Lecture Notes

Fundamentals of Control Systems

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Chapter 5

DESIGN OF CONTINUOUS CONTROL SYSTEMS

Content

- Introduction
- Effect of controllers on system performance
- Control systems design using the root locus method
- Control systems design in the frequency domain
- Design of PID controllers
- Control systems design in state-space
- Design of state estimators

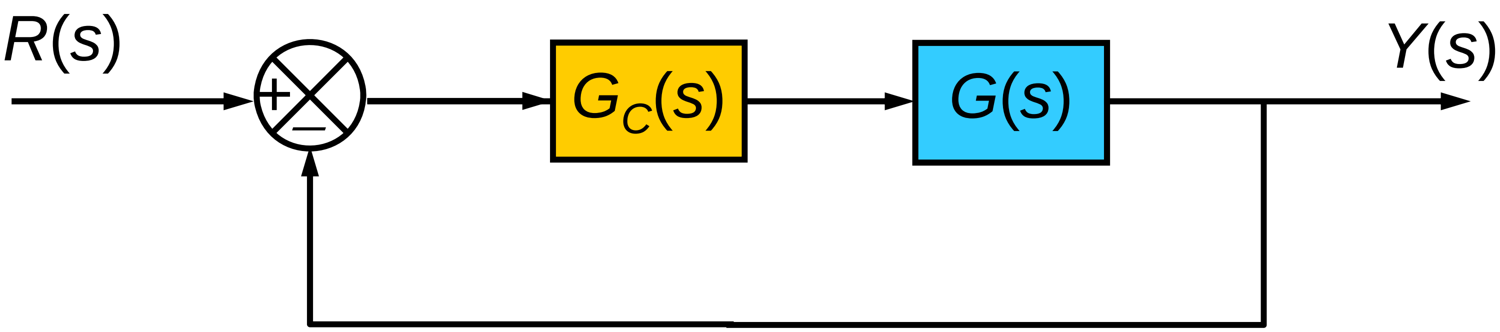
Introduction

Introduction to design process

- Design is a process of adding/configuring hardware as well as software in a system so that the new system satisfies the desired specifications.

Series compensator

✶ The controller is connected in series with the plant.

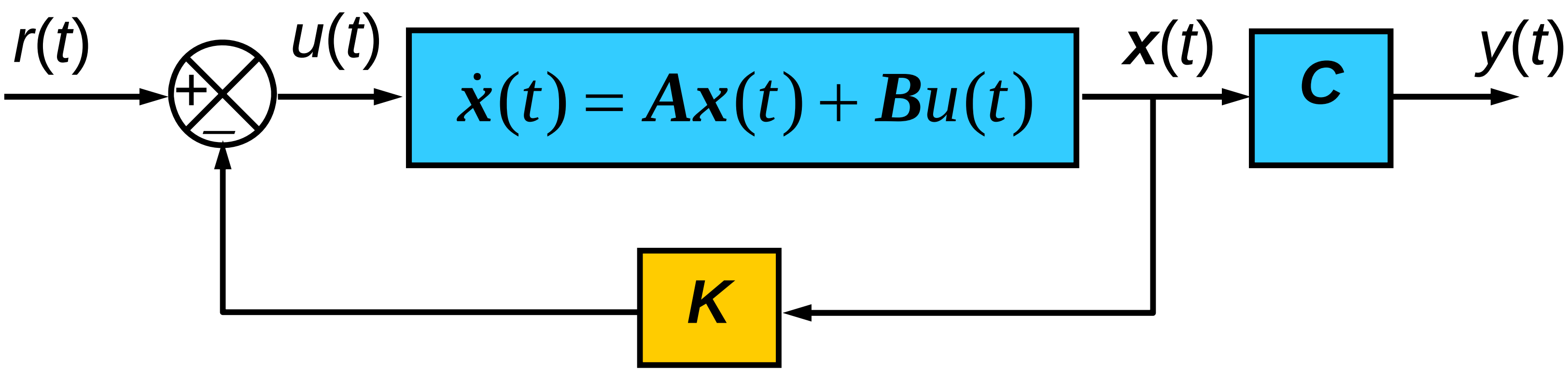


✶ Controllers: phase lead, phase lag, lead-lag compensator, P, PD, PI, PID,…

✶ Design method: root locus, frequency response

State feedback control

- All the states of the system are fed back to calculate the control rule.



- State feedback controller: $u(t) = r(t) - \boldsymbol{K}\boldsymbol{x}(t)$

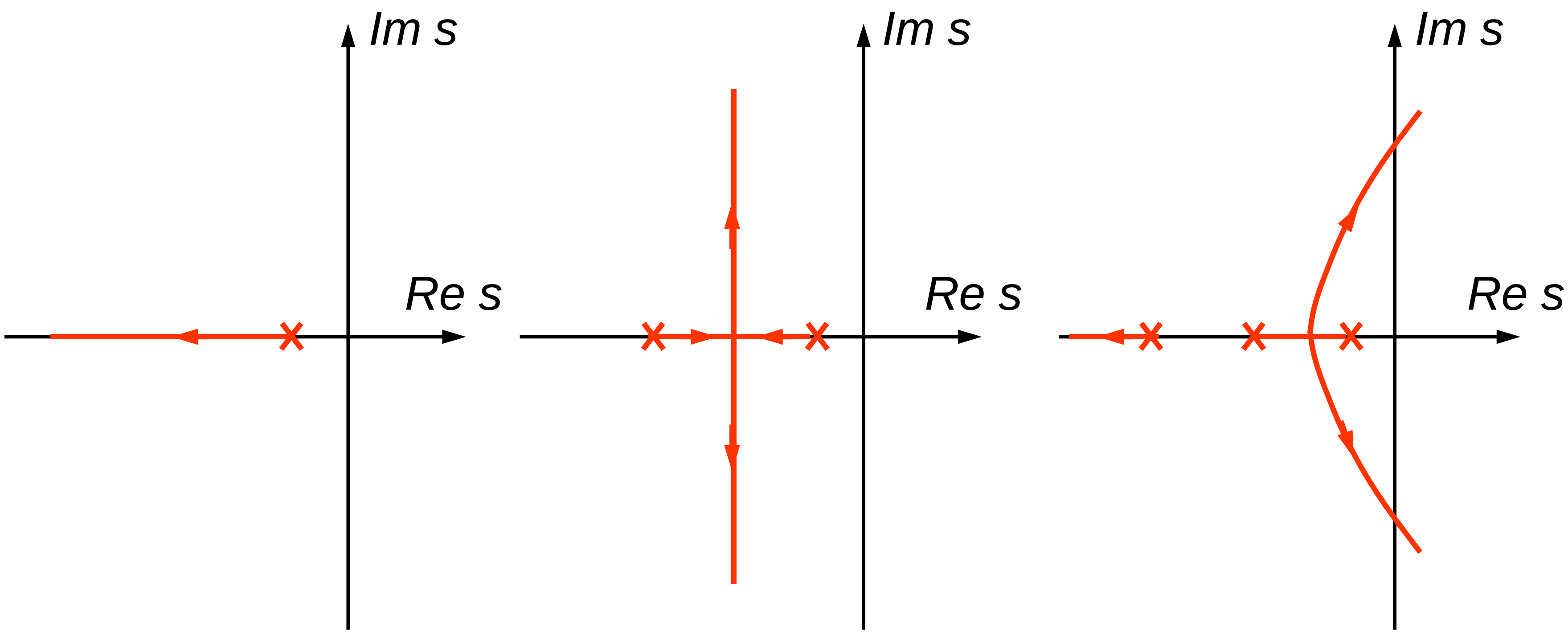
$$\boldsymbol{K} = \begin{bmatrix} k_1 & k_2 & \ldots & k_n \end{bmatrix}$$

- Design method: pole placement, LQR,…

Effects of controller on system performance

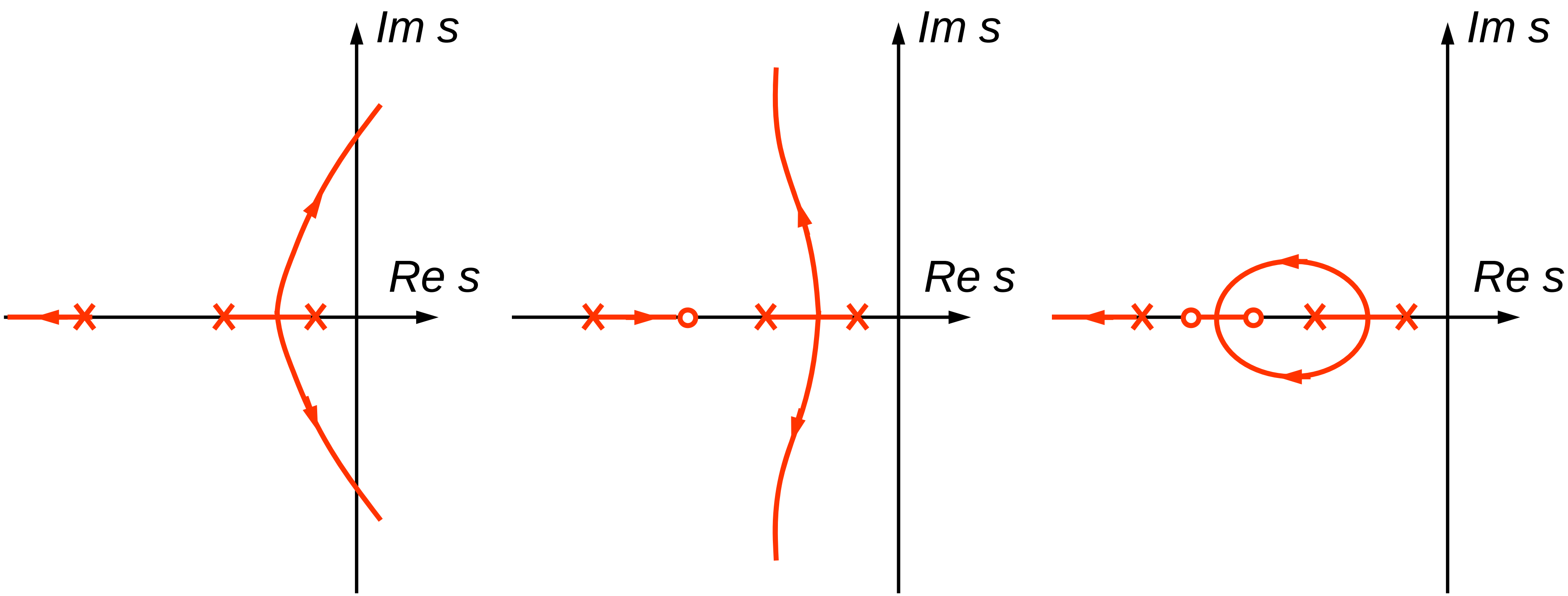
Effects of the addition of poles

- The addition of a pole (in the left-half s-plane) to the open-loop transfer function has the effect of pushing the root locus to the right, tending to lower the system's relative stability and to slow down the settling of the response.



Effects of the addition of zeros

- The addition of a zero (in the left-half s-plane) to the open-loop transfer function has the effect of pulling the root locus to the left, tending to make the system more stable and to speed up the settling of the response.



Effects of lead compensators

- Transfer function:

$$G_C(s) = K_C \frac{1+\alpha Ts}{1+Ts} \qquad (\alpha > 1)$$

- Frequency response:

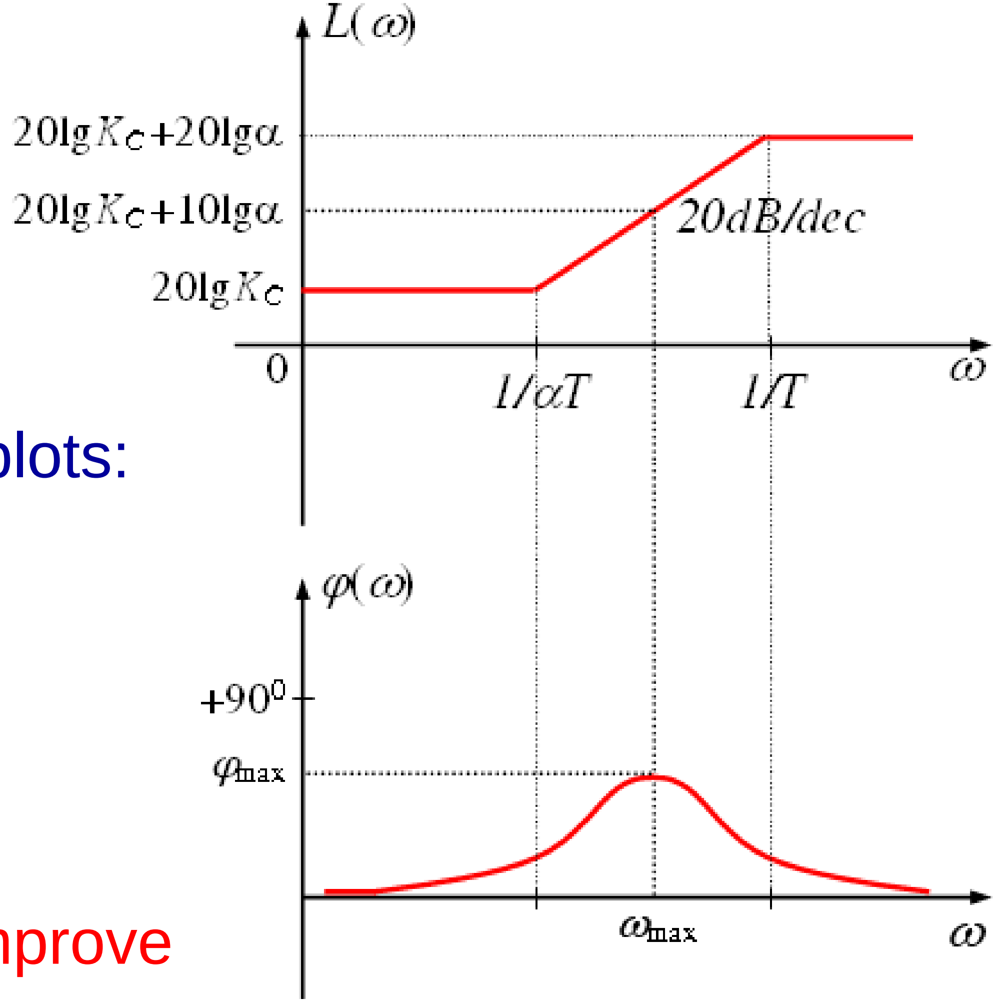
$$G_C(j\omega) = K_C \frac{1+\alpha Tj\omega}{1+Tj\omega}$$

- Characteristics of the Bode plots:

$$\varphi_{\max} = \sin^{-1}\left(\frac{\alpha - 1}{\alpha + 1}\right)$$

$$\omega_{\max} = \frac{1}{T\sqrt{\alpha}}$$

$$L(\omega_{\max}) = 20\lg K_C + 10\lg\alpha$$

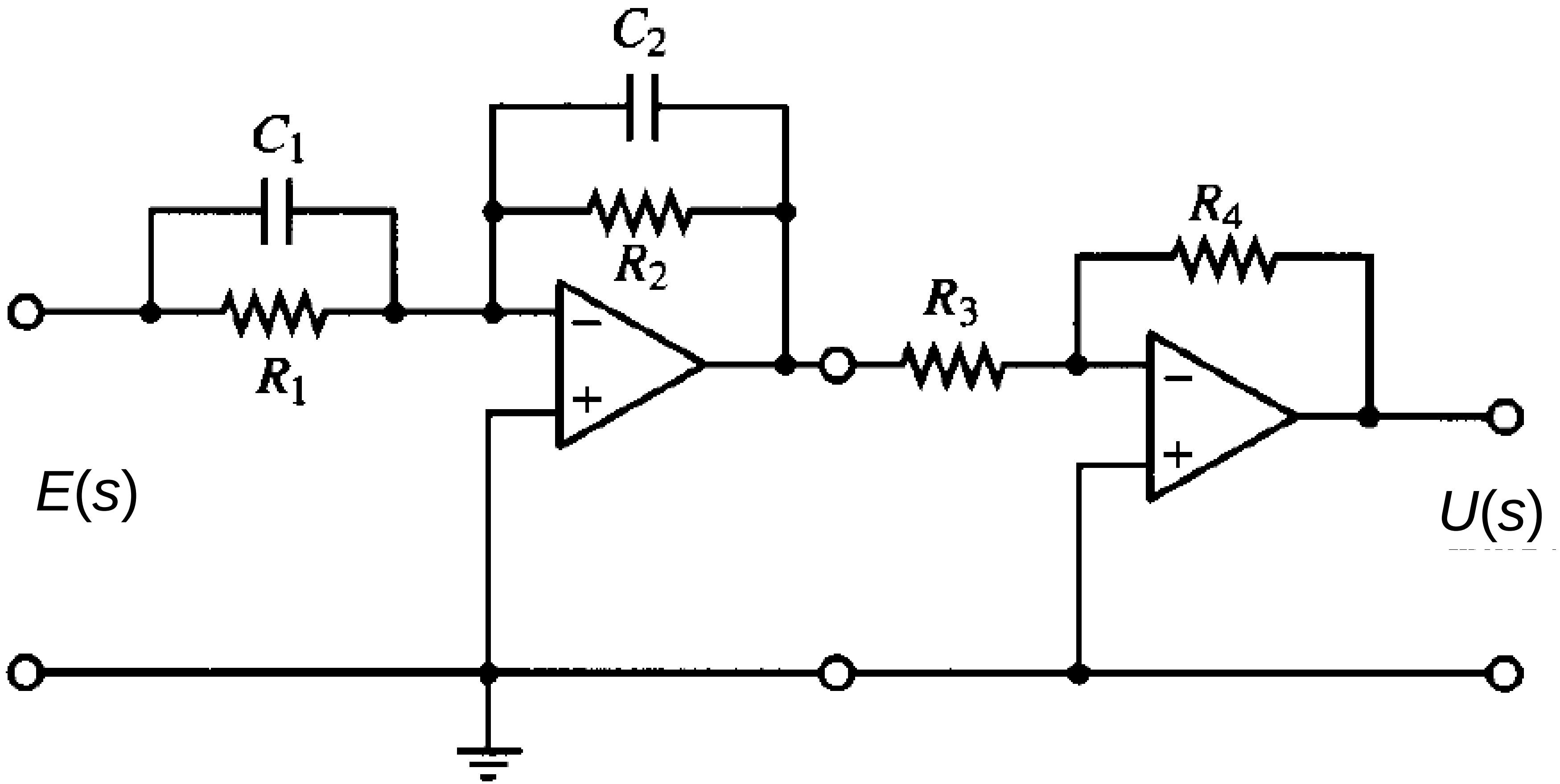


- The lead compensators improve the transient response (POT, t_s,...)

Lead compensator implementation

★ Lead compensator transfer function:

$$\frac{U(s)}{E(s)} = \frac{R_2 R_4}{R_1 R_3}\frac{1+R_1C_1s}{1+R_2C_2s} = K_C\frac{1+\alpha Ts}{1+Ts} \qquad (\alpha > 1 \Leftrightarrow R_1C_1 > R_2C_2)$$



Effects of lag compensators

- Transfer function:

$$G_C(s) = K_C \frac{1+\alpha Ts}{1+Ts} \qquad (\alpha < 1)$$

- Frequency response:

$$G_C(j\omega) = K_C \frac{1+\alpha Tj\omega}{1+Tj\omega}$$

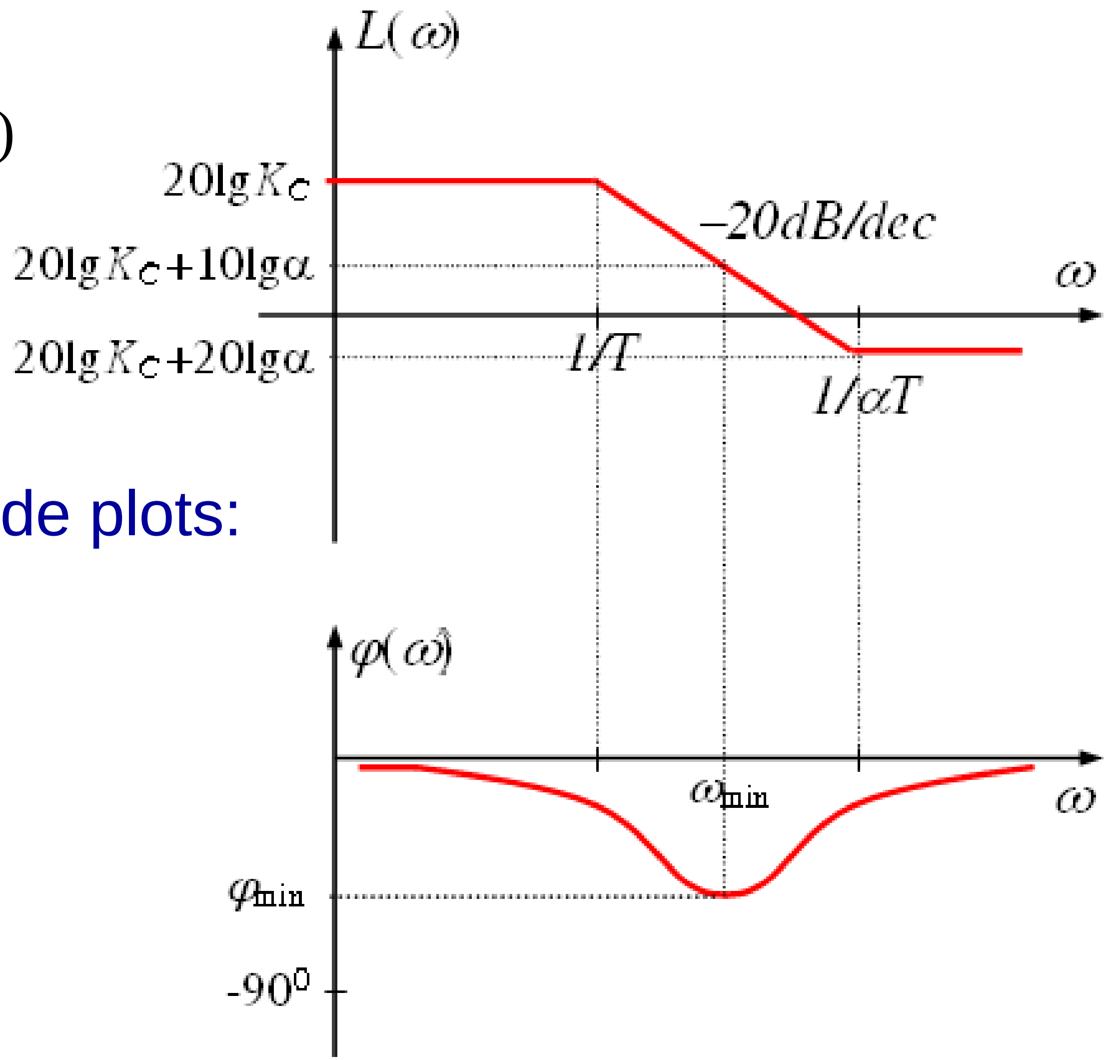
- Characteristics of the Bode plots:

$$\varphi_{\min} = \sin^{-1}\left(\frac{\alpha - 1}{\alpha + 1}\right)$$

$$\omega_{\min} = \frac{1}{T\sqrt{\alpha}}$$

$$L(\omega_{\min}) = 20\lg K_C + 10\lg\alpha$$

- The lag compensators reduce the steady-state error.



Lag compensator implementation

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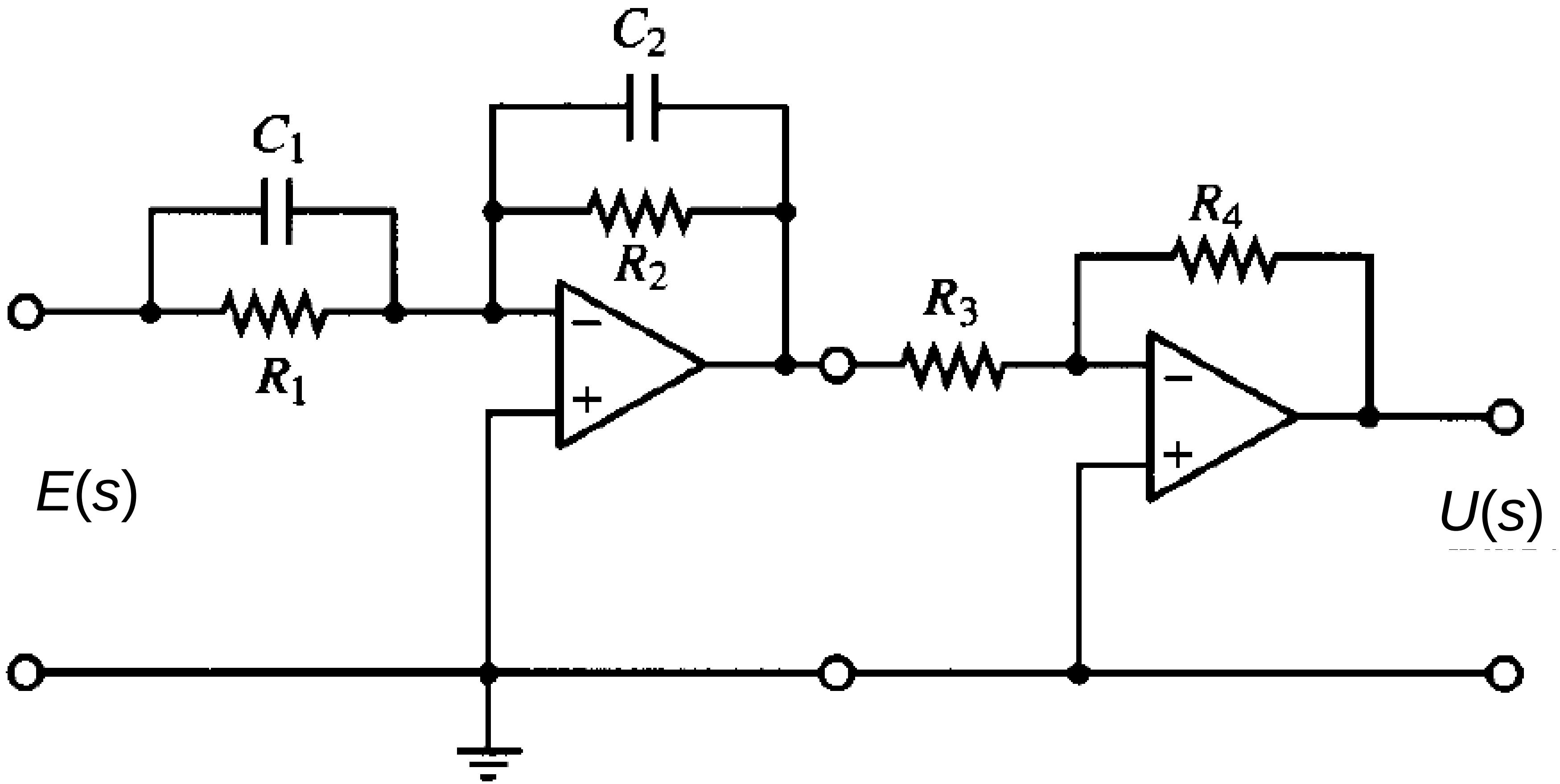
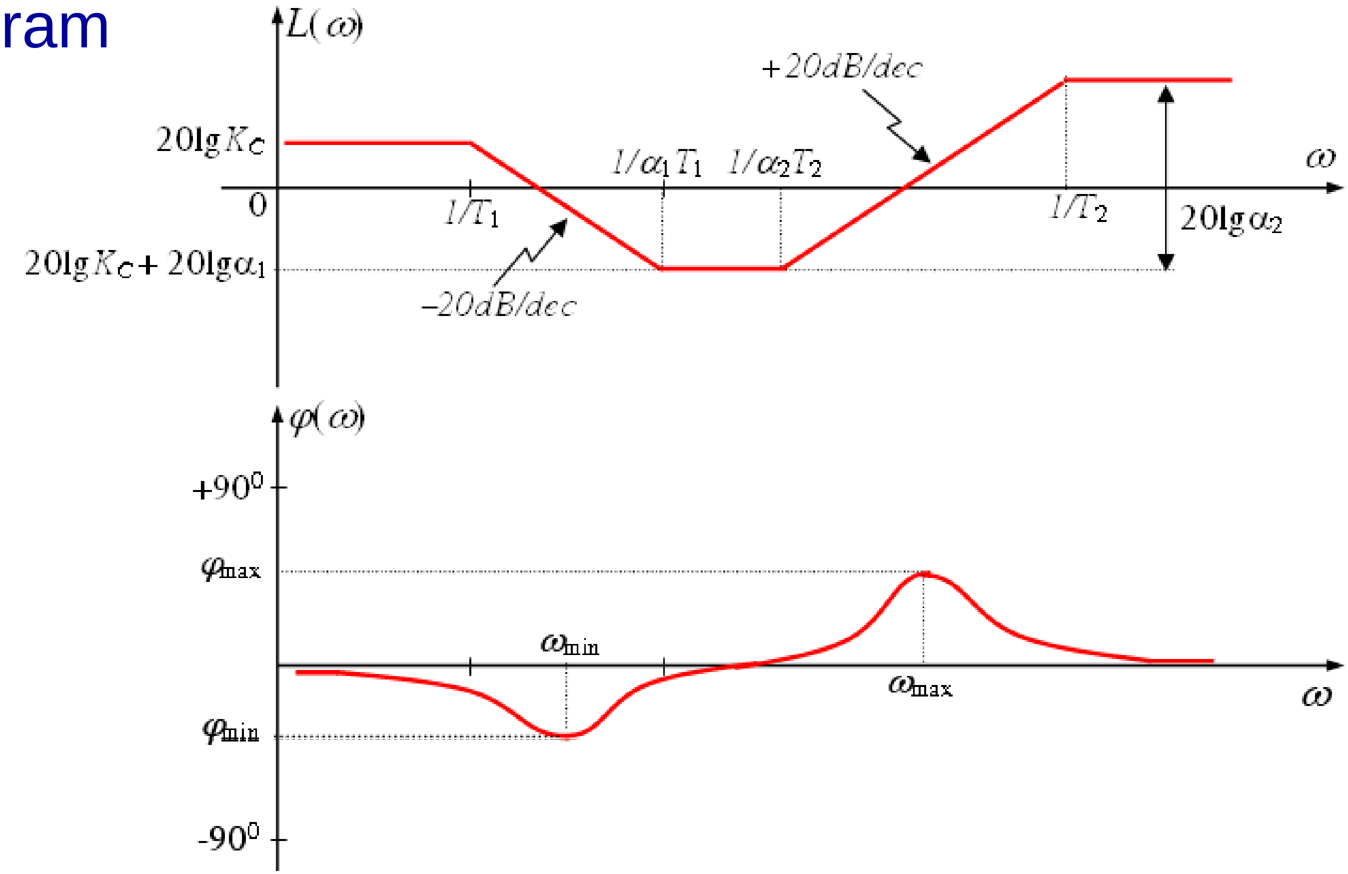


Effects of lead-lag compensators

- Transfer function: $G_C(s) = K_C\left(\dfrac{1+\alpha_1 T_1 s}{1+T_1 s}\right)\left(\dfrac{1+\alpha_2 T_2 s}{1+T_2 s}\right) \qquad (\alpha_1 < 1, \alpha_2 > 1)$
- Bode diagram

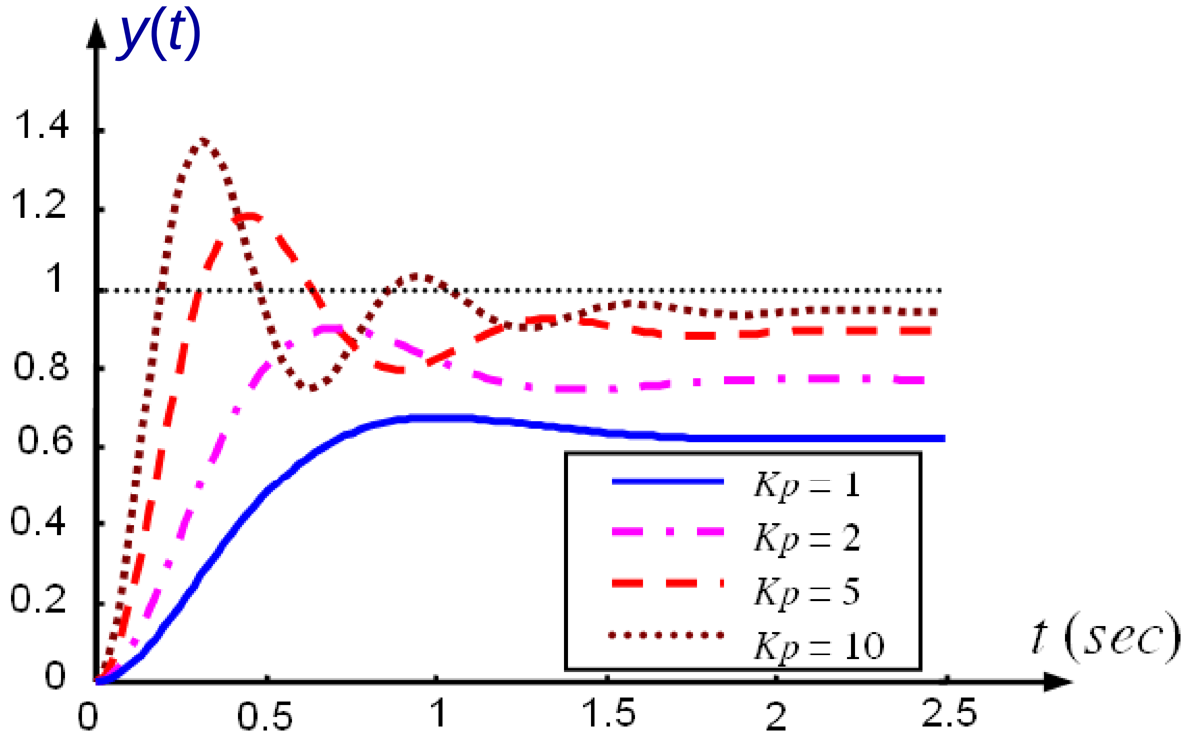


- The lead-lag compensators improve transient response and reduces the steady-state error.

Effects of proportional controller (P)

- Transfer function: $G_C(s) = K_P$
- Increasing proportional gain leads to decreasing steady-state error, however, the system become less stable, and the POT increases.
- Ex: response of a proportional control system whose plant has the transfer function below:

$$G(s) = \frac{10}{(s+2)(s+3)}$$



Effects of proportional derivative controller (PD)

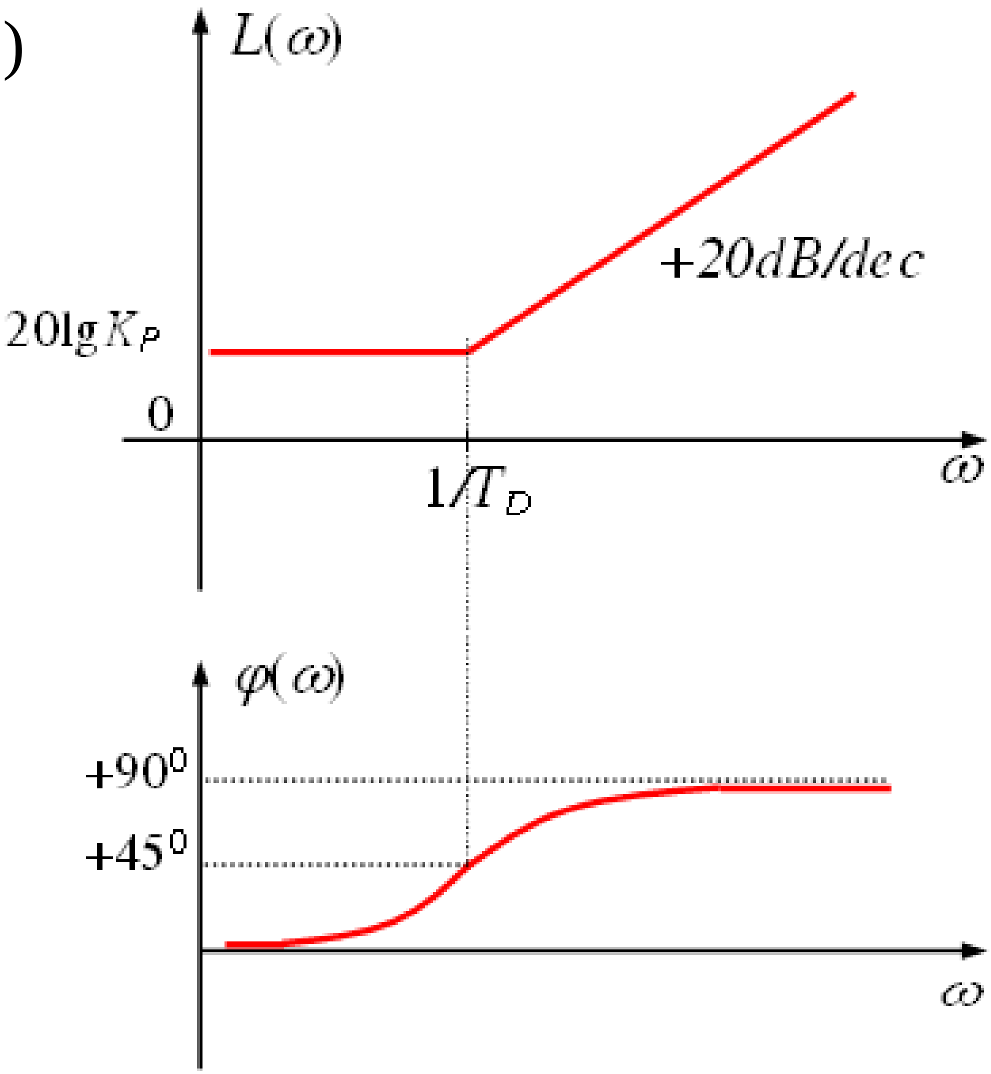
- Transfer function:

$$G_C(s) = K_P + K_D s = K_P(1 + T_D s)$$

- The PD controller is a special case of phase lead compensator, the maximum phase lead is $\varphi_{max}=90^0$ at the frequency $\omega_{max}=+\infty$.

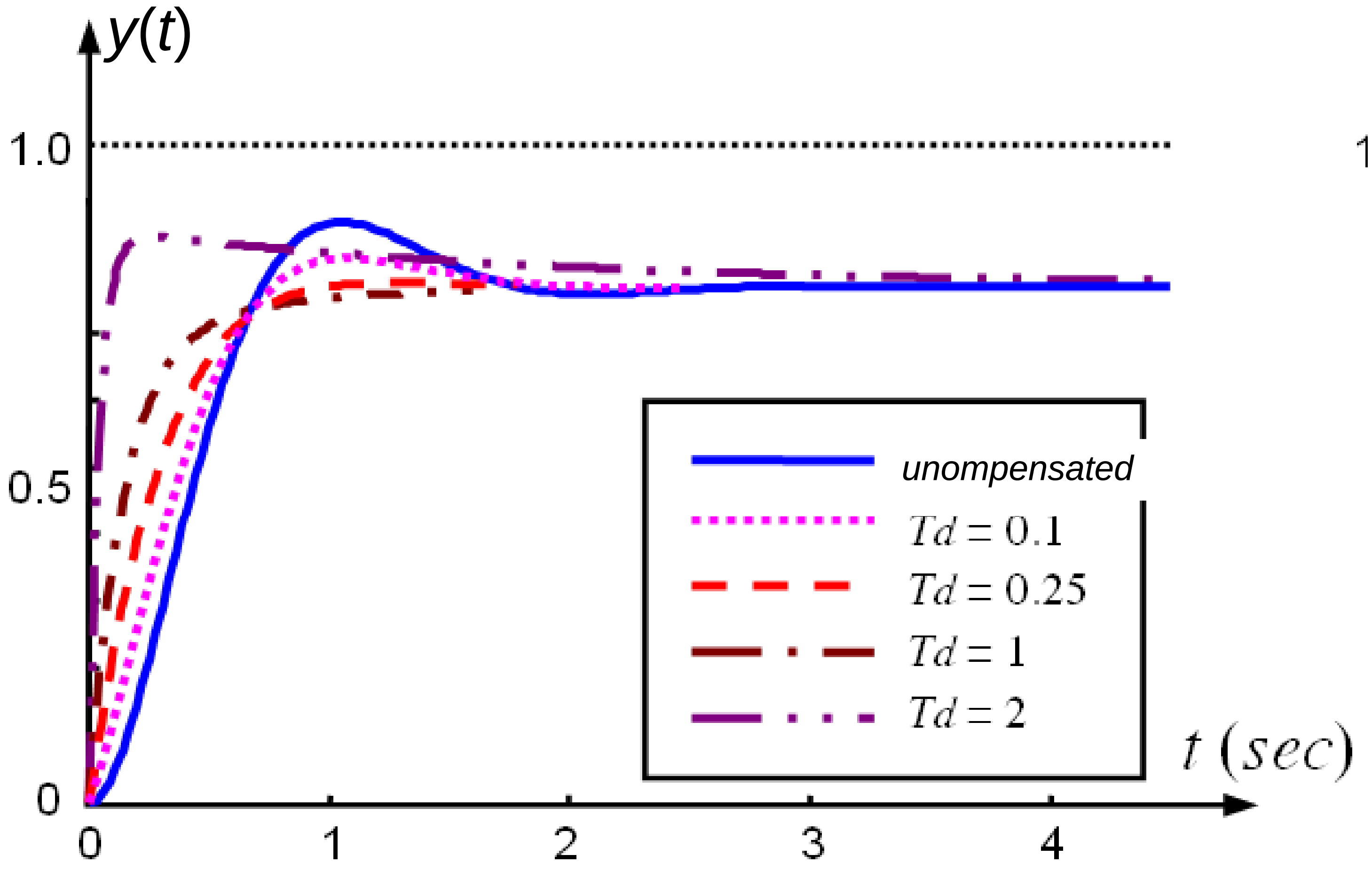
- The PD controller speed up the response of the system, however it also makes the system more sensitive to high frequency noise.

- Bode diagram



Effects of proportional derivative controller (PD)

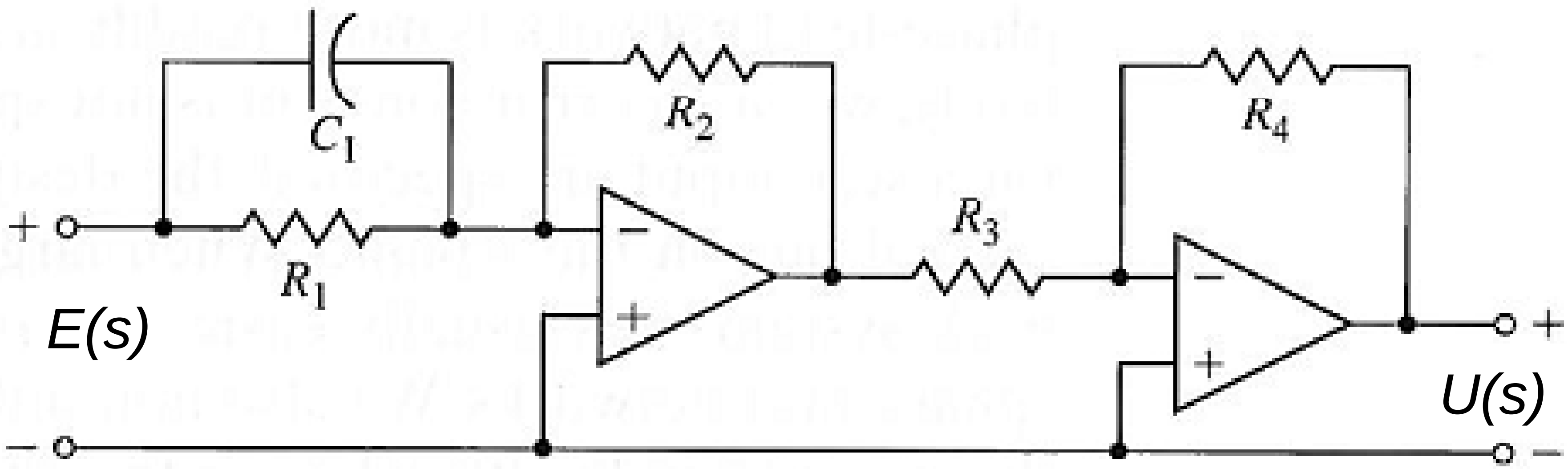
- Note: The larger the derivative constant, the faster the response of the system.



PD controller implementation

- PD controller transfer function:

$$\frac{U(s)}{E(s)} = \frac{R_2 R_4}{R_1 R_3}(1 + R_1 C_1 s) = K_P + K_D s$$



Effects of proportional integral controller (PI)

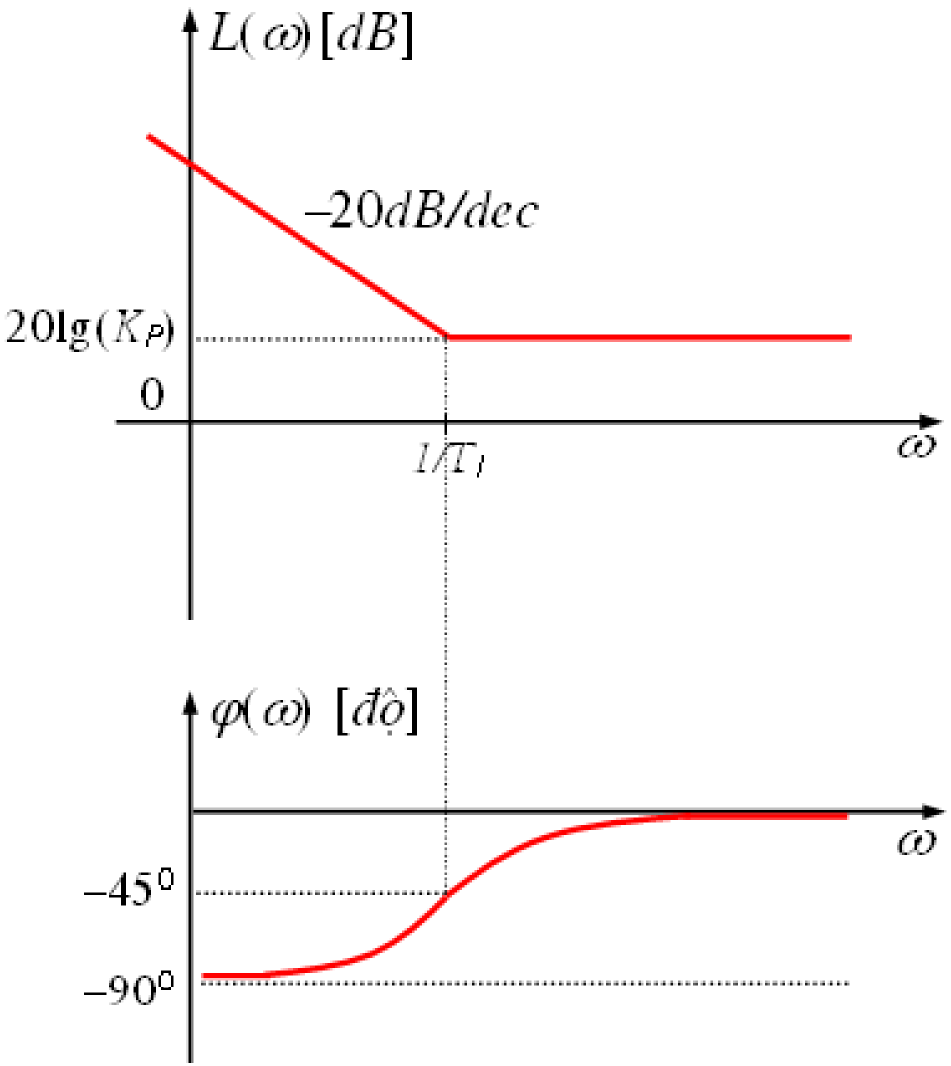
- Transfer function:

$$G_C(s) = K_P + \frac{K_I}{s} = K_P\left(1 + \frac{1}{T_I s}\right)$$

- The PI controller is a special case of phase lag compensator, the minimum phase lag is $\varphi_{min} = -90^0$ at the frequency $\omega_{min} = +\infty$.

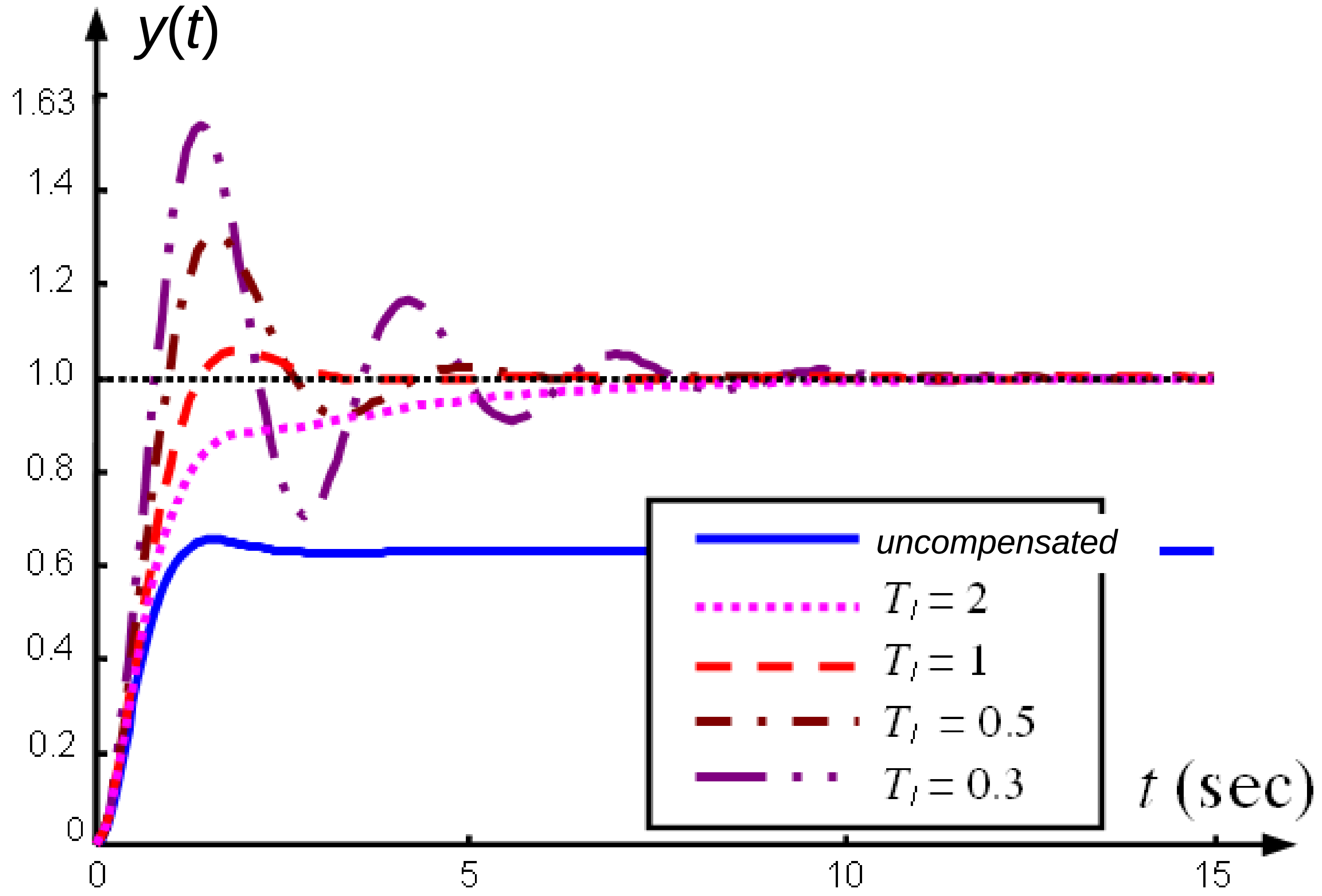
- PI controllers eliminate steady state error to step input, however it can increase POT and settling time.

- Bode diagram



Effects of proportional integral controller (PI)

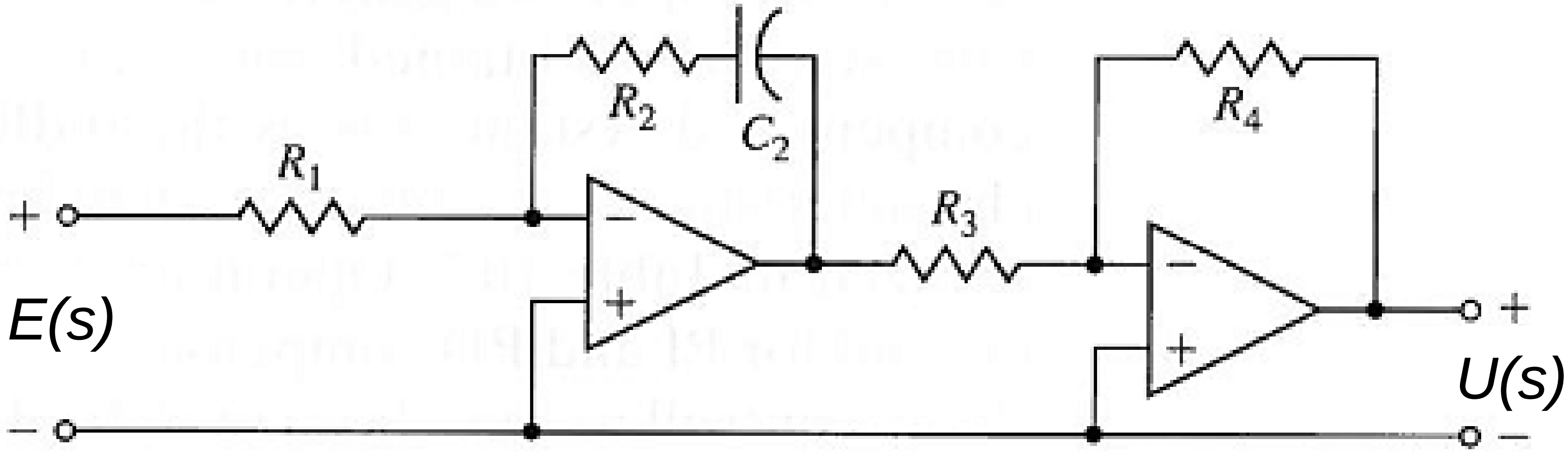
- Note: The larger the integral constant, the larger the POT of response of the system.



PI controller implementation

★ PI controller transfer function:

$$\frac{U(s)}{E(s)} = \frac{R_2 R_4}{R_1 R_3}\frac{R_2 C_2 s + 1}{R_2 C_2 s} = K_P + \frac{K_I}{s}$$



Effects of proportional integral controller (PID)

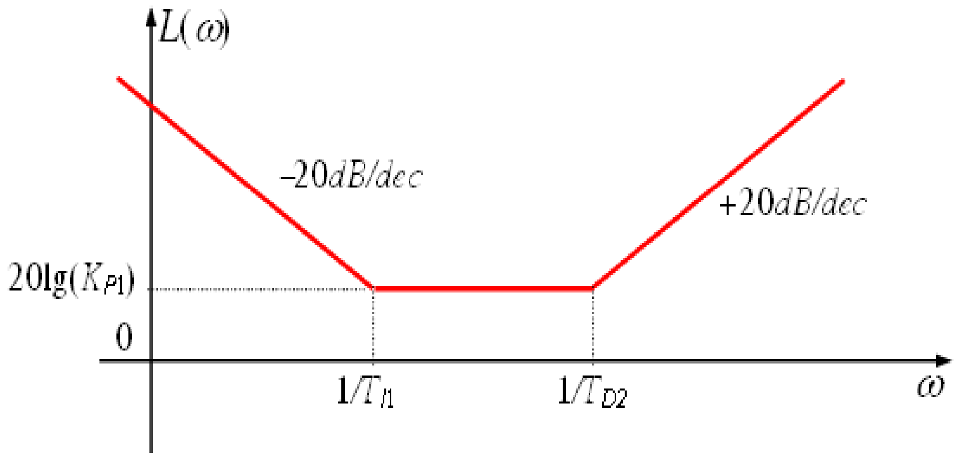
- Transfer function:

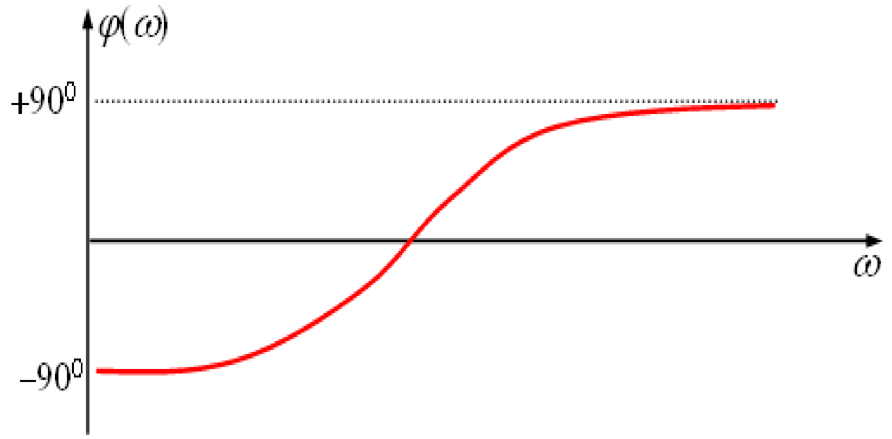
$$G_C(s) = K_P + \frac{K_I}{s} + K_D s$$

$$\Leftrightarrow G_C(s) = K_P\left(1 + \frac{1}{T_I s} + T_D s\right)$$

$$\Leftrightarrow G_C(s) = K_P\left(1 + \frac{1}{T_{I1} s}\right)(1 + T_{D2} s)$$

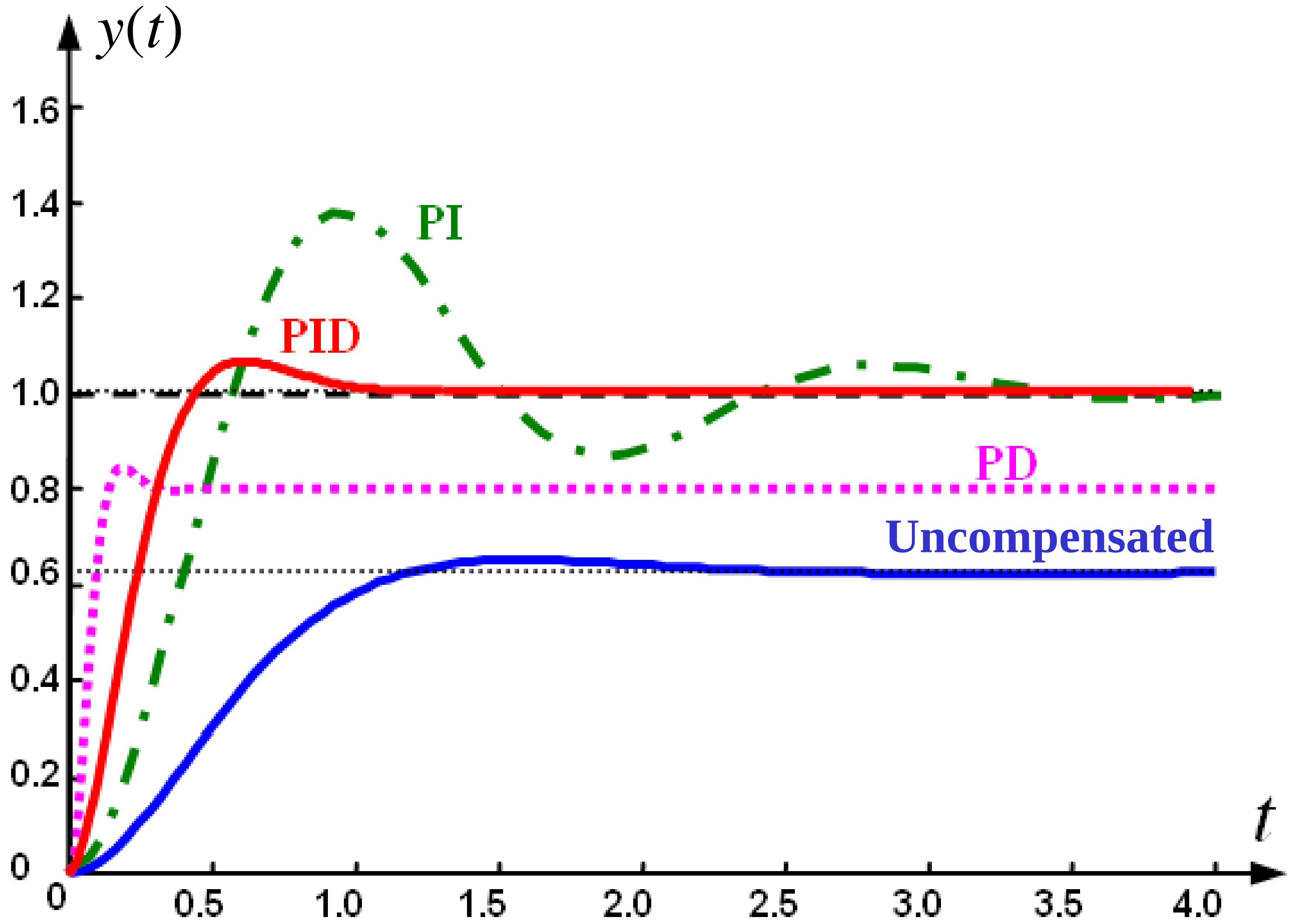
- Bode diagram





- Effects of PID controllers:
  - speed up response of the system
  - Eliminate steady-state error to step input.

Comparison of PI, PD and PID controllers



Control systems design using the root locus method

Procedure for designing lead compensator using the root locus

Lead compensator: $$G_C(s) = K_C \frac{s + (1/\alpha T)}{s + (1/T)} \qquad (\alpha > 1)$$

- ***Step 1:*** Determine the ***dominant poles*** $s_{1,2}^*$ from desired transient response specification:

$$\begin{cases} \text{Overshoot (POT)} \\ \text{Settling time } ts \end{cases} \Rightarrow \begin{cases} \xi \\ \omega_n \end{cases} \Rightarrow s_{1,2}^* = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

- ***Step 2: Determine the deficiency angle*** so that the dominant poles $s_{1,2}^*$ lie on the root locus of the compensated system:

$$\phi^* = -180^0 + \sum_{i=1}^{n} \arg(s_1^* - p_i) - \sum_{i=1}^{m} \arg(s_1^* - z_i)$$

where *pi* and *zi* are poles & zeros of *G*(*s*) before compensation.

$$\phi^* = -180^0 + \sum \text{angle from } p_i \text{ to } s_1^* - \sum \text{angle from } z_i \text{ to } s_1^*$$

Procedure for designing lead compensator using the root locus

- ***Step 3:*** ***Determine the pole & zero*** of the lead compensator

  Draw 2 arbitrarily rays starting from the dominant pole s_1^* such that the angle between the two rays equal to ϕ^*. The intersection between the two rays and the real axis are the positions of the pole and the zero of the lead compensator.

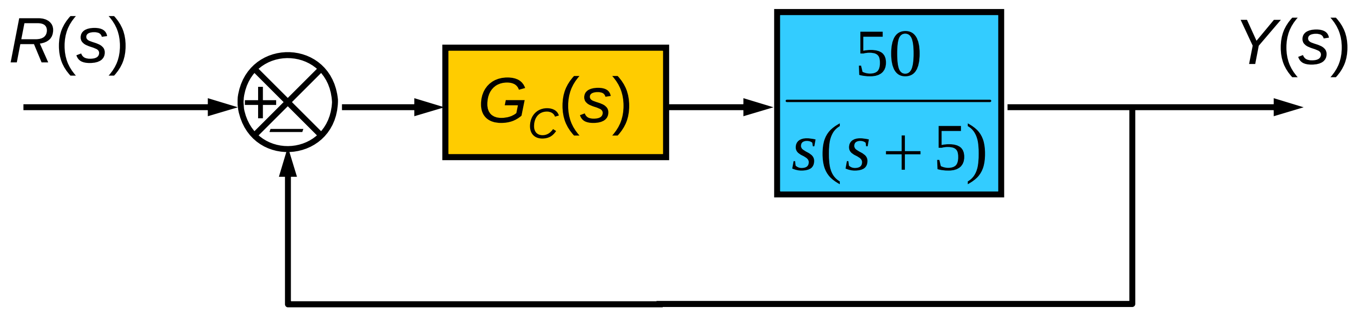
  Two methods often used for drawing the rays:

  - Bisector method
  - Pole elimination method

- ***Step 4:*** ***Calculate the gain K_C*** using the formula:

$$\left|G_C(s)G(s)\right|_{s=s_1^*} = 1$$

Example of designing a lead compensator using RL



- ***Objective:*** design the compensator $G_C(s)$ so that the response of the compensated system satisfies: POT<20%; t_s< 0,5sec (2% criterion).

- ***Solution:***

- Because the design objective is to improve the transient response, we need to design a lead compensator:

$$G_C(s) = K_C \frac{s + (1/\alpha T)}{s + (1/T)} \qquad (\alpha > 1)$$

Example of designing a lead compensator using RL (cont')

✶ ***Step 1:*** Determine the dominant poles:

$$POT = \exp\left(-\frac{\xi\pi}{\sqrt{1-\xi^2}}\right) < 0.2 \quad \Rightarrow \quad -\frac{\xi\pi}{\sqrt{1-\xi^2}} < \ln 0.2 = -1{,}6 \quad \Rightarrow \quad \xi > 0{,}45$$

Chose $\xi = 0{,}707$

$$t_{qđ} = \frac{4}{\xi\omega_n} < 0{,}5 \quad \Rightarrow \quad \omega_n > \frac{4}{0{,}5 \times \xi} \quad \Rightarrow \quad \omega_n > 11{,}4$$

Chose $\omega_n = 15$

The dominant poles are:

$$s^*_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} = -0{,}707 \times 15 \pm j15\sqrt{1-0{,}707^2}$$

$$s^*_{1,2} = -10{,}5 \pm j10{,}5$$

Example of designing a lead compensator using RL (cont')

✶ ***Step 2:*** Determine the deficiency angle:

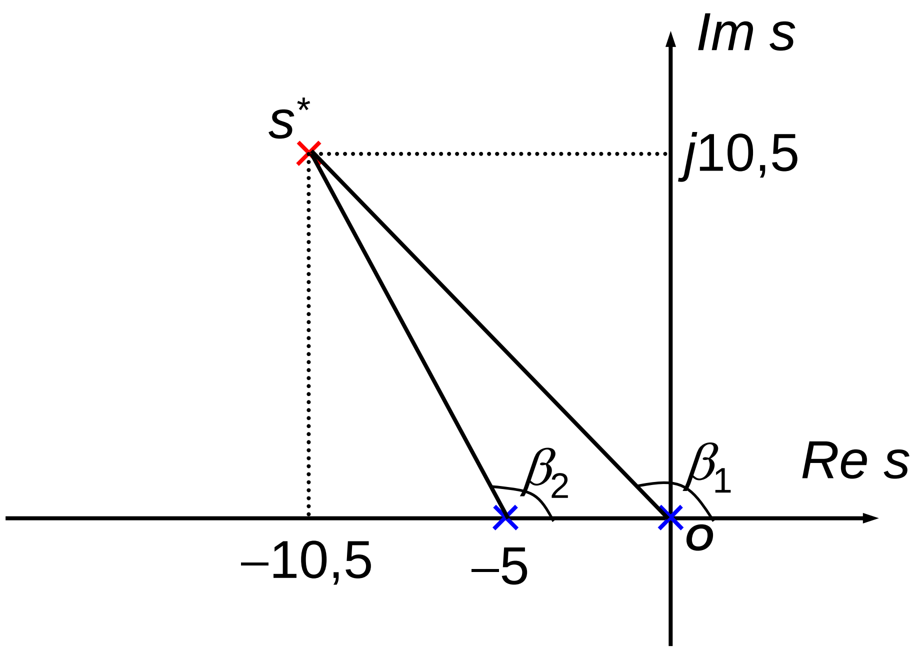
Method 1:

$$\phi^* = -180^0 + \{\arg[(-10{,}5 + j10{,}5) - 0] + \arg[(-10{,}5 + j10{,}5) - (-5)]\}$$

$$= -180^0 + \left\{\arctan\left(\frac{10{,}5}{-10{,}5}\right) + \arctan\left(\frac{10{,}5}{-5{,}5}\right)\right\}$$

$$= -180^0 + (135 + 117{,}6)$$

$$\Rightarrow \phi^* = 72{,}6^0$$



Method 2:

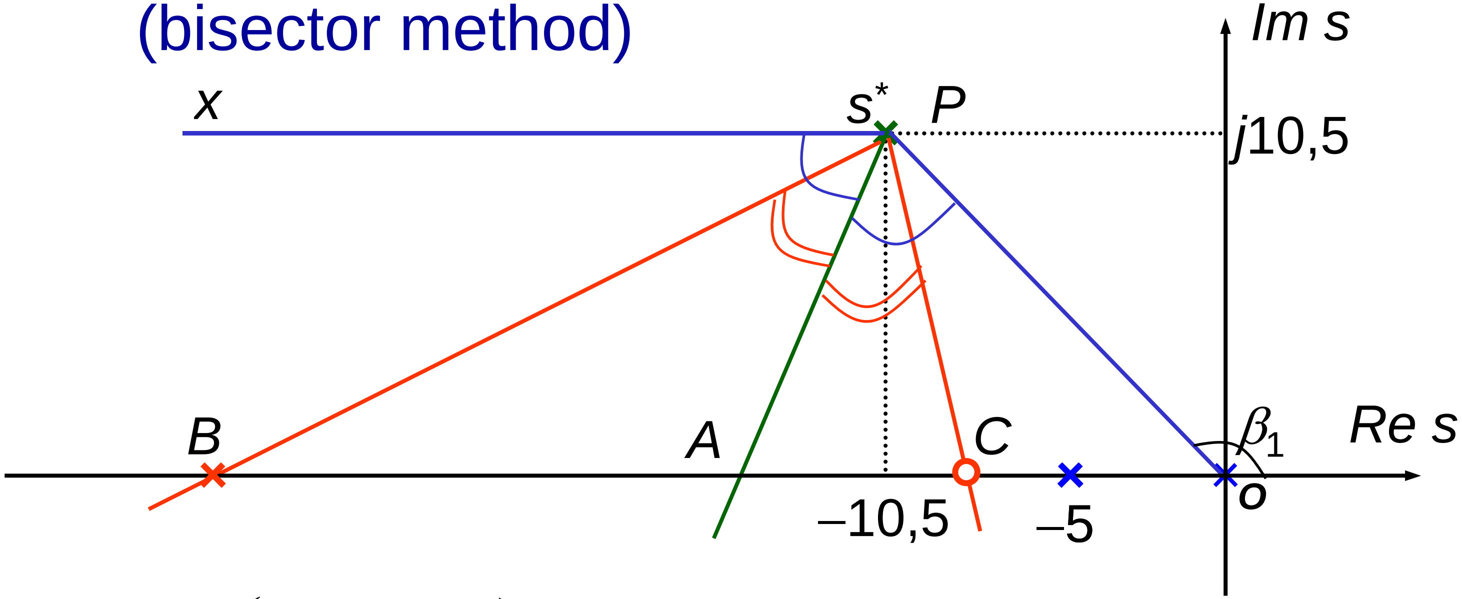
$$\phi^* = -180^0 + (\beta_1 + \beta_2)$$

$$= -180^0 + (135^0 + 117{,}6^0)$$

$$\Rightarrow \phi^* = 72{,}6^0$$

Example of designing a lead compensator using RL (cont')

✶ ***Step 3:*** Determine the pole and the zero of the compensator (bisector method)



$$OB = OP\frac{\sin\left(\frac{O\hat{P}x}{2}+\frac{\phi^*}{2}\right)}{\sin\left(\frac{O\hat{P}x}{2}-\frac{\phi^*}{2}\right)} = 28{,}12 \qquad OC = OP\frac{\sin\left(\frac{O\hat{P}x}{2}-\frac{\phi^*}{2}\right)}{\sin\left(\frac{O\hat{P}x}{2}+\frac{\phi^*}{2}\right)} = 8{,}0$$

$$\Rightarrow G_C(s) = K_C\frac{s+8}{s+28}$$

- ***Step 4:*** Determine the gain of the compensator:

$$\left|G_C(s)G(s)\right|_{s=s^*} = 1$$

$$\Leftrightarrow \left|K_C \frac{-10,5 + j10,5 + 8}{-10,5 + j10,5 + 28} \cdot \frac{50}{(-10,5 + j10,5)(-10,5 + j10,5 + 5)}\right| = 1$$

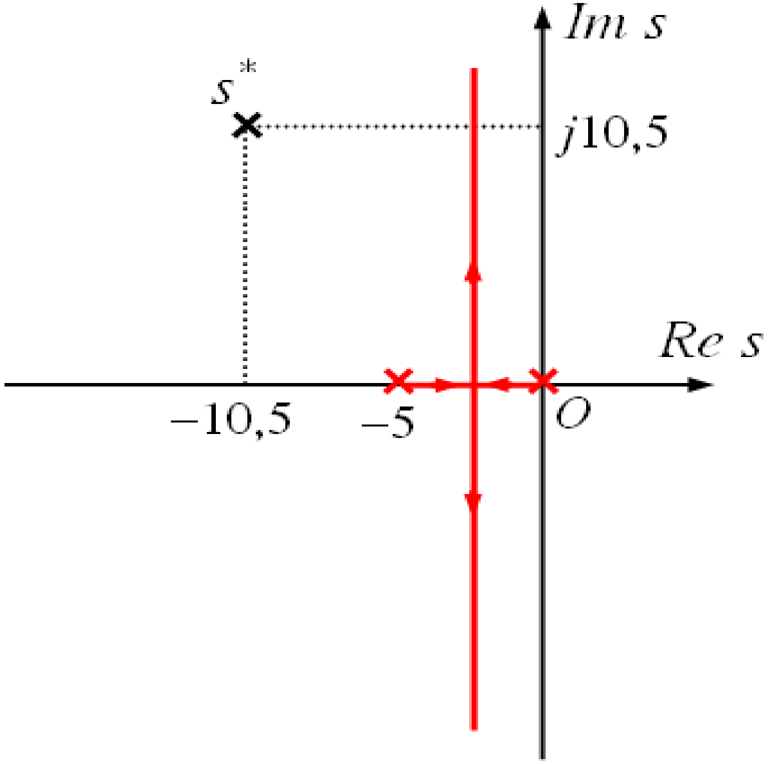
$$\Leftrightarrow K_C \frac{10,79 \times 50}{20,41 \times 15 \times 11,85} = 1$$

$$\Leftrightarrow K_C = 6,7$$

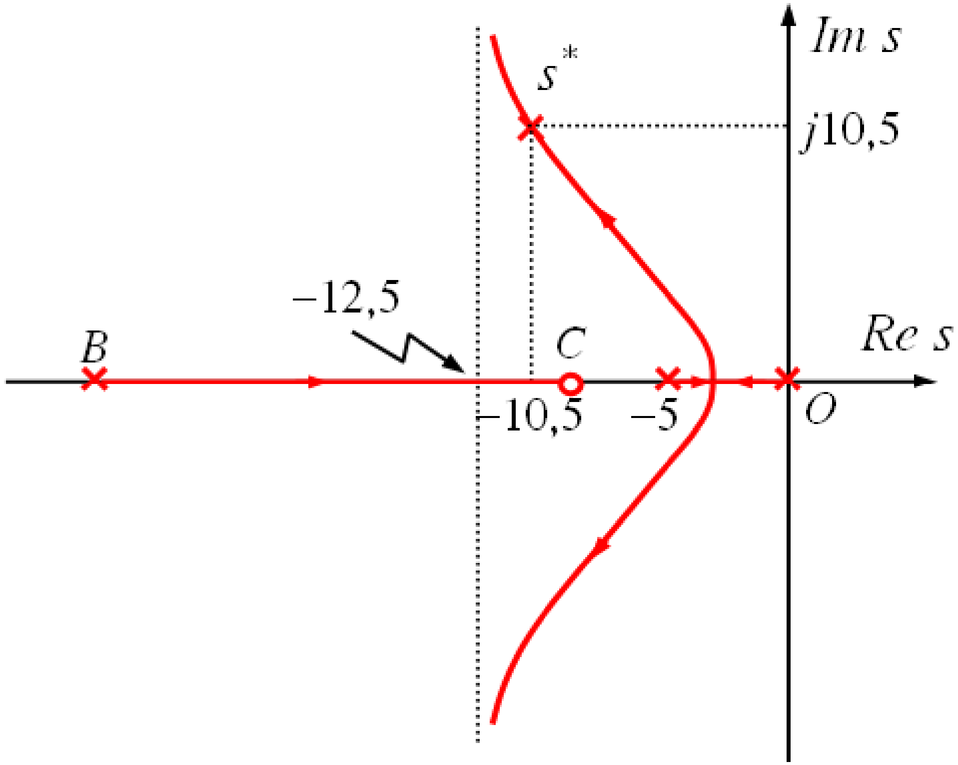
- ***Conclusion:*** The transfer function of the lead compensator is:

$$G_C(s) = 6,7 \frac{s+8}{s+28}$$

Root locus of the system

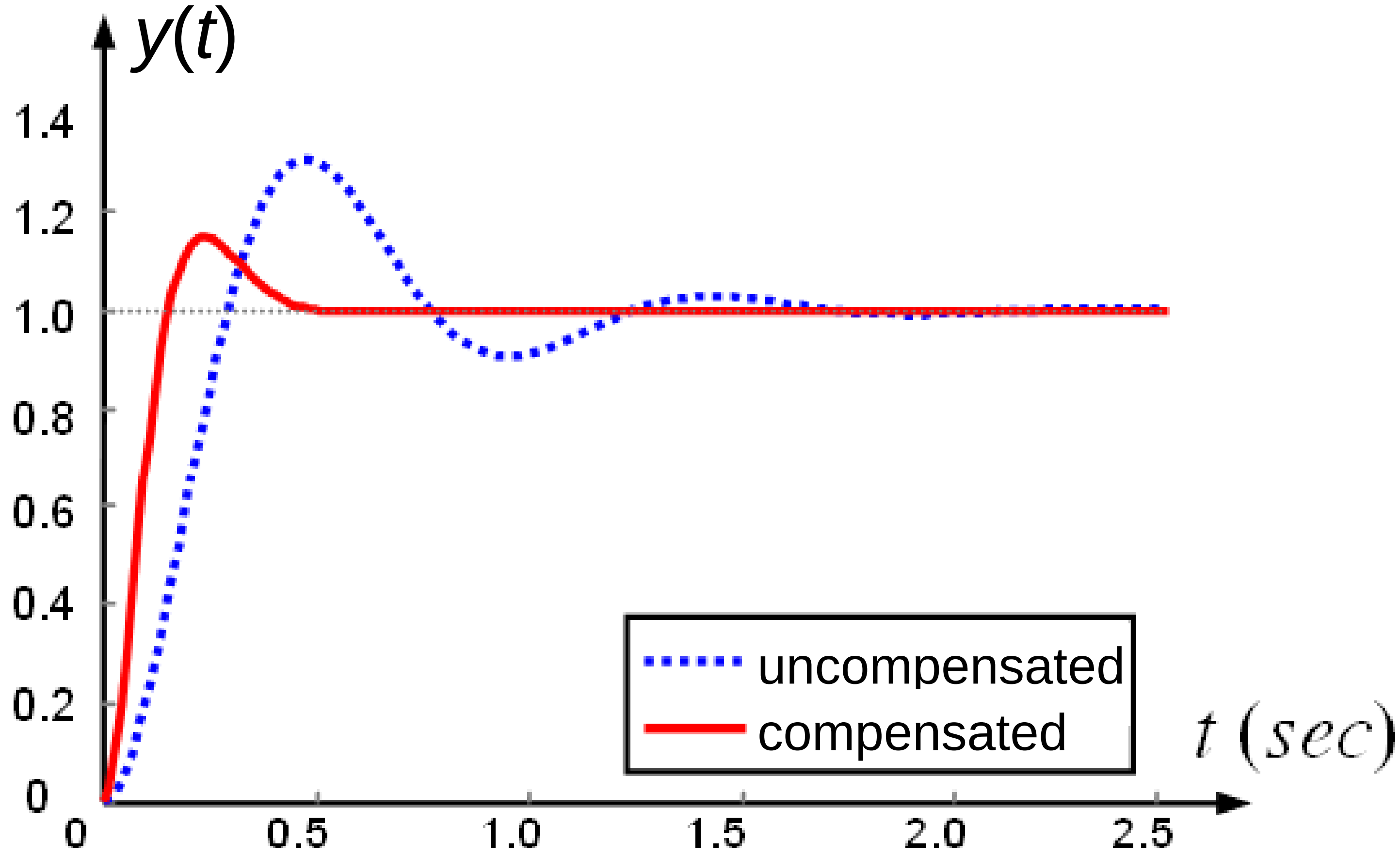


Root locus of the uncompensated system



Root locus of the compensated system

Transient response of the system



Transient response of the system

Procedure for designing lag compensator using the root locus

Lag compensator: $$G_C(s) = K_C \frac{s + (1/\beta T)}{s + (1/T)} \qquad (\beta < 1)$$

- ***Step 1:*** **Determine β** to meet the steady-state error requirement:

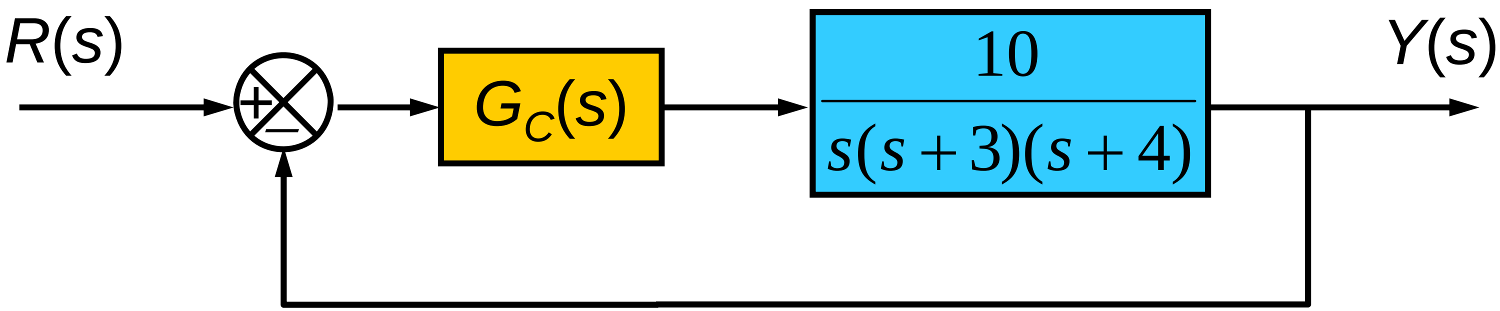
$$\beta = \frac{K_P}{K_P^*} \quad \text{or} \quad \beta = \frac{K_V}{K_V^*} \quad \text{or} \quad \beta = \frac{K_a}{K_a^*}$$

- ***Step 2:*** ***Chose the zero*** of the lag compensator: $\dfrac{1}{\beta T} << \left|\mathrm{Re}(s_{1,2}^*)\right|$

- ***Step 3:*** ***Calculate the pole*** of the compensator: $\dfrac{1}{T} = \beta . \dfrac{1}{\beta T}$

- ***Step 4:*** ***Calculate K_C*** satisfying the condition: $\left|G_C(s)G(s)\right|_{s=s_{1,2}^*} = 1$

Example of designing a lag compensator using RL



- ***Objective:*** design the compensator $G_C(s)$ so that the compensated system satisfies the following performances: steady state error to ramp input is 0,02 and transient response of the compensated system is nearly unchanged.

- ***Solution:***

- The compensator to be design is a lag compensator:

$$G_C(s) = K_C \frac{s + (1/\beta T)}{s + (1/T)} \qquad (\beta < 1)$$

Example of designing a lag compensator using RL (cont’)

- ***Step 1:*** Determine β

The velocity constant of uncompensated system :

$$K_V = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{10}{s(s+3)(s+4)} = 0.83$$

The desired velocity constant:

$$K_V^* = \frac{1}{e_{xl}^*} = \frac{1}{0,02} = 50$$

Then: $$\beta = \frac{K_V}{K_V^*} = \frac{0.83}{50}$$

$$\beta = 0,017$$

Example of designing a lag compensator using RL (cont')

- ***Step 2:*** Chose the zero of the lag compensator

The pole of the uncompensated system:

$$1+G(s)=0 \quad \Leftrightarrow \quad 1+\frac{10}{s(s+3)(s+4)}=0 \quad \Leftrightarrow \quad \begin{cases} s_{1,2}=-1\pm j \\ s_3=-5 \end{cases}$$

$\Rightarrow$ The dominant poles of the uncompensated system: $s_{1,2}=-1\pm j$

Chose: $\dfrac{1}{\beta T} << \left|\mathrm{Re}\{s_1\}\right| = 1 \quad \Rightarrow \quad \boxed{\dfrac{1}{\beta T}=0{,}1}$

- ***Step 3:*** Calculate the pole of the compensator:

$$\frac{1}{T}=\beta\frac{1}{\beta T}=(0{,}017)(0{,}1) \quad \Rightarrow \quad \boxed{\frac{1}{T}=0{,}0017}$$

$$\Rightarrow \quad \boxed{G_C(s)=K_C\frac{s+0{,}1}{s+0{,}0017}}$$

Example of designing a lag compensator using RL (cont')

- ***Step 4:*** Determine the gain of the compensator

$$\left|G_C(s)G(s)\right|_{s=s^*} = 1$$

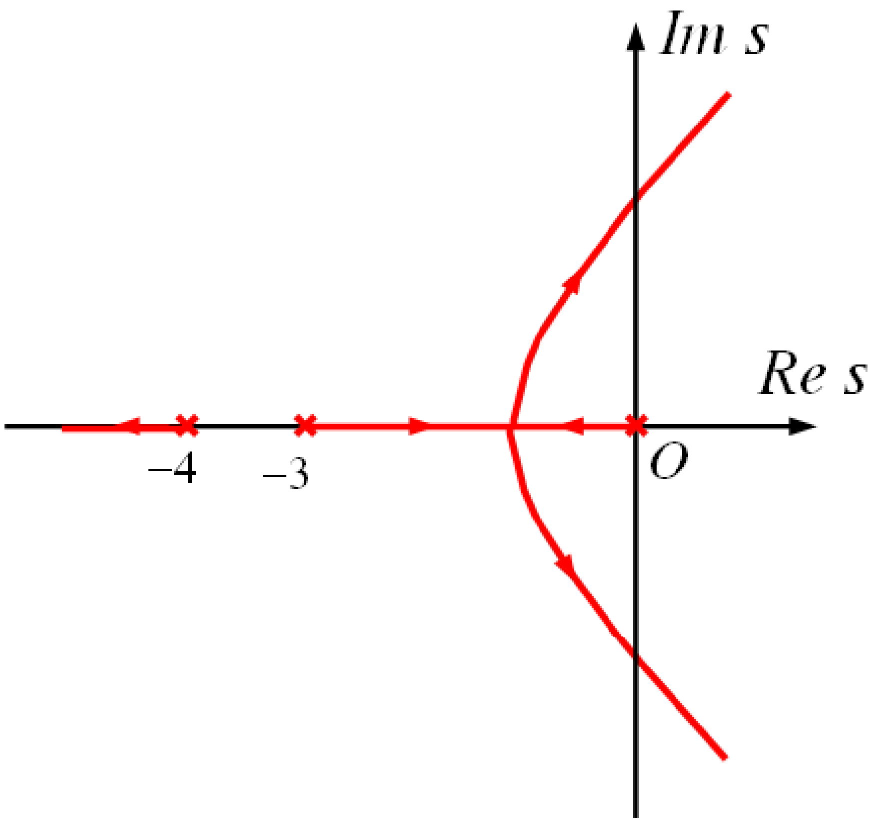
$$\Leftrightarrow \quad \left|K_C \frac{s+0{,}1}{s+0{,}0017} \cdot \frac{10}{s(s+3)(s+4)}\right|_{s=-1\pm j} = 1$$

$$\Rightarrow \quad \left|K_C \frac{(-1+j+0{,}1)}{(-1+j+0{,}0017)} \cdot \frac{10}{(-1+j)(-1+j+3)(-1+j+4)}\right| = 1$$

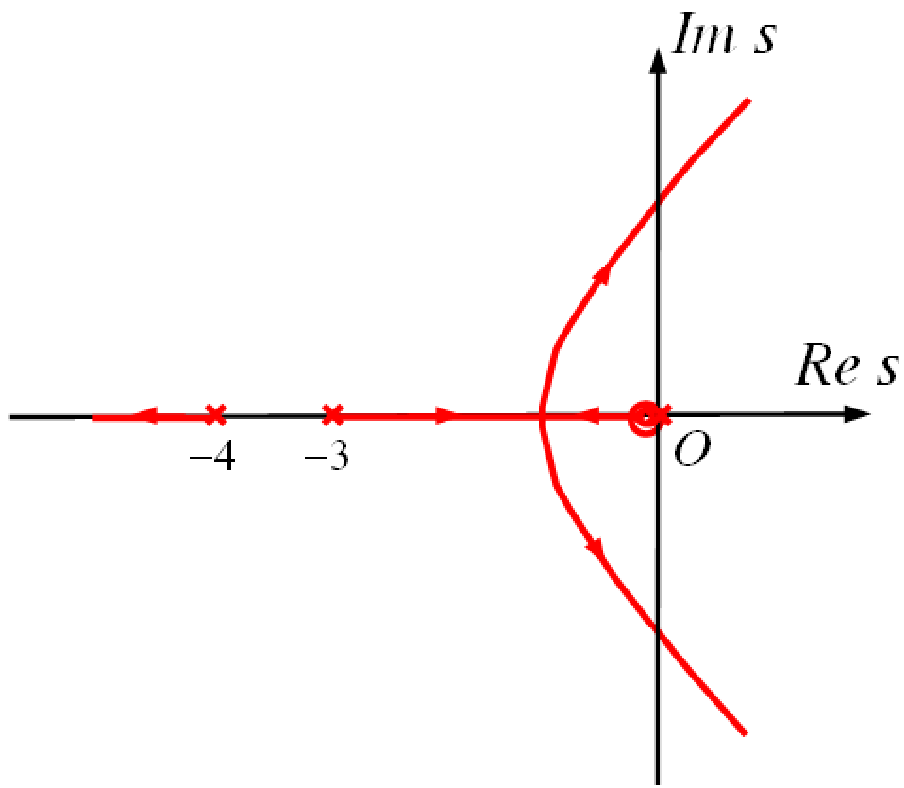
$$K_C = 1{,}0042 \approx 1$$

$$\Rightarrow \quad G_C(s) = \frac{s+0{,}1}{s+0{,}0017}$$

Root locus of the system

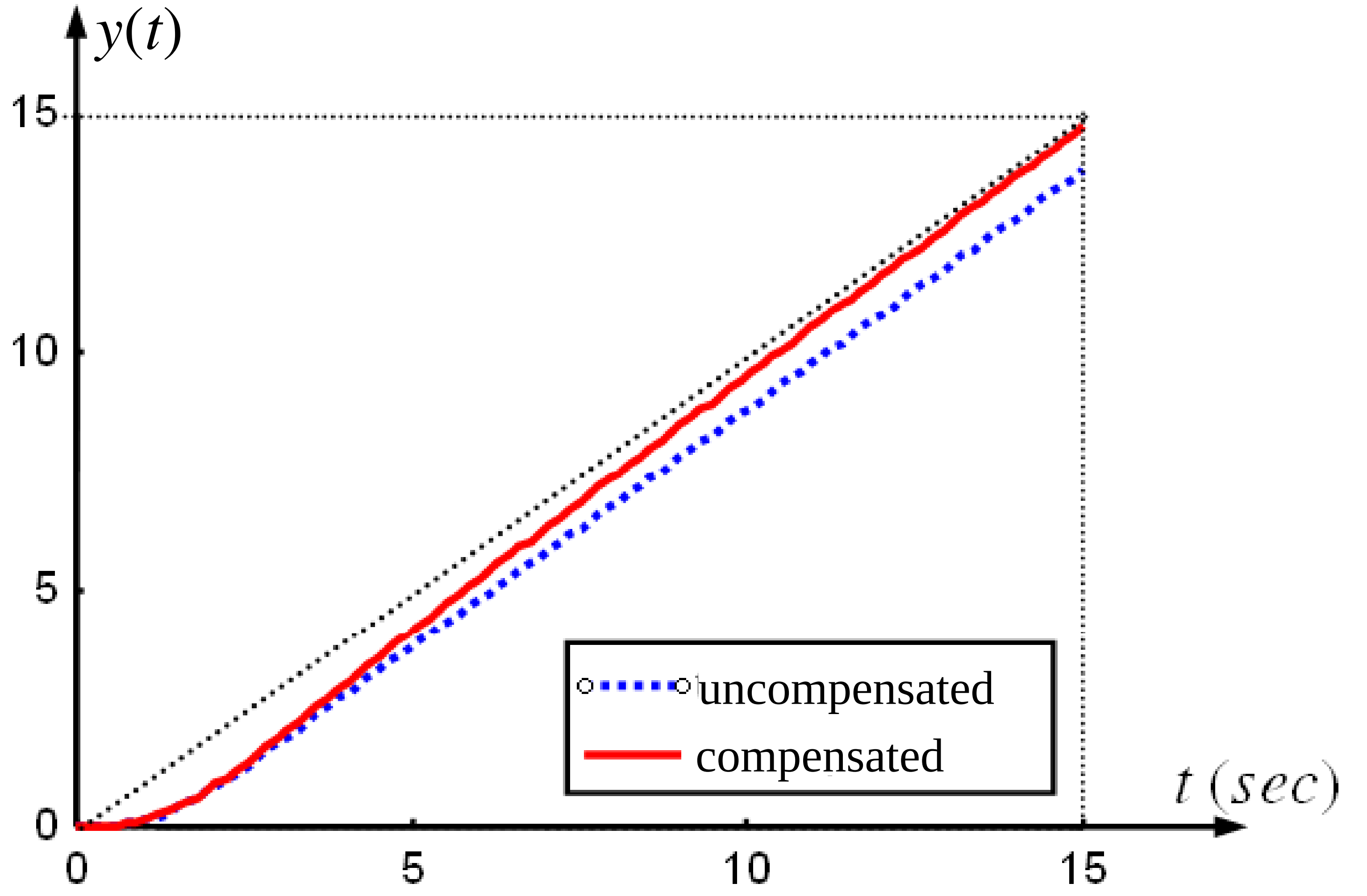


Root locus of the uncompensated system



Root locus of the compensated system

Transient response of the system



Transient response of the system

Procedure for designing lead lag compensator using the RL

The compensator to be designed

$$G_C(s) = G_{C1}(s)G_{C2}(s)$$

phase lead

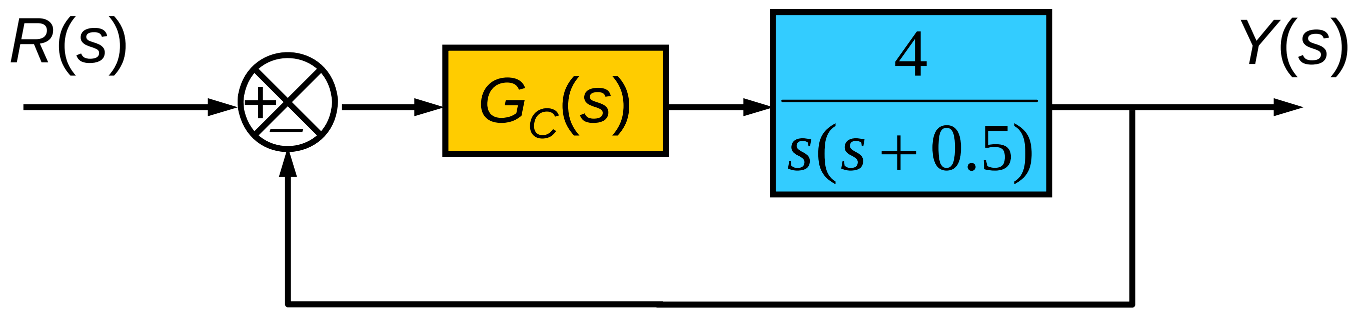
phase lag

- ***Step 1:*** Design the lead compensator $G_{C1}(s)$ to satisfy the transient response performances.

- ***Step 2:*** Let $G_1(s) = G(s).\,G_{C1}(s)$

  Design the lag compensator $G_{C2}(s)$ in series with $G_1(s)$ to satisfy the steady-state performances (and not to degrade the transient response obtained after phase lead compensating)

Example of designing a lead lag compensator using RL



- ***Objective:*** design the compensator $G_C(s)$ so that the compensated system has the dominant poles with $\xi = 0.5$, ω_n =5 (rad/sec) and the velocity constant K_V=80.

- ***Solution***

- The compensator to be designed is a lead lag compensator because the design objective is to improve the transient response and to reduce the steady-state error.

$$G_C(s) = G_{C1}(s)G_{C2}(s)$$

- ***Step 1:*** Design the lead compensator $G_{C1}(s)$

The dominant poles:

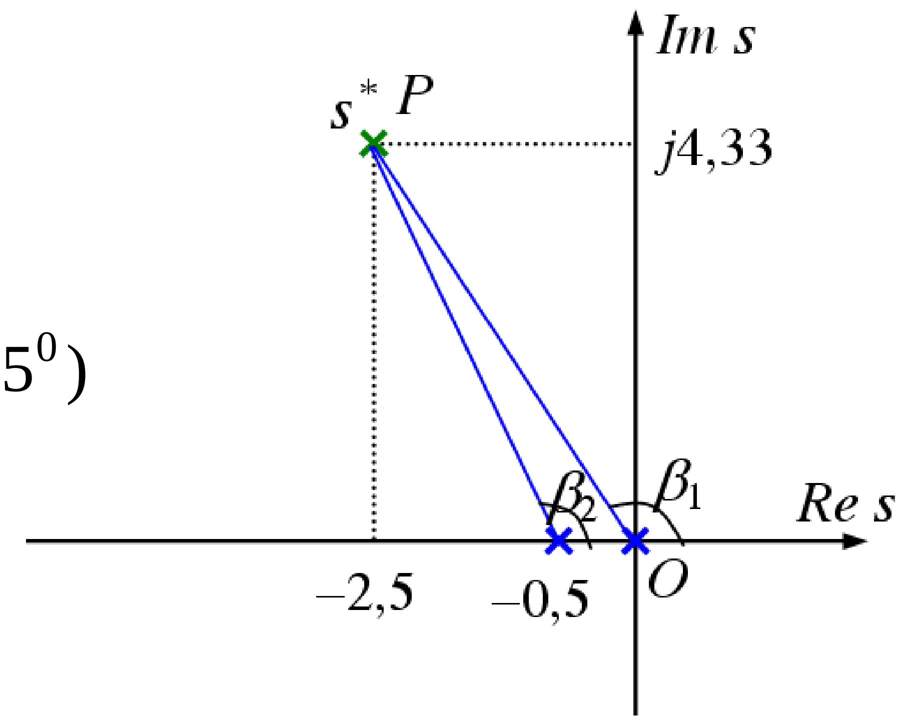
$$s^*_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} = -0{,}5\times 5 \pm j5\sqrt{1-0{,}5^2}$$

$$s^*_{1,2} = -2{,}5 \pm j4{,}33$$

The deficiency angle:

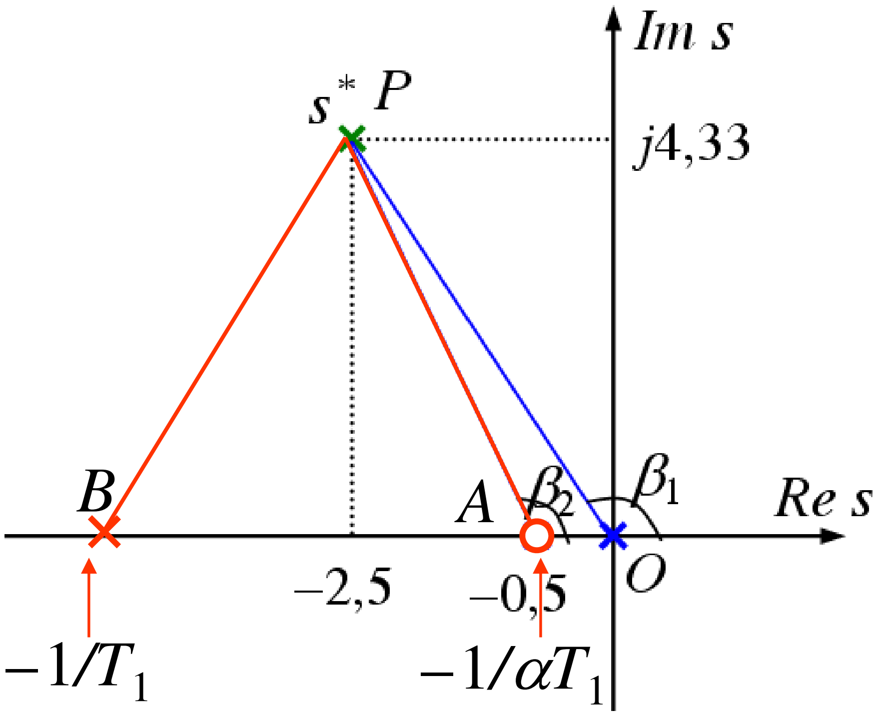
$$\phi^* = -180^0 + (\beta_1 + \beta_2)$$
$$= -180^0 + (120^0 + 115^0)$$

$$\phi^* = 55^0$$



Example of designing a lead lag compensator using RL (cont')

Chose the zero of the lead compensator so that it eliminates the pole at –0.5 of $G(s)$ (pole elimination method)



$$\frac{1}{\alpha T_1} = 0{,}5$$

$$OA = 0{,}5$$

$$AB = PA\frac{\sin A\hat{P}B}{\sin PAB} = 4.76\frac{\sin 55^0}{\sin 60^0} = 4.5$$

$$\frac{1}{T_1} = OA + AB = 5$$

$$G_{C1}(s) = K_{C1}\frac{s+0{,}5}{s+5}$$

Example of designing a lead lag compensator using RL (cont')

Calculate K_{C1}: $\left|G_{C1}(s)G(s)\right|_{s=s^*} = 1$

$$\left|K_{C1}\frac{s+0,5}{s+5}.\frac{4}{s(s+0,5)}\right|_{s=-2,5+j4,33} = 1$$

$$K_{C1} = 6,25$$

$$\Rightarrow \qquad G_{C1}(s) = 6,25\frac{s+0,5}{s+5}$$

The lead-compensated open-loop system:

$$G_1(s) = G_{C1}(s)G(s) = \frac{25}{s(s+5)}$$

Example of designing a lead lag compensator using RL (cont')

- ***Step 2:*** Design the lag compensator $G_{C2}(s)$

$$G_{C2}(s) = K_{C2}\frac{s+\dfrac{1}{\beta T_2}}{s+\dfrac{1}{T_2}}$$

– Determine β:

$$K_V = \lim_{s\to 0} sG_1(s) = \lim_{s\to 0} s\frac{25}{s(s+5)} = 5$$

$$K_V^* = 80$$

$$\Rightarrow \quad \boxed{\beta = \frac{K_V}{K_V^*} = \frac{5}{80} = \frac{1}{16}}$$

Example of designing a lead lag compensator using RL (cont')

– Determine the zero of the lag compensator:

$$\frac{1}{\beta T_2} << \left|\mathrm{Re}(s^*)\right| = \left|\mathrm{Re}(-2,5 + j4,33)\right| = 2,5$$

Chose: $\boxed{\frac{1}{\beta T_2} = 0,16}$

– Calculate the pole of the lag compensator:

$$\frac{1}{T_2} = \beta.\frac{1}{\beta T_2} = \frac{1}{16}.(0,16)$$

$$\Rightarrow \quad \boxed{\frac{1}{T_2} = 0.01}$$

Example of designing a lead lag compensator using RL (cont')

– Calculate K_{C2} using the gain condition: $\left|G_{C2}(s)G_1(s)\right|_{s=s^*} = 1$

$$\Rightarrow \quad \left(\left|G_{C2}(s)\right|_{s=s^*}\right)\left(\left|G_1(s)\right|_{s=s^*}\right) = 1$$

$$\Rightarrow \quad \left|K_{C2}\frac{-2,5+j4,33+0,16}{-2,5+j4,33+0,01}\right| = 1$$

$$\Rightarrow \quad \boxed{K_{C2} = 1.01}$$

The transfer function of the lag compensator:

$$G_{C2}(s) = 1,01\frac{(s+0,16)}{(s+0,01)}$$

Final result: $$G_C(s) = G_{C1}(s)G_{C2}(s) = 6,31\frac{(s+0,5)(s+0,16)}{(s+5)(s+0,01)}$$

Control system design in frequency domain

Procedure for designing lead compensators in frequency domain

The lead compensator: $G_C(s) = K_C \dfrac{\alpha Ts + 1}{Ts + 1} \quad (\alpha > 1)$

- ***Step 1:*** **Determine** K_C to meet the steady-state error requirement:

  $K_C = K_P^* / K_P$ or $K_C = K_V^* / K_V$ or $K_C = K_a^* / K_a$

- ***Step 2:*** Let $G_1(s) = K_C G(s)$. Plot the **Bode diagram** of $G_1(s)$
- ***Step 3:*** Determine the **gain crossover frequency** *of* $G_1(s)$:

  $L_1(\omega_C) = 0$ or $|G_1(j\omega_C)| = 1$

- ***Step 4:*** Determine the **phase margin** of $G_1(s)$ (phase margin of uncompensated system): $\Phi M = 180 + \varphi_1(\omega_C)$
- ***Step 5:*** Determine the **necessary phase lead angle** to be added to the system: $\varphi_{\max} = \Phi M^* - \Phi M + \theta$

  ΦM^* is the desired phase margin, $\theta = 5^0 \div 20^0$

Procedure for designing lead compensators in frequency domain

- ***Step 6:*** Calculate α : $$\alpha = \frac{1+\sin\varphi_{\max}}{1-\sin\varphi_{\max}}$$

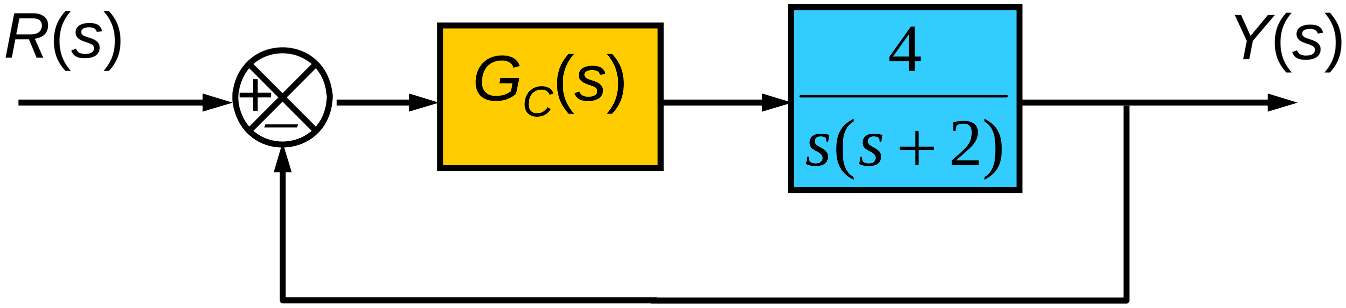
- ***Step 7:*** Determine the **new gain crossover frequency** (of the compensated open-loop system) using the conditions:

$$L_1(\omega'_C) = -10\lg\alpha \quad \text{or} \quad |G_1(j\omega'_C)| = 1/\sqrt{\alpha}$$

- ***Step 8:*** Calculate the **time constant** *T*: $$T = \frac{1}{\omega'_C\sqrt{\alpha}}$$

- ***Step 9:*** Check if the compensated system satisfies the gain margin? If not, repeat the design procedure from step 5.

- ***Note:*** It is possible to determine ω_C (step 3), ΦM (step 4) and ω'_C (step 7) by using Bode diagram instead of using analytic calculation.



- ***Objective:*** Design the compensator $G_C(s)$ so that the compensated system satisfies the performances:

$$K_V^* = 20;\quad \Phi M^* \geq 50^0;\quad GM^* \geq 10dB$$

- ***Solution:***

- The transfer function of the lead compensator to be designed:

$$G_C(s) = K_C \frac{1+\alpha Ts}{1+Ts} \qquad (\alpha > 1)$$

Design lead compensator in frequency domain – Example (cont')

✶ ***Step 1:*** Determine K_C

The velocity constant of the uncompensated system:

$$K_V = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{4}{s(s+2)} = 2$$

The desired velocity constant: $K_V^* = 20$

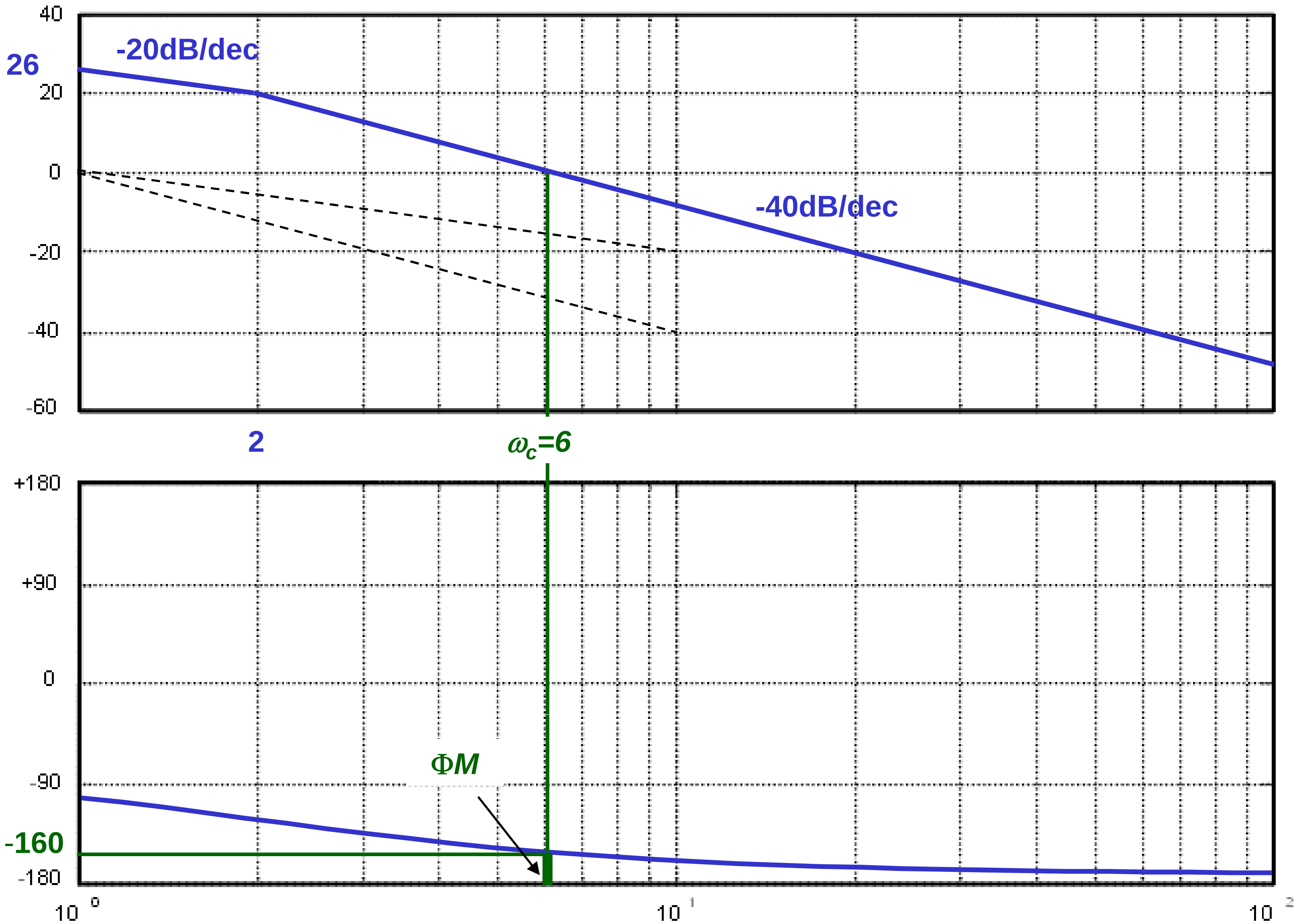
$$\Rightarrow \quad K_C = \frac{K_V^*}{K_V} = \frac{20}{2} \qquad \Rightarrow \quad \boxed{K_C = 10}$$

✶ ***Step 2:*** Denote $G_1(s) = K_C G(s) = 10.\dfrac{4}{s(s+2)}$

$$\Rightarrow \quad \boxed{G_1(s) = \frac{20}{s(0{,}5s+1)}}$$

Draw the Bode diagram of $G_1(s)$

Design lead compensator in frequency domain – Example (cont')



Design lead compensator in frequency domain – Example (cont')

- ***Step 3:*** The gain crossover frequency of $G_1(s)$

  According to the Bode diagram: $\omega_C \approx 6$ (rad/sec)

- ***Step 4:*** The phase margin of $G_1(s)$

  According to the Bode diagram:

$$\varphi_1(\omega_C) \approx -160^0$$

$$\Rightarrow \quad \Phi M = 180 + \varphi_1(\omega_C) \approx 20^0$$

- ***Step 5:*** The necessary phase lead angle to be added:

$$\varphi_{\max} = \Phi M^* - \Phi M + \theta \qquad \text{(chose } \theta\text{=7)}$$

$$\Rightarrow \quad \varphi_{\max} = 50^0 - 20^0 + 7^0$$

$$\Rightarrow \quad \varphi_{\max} = 37^0$$

Design lead compensator in frequency domain – Example (cont')

- ***Step 6:*** Calculate α

$$\alpha = \frac{1+\sin\varphi_{\max}}{1-\sin\varphi_{\max}} = \frac{1+\sin 37^0}{1-\sin 37^0} \quad \Rightarrow \quad \alpha = 4$$

- ***Step 7:*** Determine the new gain crossover frequency using Bode plot

$$L_1(\omega'_C) = -10\lg\alpha = -10\lg 4 = -6dB$$

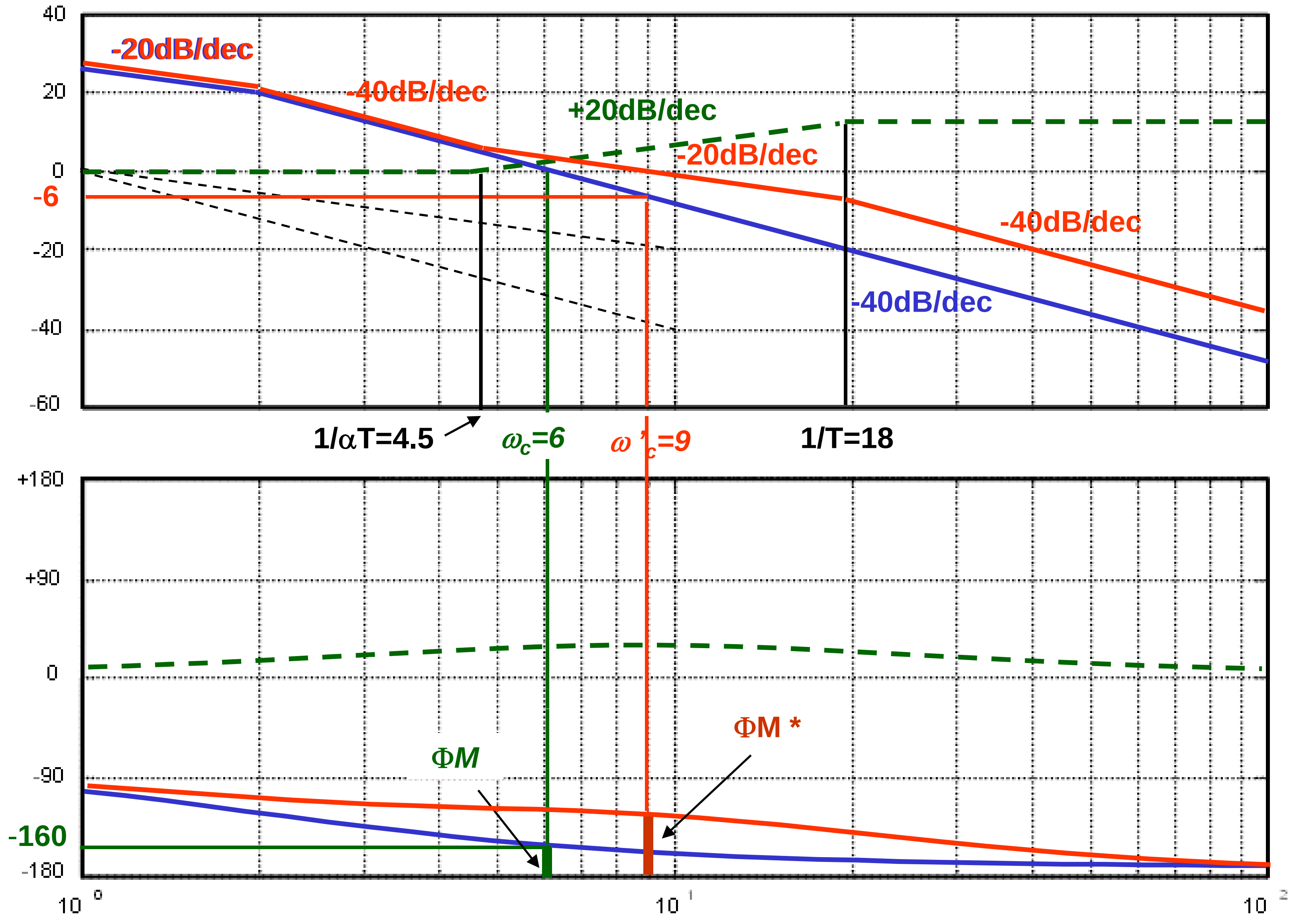
The abscissa of the intersection between Bode magnitude diagram and the horizontal line with ordinate of 6dB is the new gain crossover frequency. According to the plot (in slide 54), we have:

$$\omega'_C \approx 9 \quad \text{(rad/sec)}$$

- ***Step 8:*** Calculate *T*

$$T = \frac{1}{\omega'_C\sqrt{\alpha}} = \frac{1}{(9)(\sqrt{4})} \quad \Rightarrow \quad T = 0{,}056 \quad \Rightarrow \quad \alpha T = 0{,}224$$

Design lead compensator in frequency domain – Example (cont’)



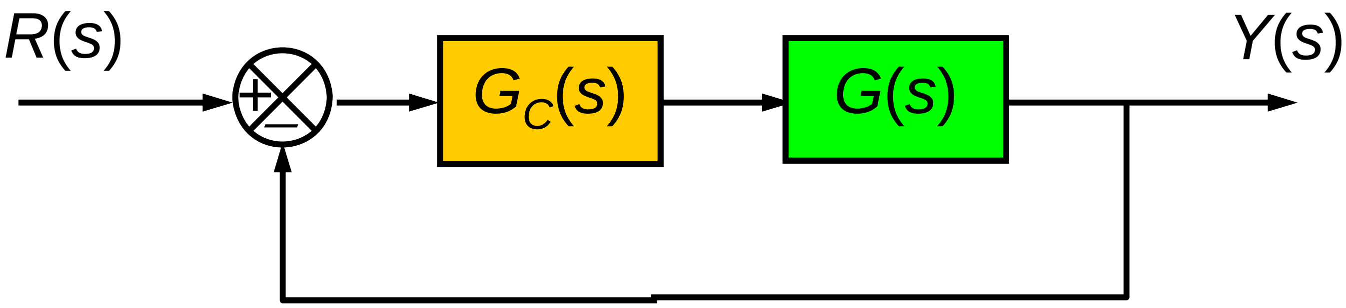
- ***Step 9:*** Check the gain margin of the compensated system

  According to the compensated Bode diagram, $GM^* = +\infty$, then the compensated system fulfills the design requirements.

- ***Conclusion:*** The designed lead compensator is:

$$G_C(s) = 10\frac{1+0{,}224s}{1+0{,}056s}$$

Design lead compensator in frequency domain – Example 2

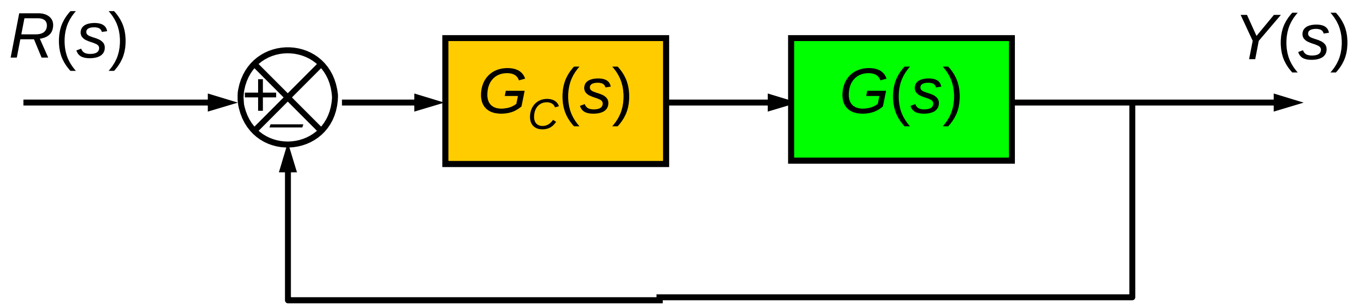


$$G(s) = \frac{20e^{-0.01s}}{s(s+4)}$$

- ***Objective:*** Design the compensator $G_C(s)$ so that the compensated system has: $\Phi M^* \geq 60^0; GM^* \geq 10dB$ and steady-state error to unit ramp input $e_{ss}^* \leq 0.05$;

- ***Solution:***

Design lead compensator in frequency domain – Example 3



$$G(s) = \frac{16e^{-0.01s}}{(s+2)(s^2+10s+25)}$$

- ***Objective:*** Design the compensator $G_C(s)$ so that the compensated system has: $\Phi M^* \geq 50^0; GM^* \geq 10dB$ and steady-state error to unit step input $e_{ss}^* \leq 0.05;$
- ***Solution:***

Procedure for designing lag compensators in frequency domain

The lag compensator: $$G_C(s) = K_C \frac{\alpha Ts + 1}{Ts + 1} \quad (\alpha < 1)$$

- ***Step 1:*** **Determine** K_C to meet the steady-state error requirement:

$$K_C = K_P^* / K_P \quad \text{or} \quad K_C = K_V^* / K_V \quad \text{or} \quad K_C = K_a^* / K_a$$

- ***Step 2:*** Let $G_1(s)=K_C G(s)$. Plot the **Bode diagram** of $G_1(s)$
- ***Step 3:*** Determine the **new gain crossover frequency** ω'_C satisfying the following condition:

$$\varphi_1(\omega'_C) = -180^0 + \Phi M^* + \theta$$

ΦM^* is the desired phase margin, $\theta = 5^0 \div 20^0$

- ***Step 4:*** Calculate α using the condition:

$$L_1(\omega'_C) = -20 \lg \alpha \quad \text{or} \quad \left| G_1(j\omega'_C) \right| = \frac{1}{\alpha}$$

- ***Step 5 :*** Chose the zero of the lag compensator so that:

$$\boxed{\frac{1}{\alpha T} << \omega'_C} \quad \Rightarrow \quad \alpha T$$

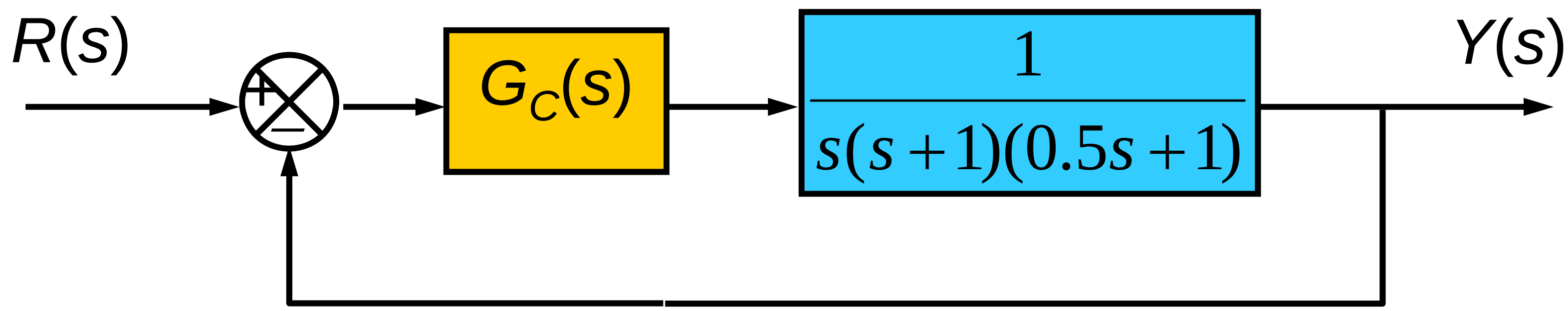
- ***Step 6:*** Calculate the **time constant** *T:*

$$\boxed{\frac{1}{T} = \alpha \frac{1}{\alpha T}} \quad \Rightarrow \quad T$$

- ***Step 7:*** Check if the compensated system satisfies the gain margin? If not, repeat the design procedure from step 3.

- ***Note:*** It is possible to determine $\varphi_1(\omega'_C)$, ω'_C (step 3), $L_1(\omega'_C)$ (step 4) by using Bode diagram instead of using analytic calculation.

Design lag compensator in frequency domain – Example



- ***Objective:*** design the lag compensator $G_C(s)$ so that compensated system satisfies the following performances:

$$K_V^* = 5; \quad \Phi M^* \geq 40^0; \; GM^* \geq 10dB$$

- ***Solution***

- The transfer function of the lag compensator to be designed:

$$G_C(s) = K_C \frac{1+\alpha Ts}{1+Ts} \qquad (\alpha < 1)$$

Design lag compensator in frequency domain – Example (cont')

- **Step 1:** Determine K_C

The velocity constant of the uncompensated system:

$$K_V = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{1}{s(s+1)(0.5s+1)} = 1$$

The desired velocity constant: $K_V^* = 5$

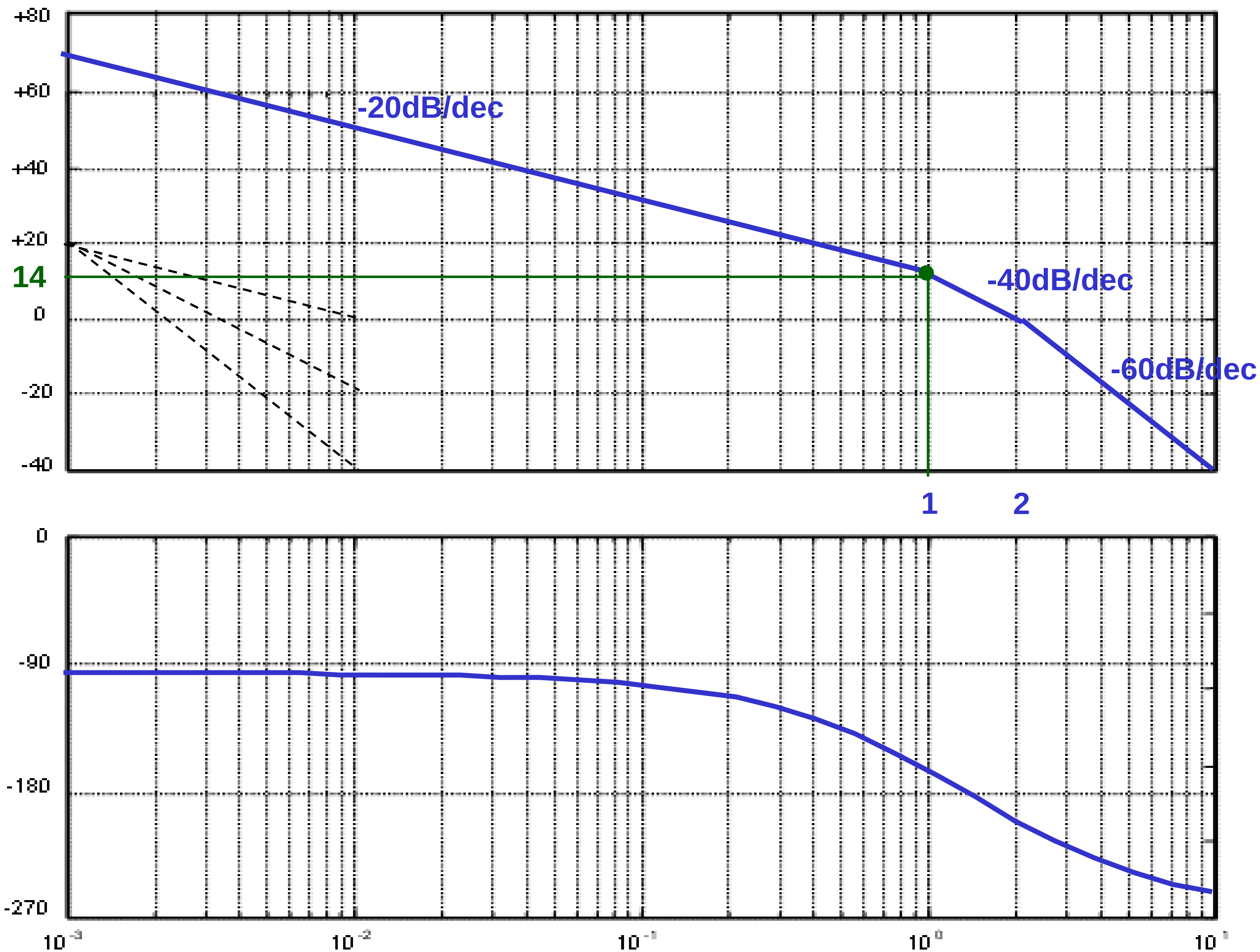
$$\Rightarrow \qquad \boxed{K_C = \frac{K_V^*}{K_V} = 5}$$

- **Step 2:** Denote $G_1(s) = K_C G(s)$

$$\Rightarrow \qquad \boxed{G_1(s) = \frac{5}{s(s+1)(0.5s+1)}}$$

Draw the Bode diagram of $G_1(s)$

Design lag compensator in frequency domain – Example (cont')



Design lag compensator in frequency domain – Example (cont')

- ***Step 3:*** Determine the new gain crossover frequency:

$$\varphi_1(\omega'_C) = -180^0 + \Phi M^* + \theta$$

$$\Rightarrow \quad \varphi_1(\omega'_C) = -180^0 + 40^0 + 5^0$$

$$\Rightarrow \quad \varphi_1(\omega'_C) = -135^0$$

According to the Bode diagram: $\omega'_C \approx 0.5$ (rad/sec)

- ***Step 4:*** Calculate α using the condition:

$$L_1(\omega'_C) = -20\lg\alpha$$

According the Bode diagram: $L_1(\omega'_C) \approx 18$ (dB)

$$\Rightarrow \quad 18 = -20\lg\alpha \quad \Rightarrow \quad \lg\alpha = -0{,}9 \quad \Rightarrow \quad \alpha = 10^{-0{,}9}$$

$$\Rightarrow \quad \alpha = 0{,}126$$

Design lag compensator in frequency domain – Example (cont')

- ***Step 5:*** Chose the zero of the lag compensator:

$$\frac{1}{\alpha T} << \omega'_C = 0.5$$

Chose $\frac{1}{\alpha T} = 0.05 \quad \Rightarrow \quad \boxed{\alpha T = 20}$

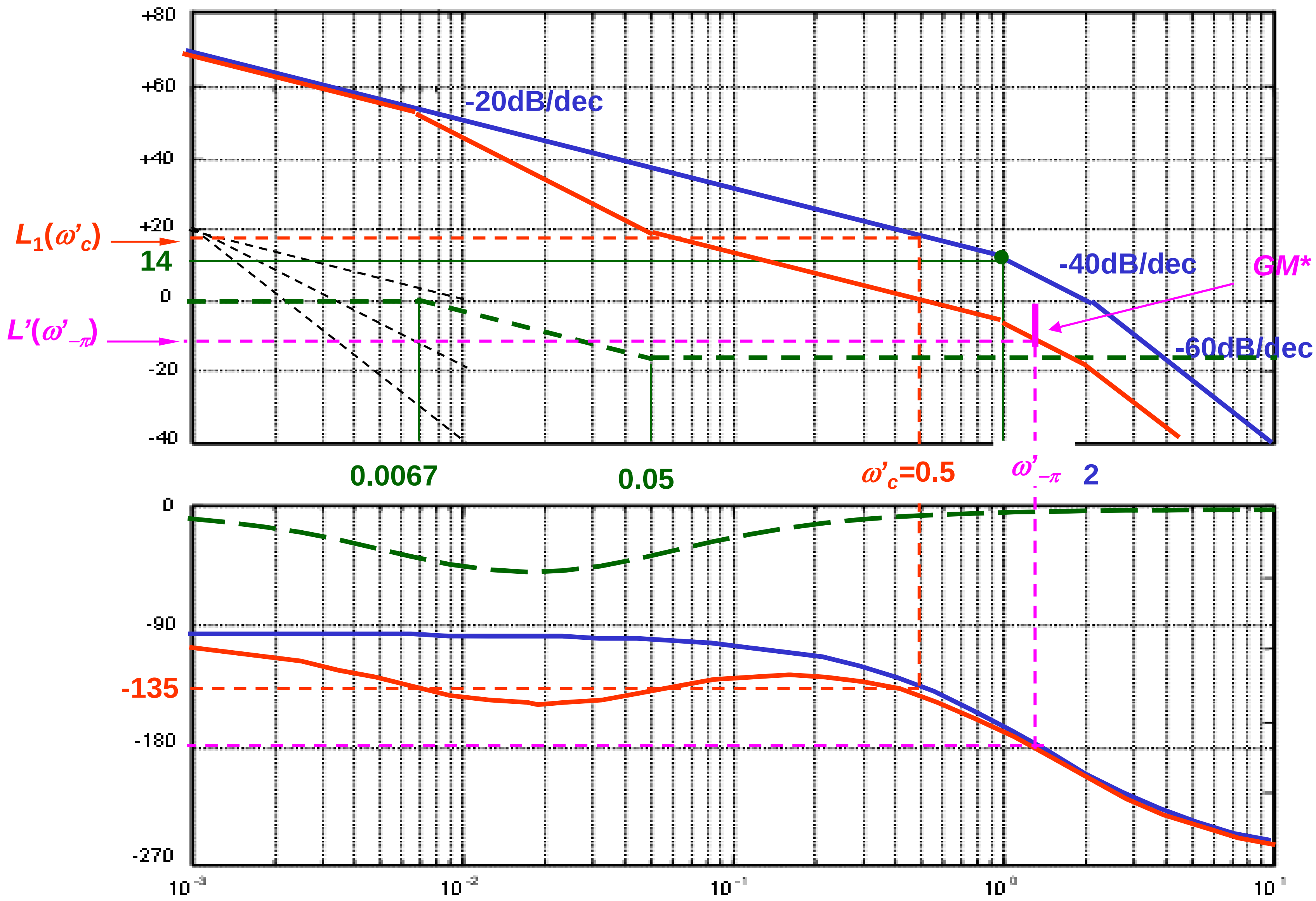
- ***Step 6:*** Calculate the time constant *T*

$$\frac{1}{T} = \alpha \frac{1}{\alpha T} = 0{,}126 \times 0{,}05 = 0{,}0063 \quad \Rightarrow \quad \boxed{T = 159}$$

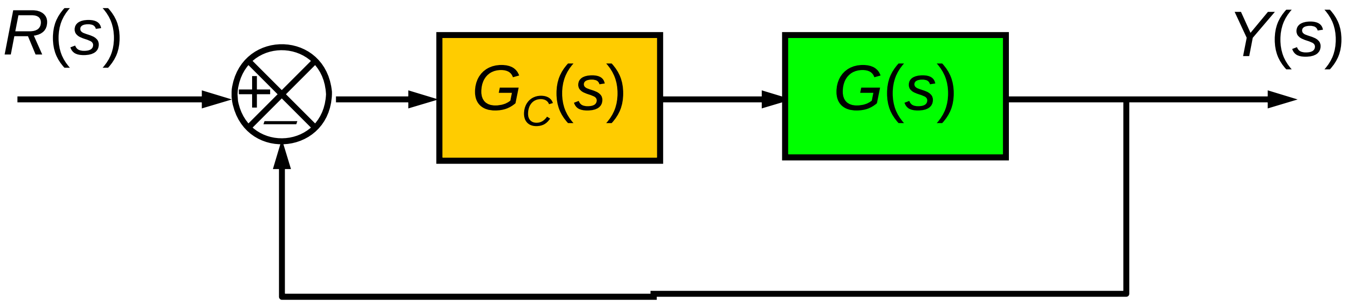
- ***Step 7:*** It can be verified in the Bode diagram that the compensated system satisfies the gain margin requirement.

Conclusion $\boxed{G_C(s) = 5\frac{(20s+1)}{(159s+1)}}$

Design lag compensator in frequency domain – Example (cont')



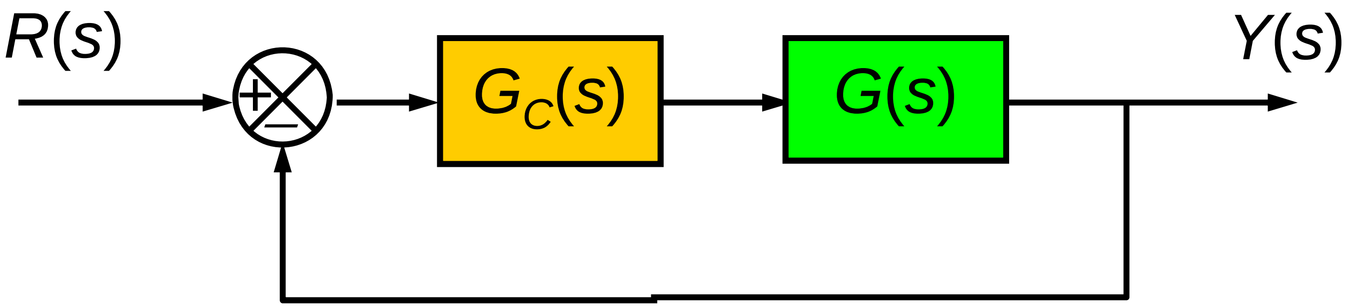
Design lag compensator in frequency domain – Example 2



$$G(s) = \frac{20e^{-0.2s}}{s(s+4)}$$

- ***Objective:*** Design the compensator $G_C(s)$ so that the compensated system has: $\Phi M^* \geq 60^0; GM^* \geq 10dB$ and steady-state error to unit ramp input $e_{ss}^* \leq 0.05$;
- ***Solution:***

Design lag compensator in frequency domain – Example 3



$$G(s)=\frac{16e^{-0.02s}}{(s+2)(s^2+10s+25)}$$

- ***Objective:*** Design the compensator $G_C(s)$ so that the compensated system has: $\Phi M^* \geq 50^0; GM^* \geq 10dB$ and steady-state error to unit step input $e_{ss}^* \leq 0.05$;
- ***Solution:***

Comparison of phase lead and phase lag compensator

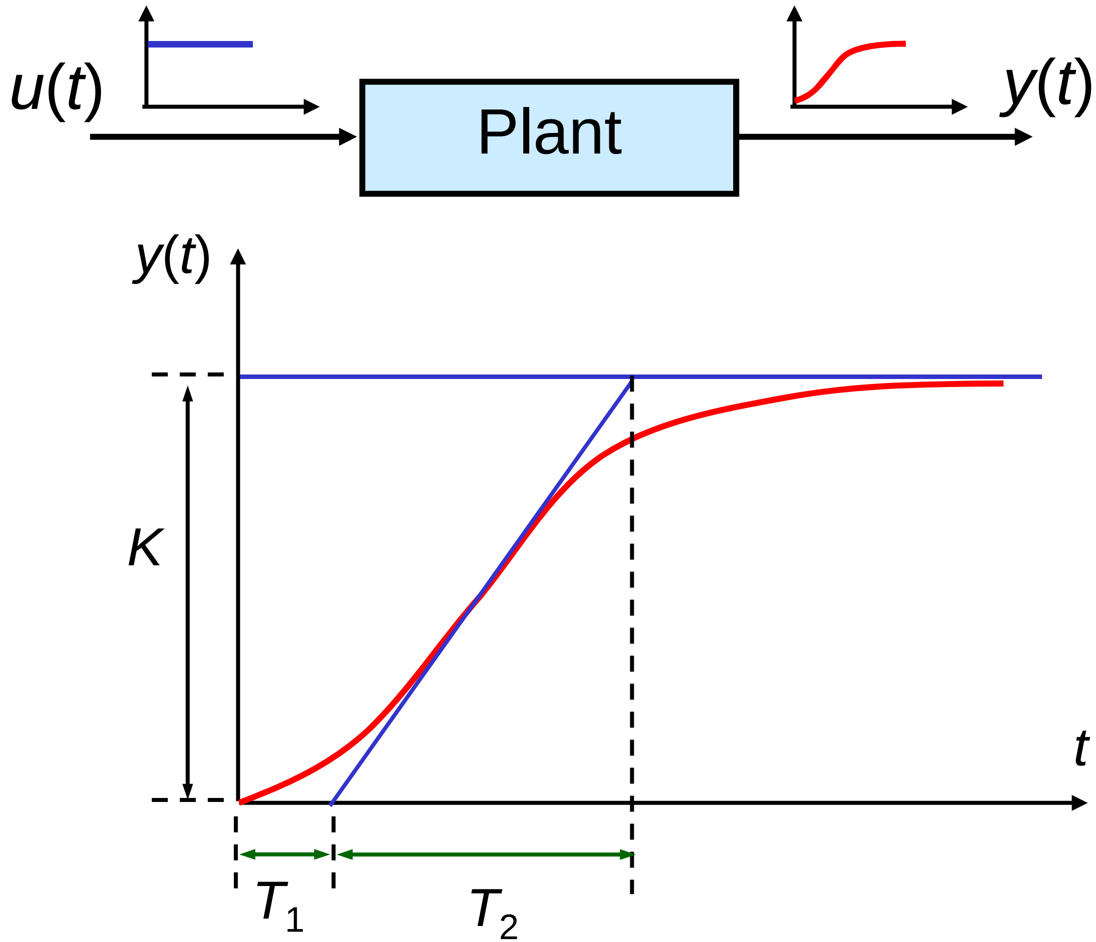
	Compensation	
	Phase-Lead	**Phase-Lag**
Approach	Addition of phase-lead angle near crossover frequency on Bode diagram. Add lead network to yield desired dominant roots in s-plane.	Addition of phase-lag to yield an increased error constant while maintaining desired dominant roots in s-plane or phase margin on Bode diagram
Results	1. Increases system bandwidth 2. Increases gain at higher frequencies	1. Decreases system bandwidth
Advantages	1. Yields desired response 2. Improves dynamic response	1. Suppresses high-frequency noise 2. Reduces steady-state error
Disadvantages	1. Requires additional amplifier gain 2. Increases bandwidth and thus susceptibility to noise 3. May require large values of components for RC network	1. Slows down transient response 2. May require large values of components for RC network
Applications	1. When fast transient response is desired	1. When error constants are specified
Situations not applicable	1. When phase decreases rapidly near crossover frequency	1. When no low-frequency range exists where phase is equal to desired phase margin

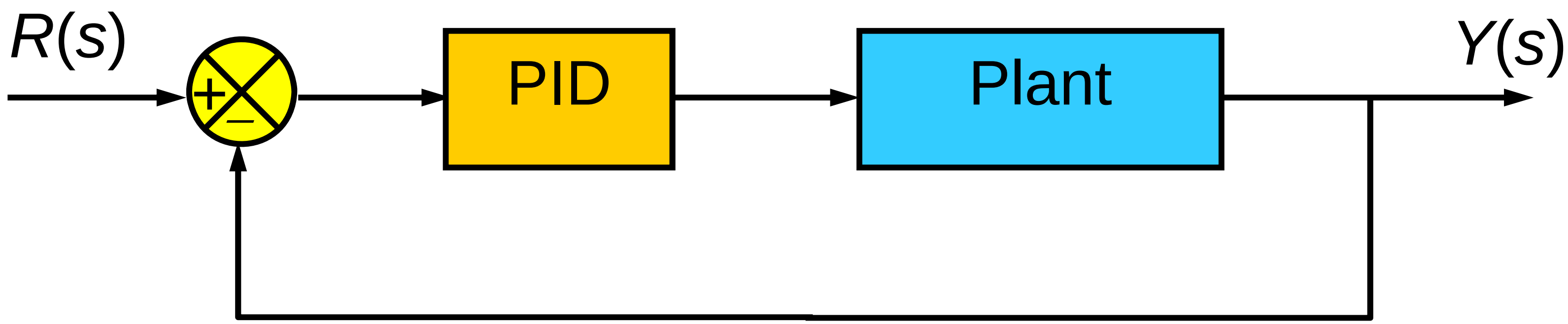
(Dorf and Bishop (2008), Modern control system –p.729)

Design of PID controllers

Zeigler – Nichols method 1

- Determine the PID parameters based on the step response of the open-loop system.





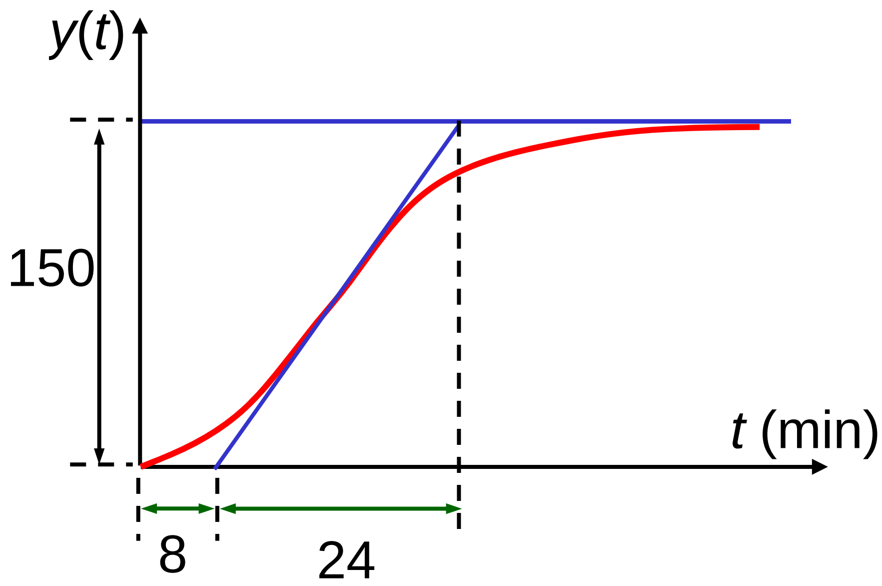
PID controller:

$$G_C(s) = K_P\left(1 + \frac{1}{T_I s} + T_D s\right)$$

Controller	K_P	T_I	T_D
P	$T_2/(T_1K)$	∞	0
PI	$0.9T_2/(T_1K)$	$0.3T_1$	0
PID	$1.2T_2/(T_1K)$	$2T_1$	$0.5T_1$

Zeigler – Nichols method 1 – Example

- ***Problem:*** Design a PID controller to control a furnace providing the open-loop characteristic of the furnace obtained from a experiment beside.



$$K = 150$$

$$T_1 = 8\,\text{min} = 480\,\text{sec}$$

$$T_2 = 24\,\text{min} = 1440\,\text{sec}$$

$$K_P = 1.2\frac{T_2}{T_1 K} = 1.2\frac{1440}{480 \times 150} = 0.024$$

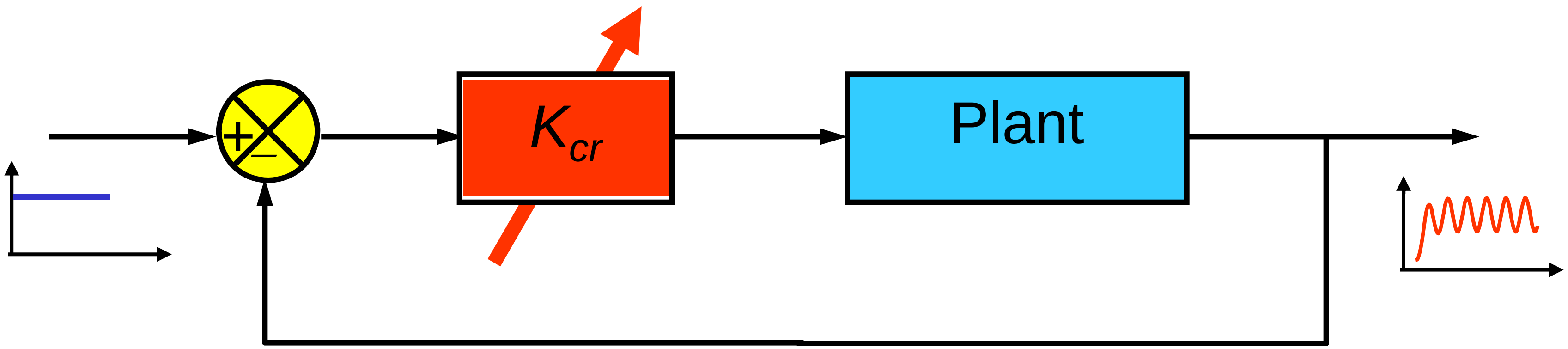
$$T_I = 2T_1 = 2 \times 480 = 960\,\text{sec}$$

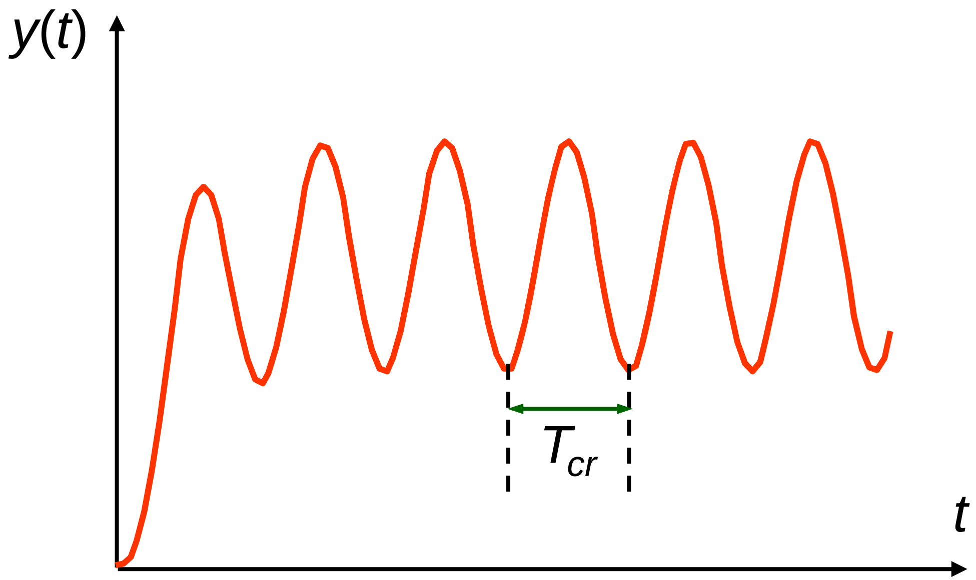
$$T_D = 0.5T_1 = 0.5 \times 480 = 240\,\text{sec}$$

$$G_{PID}(s) = 0.024\left(1 + \frac{1}{960s} + 240s\right)$$

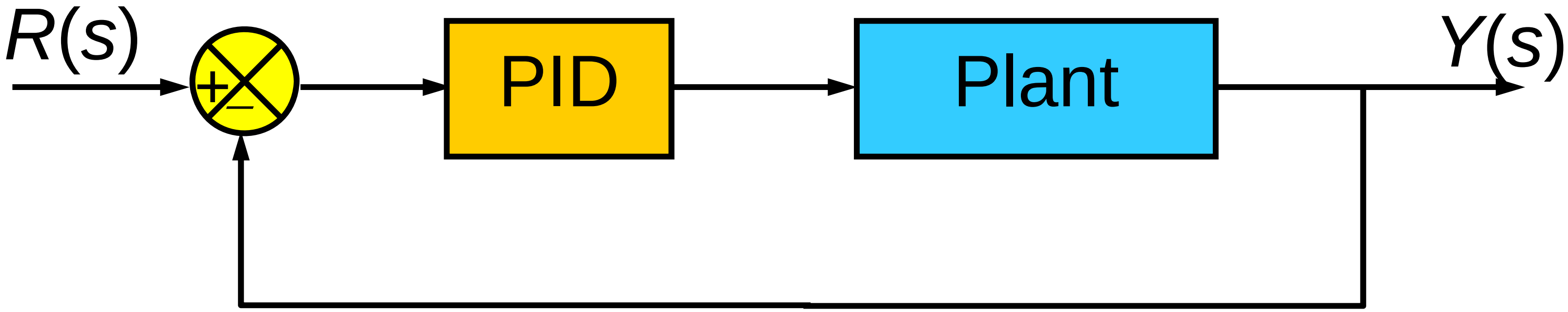
Zeigler – Nichols method 2

- Determine the PID parameters based on the response of the closed-loop system at the stability boundary.





Zeigler – Nichols method 2 (cont')



PID controller:

$$G_C(s) = K_P\left(1 + \frac{1}{T_I s} + T_D s\right)$$

Controller	K_P	T_I	T_D
P	$0.5K_{cr}$	∞	0
PI	$0.45K_{cr}$	$0.83T_{cr}$	0
PID	$0.6K_{cr}$	$0.5T_{cr}$	$0.125T_{cr}$

Zeigler – Nichols method 2 – Example

- ***Problem:*** Design a PID controller to control the angle position of a DC motor, providing that by experiment the critical gain of the system is 20 and the critical cycle is *T*= 1 sec.
- ***Solution:***
- According to the given data:

$$K_{cr} = 20$$

$$T_{cr} = 1\sec$$

- Applying Zeigler – Nichols method 2:

$$K_P = 0.6K_{cr} = 0.6 \times 20 = 12$$

$$T_I = 0.5T_{cr} = 0.5 \times 1 = 0.5\sec$$

$$T_D = 0.125T_{cr} = 0.125 \times 1 = 0.125\sec$$

$$G_{PID}(s) = 12\left(1 + \frac{1}{0.125s} + 0.5s\right)$$

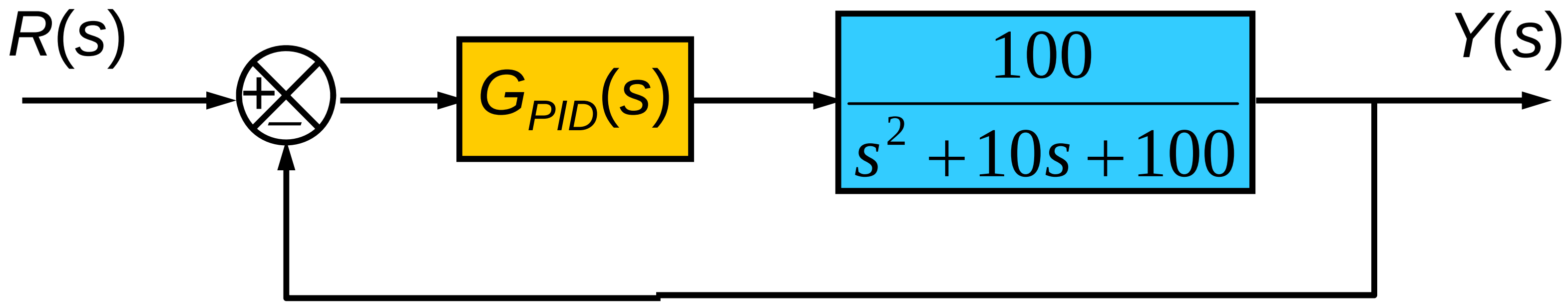
Analytical method for designing PID controller

- ***Step 1:*** Establish equation(s) representing the relationship between the controller to be designed and the desired performances.

- ***Step 2:*** Solve the equation(s) obtained in step 1 for the parameter(s) of the controller.

Analytical method for designing PID controller

- ***Example:*** Design PID controller so that the control system satisfies the following requirements:
  – Closed-loop complex poles with ξ=0.5 and ω_n=8.
  – Velocity constant K_V = 100.



- ***Solution:*** The transfer function of the PID controller to be designed

$$G_C(s) = K_P + \frac{K_I}{s} + K_D s$$

Analytical method for designing PID controller (cont')

- Velocity constant of the controlled system:

$$K_V = \lim_{s\to 0} sG_C(s)G(s) = \lim_{s\to 0} s\left(K_P + \frac{K_I}{s} + K_D s\right)\left(\frac{100}{s^2+10s+100}\right)$$

$$\Rightarrow \boxed{K_V = K_I}$$

According to the design requirement: $K_V = 100$

$$\Rightarrow \boxed{K_I = 100}$$

- The characteristic equation of the controlled system:

$$1 + \left(K_P + \frac{K_I}{s} + K_D s\right)\left(\frac{100}{s^2+10s+100}\right) = 0$$

$$\Rightarrow \boxed{s^3 + (10+100K_D)s^2 + (100+100K_P)s + 100K_I = 0} \qquad (1)$$

Analytical method for designing PID controller (cont')

- The desired characteristic equation:

$$(s+a)(s^2+2\xi\omega_n s+\omega_n^2)=0$$

$$\Rightarrow \quad (s+a)(s^2+8s+64)=0$$

$$\Rightarrow \quad s^3+(a+8)s^2+(8a+64)s+64a=0 \qquad (2)$$

- Balancing the coefficients of the equations (1) and (2), we have:

$$\begin{cases} 10+100K_D=a+8 \\ 100+100K_P=8a+64 \\ 100K_I=64a \end{cases} \Rightarrow \begin{cases} a=156.25 \\ K_P=12{,}14 \\ K_D=1{,}54 \end{cases}$$

Conclusion: $G_C(s)=12{,}64+\dfrac{100}{s}+1{,}54s$

Manual tuning of PID controllers

- Effect of increasing a parameter of PID controller independently on closed-loop performance:

Para-meter	Rise time	POT	Settling time	Steady-state error	Stability
K_P	Decrease	Increase	Small change	Decrease	Degrade
K_I	Decrease	Increase	Increase	Eliminate	Degrade
K_D	Minor change	Decrease	Decrease	No effect	Improve if K_D small

Manual tuning of PID controllers (cont.)

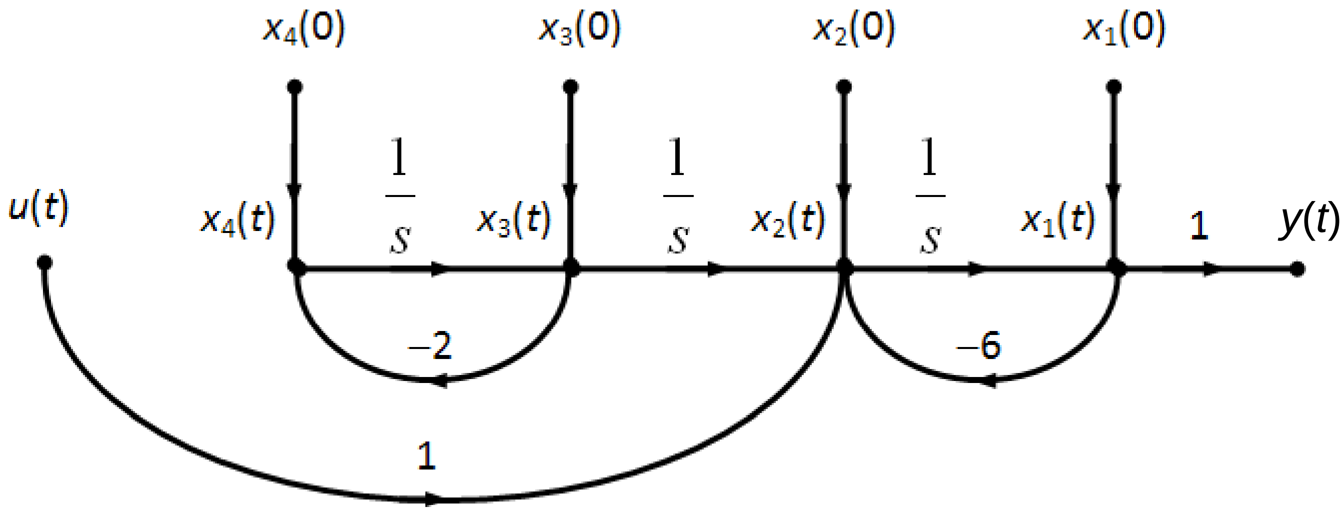
A procedure for manual tuning of PID controllers:

1. Set K_I and K_D to 0, gradually increase K_P to the critical gain K_{cr} (i.e. the gain makes the closed-loop system oscilate)
2. Set $K_P \approx K_{cr}/2$
3. Gradually increase K_I until the steady-state error is eliminated in a sufficient time for the process (Note that too much K_I will cause instability).
4. Increase K_D if needed to reduce POT and settling time (Note that too much K_D will cause excessive response and overshoot)

Control systems design in state-space using pole placement method

- Consider a system: $\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$

- The system is complete state controllable if there exists an unconstrained control law $u(t)$ that can drive the system from an initial state $\boldsymbol{x}(t_0)$ to a arbitrarily final state $\boldsymbol{x}(t_f)$ in a finite time interval $t_0 \leq t \leq t_f$. Qualitatively, the system is state controllable if each state variable can be influenced by the input



Signal flow graph of an incomplete state controllable system

Controllability condition

* System: $$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

* **Controllability** matrix

$$\mathscr{C} = [\boldsymbol{B} \quad \boldsymbol{AB} \quad \boldsymbol{A}^2\boldsymbol{B} \quad \ldots \quad \boldsymbol{A}^{n-1}\boldsymbol{B}]$$

* The necessary and sufficient condition for the controllability is:

$$rank(\mathscr{C}) = n$$

* ***Note:*** we use the term "controllable" instead of "complete state controllable" for short.

Controllability – Example

- Consider a system $\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$

  where:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

  Evaluate the controllability of the system.

- ***Solution:*** Controllability matrix:

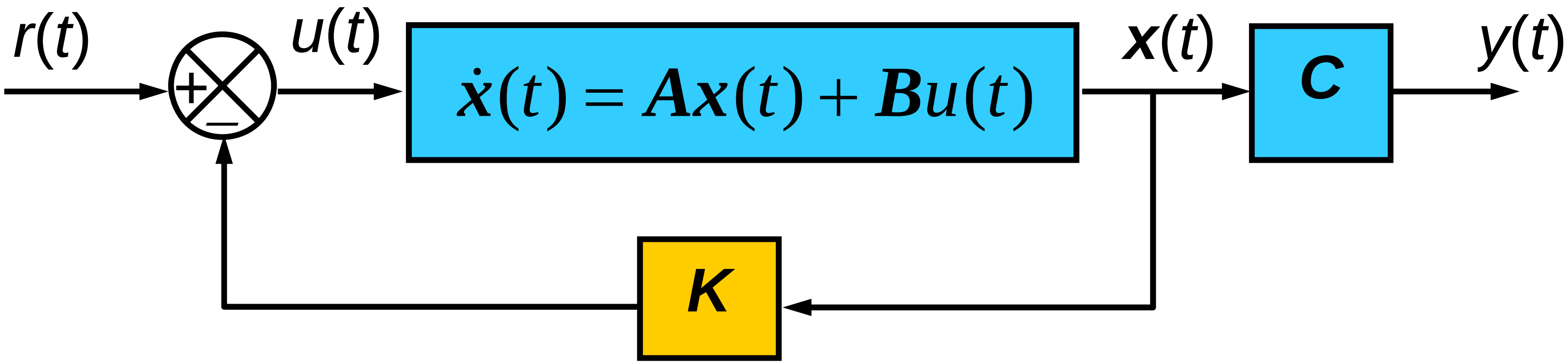
$$\mathscr{C} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} \end{bmatrix} \quad \Rightarrow \quad \mathscr{C} = \begin{bmatrix} 5 & 2 \\ 2 & -16 \end{bmatrix}$$

- Because:

$$\det(\mathscr{C}) = -84 \quad \Rightarrow \quad rank(\mathscr{C}) = 2$$

$\Rightarrow$ The system is controllable

State feedback control



- Consider a system described by the state equations:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

- The state feedback controller: $u(t) = r(t) - \boldsymbol{K}\boldsymbol{x}(t)$

- The state equations of the closed-loop system:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = [\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}]\boldsymbol{x}(t) + \boldsymbol{B}r(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

Pole placement method

If the system is controllable, then it is possible to determine the feedback gain ***K*** so that the closed-loop system has the poles at any location.

- ***Step 1***: Write the characteristic equation of the closed-loop system

$$\det[s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{BK}] = 0 \qquad (1)$$

- ***Step 2***: Write the desired characteristic equation:

$$\prod_{i=1}^{n}(s - p_i) = 0 \qquad (2)$$

$p_i,\ (i = \overline{1,n})$ are the desired poles

- ***Step 3***: Balance the coefficients of the equations (1) and (2), we can find the state feedback gain ***K***.

Pole placement method – Example

- ***Problem***: Given a system described by the state-state equation:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -7 & -3 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

- Determine the state feedback controller $u(t) = r(t) - \boldsymbol{K}\boldsymbol{x}(t)$ so that the closed-loop system has complex poles with $\xi = 0{,}6; \omega_n = 10$ and the third pole at –20.

Pole placement method – Example (cont')

- ***Solution***
- The characteristic equation of the closed-loop system:

$$\det[s\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}]=0$$

$$\Rightarrow \det\left(s\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}0&1&0\\0&0&1\\-4&-7&-3\end{bmatrix}+\begin{bmatrix}0\\3\\1\end{bmatrix}\begin{bmatrix}k_1&k_2&k_3\end{bmatrix}\right)=0$$

$$\Rightarrow s^3+(3+3k_2+k_3)s^2+(7+3k_1+10k_2-21k_3)s+(4+10k_1-12k_3)=0 \quad (1)$$

- The desired characteristic equation:

$$(s+20)(s^2+2\xi\omega_n s+\omega_n^2)=0$$

$$\Rightarrow s^3+32s^2+340s+2000=0 \quad (2)$$

Pole placement method – Example (cont')

- Balance the coefficients of the equations (1) and (2), we have:

$$\begin{cases} 3+3k_2+k_3=32 \\ 7+3k_1+10k_2-21k_3=340 \\ 4+10k_1-12k_2=2000 \end{cases}$$

- Solve the above set of equations, we have:

$$\begin{cases} k_1=220{,}578 \\ k_2=3{,}839 \\ k_3=17{,}482 \end{cases}$$

- Conclusion: $\boldsymbol{K}=\begin{bmatrix}220{,}578 & 3{,}839 & 17{,}482\end{bmatrix}$

Design of state estimators

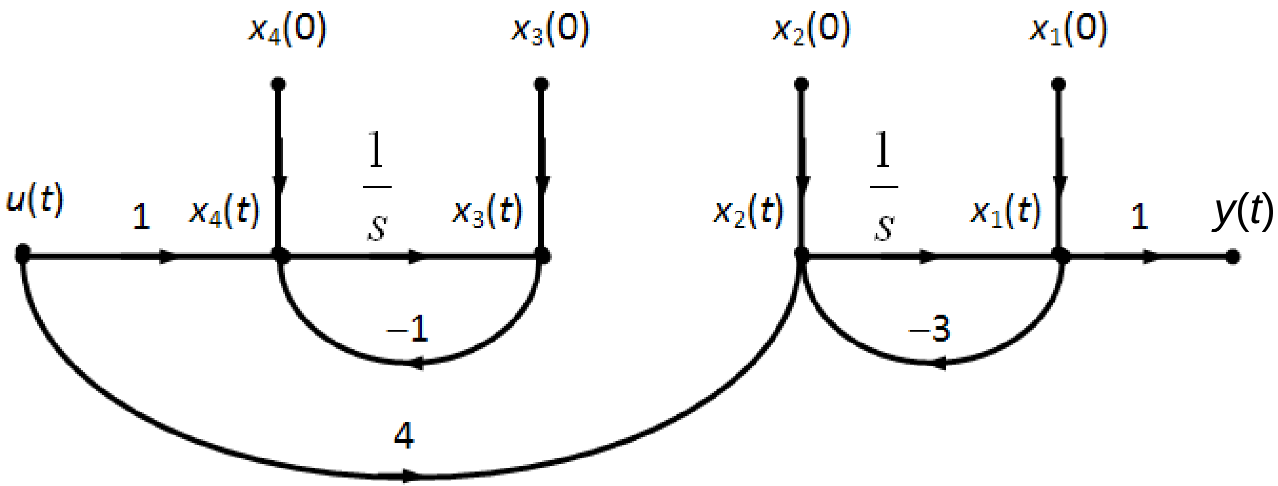
The concept of state estimation

- To be able to implement state feedback control system, it is required to measure all the states of the system.
- However, in some applications, we can only measure the output, but cannot measure the states of the system.
- The problem is to estimate the states of the system from the output measurement.

$\Rightarrow$ State estimator (or state observer)

Observability

- Consider a system: $\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$
- The system is complete state observable if given the control law $u(t)$ and the output signal $y(t)$ in a finite time interval $t_0 \leq t \leq t_f$, it is possible to determine the initial states $\boldsymbol{x}(t_0)$. Qualitatively, the system is state observable if all state variable $\boldsymbol{x}(t)$ influences the output $y(t)$.



Signal flow graph of an incomplete state observable system

Observability condition

- System
$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

  It is necessary to estimate the state $\hat{\boldsymbol{x}}(t)$ from mathematical model of the system and the input-output data.

- **<u>O</u>**bservability matrix:
$$\mathscr{O} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \boldsymbol{C}\boldsymbol{A}^2 \\ \vdots \\ \boldsymbol{C}\boldsymbol{A}^{n-1} \end{bmatrix}$$

- The necessary and sufficient condition for the observability is:
$$rank(\mathscr{O}) = n$$

Observability – Example

- Consider the system $\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$

  where: $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $\quad \boldsymbol{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \boldsymbol{C} = \begin{bmatrix} 1 & 3 \end{bmatrix}$

  Evaluate the observability of the system.

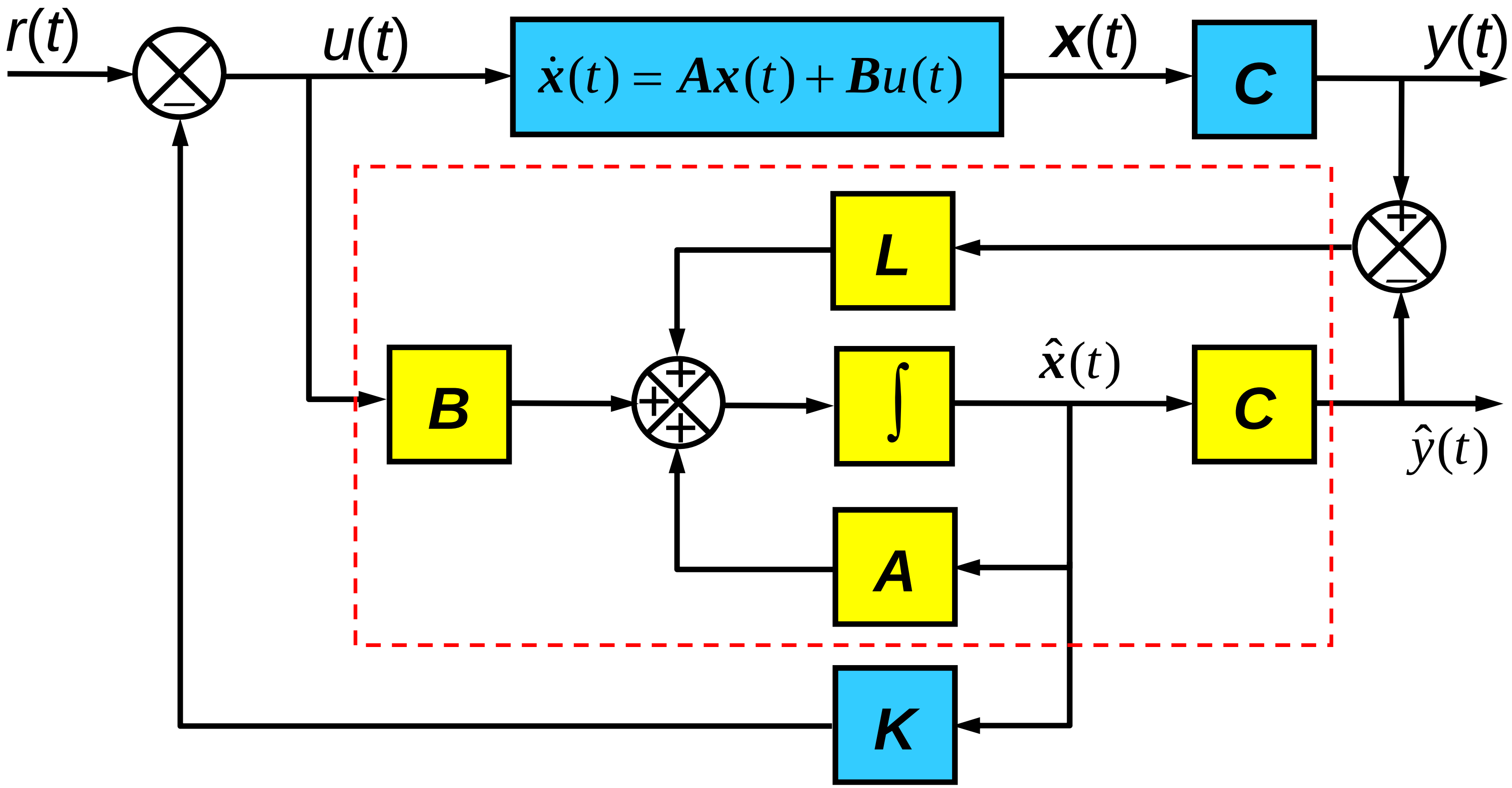
- ***Solution:*** Observability matrix:

$$\mathcal{O} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \end{bmatrix} \quad \Rightarrow \quad \mathcal{O} = \begin{bmatrix} 1 & 3 \\ -6 & -8 \end{bmatrix}$$

- Because $\det(\mathcal{O}) = 10 \quad \Rightarrow \quad rank(\mathcal{O}) = 2$

  $\Rightarrow$ The system is observable

State estimator



- State estimator:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t)) \\ \hat{y}(t) = C\hat{x}(t) \end{cases}$$

where: $L = [l_1 \quad l_2 \quad \dots \quad l_n]^T$

Design of state estimators

- Requirements:
  - The state estimator must be stable, estimation error should approach to zero.
  - Dynamic response of the state estimator should be fast enough in comparison with the dynamic response of the control loop.
- It is required to chose *L* satisfying:
  - All the roots of the equation $\det(sI - \boldsymbol{A} + \boldsymbol{LC}) = 0$ locates in the half-left s-plane.
  - The roots of the equation $\det(sI - \boldsymbol{A} + \boldsymbol{LC}) = 0$ are further from the imaginary axis than the roots of the equation $\det(sI - \boldsymbol{A} + \boldsymbol{BK}) = 0$
- Depending on the design of ***L***, we have different state estimator:
  - Luenberger state observer
  - Kalman filter

Procedure for designing the Luenberger state observer

- **Step 1**: Write the characteristic equation of the state observer

$$\det[s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{LC}] = 0 \qquad (1)$$

- **Step 1**: Write the desired characteristic equation:

$$\prod_{i=1}^{n}(s - p_i) = 0 \qquad (2)$$

$p_i, \ (i = \overline{1, n})$ are the desired poles of the state estimator

- **Step 3**: Balance the coefficients of the characteristic equations (1) and (2), we can find the gain ***L***.

Design of state estimators – Example

- ***Problem***: Given a system described by the state equation:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ y(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -7 & -3 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

- Assuming that the states of the system cannot be directly measured. Design the Luenberger state estimator so that the poles of the state estimator lying at –20, –20 and –50.

Design of state estimators – Example (cont')

- ***Solution***
- The characteristic equation of the Luenberger state estimator:

$$\det[s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{LC}] = 0$$

$$\Rightarrow \quad \det\left(s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -7 & -3 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \right) = 0$$

$$\Rightarrow \quad s^3 + (l_1 + 3)s^2 + (3l_1 + l_2 + 7)s + (7l_1 + 5l_2 + l_3 + 4) = 0 \qquad (1)$$

- The desired characteristic equation:

$$(s+20)^2(s+50) = 0$$

$$\Rightarrow \quad s^3 + 90s^2 + 2400s + 20000 = 0 \qquad (2)$$

Design of state estimators – Example (cont')

- Balancing the coefficients of the equ. (1) and (2) leads to:

$$\begin{cases} l_1 + 3 = 90 \\ 3l_1 + l_2 + 7 = 2400 \\ 7l_1 + 3l_2 + l_3 + 4 = 20000 \end{cases}$$

- Solve the above set of equations, we have:

$$\begin{cases} l_1 = 87 \\ l_2 = 2132 \\ l_3 = 12991 \end{cases}$$

- Conclusion $\boldsymbol{L} = \begin{bmatrix} 87 & 2132 & 12991 \end{bmatrix}^T$

End of Chapter 5