Exercise 5.2 - Sigmoid Function (0.5 + 1 + 0.5 + 1 points)

The commonly used activation function in hidden layers of a Neural Network is the Sigmoid function which is defined as:

$$\sigma(x) = \frac{1}{1+e^{-x}}$$

a) Prove that the derivative of the sigmoid function is $\sigma(x) - \sigma^2(x)$.

b) Sketch the gradient of the sigmoid function (please indicate ticks on the axes) and also explain what are the inherent properties that you observe from the above computed gradient?

c) Prove that the sigmoid function is point symmetric.

Hint: You can check the following wikipedia page:(https://en.wikipedia.org/wiki/Point_reflection) for point symmetric meaning.

d) We know the importance of the Taylor series in optimization from Newton's method, additionally, Taylor expansion could be beneficial in providing a cheaper computation alternative for activation functions (for further reading: http://www.yildiz.edu.tr/~tulay/publications/Tainn2003-3.pdf).

For the sigmoid function at x = 0, find the Taylor expansion to third-degree polynomial(first four terms).

Hint: You can use the derivative form proved in (a) when calculating higher derivatives.

a) $\sigma(x) = \dfrac{e^x}{e^x + 1}$ | Apply quotient rule

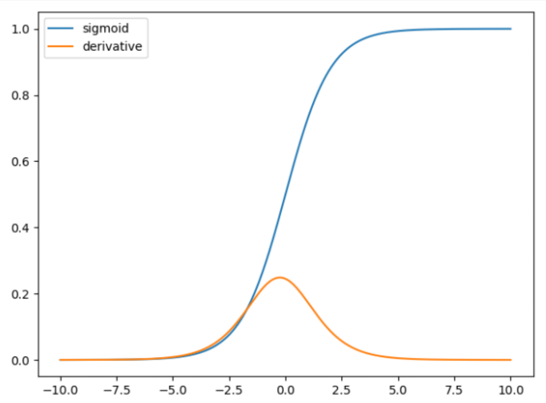
$$\sigma'(x) = \frac{e^x(1+e^x) - e^x \cdot e^x}{(1+e^x)^2}$$

$$= \frac{e^x}{1+e^x} \cdot \frac{(1+e^x) - e^x}{1+e^x}$$

$$= \sigma(x) \cdot (1 - \sigma(x))$$

$$= \sigma(x) - \sigma(x)^2$$

b) the gradient can be sketched in Python.



The derivative is symmetric around it's only maximum at $x = 0$.

c) Claim: $\forall x \in \mathbb{R}: \; 1 - \sigma(x) = \sigma(-x)$

Proof: $$1 - \sigma(x) = 1 - \frac{e^x}{1+e^x}$$

$$= \frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x}$$

$$= \frac{1}{1+e^x}$$

$$= \sigma(-x)$$

$\blacksquare$

d) The second and third derivative of $\sigma(x)$ are given by

$$\sigma^{(2)}(x) = \sigma(x) - \sigma^2(x) - 2\sigma(x)\left(\sigma(x) - \sigma^2(x)\right)$$
$$= \left(1 - 2\sigma(x)\right)\left(\sigma(x) - \sigma(x)^2\right)$$

$$\sigma^{(3)}(x) = \sigma(x) - \sigma^2(x) - 3 \cdot 2\,\sigma(x)\left(\sigma(x) - \sigma(x)^2\right) + 3 \cdot 3 \cdot \sigma^2(x)\left(\sigma(x) - \sigma(x)^2\right)$$

$$= \left(1 - 6\sigma(x) + 9\sigma^2(x)\right)\left(\sigma(x) - \sigma(x)^2\right)$$

Thus the Taylor-expansion at $x_0 = 0$ is given by

$$T_0^{(3)}(x) = \frac{1}{2} + \frac{1}{4}x + 0 + \frac{1}{3!} \cdot \left(1 - 6 \cdot \frac{1}{2} + 9 \cdot \frac{1}{4}\right) \cdot \frac{1}{2} \cdot x$$

$$= \frac{1}{2} + \frac{1}{4}x + \frac{1}{3!} \cdot \left(\frac{4}{4} - \frac{12}{4} + \frac{9}{4}\right) \cdot \frac{1}{2} \cdot x$$

$$= \frac{1}{2} + \frac{1}{4}x + \frac{1}{48}x^3$$

$\blacksquare$