Fourier and wavelet analysis

Script of “Fourier and wavelet analysis” by Prof. Mohammad Asadzadeh

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foreword — cooperation

This document is a transcript of the lecture "Fourier and wavelet analysis, WiSe 2016/2017, Term 2", by Prof. Mohammad Asadzadeh. It mainly contains the written content of the lecture. I will not assume any responsibility for the correctness of the content! For questions, remarks and mistakes please write an email to `keil.menden@web.de`. I'm grateful for every email.

Contents

1 Introduction

Definition (Fouriertransforms). We remember the Fouriertransformation for $\mathbb{R}^n$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)\,\mathrm{d}x.$$

where

$$x\cdot\xi = \sum_{i=1}^{n} x_i \xi_j$$

The course will be about

- The Fourier transforms
- Distribution theory
- Transforms related to Fouriertransforms. (Radon, Hankel z-transforms)
- The Wavelet transforms
- Multiresolution analysis
- Discrete Fouriertransforms, Sampling
- Applications (3 Lab obligus)

Notation

$$\begin{cases} \hat{f}(\xi) & = (\mathcal{F}f)(\xi) \\ f(x) & \overset{\mathcal{F}}{\supset} \hat{f}(\xi) \end{cases}$$

Basic properties Linearity

$$\begin{aligned} f+g &\supset \hat{f}+\hat{g} \\ \alpha f &\supset \alpha \hat{f} \qquad (\alpha\in\mathbb{C}, \alpha\in\mathbb{R}) \end{aligned}$$

Scaling

$$f\supset\hat{f} \qquad \Leftrightarrow \qquad \frac{1}{a} f(\frac{x}{a}) \overset{\mathcal{F}}{\supset} \hat{f}(a\xi)$$

If f is ”resonably regular/smoth” so is $\hat{f}$.

ouriertransform in $L_2(\mathbb{R})$

$$L_2(\mathbb{R}) := \left\{ f \,\middle|\, f \text{ is measurable and } \int_{\mathbb{R}} |f(x)|^2\,\mathrm{d}x < \infty \right\}$$

Scalar product With $f, g : \mathbb{R} \to \mathbb{C}$ we have

$$\langle f \, , \, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)} \, \mathrm{d}x.$$

Parsevals formula

$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \, \mathrm{d}\xi.$$

By "regular/smoth" we will mean Schwartz class S.
Often we will have that $f(x)$ is a signal where $f(t) = sin(\omega t)$.

Definition (Hevyside). We call

$$f(t) = H(x) := \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

the Hevyside-function.
We have also

$$f(t) = \delta(t)$$

which is called the Diracs "δ-function" and it is the derivative of the Hevyside function. The Dirac functions are not in L_2 (nor in S).

Fundamental 1.1 (The Fourier inversion forumula).

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x) \, \mathrm{d}x \qquad \Leftrightarrow \qquad f(x) = \int_{\mathbb{R}} e^{2\pi i x \xi} \hat{f}(\xi) \, \mathrm{d}\xi.$$

Example. Consider

$$\mathcal{F}(e^{-\pi x^2}) = e^{-\pi \xi^2}.$$

We want to scale with $a = 2$. Then we get

$$\frac{1}{2} e^{-\pi \left(\frac{x}{2}\right)^2} \overset{\mathcal{F}}{\supset} e^{-\pi (2\xi)^2}.$$

Other transforms:
Hankel:

$$\hat{f}(\xi_1, \xi_2) = \int_{\mathbb{R}^2} e^{-2\pi i (x_1 \xi_1 + x_2 \xi_2)} f(x_1, x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2$$

Let

$$f(x_1, x_2) = F(\sqrt{x_1^2 + x_2^2}) = F(r), \qquad \begin{cases} x_1 & = r\cos\theta, \\ x_2 & = r\sin\theta \end{cases}$$

Then

$$\hat{f}(\xi_1, \xi_2) = \tilde{F}(\sqrt{\xi_1^2 + \xi_2^2}) = \tilde{F}(\rho), \qquad \{$$

$$F(r) \mapsto \tilde{F}(\rho)$$

Raden IF $f(x_1, x_2)$ is the density of "the head" at point (x_1, x_2). Then

$$\int_{L_{r,\omega}} f(x_1, x_2)\,\mathrm{d}t$$

gives a measure of damping along $L_{r,\omega}$ (which can be measured).

$$\mathcal{R}_\omega = \int_{L_{r,\omega}} f(x_1, x_2)\,\mathrm{d}t$$

is the Radon transform. It has an application in computer tomography.

1.1 Functions with Fouriertransform

What functions do have a Fouriertransform?

$$L_2(\mathbb{R}) =$$

$\mathcal{S}$ (smooth functions with rapid dacay)

$$L_1(\mathbb{R}) = \left\{ f \,\middle|\, \int_{\mathbb{R}} |f(x)|\,\mathrm{d}x < \infty \right\}$$

Note that if

$$f \in L_1(\mathbb{R})$$

we get

$$\begin{aligned} |\hat{f}f(\xi)| &= |\int_{\mathbb{R}} e^{-2\pi x\xi} f(x)\,\mathrm{d}x| \\ &\le \int_{\mathbb{R}} \underbrace{|e^{-2\pi i x\xi} f(x)|}_{=|f(x)|}\,\mathrm{d}x < \infty \end{aligned}$$

So we have in general

$$f \in L_1(\mathbb{R}) \qquad \Rightarrow \qquad \hat{f} \in L_\infty(\mathbb{R})$$

And more general we have it for L_p and L_q for $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.2 (Hausdorff-Young's inequality). For $1 \le p \le 2$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$f \in L_p(\mathbb{R}) \qquad \Rightarrow \qquad \hat{f} \in L_q(\mathbb{R})$$

This is a variation of Hölder's inequality.

Theorem 1.3 (Convolution Theorem). Let

$$f(x) \overset{\mathcal{F}}{\supset} \hat{f}(\xi) \text{ and } g(x) \overset{\mathcal{F}}{\supset} \hat{g}(\xi)$$

Then we define

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\,\mathrm{d}y \qquad \Leftrightarrow \qquad \supset \hat{f}(\xi)\hat{g}(\xi)$$

with the properties

commutative: $f * g = g * f$

associative $f * (g * h) = (f * g) * h$

distributive $f * (g + h) = f * g + f * h$

Translation:
Let τ_a be a translation such that $\tau_a : f(.) \mapsto f(. - a)$. Then we have

$$\begin{aligned}
\mathcal{F}(\tau_a f)(\xi) &= \int_{\mathbb{R}} e^{-2\pi i \xi x} f(\underbrace{=y}_{x-a})\,\mathrm{d}x \\
&= \int_{\mathbb{R}} e^{-2\pi i x \xi (x+y)} f(y)\,\mathrm{d}y \\
&= e^{-2\pi i x \xi a} \int_{\mathbb{R}} e^{-2\pi i \xi y} f(y)\,\mathrm{d}y
\end{aligned}$$

and also

$$\begin{aligned}
\tau_a \hat{f}(\xi) &= \int_{\mathbb{R}} e^{-2\pi i x(\xi - a)} f(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} e^{-2\pi i x \xi} \left(e^{2\pi i x a} f(x)\right)\,\mathrm{d}x
\end{aligned}$$

Derivation:
We have

$$f \overset{\mathcal{F}}{\supset} \hat{f} \qquad \Rightarrow \qquad \begin{cases} Df & \overset{\mathcal{F}}{\supset} 2\pi i \xi \hat{f}(\xi) \\ -2\pi i(.) f & \overset{\mathcal{F}}{\supset} D\hat{f} \end{cases}$$

proof. We have

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi}\,\mathrm{d}x$$

Then

$$D\hat{f}(\xi) = \int_{\mathbb{R}} \left(-2\pi i x f(x)\right) e^{-2piix\xi}\,\mathrm{d}x.$$

And also with partial differentiation

$$\begin{aligned}
\mathcal{F}(Df) &= \int_{\mathbb{R}} e^{-2\pi i x \xi} f'(x)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}} -2\pi i \xi e^{-2\pi i x \xi} f(x)\,\mathrm{d}x.
\end{aligned}$$

□

1.2 The schwartz classes $\mathcal{S}$ and $\mathcal{S}'$

[Distributions (generalized functions)]

Definition 1.4 (Schwartz-class). The function class $\mathcal{S}$ are complex-valued functions f of a real variable such that $f : \mathbb{R} \to \mathbb{C}$ satisfies

$$\sup_{x\in\mathbb{R}} ||x|^{\alpha} D^{\beta} f(x)| < \infty.$$

for any choice of $\alpha \geq 0$ and $\beta \geq 0$.

Examples.

- For $a > 0$

$$f(x) = e^{-ax^2}$$

  What if $a = i$ or $a < 0$?

-

$$f(x) = \begin{cases} e^{-\frac{1}{(x-2)^2} - \frac{1}{(x-b)^2}}, & a < x < b \\ 0, & \text{else} \end{cases}$$

  f has compact support $f \in C^{\infty}$ ”but not! real analytic” (i.e. its power series is not convergent)

We will now state properties of $\mathcal{S}$.

Lemma 1.5 (properties of $\mathcal{S}$).

1. if $f \in \mathcal{S}$ and if for $\alpha, \beta \in \mathbb{Z}^+$ $g(x) = x^{\alpha} D^{\beta} f(x)$ then $g \in \mathcal{S}$
2. $f \in \mathcal{S} \quad \Rightarrow \quad \hat{f} \in S.$