



GÖTEBORGS UNIVERSITET



# Applied Functionalanalysis

Script of „Applied Functionalanalysis“ by Prof. Peter Kumlin

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## foreword — cooperation

This document is a transcript of the lecture „Applied Functionalanalysis, WiSe 2016/2017, Term 1“, by Prof. Peter Kumlin. It mainly contains the written content of the lecture. I will not assume any responsibility for the correctness of the content! For questions, remarks and mistakes please write an email to [keil.menden@web.de](mailto:keil.menden@web.de). I'm grateful for every email.



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# 1 Introduction

## 1.1 Introduction example

We have

$$\begin{cases} f'' + f = g, & \text{in } I = [0, 1] \\ f(0) = 1, f'(0) = 1 \end{cases}$$

where  $g$  is a known continuous function in  $I$ . We will now consider different cases:

1.  $g = 0$

$$\Rightarrow f(x) = A \cos(x) + B \sin(x), x \in I$$

where  $A, B \in \mathbb{R}$ .

2.  $g$  arbitrary. We will now introduce the Method of variation of constants. Set

$$f(x) = A(x) \cos(x) + B(x) \sin(x)$$

Differentiate

$$f'(x) = A'(x) \cos(x) + B'(x) \sin(x) - A(x) \sin(x) + B(x) \cos(x)$$

Assume (This is part of the method)

$$A'(x) \cos(x) + B'(x) \sin(x) = 0, \quad x \in I$$

Differentiate  $f'(x)$  and get

$$f''(x) = \underbrace{-A(x) \cos(x) - B(x) \sin(x)}_{=-f(x)} - A'(x) \sin(x) + B'(x) \cos(x)$$

We get

$$g(x) = f''(x) + f(x) = -A'(x) \sin(x) + B'(x) \cos(x).$$

Now:

$$\begin{cases} A'(x) \cos(x) + B'(x) \sin(x) = 0, & x \in I \\ -A'(x) \sin(x) + B'(x) \cos(x) = g(x), & x \in I \\ A(0) = 1, & B(0) = 0 \end{cases}$$

We get

$$A'(x) = -g(x) \sin(x)$$

$$A(0) = 1$$

$$B'(x) = g(x) \cos(x)$$

$$B(0) = 0$$

This implies

$$\begin{aligned} A(x) &= A(0) + \int_0^x A'(t) dt = 1 - \int_0^x g(t) \sin(t) dt \\ B(x) &= B(0) + \int_0^x B'(t) dt = 0 + \int_0^x g(t) \cos(t) dt \end{aligned}$$

Hence

$$\begin{aligned} f(x) &= \cos(x) - \int_0^x g(t) \sin(t) dt \cos(x) + \int_0^x g(t) \cos(t) dt \sin(x) \\ &= \cos(x) + \int_0^x \underbrace{(\sin(x) \cos(t) - \sin(t) \cos(x))}_{=\sin(x-t)} g(t) dt \\ &= \cos(x) + \int_0^x \sin(x-t) g(t) dt \quad (*) \end{aligned}$$

Check that  $f(x)$  in  $(*)$  satisfies the PDE.

**special case:**

Assume for  $x \in I$

$$g(x) = k(x)f(x)$$

Here  $k$  is a known continuous function in  $I$ . Insert this in  $(*)$ . We obtain

$$f(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f(t) dt, \quad x \in I \quad (**)$$

Observe that  $f$  appears both in LHS and RHS.  $(**)$  is a reformulation of the PDE with  $g = kf$ . Pick a continuous function in  $I$ . call it  $f_0$ . Set  
 $\in C(I)$

$$\begin{aligned} f_1(x) &= \cos(x) + \int_0^x \sin(x-t)k(t)f_0(t) dt \\ f_2(x) &= \cos(x) + \int_0^x \sin(x-t)k(t)f_1(t) dt \\ &\vdots \\ f_{n+1}(x) &= \cos(x) + \int_0^x \sin(x-t)k(t)f_n(t) dt, \quad n = 1, 2, 3, \dots \end{aligned}$$

**Hope:**

$f_n$  tends to some continuous function  $f$  on  $I$ , denoted  $f_n \rightarrow f$ . 'Tends to' has to be more precis!

$$\begin{array}{ccc}
 f_{n+1}(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f_n(t) dt & & \\
 \downarrow & \quad \quad \downarrow & \\
 f(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f(t) dt & & 
 \end{array}$$

for  $x \in I$ . Simplify notation set for  $v \in C(I)$

$$\begin{cases} u(x) &= \cos(x) \\ kv(x) &= \int_0^x \sin(x-t)k(t)v(t) dt \end{cases}$$

We have  $f_0 \in C(I)$ ,  $f_{n+1} = u + kf_n$  for  $n = 0, 1, 2, \dots$  (!)

Facts from previous calculus classes:

**Definition** (Sequence of continuous functions).

$$v_n \in C(I), \quad n = 1, 2, \dots$$

We say that  $(v_n)_{n=1}^\infty$  converges uniformly in  $I$  if

$$\max_{x \in I} |v_n(x) - v_m(x)| \rightarrow 0, \quad n, m \rightarrow \infty$$

i.e.

$$\forall \varepsilon > 0 \exists N : \forall n, m \geq N : \max_{x \in I} |v_n(x) - v_m(x)| < \varepsilon$$

**Lemma .** Suppose that  $(v_n)_{n=1}^\infty$  converges uniformly on  $I$ . then there exists  $v \in C(I)$  such that

$$\max_{x \in I} |v_m(x) - v(x)| \rightarrow 0 \quad \text{as } m \rightarrow \infty$$

Back to (!):

More Notation:

$$k(kv) = k^2v, \quad v \in C(I)$$

and

$$k^{n+1}v = k(k^n v), \quad n = 1, 2, \dots$$

We have  $f_0 \in C(I)$ ,  $f_1 = u + kf_0$  and

$$f_2 = u + kf_1 = u + k(u + kf_0)$$

and so on. Note that

$$k(v + w) = kv + kw$$

Then

$$f_2 = u + k(u + kf_0) = k + ku + k(kf_0) = u + ku + k^2f_0$$

$$f_3 = u + kf_2 = u + ku + k^2u + k^3f_0$$

and in general for  $n = 1, 2, \dots$

$$f_n = ku + \dots + k^{n-1}u + k^n f_0, \quad n = 1, 2, \dots$$

Assume  $n > m$  then

$$f_n - f_m = k^m u + \dots + k^{n-1}u + k^n f_0 - k^m f_0$$

Set for  $v \in C(I)$

$$\|v\| = \max_{x \in I} |v(x)|$$

Note

$$\|v + w\| \leq \|v\| + \|w\| \quad \text{for } v, w \in C(I)$$

and

$$\|-v\| = \|v\|.$$

We have

$$\begin{aligned} \|f_n - f_m\| &= \|k^m u + \dots + k^{n-1}u + k^n f_0 - k^m f_0\| \\ &\leq \|k^m u\| + \dots + \|k^{n-1}u\| + \|k^n f_0\| + \|-k^m f_0\|. \end{aligned}$$

Assumption

$$\sum_{l=1}^{\infty} \|k^l v\| < \infty \quad \text{for all } v \in C(I) \quad (***)$$

Under this assumption

$$\|f_n - f_m\| \rightarrow 0 \quad \text{as } n, m \rightarrow \infty$$

since

$$\begin{aligned} \sum_{l=1}^{\infty} \|k^l u\| &< \infty & (u(x) = \cos(x)) \\ \sum_{l=1}^{\infty} \|k^l f_0\| &< \infty & (f_0 \in C(I)) \end{aligned}$$

conclusion:  $(f_n)_{n=1}^{\infty}$  converges uniformly on  $I$ . By lemma above there exists  $f \in C(I)$  such that

$$\max_{x \in I} |f_n(x) - f(x)| \rightarrow 0, \quad n \rightarrow \infty$$

i.e.

$$\|f_n - f\| \rightarrow 0, \quad n \rightarrow \infty$$



'Back hope':  $f_n$  tends to  $f$ , denoted  $f_n \rightarrow f$  shall be interpreted as

$$\|f_n - f\| \rightarrow 0, \quad n \rightarrow \infty$$

Remember

$$f_{n+1}(x) = u(x) + kf_n(x) \rightarrow ?$$

For  $x \in I$  there is

$$\begin{aligned} |kf_n(x) - kf(x)| &= \left| \int_0^x \sin(x-t)k(t)f_n(t) dt - \int_0^x \sin(x-t)k(t)f(t) dt \right| \\ &\leq \int_0^x |\sin(x-t)k(t)| \underbrace{|f_n(t) - f(t)|}_{\leq \|f_n - f\|} dt \\ &\leq \int_0^x |\sin(x-t)k(t)| dt \|f_n - f\| \end{aligned}$$

In particular

$$\begin{aligned} \|kf_n - kf\| &\leq \max_{x \in I} \int_0^x \underbrace{|\sin(x-t)|}_{\leq 1} \underbrace{|k(t)|}_{\max_{t \in I} |k(t)| < \infty} dt \|f_n - f\| \\ &\leq \|k\| \|f_n - f\| \end{aligned}$$

We have, provided  $(***)$  holds, shown

$$\begin{aligned} f_{n+1} &= u + kf_n \\ \downarrow \\ f &= u + kf \end{aligned}$$

Let us try to prove  $(***)$ . For  $v \in C(I)$  arbitrary and for  $x \in I$

$$\begin{aligned} \|kv(x)\| &= \left| \int_0^x \sin(x-t)k(t)v(t) dt \right| \\ &\leq \int_0^x \underbrace{|\sin(x-t)|}_{\leq 1} \underbrace{|k(t)|}_{\leq \|k\|} |v(t)| dt \\ &\leq \int_0^x \underbrace{|v(t)|}_{\leq \|v\|} dt \|k\| \\ &\leq \|k\| \|v\| x \end{aligned}$$

In particular

$$\|kv\| \leq \|k\| \|v\|$$

and

$$\begin{aligned} |k^2v(x)| &\leq \int_0^x |kv(t)| dt \|k\| \\ &\leq \int_0^x \|k\| \|v\| t dt \cdot \|k\| \\ &= \|k\|^2 \|v\| \frac{x^2}{2} \end{aligned}$$

In particular

$$\|k^2 v\| \leq \|k\|^2 \|v\| \frac{1}{2}$$

By induction we get

$$\begin{aligned} |k^n v(x)| &\leq \|k\|^n \|v\| \frac{x^n}{n!} & x \in I \\ \|k^n v\| &\leq \|k\|^n \|v\| \frac{1}{n!} \end{aligned}$$

So

$$\begin{aligned} \sum_{l=1}^{\infty} \|k^l v\| &\leq \sum_{l=1}^{\infty} \|k\|^l \|v\| \frac{1}{l!} \\ &= \|v\| \sum_{l=1}^{\infty} \frac{\|k\|^l}{l!} \\ &\leq \|v\| e^{\|k\|} < \infty \end{aligned}$$

consider Taylor expansion.  $\Rightarrow (***)$  holds true.

We have now shown that  $f = u + kf$  where  $u(x) = \cos(x)$  and

$$kv = \int_0^x \sin(x-t)k(t)v(t) dt$$

$x \in I$  for  $v \in C(I)$ , has a solution  $f \in C(I)$ .

Question: Is the solution unique?

Assume  $f, \tilde{f} \in C(I)$  such that  $f = u + kf$  and  $\tilde{f} = u + k\tilde{f}$ . Set

$$v = f - \tilde{f} \in C(I)$$

$$\begin{aligned} \Rightarrow v &= (u + kf) - (u + k\tilde{f}) \\ &= kf - k\tilde{f} \\ &= k(f - \tilde{f}) \\ &= kv \end{aligned}$$

We have  $v = kv$ , implies that  $kv = k(kv) = k^2 v$ . So for  $n = 1, 2, \dots$

$$v = kv = k^2 v = \dots = k^n v$$

We know

$$\sum_{n=1}^{\infty} \|k^n \hat{v}\| < \infty \quad \text{for all } \hat{v} \in C(I).$$

Apply this to  $\hat{v} = v$ :

$$\sum_{n=1}^{\infty} \underbrace{\|k^n v\|}_{=\|v\|} < \infty$$

So  $\|v\| = 0$  with implies  $v(x) = 0$  for all  $x \in I$ . So we have  $f(x) = \tilde{f}(x)$  for  $x \in I$ .

$\Rightarrow$  Answer to the question above: YES !

We have more or less proved the following theorem:

**Theorem 1.1.** Set  $I = [0, 1]$ . Suppose  $u \in C(I)$  and  $k \in C(I \times I)$ . Consider

$$f(x) = u(x) + \int_0^x k(x, t)f(t) \, dt, \quad x \in I \quad (1)$$

Then (1) has a unique solution  $f \in C(I)$

With the same technology we can prove:

**Theorem 1.2.** Set  $I = [0, 1]$ . Suppose  $u \in C(I)$ ,  $k \in C(I \times I)$  and  $\max_{(x,t) \in I \times I} |k(x, t)| < 1$ . Consider

$$f(x) = u(x) + \int_0^1 k(x, t)f(t) \, dt, \quad x \in I \quad (2).$$

Then (2) has a unique solution  $f \in C(I)$ .

Different notions: see introductory example.

**Definition** (vector space).  $C(I)$  with the operations for  $x \in I$

**addition**  $v, w \in C(I)$ :  $(v + w)(x) = v(x) + w(x)$

**mult. by scalar**  $v \in C(I)$ ,  $\lambda \in \mathbb{R}$ :  $(\lambda v)(x) = \lambda v(x)$

Note that  $v + w, \lambda v \in C(I)$ .

**Definition** (norm). norm on  $C(I)$  for instance

$$\|v\| = \max_{x \in I} |v(x)|$$

with norm given we can talk about convergence and confirmity

**Definition** (Cauchy sequence). In our example a sequence  $(f_n)_{n=1}^\infty$  is called Cauchy sequence if  $\|f_n - f_m\| \rightarrow 0$  for  $n, m \rightarrow \infty$ .

**Definition**.  $C(I)$  with the max-norm. Lemma above says that every Cauchy sequence converges i.e.

$$\|v_n - v_m\| \rightarrow 0, \quad n, m \rightarrow \infty$$

This applies

$$\exists v \in C(I) : \|v_n - v\| \rightarrow 0, \quad n \rightarrow \infty$$

This is the defining property of a Banach space.

$K$  linear mapping  $C(I) \rightarrow C(I)$  with

$$K(v + w) = K(v) + K(w)$$

$$K(\lambda v) = \lambda K(v)$$

for  $v, w \in C(I)$ ,  $\lambda \in \mathbb{R}$ .

$K$  bounded linear:

$$\|Kv\| \leq M\|v\| \quad \forall v \in C(I)$$

where  $M > 0$  independent of  $v$ .

**Definition** (operator norm). Define

$$\|K\| = \inf\{M > 0 \mid \|Kv\| \leq M\|v\| \text{ for all } v \in C(I)\}.$$

**fixed point results:**

Our example:  $f = u + kf =: T(f)$  and  $f_0 \in C(I)$  fixed.

Form sequence of iterants  $(f_n)_{n=1}^\infty$ ,  $f_n = T(f_{n-1})$ ,  $n = 1, 2, \dots$  if

$$\|T(v) - T(w)\| \leq c\|v - w\|$$

for all  $v, w \in C(I)$  for some  $c < 1$ . Then there is a unique  $v \in C(I)$  such that  $v = T(v)$ .  
This is Banach's fixed point theorem.

**Definition** (Green's function). Our example:

$$L = \left(\frac{d}{dx}\right)^2 + 1$$

differential operator. Boundary conditions

$$f(0) = f'(0) = 0.$$

Then

$$f(x) = \int_0^1 g(x, t)h(t) dt$$

is a solution to

$$\begin{cases} f'' + f &= h, \\ f(0) = f'(0) &= 0 \end{cases}$$

**Definition** (real vector space). We say that  $E$  is a real vector space if it is a non-empty set with the operations

**addition**  $E \times E \rightarrow E$ ,  $(x, y) \mapsto x + y$

**mult. with scalar**  $\mathbb{R} \times E \rightarrow E, \quad (\lambda, x) \mapsto \lambda x$

satisfying the axioms:

- (1)  $x + y = y + x, \quad \text{for all } x, y \in E$
- (2)  $x + (y + z) = (x + y) + z, \quad \text{for all } x, y, z \in E$
- (3) For all  $x, y \in E$  there exists  $z \in E$  such that  $x + z = y$
- (4)  $\alpha(\beta x) = (\alpha \cdot \beta)x, \quad \text{for all } \alpha, \beta \in \mathbb{R}, x \in E$
- (5)  $\alpha(x + y) = \alpha x + \alpha y, \quad \text{for all } \alpha \in \mathbb{R}, x, y \in E$
- (6)  $(\alpha + \beta)x = \alpha x + \beta x, \quad \text{for all } \alpha, \beta \in \mathbb{R}, x \in E$
- (7)  $1 \cdot x = x, \quad \text{for all } x \in E.$

**Remark.**  $E$  is a complex vector space if all  $\mathbb{R}$  in the definition above are replaced by  $\mathbb{C}$ .

**Remark.** (1)

$$\exists! 0 \in E : \quad x + 0 = x \quad \text{for all } x \in E.$$

since: Fix  $x \in E$ , by (3),  $\exists 0_x$  such that  $0_x + x = x$ .

Fix  $y \in E$ . We want to show that  $y + 0_y = y$ . By (3), there exists  $z \in E$  such that  $x + z = y$ . So

$$\begin{aligned}
 y + 0_x &= (x + z) + 0_x \\
 &\stackrel{(1)}{=} (z + x) + 0_x \\
 &\stackrel{(2)}{=} z + (x + 0_x) \\
 &= z + x \\
 &\stackrel{(1)}{=} x + z \\
 &= y.
 \end{aligned}$$

Assume  $x + 0_1 = x, x + 0_2 = x$  for all  $x \in E$ . We want to show  $0_1 = 0_2$ :

$$0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2$$

(2)

$$\forall x \in E : \exists! -x \in E : x + (-x) = 0$$

proof: exercise.

(3)

$$\begin{aligned}
 0x &= 0 \quad \text{for all } x \in E \\
 (-1)x &= -x \quad \text{for all } x \in E
 \end{aligned}$$

**Examples** (Examples of real vector spaces). 1)  $\mathbb{R}$  with standard addition and mult. by scalar.

2)  $\mathbb{R}^n$ ,  $n = 2, 3, \dots$

**addition**  $(x_1, x_2, \dots) + (y_1, y_2, \dots) = (x_1 + y_1, x_2 + y_2, \dots)$

**mult.**  $\lambda(x_1, x_2, \dots) = (\lambda x_1, \lambda x_2, \dots)$

3)  $\mathbb{R}^\infty = \{(x_1, \dots, x_n, \dots) \mid x_n \in \mathbb{R}, n = 1, 2, \dots\}$

4)  $1 \leq p < \infty$ ,

$$l^p = \left\{ (x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty \mid \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty \right\}$$

with the same addition and mult. by scalar as in  $\mathbb{R}^\infty$ . We have to check:

$$(1) \quad x, y \in l^p \quad \Rightarrow \quad x + y \in l^p$$

$$(2) \quad x \in l^p, \lambda \in \mathbb{R} \quad \Rightarrow \quad \lambda x \in l^p$$

For (1) we assume  $x = (x_1, \dots, x_n, \dots)$  and  $y = (y_1, \dots, y_n, \dots)$ .

$$x \in l^p \quad \Rightarrow \quad \sum_{n=1}^{\infty} |x_n|^p < \infty$$

$$y \in l^p \quad \Rightarrow \quad \sum_{n=1}^{\infty} |y_n|^p < \infty$$

$$\Rightarrow \quad x + y = (x_1 + y_1, \dots) \stackrel{?}{\in} l^p?$$

$$\Rightarrow \sum_{n=1}^{\infty} |x_n + y_n|^p \leq \{ |x_n + y_n| \leq |x_n| + |y_n| \leq 2 \max\{|x_n|, |y_n|\} \}$$

$$\{ |x_n + y_n|^p \leq 2^p (|x_n|^p + |y_n|^p) \}$$

$$\leq \sum_{n=1}^{\infty} 2^p (|x_n|^p + |y_n|^p)$$

$$= \underbrace{2^p \sum_{n=1}^{\infty} |x_n|^p}_{< \infty} + \underbrace{2^p \sum_{n=1}^{\infty} |y_n|^p}_{< \infty} < \infty$$

and

$$\sum_{n=1}^{\infty} |\lambda x_n|^p = \sum_{n=1}^{\infty} |\lambda|^p \cdot |x_n|^p = |\lambda|^p \sum_{n=1}^{\infty} |x_n|^p < \infty$$

5) function spaces, say real-valued functions on  $I$ .

**addition:**  $(f + g)(x) = f(x) + g(x)$ ,  $x \in I$

**mult. by scalar:**  $(\lambda f)(x) = \lambda f(x)$  for functions  $f$  and  $g$

- 6)  $C(I)$  : addition and mult. by scalar as in (5).  
 $f, g$  continuous in  $I$  implies that  $f + g$  is continuous in  $I$ .  
 Also if  $f$  is continuous and  $\lambda \in \mathbb{R}$  then  $(\lambda f)$  is continuous in  $I$ .
- 7)  $P(I) =$  polynomials in  $I$ .
- 8)  $P_k(I) =$  polynomials of degree at most  $k$  in  $I$ .

**Theorem** (Hölder's inequality). Assume  $1 < p < \infty$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . Let  $(x_1, \dots, x_n, \dots)$  and  $(y_1, y_2, \dots, y_n, \dots)$  be sequences of complex numbers. Then

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{n=1}^{\infty} |y_n|^q \right)^{\frac{1}{q}}$$

Remark there the LHS can be infinity, but the RHS can also be infinity.

**proof. Step 1** We're going to proof

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \text{for all } a, b > 0$$

$$\int_0^a x^{p-1} dx = \frac{a^p}{p}$$

Note  $y = x^{p-1}$  gives

$$x = y^{\frac{1}{p-1}} = y^{\frac{1}{\frac{1}{1-\frac{1}{q}}-1}} = y^{\frac{1}{\frac{q}{q-1}-1}} = y^{q-1}$$

so

$$\int_0^b y^{q-1} dy = \frac{b^q}{q}$$

We get

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

(You also get condition for  $=$ )

**Step 2** It is enough to consider the cases LHS  $> 0$  and RHS  $< \infty$ . There consists integer  $N$  such that

$$0 < \sum_{n=1}^N |x_n|^p, \sum_{n=1}^N |y_n|^q < \infty$$

Set

$$a = \frac{|x_k|}{\left( \sum_{n=1}^N |x_n|^p \right)^{\frac{1}{p}}}, \quad k = 1, 2, \dots, N,$$

$$b = \frac{|y_k|}{\left( \sum_{n=1}^N |y_n|^q \right)^{\frac{1}{q}}}, \quad k = 1, 2, \dots, N.$$

Insert into

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

$$\frac{|x_k y_k|}{\left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^N |y_n|^q\right)^{\frac{1}{q}}} \leq \frac{|x_k|^p}{p \sum_{n=1}^N |x_n|^p} + \frac{|y_k|^q}{q \sum_{n=1}^N |y_n|^q}, \quad k = 1, 2, \dots, N.$$

We sum over  $k$  from 1 to  $N$ .

$$\sum_{k=1}^N |x_k y_k| \leq \left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{n=1}^N |y_n|^q\right)^{\frac{1}{q}}$$

Let  $N \rightarrow \infty$ . First in RHS and then in LHS.

□

**Theorem** (Minkowski's inequality). Assume  $1 \leq p < \infty$ . and  $X, Y \in l^p$ . Then

$$\|X + Y\|_{l^p} \leq \|X\|_{l^p} + \|Y\|_{l^p}$$

**proof.**  $p = 1$

$$\begin{aligned} \|X + Y\|_{l^1} &= \|(x_1, x_2, \dots, x_n, \dots) + (y_1, y_2, \dots, y_n, \dots)\|_{l^1} \\ &= \|(x_1 + y_1, \dots, x_n + y_n, \dots)\|_{l^1} \\ &= \sum_{n=1}^{\infty} |x_n + y_n| \\ &\leq \sum_{n=1}^{\infty} (|x_n| + |y_n|) \\ &= \sum_{n=1}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| \\ &= \|X\|_{l^1} + \|Y\|_{l^1} \end{aligned}$$

$1 < p < \infty$

$$\begin{aligned} \|X + Y\|_{l^p}^p &= \sum_{n=1}^{\infty} |x_n + y_n|^p \\ &= \sum_{n=1}^{\infty} |x_n + y_n| |x_n + y_n|^{p-1} \\ &\leq \sum_{n=1}^{\infty} |x_n| |x_n + y_n|^{p-1} + \sum_{n=1}^{\infty} |y_n| |x_n + y_n|^{p-1}. \end{aligned}$$



Use Hölder to get

$$\begin{aligned}
 \sum_{n=1}^{\infty} |x_n| |x_n + y_n|^{p-1} &\leq \underbrace{\left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}}_{=\|X\|_{l^p}} \cdot \left( \sum_{n=1}^{\infty} |x_n + y_n|^{(p-1)q} \right)^{\frac{1}{q}} \\
 &= \left\{ (p-1)q = (p-1) \frac{1}{1 - \frac{1}{p}} = p \right\} \\
 &= \|X\|_{l^p} \left( \sum_{n=1}^{\infty} |x_n + y_n|^p \right)^{\frac{1}{q}}.
 \end{aligned}$$

We have

$$\|X + Y\|_{l^p}^p \leq (\|X\|_{l^p} + \|Y\|_{l^p}) \|X + Y\|_{l^p}^{\frac{p}{q}}$$

If  $\|X + Y\|_{l^p} \neq 0$  then

$$\|X + Y\|_{l^p}^{p - \frac{p}{q}} \leq \|X\|_{l^p} + \|Y\|_{l^p}$$

there

$$p - \frac{p}{q} = p(1 - \frac{1}{q}) = p \frac{1}{p} = 1.$$

□

**Remark.**  $f \in C([0, 1])$  then for  $1 \leq p < \infty$

$$\|f\|_{L^p} = \left( \int_0^1 |f(t)|^p dt \right)^{\frac{1}{p}}.$$

Claim:

$$\|fg\|_{L^1} = \int_0^1 |f(t) \cdot g(t)| dt \leq \|f\|_{L^p} \cdot \|g\|_{L^q}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ . Also we have

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

This is proven with the same technique as we used for  $l^p$ .  $\sum_{n=1}^{\infty}$  is replaced by  $\int_0^1 dt$ .  
 $E$  real/complex vector space.  $x_1, \dots, x_n \in E$ ,  $\lambda_1, \dots, \lambda_n$  scalar. We say that

$$\lambda_1 x_1, \dots, \lambda_n x_n$$

is a linear combination of  $x_1, \dots, x_n$ . We say that  $x_1, \dots, x_n$  are linear independent if

$$\alpha_1 x_1 + \dots + \alpha_n x_n = 0 \quad \Rightarrow \quad \alpha_1 = \dots = \alpha_n = 0.$$

If  $A \subset E$ , we say that  $A$  is linear independant if every linear combination of vectors in  $A$  is linear independant.

**Examples.** (1) Set  $E = P([0, 1])$  and  $A = \{p_k \mid p_k(x) = x^k, x \in [0, 1], k = 0, 1, \dots\}$ .  $A$  is linear independent since:  
consider

$$\alpha_0 p_0 + \alpha_1 p_1 + \dots + \alpha_n p_n = 0$$

i.e.

$$\alpha_0 p_0(x) + \alpha_1 p_1(x) + \dots + \alpha_n p_n(x) = 0(x), \quad x \in [0, 1]$$

i.e.

$$\alpha_0 + \alpha_1 x + \dots + \alpha_n x^n = 0, \quad x \in [0, 1]$$

If  $x = 0$  then  $\alpha_0 = 0$

$$\alpha_1 x + \dots + \alpha_n x^n = 0, \quad x \in [0, 1].$$

Differentiate

$$\alpha_1 + 2\alpha_2 x + \dots + n\alpha_n x^{n-1} = 0$$

gives  $\alpha_1 = 0$ . Continue and get

$$\alpha_0 = \alpha_1 = \dots = \alpha_n = 0.$$

Set  $B \subset E$  where

$\text{span } B = \{\text{set of all linear combinations of elements in } B\}$

$$= \left\{ \sum_{k=1}^n \lambda_k x_k \mid x_k \in B, \lambda_k \in \mathbb{R}, k = 1, 2, \dots, n \text{ where } n \text{ is a positive integer} \right\}$$

**Remark.**

$$\sum_{k=1}^n \lambda_k x_k \in E$$

$$\sum_{k=1}^{\infty} \lambda_k x_k \text{ has no meaning}$$

$C \subset E$  is called a basis for  $E$  if

1)  $C$  linear independent.

2)  $\text{span } C = E$

continue of the example above:

Claim:  $A$  is a basis for  $E$ .

(2) Set  $E = l^2$  and

$$A = \{X_k \mid k = 1, 2, \dots\}$$

$$X_k = (0, 0, \dots, 0, 1, 0, 0, \dots)$$

Claim:  $A$  is linear independent since

$$\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_n X_n = 0$$

Here

$$\alpha_1 X_1 = (\alpha_1, 0, 0, \dots), \quad \text{etc}$$

and

$$0 = (0, 0, \dots)$$

So

$$(\alpha_1, \alpha_2, \dots, \alpha_n, 0, \dots) = (0, 0, \dots)$$

So  $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ .

Question: Is  $A$  a basis for  $l^2$ ?

We note: If  $X \in \text{span } A$  then

$$X = (x_1, x_2, \dots, x_n, 0, 0, \dots)$$

for some positive integer  $n$ , i.e.  $X$  has only finitely many nonzero positions.

Consider:

$$X := (1, \frac{1}{2}, \dots, \frac{1}{n}, \dots)$$

$$\|X\|_{l^2} = \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} < \infty$$

So  $X \in l^2 \setminus \text{span } A$ .

**Remark.** Every vector space has a basis (if we are allowed to use Axiom of Choice/ Zorn's lemma).

Basis = vector space basis = Hamel basis

Assume  $x_1, \dots, x_n$  is a basis for  $E$ . Then every basis for  $E$  must contain  $n$  different elements.

$$n = \dim E$$

is well-defined. (System of linear equations, homogeneous with more unknowns than equations. Then there exists a nontrivial solution.)

**Definition (norm).**  $E$  vector space. We say that  $\|\cdot\| : E \rightarrow [0, \infty)$  is a norm on  $E$  if

- 1)  $\|x\| = 0 \quad \Rightarrow \quad x = 0$
- 2)  $\|\lambda x\| = |\lambda| \|x\| \quad \text{for all } x \in E, \lambda \in \mathbb{R}$
- 3)  $\|x + y\| \leq \|x\| + \|y\| \quad \text{for all } x, y \in E$

**Remark.**

$$\|0\| = \|0 \cdot 0\| = \underbrace{|0|}_{=0} \|0\| = 0$$

**Examples.** (1)  $1 < p < \infty$  and

$$\|X\|_{l^p} = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

is a norm on  $l^p$ . Check 1), 2) and 3) above:

1)

$$0 = \|X\|_{l^p} = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

It follows

$$\begin{aligned} x_n &= 0, \quad n = 1, 2, \dots \\ \Rightarrow X &= (x_1, x_2, \dots) = (0, 0, \dots) = 0 \end{aligned}$$

2)

$$\|\lambda X\|_{l^p} = \left( \sum_{n=1}^{\infty} |\lambda x_n|^p \right)^{\frac{1}{p}} = \left( |\lambda|^p \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} = |\lambda| \|X\|_{l^p}$$

3)

$$\|X + Y\|_{l^p} \leq \{\text{Minkowski's inequality}\} \leq \|X\|_{l^p} + \|Y\|_{l^p}$$

(2)  $E = C([0, 1])$  and  $f \in E$

$$\|f\| = \max_{t \in [0, 1]} |f(t)| \in [0, \infty)$$

Check the axioms above

1) If  $\|f\| = 0$  it follows

$$|f(t)| = 0 \text{ for all } t \in [0, 1], \quad \Rightarrow \quad f = 0$$

2)

$$\|\lambda f\| = \max_{t \in [0, 1]} \underbrace{|(\lambda f)(t)|}_{|\lambda| |f(t)|} = |\lambda| \max_{t \in [0, 1]} |f(t)| = |\lambda| \|f\|$$

3)

$$\|f + g\| = \max_{t \in [0, 1]} \underbrace{|(f + g)(t)|}_{|f(t) + g(t)|} = \max_{t \in [0, 1]} (|f(t)| + |g(t)|) \leq \max_{t \in [0, 1]} |f(t)| + \max_{t \in [0, 1]} |g(t)| = \|f\| + \|g\|$$

(3)  $E = C([0, 1])$  and  $f \in E$ .

$$\|f\|_{L^1} = \int_0^1 |f(t)| dt$$

defines also a norm on  $E$ .

3)

$$\begin{aligned} \|f + g\|_{L^1} &= \int_0^1 \underbrace{|(f + g)(t)|}_{|f(t) + g(t)|} dt \\ &\leq \int_0^1 (|f(t)| + |g(t)|) dt \\ &= \int_0^1 |f(t)| dt + \int_0^1 |g(t)| dt \\ &= \|f\|_{L^1} + \|g\|_{L^1} \end{aligned}$$

2)

$$\|\lambda f\| = \int_0^1 \underbrace{|(\lambda f)(t)|}_{=|\lambda||f(t)|} dt = |\lambda| \|f\|_{L^1}$$

1)

$$0 = \|f\|_{L^1} = \int_0^1 |f(t)| dt$$

This implies  $f(t) = 0$  for  $t \in [0, 1]$  since  $f$  is continuous! i.e.  $f = 0$

**Theorem** (equivalent norm).  $E$  vector space with norms  $\|\cdot\|$  and  $\|\cdot\|_*$ . We say that  $\|\cdot\|$  and  $\|\cdot\|_*$  are equivalent if there exists  $\alpha, \beta > 0$  such that

$$\alpha \|x\|_* \leq \|x\| \leq \beta \|x\|_* \quad \text{for all } x \in E.$$

**Example.**

$E = C([0, 1])$ . Choose  $y = f(t)$  and  $y = |f(t)|$

$$\|f\| = \max_{t \in [0, 1]} |f(t)|, \quad \|f\|_* = \|f\|_{L^1} = \text{area}.$$

**Question:** Are these norms equivalent?

**Claim**  $f \in C([0, 1])$

$$\|f\|_* = \int_0^1 \underbrace{|f(t)|}_{\leq \|f\|} dt \leq \|f\|$$

Choose  $f_n(t)$  such that

$$\|f_n\| = 1, \quad \|f_n\|_* = \frac{1}{2n}$$

So

$$\frac{\|f_n\|_*}{\|f_n\|} = \frac{1}{2n} \rightarrow 0 \quad n \rightarrow \infty$$

The norms are not equivalent! Answer: NO !

**Theorem .**  $E$  vector space with  $\dim E < \infty$ .  
 $\Rightarrow$  All norms on  $E$  are equivalent.

**proof.** Assume  $n = \dim E$  with a positive integer  $n$ . Let  $x_1, x_2, \dots, x_n$  be a basis for  $E$ . For every  $x \in E$

$$x = \alpha_1(x)x_1 + \dots + \alpha_n(x)x_n$$

where  $\alpha_1(x), \dots, \alpha_n(x)$  unique. Set

$$\|x\|_* = |\alpha_1(x)| + \dots + |\alpha_n(x)|, \quad x \in E$$

Claim:  $\|\cdot\|_*$  defines a norm on  $E$  (easy proof)

Fix an arbitrary norm  $\|\cdot\|$  on  $E$ .

Claim:  $\|\cdot\|_*$  and  $\|\cdot\|$  are equivalent.

Note for  $x \in E$

$$\begin{aligned} \|x\| &= \|\alpha_1(x)x_1 + \dots + \alpha_n(x)x_n\| \\ &\leq |\alpha_1(x)|\|x_1\| + \dots + |\alpha_n(x)|\|x_n\| \\ &\leq \max_{k=1,2,\dots,n} \|x_k\| \underbrace{(|\alpha_1(x)| + \dots + |\alpha_n(x)|)}_{=\|x\|_*} \end{aligned}$$

Set  $\beta = \max_{k=1,2,\dots,n} \|x_k\|$ . Then

$$\|x\| \leq \beta \|x\|_* \quad \text{for all } x \in E.$$

Remains to prove: There exists  $\alpha > 0$  such that

$$\alpha \|x\|_* \leq \|x\| \quad \text{for all } x \in E \quad (*)$$

Let  $E$  be a vector space with norm  $\|\cdot\|$  and  $(v_m)_{m=1}^\infty$  a sequence in  $E$ . We say that  $(v_m)_{m=1}^\infty$  converges in  $(E, \|\cdot\|)$  if there exists  $v \in E$  such that  $\|v_m - v\| \rightarrow 0$  for  $n \rightarrow \infty$ .

Notation:  $v_m \rightarrow v$  in  $(E, \|\cdot\|)$ .

Note: If we have  $\|\cdot\|$  and  $\|\cdot\|_*$  are equivalent, then

$$v_n \rightarrow v \text{ in } (E, \|\cdot\|) \quad \Leftrightarrow \quad v_n \rightarrow v \text{ in } (E, \|\cdot\|_*)$$

Back to (\*): Argue by contradiction.

Assume there is no  $\alpha > 0$  such that

$$\alpha \|x\|_* \leq \|x\| \quad \text{for all } x \in E$$

For  $k = 1, 2, 3, \dots$  there are  $y_k \in E$  such that

$$\frac{1}{k} \|y_k\|_* > \|y_k\|. \quad (**)$$

We have

$$y_k = \alpha_1^{(k)} x_1 + \dots + \alpha_n^{(k)} x_n$$

where  $\alpha_1^{(k)}, \dots, \alpha_n^{(k)}$  are unique scalars and  $k = 1, 2, \dots$

(\*\*) implies that

$$k \|y_k\| < |\alpha_1^{(k)}| + \dots + |\alpha_n^{(k)}|$$

WLOG we can assume  $|\alpha_1^{(k)}| + \dots + |\alpha_n^{(k)}| = 1$ . ( If not consider

$$\begin{aligned} \lambda z &= \lambda(\alpha_1(z)x_1 + \dots + \alpha_n(z)x_n) \\ &= (\lambda\alpha_1(z))x_1 + \dots + (\lambda\alpha_n(z))x_n \\ &= \alpha_1(\lambda z)x_1 + \dots + \alpha_n(\lambda z)x_n \end{aligned}$$

We have

$$\alpha_k(\lambda z) = \lambda \alpha_k(z), \quad k = 1, 2, \dots, n)$$

We have

$$k \|y_k\| < 1 \quad k = 1, 2, \dots$$

which implies  $y_k \rightarrow 0$  in  $(E, \|\cdot\|)$ .

IF:

$$\begin{aligned} \alpha_1^{(k)} &\rightarrow \bar{\alpha}_1 \\ \alpha_2^{(k)} &\rightarrow \bar{\alpha}_2 \\ &\vdots \\ \alpha_n^{(k)} &\rightarrow \bar{\alpha}_n \end{aligned}$$

for  $k \rightarrow \infty$ . Then set

$$\bar{y} = \bar{\alpha}_1 x_1 + \dots + \bar{\alpha}_n x_n$$

and get

$$\begin{aligned} \|y_k - \bar{y}\| &= \|(\alpha_1^{(k)} - \bar{\alpha}_1)x_1 + \dots + (\alpha_n^{(k)} - \bar{\alpha}_n)x_n\| \\ &\leq \underbrace{|\alpha_1^{(k)} - \bar{\alpha}_1|}_{\rightarrow 0} \underbrace{\|x_1\|}_{< \infty} + \dots + \underbrace{|\alpha_n^{(k)} - \bar{\alpha}_n|}_{\rightarrow 0} \underbrace{\|x_n\|}_{< \infty} \rightarrow 0, \quad k \rightarrow \infty \\ \|\bar{y}\| &= \|\bar{y} - y_k + y_k\| \leq \underbrace{\|\bar{y} - y_k\|}_{\rightarrow 0} + \underbrace{\|y_k\|}_{\rightarrow 0} \rightarrow 0, \quad k \rightarrow \infty \end{aligned}$$

So  $\|\bar{y}\| = 0$  hence  $\bar{y} = 0$ . But

$$|\bar{\alpha}_1| + |\bar{\alpha}_2| + \dots + |\bar{\alpha}_n| = 1.$$

This contradicts  $x_1, \dots, x_n$  is a basis.

We have for  $k = 1, 2, \dots$  the vector  $(\alpha_1^{(k)}, \alpha_2^{(k)}, \dots, \alpha_n^{(k)})$  where

$$|\alpha_1^{(k)}| + \dots + |\alpha_n^{(k)}| = 1$$

We focus on the first one and we have

$$|\alpha_1^{(k)}| \leq 1, \quad k = 1, 2, \dots$$

By Bolzano-Weierstraß then there exists a converging subsequence  $(\alpha_{1,1}^{(k)})_{k=1}^\infty$  of  $(\alpha_1^{(k)})_{k=1}^\infty$ . Set

$$\bar{\alpha}_1 = \lim_{k \rightarrow \infty} \alpha_{1,1}^{(k)}$$

consider

$$(\alpha_{1,1}^{(k)}, \alpha_{2,1}^{(k)}, \dots, \alpha_{n,1}^{(k)}), \quad k = 1, 2, \dots$$

We have

$$|\alpha_{2,1}^{(k)}| \leq 1, \quad k = 1, 2, \dots$$

Bolzano-Weierstraß implies that there exists a converging subsequence  $(\alpha_{2,2}^{(k)})_{k=1}^\infty$  of  $(\alpha_{2,1}^{(k)})_{k=1}^\infty$ . Set

$$\bar{\alpha}_2 = \lim_{k \rightarrow \infty} \alpha_{2,2}^{(k)}$$

□

**Definition** (normed space). Let  $E$  be a vector space over  $\mathbb{R}$  or  $\mathbb{C}$ .  $\|\cdot\| : E \rightarrow \mathbb{R}$  a norm on  $E$  if

- (i)  $\|\cdot\| > 0$  for any  $x \in E \setminus \{0\}$
- (ii)  $\|\lambda x\| = |\lambda| \|x\|$  for any  $\lambda \in \mathbb{C}, x \in E$ .
- (iii)  $\|x + y\| \leq \|x\| + \|y\|$  for any  $x, y \in E$ .

Obs.  $\|x\| = 0$  if  $x = 0$ .  $(E, \|\cdot\|)$  is called a normed space. A norm generates a distance function (metric)

$$L(x, y) := \|x - y\| \quad \text{for any } x, y \in E.$$

**Examples.** •  $\mathbb{R}^n$  with  $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$  is the euclidian norm.

•  $C([0, 1])$  continuous functions in  $[0, 1]$  with

$$L(f, g) = \|f - g\|_\infty := \max_{x \in [0, 1]} |f(x) - g(x)|$$



**Definition (balls).** Let  $x \in E, r > 0$ . Define

$$\begin{aligned} B(x, r) &:= \{y \in E \mid \|x - y\| < r\} && \text{open ball} \\ \bar{B}(x, r) &:= \{y \in E \mid \|x - y\| \leq r\} && \text{closed ball} \end{aligned}$$

**Definition (open/closed).** A subset  $A \subset E$  of a normed space  $(E, \|\cdot\|)$  is called open if any point  $x$  of  $A$  is inner, i.e

$$\exists r > 0 : B(x, r) \subset A.$$

It is called closed if the complement  $E \setminus A$  is open.

**Remark.** • open balls are open sets.

- closed balls are closed.
- $(C([0, 1]), \|\cdot\|_\infty)$  with  $\|f\|_\infty = \max_{x \in [0, 1]} |f(x)|$ .

$$A := \{g \in C([0, 1]) \mid f(x) < g(x), \forall x \in [0, 1]\}$$

is an open set  $C([0, 1])$ .

$$B := \{g \in C([0, 1]) \mid f(x) \leq g(x), \forall x \in [0, 1]\}$$

is a closed set.

### Properties

- Any union of open sets is an open set.
- Any finite intersection of open sets is open.
- $\emptyset, E$  are both closed and open.
- Normed spaces are topological spaces.

**Definition (convergence in normed spaces).** Let  $(E, \|\cdot\|)$  be a normed space  $\{x_n\}_n \subset E$ . We say that  $x_n$  converges to  $x \in E$  if

$$\|x_n - x\| \rightarrow 0, \quad n \rightarrow \infty$$

One can define open and closed using the definition of convergence:

**Satz.**  $A \subseteq E$  is closed if any convergent sequence in  $A$  has a limit in  $A$ , i.e

$$\begin{matrix} x_n \rightarrow x \\ \text{for } n \rightarrow \infty \\ x_n \in A \end{matrix} \Rightarrow x \in A$$

**proof.**  $\Rightarrow$ : Assume that  $A$  is closed and  $x_n \rightarrow x$ .  $x_n \in A$ , but  $x_n \notin A$ . (try to get a contradiction).

$A$  is closed  $\Rightarrow E \setminus A$  is open and hence  $\exists r > 0$  such that

$$B(x, r) \subset E \setminus A.$$

Hence  $\|x_n - x\| \geq r$  for any  $n$ . This is a contradiction because in that case  $x_n \not\rightarrow x$

$\Leftarrow$ : Assume that for any sequence  $\{x_n\} \subset A$  such that  $x_n \rightarrow x$  we have  $x \in A$ . We try to get a contradiction and assume that  $A$  is not closed. Hence  $E \setminus A$  is not open and therefore  $\exists x \in E \setminus A$  which is not inner.

$$\Rightarrow \quad \forall B(x, \frac{1}{n}) \text{ contains points outside } E \setminus A$$

i.e.

$$\exists x_n \in B(x, \frac{1}{n}), x_n \in A.$$

We get a sequence  $\{x_n\} \subset A$  such that

$$\|x_n - x\| < \frac{1}{n} \quad \Rightarrow \quad x_n \rightarrow x$$

This is a contradiction

□

**Definition (closure).**  $A \subset E$ . The closure of  $A$  is the minimal closed subset containing  $A$ . We write  $\bar{A}$ .

**Proposition .**  $\bar{A}$  is the set of all limit points of  $A$  which means

$$\bar{A} := \{x \in E \mid \text{there exists } \{x_n\} \subseteq A \text{ such that } x_n \rightarrow x\}$$

**proof.** exercise.

□

**Definition (dense).**  $A \subset E$  is dense in  $E$  if

$$\bar{A} = E.$$

**Remark.** This definition of dense is equivalent to the following definition:

$$\forall x \in E, \forall \varepsilon > 0 \exists y \in A \text{ such that } \|x - y\| < \varepsilon.$$

**Examples.** 1)  $\mathbb{Q} \subseteq \mathbb{R}$  with  $|\cdot|$  usual absolute value function.  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .

2)  $C([a, b])$ . The Weierstrass theorem says that the set of all polynomials are dense in  $(C([a, b]), \|\cdot\|_\infty)$ :

$$\forall f \in C([a, b]), \forall \varepsilon > 0 \exists p - \text{polynomial such that } \max_{x \in [a, b]} |f(x) - p(x)| < \varepsilon.$$

Another example is  $(C_0, \|\cdot\|_\infty)$  where

$$C_0 = \{x = (x_1, x_2, \dots) \mid x_k \rightarrow 0 \text{ as } k \rightarrow \infty\}$$

$$\|x\|_\infty = \sup_i |x_i|$$

$(C_0, \|\cdot\|_\infty)$  is a normed space.

$$C_F = \{x = (x_1, x_2, \dots) \mid \text{only a finite number of } x_i \neq 0\} \subset C_0$$

**Satz.**  $C_F$  is dense in  $C_0$

**proof.**

$$\forall x \in C_0 \forall \varepsilon > 0 \text{ must find } y \in C_F \text{ such that } \|y - x\|_\infty < \varepsilon.$$

$$x \in C_0 \quad \Rightarrow \quad x_k \rightarrow 0 \text{ for } k \rightarrow \infty$$

$$\Rightarrow \quad \forall \varepsilon > 0 \exists K \text{ such that } |x_k| < \varepsilon \forall k \geq K$$

Let now  $y = (x_1, x_2, \dots, x_K, 0, \dots) \in C_F$ . Then

$$\|x - y\|_\infty = \|(0, 0, \dots, 0, x_{K+1}, x_{K+2}, \dots)\|_\infty = \sup_{k > K} |x_k| < \varepsilon$$

□

**Definition** (separable). A normed space  $(E, \|\cdot\|)$  is called separable if it contains a countable dense subset.

**Examples.** •  $(\mathbb{R}, |\cdot|)$  is separable as  $\mathbb{Q}$  is countable and dense in  $\mathbb{R}$ .

•  $(\mathbb{R}^n, \|\cdot\|_2)$  is separable,  $\mathbb{Q}^n$  is countable and dense in  $\mathbb{R}^n$ .

**Definition** (compact set). For a normed space  $(E, \|\cdot\|)$  is  $A \subset E$  a compact set if any sequence  $\{x_n\} \subset A$  has a subsequence convergent to an element  $x \in A$ .

**Example.** Any bounded and closed subset in  $\mathbb{R}, \mathbb{R}^n, \mathbb{C}^n$  is compact. A sequence  $\{x_n\}$  of a bounded set is bounded. From real Analysis one knows it has a subsequence that is convergent. If the subset is closed then the limit point is inside the set.

**Lemma .**  $S \subset \text{compact in } (E, \|\cdot\|)$  implies that  $S$  is closed and bounded. (Bounded means that  $S \subset B(0, R)$  for some  $R > 0$ )

**proof.** Let  $S$  be a compact subset of  $E$ . Assume that  $S$  is not bounded. Hence for any  $n > 0$  there exists points in  $S$  which are outside  $B(0, n)$ , i.e.

$$\exists x_n \in S : \|x_n\| > n.$$

Then  $\{x_n\}$  can not have a convergent subsequence as if  $x_{n_k} \rightarrow x$  then

$$n_k < \|x_{n_k}\| = \|x_{n_k} - x + x\| \leq \|x_{n_k} - x\| + \|x\| \rightarrow \|x\|$$

but  $n_k \rightarrow \infty$ . This is a contradiction, hence  $S$  must be bounded.

$S$  must be closed, because if  $x_n \rightarrow x$  then any subsequence converges to  $x$ . From the definition of compactness and uniqueness of the limit we have  $x \in S$ . □

**Remark.** In general,  $S$  bounded and closed doesn't imply that  $S$  is compact.

For instance let  $E = C([0, 1])$ . Then  $S = \{g \in C([0, 1]) : \|g\|_\infty \leq 1\}$  is closed and bounded, but not compact.

Take  $x_n(t) := t^n$ . Then  $x_n \in S$ .  $\{x_n\}$  does not have a subsequence convergent to a continuous function.

**Theorem .**  $(E, \|\cdot\|)$  normed space and  $\dim E < \infty$   
iff  $\{\forall A \subset E, A \text{ compact} \Leftrightarrow A \text{ is closed and bounded}\}$

**proof.**  $\Rightarrow$ : If  $\dim E < \infty$  then  $A$  is compact iff  $A$  is bounded and closed (exercise)

$\Leftarrow$ : Enough to prove the following:

If  $\dim E = \infty$  then the unit ball  $S = \{x \in E : \|x\| \leq 1\}$  is not compact.

**Lemma (Riesz's lemma).** If  $X$  is a proper closed subspace of a normed space  $(E, \|\cdot\|)$  then for every  $\varepsilon \in (0, 1)$  there exists an  $x_\varepsilon \in E$  with  $\|x_\varepsilon\| = 1$  such that

$$\|x_\varepsilon - x\| \geq \varepsilon \quad \forall x \in X.$$

**proof.** Let  $z \in E \setminus X$  ( $X$  proper and hence  $E \setminus X \neq \emptyset$ ). Set

$$d := \inf_{x \in X} \|z - x\|$$

As  $X$  is closed,  $d > 0$ , otherwise  $z$  is a limit point in  $E \setminus X$ . Fix  $\varepsilon \in (0, 1)$ . Then there exists  $x_0 \in X$  such that

$$d \leq \|z - x_0\| < \frac{d}{\varepsilon}.$$

Let  $x_\varepsilon := \frac{z-x_0}{\|z-x_0\|}$ ; We have  $\|x_\varepsilon\| = 1$  and

$$\begin{aligned}
 \|x - x_\varepsilon\| &= \left\| x - \frac{z - x_0}{\|z - x_0\|} \right\| \\
 &= \frac{\|x\|z - x_0\| - z + x_0\|}{\|z - x_0\|} \\
 &= \frac{\left\| \overbrace{x\|z - x_0\| + x_0 - z}^{\in X} \right\|}{\|z - x_0\|} \\
 &\geq \frac{d}{d}\varepsilon = \varepsilon
 \end{aligned}$$

□

Continue now proof of the theorem above:

Let  $x_1 \in S$ . Consider  $X = \text{span}\{x_1\}$  which is a proper closed subspace of  $E$ . Hence by Riesz's lemma exists  $x_2$  with  $\|x_2\| = 1$  such that

$$\|x_2 - x_1\| \geq \frac{1}{2}$$

and

$$\|x_2 - x\| \geq \frac{1}{2} \quad \forall x \in X.$$

Now consider  $\text{span}\{x_1, x_2\}$  which is a proper closed subspace of  $E$ . By Riesz's lemma follows

$$\exists x_3 \in E, \|x_3\| = 1 : \|x_3 - x_1\| \geq \frac{1}{2}, \|x_3 - x_2\| \geq \frac{1}{2}.$$

Continuing in the same fashion we get  $\{x_n\}$ ,  $\|x_n\| = 1$  such that

$$\|x_n - x_m\| \geq \frac{1}{2} \quad \forall n, m, n \neq m.$$

Clearly  $\{x_n\} \subset S$  has no convergent subsequence. Hence  $S$  is not compact. □

**Definition** (Cauchy sequence).  $(E, \|\cdot\|)$  normed space.  $\{x_n\} \subseteq E$  is called Cauchy if

$$\forall \varepsilon > 0 \exists N : \|x_n - x_m\| < \varepsilon \text{ for any } n, m \geq N.$$

**Example.**  $(C_F, \|\cdot\|_\infty)$ ,  $\|x\|_\infty = \sup_{k \in \mathbb{N}} |x_k|$  where  $x = (x_1, x_2, \dots)$ . Define

$$x_n = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, \dots)$$

Then  $\{x_n\}$  is Cauchy, as for  $n > m$

$$\begin{aligned}\|x_n - x_m\|_\infty &= \left\| (0, \dots, 0, \frac{1}{m+1}, \dots, \frac{1}{n}, 0, \dots) \right\|_\infty \\ &= \frac{1}{m+1}\end{aligned}$$

Observe that  $x_n$  is convergent in  $(C_0, \|\cdot\|_\infty)$

$$\underbrace{x_n}_{\in C_F} \rightarrow (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots) \in C_0 \setminus C_F$$

**Statement 1.3.** A convergent sequence is always a Cauchy sequence.

**Definition** (complete space). A normed vector space  $(E, \|\cdot\|)$  is called complete if any Cauchy sequence in  $E$  is convergent in  $E$ .

**Definition** (Banach space). A complete normed space is called Banach space.

**Examples.** •  $(\mathbb{R}, |\cdot|)$  is a Banach space.

- $(\mathbb{C}, |\cdot|)$  as well.
- $(l^2, \|\cdot\|_2)$  where

$$l^2 = \left\{ (x_1, x_2, \dots) \left| \sum_{i=1}^{\infty} |x_i|^2 < \infty, x_i \in \mathbb{C} \right. \right\}$$

and

$$\|(x_1, x_2, \dots)\|_2 = \left( \sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}$$

$(l^2, \|\cdot\|_2)$  is complete.

**proof.** Let  $x_n = (x_1^n, x_2^n, \dots)$  be a Cauchy sequence in  $l^2$ . We must show that it has a limit in  $l^2$ . We will do it in a few steps:

Step 1: Find a candidate for a limit  $a$

Step 2: Show that  $a \in l^2$ .

Step 3:  $\|x_n - a\|_2 \rightarrow 0$  as  $n \rightarrow \infty$ .

Step 1: Let

$$\begin{aligned} x_1 &= (x_1^1, x_2^1, \dots) \\ x_2 &= (x_1^2, x_2^2, \dots) \\ &\vdots \\ x_n &= (x_1^n, x_2^n, \dots) \end{aligned}$$

For each  $k$  consider sequence  $\{x_k^n\} \subset \mathbb{C}$  ( $k$ -th coordinates in each  $x_n$ ).  
Each sequence is Cauchy, as for all  $n, m \geq N$

$$|x_k^n - x_k^m| < \left( \sum_{k=1}^{\infty} |x_k^n - x_k^m|^2 \right)^{\frac{1}{2}} = \|x_n - x_m\|_2 < \varepsilon$$

As  $(\mathbb{C}, |\cdot|)$  is complete,  $\{x_k^n\}_n$  has a limit  $a_k \in \mathbb{C}$ . Candidate for limit of  $x_n$  is

$$a = (a_1, a_2, \dots, a_k, \dots).$$

Step 2: Write

$$a = \underbrace{x_n}_{\in l^2} - (x_n - a)$$

In order to show that  $a \in l^2$  it is enough to see that  $x_n - a \in l^2$  for some  $n$ .  
 $\{x_n\}$  Cauchy implies

$$\forall \varepsilon > 0 \exists N : \forall n, m \geq N : \|x_n - x_m\|_2 < \varepsilon.$$

Consider for some  $u > 0$

$$\sum_{i=1}^u |x_i^n - x_i^m|^2 \leq \sum_{i=1}^{\infty} |x_i^n - x_i^m|^2 = \|x_n - x_m\|_2^2 < \varepsilon^2$$

Let  $m \rightarrow \infty$ . We get

$$\sum_{i=1}^u |x_i^n - a_i|^2 \leq \varepsilon^2$$

This holds for any  $u \in \mathbb{N}$ . Hence for any  $n \geq N$

$$\underbrace{\sum_{i=1}^{\infty} |x_i^n - a_i|^2}_{= \|x_n - a\|_2^2} \leq \varepsilon^2.$$

Hence  $x_n - a \in l^2$  and moreover  $\|x_n - a\| \rightarrow 0$  as  $n \rightarrow \infty$ .

□

- $(C([a, b]), \|\cdot\|_{\infty})$  is a Banach space.

- $(l^p, \|\cdot\|_{l^p})$  for  $1 \leq p < \infty$  are all Banach spaces.
- $(C([a, b]), \|\cdot\|_2)$  with

$$\|f\|_2 = \left( \int |f(t)|^2 dt \right)^{\frac{1}{2}}$$

One can prove that  $(C([a, b]), \|\cdot\|_2)$  is not a Banach space.

**Exercise:**

$[a, b] = [0, 1]$  and

$$f_n(t) = \begin{cases} 0, & \text{falls } t < \frac{1}{2} - \frac{1}{n} \\ 1, & \text{falls } t > \frac{1}{2} \\ \text{continuous linear function} & \end{cases}$$

Show that  $\{f_n\}$  is Cauchy in  $C([0, 1], \|\cdot\|_2)$  but  $f_n \not\rightarrow f \in C([0, 1])$ .

**Definition** (Convergent and absolutely convergent series). A series  $\sum_{n=1}^{\infty} x_n$  in  $E$  is called convergent if  $\{\sum_{n=1}^m x_n\}_m$ , a sequence of partial sums, is convergent in  $E$ . If  $\sum_{n=1}^{\infty} \|x_n\| < \infty$  then we say that  $\sum_{n=1}^{\infty} x_n$  converges absolutely.

**Theorem .** A normed space  $E$  is complete iff every absolutely convergent series converges in  $E$ .

**proof.**  $\Rightarrow$ : Suppose  $X$  is complete and  $\sum_{n=1}^{\infty} \|x_n\| < \infty$ . Let

$$S_N := \sum_{n=1}^N x_n \in E.$$

For  $M > N$ :

$$\begin{aligned} \|S_N - S_M\| &= \left\| \sum_{n=N+1}^M x_n \right\| \\ &\leq \sum_{n=N+1}^M \|x_n\| \\ &\leq \sum_{n=N+1}^{\infty} \|x_n\| \rightarrow 0 \quad \text{as } N \rightarrow \infty \end{aligned}$$

Hence  $\{S_N\}$  is Cauchy. As  $E$  is complete,  $S_N$  has a limit in  $E$  i.e.  $\sum_{n=1}^{\infty} x_n$  converges in  $E$ .



$\Leftarrow$ : Assume that every absolutely convergent series is convergent in  $E$ . We want to see that  $E$  is complete.

Let  $\{x_n\}$  be a Cauchy sequence. We want to prove that  $\{x_n\}$  has a limit in  $E$ . We know that

$$\forall k \exists n_k : \|x_n - x_m\| < \frac{1}{2^k} \quad \forall n, m \geq n_k.$$

We can assume that  $\{n_k\}$  is an increasing sequence. Write

$$x_{n_k} = (x_{n_k} - x_{n_{k-1}}) + (x_{n_{k-1}} - x_{n_{k-2}}) + \dots + (x_{n_1} - \underbrace{x_{n_0}}_{=0}) = \sum_{l=1}^k (x_{n_l} - x_{n_{l-1}}).$$

$$\sum_{l=1}^{\infty} \|x_{n_l} - x_{n_{l-1}}\| \leq \sum_{l=1}^{\infty} \frac{1}{2^l} < \infty$$

Hence  $\sum_{l=1}^{\infty} (x_{n_l} - x_{n_{l-1}})$  is absolutely convergent. By assumption

$$\sum_{l=1}^{\infty} (x_{n_l} - x_{n_{l-1}})$$

is convergent in  $E$ . Hence the partial sums are convergent. Subsequence is convergent.  $\{x_{n_k}\}$  is convergent to some  $x \in E$ .

**Exercise:**

Show that the whole  $\{x_n\} \rightarrow x$ .

□

**Recall:**

converging sequences  $(x_n)_{n=1}^{\infty}$  in  $(E, \|\cdot\|)$ .  $\|x_n - x\| \rightarrow 0$  for  $n \rightarrow \infty$  for some  $x \in E$ . (Notation:  $x_n \rightarrow x$  in  $(E, \|\cdot\|)$ )

**Remark.** Assume  $x_n \rightarrow x$  in  $(E, \|\cdot\|)$ . Then

$$1) \|x_n\| \rightarrow \|x\| \text{ in } (E, \|\cdot\|).$$

$$2) \sup_n \|x_n\| < \infty.$$

because

1)

$$\|x_n\| \leq \|x_n - x\| + \|x\|$$

so

$$\|x_n\| - \|x\| \leq \|x_n - x\|$$

it follows

$$-(\|x_n\| - \|x\|) \leq \|x_n - x\|$$

So

$$\| \|x_n\| - \|x\| \| \leq \|x_n - x\| \rightarrow 0, \quad \text{for } n \rightarrow \infty$$

Cauchy sequence in  $(x_n)_{n=1}^\infty$  in  $(E, \|\cdot\|)$  if  $\|x_n - x_m\| \rightarrow 0$  for  $n, m \rightarrow \infty$ .

We obtain:  $(x_n)_{n=1}^\infty$  converges in  $(E, \|\cdot\|)$   $\Rightarrow$   $(x_n)_{n=1}^\infty$  Cauchy sequence in  $(E, \|\cdot\|)$ . ( $\Leftarrow$  in general). If  $\Leftarrow$  then we call  $(E, \|\cdot\|)$  a Banach space.

$\sum_{n=1}^\infty x_n$  converges in  $(E, \|\cdot\|)$  if  $\left(\sum_{n=1}^k x_n\right)_{k=1}^\infty$  converges in  $(E, \|\cdot\|)$ .

$\sum_{n=1}^\infty x_n$  converges absolutely in  $(E, \|\cdot\|)$  if  $\sum_{n=1}^\infty \|x_n\|$  converges  $(\mathbb{R}, \|\cdot\|)$ .

## 1.2 Mappings between normed spaces

**Definition .** Let  $(E_1, \|\cdot\|_1)$ ,  $(E_2, \|\cdot\|_2)$  be normed spaces.  $T : E_1 \rightarrow E_2$  (not necessarily linear) is called continuous at  $x_0 \in E_1$ , if

$$x_n \rightarrow x_0 \text{ in } (E_1, \|\cdot\|_1) \quad \Rightarrow \quad T(x_n) \rightarrow T(x_0) \text{ in } (E_2, \|\cdot\|_2)$$

$T$  is called continuous if it is continuous at  $x_0 \in E_1$  for all  $x_0 \in E_1$ . We say that  $T : E_1 \rightarrow E_2$  is linear if

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$$

for all scalars  $\lambda_1, \lambda_2$  and  $x_1, x_2 \in E_1$ .

$T : E_1 \rightarrow E_2$  linear is called bounded if there exists  $M > 0$  such that

$$\|T(x)\|_2 \leq M\|x\|_1 \quad \text{for all } x \in E_1.$$

If  $T$  is bounded linear  $E_1 \rightarrow E_2$  define

$$\|T\| = \|T\|_{E_1 \rightarrow E_2} := \inf\{M \geq 0 \mid \|T(x)\|_2 \leq M\|x\|_1 \text{ for all } x \in E_1\}$$

**Lemma .**

$$\|T\| = \sup_{\substack{x \in E_1 \\ x \neq 0}} \frac{\|T(x)\|_2}{\|x\|_1} = \sup_{\substack{x \in E_1 \\ \|x\|_1=1}} \|T(x)\|_2$$

**Proposition .** Assume  $T : E_1 \rightarrow E_2$  linear. Then all the following statements are equivalent:

- (1)  $T$  continuous at  $0 \in E_1$ .
- (2)  $T$  continuous at  $x_0 \in E_1$  for some  $x_0 \in E_1$ .

(3)  $T$  continuous at  $x_0 \in E_1$  for all  $x_0 \in E_1$ .

(4)  $T$  is bounded.

**proof.** (1)  $\Rightarrow$  (4): Assume  $T$  is continuous at  $0 \in E_1$ . i.e.

$$x_n \rightarrow 0 \text{ in } (E_1, \|\cdot\|_1) \quad \Rightarrow \quad T(x_n) \rightarrow T(\underbrace{0}_{\in E_1}) = \underbrace{0}_{\in E_2} \text{ in } (E_2, \|\cdot\|_2)$$

We want to prove that  $T$  is bounded. We search a  $M > 0$  such that

$$\|T(x)\|_2 \leq M\|x\|_1$$

We assume that this doesn't hold true.

For  $n = 1, 2, \dots$  there exists  $x_n \in E_1$  such that

$$\|T(x_n)\|_2 > n\|x_n\|_1.$$

Set for  $n = 1, 2, \dots$

$$z_n := \frac{1}{n\|x_n\|_1} x_n$$

(Note that  $\|x_n\|_1 > 0$ . Otherwise we would get a contradiction.)

Note

$$\|z_n\|_1 = \left\| \frac{1}{n\|x_n\|_1} \right\|_1 = \frac{1}{n\|x_n\|_1} \|x_n\|_1 = \frac{1}{n} \rightarrow 0, \quad \text{for } n \rightarrow \infty$$

We have  $z_n \rightarrow 0$  in  $(E_1, \|\cdot\|_1)$ . But

$$\|T(z_n)\|_2 = \left\| \frac{1}{n\|x_n\|_1} T(x_n) \right\|_2 = \frac{1}{n\|x_n\|_1} \|T(x_n)\|_2 > 1 \quad \text{for all } n.$$

Hence

$$T(z_n) \not\rightarrow 0 \quad \text{in } (E_2, \|\cdot\|_2).$$

This is a contradiction.

(1)  $\Leftarrow$  (4): Assume  $T$  is bounded. For some  $M > 0$

$$\|T(x)\|_2 \leq M\|x\|_1, \quad \text{for all } x \in E_1.$$

We need to show that  $T$  is continuous at  $0 \in E_1$ , i.e.

$$x_n \rightarrow 0 \text{ in } (E_1, \|\cdot\|_1) \quad \Rightarrow \quad T(x_n) \rightarrow T(0) = 0 \text{ in } (E_2, \|\cdot\|_2)$$

From

$$\|T(x_n)\|_2 \leq M\|x_n\|_1 \rightarrow 0$$

so

$$T(x_n) \rightarrow \underbrace{0}_{=T(0)} \text{ in } (E_2, \|\cdot\|_2).$$

□

**Examples.** (A)  $E_1 = E_2 = C([0, 1])$ ,  $\|\cdot\|_1 = \|\cdot\|_2 = \|\cdot\|_\infty =: \|\cdot\|$ , i.e.

$$\|f\| := \max_{x \in [0, 1]} |f(x)|.$$

$$T(f)(x) = \int_0^{1-x} \min(x, y) f(y) dy, \quad \text{for } f \in C([0, 1]), x \in [0, 1].$$

(1)  $T(f) \in C([0, 1])$  for  $f \in C([0, 1])$ ,

(2)  $T$  linear,

(3)  $T$  bounded,

(4) Calculate  $\|T\|$ .

**proof.** (1) Fix  $f \in C([0, 1])$  arbitrary and fix  $x \in [0, 1]$ . Show that  $T(f)$  is continuous at  $x$ . Consider a sequence  $(x_n)_{n=1}^\infty$  in  $[0, 1]$  such that  $x_n \rightarrow x$  in  $(\mathbb{R}, |\cdot|)$ .

To show  $T(f)(x_n) \rightarrow T(f)(x)$  in  $(\mathbb{R}, |\cdot|)$

$$\begin{aligned} |T(f)(x_n) - T(f)(x)| &= \{\text{assume that } x_n \leq x\} \\ &= \left| \int_0^{1-x_n} \min(x_n, y) f(y) dy - \int_0^{1-x} \min(x, y) f(y) dy \right| \\ &\leq \left| \int_0^{1-x} (\min(x_n, y) - \min(x, y)) f(y) dy \right| \\ &\quad + \left| \int_{1-x}^{1-x_n} \min(x_n, y) f(y) dy \right| \\ &\leq \underbrace{\int_0^{1-x} \underbrace{|\min(x_n, y) - \min(x, y)|}_{\leq |x_n - x|} \underbrace{|f(y)|}_{\leq \|f\|} dy}_{\leq |x_n - x| \|f\|} \\ &\quad + \underbrace{\int_{1-x}^{1-x_n} \underbrace{\min(x_n, y)}_{\leq 1} \underbrace{|f(y)|}_{\leq \|f\|} dy}_{0 \leq \dots \leq |x_n - x| \cdot \|f\|} \rightarrow 0, \quad \text{as } n \rightarrow \infty \end{aligned}$$

If  $x_n > x$  we get a similar calculation. Conclusion:

$$T(f)(x_n) \rightarrow T(f)(x) \text{ in } (\mathbb{R}, |\cdot|) \text{ as } n \rightarrow \infty.$$

(2) Fix  $f_1, f_2 \in C([0, 1])$  and  $\lambda_1, \lambda_2$  scalars. Then

$$\begin{aligned} T(\lambda_1 f_1 + \lambda_2 f_2)(x) &= \int_0^{1-x} \min(x, y) \underbrace{(\lambda_1 f_1 + \lambda_2 f_2)(y)}_{= \lambda_1 f_1(y) + \lambda_2 f_2(y)} dy \\ &= \lambda_1 \int_0^{1-x} \min(x, y) f_1(y) dy + \lambda_2 \int_0^{1-x} \min(x, y) f_2(y) dy \\ &= \lambda_1 T(f_1)(x) + \lambda_2 T(f_2)(x) \quad \text{for } x \in [0, 1] \end{aligned}$$

(3) Fix  $f \in C([0, 1])$ . For  $x \in [0, 1]$

$$\begin{aligned}
 |T(f)(x)| &= \left| \int_0^{1-x} \underbrace{\min(x, y) f(y)}_{\geq 0} dy \right| \\
 &\stackrel{(*_1)}{\leq} \int_0^{1-x} \min(x, y) \underbrace{|f(y)|}_{\leq \|f\|} dy \\
 &\stackrel{(*_2)}{\leq} \int_0^{1-x} \min(x, y) dy \|f\|
 \end{aligned}$$

Clearly

$$\max_{x \in [0, 1]} \int_0^{1-x} \min(x, y) dy \leq 1$$

This gives:

$$\|T(f)\| = \max_{x \in [0, 1]} |T(f)(x)| \leq 1 \cdot \|f\|, \quad \text{for all } f \in C([0, 1]).$$

Conclusion:  $T$  is bounded with ( $M = 1$ )

- (4) Consider the inequality above.  $(*_1)$  is an equality if  $f$  has a constant sign.  $(*_2)$  is an equality if  $f$  is a constant function. So we have to calculate

$$\int_0^{1-x} \min(x, y) dy \quad \text{for } x \in [0, 1].$$

**case 1:**  $1 - x \leq x$  i.e.  $\frac{1}{2} \leq x$  and we get

$$\begin{aligned}
 \int_0^{1-x} \underbrace{\min(x, y)}_{=y} dy &= \left[ \frac{1}{2} y^2 \right]_0^{1-x} \\
 &= \frac{1}{2} (1-x)^2
 \end{aligned}$$

**case 2:**  $x < 1 - x$  i.e.  $x < \frac{1}{2}$  and we get

$$\begin{aligned}
 \int_0^{1-x} \min(x, y) dy &= \int_0^x y dy + \int_x^{1-x} x dy \\
 &= \frac{1}{2} x^2 + x(1-2x) \\
 &= x - \frac{3}{2} x^2
 \end{aligned}$$

Claim

$$\|T\| = \max \left( \max_{x \in [\frac{1}{2}, 1]} \frac{1}{2} (1-x)^2, \max_{x \in [0, \frac{1}{2}]} \left( x - \frac{3}{2} x^2 \right) \right) = \dots = \frac{1}{6}$$

Note

- $\|T(f)\| \leq \|T\| \cdot \|f\|$  for all  $f \in C([0, 1])$ ,
- $\|T(1)\| = \|T\| \cdot \|1\|$  where  $1(x) = 1$  for  $x \in [0, 1]$ .

□

(B)  $E_1 = C([0, 1])$  with maximumnorm,  $E_2 = \mathbb{R}$  with absolut value.  $T : E_1 \rightarrow E_2$  with

$$T(f) = \int_0^{\frac{1}{2}} f(y) \, dy - \int_{\frac{1}{2}}^1 f(y) \, dy \quad \text{for } f \in E_1$$

$$\begin{aligned} |T(f)| &= \left| \int_0^{\frac{1}{2}} f(y) \, dy - \int_{\frac{1}{2}}^1 f(y) \, dy \right| \\ &\leq \left| \int_0^{\frac{1}{2}} f(y) \, dy \right| + \left| \int_{\frac{1}{2}}^1 f(y) \, dy \right| \\ &\leq \int_0^{\frac{1}{2}} \underbrace{|f(y)|}_{\leq \|f\|} \, dy + \int_{\frac{1}{2}}^1 \underbrace{|f(y)|}_{\leq \|f\|} \, dy \\ &\leq 1 \|f\| \end{aligned}$$

Hence  $T$  is bounded and  $\|T\| \leq 1$ .

$$T(f) = \int_0^1 k(y) f(y) \, dy$$

where

$$\begin{aligned} T(f_n) &= \left\{ \begin{array}{ll} \text{nachholen,} & \text{falls case} \end{array} \right. \\ T(f_n) &\leq 1 \left( \frac{1}{2} - \frac{1}{2n} + \frac{1}{2} - \frac{1}{2n} \right) = 1 - \frac{1}{n}, \quad n = 1, 2, \dots \end{aligned}$$

note

$$k(y) f_n(y) \geq 0 \quad \text{for } y \in [0, 1].$$

Hence  $\|T\| \leq 1 - \frac{1}{n}$  for  $n = 1, 2, \dots$ . Note  $\|f_n\| = 1$  for all  $n$ . Conclusion  $\|T\| = 1$ .  
Here

$$|T(f)| \leq \underbrace{\|T\|}_{\leq 1} \|f\| \quad \text{for all } f \in C([0, 1])$$

but

$$|T(f)| < \|T\| \|f\| \quad \text{for all } f \in C([0, 1]).$$

**Satz.**  $T_1, T_2$  bounded linear mappings  $(E_1, \|\cdot\|_1) \rightarrow (E_2, \|\cdot\|_2)$  and  $\lambda$  scalar. Set

$$\begin{aligned} (T_1 + T_2)(x) &= T_1(x) + T_2(x) \quad x \in E_1 \\ (\lambda T_1)(x) &= \lambda T_1(x) \quad x \in E_1 \end{aligned}$$

Claim:

- (1)  $T_1 + T_2$  and  $\lambda T_1$  are both linear mappings  $(E_1, \|\cdot\|_1) \rightarrow (E_2, \|\cdot\|_2)$ ,
- (2)  $T_1 + T_2$  and  $\lambda T_1$  are both bounded mappings  $(E_1, \|\cdot\|_1) \rightarrow (E_2, \|\cdot\|_2)$ .  
 $B(E_1, E_2)$  denote the vector space of all bounded linear mappings  $(E_1, \|\cdot\|_1) \rightarrow (E_2, \|\cdot\|_2)$ . ■
- (3)

$$\|T\|_{E_1 \rightarrow E_2} := \inf\{M > 0 \mid \|T(x)\|_2 \leq M\|x\|_1 \text{ for all } x \in E_1\}$$

defines a norm in  $B(E_1, E_2)$ .

**proof.** (1)  $\|T\| = 0$  implies that  $\|T(x)\|_2 = 0$  for all  $x \in E_1 \Rightarrow T(x) = 0 \in E_2$ .

$$T = 0 \in B(E_1, E_2)$$

- (2)  $T \in B(E_1, E_2)$  and  $\lambda$  scalar.

$$\begin{aligned} \|\lambda T\| &= \inf\{M > 0 \mid \|(\lambda T)(x)\|_2 \leq M\|x\|_1 \text{ for all } x \in E_1\} \\ &= \inf\{M > 0 \mid |\lambda| \|T(x)\|_2 \leq M\|x\|_1 \text{ for all } x \in E_1\} \\ &= \{\text{if } \lambda \neq 0\} \\ &= \inf\left\{ \underbrace{M}_{=|\lambda|\tilde{M}} > 0 \mid \|T(x)\|_2 \leq \underbrace{\frac{M}{|\lambda|}}_{=\tilde{M}} \|x\|_1 \text{ for all } x \in E_1 \right\} \\ &= |\lambda| \inf\{\tilde{M} > 0 \mid \|T(x)\|_2 \leq \tilde{M}\|x\|_1 \text{ for all } x \in E_1\} \\ &= |\lambda| \|T\| \end{aligned}$$

- (3) Set  $T_1, T_2 \in B(E_1, E_2)$ .

$$\begin{aligned} \|T_1 + T_2\| &= \inf\{M > 0 \mid \|(T_1 + T_2)(x)\|_2 \leq M\|x\|_1 \text{ for all } x \in E_1\} \\ &\leq \inf\{M_1 + M_2 > 0 \mid \|T_1(x)\|_2 \leq M_1\|x\|_1, \|T_2(x)\|_2 \leq M_2\|x\|_1 \text{ for all } x \in E_1\} \\ &= \|T_1\| + \|T_2\| \end{aligned}$$

□

Conclusion:  $(B(E_1, E_2), \|\cdot\|_{E_1 \rightarrow E_2})$  is a normed space.

**Satz.**  $(B(E_1, E_2), \|\cdot\|_{E_1 \rightarrow E_2})$  is a Banach space if  $(E_2, \|\cdot\|_2)$  is a Banach space.

**proof.** Assume  $(T_n)_{n=1}^\infty$  is a Cauchy sequence in  $(B(E_1, E_2), \|\cdot\|_{E_1 \rightarrow E_2})$  where  $(E_2, \|\cdot\|_2)$  is a Banach space. Fix  $x \in E_1$

$$\begin{aligned} \|T_n(x) - T_m(x)\|_2 &= \|(T_n - T_m)(x)\|_2 \\ &\leq \underbrace{\|T_n - T_m\|_{E_1 \rightarrow E_2}}_{\xrightarrow[n, m \rightarrow \infty]{0}} \cdot \|x\|_1 \rightarrow 0, \quad n, m \rightarrow \infty \end{aligned}$$

Hence  $(T_n(x))_{n=1}^\infty$  is a Cauchy sequence in  $(E_2, \|\cdot\|_2)$ . This is a Banach space which implies that  $(T_n(x))_{n=1}^\infty$  converges in  $(E_2, \|\cdot\|_2)$ . Call the limit  $T(x) \in E_2$  for all  $x \in E_1$ . Show now

- (1)  $T : E_1 \rightarrow E_2$  is linear,
- (2)  $T$  is bounded,
- (3)  $\|T_n - T\|_{E_1 \rightarrow E_2} \rightarrow 0$  for  $n \rightarrow \infty$ .

(1) Observe

$$T(\lambda_1 x_1 + \lambda_2 x_2) \leftarrow T_n(\lambda_1 x_1 + \lambda_2 x_2) = \{T \text{ linear}\} = \lambda_1 \underbrace{T_n(x_1)}_{\rightarrow T(x_1)} + \lambda_2 \underbrace{T_n(x_2)}_{\rightarrow T(x_2)} \\ \underbrace{\rightarrow \lambda_1 T(x_1)}_{\rightarrow \lambda_1 T(x_1)} + \underbrace{\rightarrow \lambda_2 T(x_2)}_{\rightarrow \lambda_2 T(x_2)} \\ \rightarrow \lambda_1 T(x_1) + \lambda_2 T(x_2)$$

So for  $n \rightarrow \infty$  it is

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2) \quad \text{in } (E_2, \|\cdot\|_2).$$

(2) Fix  $\varepsilon > 0$ . Then there exists  $N$  such that:

$$\|T_n - T_m\|_{E_1 \rightarrow E_2} < \varepsilon \quad \text{for } n, m \geq N$$

So for  $x \in E_1$

$$\|T_n(x) - T_m(x)\|_2 \leq \|T_n - T_m\|_{E_1 \rightarrow E_2} \|x\|_1 < \varepsilon \|x\|_1 \quad \text{for } n, m \geq N$$

Let  $m \rightarrow \infty$ .

$$\|T_n(x) - T(x)\|_2 \leq \varepsilon \|x\|_1 \quad \text{for } n \geq N$$

So

$$\begin{aligned} \|T(x)\|_2 &\leq \|T(x) - T_N(x)\|_2 + \|T_N(x)\|_2 \\ &\leq \varepsilon \|x\|_1 + \|T_N\|_{E_1 \rightarrow E_2} \cdot \|x\|_1 \\ &= (\varepsilon + \|T_N\|_{E_1 \rightarrow E_2}) \|x\|_1 \quad \text{for } x \in E_1 \end{aligned}$$

(3) Look above and get

$$\|T_n - T\|_{E_1 \rightarrow E_2} \rightarrow 0, \quad n \rightarrow \infty.$$

□

**Theorem** (Banach-Steinhaus theorem (uniform boundedness principle)).  $(E_1, \|\cdot\|_1)$  Banach space,  $(E_2, \|\cdot\|_2)$  normed space and  $\mathcal{F} \subset B(E_1, E_2)$ . Assume

$$\sup_{T \in \mathcal{F}} \|T(x)\|_2 < \infty \quad \text{for all } x \in E_1$$

then

$$\sup_{T \in \mathcal{F}} \|T\|_{E_1 \rightarrow E_2} < \infty.$$



**Remark.** The implication  $\Leftarrow$  is easy to prove. If  $\mathcal{F}$  is a finite set, the theorem is trivial.

**proof.** step 1: Assume

$$\exists x_0 \in E_1 \exists r > 0 \exists M > 0 : \forall x \in \overline{B(x_0, r)} \forall T \in \mathcal{F} : \|T(x)\|_2 \leq M$$

We have to show that

$$\sup_{T \in \mathcal{F}} \|T\|_{E_1 \rightarrow E_2} < \infty.$$

Fix  $T \in \mathcal{F}$ . For  $\|x\|_1 \leq r$

$$\|T(x_0 + x)\|_2 \leq M$$

Note that  $x_0 + x \in \overline{B(x_0, r)}$ .

$$\begin{aligned} \|T(x)\|_2 &= \|T(x_0 + x - x_0)\|_2 \\ &= \{T \text{ linear}\} \\ &= \|T(x_0 + x) - T(x_0)\|_2 \\ &\leq \|T(x_0 + x)\|_2 + \|T(x_0)\|_2 \\ &\leq 2M \end{aligned}$$

For  $0 \neq x \in E_1$

$$\left\| T \left( \frac{r}{\|x\|_1} x \right) \right\|_2 \leq 2M$$

$\frac{r}{\|x\|_1}$  has the  $\|\cdot\|_1$ -norm equal to  $r$ . This implies, since  $T$  linear,

$$\frac{r}{\|x\|_1} \|T(x)\|_2 \leq 2M$$

i.e.

$$\|T(x)\|_2 \leq \frac{2M}{r} \|x\|_1 \quad \text{for all } 0 \neq x \in E_1.$$

We have

$$\begin{aligned} \|T\|_{E_1 \rightarrow E_2} &\leq \underbrace{\frac{2M}{r}}_{\text{independent of } T} < \infty \\ \sup_{T \in \mathcal{F}} \|T\|_{E_1 \rightarrow E_2} &\leq \frac{2M}{r} < \infty \end{aligned}$$

step 2: Justify the assumption in step 1. This assumption is equivalent to

$$\exists x_0 \in E_1 \exists r > 0 \exists M > 0 : \forall x \in \overline{B(x_0, r)} \forall T \in \mathcal{F} : \|T(x)\|_2 \leq M$$

(Note  $\overline{B(x_0, r_1)} \subset B(x_0, r) \subset B(x_0, r_2)$  for  $0 < r_1 < r < r_2$ ).

Argue by contradiction. Assume that the assumption is false. Then it holds

$$\forall x_0 \in E_1 \forall r > 0 \forall M > 0 : \exists x \in \overline{B(x_0, r)} \exists T \in \mathcal{F} : \|T(x)\|_2 > M.$$

Idea: Find a converging sequence  $x_n \in E_1$ ,  $x_n \rightarrow x$  in  $(E_1, \|\cdot\|_1)$  and a sequence  $(T_n)_{n=1}^\infty \subset \mathcal{F}$  such that

$$\|T_n(x_n)\|_2 > n \quad \text{for all } n, \quad \text{and} \quad \|T_n(x)\|_2 > n \quad \text{for all } n.$$

We have from above  $x_1 \in B(0, 1)$  and  $T_1 \in \mathcal{F}$  such that

$$\|T_1(x_1)\|_2 > 1.$$

$T_1$  is bounded linear, hence continuous. This implies that there exists  $0 < r_1 < \frac{1}{2}$  such that

$$\|T_1(x)\|_2 > 1 \quad \text{for } x \in B(x_1, r_1)$$

and

$$\overline{B(x_1, r_1)} \subset B(0, 1).$$

□