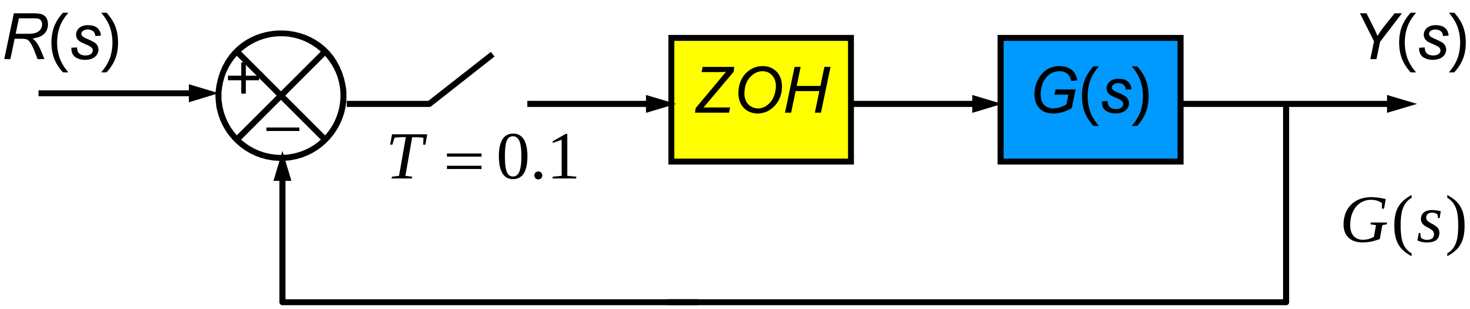
Steady state error



- Error expression: $$E(z)=\frac{R(z)}{1+G_C(z)GH(z)}$$

- Steady state error: $$e_{ss}=\lim_{k\to\infty}e(k)=\lim_{z\to 1}(1-z^{-1})E(z)$$

Performances of discrete system – Example 1



$$G(s) = \frac{10}{(s+2)(s+3)}$$

1. Calculate the closed-loop transfer function of the system.
2. Calculate the time response of the system to step input.
3. Evaluate the performance of the system: POT, settling time, steady-state error.

✶ Solution:

1. The closed-loop TF of the system:

$$G_{cl}(z) = \frac{G(z)}{1+G(z)}$$

Performance of discrete system – Example 1 (cont')

- $G(z) = (1 - z^{-1})\mathcal{Z}\left\{\dfrac{G(s)}{s}\right\}$

$$= (1 - z^{-1})\mathcal{Z}\left\{\frac{10}{s(s+2)(s+3)}\right\}$$

$$= 10(1 - z^{-1})\frac{z(Az + B)}{(z-1)(z - e^{-2\times 0.1})(z - e^{-3\times 0.1})}$$

$$\Rightarrow \quad \boxed{G(z) = \frac{0.042z + 0.036}{(z - 0.819)(z - 0.741)}}$$

Performance of discrete system – Example 1 (cont')

- $$G_{cl}(z) = \frac{G(z)}{1+G(z)}$$

$$= \frac{\dfrac{0.042z + 0.036}{(z-0.819)(z-0.741)}}{1 + \dfrac{0.042z + 0.036}{(z-0.819)(z-0.741)}}$$

$$\Rightarrow \quad \boxed{G_{cl}(z) = \frac{0.042z + 0.036}{z^2 - 1.518z + 0.643}}$$

Performance of discrete system – Example 1 (cont')

2. The time response of the system to step input.

$$Y(z) = G_k(z)R(z)$$

$$= \frac{0.042z + 0.036}{z^2 - 1.518z + 0.643} R(z)$$

$$= \frac{0.042z^{-1} + 0.036z^{-2}}{1 - 1.518z^{-1} + 0.643z^{-2}} R(z)$$

$$\Rightarrow \quad (1 - 1.518z^{-1} + 0.643z^{-2})Y(z) = (0.042z^{-1} + 0.036z^{-2})R(z)$$

$$\Rightarrow \quad y(k) - 1.518y(k-1) + 0.643y(k-2) = 0.042r(k-1) + 0.036r(k-2)$$

$$\Rightarrow \quad y(k) = 1.518y(k-1) - 0.643y(k-2) + 0.042r(k-1) + 0.036r(k-2)$$

Performance of discrete system – Example 1 (cont')

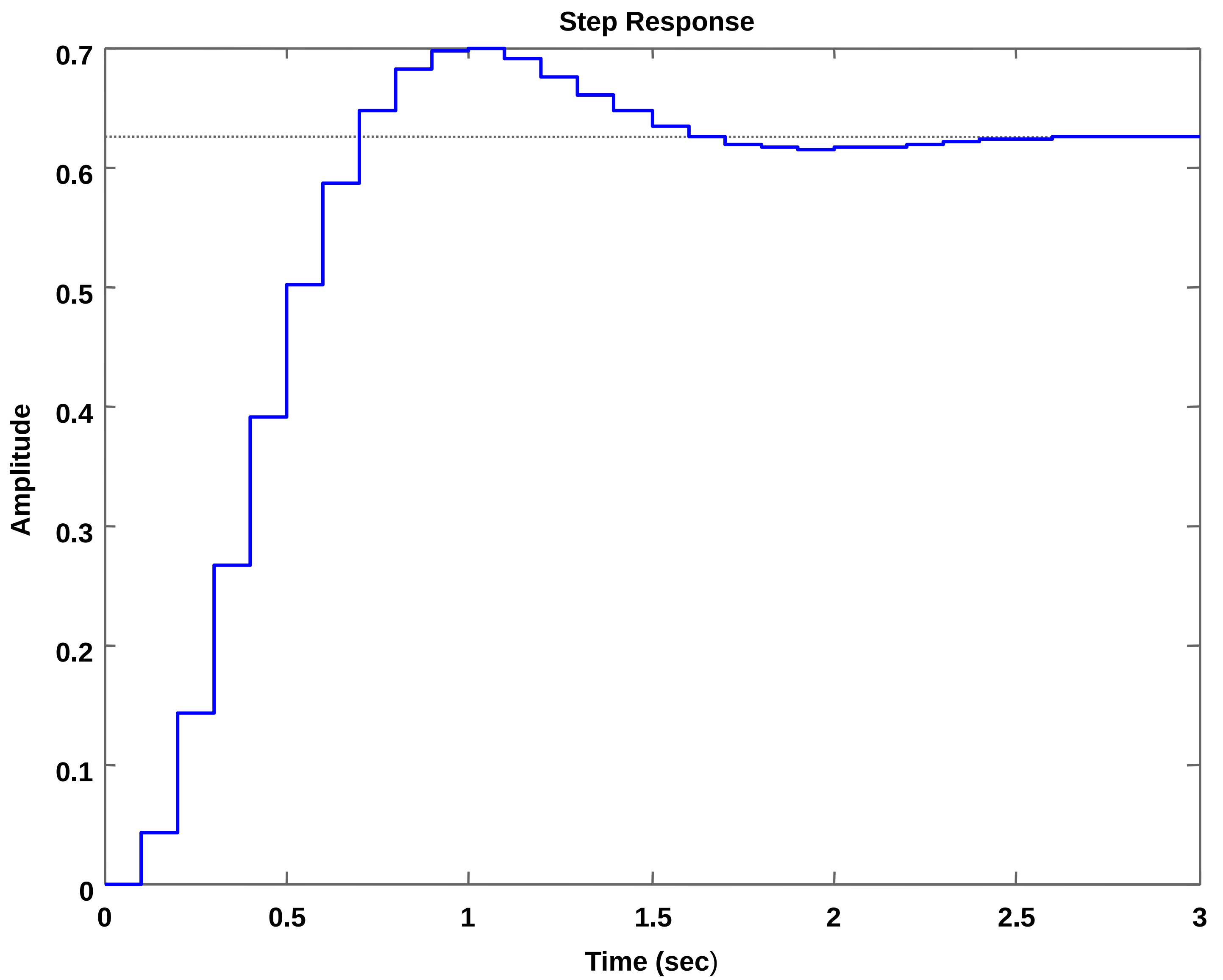
Unit step input: $r(k)=1, \forall k \geq 0$

Initial condition: $y(-1)=y(-2)=0$

Substitute the initial condition to the recursive equation of $y(k)$, we have:

$$
\begin{aligned}
y(k)=\{&0;\, 0.0420;\, 0.1418;\, 0.2662;\, 0.3909;\, 0.5003;\ldots \\
&0.5860;\; 0.6459;\; 0.6817;0.6975;\; 0.6985;\; 0.6898;\ldots \\
&0.6760;\; 0.6606;\; 0.6461;\; 0.6341;\; 0.6251;\; 0.6191;\ldots\}
\end{aligned}
$$

Performance of discrete system – Example 1 (cont’)



Performance of discrete system – Example 1 (cont')

3. Transient performances:

The steady state response:

$$y_{ss} = \lim_{z\to 1}(1-z^{-1})Y(z)$$

$$= \lim_{z\to 1}(1-z^{-1})G_k(z)R(z)$$

$$= \lim_{z\to 1}(1-z^{-1})\left(\frac{0.042z+0.036}{z^2-1.518z+0.643}\right)\left(\frac{1}{1-z^{-1}}\right)$$

$$\Rightarrow \quad y_{ss} = 0.624$$

The maximum value: $y_{\max} = 0.6985$

★ Percentage of overshoot:

$$POT = \frac{y_{\max} - y_{ss}}{y_{ss}}100\% = \frac{0.6985-0.624}{0.624}100\% = 11.94\%$$

Performance of discrete system – Example 1 (cont')

- Settling time (5% criterion):

First, we need to find k_s satisfying:

$$(1-\varepsilon)y_{ss} \le y(k) \le (1+\varepsilon)y_{ss}, \forall k \ge k_s$$

$$\Leftrightarrow \quad 0.593 \le y(k) \le 0.655, \forall k \ge k_s$$

From the time response calculated before $\Rightarrow$ $k_s = 14$

$$t_s = k_s T = 14 \times 0.1$$

$$\Rightarrow \quad t_s = 1.4\,\text{sec}$$

- Steady state error:

Since the system is unity negative feedback, we have:

$$e_{ss} = r_{ss} - y_{ss} = 1 - 0.624 \quad \Rightarrow \quad e_{ss} = 0.376$$

Performance of discrete system – Example 1 (cont')

✶ Note: It is possible to calculate *POT* and t_s based on the dominant poles .The poles of the closed-loop system are the roots of the equ.:

$$z^2 - 1.518z + 0.643 = 0$$

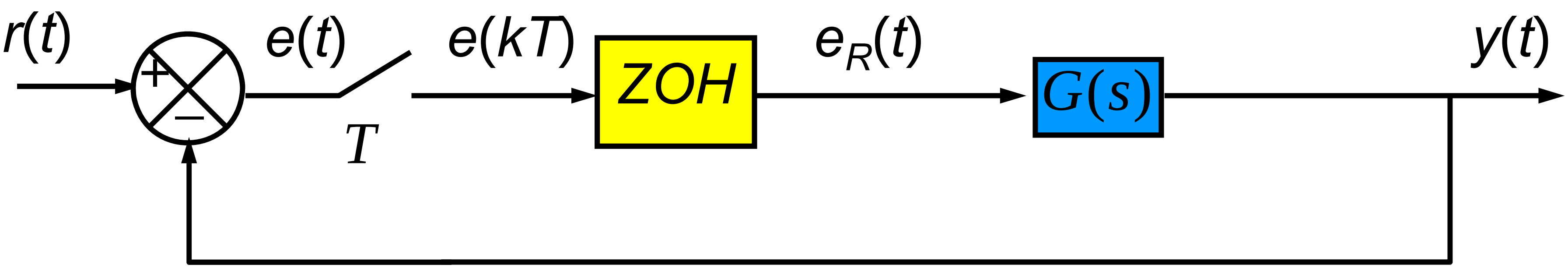
$$\Rightarrow \quad z^*_{1,2} = 0.7590 \pm j0.2587 = 0.8019\angle 0.3285$$

$$\Rightarrow \begin{cases} \xi = \dfrac{-\ln r}{\sqrt{(\ln r)^2 + \varphi^2}} = \dfrac{-\ln 0.8019}{\sqrt{(\ln 0.8019)^2 + 0.3285^2}} = 0.5579 \\ \omega_n = \dfrac{1}{T}\sqrt{(\ln r)^2 + \varphi^2} = \dfrac{1}{0.1}\sqrt{(\ln 0.8019)^2 + 0.3285^2} = 0.3958 \end{cases}$$

$$POT = \exp\left(-\frac{\xi\pi}{\sqrt{1-\xi^2}}\right).100\% = \exp\left(-\frac{0.5579\times 3.14}{\sqrt{1-0.5579^2}}\right).100\% = 12.11\%$$

$$t_{qđ} = \frac{3}{\xi\omega_n} = \frac{3}{0.5579\times 0.3958} = 1.36\,\text{sec}$$

Performance of discrete system – Example 2



with $T = 0.1$

$$G(s) = \frac{2(s+5)}{(s+2)(s+3)}$$

1. Formulate the state equations describing the system
2. Calculate the response of the system to unit step input (assuming the initial conditions are zeros) using the state equation formulated above.
3. Calculate POT, settling time, steady state error

★ **Solution:**

1. Formulate the state equation:

$$G(s) = \frac{Y(s)}{E_R(s)} = \frac{2(s+5)}{(s+2)(s+3)} = \frac{2s+10}{s^2+5s+6}$$

★ The state equation of the continuous plant:

$$\Rightarrow \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\boldsymbol{B}} e_R(t) \\ y(t) = \underbrace{\begin{bmatrix} 10 & 2 \end{bmatrix}}_{\boldsymbol{C}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

Performance of discrete system – Example 2 (cont')

★ The transient matrix:

$$\Phi(s) = (sI - A)^{-1} = \left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \right)^{-1} = \left(\begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix} \right)^{-1}$$

$$= \frac{1}{s(s+5)-6} \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix} = \begin{bmatrix} \dfrac{s+5}{(s+2)(s+3)} & \dfrac{1}{(s+2)(s+3)} \\ \dfrac{-6}{(s+2)(s+3)} & \dfrac{s}{(s+2)(s+3)} \end{bmatrix}$$

$$\Phi(t) = \mathscr{L}^{-1}[\Phi(s)] == \begin{bmatrix} \mathscr{L}^{-1}\left\{ \dfrac{3}{s+2} - \dfrac{2}{s+3} \right\} & \mathscr{L}^{-1}\left\{ \dfrac{1}{s+2} - \dfrac{1}{s+3} \right\} \\ \mathscr{L}^{-1}\left\{ -\dfrac{6}{s+2} + \dfrac{6}{s+3} \right\} & \mathscr{L}^{-1}\left\{ -\dfrac{2}{s+2} + \dfrac{3}{s+3} \right\} \end{bmatrix}$$

$$\Rightarrow \quad \Phi(t) = \begin{bmatrix} (3e^{-2t} - 2e^{-3t}) & (e^{-2t} - e^{-3t}) \\ (-6e^{-2t} + 6e^{-3t}) & (-2e^{-2t} + 3e^{-3t}) \end{bmatrix}$$

Performance of discrete system – Example 2 (cont')

- The state equation of the discrete open-loop system:

$$\begin{cases} \boldsymbol{x}[(k+1)T] = \boldsymbol{A}_d \boldsymbol{x}(kT) + \boldsymbol{B}_d e_R(kT) \\ y(kT) = \boldsymbol{C}_d \boldsymbol{x}(kT) \end{cases}$$

$$\boldsymbol{A}_d = \Phi(T) = \left.\begin{bmatrix} (3e^{-2T} - 2e^{-3T}) & (e^{-2T} - e^{-3T}) \\ (-6e^{-2T} + 6e^{-3T}) & (-2e^{-2T} + 3e^{-3T}) \end{bmatrix}\right|_{T=0.1} = \begin{bmatrix} 0.9746 & 0.0779 \\ -0.4675 & 0.5850 \end{bmatrix}$$

$$\boldsymbol{B}_d = \int_0^T \Phi(\tau)\boldsymbol{B}\,d\tau = \int_0^T \left\{ \begin{bmatrix} (3e^{-2\tau} - 2e^{-3\tau}) & (e^{-2\tau} - e^{-3\tau}) \\ (-6e^{-2\tau} + 6e^{-3\tau}) & (-2e^{-2\tau} + 3e^{-3\tau}) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau \right\}$$

$$= \int_0^T \left\{ \begin{bmatrix} (e^{-2\tau} - e^{-3\tau}) \\ (-2e^{-2\tau} + 3e^{-3\tau}) \end{bmatrix} d\tau \right\} = \left.\begin{bmatrix} (-\dfrac{e^{-2\tau}}{2} + \dfrac{e^{-3\tau}}{3}) \\ (e^{-2\tau} - e^{-3\tau}) \end{bmatrix}\right|_0^{0.1} = \begin{bmatrix} 0.0042 \\ 0.0779 \end{bmatrix}$$

$$\boldsymbol{C}_d = \boldsymbol{C} = \begin{bmatrix} 10 & 2 \end{bmatrix}$$

Performance of discrete system – Example 2 (cont')

★ The state equation of the discrete closed-loop system:

$$\begin{cases} \boldsymbol{x}[(k+1)T] = \left[\boldsymbol{A}_d - \boldsymbol{B}_d\boldsymbol{C}_d\right]\boldsymbol{x}(kT) + \boldsymbol{B}_d r(kT) \\ y(kT) = \boldsymbol{C}_d\boldsymbol{x}(kT) \end{cases}$$

with

$$\left[\boldsymbol{A}_d - \boldsymbol{B}_d\boldsymbol{C}_d\right] = \begin{bmatrix} 0.9746 & 0.0779 \\ -0.4675 & 0.5850 \end{bmatrix} - \begin{bmatrix} 0.0042 \\ 0.0779 \end{bmatrix}\begin{bmatrix} 10 & 2 \end{bmatrix} = \begin{bmatrix} 0.9326 & 0.0695 \\ -1.2465 & 0.4292 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.9326 & 0.0695 \\ -1.2465 & 0.4292 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.0042 \\ 0.0779 \end{bmatrix} r(kT)$$

$$y(k) = \begin{bmatrix} 10 & 2 \end{bmatrix}.\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Performance of discrete system – Example 2 (cont')

2. Time response of the system:

From the closed-loop state equations, we have:

$$\begin{cases} x_1(k+1) = 0.9326x_1(k) + 0.0695x_2(k) + 0.0042r(k) \\ x_2(k+1) = -1.2465x_1(k) + 0.4292x_2(k) + 0.0779r(t) \end{cases}$$

With initial condition $x_1(-1)=x_2(-1)=0$, unit step input, we can calculate the solution to the state equation:

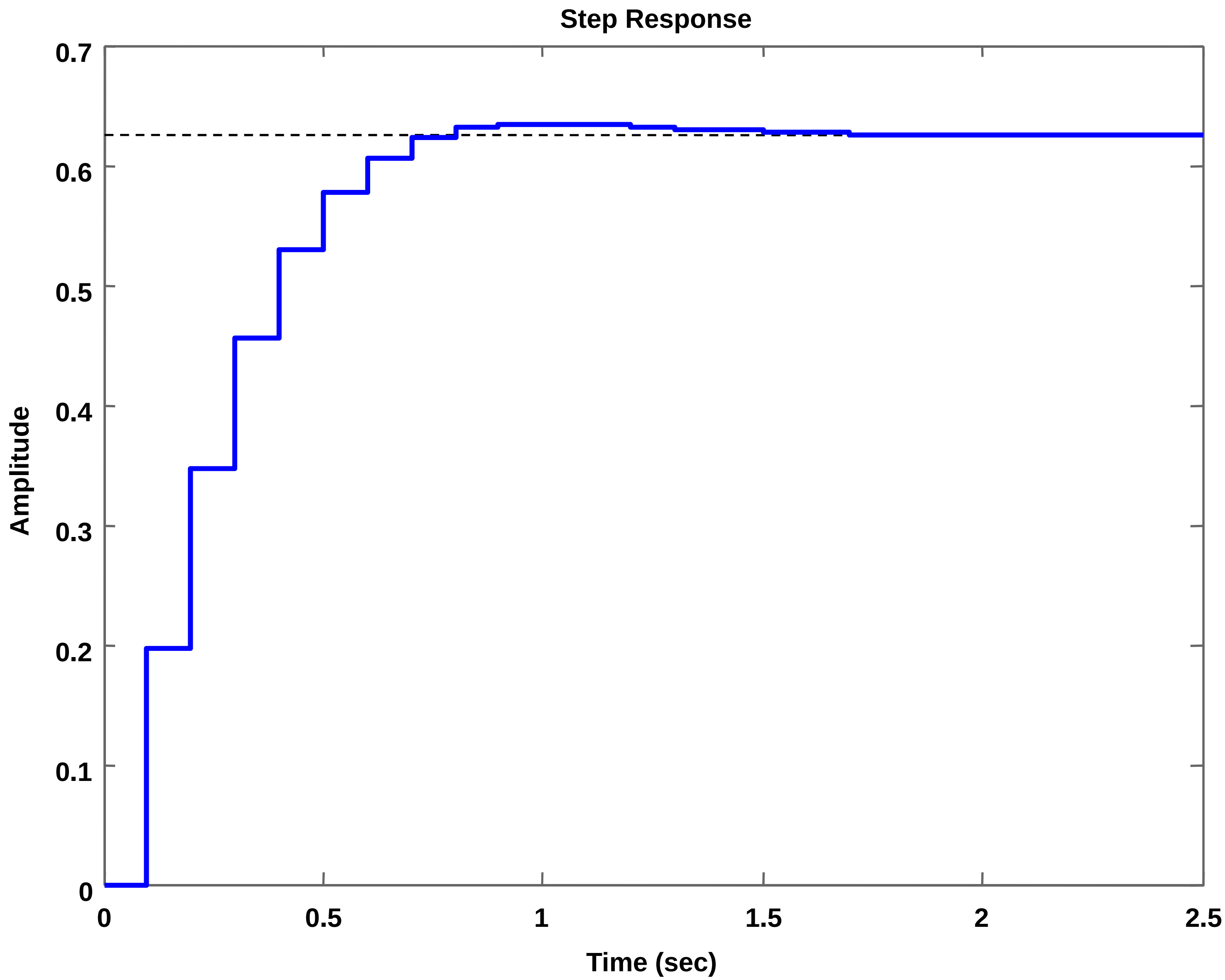
$$x_1(k) = 10^{-3} \times \{0; 4.2; 13.5; 24.2; 34.2; 42.6; 49.1; 54.0; 57.4; 59.7;\ldots$$
$$61.2; 62.0; 62.5; 62.7; 62.8; 62.8; 62.7; 62.7; 62.6; 62.6\ldots\}$$

$$x_2(k) = 10^{-3} \times \{0; 77.9; 106.1; 106.6; 93.5; 75.4; 57.2; 41.2; 28.3; 18.5;\ldots$$
$$11.4; 6.5; 3.4; 1.4; 0.3; -0.3; -0.5; -0.5; -0.5; -0.4\ldots\}$$

The closed-loop system response: $y(k) = 10x_1(k) + 2x_2(k)$

$$y(k) = \{0; 0.198; 0.348; 0.455; 0.529; 0.577; 0.606; 0.622; 0.631; 0.634;\ldots$$
$$0.635; 0.634; 0.632; 0.630; 0.629; 0.627; 0.627; 0.626; 0.625; 0.625\ldots\}$$

Performance of discrete system – Example 2 (cont')



Performance of discrete system – Example 2 (cont')

3. Performances of the system:

- Percentage of overshoot:

$$y_{max} = 0.635 \qquad\qquad \Rightarrow \qquad POT = \frac{y_{max} - y_{ss}}{y_{ss}} 100\% = 1.6\%$$

$$y_{ss} = 0.625$$

- The settling time:

$$(1 - 0.05)y \le y(k) \le (1 + 0.05)y \quad , \forall k \ge k_s$$

According to the response of the system:

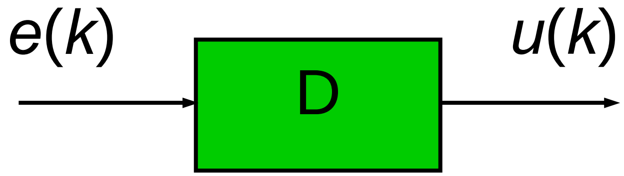
$$0.594 \le y(k) \le 0.656, \qquad \forall k \ge 6$$

$$\Rightarrow \qquad k_s = 6 \qquad \Rightarrow \qquad t_s = k_s T = 0.6\,\mathrm{sec}$$

- Steady state error: $e_{ss} = r_{ss} - y_{ss} = 1 - 0.625 = 0.375$

Discrete PID controllers

Transfer function of discrete difference term



- Differential term: $$u(t) = \frac{de(t)}{dt}$$

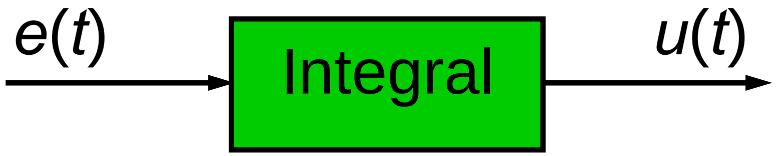
- Discrete difference: $$u(kT) = \frac{e(kT) - e[(k-1)T]}{T}$$

$$\Rightarrow \qquad U(z) = \frac{E(z) - z^{-1}E(z)}{T}$$

$\Rightarrow$ Transfer function of the discrete difference term:

$$\boxed{G_D(z) = \frac{1}{T}\frac{z-1}{z}}$$

Transfer function of discrete integral term



- Continuous integral: $u(t)=\int_0^t e(\tau)d\tau$

- Discrete integral: $u(kT)=\int_0^{kT} e(\tau)d\tau=\int_0^{(k-1)T} e(\tau)d\tau+\int_{(k-1)T}^{kT} e(\tau)d\tau$

$$\Rightarrow u(kT)=u[(k-1)T]+\int_{(k-1)T}^{kT} e(\tau)d\tau=u[(k-1)T]+\frac{T}{2}\big(e[(k-1)T]+e(kT)\big)$$

$$\Rightarrow U(z)=z^{-1}U(z)+\frac{T}{2}\left(z^{-1}E(z)+E(z)\right)$$

$\Rightarrow$ TF of discrete integral term: $\boxed{G_I(z)=\dfrac{T}{2}\dfrac{z+1}{z-1}}$

Transfer function of discrete PID controller

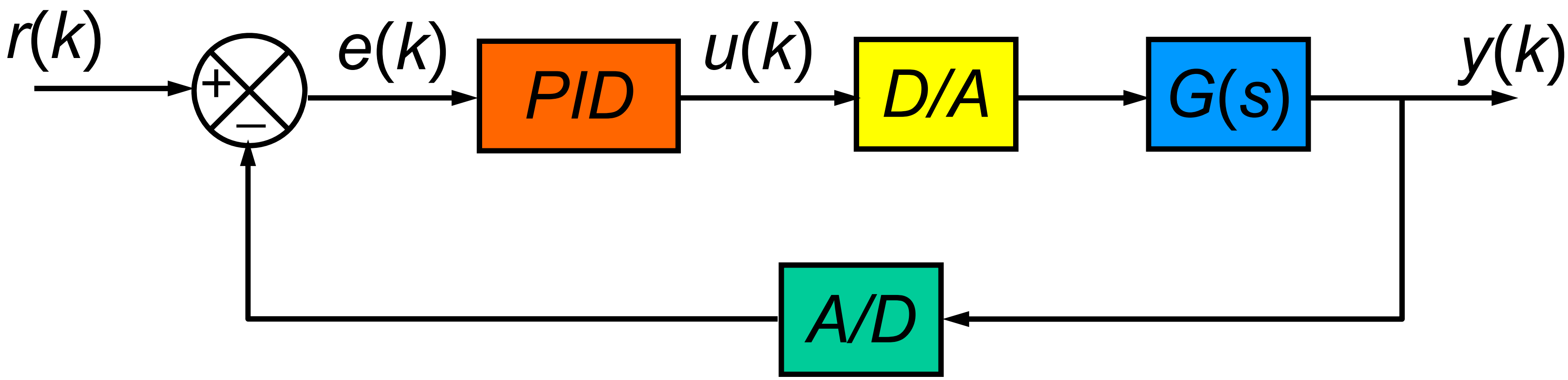
- Continuous PID controller:

$$G_{PID}(s) = K_P + \frac{K}{s} + K_D s$$

- Discrete PID controller:

$$\boxed{G_{PID}(z) = \underbrace{K_P}_{P} + \underbrace{\frac{K_I T}{2}\,\frac{z+1}{z-1}}_{I} + \underbrace{\frac{K_D}{T}\,\frac{z-1}{z}}_{D}}$$

Digital PID controller



$$G_{PID}(z) = \frac{U(z)}{E(z)} = K_P + \frac{K_I T}{2}\frac{z+1}{z-1} + \frac{K_D}{T}\frac{z-1}{z}$$

$$u(k) = u(k-1) + K_P[e(k) - e(k-1)] + \frac{K_I T}{2}[e(k) + e(k-1)] + \frac{K_D}{T}[e(k) - 2e(k-1) + e(k-2)]$$

Digital PID control programming

```
float PID_control(float setpoint, float measure)
{    ek_2 = ek_1;
     ek_1 = ek;
     ek = setpoint – measure;
     uk_1 = uk;
     uk = uk_1 + Kp*(ek-ek_1) + Ki*T/2*(ek+ek_1) +…
                    Kd/T*(ek – 2ek_1+ek_2);
     If uk > umax, uk = umax;
     If uk < umin, uk = umin;
     return(uk)
}
```

Note: Kp, Ki, Kd, uk, uk_1, ek, ek_1, ek_2 must be declared as global variables; uk_1, ek_1 and ek_e must be initialized to be zero; umax and umin are constants.

Approaches to design discrete controllers

- ***Indirect design*:** First design a continuous controller, then discretize the controller to have a discrete control system. The performances of the obtained discrete control system are approximate those of the continuous control system provided that the sample time is small enough.

- ***Direct design*:** Directly design discrete controllers in *Z* domain.

  Methods: root locus, pole placement, analytical method, …

Manual tuning of PID controllers

- Effect of increasing a parameter of PID controller independently on closed-loop performance:

Para-meter	Rise time	POT	Settling time	Steady-state error	Stability
K_P	Decrease	Increase	Small change	Decrease	Degrade
K_I	Decrease	Increase	Increase	Eliminate	Degrade
K_D	Minor change	Decrease	Decrease	No effect	Improve if K_D small

Manual tuning of PID controllers (cont.)

A procedure for manual tuning of PID controllers:

1. Set K_I and K_D to 0, gradually increase K_P to the critical gain K_{cr} (i.e. the gain makes the closed-loop system oscilate)
2. Set $K_P \approx K_{cr}/2$
3. Gradually increase K_I until the steady-state error is eliminated in a sufficient time for the process (Note that too much K_I will cause instability).
4. Increase K_D if needed to reduce POT and settling time (Note that too much K_D will cause excessive response and overshoot)

End of Chapter 6