Lecture Notes

Fundamentals of Control Systems

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Chapter 9

DESIGN OF DISCRETE CONTROL SYSTEMS

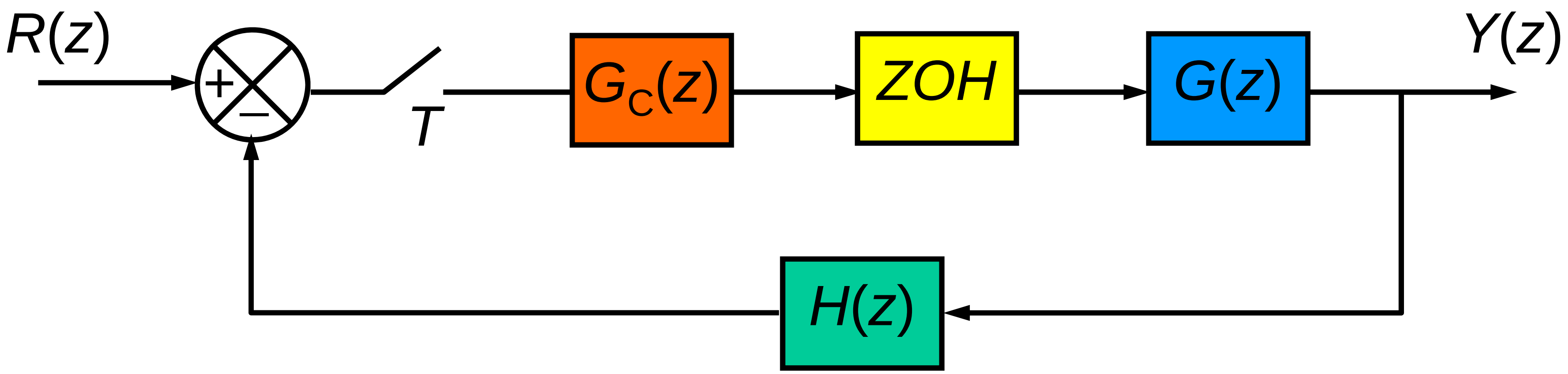
Content

- ✶ Introduction
- ✶ Discrete lead – lag compensator and PID controller
- ✶ Design discrete systems in the Z domain
- ✶ Controllability and observability of discrete systems
- ✶ Design state feedback controller using pole placement
- ✶ Design state estimator

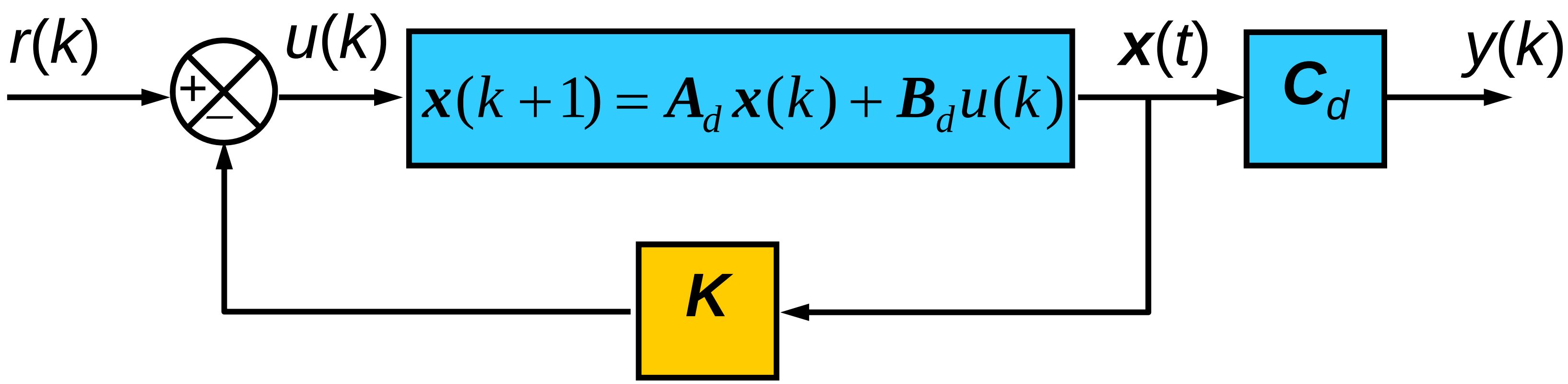
Discrete lead lag compensators and PID controllers

Control schemes

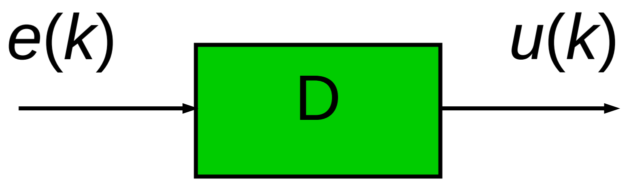
- ★ Serial compensator



- ★ State feedback control



Transfer function of discrete difference term



✶ Differential term: $$u(t) = \frac{de(t)}{dt}$$

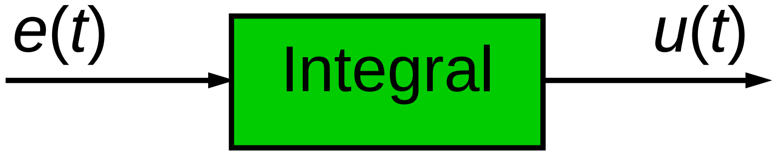
✶ Discrete difference: $$u(kT) = \frac{e(kT) - e[(k-1)T]}{T}$$

$$\Rightarrow \qquad U(z) = \frac{E(z) - z^{-1}E(z)}{T}$$

$\Rightarrow$ Transfer function of the discrete difference term:

$$\boxed{G_D(z) = \frac{1}{T}\frac{z-1}{z}}$$

Transfer function of discrete integral term



★ Continuous integral: $u(t) = \int_0^t e(\tau)d\tau$

★ Discrete integral: $u(kT) = \int_0^{kT} e(\tau)d\tau = \int_0^{(k-1)T} e(\tau)d\tau + \int_{(k-1)T}^{kT} e(\tau)d\tau$

$$\Rightarrow u(kT) = u[(k-1)T] + \int_{(k-1)T}^{kT} e(\tau)d\tau = u[(k-1)T] + \frac{T}{2}\big(e[(k-1)T] + e(kT)\big)$$

$$\Rightarrow U(z) = z^{-1}U(z) + \frac{T}{2}\left(z^{-1}E(z) + E(z)\right)$$

$\Rightarrow$ TF of discrete integral term: $\boxed{G_I(z) = \dfrac{T}{2}\dfrac{z+1}{z-1}}$

Transfer function of discrete PID controller

- Continuous PID controller:

$$G_{PID}(s) = K_P + \frac{K}{s} + K_D s$$

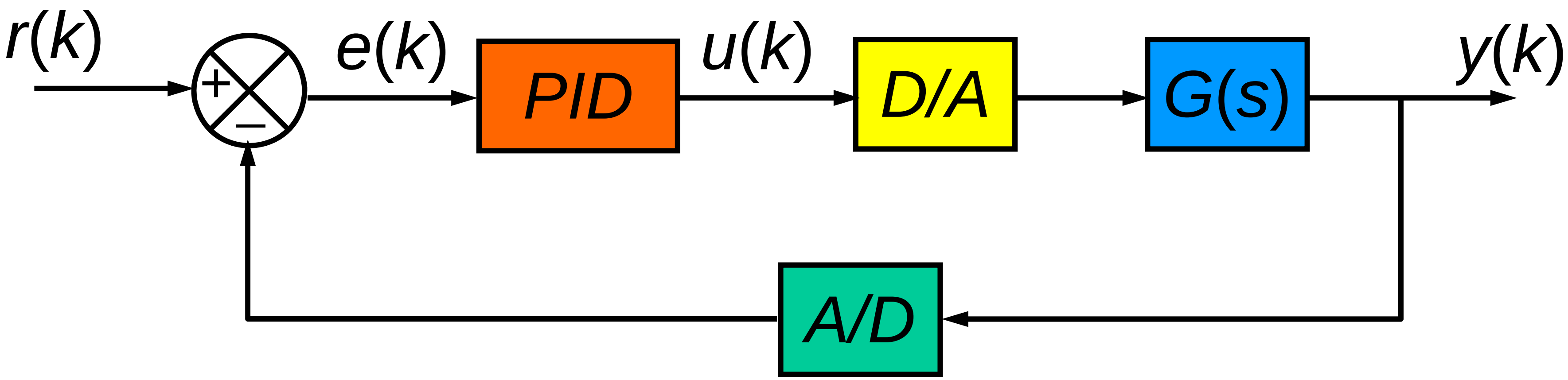
- Discrete PID controller:

$$\boxed{G_{PID}(z) = \underbrace{K_P}_{P} + \underbrace{\frac{K_I T}{2}\frac{z+1}{z-1}}_{I} + \underbrace{\frac{K_D}{T}\frac{z-1}{z}}_{D}}$$

or

$$\boxed{G_{PID}(z) = \underbrace{K_P}_{P} + \underbrace{K_I T\frac{z}{z-1}}_{I} + \underbrace{\frac{K_D}{T}\frac{z-1}{z}}_{D}}$$

Digital PID controller



$$G_{PID}(z) = \frac{U(z)}{E(z)} = K_P + \frac{K_I T}{2}\frac{z+1}{z-1} + \frac{K_D}{T}\frac{z-1}{z}$$

$$u(k) = u(k-1) + K_P[e(k) - e(k-1)] +$$
$$\frac{K_I T}{2}[e(k) + e(k-1)] + \frac{K_D}{T}[e(k) - 2e(k-1) + e(k-2)]$$

Digital PID control programming

```
float PID_control(float setpoint, float measure)
{    ek_2 = ek_1;
     ek_1 = ek;
     ek = setpoint – measure;
     uk_1 = uk;
     uk = uk_1 + Kp*(ek-ek_1) + Ki*T/2*(ek+ek_1) +…
                  Kd/T*(ek – 2ek_1+ek_2);
     If uk > umax, uk = umax;
     If uk < umin, uk = umin;
     return(uk)
}
```

Note: Kp, Ki, Kd, uk, uk_1, ek, ek_1, ek_2 must be declared as global variables; uk_1, ek_1 and ek_e must be initialized to be zero; umax and umin are constants.

TF of discrete phase lead/lag compensator

* Continuous phase lead/lag compensator:

$$G_C(s) = K\frac{s+a}{s+b} \qquad \begin{array}{ll} a<b & \text{phase lead} \\ a>b & \text{phase lag} \end{array}$$

* Discretization using trapezoidal integral:

$$G_C(z) = K\frac{(aT+2)z+(aT-2)}{(bT+2)z+(bT-2)}$$

* Denote $z_C = \dfrac{(aT-2)}{(aT+2)}$ and $p_C = \dfrac{(bT-2)}{(bT+2)}$

⇒TF of discrete phase lead/lag compensator

$$\boxed{G_C(z) = K_C\frac{z+z_C}{z+p_C}} \qquad \begin{array}{lll} |z_C|<1 & z_C<p_C & \text{phase lead} \\ |p_C|<1 & z_C>p_C & \text{phase lag} \end{array}$$

Approaches to design discrete controllers

- ***Indirect design*:** First design a continuous controller, then discretize the controller to have a discrete control system. The performances of the obtained discrete control system are approximate those of the continuous control system provided that the sample time is small enough.

- ***Direct design*:** Directly design discrete controllers in *Z* domain.

  Methods: root locus, pole placement, analytical method, …

Design discrete controllers in the Z domain

Lead compensator: $$G_C(z) = K_C \frac{z + z_C}{z + p_C} \quad (z_C < p_C)$$

- ***Step 1:*** Determine the ***dominant poles*** $z^*_{1,2}$ from desired transient response specification:

$$\begin{cases} \text{Overshoot (POT)} \\ \text{Settling time ts} \end{cases} \Rightarrow \begin{cases} \xi \\ \omega_n \end{cases} \Rightarrow s^*_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

$$\Rightarrow \boxed{\begin{cases} z^*_{1,2} = e^{Ts^*} \\ r = \left|z^*\right| = e^{-T\xi\omega_n} \\ \varphi = \angle z^* = T\omega_n\sqrt{1-\xi^2} \end{cases}}$$

- ***Step 2:* *Determine the deficiency angle*** so that the dominant poles $z^*_{1,2}$ lie on the root locus of the system after compensated:

$$\phi^* = -180^0 + \sum_{i=1}^{n} \arg(z_1^* - p_i) - \sum_{i=1}^{m} \arg(z_1^* - z_i)$$

where p_i and z_i are poles and zeros of $G(z)$

Geometry formula:

$$\phi^* = -180^0 + \sum \text{angles from poles of } G(z) \text{ to } z_1^*$$
$$- \sum \text{angles from zeros of } G(z) \text{ to } z_1^*$$

Procedure for designing discrete lead compensator using the RL

- ***Step 3:*** ***Determine the pole & zero*** of the lead compensator

  Draw 2 arbitrarily rays starting from the dominant pole z_1^* such that the angle between the two rays equal to ϕ^*. The intersection between the two rays and the real axis are the positions of the pole and the zero of the lead compensator.

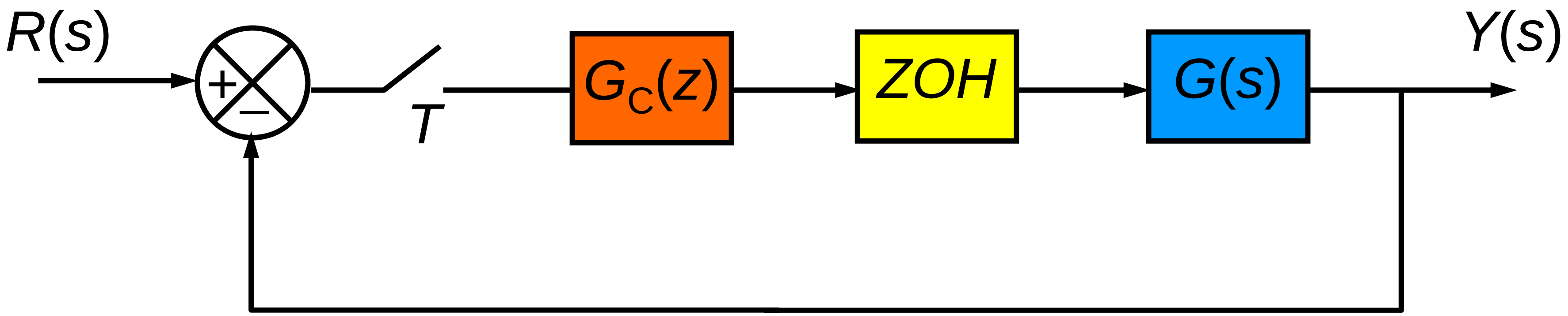
  Two methods often used for drawing the rays:

  - Bisector method
  - Pole elimination method

- ***Step 4:*** ***Calculate the gain*** K_C using the equation:

$$\left| G_C(z)G(z) \right|_{z=z_1^*} = 1$$

Design discrete lead compensator using RL – Example



$$G(s) = \frac{50}{s(s+5)} \qquad T = 0.1\text{sec}$$

- Design the compensator $G_C(z)$ so that the compensated system has dominant poles with $\xi = 0.707$, $\omega_n = 10$ (rad/sec)

★ **Solution:**

★ The open-loop discrete TF:

- $$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

$$= (1 - z^{-1})\mathcal{Z}\left\{\frac{50}{s^2(s+5)}\right\}$$

$$= 10(1 - z^{-1})\left(\frac{z[(0.5 - 1 + e^{-0.5})z + (1 - e^{-0.5} - 0.5e^{-0.5})]}{5(z-1)^2(z - e^{-0.5})}\right)$$

$$\Rightarrow G(z) = \frac{0.21z + 0.18}{(z-1)(z-0.607)}$$

Design discrete lead compensator using RL – Example (cont')

✶ The desired poles: $z^*_{1,2} = re^{\pm j\varphi}$

where

$$r = e^{-T\xi\omega_n} = e^{-0.1\times 0.707\times 10} = 0.493$$

$$\varphi = T\omega_n\sqrt{1-\xi^2} = 0.1\times 10\times\sqrt{1-0.707^2} = 0.707$$

$$\Rightarrow \quad z^*_{1,2} = 0.493e^{\pm j0.707}$$

$$\Leftrightarrow \quad z^*_{1,2} = 0.375 \pm j0.320$$

★ The deficiency angle

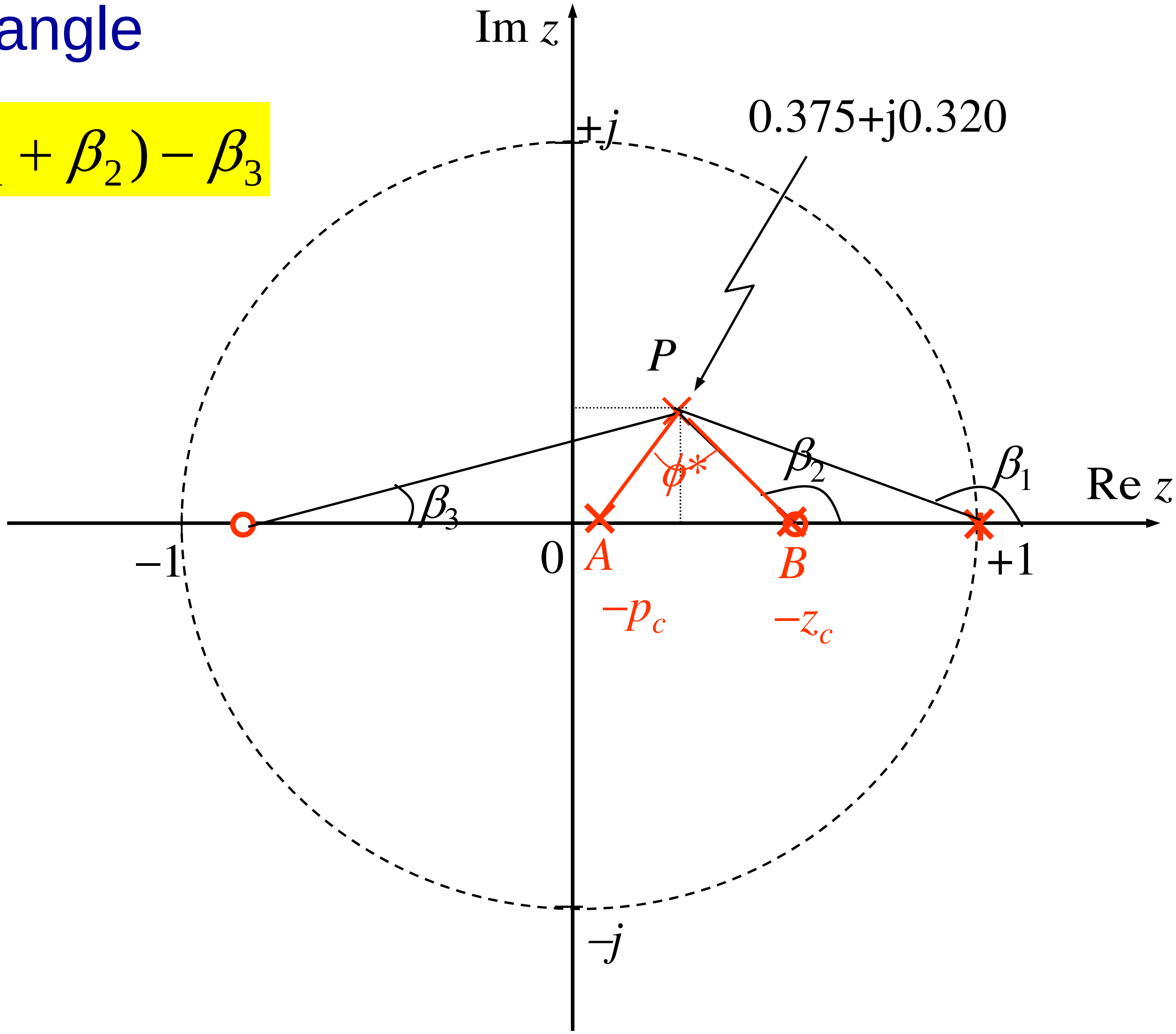
$$\phi^* = -180 + (\beta_1 + \beta_2) - \beta_3$$

$$\beta_1 = 152.9^0$$

$$\beta_2 = 125.9^0$$

$$\beta_3 = 14.6^0$$

$$\Rightarrow \quad \phi^* = 84^0$$



- Determine the pole and the zero of the compensator using the pole elimination method:

$$-z_C = 0.607$$

$$\Rightarrow \quad z_C = -0.607$$

$$-p_C = OA = OB - AB$$

$$OB = 0.607$$

$$AB = 0.578$$

$$\Rightarrow \quad p_C = -0.029$$

Design discrete lead compensator using RL – Example (cont')

- Calculate the gain K_C: $\left| G_C(z)G(z) \right|_{z=z^*} = 1$

$$\Rightarrow \left| K_C \frac{(z-0.607)}{(z-0.029)} \frac{(0.21z+0.18)}{(z-1)(z-0.607)} \right|_{z=0.375+j0.320} = 1$$

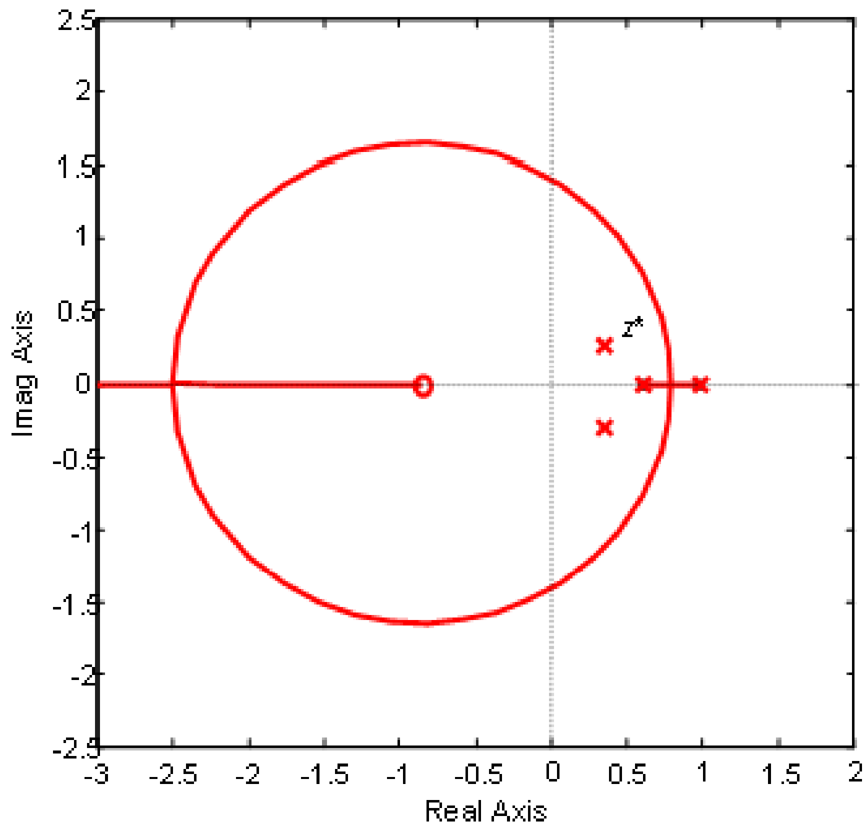
$$\Rightarrow \left| K_C \frac{[0.21(0.375+j0.320)+0.18]}{(0.375+j0.320-0.029)(0.375+j0.320-1)} \right| = 1$$

$$\Rightarrow K_C \frac{0.267}{0.471 \times 0.702} = 1 \quad \Rightarrow \quad K_C = 1.24$$

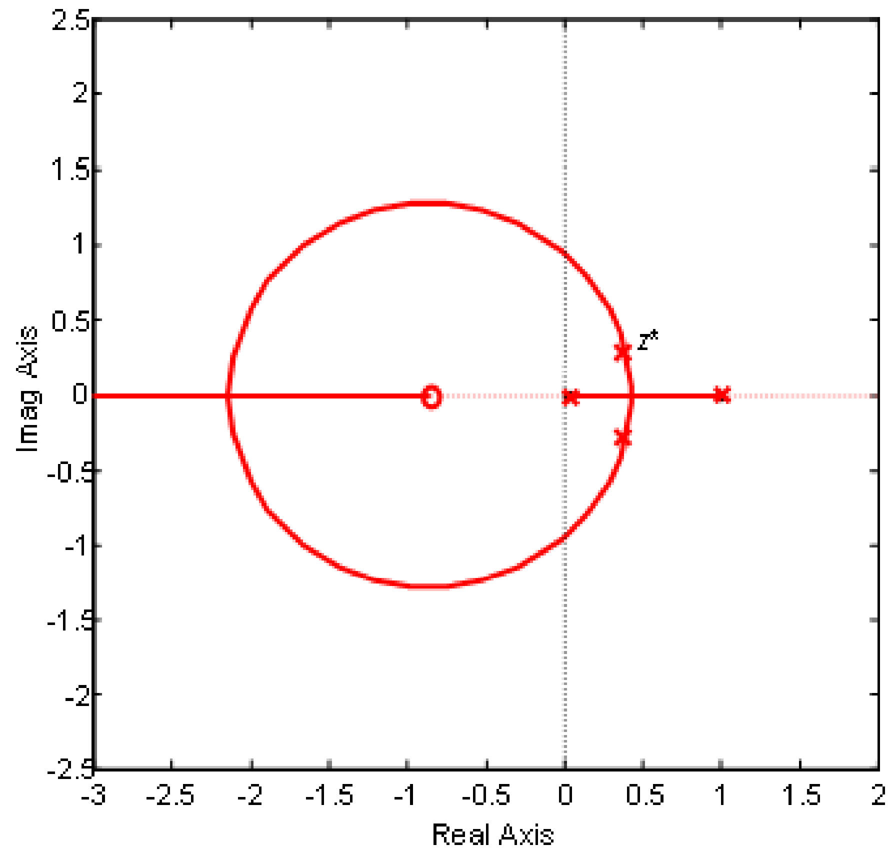
Conclusion: The TF of the lead compensator is:

$$G_C(z) = 1.24 \frac{z-0.607}{z-0.029}$$

Design discrete lead compensator using RL – Example (cont')



Root locus of the uncompensated system



Root locus of the compensated system

The discrete lag compensator: $G_C(s) = K_C \dfrac{z+z_C}{z+p_C} \quad (z_C > p_C)$

- ***Step 1:*** Denote $\beta = \dfrac{1+p_C}{1+z_C}$. **Determine β** to meet the steady-state error requirement:

$$\beta = \frac{K_P}{K_P^*} \quad \text{or} \quad \beta = \frac{K_V}{K_V^*} \quad \text{or} \quad \beta = \frac{K_a}{K_a^*}$$

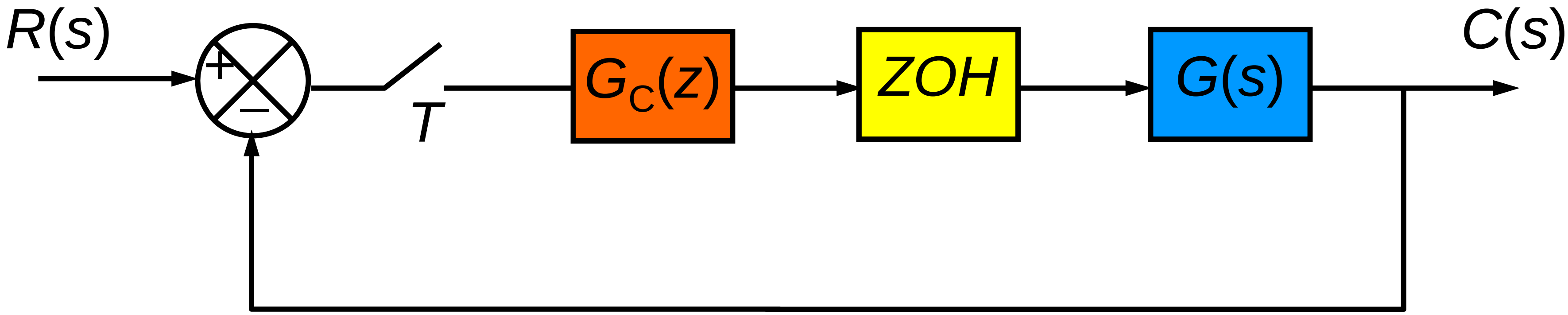
- ***Step 2: Chose the zero*** of the lag compensator: $z_C \approx -1$
- ***Step 3: Calculate the pole*** of the compensator:

$$p_C = -1 + \beta(1+z_C)$$

- ***Step 4: Calculate K_C*** satisfying the condition:

$$\left| G_C(z)GH(z) \right|_{z=z^*} = 1$$

Design discrete lag compensator using RL – Example



$$G(s) = \frac{50}{s(s+5)} \qquad T = 0.1\text{sec}$$

- Design the lag compensator $G_C(z)$ so that the compensated system has the velocity constant $K_V^* = 100$ and the closed poles are nearly unchanged.

- **Solution:**
- The discrete transfer function of the open-loop system:

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

$$= (1 - z^{-1})\mathcal{Z}\left\{\frac{50}{s^2(s+5)}\right\}$$

$$= 10(1 - z^{-1})\left(\frac{z[(0.5 - 1 + e^{-0.5})z + (1 - e^{-0.5} - 0.5e^{-0.5})]}{5(z-1)^2(z - e^{-0.5})}\right)$$

$$\Rightarrow \quad G(z) = \frac{0.21z + 0.18}{(z-1)(z-0.607)}$$

Design discrete lag compensator using RL – Example (cont')

✶ The characteristic equation of the uncompensated system:

$$1+G(z)=0$$

$$\Rightarrow \quad 1+\frac{0.21z+0.18}{(z-1)(z-0.607)}=0$$

$\Rightarrow$ Poles of the uncompensated system:

$$z_{1,2}=0.699\pm j0.547$$

Design discrete lag compensator using RL – Example (cont')

★ ***Step 1:*** Determine β

The velocity constant of the uncompensated system:

$$K_V = \frac{1}{T}\lim_{z\to 1}(1-z^{-1})G(z)$$

$$\Rightarrow K_V = \frac{1}{0.1}\lim_{z\to 1}(1-z^{-1})\frac{0.21z+0.18}{(z-1)(z-0.607)} \quad \Rightarrow \quad K_V = 9.9$$

The desired velocity constant: $K_V^* = 100$

Then: $$\beta = \frac{K_V}{K_V^*} = \frac{9.9}{100}$$

$$\Rightarrow \quad \beta = 0,099$$

- ***Step 2:*** Chose the zero of the lag compensator:

  Chose: $z_C = -0.99$

- ***Step 3:*** Calculate the pole of the lag compensator:

$$p_C = -1 + \beta(1 + z_C) = -1 + 0.099(1 - 0.99) \quad \Rightarrow \quad p_C = -0.999$$

$$\Rightarrow \quad G_C(z) = K_C \frac{z - 0,99}{s - 0,999}$$

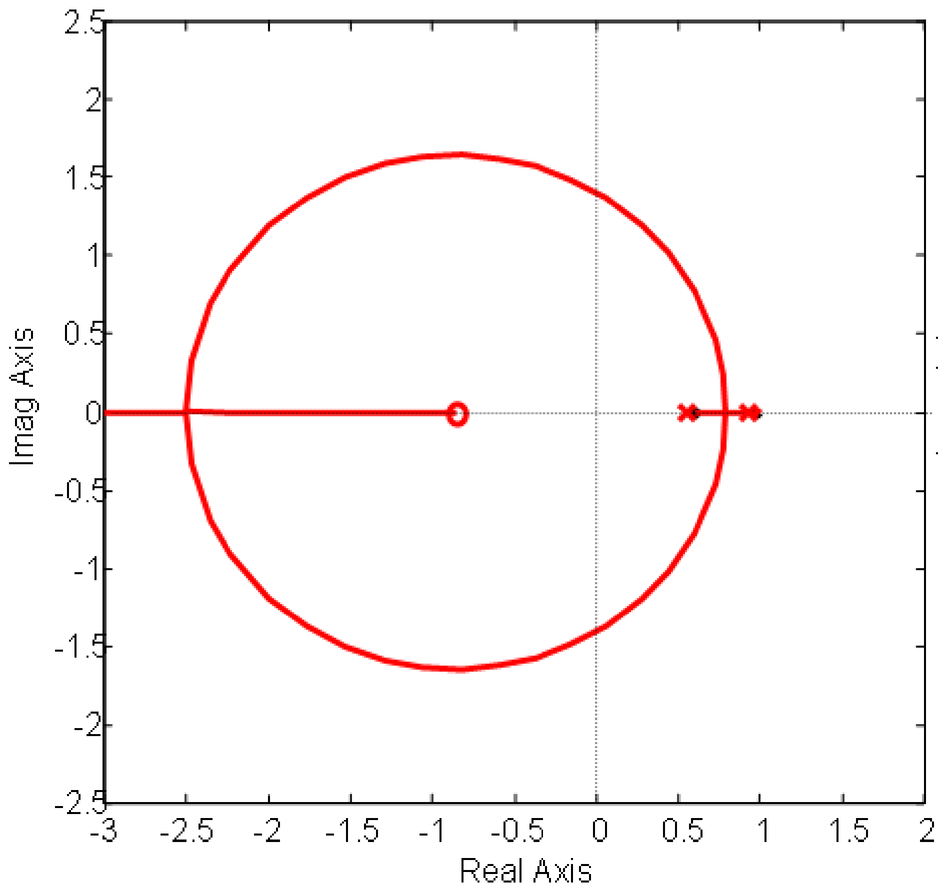
- ***Step 4:*** Determine the gain of the compensator

$$\left| G_C(z) G(z) \right|_{z = z^*} = 1$$

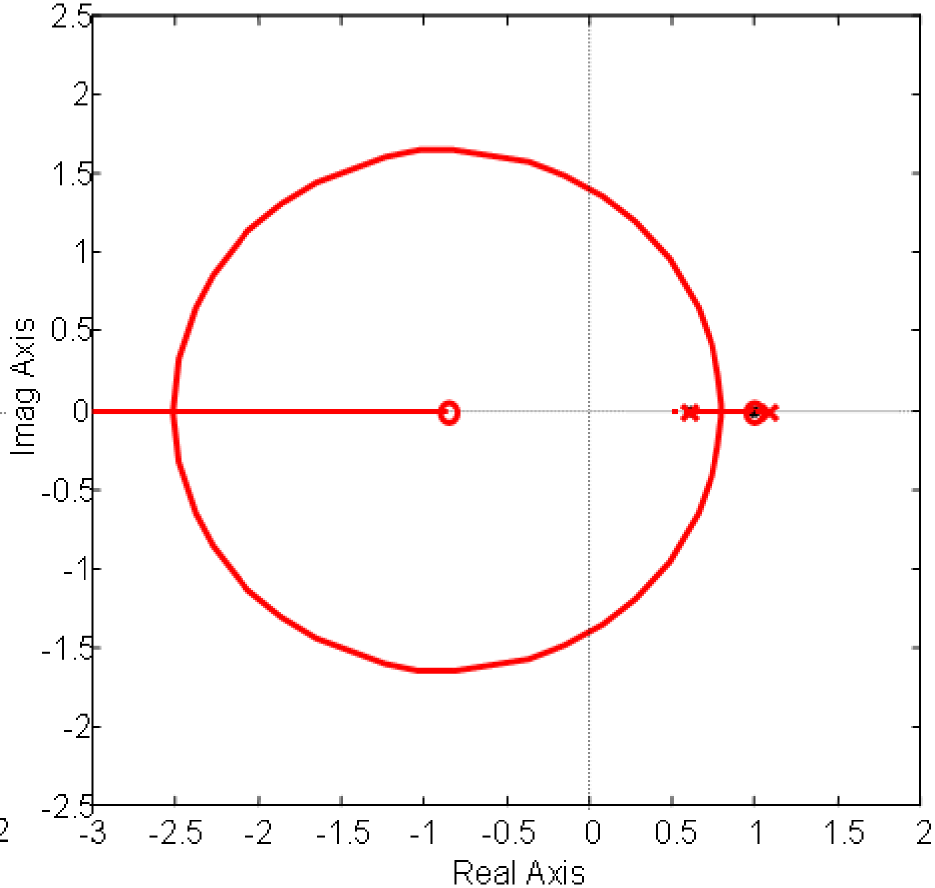
$$\Rightarrow \quad \left| K_C \frac{(z - 0.99)}{(z - 0.999)} \frac{(0.21z + 0.18)}{(z - 1)(z - 0.607)} \right|_{z = 0.699 + j0.547} = 1$$

$$\Rightarrow \quad K_C = 1.007 \approx 1$$

Design discrete lag compensator using RL – Example (cont')

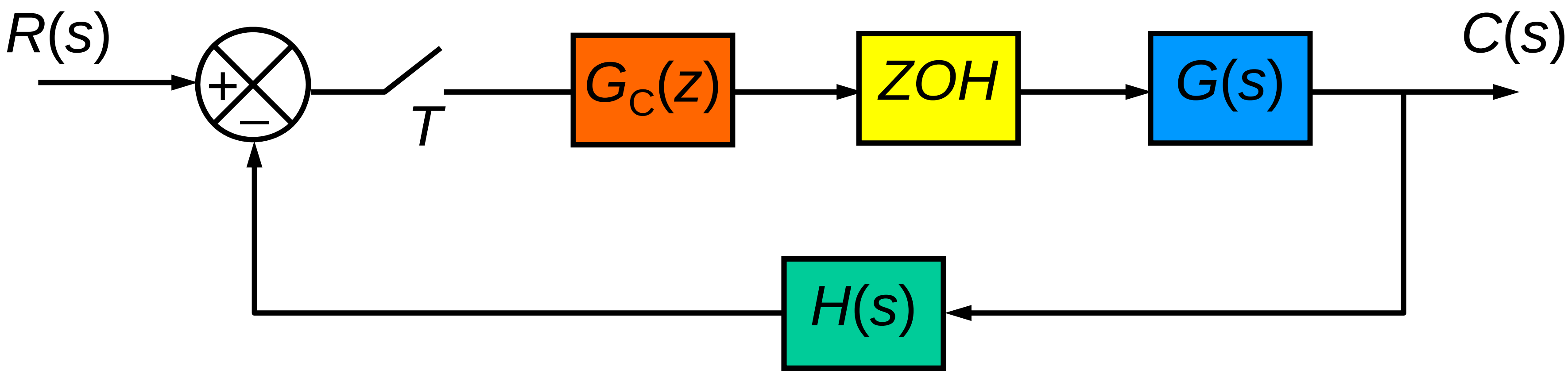


Root locus of the uncompensated system



Root locus of the compensated system

Design discrete PID controller using analytical method



$$G(s) = \frac{10}{10s+1} \qquad H(s) = 0.05 \qquad T = 2\,\text{sec}$$

Design the controller $G_C(z)$ so that the closed–loop system has the poles with ξ=0.707, ω_n=2 rad/sec and steady state error to step input is zero.

Design discrete PID controller using analytical method

- The controller to be designed is a PI controller (to meet the requirement of zero error to step input):

$$G_C(z) = K_P + \frac{K_I T}{2}\frac{z+1}{z-1}$$

- The discrete TF of the open-loop system:

$$GH(z) = (1-z^{-1})\mathcal{Z}\left\{\frac{G(s)H(s)}{s}\right\} = (1-z^{-1})\mathcal{Z}\left\{\frac{10\times 0.05}{s(10s+1)}\right\}$$

$$= (1-z^{-1})\frac{0.05z(1-e^{-0.2})}{0.1(z-1)(z-e^{-0.2})}$$

$$\Rightarrow \quad GH(z) = \frac{0.091}{(z-0.819)}$$

Design discrete PID controller using analytical method

★ The characteristic equation of the closed-loop system:

$$1+G_C(z)GH(z)=0$$

$$\Leftrightarrow \quad 1+\left(K_P+\frac{K_I T}{2}\frac{z+1}{z-1}\right)\left(\frac{0.091}{z-0.819}\right)=0$$

$$\Leftrightarrow \quad z^2+(0.091K_P+0.091K_I-1.819)z+(-0.091K_P+0.091K_I+0.819)=0$$

Design discrete PID controller using analytical method

★ The desired poles: $z_{1,2}^* = re^{\pm j\varphi}$

where

$$r = e^{-T\xi\omega_n} = e^{-2\times 0.707\times 2} = 0.059$$

$$\varphi = T\omega_n\sqrt{1-\xi^2} = 2\times 2\times\sqrt{1-0.707^2} = 2.828$$

$$\Rightarrow \quad z_{1,2}^* = 0.059e^{\pm j2.828}$$

$$\Rightarrow \quad z_{1,2}^* = -0.056 \pm j0.018$$

★ The desired characteristic equation:

$$(z+0.056+j0.018)(z+0.056-j0.018) = 0$$

$$\Leftrightarrow \quad z^2 + 0.112z + 0.0035 = 0$$

Design discrete PID controller using analytical method

- Balancing the coefficients of the system characteristic equation and the desired characteristic equation, we have:

$$\begin{cases} 0.091K_P + 0.091K_I - 1.819 = 0.112 \\ -0.091K_P + 0.091K_I + 0.819 = 0.0035 \end{cases}$$

$$\Rightarrow \quad \begin{cases} K_P = 15.09 \\ K_I = 6.13 \end{cases}$$

Conclusion

$$G_C(z) = 15.09 + 6.13\frac{z+1}{z-1}$$

Design of discrete control systems in state space domain

Controllability

- Consider a system: $\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d\boldsymbol{x}(k) + \boldsymbol{B}_d u(k) \\ y(k) = \boldsymbol{C}_d\boldsymbol{x}(k) \end{cases}$

- The system is complete state controllable if there exists an unconstrained control law $u(k)$ that can drive the system from an initial state $\boldsymbol{x}(k_0)$ to a arbitrarily final state $\boldsymbol{x}(k_f)$ in a finite time interval $k_0 \le k \le k_f$. Qualitatively, the system is state controllable if each state variable can be influenced by the input.

Controllability condition

- System: $$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d\boldsymbol{x}(k) + \boldsymbol{B}_d u(k) \\ y(k) = \boldsymbol{C}_d\boldsymbol{x}(k) \end{cases}$$

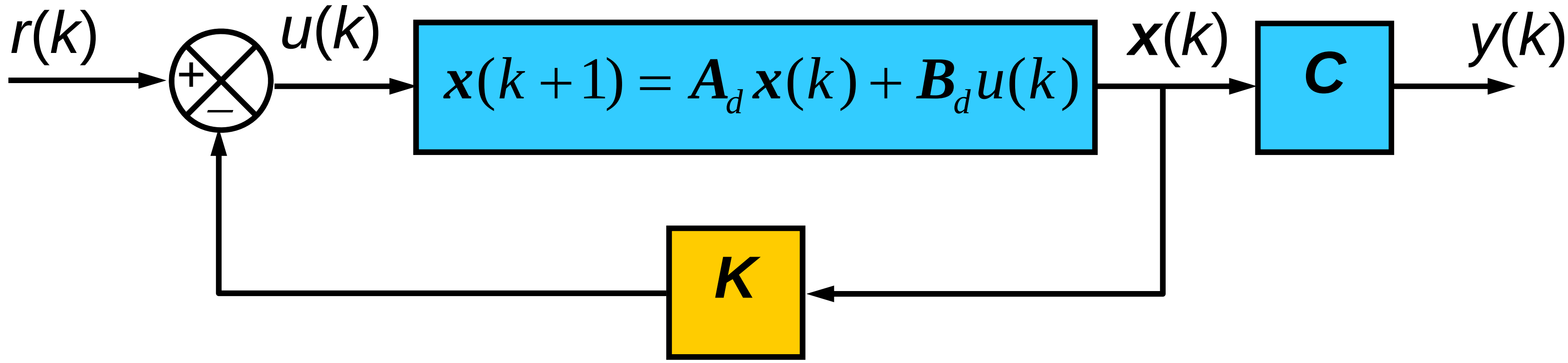
- **Controllability** matrix

$$\mathscr{C} = [\boldsymbol{B}_d \quad \boldsymbol{A}_d\boldsymbol{B}_d \quad \boldsymbol{A}_d^2\boldsymbol{B} \quad \ldots \quad \boldsymbol{A}_d^{n-1}\boldsymbol{B}_d]$$

- The necessary and sufficient condition for the controllability $$rank(\mathscr{C}) = n$$

- ***Note:*** we use the term "controllable" instead of "complete state controllable" for short.

Discrete state feedback control



- Consider a system described by the state-space equation:

$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d \boldsymbol{x}(k) + \boldsymbol{B}_d u(k) \\ y(k) = \boldsymbol{C}_d \boldsymbol{x}(k) \end{cases}$$

- The state feedback controller: $u(k) = r(k) - \boldsymbol{K}\boldsymbol{x}(k)$

- The state equation of the closed-loop system:

$$\begin{cases} \boldsymbol{x}(k+1) = [\boldsymbol{A}_d - \boldsymbol{B}_d \boldsymbol{K}]\boldsymbol{x}(k) + \boldsymbol{B}_d r(k) \\ y(k) = \boldsymbol{C}_d \boldsymbol{x}(k) \end{cases}$$

Pole placement method

If the system is controllable, then it is possible to determine the feedback gain ***K*** so that the closed-loop system has poles at any location.

- ***Step 1***: Write the characteristic equation of the closed-loop system

$$\det[z\boldsymbol{I} - \boldsymbol{A}_d + \boldsymbol{B}_d\boldsymbol{K}] = 0 \qquad (1)$$

- ***Step 2***: Write the desired characteristic equation:

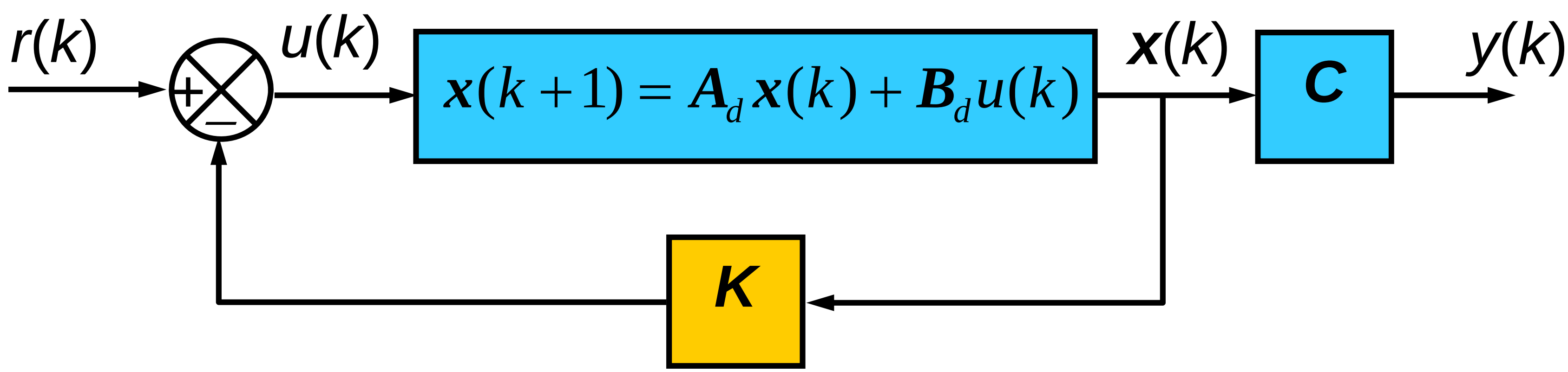
$$\prod_{i=1}^{n}(z - p_i) = 0 \qquad (2)$$

$p_i,\ (i = \overline{1,n})$ are the desired poles

- ***Step 3***: Balance the coefficients of the equations (1) and (2), we can find the state feedback gain ***K***.

Discrete pole placement design – Example 1

★ Given the control system:



$$A_d = \begin{bmatrix} 1 & 0.316 \\ 0 & 0.368 \end{bmatrix} \qquad B_d = \begin{bmatrix} 0.092 \\ 0.316 \end{bmatrix} \qquad C_d = \begin{bmatrix} 10 & 0 \end{bmatrix}$$

Determined the state feedback gain ***K*** so that the closed-loop system has a pair of complex poles with ξ=0.707, ω_n=10 rad/sec

Discrete pole placement design – Example 1 (cont')

- The closed-loop characteristic equation:

$$\det[z\boldsymbol{I} - \boldsymbol{A}_d + \boldsymbol{B}_d\boldsymbol{K}] = 0$$

$$\Leftrightarrow \quad \det\left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0.316 \\ 0 & 0.368 \end{bmatrix} + \begin{bmatrix} 0.092 \\ 0.316 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right) = 0$$

$$\Leftrightarrow \quad \det\left(\begin{bmatrix} z - 1 + 0.092k_1 & -0.316 + 0.092k_2 \\ 0.316k_1 & z - 0.368 + 0.316k_2 \end{bmatrix} \right) = 0$$

$$\Leftrightarrow \quad (z - 1 + 0.092k_1)(z - 0.368 + 0.316k_2) - 0.316k_1(-0.316 + 0.092k_2) = 0$$

$$\Leftrightarrow \quad \boxed{z^2 + (0.092k_1 + 0.316k_2 - 1.368)z + (0.066k_1 - 0.316k_2 + 0.368) = 0}$$

Discrete pole placement design – Example 1 (cont')

- The desired poles: $z_{1,2}^{*} = re^{\pm j\varphi}$

  where: $r = e^{-T\xi\omega_n} = e^{-0.1\times 0.707\times 10} = 0.493$

$$\varphi = T\omega_n\sqrt{1-\xi^2} = 0.1\times 10\times\sqrt{1-0.707^2} = 0.707$$

$$\Rightarrow \quad z_{1,2}^{*} = 0.493e^{\pm j0.707}$$

$$\Rightarrow \quad z_{1,2}^{*} = 0.375 \pm j0.320$$

- The desired characteristic equation:

$$(z-0.375-j0.320)(z-0.375+j0.320)=0$$

$$\Leftrightarrow \quad z^2 - 0.75z + 0.243 = 0$$

Discrete pole placement design – Example 1 (cont')

- Balancing the coefficients of the system characteristic equation and the desired characteristic equation, we have:

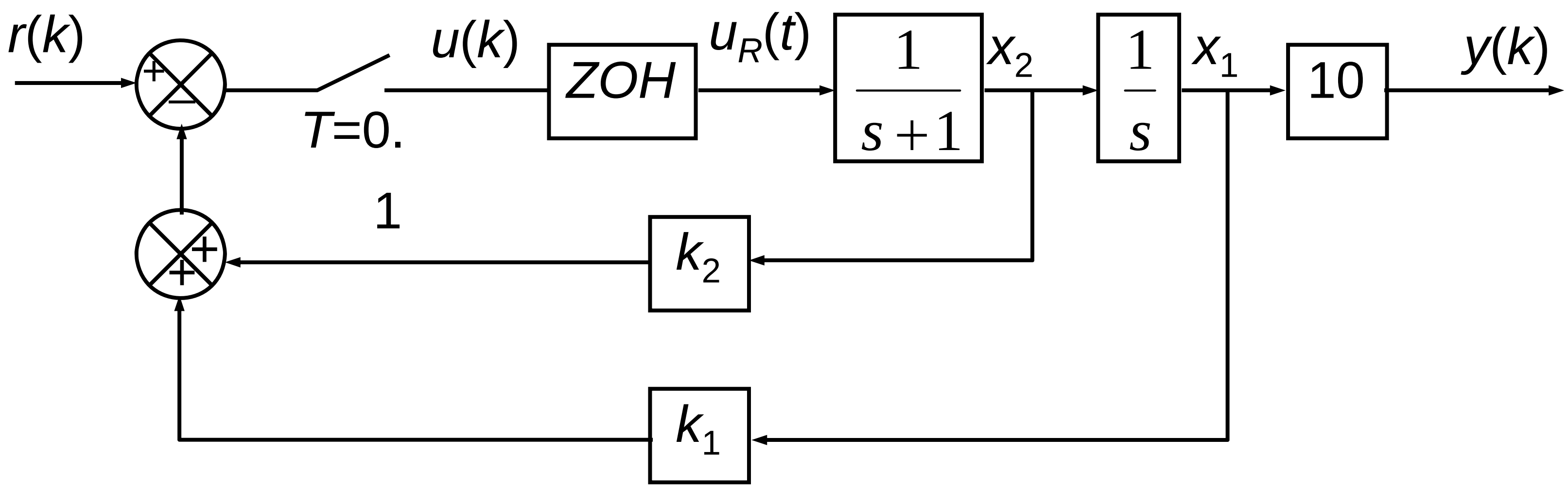
$$\begin{cases} (0.092k_1 + 0.316k_2 - 1.368) = -0.75 \\ (0.066k_1 - 0.316k_2 + 0.368) = 0.243 \end{cases}$$

$$\Rightarrow \qquad \begin{cases} k_1 = 3.12 \\ k_2 = 1.047 \end{cases}$$

Conclusion: $\boldsymbol{K} = \begin{bmatrix} 3.12 & 1.047 \end{bmatrix}$

Discrete pole placement design – Example 2

✶ Given the control system:



1. Write the state equations of the discrete open loop system
2. Determine the state feedback gain $\boldsymbol{K} = [k_1 \ k_2]$ so that the closed loop system has a pair of complex poles with ξ=0.5, ωn=8 rad/sec.
3. Calculate the response of the system to step input with the value of ***K*** obtained above. Calculate the POT and settling time.

Discrete pole placement design – Example 2 (cont')

✶ **Solution:**

1. Write the state equations of the discrete open loop system

Step 1: State space equations of open loop continuous system:

$$X_1(s)=\frac{X_2(s)}{s} \Rightarrow sX_1(s)=X_2(s) \Rightarrow \dot{x}_1(t)=x_2(t)$$

$$X_2(s)=\frac{U_R(s)}{s+1} \Rightarrow (s+1)X_2(s)=U_R(s) \Rightarrow \dot{x}_2(t)=-x_2(t)+u_R(t)$$

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u_R(t)$$

$$y(t)=10x_1(t)=\begin{bmatrix}10 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

Discrete pole placement design – Example 2 (cont')

Step 2: Transient matrix:

$$\Phi(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1} = \left(s\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1\\0 & -1\end{bmatrix}\right)^{-1} = \left(\begin{bmatrix}s & -1\\0 & s+1\end{bmatrix}\right)^{-1}$$

$$\Rightarrow \quad \Phi(s) = \begin{bmatrix}\dfrac{1}{s} & \dfrac{1}{s(s+1)}\\ 0 & \dfrac{1}{s+1}\end{bmatrix}$$

$$\Phi(t) = \mathscr{L}^{-1}[\Phi(s)] = \mathscr{L}^{-1}\left\{\begin{bmatrix}\dfrac{1}{s} & \dfrac{1}{s(s+1)}\\ 0 & \dfrac{1}{s+a}\end{bmatrix}\right\} = \begin{bmatrix}\mathscr{L}^{-1}\left\{\dfrac{1}{s}\right\} & \mathscr{L}^{-1}\left\{\dfrac{1}{s(s+1)}\right\}\\ 0 & \mathscr{L}^{-1}\left\{\dfrac{1}{s+1}\right\}\end{bmatrix}$$

$$\Rightarrow \quad \Phi(t) = \begin{bmatrix}1 & (1-e^{-t})\\ 0 & e^{-t}\end{bmatrix}$$

Discrete pole placement design – Example 2 (cont')

Step 3: State space equation of the open loop system:

$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d \boldsymbol{x}(k) + \boldsymbol{B}_d u(k) \\ c(k) = \boldsymbol{C}_d \boldsymbol{x}(k) \end{cases}$$

$$\boldsymbol{A}_d = \Phi(T) = \begin{bmatrix} 1 & (1-e^{-0.1}) \\ 0 & e^{-0.1} \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{A}_d = \begin{bmatrix} 1 & 0.095 \\ 0 & 0.905 \end{bmatrix}$$

$$\boldsymbol{B}_d = \int_0^T \Phi(\tau)\boldsymbol{B}d\tau = \int_0^{0.1} \left\{ \begin{bmatrix} 1 & (1-e^{-\tau}) \\ 0 & e^{-\tau} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau \right\} = \int_0^{0.1} \left\{ \begin{bmatrix} (1-e^{-\tau}) \\ e^{-\tau} \end{bmatrix} d\tau \right\}$$

$$= \begin{bmatrix} \left(\tau + e^{-\tau}\right) \\ -e^{-\tau} \end{bmatrix}_0^{0.1} = \begin{bmatrix} \left(0.1 + e^{-0.1} - 1\right) \\ -e^{-0.1} + 1 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{B}_d = \begin{bmatrix} 0.005 \\ 0.095 \end{bmatrix}$$

$$\boldsymbol{C}_d = \boldsymbol{C} = \begin{bmatrix} 10 & 0 \end{bmatrix}$$

Discrete pole placement design – Example 2 (cont')

2. Calculate the state feedback gain ***K***:

The closed loop characteristic equation:

$$\det[z\boldsymbol{I} - \boldsymbol{A}_d + \boldsymbol{B}_d\boldsymbol{K}] = 0$$

$$\Leftrightarrow \det\left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0.095 \\ 0 & 0.905 \end{bmatrix} + \begin{bmatrix} 0.005 \\ 0.095 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right) = 0$$

$$\Leftrightarrow \det\left(\begin{bmatrix} z - 1 + 0.005k_1 & -0.095 + 0.005k_2 \\ 0.095k_1 & z - 0.905 + 0.095k_2 \end{bmatrix} \right) = 0$$

$$\Leftrightarrow (z - 1 + 0.005k_1)(z - 0.905 + 0.095k_2) - 0.905k_1(-0.095 + 0.005k_2) = 0$$

$$\Leftrightarrow z^2 + (0.005k_1 + 0.095k_2 - 1.905)z + (0.0045k_1 - 0.095k_2 + 0.905) = 0$$

Discrete pole placement design – Example 2 (cont')

The desired dominant poles: $z_{1,2}^{*} = re^{\pm j\varphi}$

$$r = e^{-T\xi\omega_n} = e^{-0.1\times0.5\times8} = 0.67$$

$$\varphi = T\omega_n\sqrt{1-\xi^2} = 0.1\times8\sqrt{1-0.5^2} = 0.693$$

$$\Rightarrow \quad z_{1,2}^{*} = 0.67e^{\pm j0.693}$$

$$\Rightarrow \quad z_{1,2}^{*} = 0.516 \pm j0.428$$

The desired characteristic equation:

$$(z-0.516-j0.428)(z-0.516+j0.428) = 0$$

$$\Rightarrow \quad z^2 - 1.03z + 0.448 = 0$$

Discrete pole placement design – Example 2 (cont')

Balancing the coefficients of the closed loop characteristic equation and the desired characteristic equation, we have:

$$\begin{cases}(0.005k_1+0.095k_2-1.905)=-1.03\\(0.0045k_1-0.095k_2+0.905)=0.448\end{cases}$$

$$\Rightarrow \quad \begin{cases}k_1=44.0\\k_2=6.895\end{cases}$$

Conclusion: $\boldsymbol{K}=\begin{bmatrix}44.0 & 6.895\end{bmatrix}$

Discrete pole placement design – Example 2 (cont')

3. Calculate system response and performances:

State space equation of the closed-loop system:

$$\begin{cases} \boldsymbol{x}(k+1) = \left[\boldsymbol{A}_d - \boldsymbol{B}_d\boldsymbol{K}\right]\boldsymbol{x}(k) + \boldsymbol{B}_d r(k) \\ c(k) = \boldsymbol{C}_d\boldsymbol{x}(k) \end{cases}$$

Student continuous to calculate the response and performance by themselves following the method presented in the chapter 8.

Design of discrete state estimators

The concept of state estimation

- ✶ To be able to implement state feedback control system, it is required to measure all the states of the system.

- ✶ However, in some application, we can only measure the output, but cannot measure the states of the system.

- ✶ The problem is to estimate the states of the system from the output measurement.

$\Rightarrow$ State estimator (or state observer)

Observability

- Consider the system: $\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d \boldsymbol{x}(k) + \boldsymbol{B}_d u(k) \\ y(k) = \boldsymbol{C}_d \boldsymbol{x}(k) \end{cases}$

- The system is complete state observable if given the control law $u(k)$ and the output signal $y(k)$ in a finite time interval $k_0 \leq k \leq k_f$, it is possible to determine the initial states $\boldsymbol{x}(k_0)$.

- Qualitatively, the system is state observable if all state variable $\boldsymbol{x}(k)$ influences the output $y(k)$.

Observability condition

- System
$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d \boldsymbol{x}(k) + \boldsymbol{B}_d u(k) \\ y(k) = \boldsymbol{C}_d \boldsymbol{x}(k) \end{cases}$$

It is require to estimate the state $\hat{\boldsymbol{x}}(k)$ from mathematical model of the system and the input-output data.

- **Observability matrix:**
$$\mathcal{O} = \begin{bmatrix} \boldsymbol{C}_d \\ \boldsymbol{C}_d \boldsymbol{A} \\ \boldsymbol{C}_d \boldsymbol{A}_d^2 \\ \vdots \\ \boldsymbol{C}_d \boldsymbol{A}_d^{n-1} \end{bmatrix}$$

- The necessary and sufficient condition for the observability:
$$rank(\mathcal{O}) = n$$

Observability – Example

- Given the system $\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d \boldsymbol{x}(k) + \boldsymbol{B}_d u(k) \\ y(k) = \boldsymbol{C}_d \boldsymbol{x}(k) \end{cases}$

  where: $\boldsymbol{A}_d = \begin{bmatrix} 0.967 & 0.148 \\ -0.297 & 0.522 \end{bmatrix}$ $\boldsymbol{B}_d = \begin{bmatrix} 0.231 \\ 0.264 \end{bmatrix}$ $\boldsymbol{C}_d = \begin{bmatrix} 1 & 3 \end{bmatrix}$

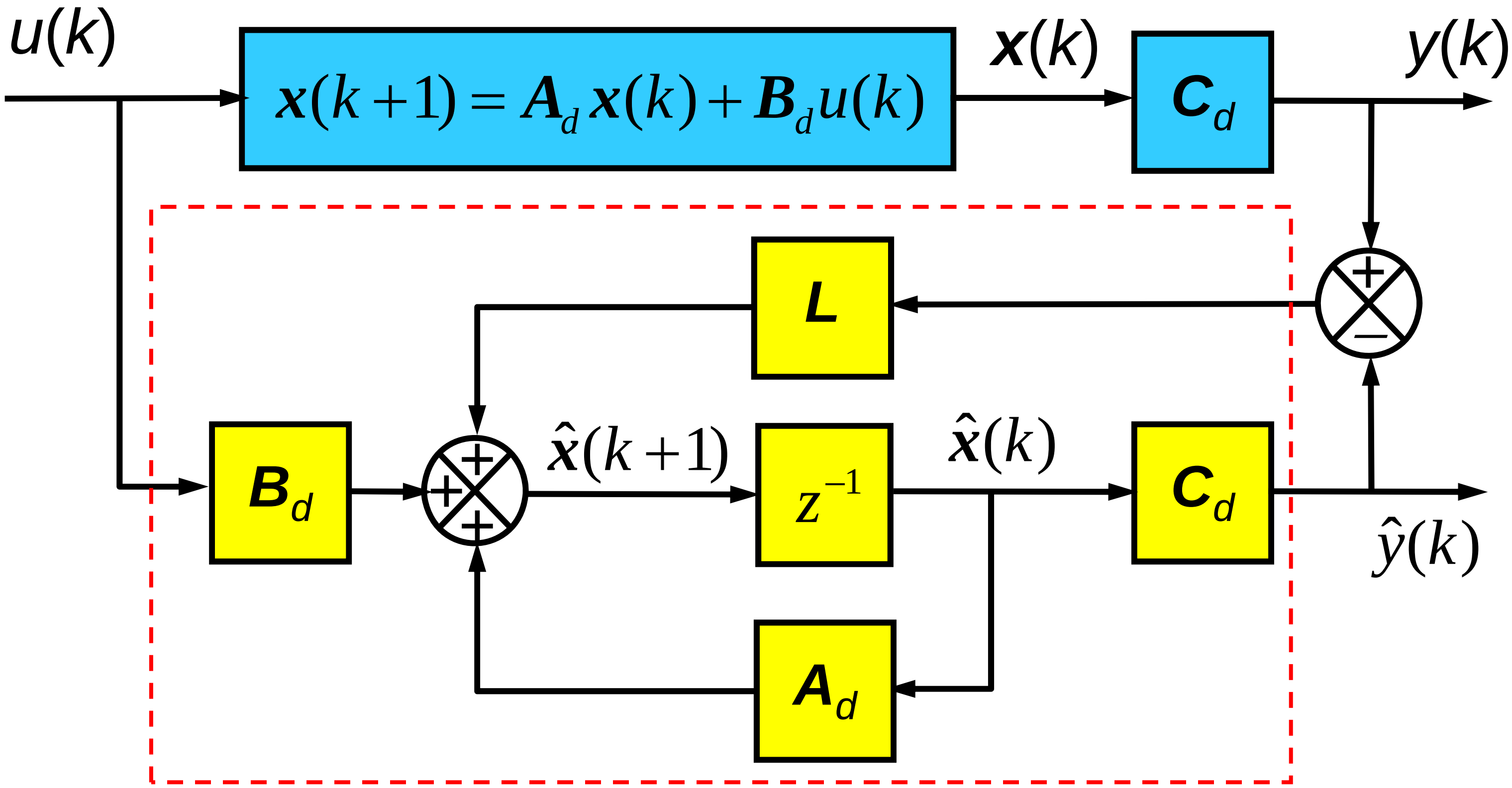
  Analyze the observability of the system.

- ***Solution:*** Observability matrix:

$$\mathcal{O} = \begin{bmatrix} \boldsymbol{C}_d \\ \boldsymbol{C}_d \boldsymbol{A}_d \end{bmatrix} \quad \Rightarrow \quad \mathcal{O} = \begin{bmatrix} 1 & 3 \\ 0.077 & 1.714 \end{bmatrix}$$

- Because $\det(\mathcal{O}) = 1.484 \quad \Rightarrow \quad rank(\mathcal{O}) = 2$

  $\Rightarrow$ The system is observable



★ State estimator: $\begin{cases} \hat{\boldsymbol{x}}(k+1) = \boldsymbol{A}_d \hat{\boldsymbol{x}}(k) + \boldsymbol{B}_d u(k) + \boldsymbol{L}(y(k) - \hat{y}(k)) \\ \hat{y}(k) = \boldsymbol{C}_d \hat{\boldsymbol{x}}(k) \end{cases}$

where: $\boldsymbol{L} = [l_1 \quad l_2 \quad \ldots \quad l_n]^T$

Design of state estimators

- Requirements:
  - The state estimator must be stable, estimation error should approach to zero.
  - Dynamic response of the state estimator should be fast enough in comparison with that of the control loop.
- It is required to chose *L* satisfying:
  - All the roots of the equation $\det(zI - \boldsymbol{A}_d + \boldsymbol{LC}_d) = 0$ locates inside the unit circle in the z-plane.
  - The roots of $\det(zI - \boldsymbol{A}_d + \boldsymbol{LC}_d) = 0$ are further from the unit circle than the roots of $\det(zI - \boldsymbol{A}_d + \boldsymbol{B}_d\boldsymbol{K}) = 0$
- Depending on the design of ***L***, we have different state estimator:
  - Luenberger state observer
  - Kalman filter

Procedure for designing the Luenberger state observer

- ***Step 1***: Write the characteristic equ. of the state observer

$$\det[z\boldsymbol{I} - \boldsymbol{A}_d + \boldsymbol{L}\boldsymbol{C}_d] = 0 \qquad (1)$$

- ***Step 1***: Write the desired characteristic equation:

$$\prod_{i=1}^{n}(z - p_i) = 0 \qquad (2)$$

$p_i,\ (i = \overline{1,n})$ are the desired poles of the state estimator

- ***Step 3***: Balance the coefficients of the characteristic equations (1) and (2), we can find the gain ***L***.

Design of state estimators – Example

- ***Problem***: Given a system described by the state equation:

$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_d \boldsymbol{x}(k) + \boldsymbol{B}_d u(k) \\ y(k) = \boldsymbol{C}_d \boldsymbol{x}(k) \end{cases}$$

$$\boldsymbol{A}_d = \begin{bmatrix} 0.967 & 0.148 \\ -0.297 & 0.522 \end{bmatrix} \qquad \boldsymbol{B}_d = \begin{bmatrix} 0.231 \\ 0.264 \end{bmatrix} \qquad \boldsymbol{C}_d = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

- Assuming that the states of the system cannot be directly measured. Design the Luenberger state estimator so that the poles of the state estimator lying at 0.13 and 0.36.

Design of state estimators – Example (cont')

★ ***Solution***

★ The characteristic equation of the Luenberger state estimator:

$$\det[z\boldsymbol{I} - \boldsymbol{A}_d + \boldsymbol{L}\boldsymbol{C}_d] = 0$$

$$\Rightarrow \quad \det\left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.967 & 0.148 \\ -0.297 & 0.522 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} \right) = 0$$

$$\Rightarrow \quad \det\left(\begin{bmatrix} z - 0.967 + l_1 & -0.148 + 3l_1 \\ 0.297 + l_2 & z - 0.522 + 3l_2 \end{bmatrix} \right) = 0$$

$$\Rightarrow \quad z^2 + (l_1 + 3l_2 - 1.489)z + (-1.413l_1 - 2.753l_2 + 0.549) = 0 \qquad (1)$$

★ The desired characteristic equation:

$$(z - 0.13)(z - 0.36) = 0 \quad \Rightarrow \quad z^2 - 0.49z + 0.0468 = 0 \qquad (2)$$

Design of state estimators – Example (cont')

- Balancing the coefficients of the equations (1) and (2):

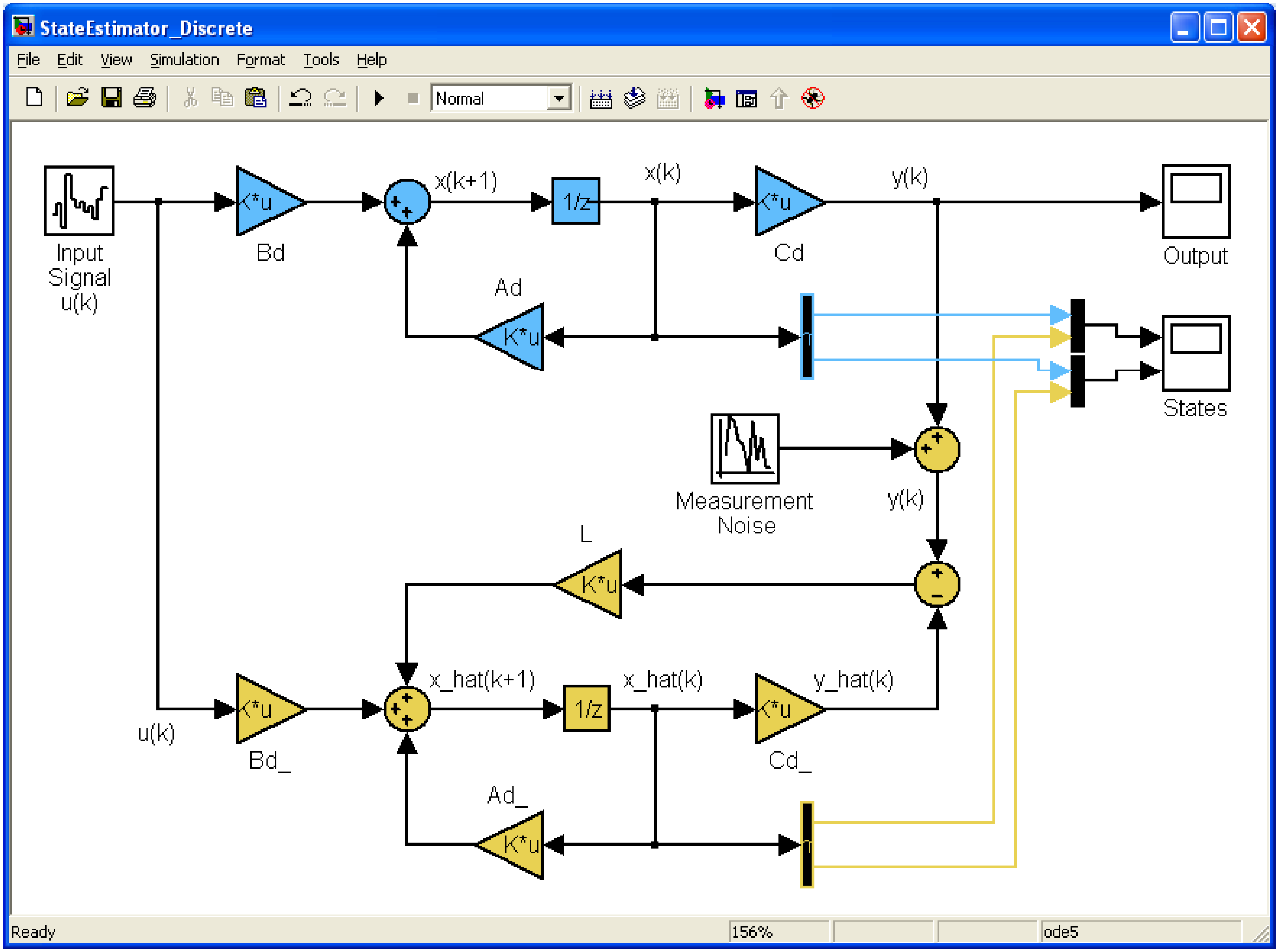
$$\begin{cases} l_1 + 3l_2 - 1.489 = 0.49 \\ -1.413l_1 - 2.753l_2 + 0.549 = 0.0468 \end{cases}$$

- Solve the above set of equations, we have:

$$\begin{cases} l_1 = -2.653 \\ l_2 = 1.544 \end{cases}$$

- Conclusion $\boldsymbol{L} = \begin{bmatrix} -2.653 & 1.544 \end{bmatrix}^T$

Simulation of discrete state estimator



State estimation simulation result