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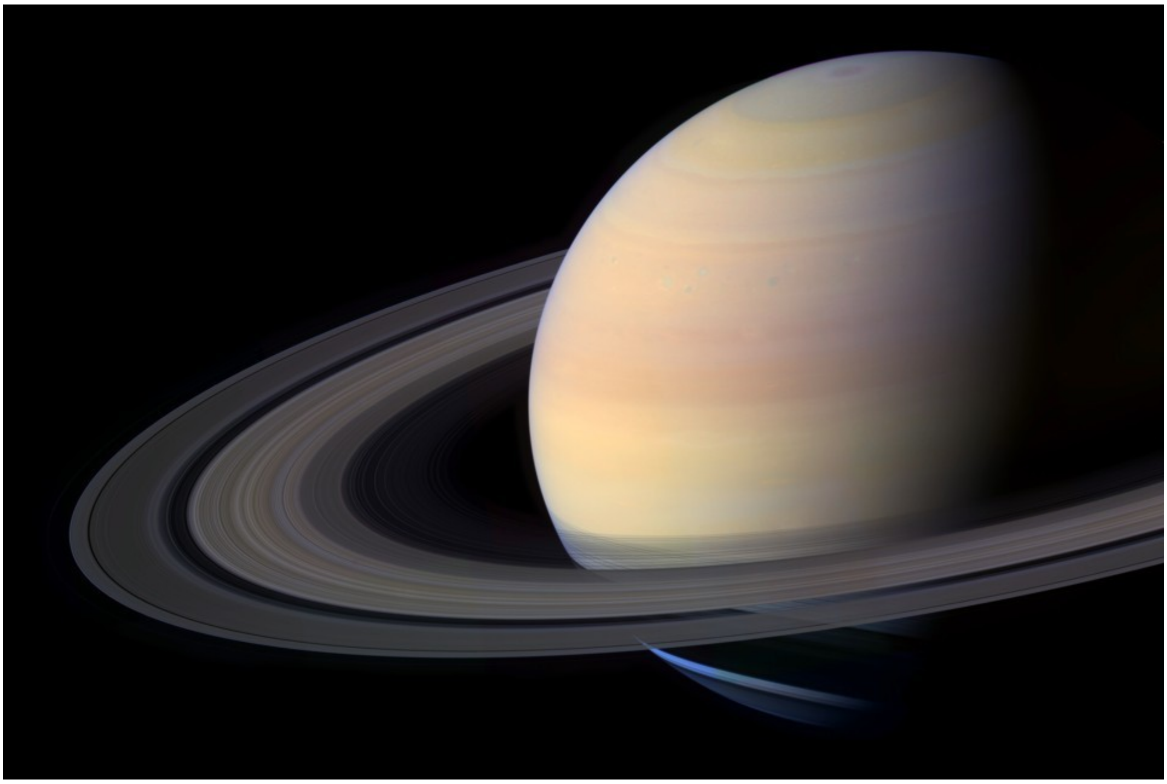
Interplanetary Orbital Transfer Proposal



AER 506
Spacecraft Dynamics and Control
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1.1 Mission Outline

A multi-purpose crew vehicle (MPCV) is in a circular orbit about the earth. The MPCV's goal is to rendezvous with the Spaceship Endurance, then enter a Cronian orbit within a time frame of 8 years from the mission start date. The energy budget available for this mission is $\Delta v_{tot} = 20\frac{km}{s}$.

Several Assumptions will be made for the orbital maneuvers performed in this assignment:

- A patched conic approximation will be used, and orbits will be treated as Keplerian.
- Orbital perturbations will be neglected.
- Thrusts are not impulsive, but rather occur instantaneously. Terrestrial Time (TT) will be used since this is the date format in JPL Horizons.
- Celestial bodies will be considered perfect spheres instead of oblate spheroids; orbital perturbations due to non-sphericity will be neglected.
- Calculations will be performed using the mass of the primary local celestial body (for example, Saturn) acting from the barycenter of the local system. The masses of local natural satellites will be considered negligible. This is to prevent the need for n-body calculations and to keep the assignment within a reasonable scope.

Numerical values for masses m_p and gravitational parameters μ, along with the symbols representing each body, are provided in the table below.

Table 1: Masses and Gravitational Parameters of Relevant Celestial Bodies

Body	$\boldsymbol{\mu}$ ($\frac{m^3}{s^2}$)	**a** (km)	**Symbol**
Sun	1.3271244×10^{20}	N/A	☉
Earth	$3.986004418 \times 10^{14}$	1.514652×10^{8}	♁
Saturn	3.7931187×10^{16}	1.427588×10^{9}	♄

1.2 Phasing Maneuver

The first maneuver that must be performed is a rendezvous with Spaceship Endurance. This requires a retrograde thrust by the MPCV in order to enter a elliptical orbit with a smaller semi-major axis than Endurance, followed by a prograde thrust of the same magnitude once the MPCV returns to its apoapsis. The initial orbital parameters of both the MPCV and Endurance are presented in Table 2.

Table 2: Initial Orbital Elements of the MPCV & Spaceship Endurance at mission start ($t = t_0$)

Spaceship	i (°)	Ω (°)	Θ (°)	ω (°)	a (km)
MPCV	30	0	30	0	6×10^4
Endurance	30	0	90	0	6×10^4

To determine the thrust vector Δv_1, we need to calculate the size of the elliptical orbit the MPCV must enter in order to dock with Endurance.

$$ToF_{circular} = 2\pi\sqrt{\frac{r^3}{\mu}}$$

$$ToF_{elliptical} = 2\pi\sqrt{\frac{a_e^3}{\mu}} = ToF_c(1 - \frac{\Delta\theta}{2\pi})$$

Here the change in true anomaly is 60°. Solving for the semi-major axis of the elliptical phasing orbit by substituting in the above expressions:

$$a_e = \sqrt[3]{\mu\Big(\frac{ToF_c}{2\pi}\Big(1 - \frac{\Delta\theta}{2\pi}\Big)\Big)^2} = \Big(\frac{5}{6}\sqrt{r^3}\Big)^{\frac{2}{3}} = 53132.93km$$

Now we can determine the thrust vector. We can use the Vis-Viva equation to determine the change in velocity at the periapsis of the elliptical orbit, and compare it to the orbital velocity at the apoapsis of the elliptical orbit (equivalent to the velocity at any point in the circular orbit).

$$\begin{aligned}
v &= \sqrt{\mu\Big(\frac{2}{r} - \frac{1}{a}\Big)} \\
|\Delta v_1| &= |v_{e_{periapsis}} - v_c| = |\sqrt{\mu\Big(\frac{2}{r_\alpha} - \frac{1}{53132.93km}\Big)} - \sqrt{\frac{\mu}{a_c}}| \\
&= |\sqrt{3.986004418 \times 10^{14}\frac{m^3}{s^2}}\left(\sqrt{\frac{2}{60000km} - \frac{1}{53132.93km}} - \sqrt{\frac{1}{60000km}}\right)| \\
\therefore |\Delta v_1| &= 172.3205\frac{m}{s}
\end{aligned}$$

So our first thrust will occur at $t_1 = 0$, and it will be a retrograde thrust with a magnitude of $172.3205\frac{m}{s}$. Our second thrust Δv_2 will be of the same magnitude in the prograde direction, and will occur after one full orbit has occurred, namely at $t_2 = 146264.138s \approx 40.62hours$. This will put the MPCV and the spaceship endeavor in the same orbit with the same true anomaly. So far we have used $0.344641\frac{km}{s}$ of our total Δv, which amounts to 1.72% of our total mission Δv.

Assuming the docking maneuver was successful, the crew of the MPCV is now on board the Endeavor. The orbit has the exact same parameters as given in Table 2, with the same position as the MPCV initially had (a true anomaly of 30°) and an elapsed mission time of $\approx 40.62hours$.

1.3 Inclination Adjustment

To make the departure trajectory calculation less complicated, we restore the orbit to an inclination of 0 with respect to the ecliptic plane. This can be achieved by a thrust at the RAAN:

$$\begin{aligned}
\Delta v_3 &= 2v_c \sin\frac{i}{2} = 2\sqrt{(\frac{\mu}{a_c})}\sin 15^\circ \\
&\approx 1334.20\frac{\mathrm{m}}{\mathrm{s}} \approx 1.334\frac{\mathrm{km}}{\mathrm{s}}
\end{aligned}$$

The thrust should be applied at an angle of $\frac{\pi + i}{2} = 105°$ with respect to the velocity vector v_c. It should be applied at the RAAN, and the first RAAN that occurs after t_2 is given via:

$$\begin{aligned}
\Theta_{♐} = 30^\circ \qquad \text{and} \qquad n_{♐} &= \sqrt{\frac{\mu_{♁}}{a_c^3}} = 4.29577 \times 10^-5\frac{\text{rad}}{s} \\
\therefore t_3 = t_2 + \frac{11\pi}{6n_{♐}} &= 146264.138 \text{ s} + 134075 \text{ s} = 280339 \text{ s} \\
&\approx 77.872 \text{ hours}
\end{aligned}$$

So if a thrust of 1.334 $\frac{\mathrm{km}}{\mathrm{s}}$ is applied at $t_3 = 280339\mathrm{s}$ after t_0, the combination of this thrust with the two phasing thrusts will put Endeavor in a circular orbit on the ecliptic plane, which is the x-y plane of our coordinate system. So far we have used $1.679\frac{km}{s}$ of our total Δv, which amounts to 8.4% of our total mission Δv.

1.4 Initial Orbital Positions

One option for the transfer to a Cronian orbit involves a direct Hohmann transfer. The orbital parameters of the Earth and Saturn are provided in Table 3 below. We will use the orbital elements as described in class, using the ICRF reference frame and the ecliptic and mean equinox of the J2000 epoch.

The screen captures of the JPL Horizon database output are provided in appendix A.

It is interesting to note that the longitude of periapsis $\bar{\omega} = \Omega + \omega$ for the earth only differs by 3 degrees. The longitude of periapsis is a quantity that expresses where the periapsis of the orbit would be if the inclination of the orbit was zero. This makes determining the phase angle between them easier, since the true anomaly of

Table 3: Orbital Elements from JPL Horizons using Solar System Barycenter at t_0 (Julian Day 2458094.5)

Planet	**e**	**i** (°)	**Ω** (°)	**ω** (°)	**a** (km)	**Θ** (°)
Earth	0.0221417	9.520198×10^{-3}	164.9007	291.0881	1.514652×10^{8}	338.8141
Saturn	0.0541041	2.486993	113.5977	339.0987	1.427588×10^{9}	176.8053

each planet is measured from its respective eccentricity vector.

The position of the planets at time t_0 is provided below in Figure 1.

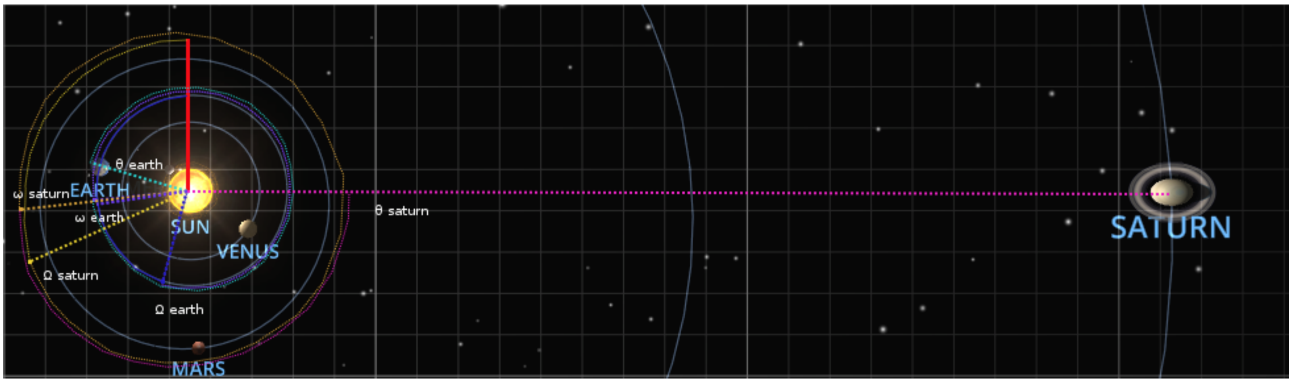

Figure 1: The positions of the Earth and Saturn at time t_0. The red line indicates the vernal equinox, and the x axis of our coordinate system.

1.5 Transfer Orbit

The elliptic Hohmann transfer orbit between Saturn and the Earth will have a semi major axis of:

$$a_{HT} = \frac{r_{\text{♁}_{t_d}} + r_{\text{♄}_{t_r}}}{2} \qquad \text{where} \qquad r_p \equiv \frac{a_p(1-e_p^2)}{1+e_p\cos\Theta_p}$$

Where t_d and t_r indicate the time that Endurance is at the periapsis and apoapsis of its interplanetary transfer orbit, departure from earth and rendezvous with saturn, respectively.

$$\begin{aligned} a_{HT} &= \frac{r_{\text{♁}_{t_d}} + r_{\text{♄}_{t_r}}}{2} \\ &= \frac{a_{\text{♁}_{t_d}}(1-e^2_{\text{♁}_{t_d}})}{2+2e_{\text{♁}_{t_d}}\cos\Theta_{\text{♁}_{t_d}}} + \frac{a_{\text{♄}_{t_r}}(1-e^2_{\text{♄}_{t_r}})}{2+2e_{\text{♄}_{t_r}}\cos\Theta_{\text{♄}_{t_r}}} \end{aligned}$$

For a Hohmann Transfer, the time of flight will be given by:

$$ToF_{HT} = \pi\sqrt{\frac{a_{HT}^3}{\mu_\odot}} = \pi\sqrt{\frac{(r_{\text{♁}_{t_d}} + r_{\text{♄}_{t_r}})^3}{8\mu_\odot}}$$

Determining what the orbital phase angle between the two planets should have at the beginning of the transfer maneuver is complicated, since the semi-major axis of the transfer orbit will be a function of the true anomaly of both planets ($a_{HT} = \frac{1}{2}(r_{\text{♁}_\pi} + r_{\text{♄}_\alpha})$) and therefore the flight time will also be a function of their true anomalies. If the flight time is dependent on true anomaly, then we cannot use it as a value of t to find the orbital phase angle.

We will use some iterative methods to try and find the correct time to begin our transfer maneuver. To aid in this iterative process, we define Γ to be the 'vernal angle', namely, the angle between a celestial body and the 1-axis of our coordinate system, the vernal equinox. With this in mind, assuming a negligible inclination, an approximate expression for Γ is given by : $\Gamma \approx \Omega + \omega + \Theta$, with Ω, ω and Θ the right ascension of the ascending node, the argument of periapsis and the true anomaly respectively.

We will begin with the assumption that both orbits are circular, and then attempt to identify a more accurate window by the following iterative process:

1. Earth's position in its orbit will be calculated with respect to the vernal equinox. Since the coordinate system uses Earth's ecliptic plane as the x-y plane, its angular position will described by the value $\Gamma_{\text{♁}}$
2. The apoapsis of the transfer orbit will occur at $\Gamma_\alpha = \Gamma_{\text{♁}} + 180°$. This gamma will be used to solve for Saturn's true anomaly at this point, which will be used to find Saturn's position vector $r_{\text{♄}}$ at this gamma value.
3. This orbital distance along with Earth's orbital distance will be used to find the semi-major axis of the transfer orbit via $a_{HT} = \frac{1}{2}(r_{\text{♁}_d} + r_{\text{♄}_r})$.
4. Endeavor's time of flight in this orbit, from periapsis to apoapsis, will be determined via $ToF_{HT} = \pi\sqrt{\frac{a_{HT}^3}{\mu_\odot}}$.
5. The time of flight will be used to look up Saturn's position from JPL Horizons. It's position will be compared to the position of the apoapsis of the Hohmann transfer orbit.
6. If Endeavor is not close to Saturn at this point, the difference in in the values of $\Gamma_{\text{♄}}$ and $\Gamma_{\text{♐}}$, will be used to choose a new $\Gamma_{\text{♁}}$, the periapsis of the Hohmann transport orbit. The iterative process will begin again at this point.

The full iterative process will be demonstrated once using the circular orbit approximation, and then summarized for following iterations.

For a circular orbit, the orbital phase angle between the planets at departure is given by:

$$\theta_{phase} = \pi - n_{\text{♄}} \cdot ToF_{HT}$$

$$n_{\text{♄}} = \frac{2\pi}{T_{\text{♄}}} \quad , \quad T_{\text{♄}} = 2\pi\sqrt{\frac{a_{\text{♄}}^3}{\mu_\odot}} \quad , \quad ToF_{HT} = \pi\sqrt{\frac{a_{HT}^3}{\mu_\odot}}$$

$$\therefore \theta_{phase} = \pi - \sqrt{\left(\frac{\mu_\odot}{a_{\text{♄}}^3}\right)} \cdot \pi\sqrt{\left(\frac{a_{HT}^3}{\mu_\odot}\right)} = \pi\left(1 - \left(\frac{a_{HT}}{a_{\text{♄}}}\right)^{\frac{3}{2}}\right)$$

Using the average of the semi-major axes of Earth and Saturn, we determine that Saturn has to be 105.97° ahead of the earth when we apply our interplanetary Hohmann transfer thrust.

At time t_0 the phase angle θ between the Earth and Saturn is given by:

$$\theta \approx \Gamma_{\text{♄}_{t_0}} - \Gamma_{\text{♁}_{t_0}} \approx 74.8029° - 269.5017 \approx -194.6988$$

Where a modulus of 360° has been applied to the sum of the angles. We need to determine when this phase angle will be equal to approximately 106°. We will again assume circular orbits, and use the mean motions of Earth and Saturn. Since there is a difference of approximately 3.2924° in the factors $\bar{\omega}_{\text{♄}}$ and $\bar{\omega}_{\text{♁}}$, we need the true anomalies of the Earth and Saturn to differ by $105.97° - 3.2924° = 102.6779°$:

$$\begin{aligned}
\theta_{desired} &= \theta_{current} - n_{\text{♁}}t + n_{\text{♄}}t \\
\therefore t &= \frac{102.6778° - 194.6988°}{n_{\text{♄}} - n_{\text{♁}}} \\
&= (102.6778° - 194.6988°)\left(\sqrt{\frac{a_{\text{♄}}^3}{\mu_\odot}} - \sqrt{\frac{a_{\text{♁}}^3}{\mu_\odot}}\right) \\
&= 8.506742 \times 10^6 \text{ s} \approx 98.458 \text{ days}
\end{aligned}$$

This is close enough to the date of the vernal equinox, March 20th, 2018, so will use this as our first estimate for the date we begin our transfer maneuver. Using JPL horizons to determine the properties of our transfer orbit:

Table 4: Hohmann Transfer orbit beginning on March 20th, 2018 at 00:00TT

Planet	$\bar{\boldsymbol{\omega}}$ (°)	$\boldsymbol{\Theta}$ (°)	$\boldsymbol{\Gamma}$ (°)	**r** (10^{11} m)
Earth$_{\pi_{HT}}$	95.1	83.6	178.7	1.51018
Saturn$_{\alpha_{HT}}$	92.7	266	358.7	14.2880

$\boldsymbol{a_{HT}}$ (10^{11} m)	$\boldsymbol{e_{HT}}$	**ToF** (days)	**Date of Rendezvous**
7.8991	0.80882	2214.40	April 11, 2024

	$\bar{\boldsymbol{\omega}}$ (°)	$\boldsymbol{\Theta}$ (°)	$\boldsymbol{\Gamma}$ (°)	**r** (10^{11} m)
Saturn	92.1	249.1	341.2	14.514
Endurance	178.7	180	358.7	14.288

So Endurance arrived too early, and missed Saturn by an angle of approximately 17.5°. Therefore, the departure time should be moved up. The vernal angle of rendezvous will be changed to 341°, making the vernal angle of departure 161°. The date that Earth has a vernal angle of 161° can be approximated as follows:

$$\Gamma_{\text{♁}_{t_0}} = 74.8029^\circ \qquad , \qquad \Gamma_{\text{♁}_{t_2}} = 161^\circ \qquad , \qquad n_{\text{♁}} = \sqrt{\frac{\mu_\odot}{a^3_{\text{♁}}}}\frac{deg}{s}$$

$$\Delta t = \frac{\Gamma_{\text{♁}_{t_2}} - \Gamma_{\text{♁}_{t_0}}}{n_{\text{♁}}} = 7.69801 \times 10^6 s \approx 89.097 \text{ days}$$

This corresponds to a date of March 6th, 2018. The subsequent iterations will be summarized in the tables below.

Table 5: Hohmann Transfer orbit beginning on March 6th, 2018 at 00:00TT

Planet	$\bar{\boldsymbol{\omega}}$ °	$\boldsymbol{\Theta}$ (°)	$\boldsymbol{\Gamma}$ (°)	**r** (10^{11} m)
Earth$_{\pi_{HT}}$	112.57	52.163	164.733	1.4924
Saturn$_{\alpha_{HT}}$	92.72	252.013	344.733	14.476

$\boldsymbol{a_{HT}}$ (10^{11} m)	$\boldsymbol{e_{HT}}$	**ToF** (days)	**Date of Rendezvous**
7.9847	0.81294	2250	May 3, 2024

At Rendezvous	$\bar{\boldsymbol{\omega}}$ °	$\boldsymbol{\Theta}$ (°)	$\boldsymbol{\Gamma}$ (°)	**r** (10^{11} m)
Saturn	92.14	249.365	341.505	14.475
Endurance	164.733	180	344.733	14.511

$$\frac{178.7^\circ - 164.733^\circ}{14 \text{ days}} = 0.99\frac{\text{deg}}{\text{day}}$$

$$\frac{164.733^\circ - 161.505^\circ}{0.99\frac{\text{deg}}{\text{day}}} = 3.24 \text{ days}$$

Table 6: Hohmann Transfer orbit beginning on March 3rd, 2018 at 00:00TT

Planet	$\bar{\boldsymbol{\omega}}$ °	$\boldsymbol{\Theta}$ (°)	$\boldsymbol{\Gamma}$ (°)	**r** (10^{11} m)
Earth$_{\pi_{HT}}$	115.37	46.356	161.726	1.4911
Saturn$_{\alpha_{HT}}$	92.72	249.006	341.726	14.515

$\boldsymbol{a_{HT}}$ (10^{11} m)	$\boldsymbol{e_{HT}}$	**ToF** (days)	**Date of Rendezvous**
8.0032	0.81369	2258.3	May 8, 2024

At Rendezvous	$\bar{\boldsymbol{\omega}}$ °	$\boldsymbol{\Theta}$ (°)	$\boldsymbol{\Gamma}$ (°)	**r** (10^{11} m)
Saturn	92.14	249.525	341.665	14.509
Endurance	161.726	180	341.726	14.515

Table 7: Hohmann Transfer orbit beginning on March 2nd, 2018 at 00:00TT

Planet	$\bar{\boldsymbol{\omega}}$ °	$\boldsymbol{\Theta}$ (°)	$\boldsymbol{\Gamma}$ (°)	**r** (10^{11} m)
$\text{Earth}_{\pi_{HT}}$	115.79	45.434	161.224	1.4907
$\text{Saturn}_{\alpha_{HT}}$	92.72	248.504	341.224	14.522

$\boldsymbol{a_{HT}}$ (10^{11} m)	$\boldsymbol{e_{HT}}$	**ToF** (days)	**Date of Rendezvous**
8.0064	0.81380	2259.67	May 8, 2024

At Rendezvous	$\bar{\boldsymbol{\omega}}$ °	$\boldsymbol{\Theta}$ (°)	$\boldsymbol{\Gamma}$ (°)	**r** (10^{11} m)
Saturn	92.14	249.59	341.73	14.508
Endurance	161.224	180	341.224	14.522

This trajectory is what we will use. We will need to make an inclination correction mid flight at the intersection of Earth's orbital plane with Saturn's.

1.6 Departure Trajectory

In order to enter into an interplanetary Hohmann transfer, we need or thrust vector to be tangent to our parking orbit. Upon entering or exiting a sphere of influence, the patch condition must be satisfied:

$$r_{♐} = r_p + \rho$$
$$v_{♐} = v_p + \nu$$

where ρ and ν are position and velocity vectors with respect to the planet, while v and r are those with respect to $\odot$.

We can rearrange these to express the hyperbolic excess speed ν_∞ applied at the boundary of an SOI:

$$\nu_\infty = v_{♐} - v_p$$

In this section, we will use *park* to denote the parking orbit around earth, and *hyp* to represent the hyperbolic trajectory out of the Earth's SOI. For our departure trajectory, the velocity of Endurance upon exiting the SOI of earth with respect to the earth and the velocity of the earth with respect to the sun can be determined by the Vis-Viva equation:

$$v_{♐ HT_\pi} = \sqrt{\mu_\odot\Big(\frac{2}{r_{♁}} - \frac{1}{a_{HT}}\Big)} = \sqrt{\mu_\odot\Big(\frac{2(1+e_{♁}\cos\Theta_{♁})}{a_{♁}(1-e_{♁}^2)} - \frac{1}{a_{HT}}\Big)}$$

and

$$v_{♁} = \sqrt{\mu_\odot\Big(\frac{2}{r_{♁}} - \frac{1}{a_{♁}}\Big)} = \sqrt{\mu_\odot\Big(\frac{2(1+e_{♁}\cos\Theta_{♁})}{a_{♁}(1-e_{♁}^2)} - \frac{1}{a_{♁}}\Big)}$$

Where $r_{♁} = \frac{l_{♁}}{1+e_{♁}\cos\Theta_{♁}}$ and $l_{♁} = a_{♁}(1-e_{♁}^2)$ have been used.

We require the velocity of Endurance at the periapsis of the interplanetary transfer orbit, while Endurance is still in its parking orbit. We can use the orbit's specific energy to determine the semi-major axis of the hyperbolic trajectory required to obtain this transfer orbit:

$$\epsilon = -\frac{\mu_{♁}}{2a_{hyp}} = \lim_{\rho\to\infty}\Big(\frac{\nu_\infty^2}{2} - \frac{\mu_{♁}}{\rho_{hyp}}\Big) = \frac{\nu_\infty^2}{2}$$
$$\therefore a_{hyp} = -\frac{\mu_{♁}}{\nu_\infty^2}$$

We can now determine the magnitude of our first interplanetary trajectory thrust Δv_3. Using the Vis-Viva equation:

$$
\begin{aligned}
\Delta v_3 = \nu_{\pi_{hyp}} - \nu_{park} &= \sqrt{\mu_{\oplus}\Big(\frac{2}{\rho_{park}} - \frac{1}{a_{hyp}}\Big)} - \sqrt{\frac{\mu_{\oplus}}{\rho_{park}}} \\
&= \sqrt{\mu_{\oplus}\Big(\frac{2}{\rho_{park}} + \frac{\nu_\infty^2}{\mu_{\oplus}}\Big)} - \sqrt{\frac{\mu_{\oplus}}{\rho_{park}}} \\
&= \sqrt{\mu_{\oplus}\Big(\frac{2}{\rho_{park}} + \frac{(v_{HT_\pi} - v_{\oplus})^2}{\mu_{\oplus}}\Big)} - \sqrt{\frac{\mu_{\oplus}}{\rho_{park}}} \\
&= \sqrt{\mu_{\oplus}\left(\frac{2}{\rho_{park}} + \frac{\left(\sqrt{\mu_{\odot}\Big(\frac{2(1+e_{\oplus}\cos\Theta_{\oplus})}{a_{\oplus}(1-e_{\oplus}^2)} - \frac{1}{a_{HT}}\Big)} - \sqrt{\mu_{\odot}\Big(\frac{2(1+e_{\oplus}\cos\Theta_{\oplus})}{a_{\oplus}(1-e_{\oplus}^2)} - \frac{1}{a_{\oplus}}\Big)}\right)^2}{\mu_{\oplus}}\right)} - \sqrt{\frac{\mu_{\oplus}}{\rho_{park}}} \\
&= \sqrt{\mu_{\oplus}\left(\frac{2}{\rho_{park}} + \frac{\mu_{\odot}}{\mu_{\oplus}}\left(\sqrt{\frac{2(1+e_{\oplus}\cos\Theta_{\oplus})}{a_{\oplus}(1-e_{\oplus}^2)} - \frac{1}{a_{HT}}} - \sqrt{\frac{2(1+e_{\oplus}\cos\Theta_{\oplus})}{a_{\oplus}(1-e_{\oplus}^2)} - \frac{1}{a_{\oplus}}}\right)^2\right)} - \sqrt{\frac{\mu_{\oplus}}{\rho_{park}}}
\end{aligned}
$$

Using the values provided in table 7, and carrying out the calculation in Matlab, the magnitude of $\Delta v_3 = 8.1778 \times 10^3 \frac{m}{s} \approx 8.18 \frac{km}{s}$.

Now we need to determine when this thrust should be made. Since this location will be the periapsis of the orbit, and at periapsis, the true anomaly will be 0, we can write:

$$\rho_{park} = \frac{a_{hyp}(1 - e_{hyp}^2)}{(1 + e_{hyp})} = a_{hyp}(1 - e_{hyp}) \rightarrow \therefore e_{hyp} = 1 - \frac{\rho_{park}}{a_{hyp}}$$

Further, as $\rho_{hyp} \rightarrow \infty$, we can find the escape true anomaly:

$$\Theta_{\infty_{hyp}} = \arccos\Big(-\frac{1}{e_{hyp}}\Big)$$

This angle relates to the half angle of the departure hyperbola by:

$$\gamma_{hyp} + \Theta_{\infty_{hyp}} = \pi \rightarrow \cos\Theta_{\infty_{hyp}} = \cos(\pi - \gamma_{hyp}) \rightarrow \gamma_{hyp} = \arccos\frac{1}{e_{hyp}}$$

And therefore, the required phase angle between the maneuver node and Earth's velocity vector ϕ_{hyp} is given by:

$$
\begin{aligned}
\phi_{hyp} &= \pi + \gamma_{hyp} \\
&= \pi + \arccos\frac{1}{1 - \frac{\rho_{park}}{a_{hyp}}} \\
&= \pi + \arccos\frac{1}{1 + \frac{\rho_{park}\nu_\infty^2}{\mu_{\oplus}}} \\
&= \pi + \arccos\frac{1}{1 + \frac{\rho_{park}(v_{hyp_{SOI}} - v_{\oplus})^2}{\mu_{\oplus}}} \\
&= \pi + \arccos\frac{1}{1 + \frac{\rho_{park}}{\mu_{\oplus}}\left(\sqrt{\mu_{\odot}\Big(\frac{2(1+e_{\oplus}\cos\Theta_{\oplus})}{a_{\oplus}(1-e_{\oplus}^2)} - \frac{1}{a_{HT}}\Big)} - \sqrt{\mu_{\odot}\Big(\frac{2(1+e_{\oplus}\cos\Theta_{\oplus})}{a_{\oplus}(1-e_{\oplus}^2)} - \frac{1}{a_{\oplus}}\Big)}\right)^2}
\end{aligned}
$$

Solving this in matlab, we obtain a value of approximately 266.5068° away from the velocity vector of earth. Now we need to find at what time this occurs.

The date that this thrust will occur is March 2nd, 2018. Using that the spacecraft begins at a true anomaly of 30° on Dec 7th, 2017 at 00:00 TT, we can use the period of orbit to determine where it will be on the date of departure.

$$85 \text{ days} = 7.344000 \times 10^6 \text{s}$$
$$T_{park} = 2\pi\sqrt{\frac{a_{park}^3}{\mu}} = 1.462641 \times 10^5 \text{s}$$
$$\therefore \frac{7.344000 \times 10^6 \text{s}}{1.462641 \times 10^5 \text{s}} = 50.21 \text{ orbits}$$

Therefore, Endurance has orbited Earth 50 times on the date of the departure. It currently has a true anomaly of $\Theta_{\text{♐}} = \Theta_0 + (0.21 \times 360°) = 105.6°$.

Making the assumption that $v_{\text{♁}_z} << \sqrt{v_{\text{♁}_x}^2 + v_{\text{♁}_y}^2}$, we can obtain the angle of Earth's velocity with respect to the Vernal axis via:

$$\therefore \Gamma_{\text{♁}} = \arctan \frac{v_{\text{♁}_y}}{v_{\text{♁}_x}}$$

Retrieving the velocity of Earth from JPL Horizons, the angle that $v_{\text{♁}}$ makes with vernal axis is approximately $70.2733° + 180° = 250.2733$. If we need our departure node to be 266.5068° away from this value, we need Endeavor to commence Δv_4 at a true anomaly of 156.7801°. Therefore, the time of our 4th thrust is given by:

$$t_4 = \frac{(\Gamma_{\text{♁}_v} + \phi_{hyp}) - \Theta_{\text{♐}}}{n_{\text{♐}_{park_{\text{♁}}}}} \approx 2.079399 \times 10^4 \text{ s} \approx 5.78 \text{ hours}$$

So we depart at 05:47 on March 2nd, 2018.

1.7 Mid-Flight Inclination Correction

In order for Endeavor to successfully make it to Saturn we need to change the inclination of Endeavor's orbit mid trajectory in order to arrive at Saturn on the Cronian orbital plane. In order to do this without changing any of the orbit's parameters, we need this adjustment to occur at the intersection of Saturn's and Earth's orbital plane.

We can determine the vernal angle of this intersection by noting that Earth's orbital plane is by definition not inclined to the coordinate axes, and therefore, the two plane's intersect at a vernal angle of $\gamma = 113.5977°$ and $\gamma = 113.5977° + 180° = 293.5977°$.

Since the periapsis of Endeavor's Hohmann transfer orbit is at the location of departure, this vernal angle corresponds to a true anomaly of $\Theta = 293.5977° - 161.224° = 132.374°$.
Solving for Endeavor's position at this true anomaly using the polar equation:

$$r_{\text{♐}} = \frac{a_{HT}(1 - e_{HT}^2)}{1 + e_{HT}\cos\Theta_{\text{♐}}} = 5.98842 \times 10^{11} \text{m}$$

The velocity of Endeavor at this point is found via the Vis-Viva equation:

$$v_{\text{♐}} = \sqrt{\mu_{\odot}\left(\frac{2}{r_{\text{♐}}} - \frac{1}{a_{HT}}\right)} = 1.666857 \times 10^4 \frac{\text{m}}{\text{s}}$$

And the required Δv_5 to adjust the inclination is given by:

$$\Delta v_5 = 2v_{\text{♐}} \sin(\frac{i_{\text{♄}}}{2}) = 723.4647 \frac{\text{m}}{\text{s}}$$

Since we want to go below the plane by an angle $i_{\saturn}$, we should direct the thrust at $\beta = \frac{180^\circ + i^\circ_{\saturn}}{2} = 91.243499^\circ$ relative to v_i.

Using a matlab script, the polar equation, and data from JPL Horizons, we can determine on which date this occurs using a quick for loop. The date this occurs is 1459 days after mission launch, on December 5th, 2021.

Therefore our fifth change in velocity occurs on December 5th, 2021 at $t_5 = 1.260576 \times 10^8$seconds after the mission commences, and has a value of .7235 km/s, pointed $\beta = 91.2435^\circ$ away from Endeavor's instantaneous velocity.

1.8 Arrival Trajectory

Saturn's sphere of influence is given by:

$$r_{\saturn_{SOI}} = r_{\saturn}\left(\frac{\mu_{\saturn}}{\mu_{\odot}}\right)^{\frac{2}{5}}$$

Endeavor's position can be determined by solving Kepler's equation for the true anomaly of Endeavor at every point, then solving for the radial distance of endeavor from the sun using the polar equation. Kepler's equation is solved iteratively, and can be found in the matlab script.

Saturn's position, on the other hand, can be determined directly from JPL Horizons, and is plotted, along with Earth's and Endeavor's position on Figure 2.

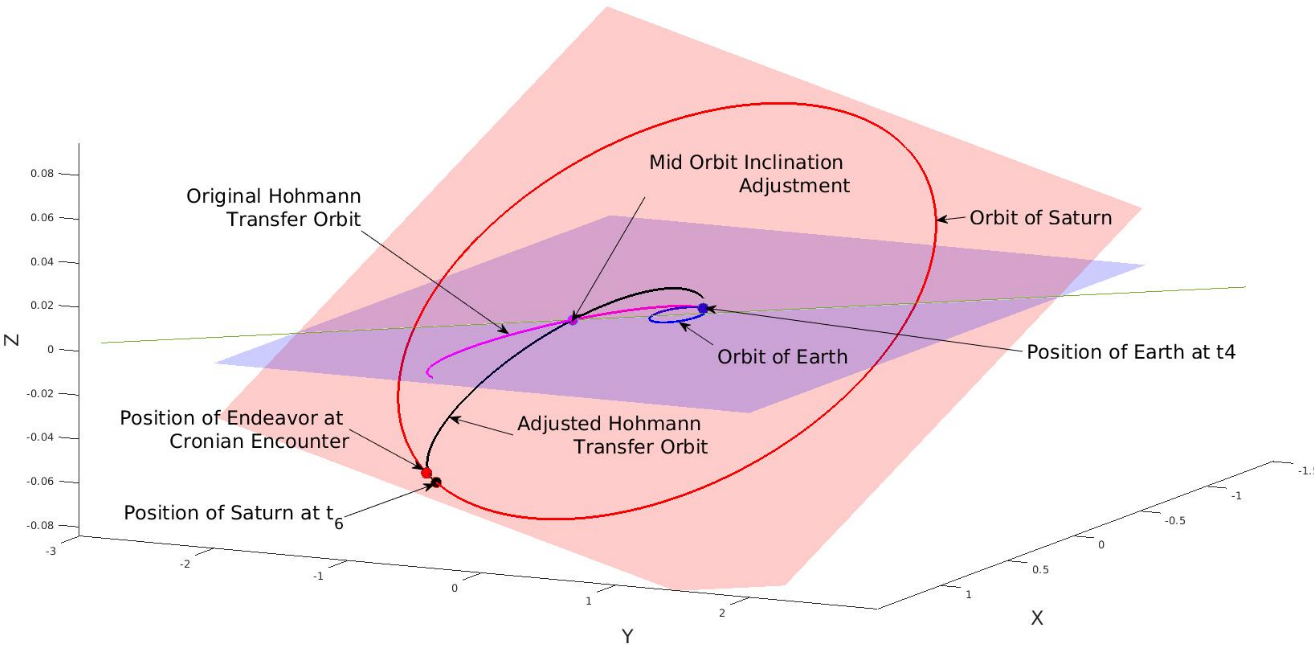

Figure 2: Proposed Mission Architecture

We can determine the date that Endeavor crosses Saturn's SOI boundary using a for loop, similar to the one used to determine the position of the mid-orbit inclination adjustment. The expression $\rho = r_{\sagittarius} - r_{\saturn}$ is solved and compared to r_{SOI} in order to determine the date the magnitude of these two values are equal, where ρ is the position of Endeavor with respect to Saturn and the positions denoted by r are with respect to the $\odot$.

The encounter of Saturn's SOI occurs on April 21nd, 2024 at approximately 09:07. We can determine the speed of Saturn and Endeavor respectively, however we can assume that their velocities are parallel, since we are very close to aphelion.

We begin by finding the semi-major and semi-minor axes of the arrival hyperbola:

$$a_{hyp} = -\frac{\mu_{\saturn}}{\nu_{\infty}^2} = -2.1583038 \times 10^9\text{m} \qquad\qquad b_{hyp} = r_{SOI} \sin\left[\arccos\Big(-\frac{\underset{\to}{\boldsymbol{\rho}_\infty} \cdot \underset{\to}{\boldsymbol{\nu}_\infty}}{\rho_\infty \nu_\infty}\Big)\right] = -7.164185 \times 10^9\text{m}$$

Where we assume that the velocity of Endeavor is in the same direction as Saturn, namely, $\underset{\to}{\hat{\boldsymbol{\nu}}}_{\nearrow} = \frac{\underset{\to}{\boldsymbol{\nu}}_{\saturn}}{|\nu_{\saturn}|}$, with a magnitude determined via the Vis-Viva equation:

$$\nu_\infty = \sqrt{\mu_{\odot}\Big(\frac{2}{r_{\saturn}} - \frac{1}{a_{HT}}\Big)}$$

Now we determine the eccentricity of the arrival hyperbola:

$$e_{hyp} = \sqrt{1 + \frac{b_{hyp}^2}{a_{hyp}^2}} = 3.46672$$

We can now determine the position of closest approach, ρ_{cap}:

$$\rho_{cap} = a_{hyp}(1 - e_{hyp}) = 5.324 \times 10^9 m$$

Using the Vis-Viva equation to solve for Δv_6, the change in velocity to enter a circular parking orbit about Saturn is given by:

$$\Delta v_6 = \nu_{\pi_{hyp}} - \nu_{cap} = \sqrt{\mu_{\saturn}\Big(\frac{2}{\rho_{cap}} - \frac{1}{a_{hyp}}\Big)} - \sqrt{\frac{\mu_{\saturn}}{\rho_{cap}}} = 2.97206 \times 10^3 \frac{\text{m}}{\text{s}}$$

We can estimate the time we make this thrust at using the angle Endeavor travels along the hyperbolic orbit:

$$\theta_\infty = \arccos\big(-\frac{1}{e_{hyp}}\big) = 106.76^\circ \qquad , \qquad \theta_{cap} = 0^\circ$$

The distance of Endeavor at each point:

$$\rho_{cap} = 5.324 \times 10^9 m \qquad , \qquad \rho_\infty = 5.40924 \times 10^{10} m$$

And the law of cosines:

$$d = \sqrt{\rho_{cap}^2 + \rho_\infty^2 - 2\rho_{cap}\rho_\infty \cos\theta_\infty} = 5.586 \times 10^{10}\text{m}$$

Using the average of Endeavor's velocity at ρ_{cap} and ρ_∞, we find a flight time along the arrival hyperbola of $ToF_{hyp} = 1.136 \times 10^7\text{s} \approx 131.4982$ days after our entry of Saturn's sphere of influence. This corresponds to a date of June 6th, 2024, at approximately 21:04.

1.9 Orbit Correction

Now that we are comfortably in orbit about Saturn, we need to get into the required orbit. First we will go into the correct orbital distance. We continue burning at ρ_{cap} with Δv_6 given by:

$$\Delta v_6 = \sqrt{\frac{\mu_{\saturn}}{\rho_{cap}}} - \sqrt{\mu_{\saturn}\Big(\frac{2}{\rho_{cap}} - \frac{1}{a_{OT}}\Big)} = 2.15665 \times 10^3 \frac{\text{m}}{\text{s}}$$

Where a_{OT} is given by $a_{OT} = \frac{\rho_{cap} + 10^8}{2}$ m.

We then wait for the periapsis of this orbit, ρ_{Orbit}, which we can determine via:

$$ToF_{♄_\pi} = \pi\sqrt{\frac{a_{OT}^3}{\mu_{♄}}} \approx 26.3672\text{hours}$$

At which point we apply Δv_7 which is given by:

$$\Delta v_7 = \sqrt{\mu_{♄}\Big(\frac{2}{\rho_{Orbit}} - \frac{1}{a_{OT}}\Big)} - \sqrt{\frac{\mu_{♄}}{\rho_{Orbit}}} = 7.81211 \times 10^3 \frac{\text{m}}{\text{s}}$$

Now we are at the correct orbital distance from Saturn, on Saturn's Ecliptic plane.

Summary

Date	$\mathbf{t}$ $\times 10^6$ (s)	$\boldsymbol{\Delta v}$	**Direction**
December 7th, 2017, 00:00	$t_1 = 0$ s	0.172	Retrograde
December 8th, 2017, 16:36	$t_2 = 0.1426414$	0.172	Prograde
December 10th, 2017, 05:52	$t_3 = 0.280339$	1.334	$-105°$
March 2nd, 2018, 05:47	$t_4 = 7.364794$	8.178	Prograde
February 28th, 2022, 14:15	$t_5 = 133.4529$	0.7235	$-91.2435°$
June 6th, 2024, 21:04	$t_6 = 205.1031$	5.129	Retrograde
July 3rd, 2024, 05:52	$t_7 = 207.3812$	7.812	Retrograde
Total	**207.3812**	**24.683**	

Assumption Discussion

I will now address how the assumptions made would alter the real life mission of entering a Cronian orbit.

- A 2-body patched conics approximation was used. Gravitational forces from non relevant bodies were neglected, but their effects would alter the orbital elements of a transfer orbit in real life. In an n-body simulation, we could calculate the effects of each gravitational body on our spacecraft, but this would be computationally expensive and complicated.
- Assuming that thrusts occur instantaneously makes calculations easier, however with near infinite accelerations, anyone on board would be killed. Instead, thrusts should impulsively, and the position the thrust begins and ends should be calculated by integrating the thrust over time. This is complicated to do, so is disregarded for the sake of brevity, however the orbits chosen would not change much by performing this added step.
- Orbital perturbations were not considered. They could be added if we created a computer program which iteratively determined the position of our spacecraft at each point, via methods discussed in this course. However, for the proposed orbital transfer, we do not pass near enough to any major bodies for this to take effect (except perhaps Earth).
- The assumption that Earth's vertical velocity is negligible is a fair one, as it is nearly 3 orders of magnitude smaller than the other velocities.
- Assuming that Endeavor's velocity was parallel to Saturn's during the Cronian encounter was not valid, since we entered its sphere of influence before aphelion. This will change the value we have for the Δv of our capture maneuver, as well as the altitude of our capture. This could be accounted for by determining Endeavor's velocity at each point on its orbit, but since we enter the Cronian parking orbit much closer to aphelion, the difference is likely not huge.

I could likely have lowered my Δv by an additional $4\frac{\text{km}}{\text{s}}$ of I had more time, however due to how many other assignments I had going on, I could not fully explore this project. If I had an additional few days, I would attempt some gravity assists.