Chapter 05
Constraint Satisfaction Problem

Instructor

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Instructor's Information

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Acknowledgment

The slides in this PPT file are composed using the materials supplied by

- **Prof. Stuart Russell and Peter Norvig**: They are currently from University of California, Berkeley. They are also the author of the book "Artificial Intelligence: A Modern Approach", which is used as the textbook for the course
- **Prof. Tom Lenaerts**, from Université Libre de Bruxelles

Outline

- CSP?
- Backtracking for CSP
- Local search for CSPs
- Problem structure and decomposition

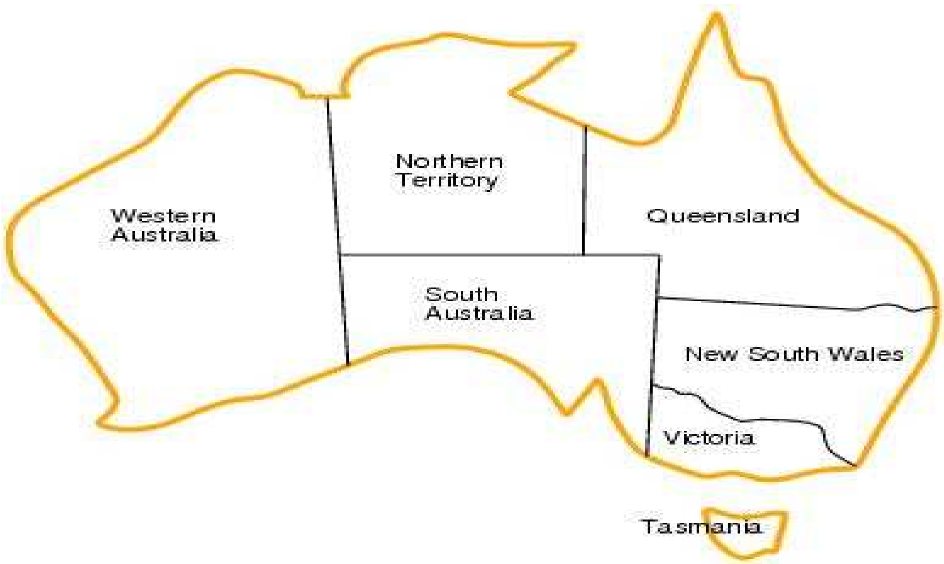
Constraint satisfaction problems

- What is a CSP?
  - Finite set of variables $V_1, V_2, \ldots, V_n$
  - Finite set of constraints $C_1, C_2, \ldots, C_m$
  - Nonemtpy domain of possible values for each variable $D_{V1}, D_{V2}, \ldots D_{Vn}$
  - Each constraint C_i limits the values that variables can take, e.g., $V_1 \neq V_2$
- A *state* is defined as an *assignment* of values to some or all variables.
- *Consistent assignment*: assignment does not not violate the constraints.

Constraint satisfaction problems

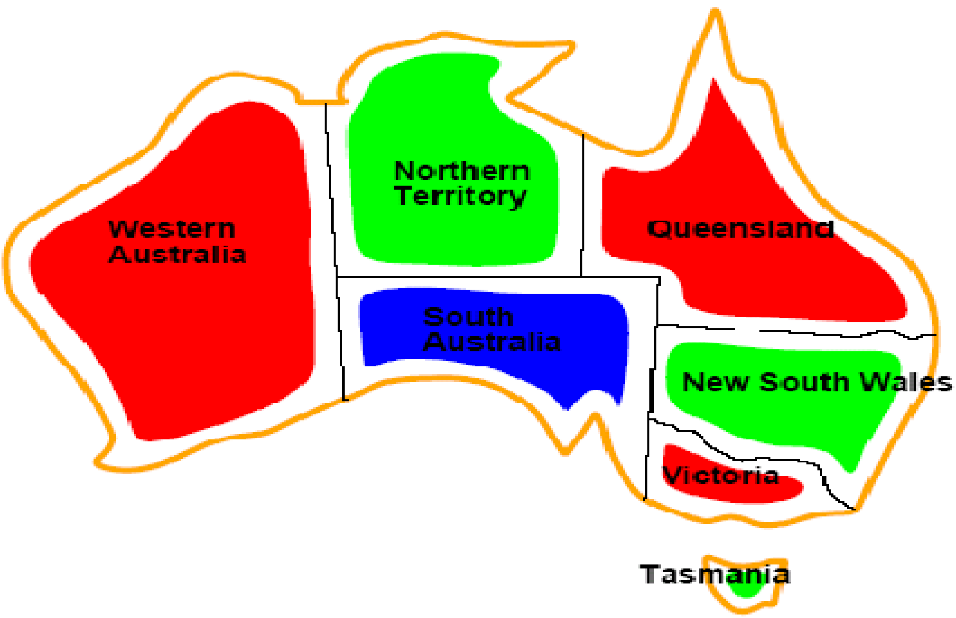
- An assignment is *complete* when every value is mentioned.
- A *solution* to a CSP is a complete assignment that satisfies all constraints.
- Some CSPs require a solution that maximizes an *objective function*.
- Applications: Scheduling the time of observations on the Hubble Space Telescope, Floor planning, Map coloring, Cryptography

CSP example: map coloring



- Variables: *WA, NT, Q, NSW, V, SA, T*
- Domains: D_i=*{red,green,blue}*
- Constraints:adjacent regions must have different colors.
    - ✓ E.g. $WA \neq NT$ (if the language allows this)
    - ✓ E.g. $(WA,NT) \neq$ *{(red,green),(red,blue),(green,red),…}*

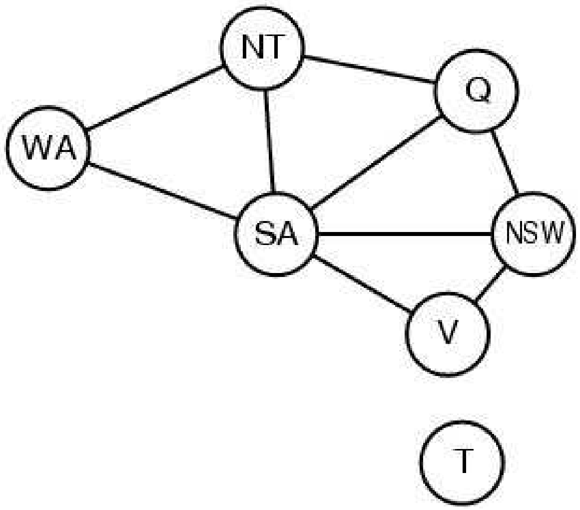
CSP example: map coloring



❖ Solutions are assignments satisfying all constraints, e.g.
{WA=red,NT=green,Q=red,NSW=green,V=red,SA=blue,T=green}

Constraint graph

- CSP benefits
  - Standard representation pattern
  - Generic goal and successor functions
  - Generic heuristics (no domain specific expertise).



- Constraint graph = nodes are variables, edges show constraints.
  - Graph can be used to simplify search.
    - ✓ e.g. Tasmania is an independent subproblem.

Varieties of CSPs

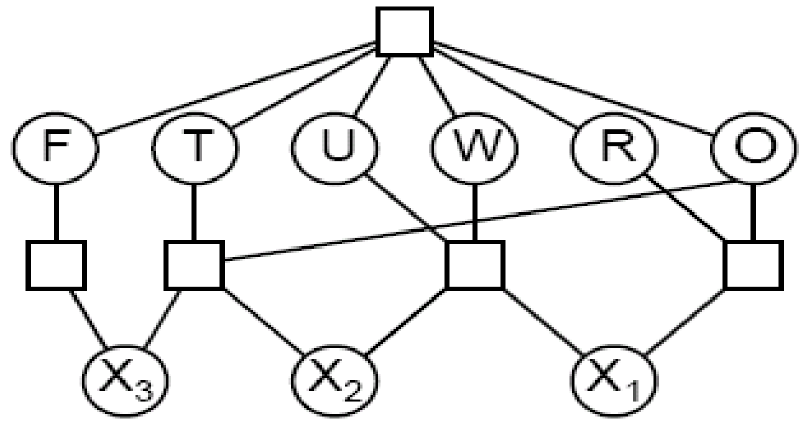
- ❖ Discrete variables
  - Finite domains; size $d \Rightarrow O(d^n)$ complete assignments.
    - ✓ E.g. Boolean CSPs, include. Boolean satisfiability (NP-complete).
  - Infinite domains (integers, strings, etc.)
    - ✓ E.g. job scheduling, variables are start/end days for each job
    - ✓ Need a constraint language e.g $StartJob_1 + 5 \leq StartJob_3$.
    - ✓ Linear constraints solvable, nonlinear undecidable.
- ❖ Continuous variables
  - e.g. start/end times for Hubble Telescope observations.
  - Linear constraints solvable in poly time by LP methods.

Varieties of constraints

- Unary constraints involve a single variable.
  - e.g. $SA \neq green$
- Binary constraints involve pairs of variables.
  - e.g. $SA \neq WA$
- Higher-order constraints involve 3 or more variables.
  - e.g. cryptharithmetic column constraints.
- Preference (soft constraints) e.g. *red* is better than *green* often representable by a cost for each variable assignment $\rightarrow$ constrained optimization problems.

Example; cryptharithmetic

$$\begin{array}{cccc} & T & W & O \\ + & T & W & O \\ \hline F & O & U & R \end{array}$$



Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$
Domains: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
Constraints

alldiff(F, T, U, W, R, O)

$O + O = R + 10 \cdot X_1$, etc.

CSP as a standard search problem

- A CSP can easily expressed as a standard search problem.
- Incremental formulation
  - *Initial State*: the empty assignment {}.
  - *Successor function*: Assign value to unassigned variable provided that there is not conflict.
  - *Goal test*: the current assignment is complete.
  - *Path cost*: as constant cost for every step.

CSP as a standard search problem

- This is the same for all CSP's !!!
- Solution is found at depth *n* (if there are *n* variables).
  - Hence depth first search can be used.
- Path is irrelevant, so complete state representation can also be used.
- Branching factor *b* at the top level is *nd*.
- $b=(n-l)d$ at depth l, hence $n!d^n$ leaves (only d^n complete assignments).

Commutativity

- CSPs are commutative.
  - The order of any given set of actions has no effect on the outcome.
  - Example: choose colors for Australian territories one at a time
    - ✓ [WA=red then NT=green] same as [NT=green then WA=red]
    - ✓ All CSP search algorithms consider a single variable assignment at a time $\Rightarrow$ there are d^n leaves.

Backtracking search

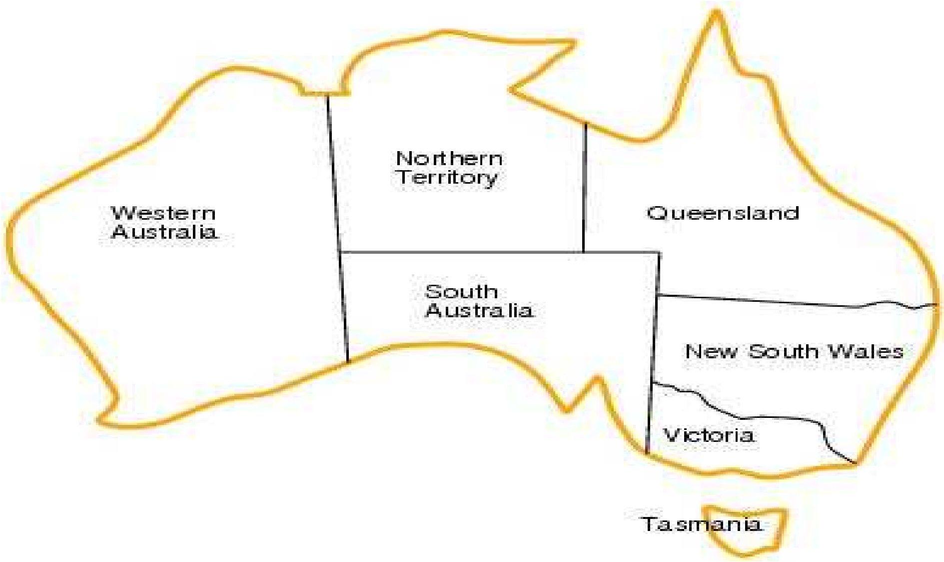
- Cfr. Depth-first search
- Chooses values for one variable at a time and backtracks when a variable has no legal values left to assign.
- Uninformed algorithm
  - No good general performance (see table p. 143)

Backtracking search

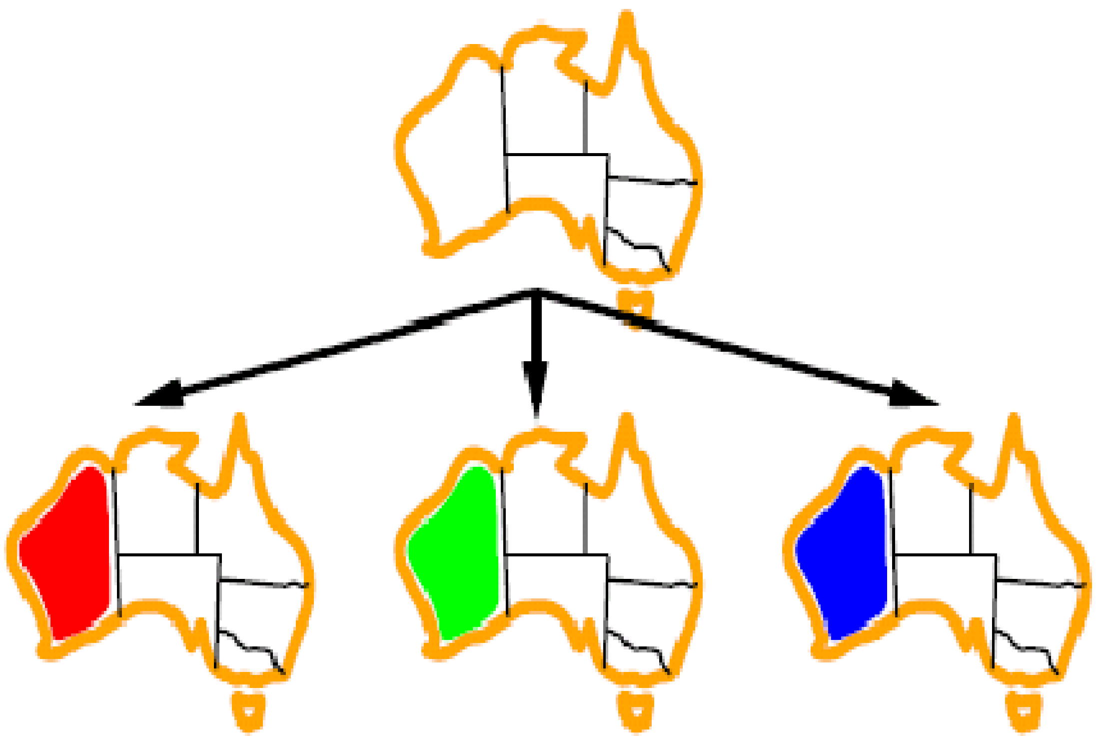
function BACKTRACKING-SEARCH(*csp*) **return** a solution or failure
    return RECURSIVE-BACKTRACKING(*{}* , *csp*)

function RECURSIVE-BACKTRACKING(*assignment*, *csp*) **return** a solution or failure
    if *assignment* is complete **then return** *assignment*
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[*csp*],*assignment*,*csp*)
    for each *value* **in** ORDER-DOMAIN-VALUES(*var*, *assignment*, *csp*) **do**
        if *value* is consistent with *assignment* according to CONSTRAINTS[*csp*] **then**
            add *{var=value}* to assignment
            result ← RRECURSIVE-BACTRACKING(*assignment*, *csp*)
            if *result* ≠ *failure* **then return** *result*
            remove *{var=value}* from *assignment*
    return *failure*

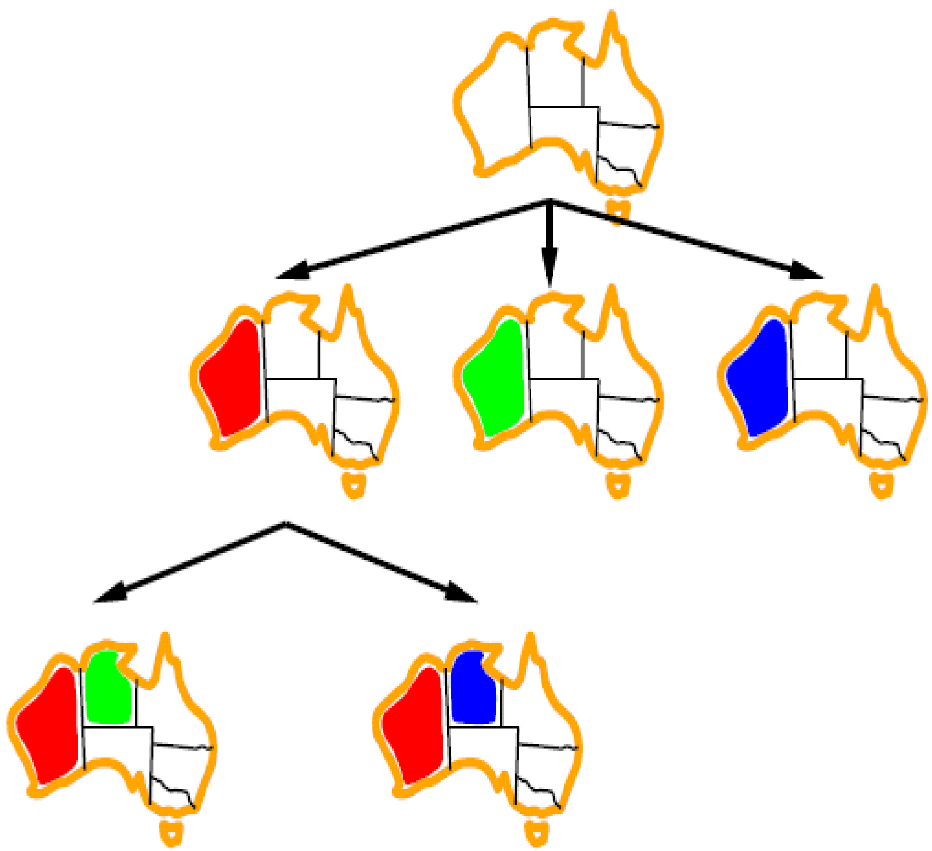
Backtracking example



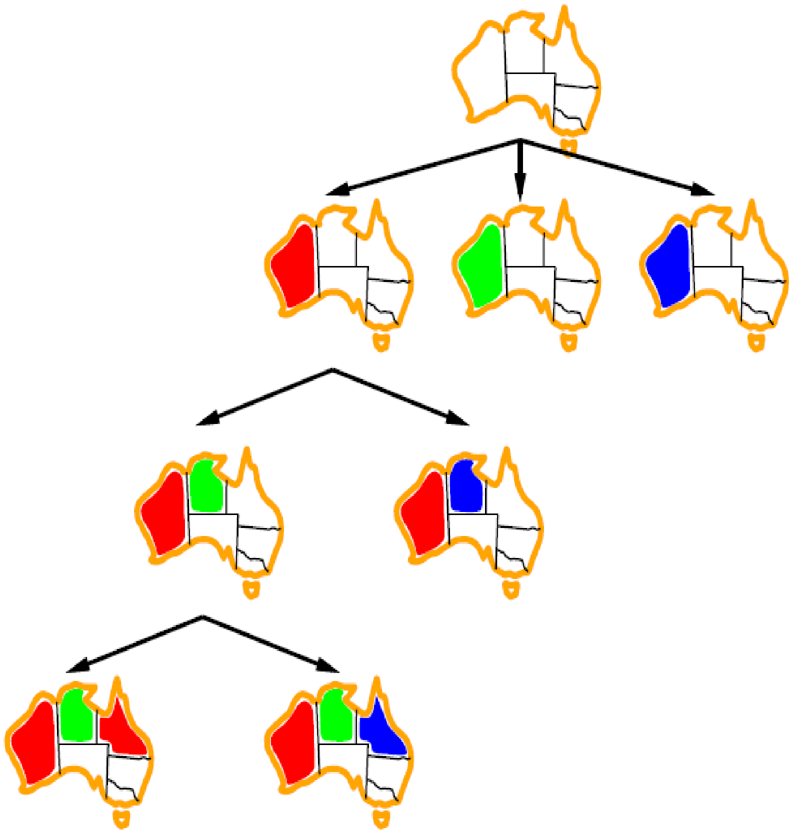
Backtracking example



Backtracking example



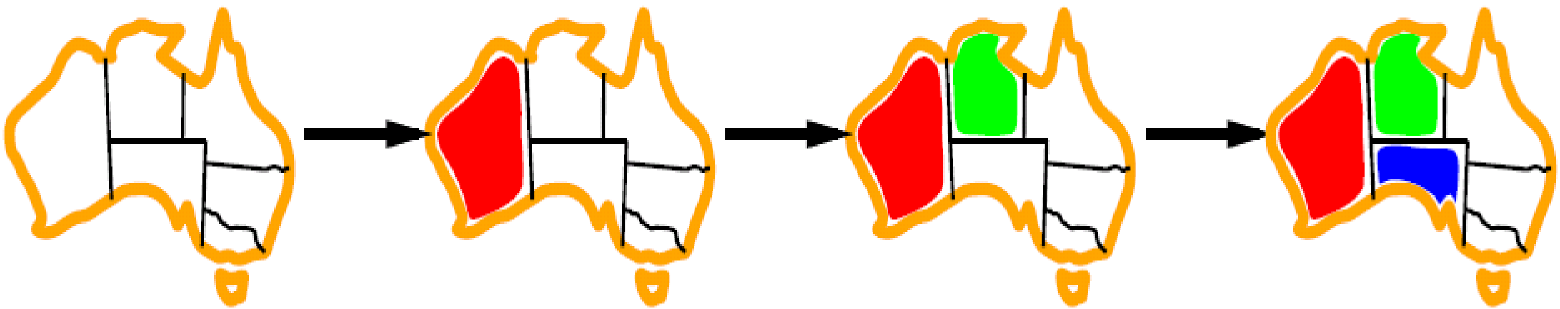
Backtracking example



Improving backtracking efficiency

- Previous improvements → introduce heuristics
- General-purpose methods can give huge gains in speed:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?
  - Can we take advantage of problem structure?

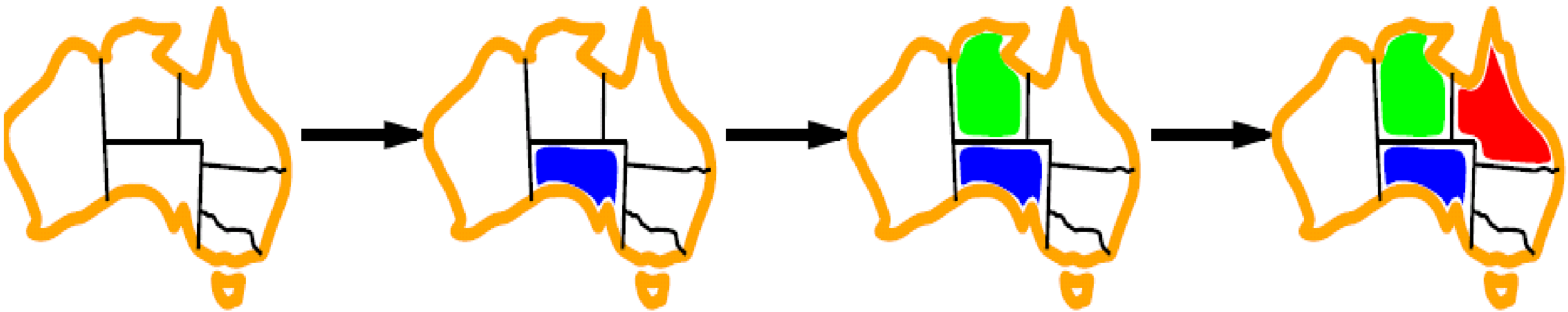
Minimum remaining values



var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[*csp*],*assignment*,*csp*)

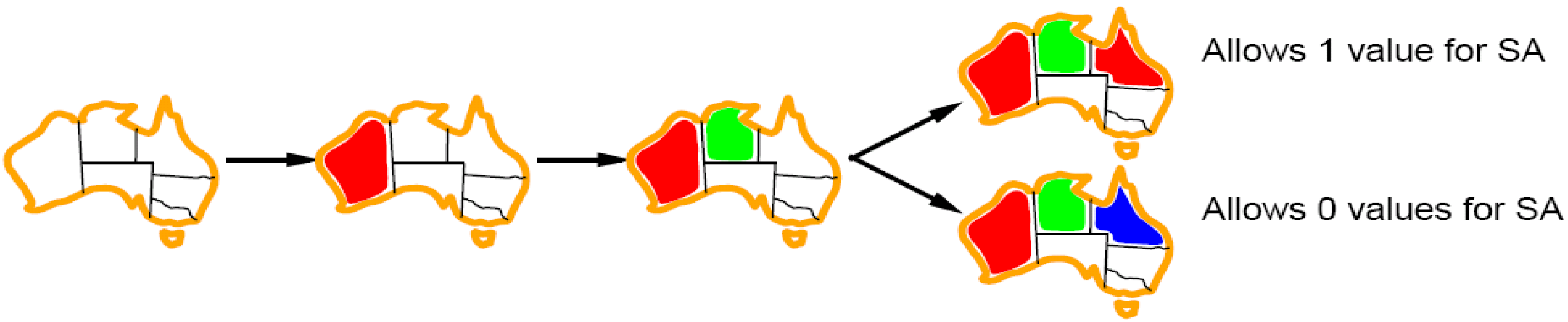
- A.k.a. most constrained variable heuristic
- *Rule*: choose variable with the fewest legal moves
- *Which variable shall we try first?*

Degree heuristic



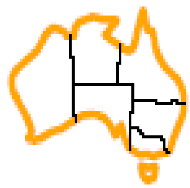
- Use degree heuristic
- *Rule*: select variable that is involved in the largest number of constraints on other unassigned variables.
- Degree heuristic is very useful as a tie breaker.
- *In what order should its values be tried?*

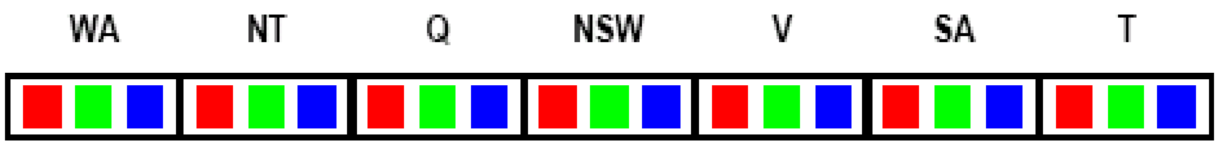
Least constraining value



- Least constraining value heuristic
- Rule: given a variable choose the least constraing value i.e. the one that leaves the maximum flexibility for subsequent variable assignments.

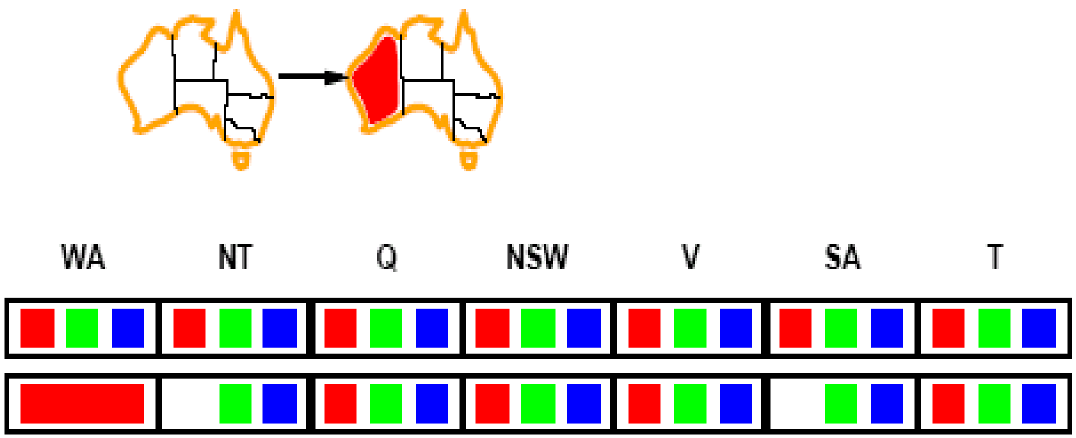
Forward checking





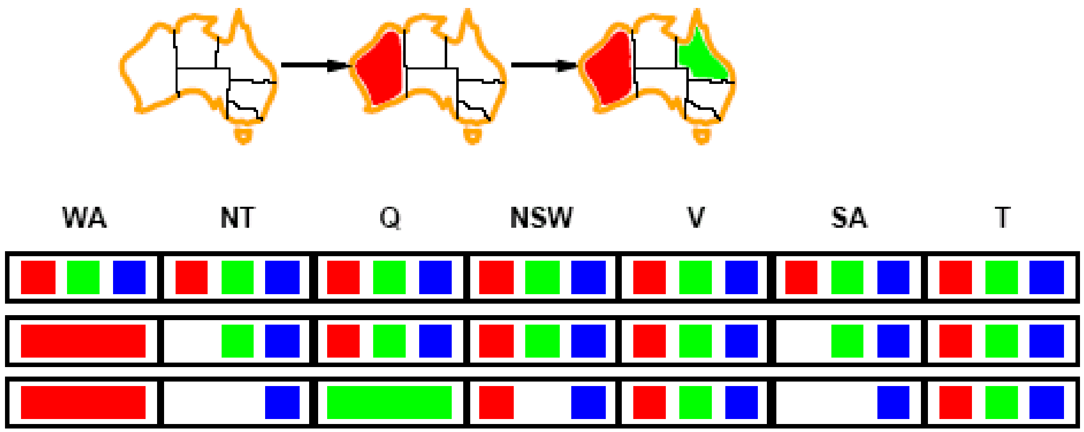
- Can we detect inevitable failure early?
  - *And avoid it later?*
- *Forward checking idea:* keep track of remaining legal values for unassigned variables.
- Terminate search when any variable has no legal values.

Forward checking



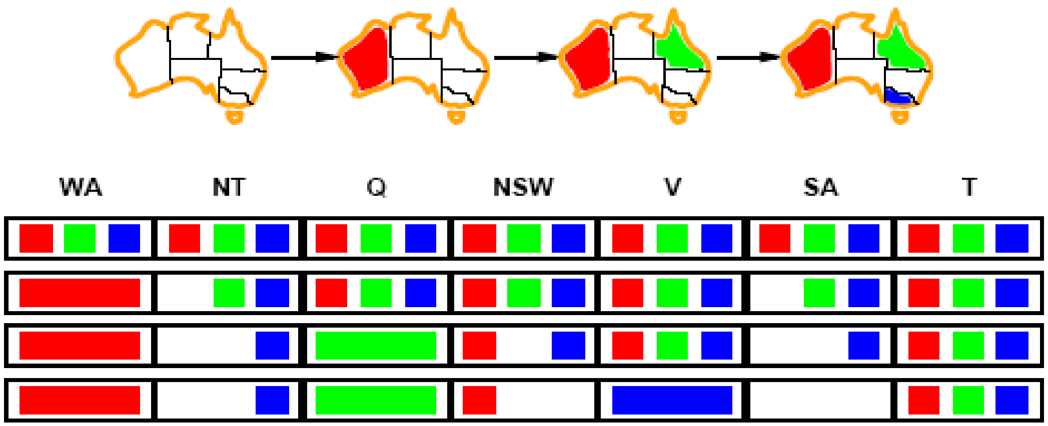
- Assign *{WA=red}*
- Effects on other variables connected by constraints with WA
  - *NT can no longer be red*
  - *SA can no longer be red*

Forward checking



- Assign *{Q=green}*
- Effects on other variables connected by constraints with WA
  - *NT can no longer be green*
  - *NSW can no longer be green*
  - *SA can no longer be green*
- *MRV heuristic* will automatically select NT and SA next, why?

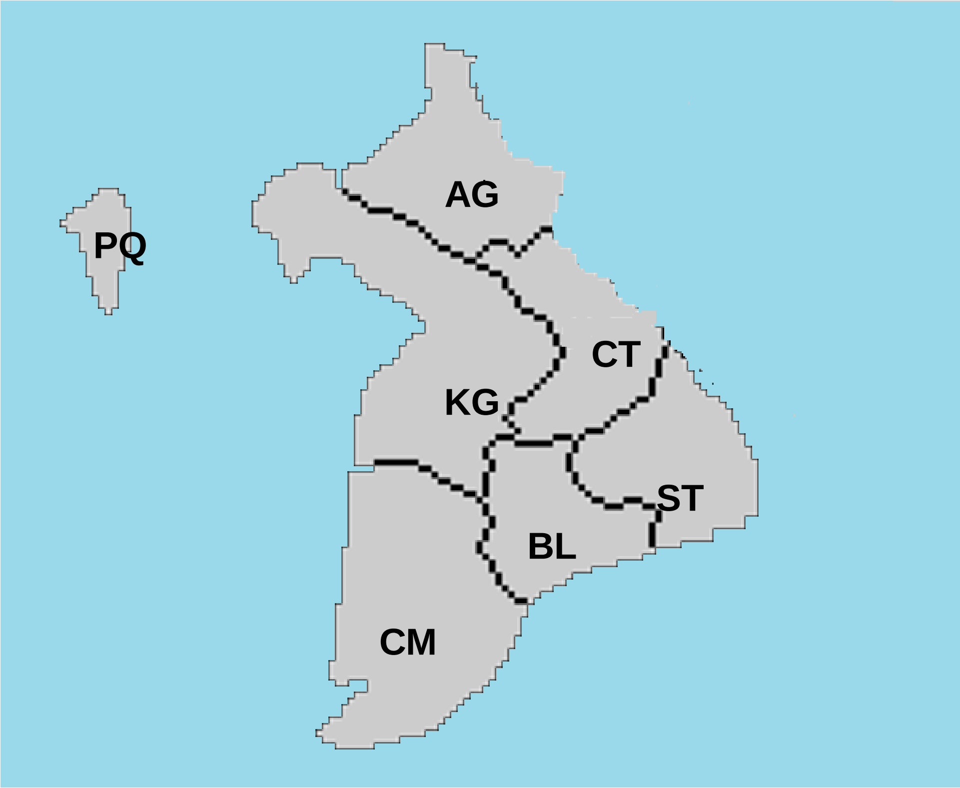
Forward checking



- If *V* is assigned *blue*
- Effects on other variables connected by constraints with WA
  - *SA is empty*
  - *NSW can no longer be blue*
- FC has detected that partial assignment is *inconsistent* with the constraints and backtracking can occur.

Map coloring

1. PQ: Phú Quốc	2. KG: Kiên Giang	3. CM: Cà Mâu
4. BL: Bạc Liêu	5. ST: Sóc Trăng	6. CT: Cần Thơ
7. AG: An Giang		



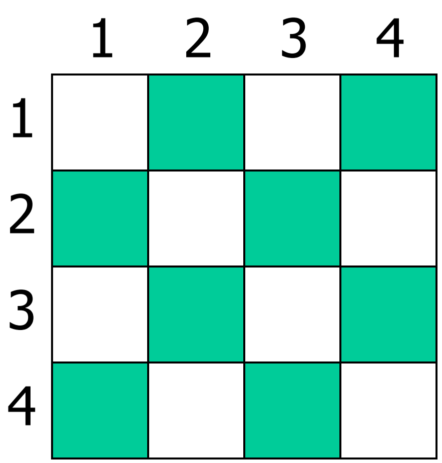
Map coloring

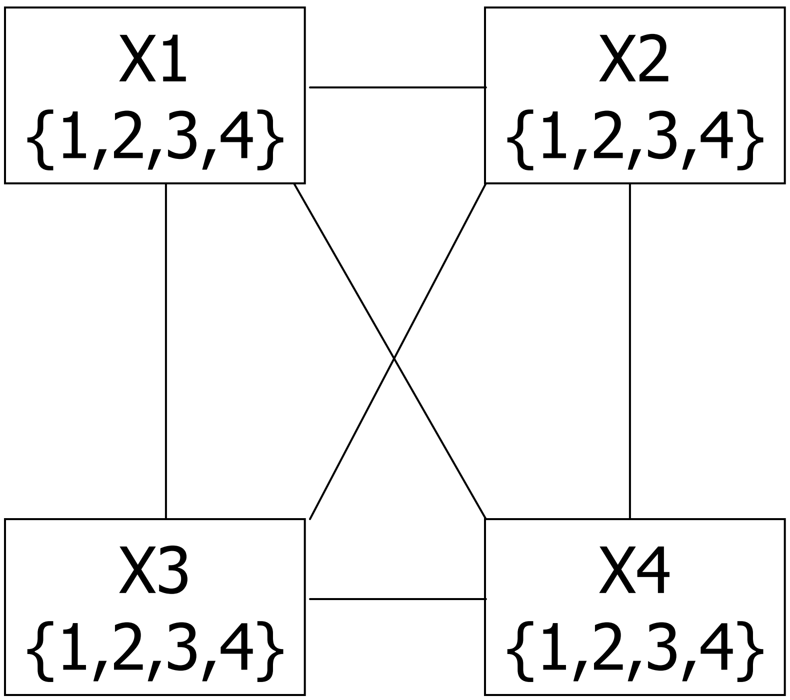
Step		PQ	KG	CM	BL	ST	CT	AG
0		RGB	RGB	RGB	RGB	RGB	RGB	RGB
1	MRV	= R						
2	FC	= R	RGB	RGB	RGB	RGB	RGB	RGB
3	MRV	= R	= R					
4	FC	= R	= R	GB	GB	RGB	GB	GB
5	MRV	= R	= R	=G				
6	FC	= R	= R	=G	B	RGB	GB	GB
7	MRV	= R	= R	=G	= B			
8	FC	= R	= R	=G	= B	RG	G	GB
9	MRV	= R	= R	=G	= B		= G	

Map coloring

Step		PQ	KG	CM	BL	ST	CT	AG
8	FC	= R	= R	=G	= B	RG	G	GB
9	MRV	= R	= R	=G	= B		= G	
10	FC	= R	= R	=G	= B	R	= G	B
11	MRV	= R	= R	=G	= B	=R	= G	
12	FC	= R	= R	=G	= B	=R	= G	B
13	MRV	= R	= R	=G	= B	=R	= G	=B

Example: 4-Queens Problem





[4-Queens slides copied from B.J. Dorr CMSC 421 course on AI]

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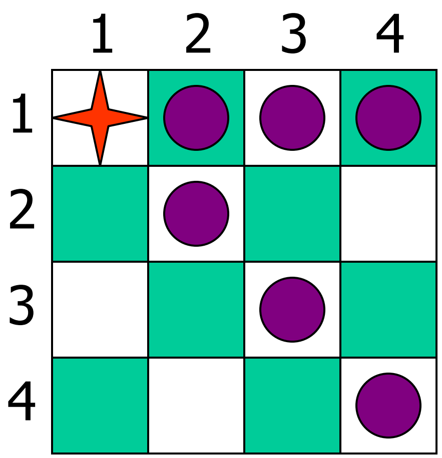

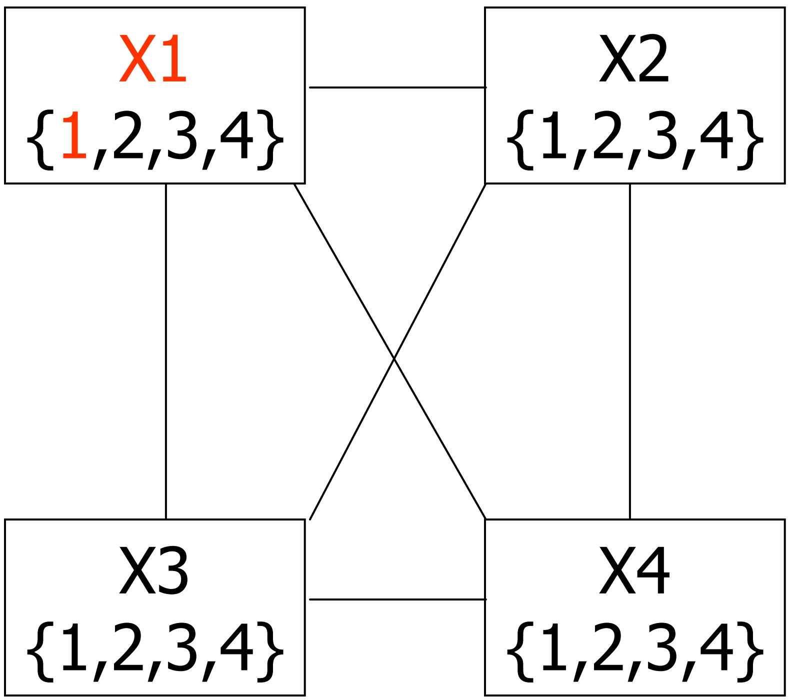

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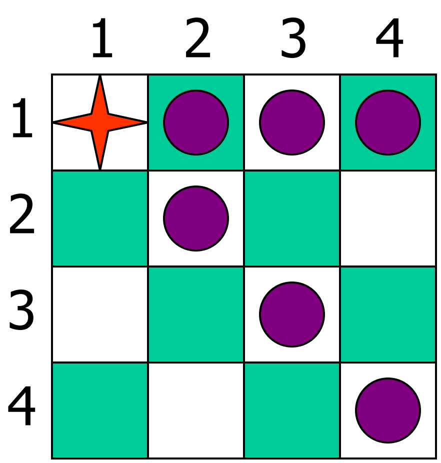
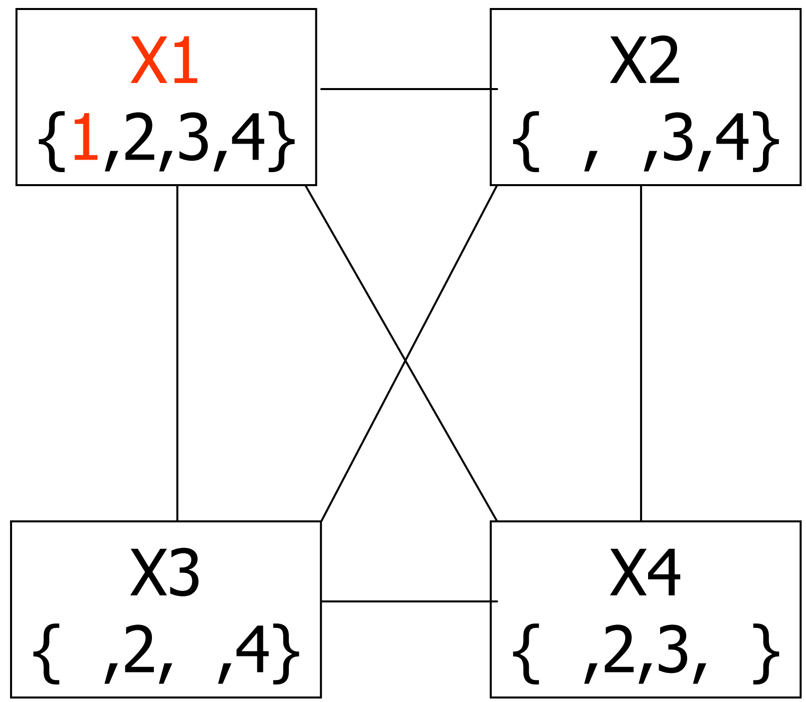
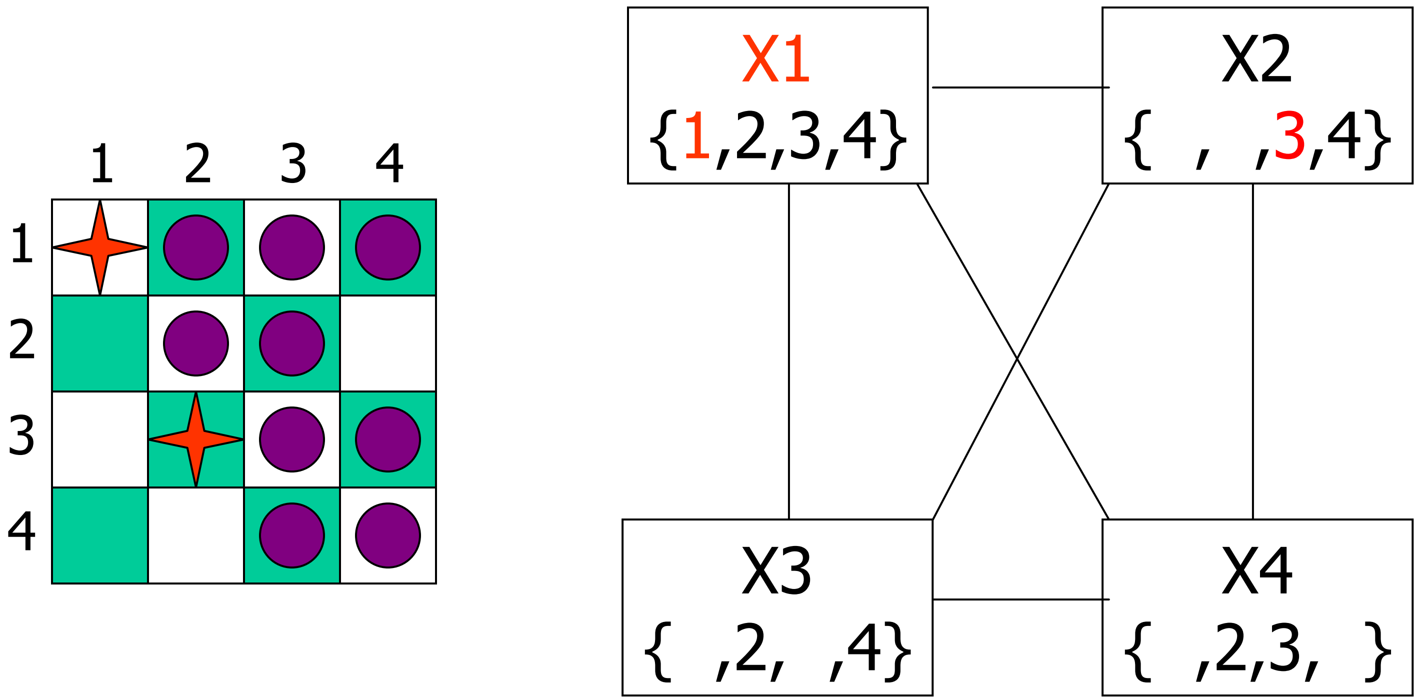




FW Checking → Remove

Example: 4-Queens Problem



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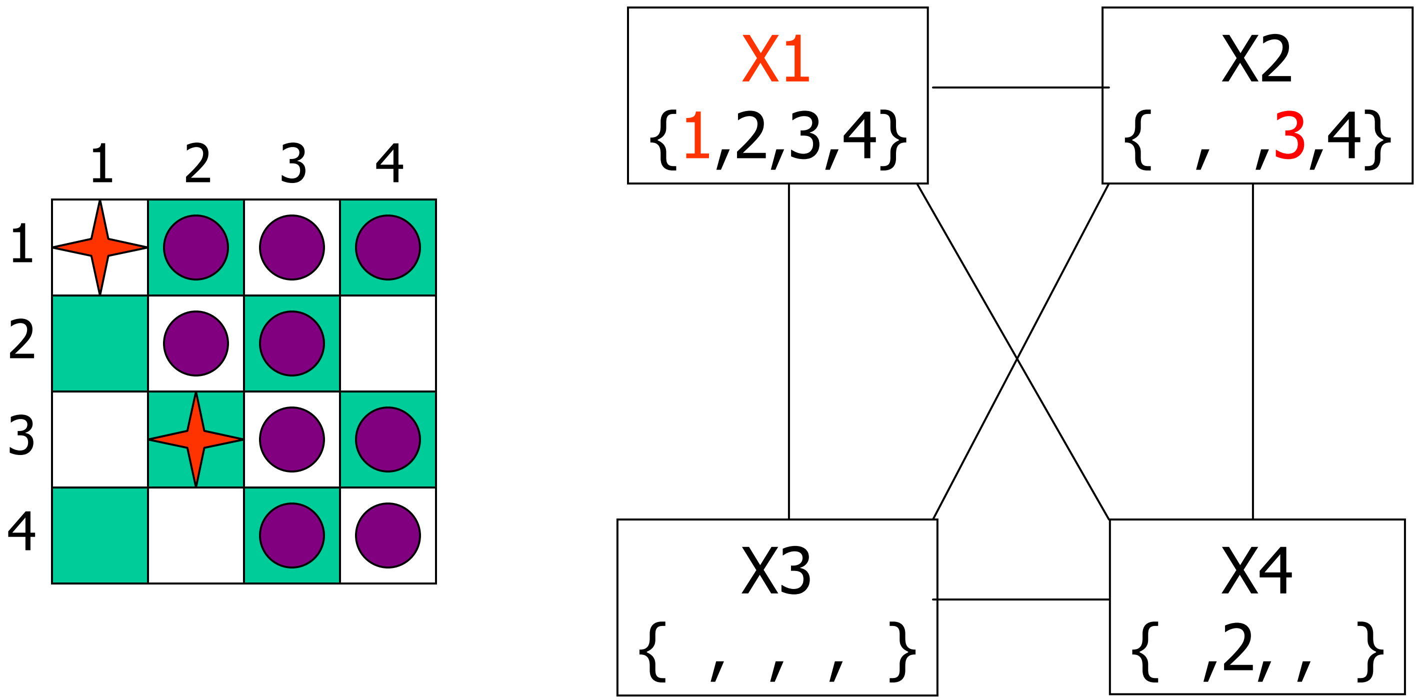

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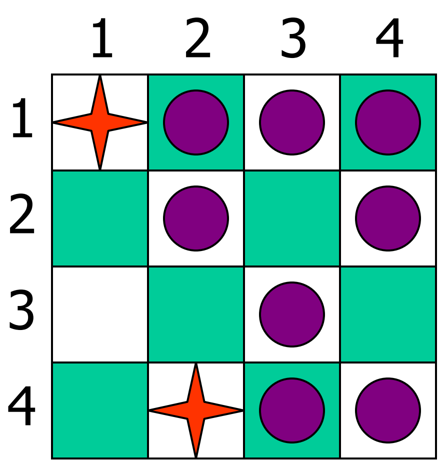

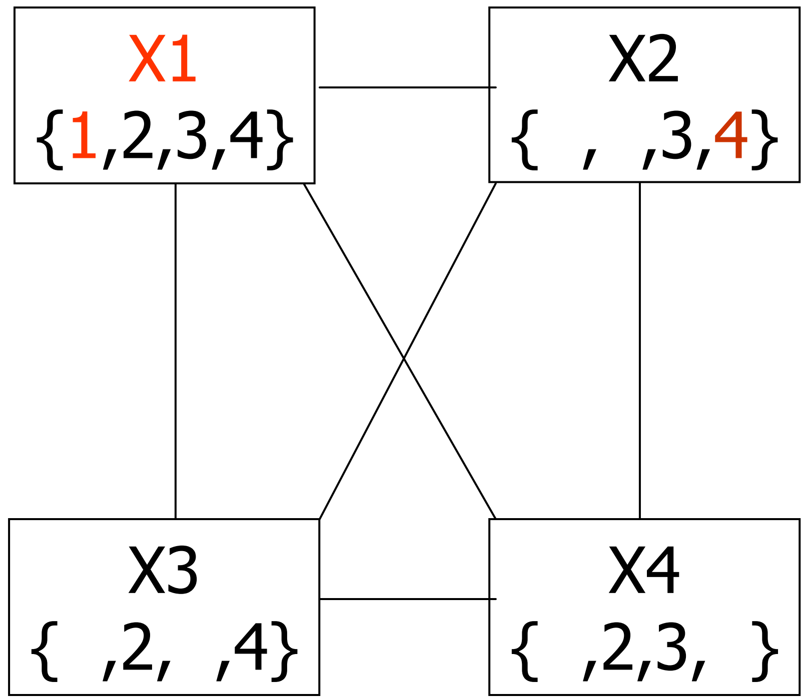

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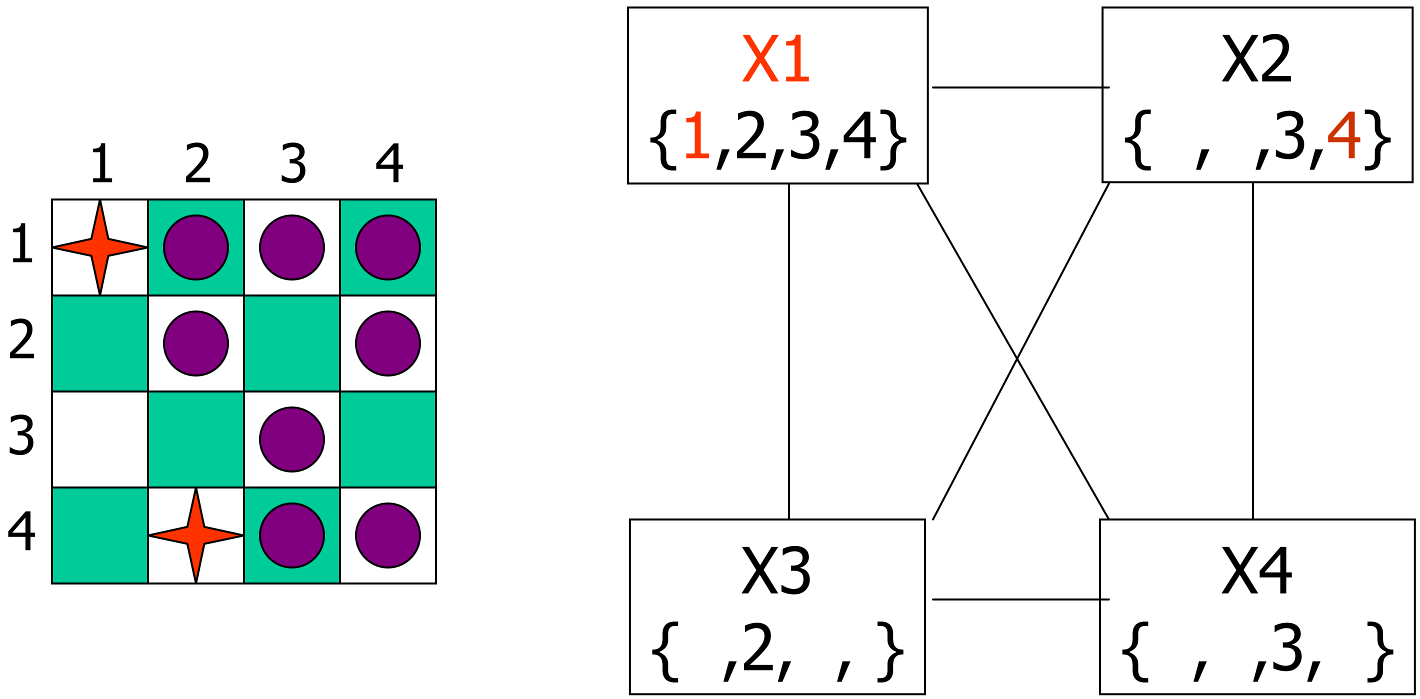

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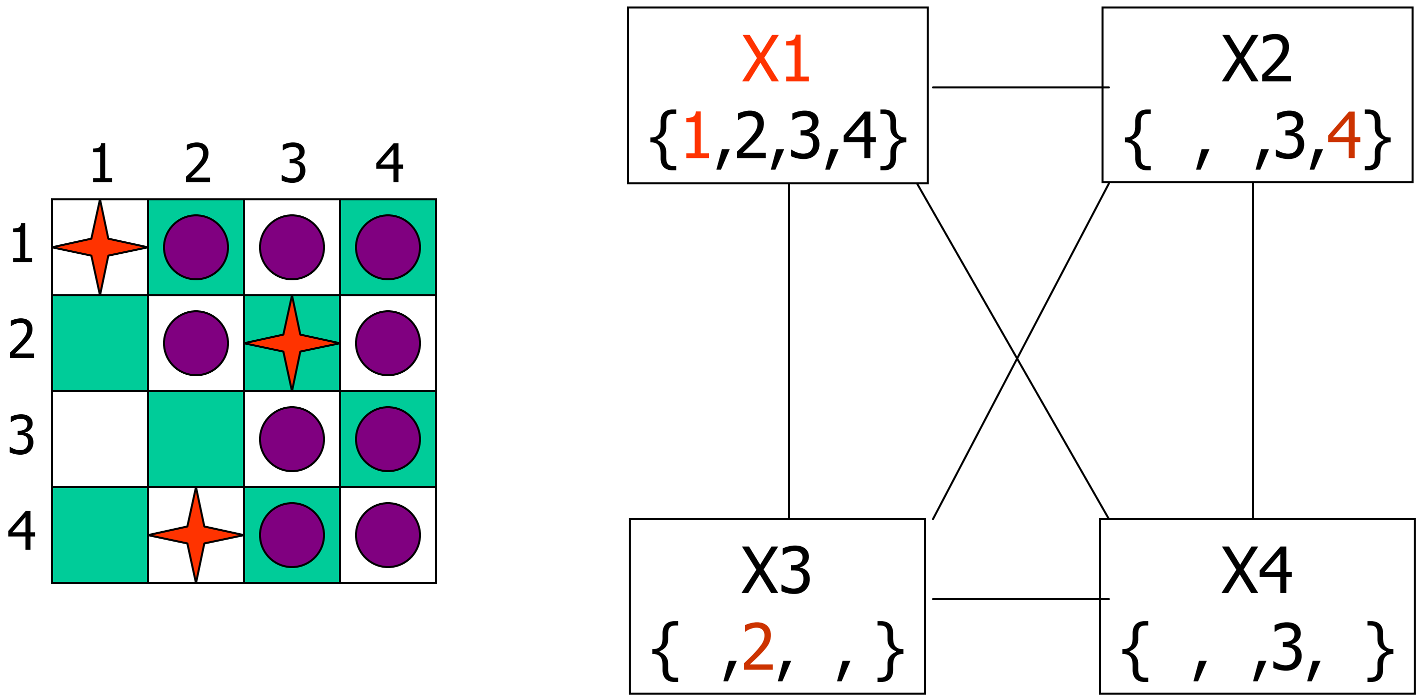

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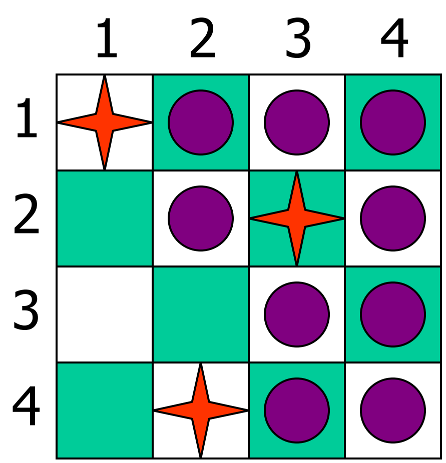

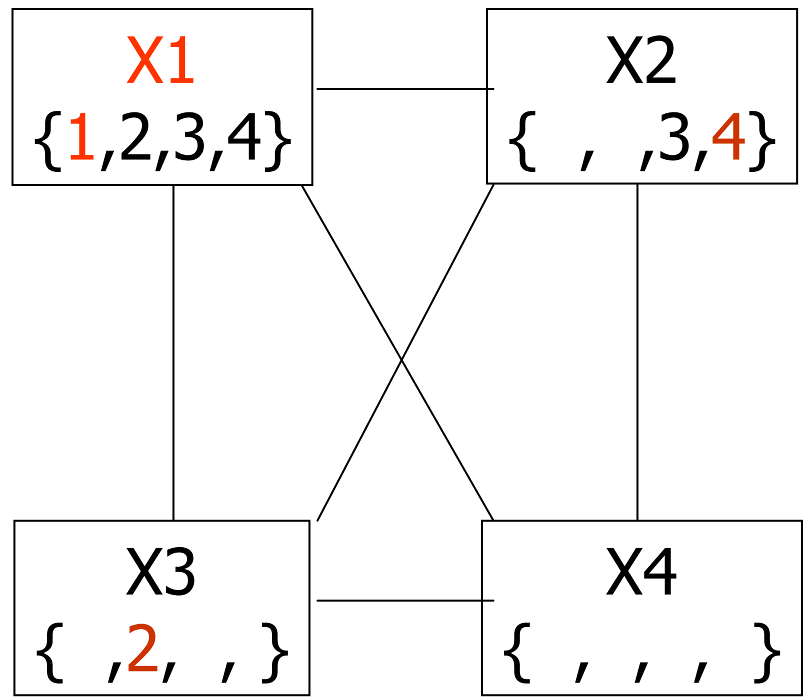

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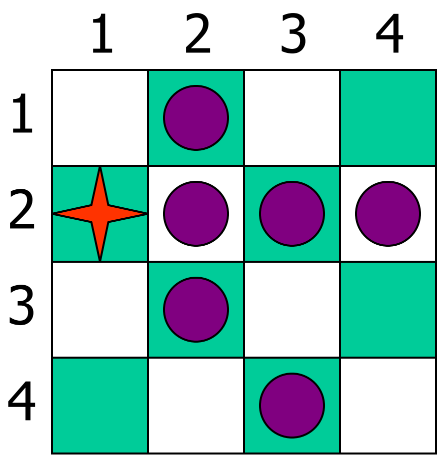
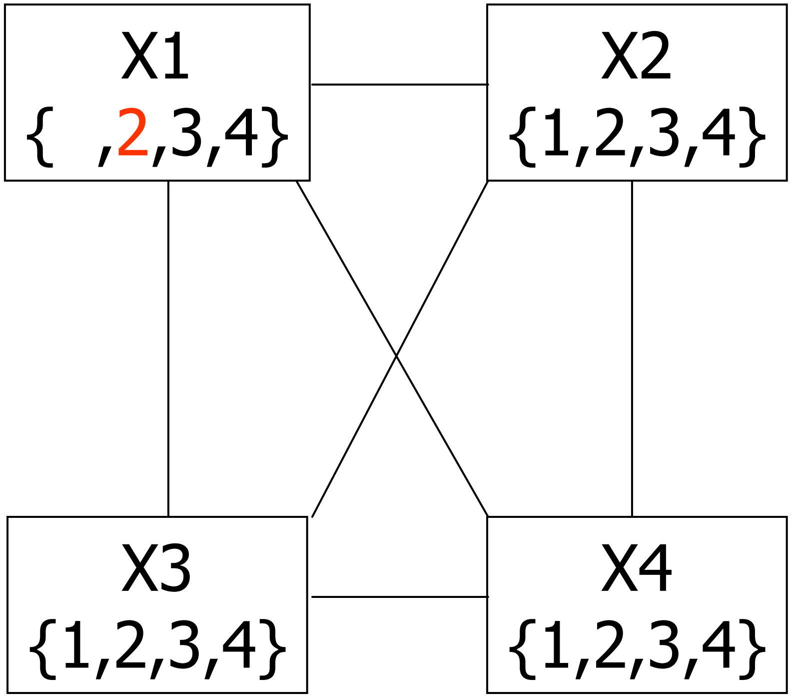
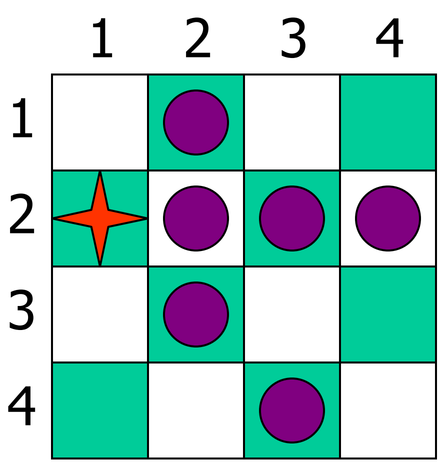
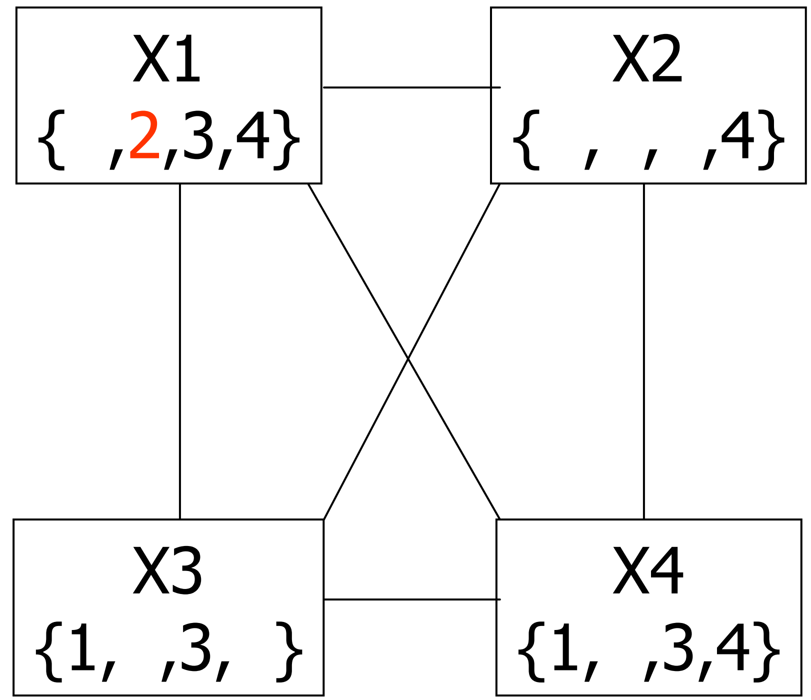




Do backtracking + Try another assignment for X1

Example: 4-Queens Problem





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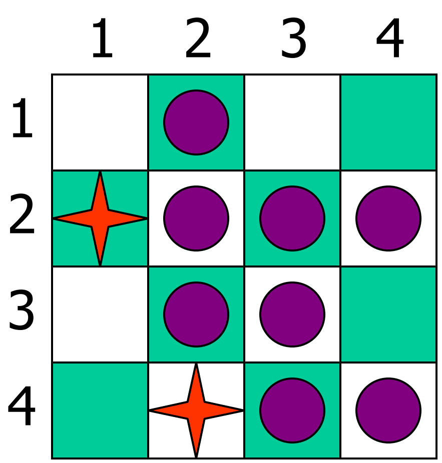

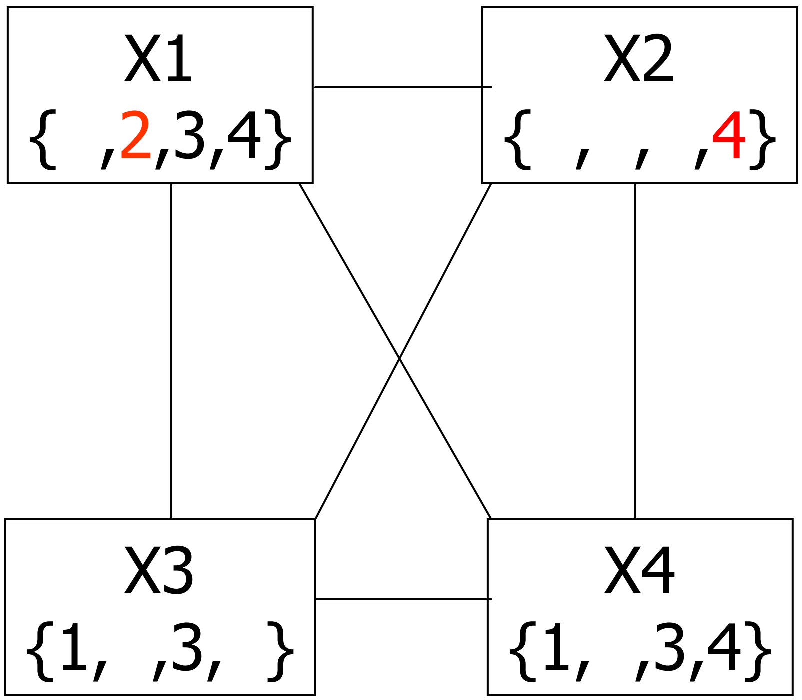

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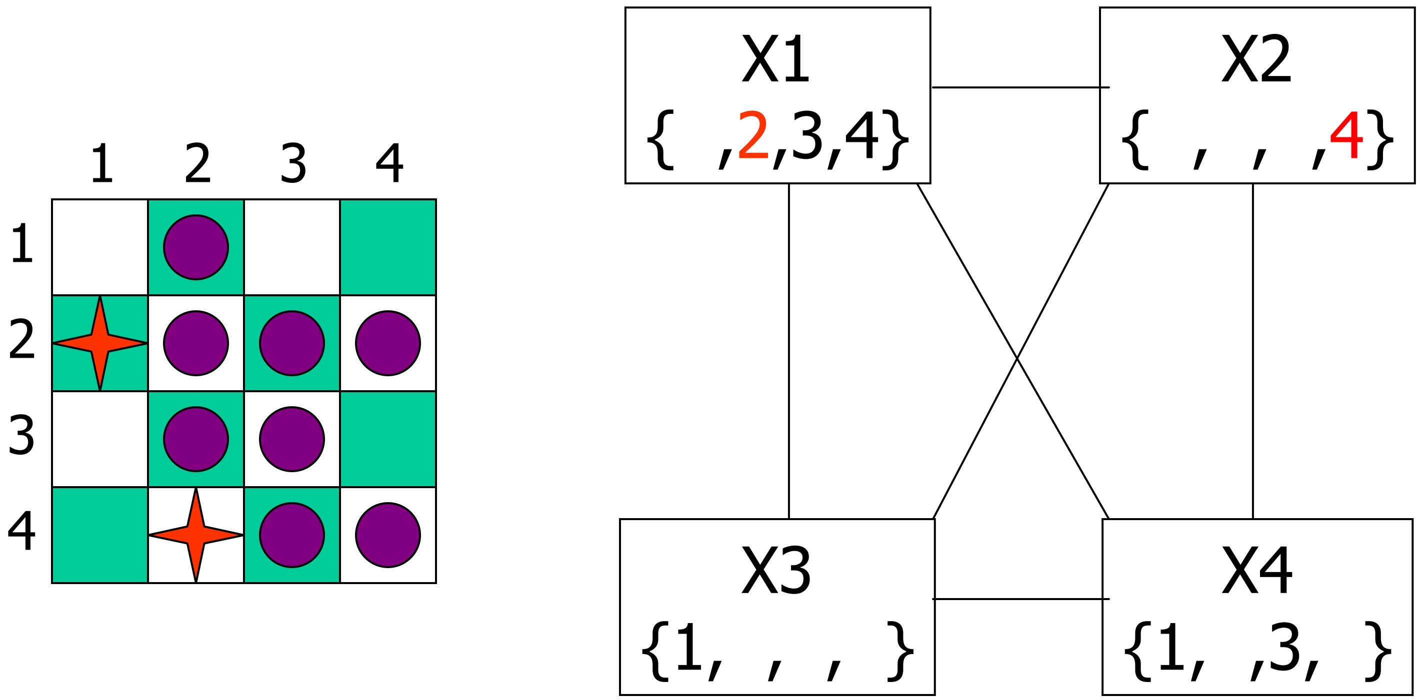

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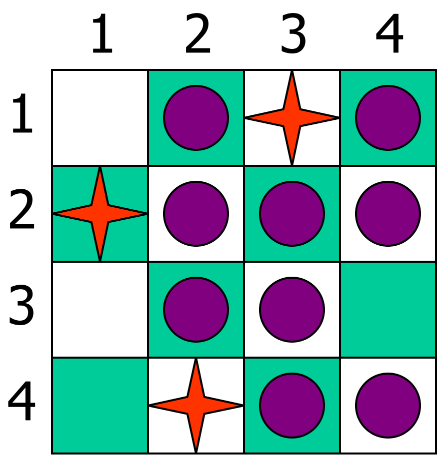

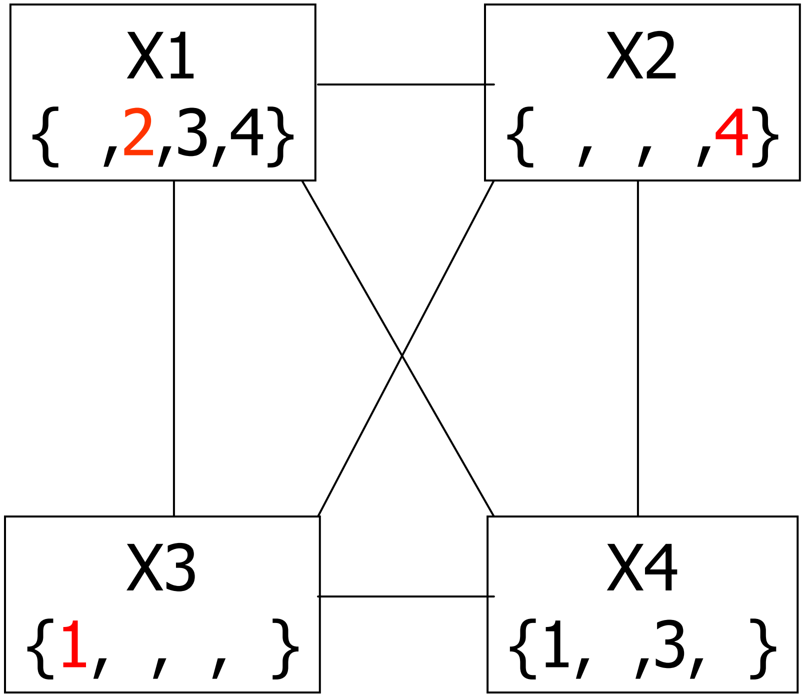

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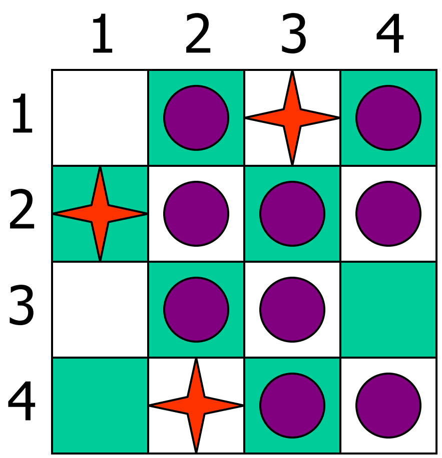

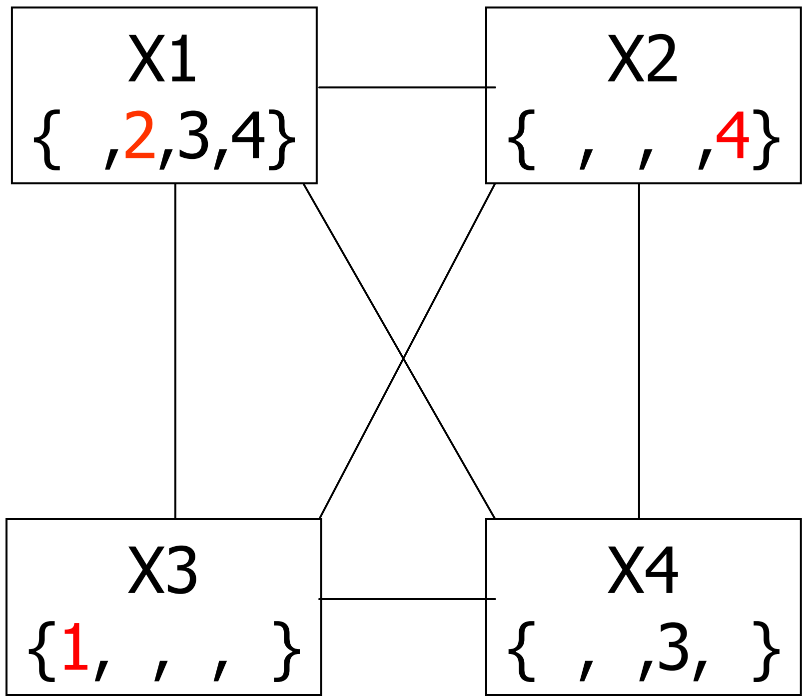

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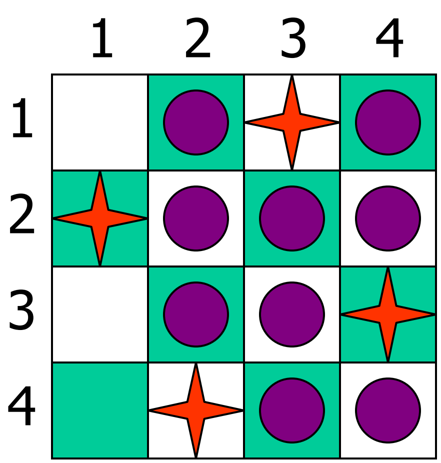
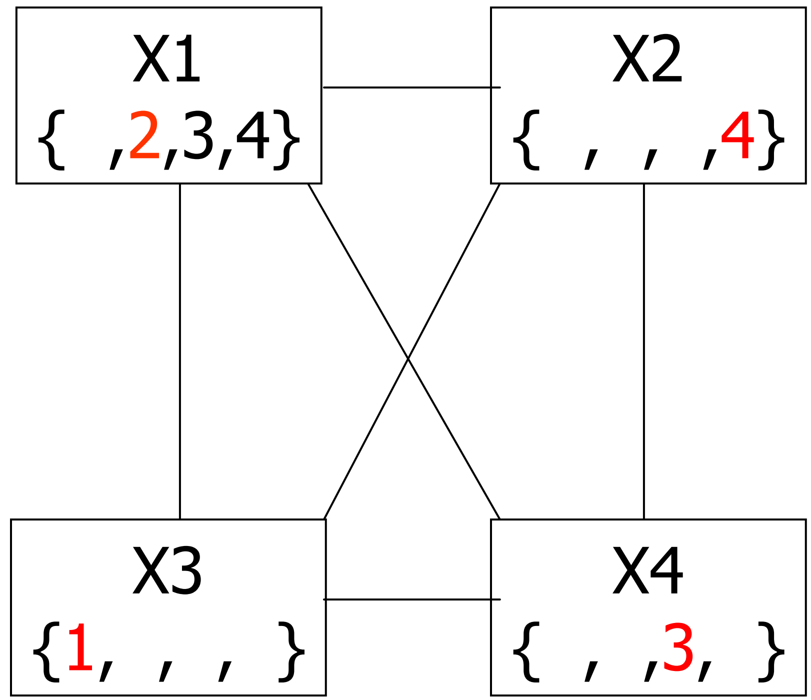
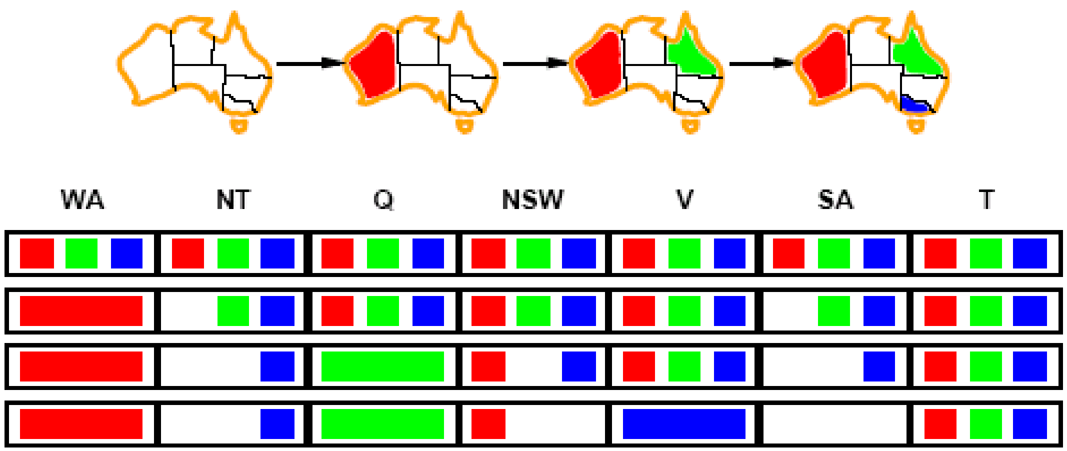


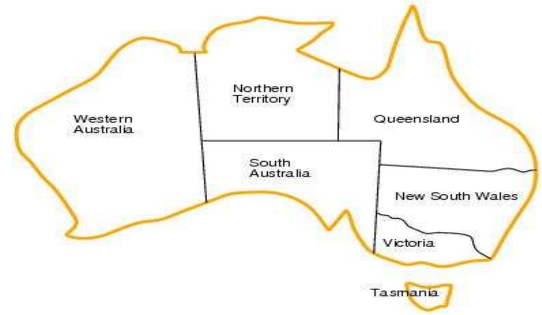


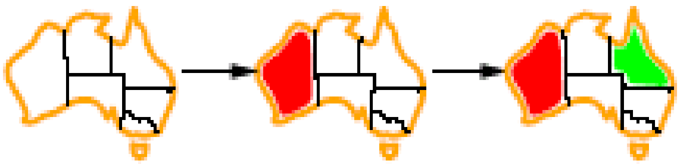
Constraint propagation

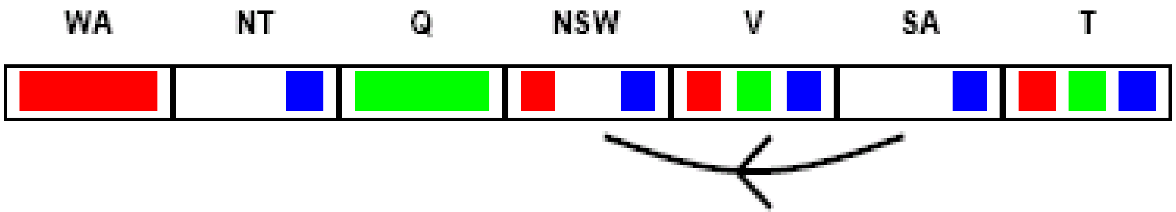


- Solving CSPs with combination of heuristics plus forward checking is more efficient than either approach alone.
- FC checking propagates information from assigned to unassigned variables but does not provide detection for all failures.
  - NT and SA cannot be blue!
- Constraint propagation repeatedly enforces constraints locally

Arc consistency





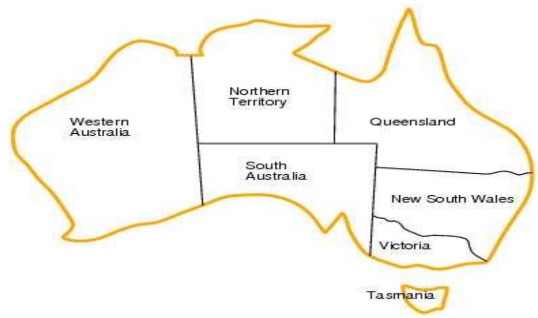


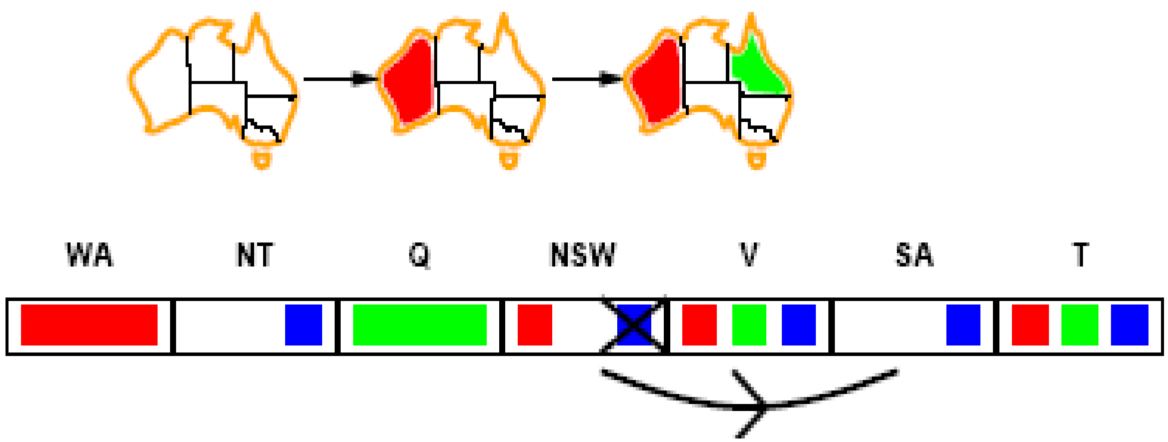
- $X \rightarrow Y$ is consistent iff

  for *every* value x of X there is some allowed y
- $SA \rightarrow NSW$ is consistent iff

  $SA = blue$ and $NSW = red$

Arc consistency





- $X \rightarrow Y$ is consistent iff

  for *every* value x of X there is some allowed y

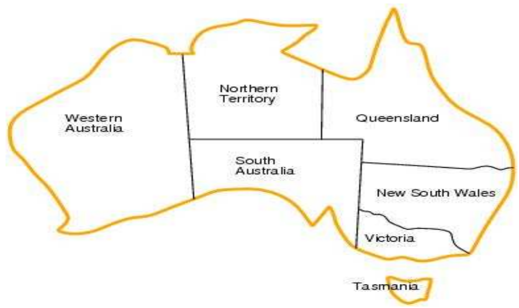
- $NSW \rightarrow SA$ is consistent iff

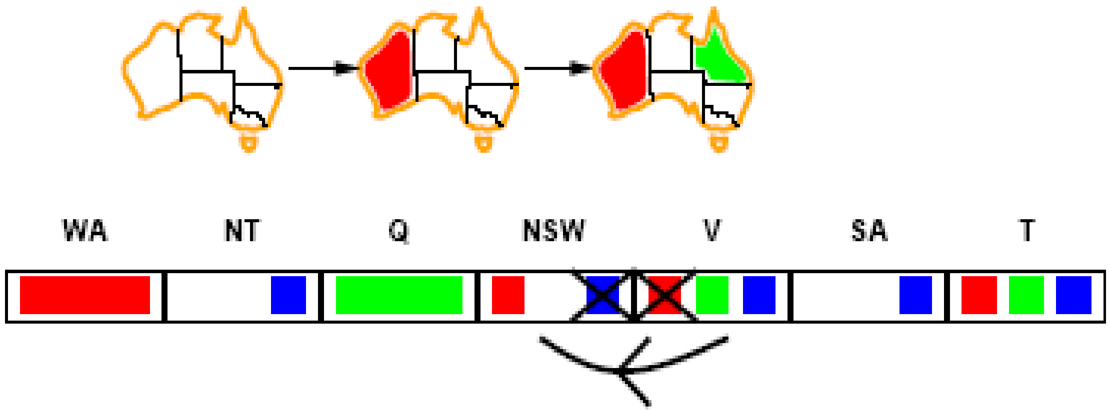
  $NSW = red$ and $SA = blue$

  $NSW = blue$ and $SA = ???$

Arc can be made consistent by removing *blue* from *NSW*

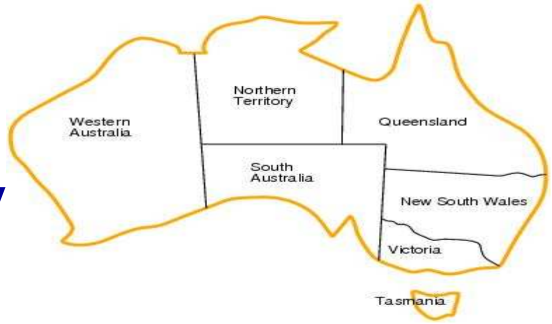
Arc consistency

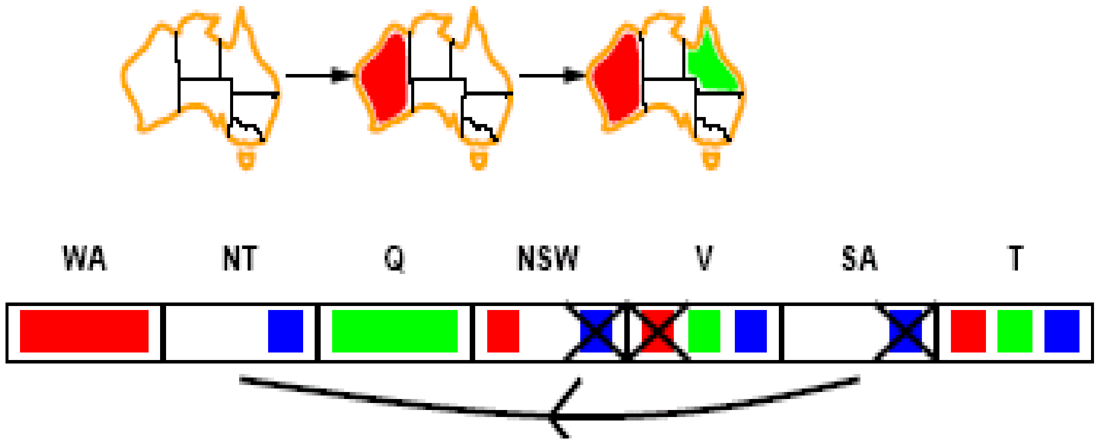




- Arc can be made consistent by removing *blue* from *NSW*
- RECHECK neighbours !!
  - Remove red from *V*

Arc consistency





- Arc can be made consistent by removing *blue* from *NSW*
- RECHECK neighbours !!
  - Remove red from *V*
- Arc consistency detects failure earlier than FC
- Can be run as a preprocessor or after each assignment.
  - Repeated until no inconsistency remains

Arc consistency algorithm

```
function AC-3(csp) return the CSP, possibly with reduced domains
    inputs: csp, a binary csp with variables {X1, X2, …, Xn}
    local variables: queue, a queue of arcs initially the arcs in csp

    while queue is not empty do
        (Xi, Xj) ← REMOVE-FIRST(queue)
        if REMOVE-INCONSISTENT-VALUES(Xi, Xj)  then
                for each Xk in NEIGHBORS[Xi ] – {Xj} do
                add (Xk, Xi) to queue

function REMOVE-INCONSISTENT-VALUES(Xi, Xj) return true iff we remove a value
    removed ← false
    for each x in DOMAIN[Xi] do
        if no value y in DOMAIN[Xj] allows (x,y) to satisfy the constraints between Xi and Xj
        then delete x from DOMAIN[Xi]; removed ← true
    return removed
```

K-consistency

- Arc consistency does not detect all inconsistencies:
  - Partial assignment *{WA=red, NSW=red}* is inconsistent.
- Stronger forms of propagation can be defined using the notion of k-consistency.
- A CSP is k-consistent if for any set of k-1 variables and for any consistent assignment to those variables, a consistent value can always be assigned to any kth variable.
  - E.g. 1-consistency or node-consistency
  - E.g. 2-consistency or arc-consistency
  - E.g. 3-consistency or path-consistency

K-consistency

- A graph is strongly k-consistent if
  - It is k-consistent and
  - Is also (k-1) consistent, (k-2) consistent, … all the way down to 1-consistent.
- This is ideal since a solution can be found in time $O(nd)$ instead of $O(n^2d^3)$
- YET *no free lunch*: any algorithm for establishing n-consistency must take time exponential in n, in the worst case.

Further improvements

- Checking special constraints
  - Checking Alldif(…) constraint
    - *E.g. {WA=red, NSW=red}*
  - Checking Atmost(…) constraint
    - *Bounds propagation for larger value domains*
- Intelligent backtracking
  - Standard form is chronological backtracking i.e. try different value for preceding variable.
  - More intelligent, backtrack to conflict set.
    - Set of variables that caused the failure or set of previously assigned variables that are connected to X by constraints.
    - Backjumping moves back to most recent element of the conflict set.
    - Forward checking can be used to determine conflict set.

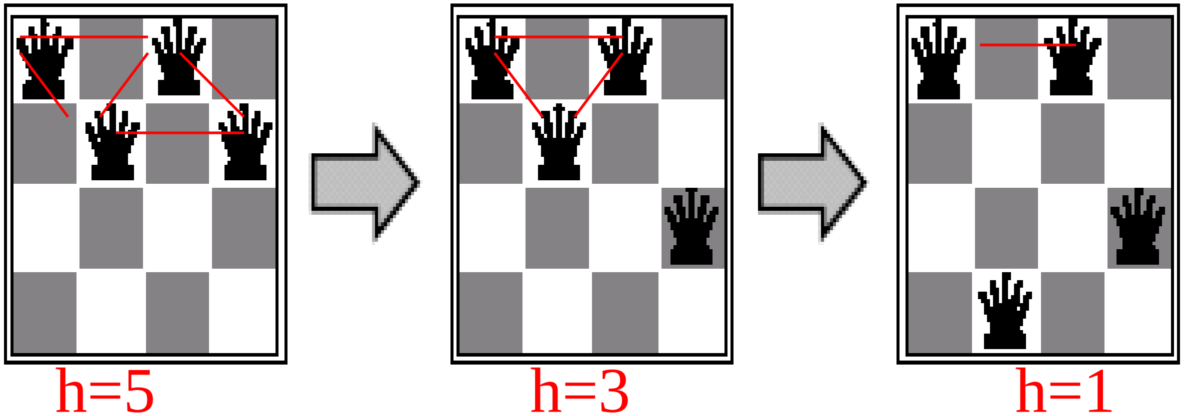
Local search for CSP

- Use complete-state representation
- For CSPs
  - allow states with unsatisfied constraints
  - operators **reassign** variable values
- Variable selection: randomly select any conflicted variable
- Value selection: *min-conflicts heuristic*
  - Select new value that results in a minimum number of conflicts with the other variables

Local search for CSP

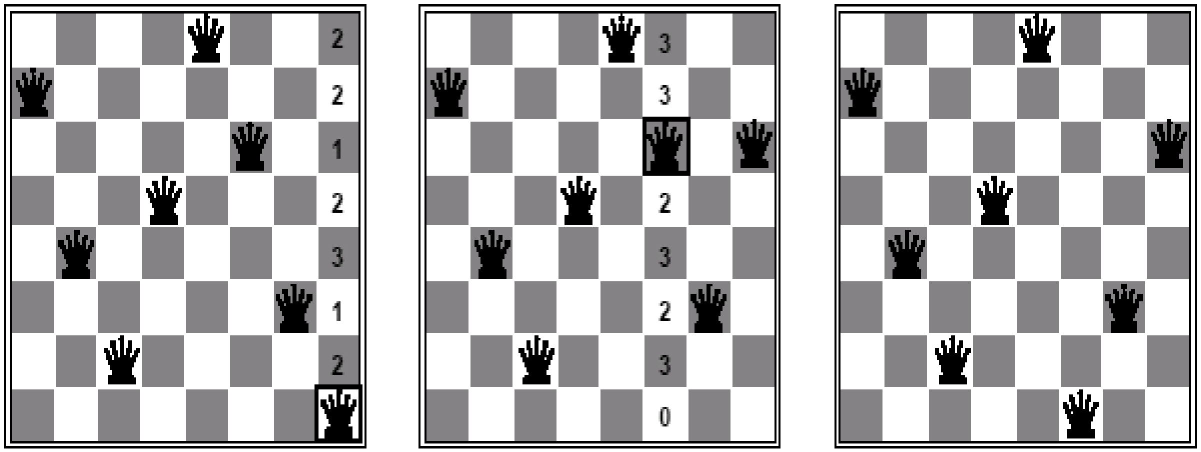
```
function MIN-CONFLICTS(csp, max_steps) return solution or failure
    inputs: csp, a constraint satisfaction problem
            max_steps, the number of steps allowed before giving up

    current ← an initial complete assignment for csp
    for i = 1 to max_steps do
            if current is a solution for csp then return current
            var ← a randomly chosen, conflicted variable from VARIABLES[csp]
            value ← the value v for var that minimizes CONFLICTS(var,v,current,csp)
            set var = value in current
    return failure
```

Min-conflicts example 1



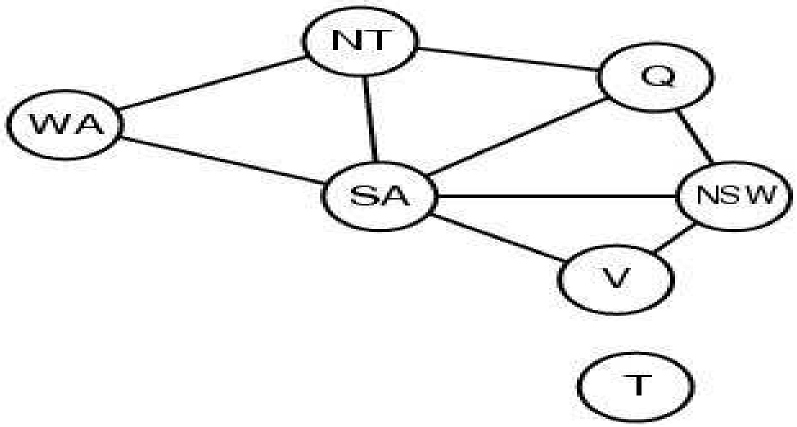
❖ Use of min-conflicts heuristic in hill-climbing.

Min-conflicts example 2



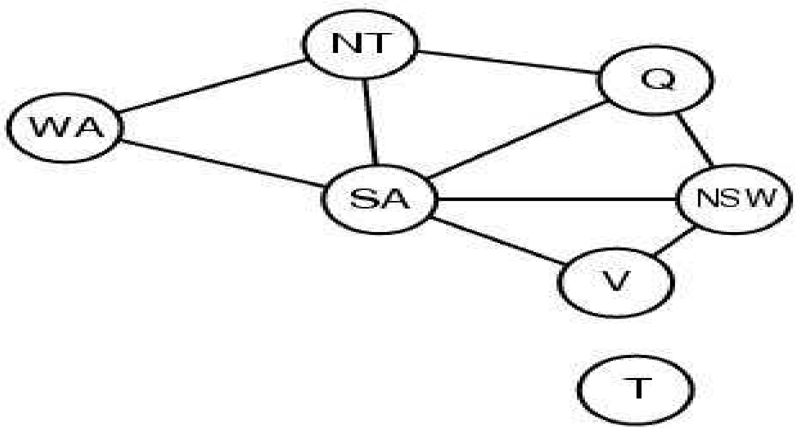
- A two-step solution for an 8-queens problem using min-conflicts heuristic.
- At each stage a queen is chosen for reassignment in its column.
- The algorithm moves the queen to the min-conflict square breaking ties randomly.

Problem structure



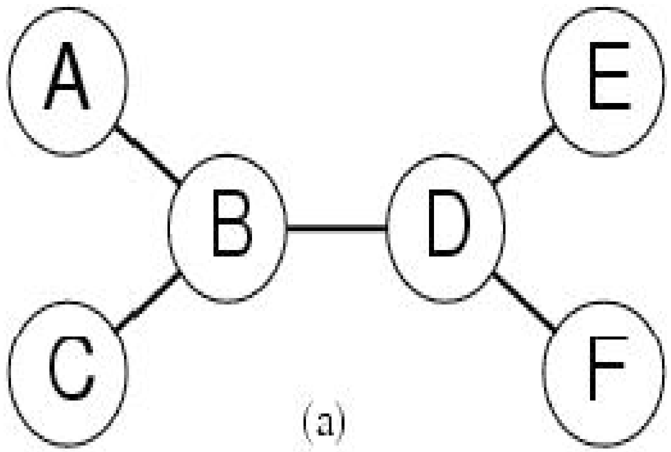
- *How can the problem structure help to find a solution quickly?*
- Subproblem identification is important:
  - Coloring Tasmania and mainland are independent subproblems
  - Identifiable as connected components of constrained graph.
- Improves performance

Problem structure



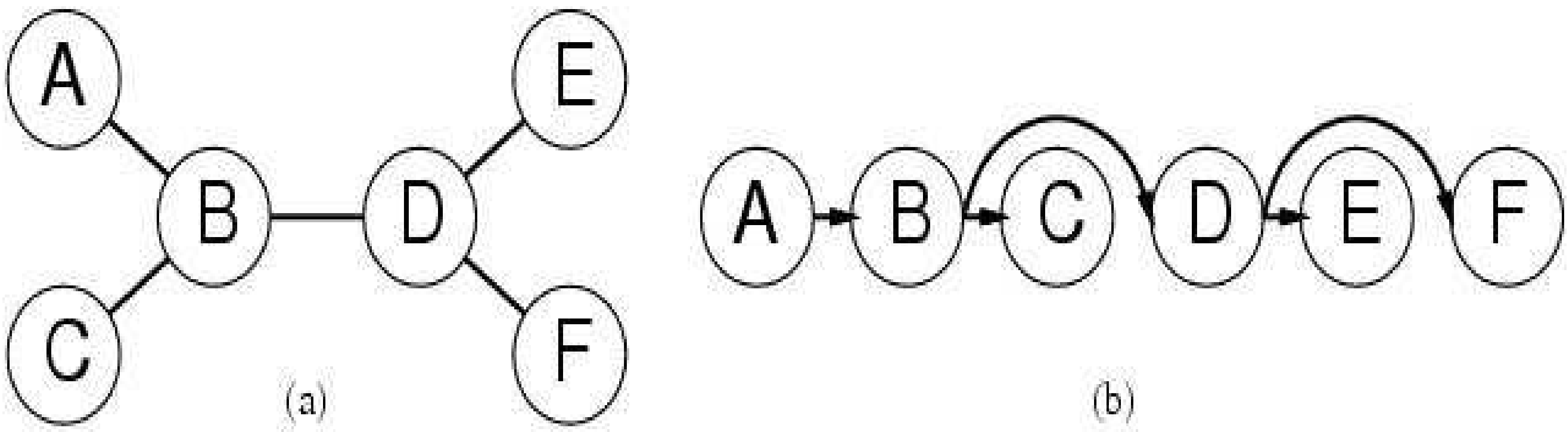
- Suppose each problem has *c* variables out of a total of *n*.
- Worst case solution cost is $O(n/c\ d^c)$, i.e. linear in *n*
  - Instead of $O(d^n)$, exponential in *n*
- E.g. $n= 80$, $c= 20$, $d=2$
  - 2^{80} = 4 billion years at 1 million nodes/sec.
  - $4 * 2^{20}$= .4 second at 1 million nodes/sec

Tree-structured CSPs



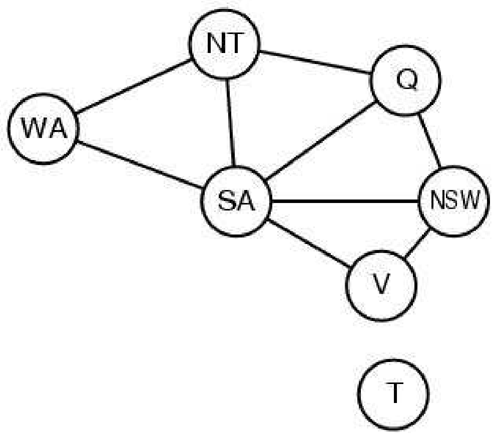
- Theorem: if the constraint graph has no loops then CSP can be solved in $O(nd^2)$ time
- Compare difference with general CSP, where worst case is $O(d^n)$

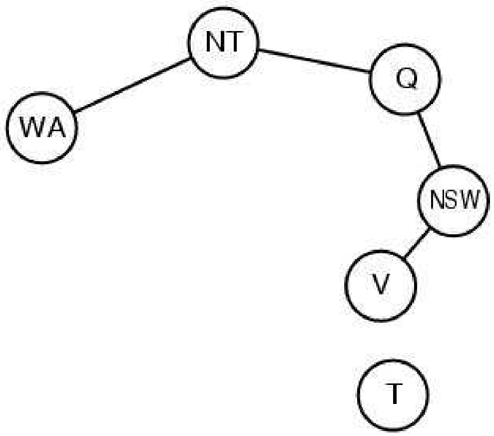
Tree-structured CSPs



- In most cases subproblems of a CSP are connected as a tree
- Any tree-structured CSP can be solved in time linear in the number of variables.
  - Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering.
  - For j from n down to 2, apply REMOVE-INCONSISTENT-VALUES(Parent(X_j),X_j)
  - For j from 1 to n assign X_j consistently with Parent(X_j)

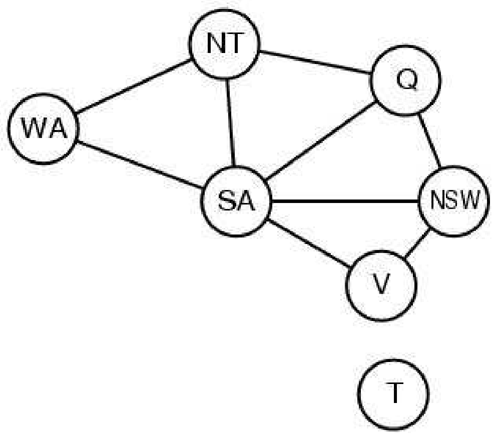
Nearly tree-structured CSPs

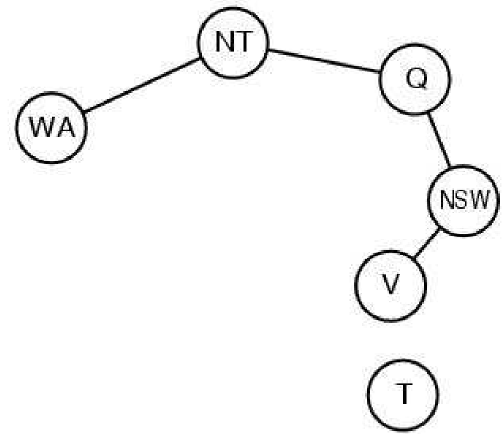




- *Can more general constraint graphs be reduced to trees?*
- Two approaches:
  - Remove certain nodes
  - Collapse certain nodes

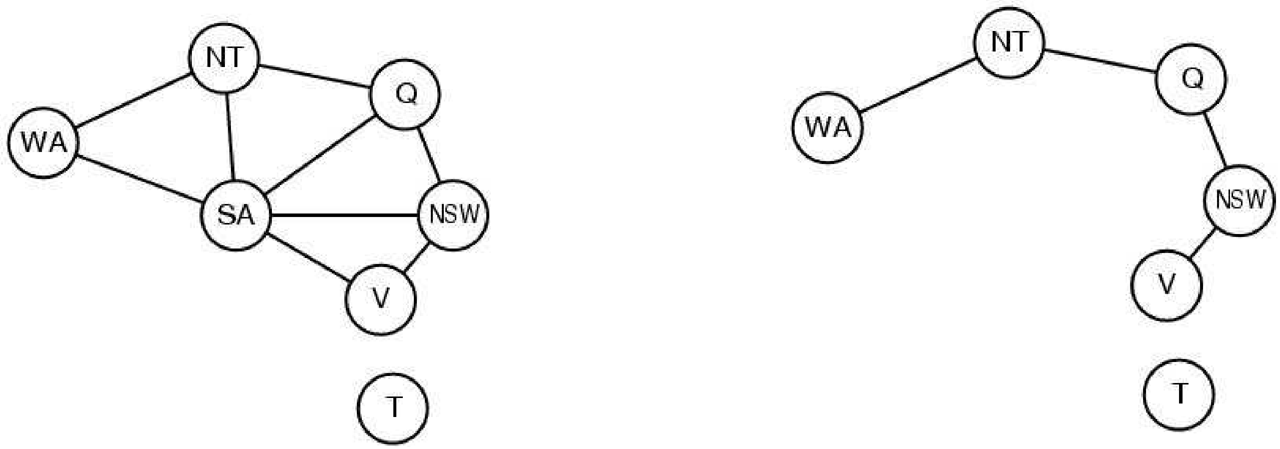
Nearly tree-structured CSPs





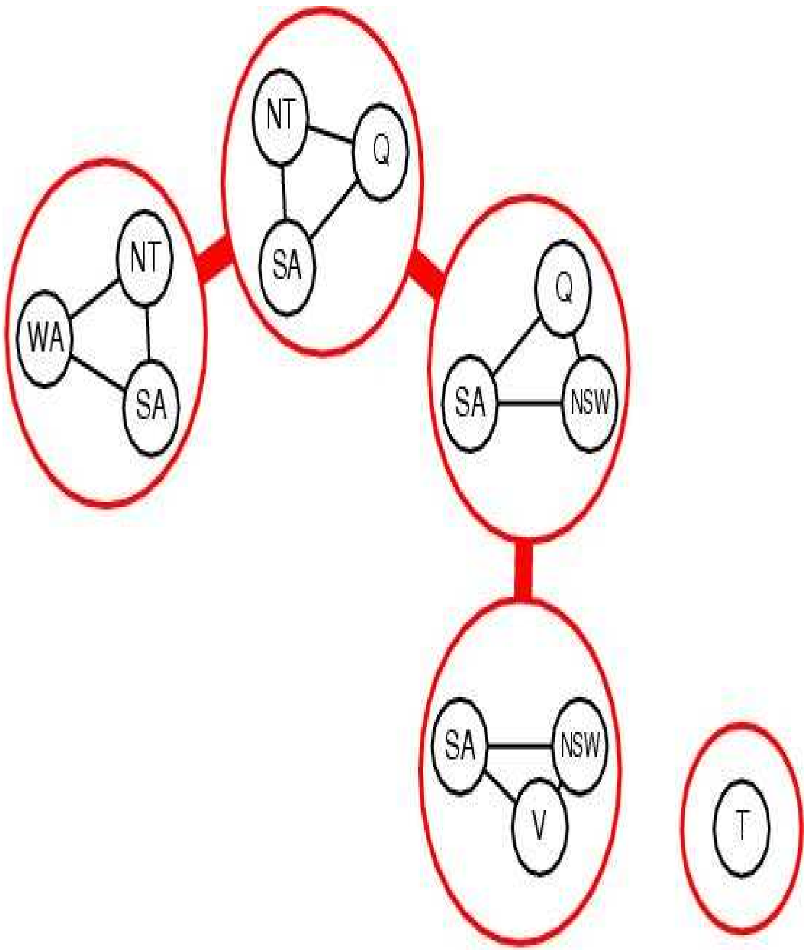
- Idea: assign values to some variables so that the remaining variables form a tree.
- Assume that we assign *{SA=x}* ← *cycle cutset*
  - And remove any values from the other variables that are inconsistent.
  - The selected value for SA could be the wrong one so we have to try all of them

Nearly tree-structured CSPs



- This approach is worthwhile if cycle cutset is small.
- Finding the smallest cycle cutset is NP-hard
  - Approximation algorithms exist
- This approach is called *cutset conditioning*.

Nearly tree-structured CSPs



- Tree decomposition of the constraint graph in a set of connected subproblems.
- Each subproblem is solved independently
- Resulting solutions are combined.
- Necessary requirements:
  - Every variable appears in ar least one of the subproblems.
  - If two variables are connected in the original problem, they must appear together in at least one subproblem.
  - If a variable appears in two subproblems, it must appear in eacht node on the path.

Summary

- CSPs are a special kind of problem: states defined by values of a fixed set of variables, goal test defined by constraints on variable values
- Backtracking=depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that lead to failure.
- Constraint propagation does additional work to constrain values and detect inconsistencies.
- The CSP representation allows analysis of problem structure.
- Tree structured CSPs can be solved in linear time.
- Iterative min-conflicts is usually effective in practice.