



GÖTEBORGS UNIVERSITET



Applied Functionalanalysis

Script of “Applied Functionalanalysis” by Prof. Peter Kumlin

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foreword — cooperation

This document is a transcript of the lecture “Applied Functionalanalysis, WiSe 2016/2017, Term 1”, by Prof. Peter Kumlin. It mainly contains the written content of the lecture. I will not assume any responsibility for the correctness of the content! For questions, remarks and mistakes please write an email to keil.menden@web.de. I’m grateful for every email.



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1 Introduction

1.1 Introduction example

We have

$$\begin{cases} f'' + f = g, & \text{in } I = [0, 1] \\ f(0) = 1, f'(0) = 1 \end{cases},$$

where g is a known continuous function on I . We will now consider different cases:

1. $g = 0$

$$\Rightarrow f(x) = A \cos(x) + B \sin(x), x \in I,$$

where $A, B \in \mathbb{R}$.

2. g arbitrary. We will now introduce the Method of variation of constants. Set

$$f(x) = A(x) \cos(x) + B(x) \sin(x).$$

Differentiate

$$f'(x) = A'(x) \cos(x) + B'(x) \sin(x) - A(x) \sin(x) + B(x) \cos(x).$$

Assume (this is part of the method)

$$A'(x) \cos(x) + B'(x) \sin(x) = 0, \quad x \in I.$$

Differentiate $f'(x)$ and get

$$f''(x) = \underbrace{-A(x) \cos(x) - B(x) \sin(x)}_{=-f(x)} - A'(x) \sin(x) + B'(x) \cos(x).$$

We get

$$g(x) = f''(x) + f(x) = -A'(x) \sin(x) + B'(x) \cos(x).$$

Now:

$$\begin{cases} A'(x) \cos(x) + B'(x) \sin(x) = 0, & x \in I \\ -A'(x) \sin(x) + B'(x) \cos(x) = g(x), & x \in I. \\ A(0) = 1, & B(0) = 0 \end{cases}$$

We get

$$\begin{aligned} A'(x) &= -g(x) \sin(x), \\ A(0) &= 1, \\ B'(x) &= g(x) \cos(x), \\ B(0) &= 0. \end{aligned}$$

This implies

$$\begin{aligned}A(x) &= A(0) + \int_0^x A'(t) dt = 1 - \int_0^x g(t) \sin(t) dt, \\B(x) &= B(0) + \int_0^x B'(t) dt = 0 + \int_0^x g(t) \cos(t) dt.\end{aligned}$$

Hence

$$\begin{aligned}f(x) &= \cos(x) - \int_0^x g(t) \sin(t) dt \cos(x) + \int_0^x g(t) \cos(t) dt \sin(x) \\&= \cos(x) + \int_0^x \underbrace{(\sin(x) \cos(t) - \sin(t) \cos(x))}_{=\sin(x-t)} g(t) dt \\&= \cos(x) + \int_0^x \sin(x-t) g(t) dt \quad (*).\end{aligned}$$

Check that $f(x)$ in $(*)$ satisfies the PDE.

special case:

Assume for $x \in I$

$$g(x) = k(x)f(x).$$

Here k is a known continuous function on I . Insert this in $(*)$. We obtain

$$f(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f(t) dt, \quad x \in I \quad (**).$$

Observe that f appears both in LHS and RHS. $(**)$ is a reformulation of the PDE with $g = kf$. Pick a continuous function in I . call it f_0 . Set
 $\in C(I)$

$$\begin{aligned}f_1(x) &= \cos(x) + \int_0^x \sin(x-t)k(t)f_0(t) dt, \\f_2(x) &= \cos(x) + \int_0^x \sin(x-t)k(t)f_1(t) dt, \\&\vdots \\f_{n+1}(x) &= \cos(x) + \int_0^x \sin(x-t)k(t)f_n(t) dt, \quad n = 1, 2, 3, \dots\end{aligned}$$

Hope:

f_n tends to some continuous function f on I , denoted $f_n \rightarrow f$. 'Tends to' has to be more precis!

$$\begin{array}{ccc}
 f_{n+1}(x) & = & \cos(x) + \int_0^x \sin(x-t)k(t)f_n(t) dt \\
 \downarrow & & \downarrow \\
 f(x) & = & \cos(x) + \int_0^x \sin(x-t)k(t)f(t) dt
 \end{array}$$

for $x \in I$. Simplify notation set for $v \in C(I)$

$$\begin{cases} u(x) & = \cos(x) \\ kv(x) & = \int_0^x \sin(x-t)k(t)v(t) dt \end{cases} .$$

We have $f_0 \in C(I)$, $f_{n+1} = u + kf_n$ for $n = 0, 1, 2, \dots$ (!)

Facts from previous calculus classes:

Definition (Sequence of continuous functions).

$$v_n \in C(I), \quad n = 1, 2, \dots$$

We say that $(v_n)_{n=1}^\infty$ converges uniformly in I if

$$\max_{x \in I} |v_n(x) - v_m(x)| \rightarrow 0, \quad n, m \rightarrow \infty,$$

i.e.

$$\forall \varepsilon > 0 \exists N : \forall n, m \geq N : \max_{x \in I} |v_n(x) - v_m(x)| < \varepsilon.$$

Lemma . Suppose that $(v_n)_{n=1}^\infty$ converges uniformly on I . then there exists $v \in C(I)$ such that

$$\max_{x \in I} |v_m(x) - v(x)| \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Back to (!):

More Notation:

$$k(kv) = k^2v, \quad v \in C(I)$$

and

$$k^{n+1}v = k(k^n v), \quad n = 1, 2, \dots$$

We have

$$\begin{aligned}f_0 &\in C(I) \\f_1 &= u + kf_0 \\ \text{and } f_2 &= u + kf_1 = u + k(u + kf_0)\end{aligned}$$

and so on. Note that

$$k(v + w) = kv + kw.$$

Then

$$\begin{aligned}f_2 &= u + k(u + kf_0) = u + ku + k(kf_0) = u + ku + k^2f_0 \\f_3 &= u + kf_2 = u + ku + k^2u + k^3f_0\end{aligned}$$

and in general for $n = 1, 2, \dots$

$$f_n = ku + \dots + k^{n-1}u + k^n f_0, \quad n = 1, 2, \dots$$

Assume $n > m$ then

$$f_n - f_m = k^m u + \dots + k^{n-1}u + k^n f_0 - k^m f_0.$$

Set for $v \in C(I)$

$$\|v\| = \max_{x \in I} |v(x)|.$$

Note

$$\|v + w\| \leq \|v\| + \|w\| \quad \text{for } v, w \in C(I)$$

and

$$\|-v\| = \|v\|.$$

We have

$$\begin{aligned}\|f_n - f_m\| &= \|k^m u + \dots + k^{n-1}u + k^n f_0 - k^m f_0\| \\&\leq \|k^m u\| + \dots + \|k^{n-1}u\| + \|k^n f_0\| + \|-k^m f_0\|.\end{aligned}$$

Assumption:

$$\sum_{l=1}^{\infty} \|k^l v\| < \infty \quad \text{for all } v \in C(I) \quad (**).$$

Under this assumption

$$\|f_n - f_m\| \rightarrow 0 \quad \text{as } n, m \rightarrow \infty$$

since

$$\begin{aligned}\sum_{l=1}^{\infty} \|k^l u\| &< \infty & (u(x) = \cos(x)) \\ \sum_{l=1}^{\infty} \|k^l f_0\| &< \infty & (f_0 \in C(I)).\end{aligned}$$

Conclusion: $(f_n)_{n=1}^{\infty}$ converges uniformly on I . By lemma above there exists $f \in C(I)$ such that

$$\max_{x \in I} |f_n(x) - f(x)| \rightarrow 0, \quad n \rightarrow \infty,$$

i.e.

$$\|f_n - f\| \rightarrow 0, \quad n \rightarrow \infty.$$

'Back hope': f_n tends to f , denoted $f_n \rightarrow f$ shall be interpreted as

$$\|f_n - f\| \rightarrow 0, \quad n \rightarrow \infty.$$

Remember

$$f_{n+1}(x) = u(x) + k f_n(x) \rightarrow ?.$$

For $x \in I$ there is

$$\begin{aligned} |k f_n(x) - k f(x)| &= \left| \int_0^x \sin(x-t) k(t) f_n(t) dt - \int_0^x \sin(x-t) k(t) f(t) dt \right| \\ &\leq \int_0^x |\sin(x-t) k(t)| \underbrace{|f_n(t) - f(t)|}_{\leq \|f_n - f\|} dt \\ &\leq \int_0^x |\sin(x-t) k(t)| dt \|f_n - f\|. \end{aligned}$$

In particular

$$\begin{aligned} \|k f_n - k f\| &\leq \max_{x \in I} \int_0^x \underbrace{|\sin(x-t)|}_{\leq 1} \underbrace{|k(t)|}_{\max_{t \in I} |k(t)| < \infty} dt \|f_n - f\| \\ &\leq \|k\| \|f_n - f\|. \end{aligned}$$

We have, provided $(***)$ holds, shown

$$\begin{aligned} f_{n+1} &= u + k f_n \\ \downarrow \\ f &= u + k f. \end{aligned}$$

Let us try to prove $(***)$. For $v \in C(I)$ arbitrary and for $x \in I$

$$\begin{aligned} \|k v(x)\| &= \left| \int_0^x \sin(x-t) k(t) v(t) dt \right| \\ &\leq \int_0^x \underbrace{|\sin(x-t)|}_{\leq 1} \underbrace{|k(t)|}_{\leq \|k\|} |v(t)| dt \\ &\leq \int_0^x \underbrace{|v(t)|}_{\leq \|v\|} dt \|k\| \\ &\leq \|k\| \|v\| x. \end{aligned}$$

In particular

$$\|kv\| \leq \|k\|\|v\|$$

and

$$\begin{aligned} |k^2v(x)| &\leq \int_0^x |kv(t)| \, dt \|k\| \\ &\leq \int_0^x \|k\|\|v\|t \, dt \cdot \|k\| \\ &= \|k\|^2\|v\|\frac{x^2}{2}. \end{aligned}$$

In particular

$$\|k^2v\| \leq \|k\|^2\|v\|\frac{1}{2}.$$

By induction we get

$$\begin{aligned} |k^n v(x)| &\leq \|k\|^n \|v\| \frac{x^n}{n!} \quad x \in I \\ \|k^n v\| &\leq \|k\|^n \|v\| \frac{1}{n!}. \end{aligned}$$

So

$$\begin{aligned} \sum_{l=1}^{\infty} \|k^l v\| &\leq \sum_{l=1}^{\infty} \|k\|^l \|v\| \frac{1}{l!} \\ &= \|v\| \sum_{l=1}^{\infty} \frac{\|k\|^l}{l!} \\ &\leq \|v\| e^{\|k\|} < \infty. \end{aligned}$$

Consider Taylor expansion. $\Rightarrow (***)$ holds true.

We have now shown that $f = u + kf$ where $u(x) = \cos(x)$ and

$$kv = \int_0^x \sin(x-t)k(t)v(t) \, dt.$$

$x \in I$ for $v \in C(I)$, has a solution $f \in C(I)$.

Question:

Is the solution unique?

Assume $f, \tilde{f} \in C(I)$ such that $f = u + kf$ and $\tilde{f} = u + k\tilde{f}$. Set

$$v = f - \tilde{f} \in C(I)$$

$$\begin{aligned} \Rightarrow v &= (u + kf) - (u + k\tilde{f}) \\ &= kf - k\tilde{f} \\ &= k(f - \tilde{f}) \\ &= kv. \end{aligned}$$

We have $v = kv$, implies that $kv = k(kv) = k^2v$. So for $n = 1, 2, \dots$

$$v = kv = k^2v = \dots = k^nv.$$

We know

$$\sum_{n=1}^{\infty} \|k^n \hat{v}\| < \infty \quad \text{for all } \hat{v} \in C(I).$$

Apply this to $\hat{v} = v$:

$$\sum_{n=1}^{\infty} \underbrace{\|k^n v\|}_{=\|v\|} < \infty.$$

So $\|v\| = 0$ with implies $v(x) = 0$ for all $x \in I$. So we have $f(x) = \tilde{f}(x)$ for $x \in I$.
 \Rightarrow Answer to the question above: YES !

We have more or less proved the following theorem:

Theorem 1.1. Set $I = [0, 1]$. Suppose $u \in C(I)$ and $k \in C(I \times I)$. Consider

$$f(x) = u(x) + \int_0^x k(x, t)f(t) dt, \quad x \in I \quad (1).$$

Then (1) has a unique solution $f \in C(I)$

With the same technology we can prove:

Theorem 1.2. Set $I = [0, 1]$. Suppose $u \in C(I)$, $k \in C(I \times I)$ and $\max_{(x,t) \in I \times I} |k(x, t)| < 1$. Consider

$$f(x) = u(x) + \int_0^1 k(x, t)f(t) dt, \quad x \in I \quad (2).$$

Then (2) has a unique solution $f \in C(I)$.

Different notions: see introductional example.

Definition (vector space). $C(I)$ with the operations for $x \in I$:

addition $v, w \in C(I)$: $(v + w)(x) = v(x) + w(x)$,

mult. by scalar $v \in C(I)$, $\lambda \in \mathbb{R}$: $(\lambda v)(x) = \lambda v(x)$.

Note that $v + w, \lambda v \in C(I)$.

Definition (norm). Norm on $C(I)$ for instance

$$\|v\| = \max_{x \in I} |v(x)|$$

with norm given we can talk about convergence and continuity.

Definition (Cauchy sequence). In our example a sequence $(f_n)_{n=1}^{\infty}$ is called Cauchy sequence if $\|f_n - f_m\| \rightarrow 0$ for $n, m \rightarrow \infty$.

Definition . $C(I)$ with the max-norm. Lemma above says that every Cauchy sequence converges i.e.

$$\|v_n - v_m\| \rightarrow 0, \quad n, m \rightarrow \infty.$$

This applies

$$\exists v \in C(I) : \|v_n - v\| \rightarrow 0, \quad n \rightarrow \infty.$$

This is the defining property of a Banach space.

K linear mapping $C(I) \rightarrow C(I)$ with

$$K(v + w) = K(v) + K(w)$$

$$K(\lambda v) = \lambda K(v)$$

for $v, w \in C(I)$, $\lambda \in \mathbb{R}$.

K bounded linear:

$$\|Kv\| \leq M\|v\| \quad \forall v \in C(I),$$

where $M > 0$ independent of v .

Definition (operator norm). Define

$$\|K\| := \inf\{M > 0 \mid \|Kv\| \leq M\|v\| \text{ for all } v \in C(I)\}.$$

fixed point results:

Our example: $f = u + kf =: T(f)$ and $f_0 \in C(I)$ fixed.

Form sequence of iterants $(f_n)_{n=1}^{\infty}$, $f_n = T(f_{n-1})$, $n = 1, 2, \dots$ if

$$\|T(v) - T(w)\| \leq c\|v - w\|$$

for all $v, w \in C(I)$ for some $c < 1$. Then there is a unique $v \in C(I)$ such that $v = T(v)$.

This is Banach's fixed point theorem.

Definition (Green's function). Our example:

$$L = \left(\frac{d}{dx}\right)^2 + 1$$

differential operator. Boundary conditions

$$f(0) = f'(0) = 0.$$

Then

$$f(x) = \int_0^1 g(x, t)h(t) \, dt$$

is a solution to

$$\begin{cases} f'' + f &= h, \\ f(0) = f'(0) &= 0. \end{cases}$$

Definition (real vector space). We say that E is a real vector space if it is a non-empty set with the operations

addition $E \times E \rightarrow E$, $(x, y) \mapsto x + y$

mult. with scalar $\mathbb{R} \times E \rightarrow E$, $(\lambda, x) \mapsto \lambda x$

satisfying the axioms:

- (1) $x + y = y + x$, for all $x, y \in E$,
- (2) $x + (y + z) = (x + y) + z$, for all $x, y, z \in E$,
- (3) For all $x, y \in E$ there exists $z \in E$ such that $x + z = y$,
- (4) $\alpha(\beta x) = (\alpha \cdot \beta)x$, for all $\alpha, \beta \in \mathbb{R}, x \in E$,
- (5) $\alpha(x + y) = \alpha x + \alpha y$, for all $\alpha \in \mathbb{R}, x, y \in E$,
- (6) $(\alpha + \beta)x = \alpha x + \beta x$, for all $\alpha, \beta \in \mathbb{R}, x \in E$,
- (7) $1 \cdot x = x$, for all $x \in E$.

Remark. E is a complex vector space if all \mathbb{R} in the definition above are replaced by \mathbb{C} .

Remark. (1)

$$\exists ! 0 \in E : \quad x + 0 = x \quad \text{for all } x \in E.$$

Since: Fix $x \in E$, by (3), $\exists 0_x$ such that $0_x + x = x$.

Fix $y \in E$. We want to show that $y + 0_y = y$. By (3), there exists $z \in E$ such that $x + z = y$. So

$$\begin{aligned} y + 0_x &= (x + z) + 0_x \\ &\stackrel{(1)}{=} (z + x) + 0_x \\ &\stackrel{(2)}{=} z + (x + 0_x) \\ &= z + x \\ &\stackrel{(1)}{=} x + z \\ &= y. \end{aligned}$$

Assume $x + 0_1 = x$, $x + 0_2 = x$ for all $x \in E$. We want to show $0_1 = 0_2$:

$$0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2,$$

(2)

$$\forall x \in E : \exists ! -x \in E : x + (-x) = 0.$$

proof: exercise.

(3)

$$\begin{aligned} 0x &= 0 && \text{for all } x \in E \\ (-1)x &= -x && \text{for all } x \in E. \end{aligned}$$

Examples (Examples of real vector spaces). 1) \mathbb{R} with standard addition and mult. by scalar.

2) \mathbb{R}^n , $n = 2, 3, \dots$

addition $(x_1, x_2, \dots) + (y_1, y_2, \dots) = (x_1 + y_1, x_2 + y_2, \dots)$

mult. $\lambda(x_1, x_2, \dots) = (\lambda x_1, \lambda x_2, \dots)$

3) $\mathbb{R}^\infty = \{(x_1, \dots, x_n, \dots) \mid x_n \in \mathbb{R}, n = 1, 2, \dots\}$

4) $1 \leq p < \infty$,

$$l^p = \left\{ (x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty \left| \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty \right. \right\}$$

with the same addition and mult. by scalar as in \mathbb{R}^∞ . We have to check:

(1) $x, y \in l^p \Rightarrow x + y \in l^p$

(2) $x \in l^p, \lambda \in \mathbb{R} \Rightarrow \lambda x \in l^p$.

For (1) we assume $x = (x_1, \dots, x_n, \dots)$ and $y = (y_1, \dots, y_n, \dots)$.

$$x \in l^p \Rightarrow \sum_{n=1}^{\infty} |x_n|^p < \infty$$

$$y \in l^p \Rightarrow \sum_{n=1}^{\infty} |y_n|^p < \infty$$

$$\Rightarrow x + y = (x_1 + y_1, \dots) \stackrel{?}{\in} l^p?$$

$$\begin{aligned}
\Rightarrow \sum_{n=1}^{\infty} |x_n + y_n|^p &\leq \{|x_n + y_n| \leq |x_n| + |y_n| \leq 2 \max\{|x_n|, |y_n|\}\} \\
&\leq \sum_{n=1}^{\infty} 2^p (|x_n|^p + |y_n|^p) \\
&= 2^p \underbrace{\sum_{n=1}^{\infty} |x_n|^p}_{< \infty} + 2^p \underbrace{\sum_{n=1}^{\infty} |y_n|^p}_{< \infty} < \infty
\end{aligned}$$

and

$$\sum_{n=1}^{\infty} |\lambda x_n|^p = \sum_{n=1}^{\infty} |\lambda|^p \cdot |x_n|^p = |\lambda|^p \sum_{n=1}^{\infty} |x_n|^p < \infty.$$

5) Function spaces, say real-valued functions on I .

addition: $(f + g)(x) = f(x) + g(x), \quad x \in I$

mult. by scalar: $(\lambda f)(x) = \lambda f(x) \quad \text{for functions } f \text{ and } g$

6) $C(I)$: addition and mult. by scalar as in (5).

f, g continuous in I implies that $f + g$ is continuous in I .

Also if f is continuous and $\lambda \in \mathbb{R}$ then (λf) is continuous in I .

7) $P(I)$ = polynomials in I .

8) $P_k(I)$ = polynomials of degree at most k in I .

Theorem 1.3 (Hölder's inequality). Assume $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Let (x_1, \dots, x_n, \dots) and $(y_1, y_2, \dots, y_n, \dots)$ be sequences of complex numbers. Then

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{n=1}^{\infty} |y_n|^q \right)^{\frac{1}{q}}.$$

Remark there the LHS can be infinity, but the RHS can also be infinity.

proof. Step 1 We're going to proof

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \text{for all } a, b > 0.$$

$$\int_0^a x^{p-1} dx = \frac{a^p}{p}.$$

Note $y = x^{p-1}$ gives

$$x = y^{\frac{1}{p-1}} = y^{\frac{1}{\frac{1}{1-\frac{1}{q}}-1}} = y^{\frac{1}{\frac{q}{q-1}-1}} = y^{q-1}$$

so

$$\int_0^b y^{q-1} dy = \frac{b^q}{q}.$$

We get

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

(You also get condition for =)

Step 2 It is enough to consider the cases $\text{LHS} > 0$ and $\text{RHS} < \infty$. There exists an integer N such that

$$0 < \sum_{n=1}^N |x_n|^p, \sum_{n=1}^N |y_n|^q < \infty.$$

Set

$$a = \frac{|x_k|}{\left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}}}, \quad k = 1, 2, \dots, N,$$
$$b = \frac{|y_k|}{\left(\sum_{n=1}^N |y_n|^q\right)^{\frac{1}{q}}}, \quad k = 1, 2, \dots, N.$$

Insert into

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

$$\frac{|x_k y_k|}{\left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^N |y_n|^q\right)^{\frac{1}{q}}} \leq \frac{|x_k|^p}{p \sum_{n=1}^N |x_n|^p} + \frac{|y_k|^q}{q \sum_{n=1}^N |y_n|^q}, \quad k = 1, 2, \dots, N.$$

We sum over k from 1 to N :

$$\sum_{k=1}^N |x_k y_k| \leq \left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{n=1}^N |y_n|^q\right)^{\frac{1}{q}}.$$

Let $N \rightarrow \infty$. First in RHS and then in LHS.

□

Theorem 1.4 (Minkowski's inequality). Assume $1 \leq p < \infty$. and $X, Y \in l^p$. Then

$$\|X + Y\|_{l^p} \leq \|X\|_{l^p} + \|Y\|_{l^p}.$$

proof. $p = 1$:

$$\begin{aligned}
 \|X + Y\|_{l^1} &= \|(x_1, x_2, \dots, x_n, \dots) + (y_1, y_2, \dots, y_n, \dots)\|_{l^1} \\
 &= \|(x_1 + y_1, \dots, x_n + y_n, \dots)\|_{l^1} \\
 &= \sum_{n=1}^{\infty} |x_n + y_n| \\
 &\leq \sum_{n=1}^{\infty} (|x_n| + |y_n|) \\
 &= \sum_{n=1}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| \\
 &= \|X\|_{l^1} + \|Y\|_{l^1}
 \end{aligned}$$

$1 < p < \infty$:

$$\begin{aligned}
 \|X + Y\|_{l^p}^p &= \sum_{n=1}^{\infty} |x_n + y_n|^p \\
 &= \sum_{n=1}^{\infty} |x_n + y_n| |x_n + y_n|^{p-1} \\
 &\leq \sum_{n=1}^{\infty} |x_n| |x_n + y_n|^{p-1} + \sum_{n=1}^{\infty} |y_n| |x_n + y_n|^{p-1}.
 \end{aligned}$$

Use Hölder to get

$$\begin{aligned}
 \sum_{n=1}^{\infty} |x_n| |x_n + y_n|^{p-1} &\leq \underbrace{\left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}}_{=\|X\|_{l^p}} \cdot \left(\sum_{n=1}^{\infty} |x_n + y_n|^{(p-1)q} \right)^{\frac{1}{q}} \\
 &= \left\{ (p-1)q = (p-1) \frac{1}{1 - \frac{1}{p}} = p \right\} \\
 &= \|X\|_{l^p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^p \right)^{\frac{1}{q}}.
 \end{aligned}$$

We have

$$\|X + Y\|_{l^p}^p \leq (\|X\|_{l^p} + \|Y\|_{l^p}) \|X + Y\|_{l^p}^{\frac{p}{q}}.$$

If $\|X + Y\|_{l^p} \neq 0$ then

$$\|X + Y\|_{l^p}^{p - \frac{p}{q}} \leq \|X\|_{l^p} + \|Y\|_{l^p}$$

there

$$p - \frac{p}{q} = p \left(1 - \frac{1}{q}\right) = p \frac{1}{p} = 1.$$

□

Remark. $f \in C([0, 1])$ then for $1 \leq p < \infty$

$$\|f\|_{L^p} = \left(\int_0^1 |f(t)|^p dt \right)^{\frac{1}{p}}.$$

Claim:

$$\|fg\|_{L^1} = \int_0^1 |f(t) \cdot g(t)| dt \leq \|f\|_{L^p} \cdot \|g\|_{L^q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Also we have

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

This is proven with the same technique as we used for l^p . $\sum_{n=1}^{\infty}$ is replaced by $\int_0^1 dt$. E real/complex vector space. $x_1, \dots, x_n \in E$, $\lambda_1, \dots, \lambda_n$ scalar. We say that

$$\lambda_1 x_1, \dots, \lambda_n x_n$$

is a linear combination of x_1, \dots, x_n . We say that x_1, \dots, x_n are linear independent if

$$\alpha_1 x_1 + \dots + \alpha_n x_n = 0 \quad \Rightarrow \quad \alpha_1 = \dots = \alpha_n = 0.$$

If $A \subset E$, we say that A is linear independent if every linear combination of vectors in A is linear independent.

Examples. (1) Set $E = P([0, 1])$ and $A = \{p_k \mid p_k(x) = x^k, x \in [0, 1], k = 0, 1, \dots\}$. A is linear independent since:

Consider

$$\alpha_0 p_0 + \alpha_1 p_1 + \dots + \alpha_n p_n = 0,$$

i.e.

$$\alpha_0 p_0(x) + \alpha_1 p_1(x) + \dots + \alpha_n p_n(x) = 0(x), \quad x \in [0, 1],$$

i.e.

$$\alpha_0 + \alpha_1 x + \dots + \alpha_n x^n = 0, \quad x \in [0, 1].$$

If $x = 0$ then $\alpha_0 = 0$

$$\alpha_1 x + \dots + \alpha_n x^n = 0, \quad x \in [0, 1].$$

Differentiate

$$\alpha_1 + 2\alpha_2 x + \dots + n\alpha_n x^{n-1} = 0$$

gives $\alpha_1 = 0$. Continue and get

$$\alpha_0 = \alpha_1 = \dots = \alpha_n = 0.$$

Set $B \subset E$ where

$\text{span } B = \{\text{set of all linear combinations of elements in } B\}$

$$= \left\{ \sum_{k=1}^n \lambda_k x_k \mid x_k \in B, \lambda_k \in \mathbb{R}, k = 1, 2, \dots, n \text{ where } n \text{ is a positive integer} \right\}.$$

Remark.

$$\sum_{k=1}^n \lambda_k x_k \in E,$$

$$\sum_{k=1}^{\infty} \lambda_k x_k \text{ has no meaning.}$$

$C \subset E$ is called a basis for E if

- 1) C linear independent,
- 2) $\text{span } C = E$.

Continue of the example above:

Claim: A is a basis for E .

(2) Set $E = l^2$ and

$$A = \{X_k \mid k = 1, 2, \dots\},$$

$$X_k = (0, 0, \dots, 0, 1, 0, 0, \dots).$$

Claim: A is linear independent since

$$\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_n X_n = 0.$$

Here

$$\alpha_1 X_1 = (\alpha_1, 0, 0, \dots), \quad \text{etc}$$

and

$$0 = (0, 0, \dots).$$

So

$$(\alpha_1, \alpha_2, \dots, \alpha_n, 0, \dots) = (0, 0, \dots).$$

So $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

Question: Is A a basis for l^2 ?

We note: If $X \in \text{span } A$ then

$$X = (x_1, x_2, \dots, x_n, 0, 0, \dots)$$

for some positive integer n , i.e. X has only finitely many nonzero positions.

Consider:

$$X := (1, \frac{1}{2}, \dots, \frac{1}{n}, \dots),$$

$$\|X\|_{l^2} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} < \infty.$$

So $X \in l^2 \setminus \text{span } A$.

Remark. Every vector space has a basis (if we are allowed to use Axiom of Choice/ Zorn's lemma).

Basis = vector space basis = Hamel basis

Assume x_1, \dots, x_n is a basis for E . Then every basis for E must contain n different elements.

$$n = \dim E$$

is well-defined. (System of linear equations, homogeneous with more unknowns than equations. Then there exists a nontrivial solution.)

2 Normed Spaces and Banach Spaces

Definition (norm). E vector space. We say that $\|\cdot\| : E \rightarrow [0, \infty)$ is a norm on E if

- 1) $\|x\| = 0 \Rightarrow x = 0$,
- 2) $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in E, \lambda \in \mathbb{R}$,
- 3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in E$.

Remark.

$$\|0\| = \|0 \cdot 0\| = \underbrace{|0|}_{=0} \|0\| = 0.$$

Examples. (1) $1 < p < \infty$ and

$$\|X\|_{l^p} = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

is a norm on l^p . Check 1), 2) and 3) above:

1)

$$0 = \|X\|_{l^p} = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}.$$

It follows

$$\begin{aligned} x_n &= 0, \quad n = 1, 2, \dots, \\ \Rightarrow X &= (x_1, x_2, \dots) = (0, 0, \dots) = 0. \end{aligned}$$

2)

$$\|\lambda X\|_{l^p} = \left(\sum_{n=1}^{\infty} |\lambda x_n|^p \right)^{\frac{1}{p}} = \left(|\lambda|^p \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} = |\lambda| \|X\|_{l^p}$$

3)

$$\|X + Y\|_{l^p} \leq \{\text{Minkowski's inequality}\} \leq \|X\|_{l^p} + \|Y\|_{l^p}$$

(2) $E = C([0, 1])$ and $f \in E$

$$\|f\| = \max_{t \in [0, 1]} |f(t)| \in [0, \infty).$$

Check the axioms above

1) If $\|f\| = 0$ it follows

$$|f(t)| = 0 \text{ for all } t \in [0, 1], \quad \Rightarrow \quad f = 0$$

2)

$$\|\lambda f\| = \max_{t \in [0, 1]} \underbrace{|(\lambda f)(t)|}_{\substack{\lambda f(t) \\ |\lambda| |f(t)|}} = |\lambda| \max_{t \in [0, 1]} |f(t)| = |\lambda| \|f\|$$

3)

$$\|f + g\| = \max_{t \in [0, 1]} \underbrace{|(f + g)(t)|}_{f(t)+g(t)} = \max_{t \in [0, 1]} (|f(t)| + |g(t)|) \leq \max_{t \in [0, 1]} |f(t)| + \max_{t \in [0, 1]} |g(t)| = \|f\| + \|g\|$$

(3) $E = C([0, 1])$ and $f \in E$.

$$\|f\|_{L^1} = \int_0^1 |f(t)| dt$$

defines also a norm on E .

3)

$$\begin{aligned} \|f + g\|_{L^1} &= \int_0^1 \underbrace{|(f + g)(t)|}_{f(t)+g(t)} dt \\ &\leq \int_0^1 (|f(t)| + |g(t)|) dt \\ &= \int_0^1 |f(t)| dt + \int_0^1 |g(t)| dt \\ &= \|f\|_{L^1} + \|g\|_{L^1} \end{aligned}$$

2)

$$\|\lambda f\| = \int_0^1 \underbrace{|(\lambda f)(t)|}_{=|\lambda| |f(t)|} dt = |\lambda| \|f\|_{L^1}$$

1)

$$0 = \|f\|_{L^1} = \int_0^1 |f(t)| dt$$

This implies $f(t) = 0$ for $t \in [0, 1]$ since f is continuous! i.e. $f = 0$.

Theorem 2.1 (equivalent norm). E vector space with norms $\|\cdot\|$ and $\|\cdot\|_*$. We say that $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent if there exists $\alpha, \beta > 0$ such that

$$\alpha\|x\|_* \leq \|x\| \leq \beta\|x\|_* \quad \text{for all } x \in E.$$

Example.

$E = C([0, 1])$. Choose $y = f(t)$ and $y = |f(t)|$

$$\|f\| = \max_{t \in [0, 1]} |f(t)|, \quad \|f\|_* = \|f\|_{L^1} = \text{area}.$$

Question: Are these norms equivalent?

Claim: $f \in C([0, 1])$

$$\|f\|_* = \int_0^1 \underbrace{|f(t)|}_{\leq \|f\|} dt \leq \|f\|.$$

Choose $f_n(t)$ such that

$$\|f_n\| = 1, \quad \|f_n\|_* = \frac{1}{2n}.$$

So

$$\frac{\|f_n\|_*}{\|f_n\|} = \frac{1}{2n} \rightarrow 0 \quad n \rightarrow \infty.$$

The norms are not equivalent! Answer: NO !

Theorem 2.2. E vector space with $\dim E < \infty$.

\Rightarrow All norms on E are equivalent.

proof. Assume $n = \dim E$ with a positive integer n . Let x_1, x_2, \dots, x_n be a basis for E . For every $x \in E$

$$x = \alpha_1(x)x_1 + \dots + \alpha_n(x)x_n,$$

where $\alpha_1(x), \dots, \alpha_n(x)$ unique. Set

$$\|x\|_* = |\alpha_1(x)| + \dots + |\alpha_n(x)|, \quad x \in E$$

Claim: $\|\cdot\|_*$ defines a norm on E (easy proof)

Fix an arbitrary norm $\|\cdot\|$ on E .

Claim: $\|\cdot\|_*$ and $\|\cdot\|$ are equivalent.

Note for $x \in E$

$$\begin{aligned} \|x\| &= \|\alpha_1(x)x_1 + \dots + \alpha_n(x)x_n\| \\ &\leq |\alpha_1(x)|\|x_1\| + \dots + |\alpha_n(x)|\|x_n\| \\ &\leq \max_{k=1,2,\dots,n} \|x_k\| \underbrace{(|\alpha_1(x)| + \dots + |\alpha_n(x)|)}_{=\|x\|_*}. \end{aligned}$$

Set $\beta = \max_{k=1,2,\dots,n} \|x_k\|$.

Then

$$\|x\| \leq \beta \|x\|_* \quad \text{for all } x \in E.$$

Remains to prove: There exists $\alpha > 0$ such that

$$\alpha \|x\|_* \leq \|x\| \quad \text{for all } x \in E \quad (*).$$

Let E be a vector space with norm $\|\cdot\|$ and $(v_m)_{m=1}^\infty$ a sequence in E . We say that $(v_m)_{m=1}^\infty$ converges in $(E, \|\cdot\|)$ if there exists $v \in E$ such that $\|v_m - v\| \rightarrow 0$ for $n \rightarrow \infty$.

Notation: $v_m \rightarrow v$ in $(E, \|\cdot\|)$.

Note: If we have $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent, then

$$v_n \rightarrow v \text{ in } (E, \|\cdot\|) \quad \Leftrightarrow \quad v_n \rightarrow v \text{ in } (E, \|\cdot\|_*).$$

Back to (*): Argue by contradiction.

Assume there is no $\alpha > 0$ such that

$$\alpha \|x\|_* \leq \|x\| \quad \text{for all } x \in E.$$

For $k = 1, 2, 3, \dots$ there are $y_k \in E$ such that

$$\frac{1}{k} \|y_k\|_* > \|y_k\|. \quad (**).$$

We have

$$y_k = \alpha_1^{(k)} x_1 + \dots + \alpha_n^{(k)} x_n,$$

where $\alpha_1^{(k)}, \dots, \alpha_n^{(k)}$ are unique scalars and $k = 1, 2, \dots$

(**) implies that

$$k \|y_k\| < |\alpha_1^{(k)}| + \dots + |\alpha_n^{(k)}|,$$

WLOG we can assume $|\alpha_1^{(k)}| + \dots + |\alpha_n^{(k)}| = 1$. (If not consider

$$\begin{aligned} \lambda z &= \lambda(\alpha_1(z)x_1 + \dots + \alpha_n(z)x_n) \\ &= (\lambda\alpha_1(z))x_1 + \dots + (\lambda\alpha_n(z))x_n \\ &= \alpha_1(\lambda z)x_1 + \dots + \alpha_n(\lambda z)x_n. \end{aligned}$$

We have

$$\alpha_k(\lambda z) = \lambda \alpha_k(z), \quad k = 1, 2, \dots, n).$$

We have

$$k \|y_k\| < 1 \quad k = 1, 2, \dots$$

which implies $y_k \rightarrow 0$ in $(E, \|\cdot\|)$.

IF:

$$\begin{aligned} \alpha_1^{(k)} &\rightarrow \bar{\alpha}_1 \\ \alpha_2^{(k)} &\rightarrow \bar{\alpha}_2 \\ &\vdots \\ \alpha_n^{(k)} &\rightarrow \bar{\alpha}_n \end{aligned}$$

for $k \rightarrow \infty$. Then set

$$\bar{y} = \bar{\alpha}_1 x_1 + \dots + \bar{\alpha}_n x_n$$

and get

$$\begin{aligned} \|y_k - \bar{y}\| &= \left\| (\alpha_1^{(k)} - \bar{\alpha}_1)x_1 + \dots + (\alpha_n^{(k)} - \bar{\alpha}_n)x_n \right\| \\ &\leq \underbrace{|\alpha_1^{(k)} - \bar{\alpha}_1|}_{\rightarrow 0} \underbrace{\|x_1\|}_{< \infty} + \dots + \underbrace{|\alpha_n^{(k)} - \bar{\alpha}_n|}_{\rightarrow 0} \underbrace{\|x_n\|}_{< \infty} \rightarrow 0, \quad k \rightarrow \infty \\ \|\bar{y}\| &= \|\bar{y} - y_k + y_k\| \leq \underbrace{\|\bar{y} - y_k\|}_{\rightarrow 0} + \underbrace{\|y_k\|}_{\rightarrow 0} \rightarrow 0, \quad k \rightarrow \infty. \end{aligned}$$

So $\|\bar{y}\| = 0$ hence $\bar{y} = 0$. But

$$|\bar{\alpha}_1| + |\bar{\alpha}_2| + \dots + |\bar{\alpha}_n| = 1.$$

This contradicts x_1, \dots, x_n is a basis.

We have for $k = 1, 2, \dots$ the vector $(\alpha_1^{(k)}, \alpha_2^{(k)}, \dots, \alpha_n^{(k)})$ where

$$|\alpha_1^{(k)}| + \dots + |\alpha_n^{(k)}| = 1.$$

We focus on the first one and we have

$$|\alpha_1^{(k)}| \leq 1, \quad k = 1, 2, \dots$$

By Bolzano-Weierstraß then there exists a converging subsequence $(\alpha_{1,1}^{(k)})_{k=1}^\infty$ of $(\alpha_1^{(k)})_{k=1}^\infty$. Set

$$\bar{\alpha}_1 = \lim_{k \rightarrow \infty} \alpha_{1,1}^{(k)}$$

and consider

$$(\alpha_{1,1}^{(k)}, \alpha_{2,1}^{(k)}, \dots, \alpha_{n,1}^{(k)}), \quad k = 1, 2, \dots$$

We have

$$|\alpha_{2,1}^{(k)}| \leq 1, \quad k = 1, 2, \dots$$

Bolzano-Weierstraß implies that there exists a converging subsequence $(\alpha_{2,2}^{(k)})_{k=1}^\infty$ of $(\alpha_{2,1}^{(k)})_{k=1}^\infty$. Set

$$\bar{\alpha}_2 = \lim_{k \rightarrow \infty} \alpha_{2,2}^{(k)}.$$

□

Definition (normed space). Let E be a vector space over \mathbb{R} or \mathbb{C} . $\|\cdot\| : E \rightarrow \mathbb{R}$ a norm on E if

- (i) $\|x\| > 0$ for any $x \in E \setminus \{0\}$,
- (ii) $\|\lambda x\| = |\lambda| \|x\|$ for any $\lambda \in \mathbb{C}, x \in E$,
- (iii) $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in E$.

Obs. $\|x\| = 0$ if $x = 0$. $(E, \|\cdot\|)$ is called a normed space. A norm generates a distance

function (metric)

$$L(x, y) := \|x - y\| \quad \text{for any } x, y \in E.$$

Examples. • \mathbb{R}^n with $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$ is the eukledian norm.

- $C([0, 1])$ continuous functions in $[0, 1]$ with

$$L(f, g) = \|f - g\|_\infty := \max_{x \in [0, 1]} |f(x) - g(x)|$$

Definition (balls). Let $x \in E, r > 0$. Define

$$\begin{aligned} B(x, r) &:= \{y \in E \mid \|x - y\| < r\} && \text{open ball,} \\ \bar{B}(x, r) &:= \{y \in E \mid \|x - y\| \leq r\} && \text{closed ball.} \end{aligned}$$

Definition (open/closed). A subset $A \subset E$ of a normed space $(E, \|\cdot\|)$ is called open if any point x of A is inner, i.e

$$\exists r > 0 : B(x, r) \subset A.$$

It is called closed if the complement $E \setminus A$ is open.

Remark. • open balls are open sets.

- closed balls are closed.
- $(C([0, 1]), \|\cdot\|_\infty)$ with $\|f\|_\infty = \max_{x \in [0, 1]} |f(x)|$.

$$A := \{g \in C([0, 1]) \mid f(x) < g(x), \forall x \in [0, 1]\}$$

is an open set $C([0, 1])$.

$$B := \{g \in C([0, 1]) \mid f(x) \leq g(x), \forall x \in [0, 1]\}$$

is a closed set.

Properties

- Any union of open sets is an open set.
- Any finite intersection of open sets is open.
- \emptyset, E are both closed and open.
- Normed spaces are topological spaces.

Definition (convergence in normed spaces). Let $(E, \|\cdot\|)$ be a normed space $\{x_n\}_n \subset E$. We say that x_n converges to $x \in E$ if

$$\|x_n - x\| \rightarrow 0, \quad n \rightarrow \infty.$$

One can define open and closed using the definition of convergence:

Statement 2.3. $A \subseteq E$ is closed if any convergent sequence in A has a limit in A , i.e

$$\begin{matrix} x_n \rightarrow x \\ \text{for } n \rightarrow \infty \\ x_n \in A \end{matrix} \Rightarrow x \in A.$$

proof. \Rightarrow : Assume that A is closed and $x_n \rightarrow x$. $x_n \in A$, but $x_n \notin A$. (try to get a contradiction).

A is closed $\Rightarrow E \setminus A$ is open and hence $\exists r > 0$ such that

$$B(x, r) \subset E \setminus A.$$

Hence $\|x_n - x\| \geq r$ for any n . This is a contradiction because in that case $x_n \not\rightarrow x$.

\Leftarrow : Assume that for any sequence $\{x_n\} \subset A$ such that $x_n \rightarrow x$ we have $x \in A$. We try to get a contradiction and assume that A is not closed. Hence $E \setminus A$ is not open and therefore $\exists x \in E \setminus A$ which is not inner.

$$\Rightarrow \quad \forall B(x, \frac{1}{n}) \text{ contains points outside } E \setminus A,$$

i.e.

$$\exists x_n \in B(x, \frac{1}{n}), x_n \in A.$$

We get a sequence $\{x_n\} \subset A$ such that

$$\|x_n - x\| < \frac{1}{n} \quad \Rightarrow \quad x_n \rightarrow x.$$

This is a contradiction. □

Definition (closure). $A \subset E$. The closure of A is the minimal closed subset containing A . We write \bar{A} .

Proposition 2.4. \bar{A} is the set of all limit points of A which means

$$\bar{A} := \{x \in E \mid \text{there exists } \{x_n\} \subseteq A \text{ such that } x_n \rightarrow x\}.$$

proof. Exercise. □

Definition (dense). $A \subset E$ is dense in E if

$$\bar{A} = E.$$

Remark. This definition of dense is equivalent to the following definition:

$$\forall x \in E, \forall \varepsilon > 0 \exists y \in A \text{ such that } \|x - y\| < \varepsilon.$$

Examples. 1) $\mathbb{Q} \subseteq \mathbb{R}$ with $|\cdot|$ usual absolute value function. \mathbb{Q} is dense in \mathbb{R} .

2) $C([a, b])$. The Weierstraß-Theorem says that the set of all polynomials are dense in $(C([a, b], \|\cdot\|_\infty))$:

$$\forall f \in C([a, b]), \forall \varepsilon > 0 \exists p - \text{polynomial such that } \max_{x \in [a, b]} |f(x) - p(x)| < \varepsilon.$$

Another example is $(C_0, \|\cdot\|_\infty)$ where

$$C_0 = \{x = (x_1, x_2, \dots) \mid x_k \rightarrow 0 \text{ as } k \rightarrow \infty\},$$

$$\|x\|_\infty = \sup_i |x_i|.$$

$(C_0, \|\cdot\|_\infty)$ is a normed space.

$$C_F = \{x = (x_1, x_2, \dots) \mid \text{only a finite number of } x_i \neq 0\} \subset C_0.$$

Statement 2.5. C_F is dense in C_0 .

proof.

$$\forall x \in C_0 \forall \varepsilon > 0 \text{ must find } y \in C_F \text{ such that } \|y - x\|_\infty < \varepsilon.$$

$$x \in C_0 \quad \Rightarrow \quad x_k \rightarrow 0 \text{ for } k \rightarrow \infty$$

$$\Rightarrow \quad \forall \varepsilon > 0 \exists K \text{ such that } |x_k| < \varepsilon \forall k \geq K.$$

Let now $y = (x_1, x_2, \dots, x_K, 0, \dots) \in C_F$. Then

$$\|x - y\|_\infty = \|(0, 0, \dots, 0, x_{K+1}, x_{K+2}, \dots)\|_\infty = \sup_{k > K} |x_k| < \varepsilon.$$

□

Definition (separable). A normed space $(E, \|\cdot\|)$ is called separable if it contains a countable dense subset.

Examples. • $(\mathbb{R}, |\cdot|)$ is separable as \mathbb{Q} is countable and dense in \mathbb{R} .

• $(\mathbb{R}^n, \|\cdot\|_2)$ is separable, \mathbb{Q}^n is countable and dense in \mathbb{R}^n .

Definition (compact set). For a normed space $(E, \|\cdot\|)$ is $A \subset E$ a compact set if any sequence $\{x_n\} \subset A$ has a subsequence convergent to an element $x \in A$.

Example. Any bounded and closed subset in $\mathbb{R}, \mathbb{R}^n, \mathbb{C}^n$ is compact. A sequence $\{x_n\}$ of a bounded set is bounded. From real Analysis one knows it has a subsequence that is convergent. If the subset is closed then the limit point is inside the set.

Lemma . $S \subset \text{compact in } (E, \|\cdot\|)$ implies that S is closed and bounded. (Bounded means that $S \subset B(0, R)$ for some $R > 0$).

proof. Let S be a compact subset of E . Assume that S is not bounded. Hence for any $n > 0$ there exists points in S which are outside $B(0, n)$, i.e.

$$\exists x_n \in S : \|x_n\| > n.$$

Then $\{x_n\}$ can not have a convergent subsequence as if $x_{n_k} \rightarrow x$ then

$$n_k < \|x_{n_k}\| = \|x_{n_k} - x + x\| \leq \|x_{n_k} - x\| + \|x\| \rightarrow \|x\|$$

but $n_k \rightarrow \infty$. This is a contradiction, hence S must be bounded.

S must be closed, because if $x_n \rightarrow x$ then any subsequence converges to x . From the definition of compactness and uniqueness of the limit we have $x \in S$.

□

Remark. In general, S bounded and closed doesn't imply that S is compact.

For instance let $E = C([0, 1])$. Then $S = \{g \in C([0, 1]) : \|g\|_\infty \leq 1\}$ is closed and bounded, but not compact.

Take $x_n(t) := t^n$. Then $x_n \in S$. $\{x_n\}$ does not have a subsequence convergent to a continuous function.

Theorem 2.6. $(E, \|\cdot\|)$ normed space and $\dim E < \infty$
iff

$$\forall A \subset E, A \text{ compact} \Leftrightarrow A \text{ is closed and bounded.}$$

proof. \Rightarrow : If $\dim E < \infty$ then A is compact iff A is bounded and closed (exercise).

\Leftarrow : Enough to prove the following:

If $\dim E = \infty$ then the unit ball $S = \{x \in E : \|x\| \leq 1\}$ is not compact.

Lemma 2.7 (Riesz's lemma). If X is a proper closed subspace of a normed space $(E, \|\cdot\|)$ then for every $\varepsilon \in (0, 1)$ there exists an $x_\varepsilon \in E$ with $\|x_\varepsilon\| = 1$ such that

$$\|x_\varepsilon - x\| \geq \varepsilon \quad \forall x \in X.$$

proof. Let $z \in E \setminus X$ (X proper and hence $E \setminus X \neq \emptyset$). Set

$$d := \inf_{x \in X} \|z - x\|.$$

As X is closed, $d > 0$, otherwise z is a limit point in $E \setminus X$. Fix $\varepsilon \in (0, 1)$. Then there exists $x_0 \in X$ such that

$$d \leq \|z - x_0\| < \frac{d}{\varepsilon}.$$

Let $x_\varepsilon := \frac{z - x_0}{\|z - x_0\|}$; We have $\|x_\varepsilon\| = 1$ and

$$\begin{aligned} \|x - x_\varepsilon\| &= \left\| x - \frac{z - x_0}{\|z - x_0\|} \right\| \\ &= \frac{\|x\|z - x_0\| - z + x_0\|}{\|z - x_0\|} \\ &= \frac{\left\| \overbrace{x\|z - x_0\|}^{\in X} + x_0 - z \right\|}{\|z - x_0\|} \\ &\geq \frac{d}{d} \varepsilon = \varepsilon. \end{aligned}$$

□

Continue now the proof of the theorem above:

Let $x_1 \in S$. Consider $X = \text{span}\{x_1\}$ which is a proper closed subspace of E . Hence by Riesz's lemma exists x_2 with $\|x_2\| = 1$ such that

$$\|x_2 - x_1\| \geq \frac{1}{2}$$

and

$$\|x_2 - x\| \geq \frac{1}{2} \quad \forall x \in X.$$

Now consider $\text{span}\{x_1, x_2\}$ which is a proper closed subspace of E . By Riesz's lemma follows

$$\exists x_3 \in E, \|x_3\| = 1 : \|x_3 - x_1\| \geq \frac{1}{2}, \|x_3 - x_2\| \geq \frac{1}{2}.$$

Continuing in the same fashion we get $\{x_n\}$, $\|x_n\| = 1$ such that

$$\|x_n - x_m\| \geq \frac{1}{2} \quad \forall n, m, n \neq m.$$

Clearly $\{x_n\} \subset S$ has no convergent subsequence. Hence S is not compact. □

Definition (Cauchy sequence). $(E, \|\cdot\|)$ normed space. $\{x_n\} \subseteq E$ is called Cauchy if

$$\forall \varepsilon > 0 \exists N : \|x_n - x_m\| < \varepsilon \text{ for any } n, m \geq N.$$

Example. $(C_F, \|\cdot\|_\infty)$, $\|x\|_\infty = \sup_{k \in \mathbb{N}} |x_k|$ where $x = (x_1, x_2, \dots)$. Define

$$x_n = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, \dots).$$

Then $\{x_n\}$ is Cauchy, as for $n > m$

$$\begin{aligned} \|x_n - x_m\|_\infty &= \left\| (0, \dots, 0, \frac{1}{m+1}, \dots, \frac{1}{n}, 0, \dots) \right\|_\infty \\ &= \frac{1}{m+1}. \end{aligned}$$

Observe that x_n is convergent in $(C_0, \|\cdot\|_\infty)$

$$\underbrace{x_n}_{\in C_F} \rightarrow (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots) \in C_0 \setminus C_F.$$

Statement 2.8. A convergent sequence is always a Cauchy sequence.

Definition (complete space). A normed vector space $(E, \|\cdot\|)$ is called complete if any Cauchy sequence in E is convergent in E .

$(C_F, \|\cdot\|_\infty)$ is not complete.

Definition (Banach space). A complete normed space is called Banach space.

Examples. • $(\mathbb{R}, |\cdot|)$ is a Banach space.

• $(\mathbb{C}, |\cdot|)$ is a Banach space.

• $(l^2, \|\cdot\|_2)$ where

$$l^2 = \left\{ (x_1, x_2, \dots) \left| \sum_{i=1}^{\infty} |x_i|^2 < \infty, x_i \in \mathbb{C} \right. \right\}$$

and

$$\|(x_1, x_2, \dots)\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}.$$

$(l^2, \|\cdot\|_2)$ is complete.

proof. Let $x_n = (x_1^n, x_2^n, \dots)$ be a Cauchy sequence in l^2 . We must show that it has a limit in l^2 . We will do it in a few steps:

Step 1: Find a candidate for a limit a .

Step 2: Show that $a \in l^2$.

Step 3: $\|x_n - a\|_2 \rightarrow 0$ as $n \rightarrow \infty$.

Step 1: Let

$$\begin{aligned} x_1 &= (x_1^1, x_2^1, \dots) \\ x_2 &= (x_1^2, x_2^2, \dots) \\ &\vdots \\ x_n &= (x_1^n, x_2^n, \dots). \end{aligned}$$

For each k consider sequence $\{x_k^n\} \subset \mathbb{C}$ (k -th coordinates in each x_n).
Each sequence is Cauchy, as for all $n, m \geq N$

$$|x_k^n - x_k^m| < \left(\sum_{k=1}^{\infty} |x_k^n - x_k^m|^2 \right)^{\frac{1}{2}} = \|x_n - x_m\|_2 < \varepsilon.$$

As $(\mathbb{C}, |\cdot|)$ is complete, $\{x_k^n\}_n$ has a limit $a_k \in \mathbb{C}$. Candidate for limit of x_n is

$$a = (a_1, a_2, \dots, a_k, \dots).$$

Step 2: Write

$$a = \underbrace{x_n}_{\in l^2} - (x_n - a).$$

In order to show that $a \in l^2$ it is enough to see that $x_n - a \in l^2$ for some n .
 $\{x_n\}$ Cauchy implies

$$\forall \varepsilon > 0 \exists N : \forall n, m \geq N : \|x_n - x_m\|_2 < \varepsilon.$$

Consider for some $u > 0$

$$\sum_{i=1}^u |x_i^n - x_i^m|^2 \leq \sum_{i=1}^{\infty} |x_i^n - x_i^m|^2 = \|x_n - x_m\|_2^2 < \varepsilon^2.$$

Let $m \rightarrow \infty$. We get

$$\sum_{i=1}^u |x_i^n - a_i|^2 \leq \varepsilon^2.$$

This holds for any $u \in \mathbb{N}$. Hence for any $n \geq N$

$$\underbrace{\sum_{i=1}^{\infty} |x_i^n - a_i|^2}_{=\|x_n - a\|_2^2} \leq \varepsilon^2.$$

Hence $x_n - a \in l^2$ and moreover $\|x_n - a\| \rightarrow 0$ as $n \rightarrow \infty$.



- $(C([a, b]), \|\cdot\|_\infty)$ is a Banach space.
- $(l^p, \|\cdot\|_{l^p})$ for $1 \leq p < \infty$ are all Banach spaces.
- $(C([a, b]), \|\cdot\|_2)$ with

$$\|f\|_2 = \left(\int |f(t)|^2 dt \right)^{\frac{1}{2}}.$$

One can prove that $(C([a, b]), \|\cdot\|_2)$ is not a Banach space.

Exercise:

$[a, b] = [0, 1]$ and

$$f_n(t) = \begin{cases} 0, & \text{falls } t < \frac{1}{2} - \frac{1}{n} \\ 1, & \text{falls } t > \frac{1}{2} \\ \text{continuous linear function} & \end{cases}$$

Show that $\{f_n\}$ is Cauchy in $C([0, 1], \|\cdot\|_2)$ but $f_n \not\rightarrow f \in C([0, 1])$.

Definition (Convergent and absolutely convergent series). A series $\sum_{n=1}^{\infty} x_n$ in E is called convergent if $\{\sum_{n=1}^m x_n\}_m$, a sequence of partial sums, is convergent in E . If $\sum_{n=1}^{\infty} \|x_n\| < \infty$ then we say that $\sum_{n=1}^{\infty} x_n$ converges absolutely.

Theorem 2.9. A normed space E is complete iff every absolutely convergent series converges in E .

proof. \Rightarrow : Suppose X is complete and $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Let

$$S_N := \sum_{n=1}^N x_n \in E.$$

For $M > N$:

$$\begin{aligned} \|S_N - S_M\| &= \left\| \sum_{n=N+1}^M x_n \right\| \\ &\leq \sum_{n=N+1}^M \|x_n\| \\ &\leq \sum_{n=N+1}^{\infty} \|x_n\| \rightarrow 0 \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Hence $\{S_N\}$ is Cauchy. As E is complete, S_N has a limit in E i.e. $\sum_{n=1}^{\infty} x_n$ converges in E .

\Leftarrow : Assume that every absolutely convergent series is convergent in E . We want to see that E is complete.

Let $\{x_n\}$ be a Cauchy sequence. We want to prove that $\{x_n\}$ has a limit in E . We know that

$$\forall k \exists n_k : \|x_n - x_m\| < \frac{1}{2^k} \quad \forall n, m \geq n_k.$$

We can assume that $\{n_k\}$ is an increasing sequence. Write

$$x_{n_k} = (x_{n_k} - x_{n_{k-1}}) + (x_{n_{k-1}} - x_{n_{k-2}}) + \dots + (x_{n_1} - \underbrace{x_{n_0}}_{=0}) = \sum_{l=1}^k (x_{n_l} - x_{n_{l-1}}).$$

$$\sum_{l=1}^{\infty} \|x_{n_l} - x_{n_{l-1}}\| \leq \sum_{l=1}^{\infty} \frac{1}{2^l} < \infty.$$

Hence $\sum_{l=1}^{\infty} (x_{n_l} - x_{n_{l-1}})$ is absolutely convergent. By assumption

$$\sum_{l=1}^{\infty} (x_{n_l} - x_{n_{l-1}})$$

is convergent in E . Hence the partial sums are convergent. Subsequence is convergent. $\{x_{n_k}\}$ is convergent to some $x \in E$.

Exercise:

Show that the whole $\{x_n\} \rightarrow x$.

□

Recall:

converging sequences $(x_n)_{n=1}^{\infty}$ in $(E, \|\cdot\|)$. $\|x_n - x\| \rightarrow 0$ for $n \rightarrow \infty$ for some $x \in E$. (Notation: $x_n \rightarrow x$ in $(E, \|\cdot\|)$)

Remark. Assume $x_n \rightarrow x$ in $(E, \|\cdot\|)$. Then

$$1) \|x_n\| \rightarrow \|x\| \text{ in } (E, \|\cdot\|).$$

$$2) \sup_n \|x_n\| < \infty.$$

because

1)

$$\|x_n\| \leq \|x_n - x\| + \|x\|,$$

so

$$\|x_n\| - \|x\| \leq \|x_n - x\|.$$

It follows

$$-(\|x_n\| - \|x\|) \leq \|x_n - x\|.$$

So

$$\|x_n\| - \|x\| \leq \|x_n - x\| \rightarrow 0, \quad \text{for } n \rightarrow \infty.$$

Cauchy sequence in $(x_n)_{n=1}^\infty$ in $(E, \|\cdot\|)$ if $\|x_n - x_m\| \rightarrow 0$ for $n, m \rightarrow \infty$.

We obtain: $(x_n)_{n=1}^\infty$ converges in $(E, \|\cdot\|) \Rightarrow (x_n)_{n=1}^\infty$ Cauchy sequence in $(E, \|\cdot\|)$. (\Leftarrow in general). If \Leftarrow then we call $(E, \|\cdot\|)$ a Banach space.

$\sum_{n=1}^\infty x_n$ converges in $(E, \|\cdot\|)$ if $\left(\sum_{n=1}^k x_n\right)_{k=1}^\infty$ converges in $(E, \|\cdot\|)$.

$\sum_{n=1}^\infty x_n$ converges absolutely in $(E, \|\cdot\|)$ if $\sum_{n=1}^\infty \|x_n\|$ converges $(\mathbb{R}, \|\cdot\|)$.

2.1 Mappings between normed spaces

Definition . Let $(E_1, \|\cdot\|_1)$, $(E_2, \|\cdot\|_2)$ be normed spaces. $T : E_1 \rightarrow E_2$ (not necessarily linear) is called continuous at $x_0 \in E_1$, if

$$x_n \rightarrow x_0 \text{ in } (E_1, \|\cdot\|_1) \Rightarrow T(x_n) \rightarrow T(x_0) \text{ in } (E_2, \|\cdot\|_2).$$

T is called continuous if it is continuous at $x_0 \in E_1$ for all $x_0 \in E_1$. We say that $T : E_1 \rightarrow E_2$ is linear if

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$$

for all scalars λ_1, λ_2 and $x_1, x_2 \in E_1$.

$T : E_1 \rightarrow E_2$ linear is called bounded if there exists $M > 0$ such that

$$\|T(x)\|_2 \leq M\|x\|_1 \quad \text{for all } x \in E_1.$$

If T is bounded linear $E_1 \rightarrow E_2$ define

$$\|T\| = \|T\|_{E_1 \rightarrow E_2} := \inf\{M \geq 0 \mid \|T(x)\|_2 \leq M\|x\|_1 \text{ for all } x \in E_1\}.$$

Lemma .

$$\|T\| = \sup_{\substack{x \in E_1 \\ x \neq 0}} \frac{\|T(x)\|_2}{\|x\|_1} = \sup_{\substack{x \in E_1 \\ \|x\|_1 = 1}} \|T(x)\|_2.$$

Proposition 2.10. Assume $T : E_1 \rightarrow E_2$ linear. Then all the following statements are equivalent:

- (1) T continuous at $0 \in E_1$.
- (2) T continuous at $x_0 \in E_1$ for some $x_0 \in E_1$.
- (3) T continuous at $x_0 \in E_1$ for all $x_0 \in E_1$.

(4) T is bounded.

proof. (1) \Rightarrow (4): Assume T is continuous at $0 \in E_1$, i.e.

$$x_n \rightarrow 0 \text{ in } (E_1, \|\cdot\|_1) \quad \Rightarrow \quad T(x_n) \rightarrow T(\underbrace{0}_{\in E_1}) = \underbrace{0}_{\in E_2} \text{ in } (E_2, \|\cdot\|_2).$$

We want to prove that T is bounded. We search a $M > 0$ such that

$$\|T(x)\|_2 \leq M\|x\|_1.$$

We assume that this doesn't hold true.

For $n = 1, 2, \dots$ there exists $x_n \in E_1$ such that

$$\|T(x_n)\|_2 > n\|x_n\|_1.$$

Set for $n = 1, 2, \dots$

$$z_n := \frac{1}{n\|x_n\|_1} x_n.$$

(Note that $\|x_n\|_1 > 0$. Otherwise we would get a contradiction.)

Note

$$\|z_n\|_1 = \left\| \frac{1}{n\|x_n\|_1} \right\|_1 = \frac{1}{n\|x_n\|_1} \|x_n\|_1 = \frac{1}{n} \rightarrow 0, \quad \text{for } n \rightarrow \infty.$$

We have $z_n \rightarrow 0$ in $(E_1, \|\cdot\|_1)$. But

$$\|T(z_n)\|_2 = \left\| \frac{1}{n\|x_n\|_1} T(x_n) \right\|_2 = \frac{1}{n\|x_n\|_1} \|T(x_n)\|_2 > 1 \quad \text{for all } n.$$

Hence

$$T(z_n) \not\rightarrow 0 \quad \text{in } (E_2, \|\cdot\|_2).$$

This is a contradiction.

(1) \Leftarrow (4): Assume T is bounded. For some $M > 0$

$$\|T(x)\|_2 \leq M\|x\|_1, \quad \text{for all } x \in E_1.$$

We need to show that T is continuous at $0 \in E_1$, i.e.

$$x_n \rightarrow 0 \text{ in } (E_1, \|\cdot\|_1) \quad \Rightarrow \quad T(x_n) \rightarrow T(0) = 0 \text{ in } (E_2, \|\cdot\|_2).$$

From

$$\|T(x_n)\|_2 \leq M\|x_n\|_1 \rightarrow 0$$

so

$$T(x_n) \rightarrow \underbrace{0}_{=T(0)} \text{ in } (E_2, \|\cdot\|_2).$$

□

Examples. (A) $E_1 = E_2 = C([0, 1])$, $\|\cdot\|_1 = \|\cdot\|_2 = \|\cdot\|_\infty =: \|\cdot\|$, i.e.

$$\|f\| := \max_{x \in [0, 1]} |f(x)|.$$

$$T(f)(x) = \int_0^{1-x} \min(x, y) f(y) dy, \quad \text{for } f \in C([0, 1]), x \in [0, 1].$$

(1) $T(f) \in C([0, 1])$ for $f \in C([0, 1])$,

(2) T linear,

(3) T bounded,

(4) Calculate $\|T\|$.

proof. (1) Fix $f \in C([0, 1])$ arbitrary and fix $x \in [0, 1]$. Show that $T(f)$ is continuous at x . Consider a sequence $(x_n)_{n=1}^\infty$ in $[0, 1]$ such that $x_n \rightarrow x$ in $(\mathbb{R}, |\cdot|)$.

To show $T(f)(x_n) \rightarrow T(f)(x)$ in $(\mathbb{R}, |\cdot|)$.

$$\begin{aligned} |T(f)(x_n) - T(f)(x)| &= \{\text{assume that } x_n \leq x\} \\ &= \left| \int_0^{1-x_n} \min(x_n, y) f(y) dy - \int_0^{1-x} \min(x, y) f(y) dy \right| \\ &\leq \left| \int_0^{1-x} (\min(x_n, y) - \min(x, y)) f(y) dy \right| \\ &\quad + \left| \int_{1-x}^{1-x_n} \min(x_n, y) f(y) dy \right| \\ &\leq \underbrace{\int_0^{1-x} \underbrace{|\min(x_n, y) - \min(x, y)|}_{\leq |x_n - x|} \underbrace{|f(y)|}_{\leq \|f\|} dy}_{\leq |x_n - x| \|f\|} \\ &\quad + \underbrace{\int_{1-x}^{1-x_n} \underbrace{\min(x_n, y)}_{\leq 1} \underbrace{|f(y)|}_{\leq \|f\|} dy}_{0 \leq \dots \leq |x_n - x| \cdot \|f\|} \rightarrow 0, \quad \text{as } n \rightarrow \infty \end{aligned}$$

If $x_n > x$ we get a similar calculation. Conclusion:

$$T(f)(x_n) \rightarrow T(f)(x) \text{ in } (\mathbb{R}, |\cdot|) \text{ as } n \rightarrow \infty.$$

(2) Fix $f_1, f_2 \in C([0, 1])$ and λ_1, λ_2 scalars. Then

$$\begin{aligned} T(\lambda_1 f_1 + \lambda_2 f_2)(x) &= \int_0^{1-x} \min(x, y) \underbrace{(\lambda_1 f_1 + \lambda_2 f_2)(y)}_{= \lambda_1 f_1(y) + \lambda_2 f_2(y)} dy \\ &= \lambda_1 \int_0^{1-x} \min(x, y) f_1(y) dy + \lambda_2 \int_0^{1-x} \min(x, y) f_2(y) dy \\ &= \lambda_1 T(f_1)(x) + \lambda_2 T(f_2)(x) \quad \text{for } x \in [0, 1] \end{aligned}$$

(3) Fix $f \in C([0, 1])$. For $x \in [0, 1]$

$$\begin{aligned}
 |T(f)(x)| &= \left| \int_0^{1-x} \underbrace{\min(x, y)f(y)}_{\geq 0} dy \right| \\
 &\stackrel{(*_1)}{\leq} \int_0^{1-x} \min(x, y) \underbrace{|f(y)|}_{\leq \|f\|} dy \\
 &\stackrel{(*_2)}{\leq} \int_0^{1-x} \min(x, y) dy \|f\|.
 \end{aligned}$$

Clearly

$$\max_{x \in [0, 1]} \int_0^{1-x} \min(x, y) dy \leq 1.$$

This gives:

$$\|T(f)\| = \max_{x \in [0, 1]} |T(f)(x)| \leq 1 \cdot \|f\|, \quad \text{for all } f \in C([0, 1]).$$

Conclusion: T is bounded with $(M = 1)$

- (4) Consider the inequality above. $(*_1)$ is an equality if f has a constant sign. $(*_2)$ is an equality if f is a constant function. So we have to calculate

$$\int_0^{1-x} \min(x, y) dy \quad \text{for } x \in [0, 1].$$

case 1: $1 - x \leq x$ i.e. $\frac{1}{2} \leq x$ and we get

$$\begin{aligned}
 \int_0^{1-x} \underbrace{\min(x, y)}_{=y} dy &= \left[\frac{1}{2} y^2 \right]_0^{1-x} \\
 &= \frac{1}{2} (1-x)^2.
 \end{aligned}$$

case 2: $x < 1 - x$ i.e. $x < \frac{1}{2}$ and we get

$$\begin{aligned}
 \int_0^{1-x} \min(x, y) dy &= \int_0^x y dy + \int_x^{1-x} x dy \\
 &= \frac{1}{2} x^2 + x(1-2x) \\
 &= x - \frac{3}{2} x^2.
 \end{aligned}$$

Claim:

$$\|T\| = \max \left(\max_{x \in [\frac{1}{2}, 1]} \frac{1}{2} (1-x)^2, \max_{x \in [0, \frac{1}{2}]} \left(x - \frac{3}{2} x^2 \right) \right) = \dots = \frac{1}{6}.$$

Note

- $\|T(f)\| \leq \|T\| \cdot \|f\|$ for all $f \in C([0, 1])$,
- $\|T(1)\| = \|T\| \cdot \|1\|$ where $1(x) = 1$ for $x \in [0, 1]$.

□

(B) $E_1 = C([0, 1])$ with maximumnorm, $E_2 = \mathbb{R}$ with absolut value. $T : E_1 \rightarrow E_2$ with

$$T(f) = \int_0^{\frac{1}{2}} f(y) dy - \int_{\frac{1}{2}}^1 f(y) dy \quad \text{for } f \in E_1$$

$$\begin{aligned} |T(f)| &= \left| \int_0^{\frac{1}{2}} f(y) dy - \int_{\frac{1}{2}}^1 f(y) dy \right| \\ &\leq \left| \int_0^{\frac{1}{2}} f(y) dy \right| + \left| \int_{\frac{1}{2}}^1 f(y) dy \right| \\ &\leq \int_0^{\frac{1}{2}} \underbrace{|f(y)|}_{\leq \|f\|} dy + \int_{\frac{1}{2}}^1 \underbrace{|f(y)|}_{\leq \|f\|} dy \\ &\leq 1 \|f\|. \end{aligned}$$

Hence T is bounded and $\|T\| \leq 1$.

$$T(f) = \int_0^1 k(y) f(y) dy,$$

where

$$T(f_n) = \left\{ \begin{array}{l} \text{to be completed,} \\ \text{falls case .} \end{array} \right.$$

$$T(f_n) \leq 1 \left(\frac{1}{2} - \frac{1}{2n} + \frac{1}{2} - \frac{1}{2n} \right) = 1 - \frac{1}{n}, \quad n = 1, 2, \dots$$

Note

$$k(y) f_n(y) \geq 0 \quad \text{for } y \in [0, 1].$$

Hence $\|T\| \leq 1 - \frac{1}{n}$ for $n = 1, 2, \dots$. Note $\|f_n\| = 1$ for all n . Conclusion $\|T\| = 1$.
Here

$$|T(f)| \leq \underbrace{\|T\|}_{\leq 1} \|f\| \quad \text{for all } f \in C([0, 1])$$

but

$$|T(f)| < \|T\| \|f\| \quad \text{for all } f \in C([0, 1]).$$

Statement 2.11. T_1, T_2 bounded linear mappings $(E_1, \|\cdot\|_1) \rightarrow (E_2, \|\cdot\|_2)$ and λ scalar. Set

$$\begin{aligned} (T_1 + T_2)(x) &= T_1(x) + T_2(x) \quad x \in E_1 \\ (\lambda T_1)(x) &= \lambda T_1(x) \quad x \in E_1. \end{aligned}$$

Claim:

- (1) $T_1 + T_2$ and λT_1 are both linear mappings $(E_1, \|\cdot\|_1) \rightarrow (E_2, \|\cdot\|_2)$,
- (2) $T_1 + T_2$ and λT_1 are both bounded mappings $(E_1, \|\cdot\|_1) \rightarrow (E_2, \|\cdot\|_2)$.
 $B(E_1, E_2)$ denote the vector space of all bounded linear mappings $(E_1, \|\cdot\|_1) \rightarrow (E_2, \|\cdot\|_2)$.
- (3)
- $$\|T\|_{E_1 \rightarrow E_2} := \inf\{M > 0 \mid \|T(x)\|_2 \leq M\|x\|_1 \text{ for all } x \in E_1\}$$
- defines a norm in $B(E_1, E_2)$.

proof. (1) $\|T\| = 0$ implies that $\|T(x)\|_2 = 0$ for all $x \in E_1 \Rightarrow T(x) = 0 \in E_2$.

$$T = 0 \in B(E_1, E_2)$$

(2) $T \in B(E_1, E_2)$ and λ scalar.

$$\begin{aligned} \|\lambda T\| &= \inf\{M > 0 \mid \|(\lambda T)(x)\|_2 \leq M\|x\|_1 \text{ for all } x \in E_1\} \\ &= \inf\{M > 0 \mid |\lambda| \|T(x)\|_2 \leq M\|x\|_1 \text{ for all } x \in E_1\} \\ &= \{\text{if } \lambda \neq 0\} \\ &= \inf\left\{ \underbrace{M}_{=|\lambda|\tilde{M}} > 0 \mid \|T(x)\|_2 \leq \underbrace{\frac{M}{|\lambda|}}_{=\tilde{M}} \|x\|_1 \text{ for all } x \in E_1 \right\} \\ &= |\lambda| \inf\left\{ \tilde{M} > 0 \mid \|T(x)\|_2 \leq \tilde{M}\|x\|_1 \text{ for all } x \in E_1 \right\} \\ &= |\lambda| \|T\| \end{aligned}$$

(3) Set $T_1, T_2 \in B(E_1, E_2)$.

$$\begin{aligned} \|T_1 + T_2\| &= \inf\{M > 0 \mid \|(T_1 + T_2)(x)\|_2 \leq M\|x\|_1 \text{ for all } x \in E_1\} \\ &\leq \inf\{M_1 + M_2 > 0 \mid \|T_1(x)\|_2 \leq M_1\|x\|_1, \|T_2(x)\|_2 \leq M_2\|x\|_1 \text{ for all } x \in E_1\} \\ &= \|T_1\| + \|T_2\| \end{aligned}$$

□

Conclusion: $(B(E_1, E_2), \|\cdot\|_{E_1 \rightarrow E_2})$ is a normed space.

Statement 2.12. $(B(E_1, E_2), \|\cdot\|_{E_1 \rightarrow E_2})$ is a Banach space if $(E_2, \|\cdot\|_2)$ is a Banach space.

proof. Assume $(T_n)_{n=1}^\infty$ is a Cauchy sequence in $(B(E_1, E_2), \|\cdot\|_{E_1 \rightarrow E_2})$ where $(E_2, \|\cdot\|_2)$ is a Banach space. Fix $x \in E_1$

$$\begin{aligned} \|T_n(x) - T_m(x)\|_2 &= \|(T_n - T_m)(x)\|_2 \\ &\leq \underbrace{\|T_n - T_m\|_{E_1 \rightarrow E_2}}_{\substack{\rightarrow 0 \\ n, m \rightarrow \infty}} \cdot \|x\|_1 \rightarrow 0, \quad n, m \rightarrow \infty. \end{aligned}$$

Hence $(T_n(x))_{n=1}^\infty$ is a Cauchy sequence in $(E_2, \|\cdot\|_2)$. This is a Banach space which implies that $(T_n(x))_{n=1}^\infty$ converges in $(E_2, \|\cdot\|_2)$. Call the limit $T(x) \in E_2$ for all $x \in E_1$. Show now

- (1) $T : E_1 \rightarrow E_2$ is linear,
- (2) T is bounded,
- (3) $\|T_n - T\|_{E_1 \rightarrow E_2} \rightarrow 0$ for $n \rightarrow \infty$.

(1) Observe

$$\begin{aligned} T(\lambda_1 x_1 + \lambda_2 x_2) &\leftarrow T_n(\lambda_1 x_1 + \lambda_2 x_2) = \{T \text{ linear}\} = \lambda_1 \underbrace{T_n(x_1)}_{\rightarrow T(x_1)} + \lambda_2 \underbrace{T_n(x_2)}_{\rightarrow T(x_2)} \\ &\quad \underbrace{\rightarrow \lambda_1 T(x_1)}_{\rightarrow \lambda_1 T(x_1)} + \underbrace{\rightarrow \lambda_2 T(x_2)}_{\rightarrow \lambda_2 T(x_2)} \\ &\quad \rightarrow \lambda_1 T(x_1) + \lambda_2 T(x_2) \end{aligned}$$

So for $n \rightarrow \infty$ it is

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2) \quad \text{in } (E_2, \|\cdot\|_2).$$

(2) Fix $\varepsilon > 0$. Then there exists N such that:

$$\|T_n - T_m\|_{E_1 \rightarrow E_2} < \varepsilon \quad \text{for } n, m \geq N$$

So for $x \in E_1$

$$\|T_n(x) - T_m(x)\|_2 \leq \|T_n - T_m\|_{E_1 \rightarrow E_2} \|x\|_1 < \varepsilon \|x\|_1 \quad \text{for } n, m \geq N.$$

Let $m \rightarrow \infty$.

$$\|T_n(x) - T(x)\|_2 \leq \varepsilon \|x\|_1 \quad \text{for } n \geq N$$

So

$$\begin{aligned} \|T(x)\|_2 &\leq \|T(x) - T_N(x)\|_2 + \|T_N(x)\|_2 \\ &\leq \varepsilon \|x\|_1 + \|T_N\|_{E_1 \rightarrow E_2} \cdot \|x\|_1 \\ &= (\varepsilon + \|T_N\|_{E_1 \rightarrow E_2}) \|x\|_1 \quad \text{for } x \in E_1. \end{aligned}$$

(3) Look above and get

$$\|T_n - T\|_{E_1 \rightarrow E_2} \rightarrow 0, \quad n \rightarrow \infty.$$

□

Theorem 2.13 (Banach-Steinhaus Theorem (uniform boundedness principle)). Set $(E_1, \|\cdot\|_1)$ Banach space, $(E_2, \|\cdot\|_2)$ normed space and $\mathcal{F} \subset B(E_1, E_2)$. Assume

$$\sup_{T \in \mathcal{F}} \|T(x)\|_2 < \infty \quad \text{for all } x \in E_1$$

then

$$\sup_{T \in \mathcal{F}} \|T\|_{E_1 \rightarrow E_2} < \infty.$$

Remark. The implication \Leftarrow is easy to prove. If \mathcal{F} is a finite set, the theorem is trivial.

proof. Step 1: Assume

$$\exists x_0 \in E_1 \exists r > 0 \exists M > 0 : \forall x \in \overline{B(x_0, r)} \forall T \in \mathcal{F} : \|T(x)\|_2 \leq M.$$

We have to show that

$$\sup_{T \in \mathcal{F}} \|T\|_{E_1 \rightarrow E_2} < \infty.$$

Fix $T \in \mathcal{F}$. For $\|x\|_1 \leq r$

$$\|T(x_0 + x)\|_2 \leq M.$$

Note that $x_0 + x \in \overline{B(x_0, r)}$.

$$\begin{aligned} \|T(x)\|_2 &= \|T(x_0 + x - x_0)\|_2 \\ &= \{T \text{ linear}\} \\ &= \|T(x_0 + x) - T(x_0)\|_2 \\ &\leq \|T(x_0 + x)\|_2 + \|T(x_0)\|_2 \\ &\leq 2M. \end{aligned}$$

For $0 \neq x \in E_1$

$$\left\| T \left(\frac{r}{\|x\|_1} x \right) \right\|_2 \leq 2M.$$

$\frac{r}{\|x\|_1}$ has the $\|\cdot\|_1$ -norm equal to r . This implies, since T linear,

$$\frac{r}{\|x\|_1} \|T(x)\|_2 \leq 2M,$$

i.e.

$$\|T(x)\|_2 \leq \frac{2M}{r} \|x\|_1 \quad \text{for all } 0 \neq x \in E_1.$$

We have

$$\begin{aligned} \|T\|_{E_1 \rightarrow E_2} &\leq \underbrace{\frac{2M}{r}}_{\text{independent of } T} < \infty \\ \sup_{T \in \mathcal{F}} \|T\|_{E_1 \rightarrow E_2} &\leq \frac{2M}{r} < \infty. \end{aligned}$$

Step 2: Justify the assumption in step 1. This assumption is equivalent to

$$\exists x_0 \in E_1 \exists r > 0 \exists M > 0 : \forall x \in B(x_0, r) \forall T \in \mathcal{F} : \|T(x)\|_2 \leq M.$$

(Note $\overline{B(x_0, r_1)} \subset B(x_0, r) \subset B(x_0, r_2)$ for $0 < r_1 < r < r_2$).

Argue by contradiction. Assume that the assumption is false. Then it holds

$$\forall x_0 \in E_1 \forall r > 0 \forall M > 0 : \exists x \in B(x_0, r) \exists T \in \mathcal{F} : \|T(x)\|_2 > M.$$

Idea: Find a converging sequence $x_n \in E_1$, $x_n \rightarrow x$ in $(E_1, \|\cdot\|_1)$ and a sequence $(T_n)_{n=1}^\infty \subset \mathcal{F}$ such that

$$\|T_n(x_n)\|_2 > n \quad \text{for all } n, \quad \text{and} \quad \|T_n(x)\|_2 > n \quad \text{for all } n.$$

We have from above $x_1 \in B(0, 1)$ and $T_1 \in \mathcal{F}$ such that

$$\|T_1(x_1)\|_2 > 1.$$

T_1 is bounded linear, hence continuous. This implies that there exists $0 < r_1 < \frac{1}{2}$ such that

$$\|T_1(x)\|_2 > 1 \quad \text{for } x \in B(x_1, r_1)$$

and

$$\overline{B(x_1, r_1)} \subset B(0, 1).$$

□

2.2 Fixed point theory

Example. Consider

$$f(x) + 5 \int_0^{1-x} \min(x, y) f(y) dy = g(x), \quad x \in [0, 1] \quad (*)$$

where $g \in C([0, 1])$.

Claim: There exists an unique solution $f \in C([0, 1])$ that (*).

Idea:

$$f(x) = f(x) - 5 \int_0^{1-x} \min(x, y) f(y) dy, \quad x \in [0, 1].$$

Set for $x \in [0, 1]$

$$\tilde{T}(f)(x) = RHS(x).$$

To find a solution to (*) is the same finding $f \in C([0, 1])$ such that

$$f = \tilde{T}(f).$$

Clearly $\tilde{T} : C([0, 1]) \rightarrow C([0, 1])$. (continual later).

Theorem 2.14 (Banach's fixed point theorem). $(E, \|\cdot\|)$ Banach space. $T : E \rightarrow E$ (no assumption on linearity) is a contraction on E , i.e. there exists $c < 1$ such that

$$\|T(x) - T(\tilde{x})\| \leq c\|x - \tilde{x}\| \quad \text{for all } x, \tilde{x} \in E.$$

Then there exists a unique $\bar{x} \in E$ such that

$$\bar{x} = T(\bar{x}).$$

(\bar{x} is a fixed point)

proof. Uniqueness: Assume $T(\bar{x}) = \bar{x}$ and $T(\tilde{x}) = \tilde{x}$. Then

$$\underbrace{\|\bar{x} - \tilde{x}\|}_{\geq 0} = \|T(\bar{x}) - T(\tilde{x})\| \leq \underbrace{c}_{< 1} \|\bar{x} - \tilde{x}\|.$$

Thus $\|\bar{x} - \tilde{x}\| = 0$, i.e. $\bar{x} = \tilde{x}$.

Existence: Pick an arbitrary $x_0 \in E$. Set

$$x_{n+1} = T(x_n), \quad n = 0, 1, 2, \dots$$

Claim: $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(E, \|\cdot\|)$. Note:

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|T(x_n) - T(x_{n-1})\| \\ &\leq c\|x_n - x_{n-1}\| \\ &\leq \dots \\ &\leq c^n \|x_1 - x_0\|, \quad n = 1, 2, \dots \end{aligned}$$

For $n > m$

$$\begin{aligned} \|x_n - x_m\| &= \|x_n - x_{n-1} + x_{n-1} - \dots + x_{m+1} - x_m\| \\ &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \dots + \|x_{m+1} - x_m\| \\ &\leq (c^{n-1} + c^{n-2} + \dots + c^m) \|x_1 - x_0\| \\ &\leq \frac{c^m}{1-c} \|x_1 - x_0\| \rightarrow 0 \quad \text{as } n, m \rightarrow \infty. \end{aligned}$$

Hence $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(E, \|\cdot\|)$. $(E, \|\cdot\|)$ is a Banach space. So $(x_n)_{n=1}^{\infty}$ converges in $(E, \|\cdot\|)$. Call the limit \bar{x} .

Claim: \bar{x} is a fixed point for T .

$$\begin{aligned} \|\bar{x} - T(\bar{x})\| &= \|\bar{x} - x_{n+1} + x_{n+1} - T(\bar{x})\| \\ &\leq \|\bar{x} - x_{n+1}\| + \left\| \underbrace{x_{n+1}}_{T(x_n)} - T(\bar{x}) \right\| \\ &\leq \underbrace{\|\bar{x} - x_{n+1}\|}_{\rightarrow 0} + c \underbrace{\|x_n - \bar{x}\|}_{\rightarrow 0} \rightarrow 0, \quad n \rightarrow \infty \end{aligned}$$

□

Remark. (1) $x_n \rightarrow \bar{x}$ for $n \rightarrow \infty$ independent of the choice of x_0

(2) Fix $z \in E$

$$\begin{aligned}\|\bar{x} - z\| &= \|T(\bar{x}) - T(z) + T(z) - z\| \\ &\leq \|T(\bar{x}) - T(z)\| + \|T(z) - z\| \\ &\leq c\|\bar{x} - z\| + \|T(z) - z\|.\end{aligned}$$

Hence

$$\|\bar{x} - z\| \leq \frac{1}{1-c}\|T(z) - z\|.$$

Example. Consider now the example from above: $(C([0, 1]), \|\cdot\|)$ with $\|f\| = \max_{x \in [0, 1]} |f(x)|$ is a Banach space! To apply Banach's fixed point theorem we need \tilde{T} to be a contraction. Fix $f_1, f_2 \in C([0, 1])$ and get for $x \in [0, 1]$

$$\begin{aligned}|(\tilde{T}(f_1) - \tilde{T}(f_2))(x)| &= |5 \int_0^{1-x} \min(x, y) f_2(y) dy - 5 \int_0^{1-x} \min(x, y) f_1(y) dy| \\ &= |5 \int_0^{1-x} \min(x, y) (f_2(y) - f_1(y)) dy| \\ &\leq 5 \int_0^{1-x} \min(x, y) \underbrace{|f_2(y) - f_1(y)|}_{\leq \|f_2 - f_1\|} dy \\ &\leq 5 \underbrace{\int_0^{1-x} \min(x, y) dy}_{0 \leq \dots \leq \frac{1}{6}} \|f_2 - f_1\| \\ &\leq \frac{5}{6} \|f_2 - f_1\|.\end{aligned}$$

Hence

$$\|\tilde{T}(f_1) - \tilde{T}(f_2)\| \leq \frac{5}{6} \|f_1 - f_2\|.$$

We conclude that \tilde{T} is a contraction. We can take $c = \frac{5}{6}$. By Banach's fixed point theorem \tilde{T} has a unique fixed point. Finally (*) has a unique solution $f \in C([0, 1])$ which is the fixed point.

Theorem 2.15 (Banach's fixed point theorem (generalization)). $(E, \|\cdot\|)$ Banach space. $T : F \rightarrow F$ where F is a closed set in E . N positive integer. Assume $T^N = \underbrace{T \circ T \circ \dots \circ T}_{N\text{-times}}$

is a contraction on F , i.e. there exists $c > 1$ such that

$$\|T^N(x) - T^N(\tilde{x})\| \leq c\|x - \tilde{x}\|, \quad \text{for all } x, \tilde{x} \in F.$$

Then T has unique fixed point \bar{x} , i.e.

$$\bar{x} = T(\bar{x}) \in F.$$

proof. $N = 1$: Fix $x_0 \in F$ and consider $(x_n)_{n=1}^\infty$ where $x_{n+1} = T(x_n)$ for $n = 0, 1, 2, \dots$. There $(x_n)_{n=1}^\infty$ is a Cauchy sequence and hence this converges in E since this is a Banach space. Call the limit \bar{x} . Note

$$\underbrace{x_n}_{\in F} \rightarrow \bar{x} \text{ in } E \text{ and } F \text{ is closed}$$

implies $\bar{x} \in F$. The rest of the argument is the same as before.

$N > 1$: By previous result we know that T^N has a unique fixpoint $\bar{x} \in F$, i.e. $\bar{x} = T^N(\bar{x})$.

Claim: \bar{x} is a fixed point for T .

$$\begin{aligned} \|T(\bar{x}) - \bar{x}\| &= \|T(T^N(\bar{x})) - T^N(\bar{x})\| \\ &= \|T^N(T(\bar{x})) - T^N(\bar{x})\| \\ &\leq c\|T(\bar{x}) - \bar{x}\|. \end{aligned}$$

This gives

$$\|T(\bar{x} - \bar{x})\| = 0, \quad \text{i.e. } \bar{x} = T(\bar{x}).$$

Existence of a fixed point for T done. For the uniqueness assume $\bar{x} = T(\bar{x})$ and $\tilde{x} = T(\tilde{x})$. Then

$$\begin{aligned} \bar{x} &= T(\bar{x}) = T^2(\bar{x}) = \dots = T^N(\bar{x}) \\ \tilde{x} &= T(\tilde{x}) = T^2(\tilde{x}) = \dots = T^N(\tilde{x}). \end{aligned}$$

But T^N has a unique fixed point so

$$\bar{x} = \tilde{x}.$$

□

Remark. (1) $T : (0, 1] \rightarrow (0, 1]$ where $T(x) = \frac{x}{2}$. Clearly T is a contraction on $(0, 1]$ but has no fixed point. Note that $(0, 1]$ is not a closed interval.

(2) $T : [0, \infty) \rightarrow [0, \infty)$, where $T(x) = x + \frac{1}{x}$. Clearly $[0, \infty)$ is a closed interval in \mathbb{R} but T has no fixed point.

Claim: T is not a contraction but 'close' to be a contraction.

$$|T(x) - T(\tilde{x})| < |x - \tilde{x}| \quad \text{for } x, \tilde{x} \in [1, \infty), x \neq \tilde{x}$$

Note

$$|T(x) - T(\tilde{x})| = \underbrace{|T'(t)|}_{(1-\frac{1}{t}) \leq 1 \text{ for } t \in [1, \infty)} |x - \tilde{x}|$$

for some t between x and \tilde{x} .

Example. $(E, \|\cdot\|)$ Banach space. K compact set in E and $T : K \rightarrow K$ where

$$\|T(x) - T(\bar{x})\| < \|x - \bar{x}\| \quad \text{for all } x, \bar{x} \in K, x \neq \bar{x}.$$

Show: T has a unique fixed point in K .

Uniqueness: Assume $\bar{x} = T(\bar{x})$ and $\tilde{x} = T(\tilde{x})$ and $\bar{x} \neq \tilde{x}$ for $\bar{x}, \tilde{x} \in K$. Then

$$\|\bar{x} - \tilde{x}\| = \|T(\bar{x}) - T(\tilde{x})\| < \|\bar{x} - \tilde{x}\|.$$

Contradiction because then $\bar{x} = \tilde{x}$.

Existence: To show: There exists $x \in K$ such that $x = T(x)$, i.e.

$$\|T(x) - x\| = 0.$$

Set $d := \inf_{x \in K} \|T(x) - x\|$. Let $(x_n)_{n=1}^\infty$ be a sequence in K such that

$$\|T(x_n) - x_n\| \rightarrow d, \quad \text{as } n \rightarrow \infty.$$

K compact implies that there exists a subsequence $(\tilde{x}_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $(\tilde{x}_n)_{n=1}^\infty$ converges in K . Call the limit element $\bar{x} \in K$. We know

$$\tilde{x}_n \rightarrow \bar{x} \quad \text{in } K$$

and

$$\|T(\tilde{x}_n) - \tilde{x}_n\| \rightarrow d.$$

Question:

$$T(\tilde{x}_n) \rightarrow T(\bar{x}) \quad \text{in } K?$$

But since

$$\|T(x) - T(\tilde{x})\| \leq \|x - \tilde{x}\| \quad \text{for all } x, \tilde{x} \in K$$

we have

$$\tilde{x}_n \rightarrow \bar{x} \quad \text{in } K$$

which implies

$$T(\tilde{x}_n) \rightarrow T(\bar{x}) \text{ in } K.$$

Hence:

$$\|T(\bar{x}) - \bar{x}\| \leftarrow \|T(\tilde{x}_n) - \tilde{x}_n\| \rightarrow d, \quad n \rightarrow \infty.$$

We obtain

$$\|T(\bar{x}) - \bar{x}\| = d.$$

Question: Is $d = 0$?

If $d > 0$ then $\bar{x} \neq T(\bar{x})$, $\bar{x}, T(\bar{x}) \in K$

$$\|T(\bar{x}) - T(T(\bar{x}))\| < \|\bar{x} - T(\bar{x})\| = d = \inf_{x \in K} \|x - T(x)\|.$$

This is a contradiction which gives $d = 0$ and so $\bar{x} = T(\bar{x})$.

Example. Consider

$$f(x) = \int_0^x k(x, y)h(y, f(y)) \, dy + g(x), \quad x \in [0, 1] \quad (*),$$

where $g \in C([0, 1])$, $k \in C([0, 1] \times [0, 1])$ and $h : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ continuous and satisfies:
There exists $M > 0$ such that

$$|h(x, z_1) - h(x, z_2)| \leq M|z_1 - z_2| \quad \text{for all } x \in [0, 1], z_1, z_2 \in \mathbb{R}.$$

Claim: (*) has a unique solution $f \in C([0, 1])$.

For $f \in C([0, 1])$ set

$$T(f)(x) = \int_0^x k(x, y)h(y, f(y)) \, dy + g(x) \quad x \in [0, 1].$$

Here $T(f)(x) \in C([0, 1])$.

Want to show: $T : C([0, 1]) \rightarrow C([0, 1])$ has a unique fixed point.

Start with the Banach space $(C([0, 1]), \text{max-norm})$. Check if T is a contraction in $C([0, 1])$.

Fix $f_1, f_2 \in C([0, 1])$

$$T(f_1)(x) - T(f_2)(x) = \int_0^x k(x, y)(h(y, f_1(y)) - h(y, f_2(y))) \, dy.$$

k is continuous on the compact set $[0, 1] \times [0, 1]$ so

$$\sup_{(x,y) \in [0,1] \times [0,1]} |k(x, y)| =: N < \infty.$$

We obtain

$$\begin{aligned} |(T(f_1) - T(f_2))(x)| &\leq \int_0^x \underbrace{|k(x, y)|}_{\leq N} \underbrace{|h(y, f_1(y)) - h(y, f_2(y))|}_{\leq M|f_1(y) - f_2(y)|} \, dy \\ &\leq \int_0^x NM \, dy \|f_1 - f_2\| \\ &\leq NM \|f_1 - f_2\|. \end{aligned}$$

This yields

$$\|T(f_1) - T(f_2)\| \leq NM \|f_1 - f_2\|.$$

IF: $NM < 1$ Then T is a contraction.

Trick: For $a > 0$ set

$$\|f\|_a = \max_{x \in [0,1]} e^{-ax} |f(x)|$$

for $f \in C([0, 1])$.

Claim: $\|\cdot\|_a$ defines a norm on $C([0, 1])$. This is easy to check.

Claim: $\|\cdot\|$ and $\|\cdot\|_a$ are equivalent.

This follows from

$$e^{-a}\|f\| \leq \|f\|_a \leq \|f\|$$

for all $f \in C([0, 1])$ (note that $\|\cdot\|$ is the max-norm).

Claim: $(C([0, 1]), \|\cdot\|_a)$ is a Banach space.

This follows from the fact that $\|\cdot\|$ and $\|\cdot\|_a$ are equivalent and $(C([0, 1]), \|\cdot\|)$ is a Banach space.

Claim: T is a contraction on $(C([0, 1]), \|\cdot\|_a)$ for $a > 0$ large enough.

For $f_1, f_2 \in C([0, 1])$ and $x \in [0, 1]$ we have

$$\begin{aligned} |(T(f_1) - T(f_2))(x)| &\leq \int_0^x NM |(f_1 - f_2)(y)| dy \\ &= \int_0^x NM e^{ay} \cdot \underbrace{e^{-ay} |(f_1 - f_2)(x)|}_{\leq \|f_1 - f_2\|_a} dy \\ &\leq NM \underbrace{\int_0^x e^{ay} dy}_{\frac{1}{a}(e^{ax} - 1)} \|f_1 - f_2\|_a. \end{aligned}$$

So

$$e^{-ax} |(T(f_1) - T(f_2))(x)| \leq \frac{NM}{a} (1 - e^{-ax}) \|f_1 - f_2\|_a$$

and

$$\|T(f_1) - T(f_2)\|_a \leq \frac{NM}{a} \|f_1 - f_2\|_a$$

For $a > NM$ is T a contraction on $(C([0, 1]), \|\cdot\|_a)$. Banach fixed point theorem implies that there is a unique $f \in C([0, 1])$ that solves (*).

Theorem 2.16. $(E, \|\cdot\|)$ Banach space, $(Y, \|\cdot\|)$ normed space. $T : E \times Y \rightarrow E$ where

(1) There exists a $C > 1$ such that

$$\|T(x, y) - T(\tilde{x}, y)\| \leq C \|x - \tilde{x}\| \quad \text{for all } x, \tilde{x} \in E, y \in Y.$$

(2) $T_x : Y \rightarrow E$ where $T_x(y) = T(x, y)$ is continuous for all $x \in E$.

\Rightarrow For every $y \in Y$ there exists a unique $g(y) \in E$ such that

$$g(y) = T(g(y), y)$$

and $g : Y \rightarrow E$ is continuous.

proof. The existence of a unique element $g(y) \in E$ for every $y \in Y$ follows from Banach's fixed point theorem.

Assume $y_n \rightarrow \tilde{y}$ in $(Y, \|\cdot\|_*)$, i.e.

$$\|y_n - \tilde{y}\|_* \rightarrow 0, \quad n \rightarrow \infty.$$

Remains to show

$$g(y_m) \rightarrow g(\tilde{y}) \quad \text{in } (E, \|\cdot\|).$$

$$\begin{aligned} \|g(y_n) - g(\tilde{y})\| &= \|T(g(y_n), y_n) - T(g(\tilde{y}), \tilde{y})\| \\ &\leq \underbrace{\|T(g(y_n), y_n) - T(g(\tilde{y}), y_n)\|}_{\stackrel{(1)}{\leq c\|g(y_n) - g(\tilde{y})\|}} + \underbrace{\|T(g(\tilde{y}), y_n) - T(g(\tilde{y}), \tilde{y})\|}_{\stackrel{(2)}{\rightarrow 0, n \rightarrow \infty}} \end{aligned}$$

We obtain

$$\|g(y_n) - g(\tilde{y})\| \leq \frac{1}{1-c} \|T(g(\tilde{y}), y_n) - T(g(\tilde{y}), \tilde{y})\| \rightarrow 0, \quad n \rightarrow \infty.$$

□

Theorem 2.17 (Brouwer's fixed point theorem). K compact (= closed and bounded) convex subset of \mathbb{R}^n and $T : K \rightarrow K$ continuous. Then T has a fixed point, i.e. there exists $\bar{x} \in K$ with

$$T(\bar{x}) = \bar{x}.$$

Remark. • No uniqueness! Consider the case $T = \text{id}_K$.

- Set $K \subseteq \mathbb{R}^n$ (in general) is convex if

$$x, \tilde{x} \in K \text{ and } \lambda \in [0, 1] \quad \Rightarrow \quad \lambda x + (1 - \lambda)\tilde{x} \in K.$$

Theorem 2.18 (Perron's theorem). A real-valued $n \times n$ -Matrix with positive entries.

$A = [a_{ij}]_{i,j=1,\dots,n}$ all $a_{ij} > 0$.

\Rightarrow The mapping for $x \in \mathbb{R}^n$

$$x \mapsto Ax$$

has an eigenvalue > 0 with an eigenvector with positive entries, i.e. there exists $\lambda > 0$ and $\tilde{x} \in \mathbb{R}^n$ with $A\tilde{x} = \lambda\tilde{x}$ and all entries in \tilde{x} are positive.

proof. We use Brouwer's fixed point theorem. Set

$$K := \left\{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_k \geq 0, \sum_{i=1}^n x_i = 1 \right\}.$$

Claim: K is closed, bounded and a convex set in \mathbb{R}^n . Thus K is compact (since $K \subseteq \mathbb{R}^n$). Set

$$T(x_1, \dots, x_n) = \underbrace{\frac{1}{\|Ax\|_{l^1}} A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\in K} \quad \text{for all } (x_1, \dots, x_n) \in K$$

Claim: $T : K \rightarrow K$ is continuous.

Since

$$x_k \rightarrow x \quad \text{in } K \text{ w.r.t. } l^1 - \text{norm.}$$

To show:

$$T(x_k) \rightarrow T(x) \quad \text{in } K \text{ w.r.t. } l^1 - \text{norm.}$$

Set

$$\begin{aligned} x &= (x_1, x_2, \dots, x_n) \\ x_k &= (x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}) \quad k = 1, 2, \dots \end{aligned}$$

Consider

$$\begin{aligned} \|T(x_k) - T(x)\|_{l^1} &= \left\| \frac{1}{\|Ax_k\|_{l^1}} Ax_k - \frac{1}{\|Ax\|_{l^1}} Ax \right\|_{l^1} \\ &\leq \left\| \frac{1}{\|Ax_k\|_{l^1}} Ax_k - \frac{1}{\|Ax\|_{l^1}} Ax_k \right\|_{l^1} + \left\| \frac{1}{\|Ax\|_{l^1}} Ax_k - \frac{1}{\|Ax\|_{l^1}} Ax \right\|_{l^1} \\ &= \left| \frac{1}{\|Ax_k\|_{l^1}} - \frac{1}{\|Ax\|_{l^1}} \right| \|Ax_k\|_{l^1} + \frac{1}{\|Ax\|_{l^1}} \|A(x - x_k)\|_{l^1} \end{aligned}$$

and

$$\begin{aligned} \|A(x - x_k)\|_{l^1} &= \sum_{i=1}^n \left| \sum_{j=1}^n a_{ij} (x_j - x_j^{(k)}) \right| \\ &\leq \sum_{i=1}^n \sum_{j=1}^n a_{ij} |x_j - x_j^{(k)}| \\ &\leq \underbrace{n \cdot \max_{i,j} a_{ij}}_{< \infty} \underbrace{\|x - x_k\|_{l^1}}_{\rightarrow 0} \rightarrow 0, \quad k \rightarrow \infty. \end{aligned}$$

So

$$Ax_k \rightarrow Ax \quad \text{in } l^1.$$

This implies

$$\|Ax_k\|_{l^1} \rightarrow \|Ax\|_{l^1} \quad \text{in } \mathbb{R}.$$

Brouwer's fixed point theorem implies that T has a fixed point $\bar{x} \in K$.

$$\begin{aligned} \bar{x} &= (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \\ \bar{x} &= T(\bar{x}) = \frac{1}{\|A\bar{x}\|_{l^1}} A\bar{x} \end{aligned}$$

Hence $A\bar{x} = \|A\bar{x}\|_{l^1} \bar{x}$ where $|A\bar{x}|_l^1 > 0$ and \bar{x} has all entries > 0 . □

Theorem 2.19 (Schander's fixed point theorem). $(E, \|\cdot\|)$ Banach space. K compact, convex set in E . $T : K \rightarrow K$ continuous.
 $\Rightarrow T$ has a fixed point in K .

Example.

$$S = \{f \in C([0, 1]) \mid f(0) = 0, f(1) = 1, \|f\| = \max_{x \in [0, 1]} |f(x)| \leq 1\}$$

$T : S \rightarrow S$ defined by

$$T(f)(x) = f(x^2), \quad x \in [0, 1].$$

$C([0, 1])$ is equipped with the max-norm.

Claim:

- S is closed, bounded and convex in $C([0, 1])$.
- $T : S \rightarrow S$ is continuous.
- T has no fixed point in S .
- S bounded: $f \in S$ implies $\|f\| \leq 1$.
- S closed: $f_n \rightarrow f$ in $(C([0, 1]), \|\cdot\|)$.
 To show: $f \in S$.

Note

$$\max_{x \in [0, 1]} |f_n(x) - f(x)| \rightarrow 0, \quad n \rightarrow \infty.$$

This implies

$$|f(0)| = |f_n(0) - f(0)| \rightarrow 0, \quad n \rightarrow \infty.$$

So $f(0) = 0$.

$$|1 - f(1)| = \|f_n(1) - f(1)\| \rightarrow 0, \quad n \rightarrow \infty.$$

So $f(1) = 1$. For $x \in [0, 1]$ we get

$$\begin{aligned} |f(x)| &\leq \|f(x) - f_n(x)\| + |f_n(x)| \\ &\leq \underbrace{\|f - f_n\|}_{\rightarrow 0} + \underbrace{\|f_n\|}_{\leq 1}. \end{aligned}$$

Conclusion $f \in S$

$$\|f\| = \max_{x \in [0, 1]} |f(x)| \leq 1.$$

- $f, \tilde{f} \in S$ and $\lambda \in [0, 1]$.
 To show:

$$\lambda f + (1 - \lambda)\tilde{f} \in S.$$

Trivial since

$$(\lambda f + (1 - \lambda)\tilde{f})(0) = 0$$

$$(\lambda f + (1 - \lambda)\tilde{f})(1) = \lambda f(1) + (1 - \lambda)\tilde{f}(1) = 1$$

and

$$\|\lambda f + (1 - \lambda)\tilde{f}\| \leq |\lambda|\|f\| + |1 - \lambda|\|\tilde{f}\| \leq 1.$$

We want to show that $T : S \rightarrow S$ is continuous. (obvious that $T(S) \subseteq S$)
Assume $f_n \rightarrow f$ in S in max-norm, i.e.

$$\max_{x \in [0,1]} |f_n(x) - f(x)| \rightarrow 0, \quad n \rightarrow \infty.$$

To show: $T(f_n) \rightarrow T(f)$ in S in max-norm.

$$\begin{aligned} \|T(f_n) - T(f)\| &= \max_{x \in [0,1]} |T(f_n)(x) - T(f)(x)| \\ &= \max_{x \in [0,1]} |f_n(x^2) - f(x^2)| \\ &= \|f_n - f\| \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

$T : S \rightarrow S$ has no fixed point.

If $f \in S$ is a fixed point for T then

$$f(x^2) = T(f)(x) = f(x), \quad x \in [0, 1].$$

To show: there can be no such $f \in S$.

Set $a = \inf\{x \in [0, 1] \mid f(x) = \frac{1}{2}\} \neq \emptyset$ since f is continuous. $a \in (0, 1)$ since if $a = 0$ then there exists a sequence

$$a_n \in \{x \in [0, 1] \mid f(x) = \frac{1}{2}\}$$

such that $a_n \rightarrow a$ in \mathbb{R} as $n \rightarrow \infty$. Contradiction since

$$\frac{1}{2} = f(a_n) \rightarrow f(a) = f(0) = 0$$

since f is continuous.

But $0 < a^2 < a$ and $f(a^2) = f(a) = \frac{1}{2}$. This is a contradiction.

If we believe in Schauder then we can conclude that $S \subseteq C([0, 1])$ is not compact.

Theorem 2.20 (Arzela-Ascoli theorem). Assume K is a compact set in \mathbb{R}^n (e.g. $K = [0, 1]$ in \mathbb{R} $n = 1$) and $S \subseteq C(K)$ where $C(K)$ is equipped with the max-norm.
 $\Rightarrow S$ is relatively compact in $C(K)$ iff

- (1) S uniformly bounded.
- (2) S is equicontinuous.

Definition . (i) S is uniformly bounded if

$$\sup_{f \in S} \|f\| < \infty.$$

(ii) S is equicontinuous if: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - \tilde{x}| < \delta, x, \tilde{x} \in K \quad \Rightarrow \quad |f(x) - f(\tilde{x})| < \varepsilon.$$

$\delta = \delta(\varepsilon)$ must not depend on f .

S is relatively compact in $C(K)$ if for every sequence $(f_n)_{n=1}^{\infty}$ in S there exists a converging subsequence in $C(K)$.

To show: S is relatively compact in $C(K)$ iff the closure \bar{S} is compact in $C(K)$.

Things to do:

- (1) Proof of Schander's theorem.
- (2) Proof of Arzela-Ascoli theorem.
- (3) Application with Schander.
- (4) Proof of Brouwer's theorem (special case).
- (5) Completion of normed spaces.

For (4) we consider the following lemma.

Lemma 2.21 (Sperner's lemma). Big triangle T

$$T = \bigcup_{a \in A} T_a.$$

$\{T_a\}_{a \in A}$ is triangle of T , i.e. for any pair $T_a, T_{\tilde{a}}$ in the triangulation

$$T_a \cup T_{\tilde{a}} = \{\emptyset \text{ or common vertex or common side or } T_a = T_{\tilde{a}}\}.$$

\Rightarrow There must exist a triangle T_a with all vertices colored differently. MISSING FIGURE!

Proof of Schander's fixed point theorem: To prove: $(E, \|\cdot\|)$ Banach space, K compact convex set in E and $T : K \rightarrow K$ continuous.

Claim: T has a fixed point.

Lemma . Assume $(x_n)_{n=1}^\infty$ sequence in K such that

$$\|T(x_n) - x_n\| \rightarrow 0, \quad n \rightarrow \infty.$$

T has a fixed point in K .

proof. Consider $(T(x_n))_{n=1}^\infty$ in K . K compact implies that there exists a $z \in K$ and a subsequence $(T(\tilde{x}_n))_{n=1}^\infty$ of $(T(x_n))_{n=1}^\infty$ such that

$$T(\tilde{x}_n) \rightarrow z \quad \text{in } K \text{ as } n \rightarrow \infty.$$

Then

$$\left\| \underbrace{T(\tilde{x}_n)}_{\rightarrow z} - \tilde{x}_n \right\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

So $\tilde{x}_n \rightarrow z$ for $n \rightarrow \infty$. But T continuous implies

$$z \leftarrow T(\tilde{x}_n) \rightarrow T(z), \quad n \rightarrow \infty.$$

Conclusion: $z = T(z)$ so z is a fixed point. □

Lemma . K compact set in E . Let $\varepsilon > 0$. Then there exists a finite set $x_1, \dots, x_N \in K$ such that for all $x \in K$

$$\min_{k=1, \dots, N} \|x - x_k\| < \varepsilon.$$

proof. Assume there is no finite sequence x_1, \dots, x_N . Then there exists a sequence $(x_n)_{n=1}^\infty$ such that

$$\|x_k - x_l\| \geq \varepsilon, \quad \text{for } k \neq l.$$

Clearly $(x_n)_{n=1}^\infty$ has no converging subsequence. This contradicts K being compact. □

Fix positive integer n . Apply previous lemma with $\varepsilon = \frac{1}{n}$. then there exists a finite set x_1, \dots, x_N such that

$$K \subset \bigcup_{k=1}^N B\left(x_k, \frac{1}{n}\right).$$

Set

$$\begin{aligned} K_n &= \{\text{set of all convex combinations of } x_1, \dots, x_N\} \\ &= \left\{ \sum_{k=1}^N \lambda_k x_k \mid \lambda_k \geq 0 \text{ for all } k, \sum_{k=1}^N \lambda_k = 1 \right\}. \end{aligned}$$

This set is a closed and bounded set in $\text{span}(K_n)$ finite dimensional. Also K_n is convex. (want $T_n : K_n \rightarrow K_n$ where T_n close to T).

Set $f_k(x) = \max(0, \frac{1}{n} - \|x - x_k\|)$ for $x \in K$ and $k = 1, 2, \dots, N$.
For each $x \in K$ there exists a k such that $f_k(x) > 0$. Set

$$P_n(x) = \frac{f_1(x)x_1 + f_2(x)x_2 + \dots + f_N(x)x_N}{f_1(x) + f_2(x) + \dots + f_N(x)}, \quad x \in K.$$

P_n is a convex combination of x_1, \dots, x_N for every $x \in K$. So $P_n(x) \in K_n$ for every $x \in K$.

Claim: $\|P_n(x) - x\| < \frac{1}{n}$ for all $x \in K$. Set T_n to be defined like

$$T_n := P_n T : K_n \rightarrow K_n.$$

Here T_n is continuous since T and P_n are continuous. K_n is compact and convex in a finite dimensional space. Brouwer's fixed point theorem implies that T_n has a fixed point in K_n , i.e. there exists $x_n \in K_n$ such that

$$x_n = T_n(x_n) = P_n(x_n).$$

But then

$$\|x_n - T(x_n)\| \leq \underbrace{\left\| x_n - \underbrace{P_n T(x_n)}_{=T_n} \right\|}_{=0} + \underbrace{\|P_n T(x_n) - T(x_n)\|}_{< \frac{1}{n}}.$$

The first lemma above gives that T has a fixed point in K . □

Example. Assume $k(x, y)$ continuous on $[0, 1] \times [0, 1]$ and $h(y, z)$ continuous on $[0, 1] \times \mathbb{R}$ and

$$\sup_{(y,z) \in [0,1] \times \mathbb{R}} |h(y, z)| \equiv B < \infty.$$

Then there exists a solution $f \in C([0, 1])$ to

$$f(x) = \int_0^1 k(x, y) h(y, f(y)) dy, \quad x \in [0, 1].$$

Method: Set $f \in C([0, 1])$ and

$$T(f)(x) = \int_0^1 k(x, y) h(y, f(y)) dy, \quad x \in [0, 1] \quad (*).$$

We want to apply (a generalized version of) Schander's fixed point theorem. Assume $(E, \|\cdot\|)$ is a Banach space and F closed convex subset of E . Moreover assume $T : E \rightarrow E$ continuous and $T(F)$ relatively compact in $(E, \|\cdot\|)$. Then T has a fixed point in F .

Step 1: T as in $(*)$.

Claim: $T(C([0, 1])) \subseteq C([0, 1])$.

To proof this we note that k is continuous on $[0, 1] \times [0, 1]$ which is compact in \mathbb{R}^2 .

This implies that k is uniformly continuous on $[0, 1] \times [0, 1]$. Fix now $\varepsilon > 0$. Then there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|k(x_1, y_1) - k(x_2, y_2)| < \frac{\varepsilon}{B}$$

for $|(x_1, y_1) - (x_2, y_2)| < \delta$.

Fix $f \in C([0, 1])$

$$\begin{aligned} |T(f)(x_1) - T(f)(x_2)| &= \left| \int_0^1 (k(x_1, y) - k(x_2, y)) h(y, f(y)) \, dy \right| \\ &\leq \int_0^1 \underbrace{|k(x_1, y) - k(x_2, y)|}_{< \frac{\varepsilon}{B} \text{ if } |x_1 - x_2| < \delta} \underbrace{|h(y, f(y))|}_{\leq B} \, dy < \varepsilon, \quad \text{provided } |x_1 - x_2| < \delta \end{aligned}$$

Conclusion: $T(f) \in C([0, 1])$ for $f \in C([0, 1])$

Step 2: Choose F .

k is a continuous function on a compact set $[0, 1] \times [0, 1]$ implies

$$\sup_{(x,y) \in [0,1] \times [0,1]} |k(x, y)| \equiv A < \infty.$$

Hence

$$|T(f)(x)| \leq AB \quad \text{for all } f \in C([0, 1]).$$

Set

$$F := \{f \in C([0, 1])\} \quad \|f\| = \max_{x \in [0,1]} |f(x)| \leq AB.$$

Clearly F is closed convex in $(C([0, 1]), \|\cdot\|)$ which is a Banach space.

Step 3: Claim: $T(F)$ is relatively compact.

To prove this we use the Arzela-Ascoli Theorem.

Let K be a compact set in \mathbb{R}^n . Let $\mathcal{S} \subset C(K)$ (realvalued continuous functions on K). Then \mathcal{S} is relatively compact in $(C(K), \|\cdot\|_\infty)$ if

(1) \mathcal{S} uniformly bounded, i.e.

$$\sup_{f \in \mathcal{S}} \|f\| < \infty.$$

(2) Equicontinuity of $f \in \mathcal{S}$, i.e.

$$\begin{aligned} \forall \varepsilon > 0 \exists \delta = \delta(\varepsilon) > 0 : \forall f \in \mathcal{S} : \\ |x_1 - x_2| < \delta, x_1, x_2 \in K \quad \Rightarrow \quad |f(x_2) - f(x_1)| < \varepsilon. \end{aligned}$$

In our example it is $\mathcal{S} = F$, $K = [0, 1]$ in \mathbb{R} . Check that (1) and (2) in AA-Theorem are satisfied.

(1) F is uniformly bounded since

$$\sup_{f \in F} \|f\| \leq AB < \infty.$$

(2) Equicontinuity follows from calculations in Step 1.

Conclusion: $T(F)$ is relatively compact.

Step 4: Claim: $T : F \rightarrow F$ continuous

In step 1 we had $f \in F$ and $x_n \rightarrow x$ in $[0, 1]$. We have shown that $T(f)(x_n) \rightarrow T(f)(x)$ in \mathbb{R} . So $T(f)$ is a continuous function.

Now we want to show that for $f_n \rightarrow f$ in F we've got $T(f_n) \rightarrow T(f)$ in $C([0, 1])$.

Note that $h : [0, 1] \times [-AB, AB] \rightarrow \mathbb{R}$ is continuous and $[0, 1] \times [-AB, AB]$ is compact set in \mathbb{R}^2 . So $h : [0, 1] \times [-AB, AB] \rightarrow \mathbb{R}$ is uniformly continuous.

Fix $\varepsilon > 0$. Then there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$|h(y_1, z_1) - h(y_2, z_2)| < \frac{\varepsilon}{A}$$

for $|(y_1, z_1) - (y_2, z_2)| < \delta$. For $f_1, f_2 \in F$ with

$$\|f_1 - f_2\| < \delta.$$

We have

$$\begin{aligned} |T(f_1)(x) - T(f_2)(x)| &= \left| \int_0^1 k(x, y) (h(y, f_1(y)) - h(y, f_2(y))) \, dy \right| \\ &\leq \int_0^1 \underbrace{|k(x, y)|}_{\leq A} \underbrace{|h(y, f_1(y)) - h(y, f_2(y))|}_{< \frac{\varepsilon}{A}} \, dy < \varepsilon. \end{aligned}$$

Conclusion: $T : F \rightarrow F$ is continuous.

Step 5: Apply Schander's fixed point theorem.

2.3 Completion of normed spaces

$(E, \|\cdot\|)$ normed spaces. We say that $(\tilde{E}, \|\cdot\|_*)$ is a completion of $(E, \|\cdot\|)$ if $(\tilde{E}, \|\cdot\|_*)$ is a normed space such that

- (1) $\exists \Phi : E \rightarrow \tilde{E}$ injective and linear.
- (2) $\|x\| = \|\Phi(x)\|_*$ for all $x \in E$.
- (3) $\Phi(E)$ is dense in \tilde{E} .
- (4) $(\tilde{E}, \|\cdot\|_*)$ is a Banach space.

Construction:

Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be Cauchy sequences in $(E, \|\cdot\|)$. We say that $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are equivalent, denoted by $(x_n) \sim (y_n)$, if

$$\|x_n - y_n\| \rightarrow 0, \quad n \rightarrow \infty.$$

Set

$$\tilde{E} = \{((x_n))_N \mid (x_n)_{n=1}^\infty \text{ Cauchy sequence in } (E, \|\cdot\|)\}.$$

Vector space structure:

$$\begin{cases} [(x_n)]_N + [(\tilde{x}_n)]_N &= [(x_n + \tilde{x}_n)]_N \\ \lambda[(x_n)]_N &= [(\lambda x_n)]_N. \end{cases}$$

Show that these definitions are well-defined, i.e. independent of the choice of representative norm

$$\|[(x_n)]_N\|_* = \lim_{n \rightarrow \infty} \|x_n\|.$$

Note

$$(x_n) \sim (y_n)$$

implies

$$\lim_{n \rightarrow \infty} \|x_n\| = \lim_{n \rightarrow \infty} \|y_n\|.$$

Since

$$\| \|x_n\| - \|y_n\| \| \leq \|x_n - y_n\| \rightarrow 0, \quad n \rightarrow \infty$$

Check that the axioms for being a norm are satisfied.

Now we have $(\tilde{E}, \|\cdot\|_*)$ is a normed space.

Define Φ : For $x \in E$ set $\Phi(x) = [(x)_{n=1}^\infty]_N$ where

$$(x)_{n=1}^\infty = (x, x, x, \dots).$$

Claim 1 & 2: easy to prove.

Claim 3: item $\Phi(E)$ dense in $(\tilde{E}, \|\cdot\|_*)$. Fix $[(x_n)]_N \in \tilde{E}$. Consider $\Phi(x_k)$ where x_k is the element in the k -th position in the sequence $(x_1, x_2, \dots, x_n, \dots)$.

$$\|[(x_n)]_N - \Phi(x_k)\|_* = \lim_{n \rightarrow \infty} \|x_n - x_k\| \rightarrow 0 \quad k \rightarrow \infty.$$

Since $(x_n)_{n=1}^\infty$ is a Cauchy sequence.

Claim 4: item $(\tilde{E}, \|\cdot\|_*)$ is a Banach space.

Consider a Cauchy sequence $z_n \in \tilde{E}$ such that $\|z_n - z\| \rightarrow 0$ as $n \rightarrow \infty$.

To show: There exists $z \in \tilde{E}$ such that

$$\|z_n - z\| \rightarrow 0, \quad n \rightarrow \infty.$$

By 3 we have that $\Phi(E)$ is dense in \tilde{E} so for $n = 1, 2, \dots$ there exists $x_n \in E, n = 1, 2, \dots$ such that

$$\|z_n - \Phi(z_n)\| < \frac{1}{n}, \quad n = 1, 2, \dots$$

Set $z =: [(x_n)]_N$.

Need to show that $(x_n)_{n=1}^\infty$ is a Cauchy sequence

$$\begin{aligned} \|x_n - x_m\| &= \|\Phi(x_n) - \Phi(x_m)\|_* \\ &\leq \|\Phi(x_n) - z_n\|_* + \|z_n - z_m\|_* + \|z_m - \Phi(x_m)\|_* \\ &< \frac{1}{n} + \|z_n - z_m\| + \frac{1}{m} \rightarrow 0, \quad n, m \rightarrow \infty. \end{aligned}$$

Conclusion: $(x_n)_{n=1}^\infty$ is a Cauchy sequence in $(E, \|\cdot\|)$. Remains to show:

$$\begin{aligned} \|z_n - z\|_* &\rightarrow 0, \quad n \rightarrow \infty \\ \|z_n - z\|_* &\leq \underbrace{\|z_n - \Phi(x_n)\|_*}_{< \frac{1}{n}} + \underbrace{\|\Phi(x_n) - z\|_*}_{=\lim_{n \rightarrow \infty} \|x_n - x_m\|} \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Consider $f \in C([0, 1])$

- max-norm: $\|f\| = \max_{x \in [0, 1]} |f(x)|$. Then $(C([0, 1]), \|\cdot\|)$ is a Banach space.
- $p \geq 1$:

$$\|f\|_{L^p} = \left(\int_0^1 |f(x)|^p dx \right)^{\frac{1}{p}}$$

defines a norm for $C([0, 1])$.

Remark. • Consider piecewise linear $f_n \in C([0, 1])$ for $n = 1, 2, \dots$

$$f_n(x) = \begin{cases} 1, & \text{if } \frac{1}{2} \leq x \leq 1 \\ 0, & \text{if } x \leq \frac{1}{2} - \frac{1}{2n} \end{cases}$$

with

$$\|f_n - f_m\|_{L^1} \leq \frac{1}{2 \min(m, n)} \rightarrow 0, \quad n, m \rightarrow \infty.$$

So $(f_n)_{n=1}^\infty$ is a Cauchy sequence in $(C([0, 1]), \|\cdot\|_{L^1})$ but $(f_n)_{n=1}^\infty$ does not converge in $(C([0, 1]), \|\cdot\|_{L^1})$ since if $\|f_n - f\|_{L^1} \rightarrow 0$ as $n \rightarrow \infty$ and $f \in C([0, 1])$ then

$$f(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}) \\ 1, & \text{if } x \in [\frac{1}{2}, 1] \end{cases}.$$

Conclusion: $(C([0, 1]), \|\cdot\|_{L^1})$ is not a Banach space.

- Consider:

$$f(x) = \begin{cases} 1, & \text{if } x = \frac{1}{2} \\ 0, & \text{if } x \in [0, 1] \setminus \{\frac{1}{2}\} \end{cases}.$$

Then

$$\|f\|_{L^1} = 0 = \|0\|_{L^1}.$$

Compare this with the first axiom for a norm function.

- Replace $[0, 1]$ with \mathbb{R} . For $f : \mathbb{R} \rightarrow \mathbb{R}$ set

$$\text{supp}(f) = \{x \in \mathbb{R} \mid f(x) \neq 0\}.$$

Set

$$C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) \mid \text{supp}(f) \text{ is compact in } \mathbb{R}\}.$$

Claim: $C_0(\mathbb{R})$ forms a vector space and for every $p \geq 1$ and $f \in C_0(\mathbb{R})$

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{\frac{1}{p}}$$

defines a norm on $C_0(\mathbb{R})$.

Problem: $(C_0(\mathbb{R}), \|\cdot\|_{L^p})$ for $p \geq 1$ are not Banach spaces.

$(L^1(\mathbb{R}), \|\cdot\|_{L^1})$ is a completion of $(C_0(\mathbb{R}), \|\cdot\|_{L^1})$.

Note $A \subset \mathbb{R}$ and A bounded. Define

$$f_A(x) = \begin{cases} 1, & x \in A \\ 0, & \text{elsewhere} \end{cases}.$$

Lebesguesmeasure of $A = \|f_A\|_{L^1} = \mu(f_A)$. $A \subset \mathbb{R}$ and A unbounded

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A \cap [-n, n]).$$

We say that $A \subset \mathbb{R}$ is a 0- set if for all $\varepsilon > 0$ there exist open intervals I_n , $n = 1, 2, \dots$ such that

- (1) $A \subseteq \bigcup_{n=1}^{\infty} I_n$,
- (2) $\sum_{n=1}^{\infty} \text{lengths of } I_n < \varepsilon$.

In particular

$$A = \mathbb{Q} = \{r_n \mid n = 1, 2, \dots\} \quad \text{is a 0-set.}$$

3 Hilbert spaces

Example. Consider $\mathbb{C}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{C}\}$ and $x, y \in \mathbb{C}^n$ with $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$. Define the inner product of x, y (scalar product)

$$\langle x, y \rangle = \sum_{i=1}^n x_i \bar{y}_i \in \mathbb{C}.$$

We have a map

$$\begin{aligned} \mathbb{C}^n \times \mathbb{C}^n &\rightarrow \mathbb{C} \\ (x, y) &\mapsto \langle x, y \rangle. \end{aligned}$$

This mapping has properties:

- $x \neq 0$ folgt $\langle x, x \rangle = \sum_{i=1}^n x_i \bar{x}_i = \sum_{i=1}^n |x_i|^2 > 0$
- $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ for $x, y \in \mathbb{C}^n, \lambda \in \mathbb{C}$.
- $\langle x, y \rangle = \sum_{i=1}^n x_i \bar{y}_i = \overline{\sum_{i=1}^n y_i \bar{x}_i}$ for $x, y \in \mathbb{C}^n$.
In particular $\langle x, \lambda y \rangle = \bar{\lambda} \langle x, y \rangle$ for $\lambda \in \mathbb{C}$.
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for $x, y, z \in \mathbb{C}^n$.

Definition . An inner product space V is a complex vector space with an inner product which is a map

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}.$$

Satisfying

- $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ for any $x, y \in V, \lambda \in \mathbb{C}$.
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for any $x, y, z \in V$.
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for any $x, y \in V$.
- $\langle x, x \rangle > 0$ for any $x \in V, x \neq 0$.

Can we generalize \mathbb{C}^n ?

$$\mathbb{C}^{\mathbb{N}} \{(x_1, x_2, \dots) \mid x_i \in \mathbb{C}\}$$

with

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i.$$

This is not necessarily convergent.

Examples. (1)

$$l^2 = \left\{ (x_1, x_2, \dots) \mid \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}.$$

We have with Cauchy Schwarz

$$\sum_{i=1}^n |x_i \bar{y}_i| \leq \left(\sum_{i=1}^n |x_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^n |y_i|^2 \right)^{\frac{1}{2}}$$

if $x \in l^2$ and $y \in l^2$ we get

$$\sum_{i=1}^n |x_i \bar{y}_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |y_i|^2 \right)^{\frac{1}{2}} < \infty.$$

It follows that $\sum_{i=1}^{\infty} x_i \bar{y}_i$ converges absolutely and hence it is convergent. The following

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i$$

is well-defined for vectors $x, y \in l^2$. Like for \mathbb{C}^n one can easily check that $\langle \cdot, \cdot \rangle$ satisfies the axioms for inner products.

$(l^2, \langle \cdot, \cdot \rangle)$ is an inner product space.

(2) Consider $C([0, 1])$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} dt \quad \forall f, g \in C([0, 1]).$$

•

$$\langle \lambda f, g \rangle = \int_0^1 \lambda f(t) \overline{g(t)} dt = \lambda \int_0^1 f(t) \overline{g(t)} dt = \lambda \langle f, g \rangle.$$

•

$$\langle f, f \rangle = \int_0^1 f(t) \overline{f(t)} dt = \int_0^1 |f(t)|^2 dt > 0.$$

• ...

If we take \mathbb{R}^3 with the Eukledian norm on \mathbb{R}^3

$$\|(x_1, x_2, x_3)\| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \left(\sum_{i=1}^3 |x_i|^2 \right)^{\frac{1}{2}} = \langle x, x \rangle^{\frac{1}{2}}.$$

Let V be an inner product space with $\langle \cdot, \cdot \rangle$ as the inner product. Let for $x \in V$

$$\|x\| := \langle x, x \rangle^{\frac{1}{2}}.$$

Statement 3.1. The $x \mapsto \|x\|$ with $\|\cdot\|$ defined above is a norm.

We are going to prove the norm axioms but first we need another theorem.

Theorem 3.2 (Cauchy-Schwarz inequality). For any $x, y \in V$ (inner product space)

$$|\langle x, y \rangle| \leq \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}}.$$

The equality holds iff x, y are linearly dependent.

proof. Assume x, y linearly dependent. We can assume that $x = \lambda y$ for some $\lambda \in \mathbb{C}$.

$$|\langle x, y \rangle| = |\langle \lambda y, y \rangle| = |\lambda| \langle y, y \rangle$$

and

$$\begin{aligned} \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}} &= \langle \lambda y, \lambda y \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}} \\ &= |\lambda| \langle y, y \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}} \\ &= |\lambda| \langle y, y \rangle. \end{aligned}$$

Hence

$$|\langle x, y \rangle| = \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}}.$$

Assume x, y are linearly independent. Hence $x + \lambda y \neq 0$ for any $\lambda \in \mathbb{C}$. By an axiom for inner product we get

$$0 < \langle x + \lambda y, x + \lambda y \rangle = \langle x, x \rangle + \lambda \langle y, x \rangle + \bar{\lambda} \langle x, y \rangle + |\lambda|^2 \langle y, y \rangle.$$

Pick now

$$\lambda = -\frac{\langle x, y \rangle}{\langle y, y \rangle}.$$

(Note that $y \neq 0$ as x, y linearly independent.) We have

$$\begin{aligned} 0 &< \langle x, x \rangle - \frac{\overbrace{\langle x, y \rangle \langle y, x \rangle}^{=|\langle x, y \rangle|^2}}{\langle y, y \rangle} - \frac{\overbrace{\langle x, y \rangle \langle x, y \rangle}^{=|\langle x, y \rangle|^2}}{\langle y, y \rangle} + \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle^2} \langle y, y \rangle \\ &= \langle x, x \rangle - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle}. \end{aligned}$$

This gives

$$\frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} < \langle x, x \rangle$$

and it follows

$$|\langle x, y \rangle|^2 < \langle x, x \rangle \langle y, y \rangle.$$

□

Now we can use this inequality to proof the statement above:

proof. (i) $\|x\| > 0$ for all $x \neq 0$ in V (Exercise).

(ii) $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in V, \lambda \in \mathbb{C}$ (Exercise).

(iii) Let $x, y \in V$. Then

$$\begin{aligned}\|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \langle x, x \rangle + 2\operatorname{Re}(\langle x, y \rangle) + \langle y, y \rangle \\ &\leq \langle x, x \rangle + 2|\langle x, y \rangle| + \langle y, y \rangle \\ &\leq \langle x, x \rangle + 2\langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}} + \langle y, y \rangle \\ &= \left(\langle x, x \rangle^{\frac{1}{2}} + \langle y, y \rangle^{\frac{1}{2}} \right)^2.\end{aligned}$$

So

$$\|x + y\|^2 \leq (\|x\| + \|y\|)^2.$$

□

Theorem 3.3 (The Parallelogram Law). Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Let $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$. Then

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \forall x, y \in V.$$

Statement 3.4. l^p has inner product $\langle \cdot, \cdot \rangle_{l^p}$ such that

$$\|x\|_p = \sqrt{\langle x, x \rangle_{l^p}}$$

iff $p = 2$.

proof. Enough to show that $\|\cdot\|_p$ -norm does not satisfy the parallelogram law for some $x, y \in l^p$ if $p \neq 2$. Take for example $x = (1, 0, 0, \dots)$ and $y = (0, 1, 0, \dots)$. Note that $\|x\|_{l^p} = \|y\|_{l^p} = 1$

$$\begin{aligned}\|x + y\|_{l^p}^2 &= \|(1, 1, 0, \dots)\|_{l^p}^2 = 2^{\frac{2}{p}} \\ \|x - y\|_{l^p}^2 &= \|(1, -1, 0, \dots)\|_{l^p}^2 = 2^{\frac{2}{p}} \\ \|x + y\|_{l^p}^2 + \|x - y\|_{l^p}^2 &= 2 \cdot 2^{\frac{2}{p}} = 2(\|x\|_{l^p}^2 + \|y\|_{l^p}^2) = 2 \cdot 2.\end{aligned}$$

□

All l^p with $p \neq 2$ are not inner product spaces.

Exercise:

Show that $(C([0, 1]), \|\cdot\|_\infty)$ is not an inner product space.

Remark. Whenever a norm satisfies the parallelogram law then there exists an inner product on V such that

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}.$$

Theorem 3.5 (The Polarization Identity). Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Then

$$4\langle x, y \rangle = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2.$$

Definition 3.6. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. We say that x, y in V are orthogonal if $\langle x, y \rangle = 0$ (We write $x \perp y$). Let $M \subseteq V$ Define the orthogonal complement

$$M^\perp = \{x \in V \mid x \perp y \text{ for any } y \in M\}.$$

Proposition 3.7. If $M \subseteq V$ then M^\perp is a subspace of V .

Theorem 3.8 (Pythagorean formula). $x, y \in V$ (inner product space). Then

$$x \perp y \quad \text{iff} \quad \|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

3.1 Orthogonal Systems

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space $\{u_n\} \subseteq V$ is called orthogonal system (with n finite or infinite) if $u_n \perp u_m$ for all $n \neq m$. It is an orthonormal system if in addition $\|u_n\| = 1$.

Examples. 1) $\{e_k\}_{k=1}^\infty \subseteq \ell^2$ with

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i$$

with

$$e_k = (0, \dots, 1, 0, \dots).$$

$\Rightarrow \{e_k\}$ is an ON-system.

2) $C([-\pi, \pi])$ with

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt.$$

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{-int} \mid n \in \mathbb{Z} \right\}$$

is an orthonormal system.

Definition 3.9. Let $\{a_n \mid n \in \mathbb{N}\}$ be an orthonormal system in V . The formal series

$$\sum_{n=1}^{\infty} \langle x, a_n \rangle a_n$$

is called a fourier series of x corresponding $\{a_n \mid n \in \mathbb{N}\}$ and $\langle x, a_n \rangle$ are called fourier coefficients of x corresponding to $\{a_n \mid n \in \mathbb{N}\}$.

Theorem 3.10 (Bessel's Equality and Inequality). If $\{u_n\}$ orthonormal system in an inner product space V , then for all $x \in V$

$$\left\| x - \sum_{k=1}^n \langle x, a_k \rangle a_k \right\|^2 = \|x\|^2 - \sum_{k=1}^n |\langle x, a_k \rangle|^2$$

and

$$\sum_{k=1}^{\infty} |\langle x, a_k \rangle|^2 \leq \|x\|^2.$$

proof.

$$\begin{aligned} \left\| x - \sum_{k=1}^n \langle x, a_k \rangle a_k \right\|^2 &= \left\langle x - \sum_{k=1}^n \langle x, a_k \rangle a_k, x - \sum_{k=1}^n \langle x, a_k \rangle a_k \right\rangle \\ &= \langle x, x \rangle - \sum_{k=1}^n \overline{\langle x, a_k \rangle} \langle x, a_k \rangle - \sum_{k=1}^n \langle x, a_k \rangle \langle a_k, x \rangle \\ &\quad + \left\langle \sum_{k=1}^n \langle x, a_k \rangle a_k, \sum_{k=1}^n \langle x, a_k \rangle a_k \right\rangle \\ &= \|x\|^2 - \sum_{k=1}^n |\langle x, a_k \rangle|^2 - \sum_{k=1}^n |\langle x, a_k \rangle|^2 + \sum_{k=1}^n |\langle x, a_k \rangle|^2 \\ &= \|x\|^2 - \sum_{k=1}^n |\langle x, a_k \rangle|^2. \end{aligned}$$

This gives also:

$$\sum_{k=1}^n |\langle x, a_k \rangle|^2 = \|x\|^2 - \left\| x - \sum_{k=1}^n \langle x, a_k \rangle a_k \right\|^2 \leq \|x\|^2$$

for all $n \in \mathbb{N}$. Hence

$$\sum_{k=1}^{\infty} |\langle x, a_k \rangle|^2 \leq \|x\|^2.$$

□

Definition 3.11 (Hilbert space). A Hilbert space is an inner product space which is complete w.r.t. the norm is defined through the inner product.

Examples. • \mathbb{C}^n is an inner product space and complete w.r.t the Eukledean norm. Hence \mathbb{C}^n is a Hilbert space.

- l^2 is a Banach space w.r.t.

$$\|x\|_{l^2} = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}$$

and

$$\|x\|_{l^2} = \langle x, x \rangle^{\frac{1}{2}},$$

where

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i.$$

- $(C([0, 1]), \|\cdot\|_{\infty})$ is a Banach space but not an inner product space. Hence it is no Hilbert space.
- $(C([0, 1]), \langle \cdot, \cdot \rangle)$ is an inner product space $f, g \in C([0, 1])$ with

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} dt$$

and the corresponding

$$\|f\|_2 = \langle f, f \rangle = \int_0^1 |f(t)|^2 dt.$$

Remark. Other l^p spaces are not Hilbert spaces!!!! They are not inner product spaces.

Statement 3.12. $(C([0, 1]), \langle \cdot, \cdot \rangle)$ is not a Hilbert space since $(C([0, 1]), \|\cdot\|_2)$ is not complete.

proof. Sketch: Show that $f_n(t)$, which is defined as a piecewise continuous function for example

$$f_n(x) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}] \\ 0, & \text{if } x \in [\frac{1}{2} + \frac{1}{n}, 1] \\ \text{continuous,} & \text{else} \end{cases}$$

is a Cauchy sequence w.r.t $\|\cdot\|_2$ but has no limit in $C([0, 1])$. □

Consider

$$C_F = \{(x_1, x_2, \dots) \mid \text{only finite } x_i \neq 0\}$$

with

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i.$$

Show that $(C_F, \langle \cdot, \cdot \rangle)$ is not a Hilbert space.

Definition 3.13 (strongly and weakly convergent). A sequence $\{x_n\} \subseteq H$, where H is a Hilbert space, is called strongly convergent ($x_n \rightarrow x \in H$) if

$$\|x_n - x\| \rightarrow 0, \quad n \rightarrow \infty.$$

(Norm induced by an inner product)

We say that x_n is weakly convergent ($x_n \rightharpoonup x$) if

$$\langle x_n, y \rangle \rightarrow \langle x, y \rangle, \quad \forall y \in H.$$

Statement 3.14. $x_n \rightarrow x \Rightarrow x_n \rightharpoonup x.$

proof. Assume strong convergence for $(x_n)_{n \in \mathbb{N}}$. Then

$$\begin{aligned} |\langle x_n, y \rangle - \langle x, y \rangle| &= |\langle x_n - x, y \rangle| \\ &\leq \underbrace{\langle x_n - x, x_n - x \rangle^{\frac{1}{2}}}_{=\|x_n - x\|} \underbrace{\langle y, y \rangle^{\frac{1}{2}}}_{=\|y\|} \\ &= \underbrace{\|x_n - x\|}_{\rightarrow 0} \|y\| \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Hence $\langle x_n, y \rangle \rightarrow \langle x, y \rangle$. □

Remark. The converse is not true in general:

Take $H = l^2$ and

$$\begin{aligned} x_n &= e_n = (0, \dots, 1, 0, \dots) \\ y &= (y_1, y_2, \dots) \in l^2. \end{aligned}$$

We have for all $y \in H$

$$\langle e_n, y \rangle = y_n \rightarrow 0, \quad n \rightarrow \infty$$

as

$$\|e_n - 0\|_{l^2} = \|e_n\|_{l^2} = 1.$$

Statement 3.15. $x_n \rightarrow x$ and $y_n \rightarrow y$ yields

$$\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle.$$

In particular

$$x_n \rightarrow x \quad \Rightarrow \quad \|x_n\| \rightarrow \|x\|.$$

proof.

$$\begin{aligned}
 |\langle x_n, y_n \rangle - \langle x, y \rangle| &= |\langle x_n, y_n \rangle - \langle x, y_n \rangle + \langle x, y_n \rangle - \langle x, y \rangle| \\
 &= |\langle x_n - x, y_n \rangle + \langle x, y_n - y \rangle| \\
 &\leq |\langle x_n - x, y \rangle| + |\langle x, y_n - y \rangle| \\
 &\leq \underbrace{\|x_n - x\|}_{\rightarrow 0} \underbrace{\|y_n\|}_{< \infty} + \underbrace{\|x\|}_{< \infty} \underbrace{\|y_n - y\|}_{\rightarrow 0} \rightarrow 0, \quad n \rightarrow \infty.
 \end{aligned}$$

Check $\{\|y_n\|\}$ is bounded

$$\|y_n\| = \|y_n - y + y\| \leq \underbrace{\|y_n - y\|}_{\rightarrow 0} + \underbrace{\|y\|}_{< \infty} \rightarrow 0, \quad n \rightarrow \infty.$$

□

Statement 3.16. $x_n \rightarrow x$ and $\|x_n\| \rightarrow \|x\|$ yields

$$x_n \rightarrow x.$$

proof.

$$\begin{aligned}
 \|x_n - x\|^2 &= \langle x_n - x, x_n - x \rangle \\
 &= \underbrace{\langle x_n, x_n \rangle}_{=\|x_n\|^2} - \langle x, x_n \rangle - \langle x_n, x \rangle + \langle x, x \rangle \\
 &= \|x_n\|^2 - \overline{\langle x_n, x \rangle} - \langle x_n, x \rangle + \|x\|^2 \\
 &\rightarrow \|x\|^2 - \|x\|^2 - \|x\|^2 + \|x\|^2 = 0.
 \end{aligned}$$

□

We have proved

$$x_n \rightarrow x \quad \Rightarrow \quad \{\|x_n\|\} \text{ is bounded.}$$

Theorem 3.17.

$$x_n \rightarrow x \quad \Rightarrow \quad \sup_{n \in \mathbb{N}} \|x_n\| < \infty.$$

proof. Let $x_n \rightarrow x$. Consider $f_n : H \rightarrow \mathbb{C}$ where

$$f_n(y) = \langle y, x_n \rangle, \quad y \in H.$$

- f_n is a linear functional for every $n \in \mathbb{N}$.

- $\forall n \in \mathbb{N}$ f_n is a bounded (\Leftrightarrow continuous) linear functional as if

$$y_k \xrightarrow{k \rightarrow \infty} y \quad \Rightarrow \quad f_n(y_k) = \langle y_k, x_n \rangle \rightarrow \langle y, x_n \rangle = f_n(y), \quad k \rightarrow \infty.$$

- $f_n(y) \rightarrow \langle y, x \rangle$.
 $\{f_n(y)\}_n$ is a convergent sequence in \mathbb{C} and hence bounded for all $y \in H$.
Hence it exists M_y such that

$$|f_n(y)| \leq M_y.$$

By Banach-Steinhaus-Theorem it holds

$$\|f_n\| \leq M \text{ for some } M > 0.$$

We are done if we proof that $\|f_n\| = \|x_n\|$.

$$|f_n(y)| = |\langle y, x_n \rangle| \leq \|y\| \|x_n\|, \quad \forall y \in H.$$

Hence

$$\|f_n\| \leq \|x_n\| \quad (1).$$

On the other Hand we have

$$f_n(x_n) = \langle x_n, x_n \rangle = \|x_n\|^2$$

and thus

$$\|f_n\| = \sup_{x \in H} \frac{|f_n(x)|}{\|x\|} \geq \frac{|f_n(x_n)|}{\|x_n\|} = \|x_n\| \quad (2)$$

With (1) and (2) we are finished.

□

3.2 Orthogonal decomposition in Hilbert spaces

Remember Linear Algebra. Take \mathbb{R}^n and a subspace $M \subseteq \mathbb{R}^n$

$$\Rightarrow \quad \forall x \in \mathbb{R}^n \quad x = z + y, \quad \text{where } z \in M, y \in M^\perp.$$

This can be done in a unique way

$$M = \text{span}\{e_z\}$$

$$M^\perp = \text{span}\{e_y\}$$

and

$$z = \text{proj}_{M^\perp} x, \quad \|x - \text{proj}_M x\| = \min_{y \in M} \|x - y\|.$$

General Hilbert space case

Proposition 3.18. $M \subseteq H$, then M^\perp is a closed subspace and

$$(M^\perp)^\perp = \overline{\text{span } M}.$$

Statement 3.19. H Hilbert space and M -closed subspace of H and $x \in H$. Then there exists a unique $z \in M$ such that

$$\|x - z\| = \text{dist}(x, M) := \inf_{y \in M} \|x - y\|.$$

(z analog of the $\text{proj}_M x$ in the other case).

Proposition 3.20. Taking $z \in M$ from the previous proposition. We have $x - z \in M^\perp$, i.e.

$$x = \underbrace{z}_{\in M} + \underbrace{(x - z)}_{\in M^\perp}.$$

Theorem 3.21 (Orthogonal Decomposition Theorem). Let $(E, \langle \cdot, \cdot \rangle)$ be a Hilbert space and S be a closed subspace of E .

$$\Rightarrow E = S \oplus S^\perp$$

which means that for every $x \in E$ there exists a unique decomposition

$$x = y + z$$

with $y \in S$ and $z \in S^\perp$.

Example. Let $A \subseteq E$ where E is a Hilbert space. It follows

$$\overline{\text{span } A} = (A^\perp)^\perp.$$

Note

$$A \subseteq \underbrace{(A^\perp)^\perp}_{\text{subspace of } E} \Rightarrow \text{span } A \subseteq \underbrace{(A^\perp)^\perp}_{\text{closed}} \Rightarrow \overline{\text{span } A} \subseteq (A^\perp)^\perp$$

$$A \subseteq \overline{\text{span } A} \Rightarrow \overline{\text{span } A}^\perp \subseteq A^\perp \Rightarrow (A^\perp)^\perp \subseteq (\overline{\text{span } A}^\perp)^\perp.$$

Hence

$$\overline{\text{span } A} \subseteq (A^\perp)^\perp \subseteq (\overline{\text{span } A}^\perp)^\perp.$$

By the Orthogonal Decomposition Theorem we get

$$E = \overline{\text{span } A} \oplus \overline{\text{span } A}^\perp = \overline{\text{span } A}^\perp \oplus \left(\overline{\text{span } A}^\perp \right)^\perp,$$

which implies

$$\begin{aligned} \overline{\text{span } A} &= \left(\overline{\text{span } A}^\perp \right)^\perp, \\ \Rightarrow \quad \left(A^\perp \right)^\perp &= \overline{\text{span } A}. \end{aligned}$$

Now we are going to prove the Orthogonal Decomposition Theorem.

proof. Step 1: S is a closed convex set in a Hilbert space E . This implies that

$$\forall x \in E \exists! y \in S : \quad \|x - y\| \leq \|x - \tilde{y}\| \quad \forall \tilde{y} \in S.$$

which means

$$\|x - y\| = \inf_{\tilde{y} \in S} \|x - \tilde{y}\|.$$

Fix $x \notin S$ with

$$\inf_{\tilde{y} \in S} \|x - \tilde{y}\| = d > 0.$$

Take a sequence $(y_n)_{n=1}^\infty$ in S such that

$$\|x - y_n\| \rightarrow d, \quad n \rightarrow \infty.$$

Claim: This is a Cauchy sequence.

(use Parallelogram-law for $\|\cdot\|$)

Step 2: S as in ODT.

Note: S must be convex.

Fix $x \in E$, choose $y \in S$ with

$$\|x - y\| \leq \|x - \tilde{y}\|, \quad \forall \tilde{y} \in S.$$

Set

$$\underbrace{x}_{\in E} = \underbrace{y}_{\in S} + (x - y).$$

To show: $x - y \in S^\perp$. A variational argument of this is

$$\langle x - y, v \rangle = 0, \quad \forall v \in S.$$

We know

$$\begin{aligned} \|x - y\|^2 &\leq \|x - y + \alpha v\|^2 \quad \forall \text{ scalars } \alpha \\ \|x - y\|^2 &\leq \langle x - y + \alpha v, x - y + \alpha v \rangle \\ &= \|x - y\|^2 + \alpha \langle v, x - y \rangle + \bar{\alpha} \langle x - y, v \rangle + |\alpha|^2 \|v\|^2 \end{aligned}$$

and

$$0 \leq 2 \operatorname{Re}(\alpha \langle x + y, v \rangle) + |\alpha|^2 \|v\|.$$

Set

$$\begin{aligned} \alpha &= t \overline{\langle x - y, v \rangle}, \quad t \in \mathbb{R}, \\ \Rightarrow \quad 0 &\leq 2t |\langle x - y, v \rangle|^2 + t^2 |\langle x - y, v \rangle|^2 \|v\|^2. \end{aligned}$$

Assume $\langle x - y, v \rangle \neq 0$:

We have

$$\begin{aligned} 0 &\leq 2t + t^2 \|v\|^2 \quad \forall t \in \mathbb{R} \\ \Rightarrow \quad -2t &\leq t^2 \|v\|^2, \quad \text{Let } t < 0 \\ \Leftrightarrow \quad 2 &\leq -t \|v\|^2, \quad t < 0. \end{aligned}$$

Let $t \rightarrow 0$, then

$$2 \leq 0$$

which is a contradiction. □

3.3 Bounded linear functionals on Hilbert spaces

Consider $(H, \langle \cdot, \cdot \rangle)$ - Hilbert space (inner product space which is complete w.r.t. to a norm $\|x\| = \sqrt{\langle x, x \rangle}$).

Let M be a closed subspace of H .

$$M^\perp = \{y \in H \mid \langle x, y \rangle = 0, \forall x \in M\}.$$

Then we know $H = M + M^\perp$, i.e. for any $x \in H$ there exists a unique $y \in M$ and $z \in M^\perp$ such that

$$x = y + z.$$

Theorem 3.22 (Riesz-Freché representation theorem). Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbertspace. Let f be a bounded linear functional on H . Then there exists a unique $x_f \in H$ such that

$$f(x) = \langle x, x_f \rangle, \quad \forall x \in H.$$

Moreover

$$\|f\| = \|x_f\|_H.$$

Remark. If $f : H \rightarrow \mathbb{C}$ is of the form

$$f(x) = \langle x, y \rangle, \quad \text{for all } x \in H \text{ and some } y \in H.$$

Then f is bounded and linear (easy with Cauchy-Schwarz and properties of the scalar product).

proof. Existence of x_f : If f is a zero linear functional, i.e. $f(x) = 0$ for all $x \in H$ take $x_f = 0$. Assume now that f is not the zero functional. Consider

$$N(f) := \ker f = \{x \in H \mid f(x) = 0\}.$$

Then $N(f)$ is a closed subspace of H :

For $x_1, x_2 \in N(f)$, $\alpha, \beta \in \mathbb{C}$ it holds

$$f(\alpha x_1 + \beta x_2) \stackrel{\text{lin}}{=} \alpha f(x_1) + \beta f(x_2).$$

Hence $\alpha x_1 + \beta x_2 \in N(f)$ and $N(f)$ is a subspace. $N(f)$ is closed since if $x_n \in N(f)$ with $x_n \rightarrow x$ strongly. Then

$$f(x_n) \rightarrow f(x)$$

because of bounded and hence continuous. But we know that $f(x_n) = 0$ so the limit has to be $f(x) = 0$, i.e. $x \in N(f)$. $N(f)$ is a proper closed subspace. ($N(f) \neq H$). Consider now $N(f)^\perp$ which is non-zero.

- $\dim N(f)^\perp = 1$.

Assume that $x_1 \neq 0, x_2 \neq 0 \in N(f)^\perp$. Then we have $f(x_1), f(x_2) \neq 0$. It exists $a \in \mathbb{C}$ such that

$$f(x_1) + a f(x_2) = 0.$$

And also

$$f(x_1 + a x_2) = 0$$

which gives

$$x_1 + a x_2 \in N(f) \cap N(f)^\perp = \{0\}.$$

Hence

$$x_1 + a x_2 = 0.$$

Any two vectors are linearly dependent in $N(f)^\perp$ which gives

$$\dim N(f)^\perp = 1.$$

Take $y' \in N(f)^\perp$ with $\|y'\| = 1$ and let

$$x_f = \overline{f(y')} y'.$$

We get

$$\langle x, x_f \rangle = \begin{cases} 0, & \text{if } x \in N(f) \\ \langle \lambda y', \overline{f(y')} y' \rangle = f(y') \lambda \underbrace{\langle y', y' \rangle}_{=1}, & \text{if } x = \lambda y' \end{cases}.$$

Furthermore

$$\langle x, x_f \rangle = \begin{cases} f(x), & \text{if } x \in N(f) \\ f(\lambda y') = f(x), & \text{if } x = \lambda y' \end{cases}.$$

Since every element in H is given by $x + \lambda y'$. For $x \in N(f)$ and $\lambda \in \mathbb{C}$. Using linearity we get

$$f(x + \lambda y') = f(x) + f(\lambda y') = \langle x, x_f \rangle + \langle \lambda y', x_f \rangle = \langle x + \lambda y', x_f \rangle$$

uniqueness: Assume there exists $x_1, x_2 \in H$ such that

$$f(x) = \langle x, x_1 \rangle = \langle x, x_2 \rangle, \quad \forall x \in H.$$

We get

$$\langle x, x_1 - x_2 \rangle = 0, \quad \forall x \in H.$$

It holds in particular for $x = x_1 - x_2$ the following equality

$$\langle x_1 - x_2, x_1 - x_2 \rangle = 0 \quad \Rightarrow \quad x_1 - x_2 = 0.$$

norm equality We must see that

$$\|f\| = \|x_f\|_H.$$

From remark we have

$$f(x) = \langle x, x_f \rangle \quad \Rightarrow \quad \|f\| \leq \|x_f\|.$$

We have for $x_f \neq 0$:

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} \geq \frac{|f(x_f)|}{\|x_f\|} = \frac{\|x_f\|^2}{\|x_f\|} = \|x_f\|.$$

This gives the desired result. □

Example.

$$E = C_F = \{(x_1, x_2, \dots) \mid \text{only finite number of } x_i \neq 0\} \subseteq l^2.$$

On C_F consider l^2 -inner-product

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i \quad \text{for } x, y \in C_F.$$

1. C_F is not a Hilbert space as it is not complete w.r.t

$$\|x\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}.$$

Find a Cauchy sequence that is not convergent to an element in C_F .

Find a proper closed subspace M such that $M^\perp = \{0\}$ (This would mean in particular that $C_F \neq M + M^\perp$)

Consider

$$M = \left\{ (x_1, x_2, \dots) \in C_F \mid \sum_{k=1}^{\infty} x_k \frac{1}{k} = 0 \right\},$$

$$x_f = (1, \frac{1}{2}, \frac{1}{3}, \dots) \in l^2,$$

$$M = \ker f \cap C_F$$

where

$$f : l^2 \rightarrow \mathbb{C}$$

$$f(x) = \langle x, x_f \rangle = \sum_{k=1}^{\infty} x_k \frac{1}{k},$$

$$M^{\perp} = \text{all elements in } C_F \text{ which are in } (\ker f)^{\perp}.$$

From the proof of Riesz-Frechet theorem we have $(\ker f)^{\perp}$ is 1-dimensional and

$$x_f \in (\ker f)^{\perp}.$$

Hence

$$(\ker f)^{\perp} = \{\lambda x_f \mid \lambda \in \mathbb{C}\}.$$

We have

$$\underbrace{(\ker f)^{\perp} \cap C_F}_{=M^{\perp}} = \{0\}.$$

2. $(H, \langle \cdot, \cdot \rangle)$ Hilbert space and $\{u_i\} \subseteq H$ finite or infinite i . $\{u_i\}$ is an orthogonal system if

$$\langle u_i, u_j \rangle = 0, \quad \forall i \neq j$$

and an orthonormal system if

$$\langle u_i, u_j \rangle = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}.$$

Proposition 3.23. Orthogonal system of non-zero vectors are linearly independent. (See linear algebra)

Having linearly independent family of vectors we can make it orthogonal with for example using Gram-Schmidt orthogonalization procedure. (See linear algebra for details).

Recall that we can write a Fourier series of x with $\langle x, u_i \rangle$ Fourier coefficients

$$x \in H \quad \Rightarrow \quad x = \sum_{i=1}^{\infty} \langle x, u_i \rangle u_i$$

with $\{u_i\}$ -ON-system.

$C([-\pi, \pi])$ and $\{u_k\} = \left\{ \frac{1}{\sqrt{2\pi}} e^{ikt} \mid k \in \mathbb{Z} \right\}$ equipped with the scalar product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt.$$

It holds for the Fourier-series

$$\langle f, u_k \rangle = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt.$$

We want to see when

$$\sum_{i=1}^{\infty} \langle x, u_i \rangle u_i$$

is convergent to x .

Definition 3.24. \mathcal{A}_n ON-system is called an ON-basis for H if its span is dense in H . We say that an ON-system is complete if every $x \in H$ is

$$\sum_{i=1}^{\infty} \langle x, u_i \rangle u_i.$$

Theorem 3.25. $(H, \langle \cdot, \cdot \rangle)$ - Hilbert space, $\{u_k\}$ is ON-system in H . The following statements are equivalent.

- (1) $\{u_n\}$ is a complete ON-system.
- (2) $\{u_n\}$ is an ON-basis for H .
- (3) (Parseval's Identity)

$$\|x\| = \left(\sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2 \right)^{\frac{1}{2}}, \quad \forall x \in H.$$

- (4) $\langle x, y \rangle = \sum_{k=1}^{\infty} \langle x, u_k \rangle \overline{\langle y, u_k \rangle}$ for all $x, y \in H$.
- (5) $\langle x, u_k \rangle = 0$ for all $k \in \mathbb{N}$ follows $x = 0$.

proof. (1) \Rightarrow (2): We have

$$x = \sum_{i=1}^{\infty} \langle x, u_i \rangle u_i$$

it means

$$x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \langle x, u_i \rangle u_i \in \overline{\text{span}\{u_i \mid i \geq 1\}}.$$

This implies that any $x \in H$ is in $\overline{\text{span}\{u_i \mid i \geq 1\}}$, i.e. $\{u_i\}$ is ON-basis.

(2) \Rightarrow (5): Let $\{u_i\}$ be a ON-basis. Assume

$$\langle x, u_k \rangle = 0, \quad \forall k \in \mathbb{N}.$$

Then

$$\langle x, u \rangle = 0, \quad \forall u \in \text{span}\{u_k \mid k \geq 1\}.$$

By the property that strong convergence implies weak convergence we will have

$$\langle x, u \rangle = 0, \quad \forall u \in \text{span}\{u_k \mid k \geq 1\} = H.$$

In particular

$$\langle x, u \rangle = 0, \quad \text{for } u = x$$

which means

$$\langle x, x \rangle = 0 \quad \Leftrightarrow \quad x = 0.$$

(5) \Rightarrow (1) Recall Bessel's equality. For $\{u_k\}$ - ON-system then

$$\left\| x - \sum_{i=1}^k \langle x, u_i \rangle u_i \right\|^2 = \|x\|^2 - \sum_{i=1}^k |\langle x, u_i \rangle|^2$$

Assume (5), i.e.

$$\langle x, u_k \rangle = 0, \quad \forall k \quad \Rightarrow \quad x = 0$$

We must see

$$x = \sum_{k=1}^n \langle x, u_k \rangle u_k \quad \forall x \in H.$$

From Bessel's equality we have

$$\sum_{k=1}^n |\langle x, u_k \rangle|^2 = \|x\|^2 - \left\| x - \sum_{k=1}^n \langle x, u_k \rangle u_k \right\|^2 \leq \|x\|^2, \quad \forall n \in \mathbb{N}$$

and hence $\sum_{k=1}^n |\langle x, u_k \rangle|^2$ is convergent. It implies that for $n > m$ we have

$$\begin{aligned} \left\| \sum_{k=1}^n \langle x, u_k \rangle u_k - \sum_{k=1}^m \langle x, u_k \rangle u_k \right\|^2 &= \left\| \sum_{k=m+1}^n \langle x, u_k \rangle u_k \right\|^2 \\ &\stackrel{\text{pythagorian thm}}{=} \sum_{k=m+1}^n |\langle x, u_k \rangle|^2 \|u_k\|^2 \\ &\rightarrow 0, \quad n, m \rightarrow \infty \quad (*). \end{aligned}$$

Note that if $\{x_i\}$ are pairwise orthogonal it holds

$$\left\| \sum_{i=1}^n x_i \right\|^2 = \sum_{i=1}^n \|x_i\|^2.$$

From (*) we know that the partial sum

$$S_n := \sum_{k=1}^n \langle x, u_k \rangle u_k$$

is a Cauchy sequence. As H is a Hilbert space, H is complete and hence S_n has a limit in H . Write

$$\sum_{i=1}^{\infty} \langle x, u_i \rangle w_i$$

for the limit. We must see that the limit is x . Consider

$$y := x - \sum_{i=1}^{\infty} \langle x, u_i \rangle w_i.$$

Then

$$\langle y, u_i \rangle = \langle x, u_i \rangle - \langle x, u_i \rangle = 0, \quad \forall i.$$

By assumption (5) it follows

$$y = 0 \quad \Leftrightarrow \quad x = \sum_{i=1}^{\infty} \langle x, u_i \rangle w_i.$$

(1) \Rightarrow (3): From Bessel's equality we have again

$$\left\| x - \sum_{i=1}^n \langle x, u_i \rangle w_i \right\|^2 = \|x\|^2 - \sum_{i=1}^n |\langle x, u_i \rangle|^2.$$

By assumption (1) the LHS tends to 0 as $n \rightarrow \infty$. On the other hand the RHS goes to

$$\rightarrow \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, u_i \rangle|^2, \quad n \rightarrow \infty.$$

This gives

$$\|x\|^2 - \sum_{i=1}^{\infty} |\langle x, u_i \rangle|^2 = 0.$$

(3) \Rightarrow (5) trivial.

(4) \Rightarrow (5) trivial (take $y = x$).

(1) \Rightarrow (4) We have

$$x = \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k.$$

Then

$$\langle x, y \rangle = \sum_{k=1}^{\infty} \langle x, u_k \rangle \langle u_k, y \rangle = \sum_{k=1}^{\infty} \langle x, u_k \rangle \overline{\langle y, u_k \rangle}.$$

□

Example. $L^2([-\pi, \pi])$ with

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{int} \mid n \in \mathbb{Z} \right\}$$

is an ON-system in $L^2([-\pi, \pi])$ where

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt.$$

Statement 3.26. The system above is an ON-basis for $L^2([-\pi, \pi])$. In particular, for any $f \in L^2([-\pi, \pi])$

$$f = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikt}$$

convergent in the L^2 -norm.

$$\|f\|_{L^2} = \left(\int_{-\pi}^{\pi} |f(t)|^2 dt \right)^{\frac{1}{2}}$$

which is equivalent to

$$\left\| f - \sum_{k=-n}^n \hat{f}(k) e^{ikt} \right\|_{L^2}^2 \rightarrow 0.$$

Sketch of the proof:

(1) Stein-Weierstraß-Theorem. X compact set $C(X, \mathbb{C})$ continuous functions with complex values. Let $M \subseteq C(X, \mathbb{C})$ be a subspace that satisfies:

(a) it separates points of X , i.e.

$$\forall x_1, x_2 \in X, x_1 \neq x_2 \exists f \in M : f(x_1) \neq f(x_2).$$

(b) M contains the constant function 1 ($f(x) = 1$ for all $x \in X$).

(c) It is closed under complex conjugation, i.e.

$$f \in M \Rightarrow \bar{f} \in M$$

and closed under product, i.e.

$$f_1, f_2 \in M \Rightarrow f_1 \cdot f_2 \in M.$$

Then M is dense in $C(X, \mathbb{C})$ w.r.t. $\|\cdot\|_{\infty}$ (Continuous function by Polynomials) From this it follows

$$M = \{\text{all complex polynomials}\}$$

are dense in $C([a, b])$.

(2) $C([a, b])$ is dense in $L^2([a, b])$ w.r.t. $\|\cdot\|_{L^2}$ -norm.

We will use (1) and (2) to show that $\text{span}\left\{\frac{1}{\sqrt{2\pi}}e^{int} \mid n \in \mathbb{Z}\right\}$ is dense in $L^2([-\pi, \pi])$.

proof. Let

$$M := \text{span}\left\{\frac{1}{\sqrt{2\pi}}e^{int} \mid n \in \mathbb{Z}\right\} \subseteq \{f \in C([-\pi, \pi]) \mid f(\pi) = f(-\pi)\}.$$

M separates points, it contains the constant function 1 and it is closed under complex conjugation. Furthermore M is closed under taking products. With Stein-Weierstraß it follows that M is dense in

$$\{f \in C([-\pi, \pi]) \mid f(\pi) = f(-\pi)\}.$$

By (2) we have $C([-\pi, \pi])$ is dense in $L^2([-\pi, \pi])$ w.r.t. the L^2 -norm. From this one can see that even $\{f \in C([-\pi, \pi]) \mid f(\pi) = f(-\pi)\}$ is dense in $L^2([-\pi, \pi])$:

$$\forall \varepsilon > 0, \forall f \in L^2 \exists g \in C([-\pi, \pi]) : \quad \|f - g\|_{L^2}^2 = \int_{-\pi}^{\pi} |f(t) - g(t)|^2 dt < \varepsilon.$$

Define g_ε such that it has a pike in $x = \pi - \varepsilon$ but it is continuous and is equal to g for $x < \pi - \varepsilon$. Then

$$g_\varepsilon \in C([-\pi, \pi]), \quad g_\varepsilon(-\pi) = g_\varepsilon(\pi).$$

It holds

$$\begin{aligned} \|f - g_\varepsilon\|_{L^2} &\leq \underbrace{\|f - g\|_{L^2}}_{< \sqrt{\varepsilon}} + \|g - g_\varepsilon\|_{L^2} \\ &\leq \sqrt{\varepsilon} + \left(\int_{\pi-\varepsilon}^{\pi} |g(t) - g_\varepsilon(t)|^2 dt \right)^{\frac{1}{2}} \\ &\leq \sqrt{\varepsilon} + \sqrt{\max_{x \in [\pi-\varepsilon, \pi]} |g - g_\varepsilon| \varepsilon} \\ &= \sqrt{\varepsilon} + \sqrt{C} \sqrt{\varepsilon}. \end{aligned}$$

We conclude: any $f = L^2$ -limit g_n with $g_n \in C([-\pi, \pi])$ and $g_n(-\pi) = g_n(\pi)$. Each $g_n = \|\cdot\|_\infty$ -norm limit of an element in $\text{span}\left\{\frac{1}{\sqrt{2\pi}}e^{int} \mid n \in \mathbb{Z}\right\}$ as

$$\|g - f\|_{L^2} \leq \|g - f\|_\infty^{\frac{1}{2}} (2\pi)^{\frac{1}{2}}.$$

Note that

$$\left(\int_{-\pi}^{\pi} |g(t) - f(t)|^2 dt \right)^{\frac{1}{2}} \leq \max_{x \in [-\pi, \pi]} |g(t) - f(t)| \left(\int_{-\pi}^{\pi} dt \right)^{\frac{1}{2}}.$$

We get that each g_n can be approximated in the L^2 -norm by elements in $\text{span}\left\{\frac{1}{\sqrt{2\pi}}e^{int} \mid n \in \mathbb{Z}\right\}$ hence

$$\text{span}\left\{\frac{1}{\sqrt{2\pi}}e^{int} \mid n \in \mathbb{Z}\right\} \subseteq L^2([-\pi, \pi]).$$

□

3.4 Linear operators on Hilbert spaces

Set $(H_1, \langle \cdot, \cdot \rangle_1)$ and $(H_2, \langle \cdot, \cdot \rangle_2)$ Hilbert spaces. A bounded linear mapping $A : H_1 \rightarrow H_2$ is called bounded linear operator.

Bounded means in our case

$$\|Ax\|_2 \leq C\|x\|_1 \quad \forall x \in H \text{ and some constant } C$$

Example. Set $H_1 = H_2 = L^2([0, 1])$ and $K : [0, 1] \times [0, 1] \rightarrow \mathbb{C}$. Assume that K is continuous. Consider

$$(Af)(x) = \int_0^1 K(x, y)f(y) dy.$$

A is linear (trivial). Show that A is bounded:

$$\begin{aligned} \|Af\|_2 &= \int_0^1 \left| \int_0^1 K(x, y)f(y) dy \right|^2 dx \\ &\stackrel{\text{CS}}{\leq} \int_0^1 \left(\int_0^1 |K(x, y)|^2 dy \cdot \int_0^1 |f(y)|^2 dy \right) dx \\ &= \underbrace{\int_0^1 \left(\int_0^1 |K(x, y)|^2 dy \right) dx}_{< \infty} \cdot \underbrace{\int_0^1 |f(y)|^2 dy}_{=\|f\|_2^2}. \end{aligned}$$

Hence

$$\|A\| \leq \left(\int_0^1 \int_0^1 |K(x, y)|^2 dx dy \right)^{\frac{1}{2}}.$$

Products $(A \cdot B)$ of operators $H \rightarrow H$ with $A : H \rightarrow H$ and $B : H \rightarrow H$ are defined by

$$(A \cdot B)(f) := A(Bf).$$

Statement 3.27. If A and B are bounded, then $A \cdot B$ is also bounded and

$$\|AB\| \leq \|A\|\|B\|.$$

In particular: for all $n \in \mathbb{N}$ A^n is bounded and

$$\|A^n\| \leq \|A\|^n.$$

Example. $E = L^2([0, 1])$ and $f, g \in E$ with

$$\langle f, g \rangle_{L^2} = \int_0^1 f(x) \overline{g(x)} dx, \quad \|f\|_{L^2} = \left(\int_0^1 |f(x)|^2 dx \right)^{\frac{1}{2}}.$$

Set $h \in C([0, 1] \times [0, 1])$ and for $f \in L^2([0, 1])$

$$A(f)(x) = \int_0^1 h(x, y)f(y) \, dy, \quad x \in [0, 1].$$

Then

$$\|A\| \leq \left(\int_0^1 \left(\int_0^1 |h(x, y)|^2 \, dy \right) \, dx \right)^{\frac{1}{2}} < \infty.$$

Example. Let $(E, \|\cdot\|)$ be a normed space. Then there are no $A, B \in B(E, E)$ such that

$$AB - BA = I$$

where I is the identity ($I(x) = x$ for $x \in E$).

Remark. Consider $f \in E = C^\infty([0, 1])$ and

$$A = \frac{d}{dx}, \quad B = x.$$

Then

$$(AB - BA)(f)(x) = \frac{d}{dx}(x(f(x))) - x \frac{d}{dx}f(x) = f(x).$$

Argue by contradiction.

Assume $A, B \in B(E, E)$ with $AB - BA = I$.

Hint: Consider $A^n B - BA^n$ for $n = 1, 2, \dots$. For $n = 2$ we have

$$\begin{aligned} A^2 B - BA^2 &= A^2 B - ABA + ABA - BA^2 \\ &= A(AB - BA) + (AB - BA)A \\ &= 2A. \end{aligned}$$

For $n = 3$ we have

$$\begin{aligned} A^3 B - BA^3 &= A^3 B - A^2 BA + A^2 BA - BA^3 \\ &= A^2(AB - BA) + (A^2 B - BA^2)A \\ &= 3A^2. \end{aligned}$$

In general

$$A^n B - BA^n = nA^{n-1}, \quad n = 2, 3, 4, \dots \quad (*)$$

Check using an induction argument. We obtain

$$n\|A^{n-1}\| = \|A^n B - BA^n\| \leq \|A^n B\| + \|BA^n\| \leq 2\|A^{n-1}\|\|A\|\|B\|$$

Hence

$$(2\|A\|\|B\| - n)\|A^{n-1}\| \geq 0, \quad \forall n = 2, 3, \dots$$

We conclude that $\|A^{n-1}\| = 0$ for n large enough. Clearly the same for $\|A^n\|$. This yields $A^n = 0$ for n large enough. Repeated use of $(*)$ gives $A = 0$. This contradicts $AB - BA = I$ so the implication in the example is proven.

Recall a important theorem:

Theorem 3.28 (Riesz representation theorem). $(E, \langle \cdot, \cdot \rangle)$ Hilbert space $f \in B(E, \mathbb{C})$. f is bounded linear functional on E . This yields

$$\exists ! x_f \in E : \quad f(x) = \langle x, x_f \rangle, \quad \forall x \in E.$$

Also it holds

$$\underbrace{\|f\|}_{\text{operator norm of } f} = \underbrace{\|x_f\|}_{\text{norm of } x_f \text{ in } E}.$$

Definition 3.29. $\varphi : E \times E \rightarrow \mathbb{C}$ is called:

- Bilinear, if for scalars α and β it holds

$$\begin{aligned} \varphi(\alpha x, \beta y, z) &= \alpha \varphi(x, z) + \beta \varphi(y, z) & \forall x, y, z \in E \\ \varphi(x, \alpha y + \beta z) &= \bar{\alpha} \varphi(x, z) + \bar{\beta} \varphi(y, z) & \forall x, y, z \in E. \end{aligned}$$

- Bounded, if there exists $M > 0$ such that

$$|\varphi(x, y)| \leq M \|x\| \|y\|, \quad \forall x, y \in E.$$

- Coercive, if there exists $K > 0$ such that

$$\varphi(x, x) \geq K \|x\|^2, \quad \forall x \in E.$$

Clearly $\langle \cdot, \cdot \rangle$ in E is a bilinear, bounded and coercive functional in E (with $M = K = 1$). We will now introduce a Generalization of the Riesz representation theorem.

Theorem 3.30 (Lax-Milgram). $(E, \langle \cdot, \cdot \rangle)$ Hilbert space. Let $\varphi : E \times E \rightarrow \mathbb{C}$ be a bilinear, bounded and coercive functional. $f : E \rightarrow \mathbb{C}$ bounded linear functional in E . Then there exists an unique $x_f \in E$ such that

$$f(x) = \varphi(x, x_f), \quad \forall x \in E.$$

proof. Step 1: $\exists ! A \in B(E, E)$ with

$$\varphi(x, y) = \langle x, A(y) \rangle, \quad \forall x, y \in E.$$

Step 2: A is injective and surjective.

Step 3: Apply RRT with $\tilde{x}_f = A^{-1}(x_f)$

$$\begin{aligned} f(x) &= \langle x, x_f \rangle \\ &= \langle x, A(A^{-1}(x_f)) \rangle \\ &= \varphi(x, \tilde{x}_f), \quad \forall x \in E. \end{aligned}$$

Step 1: Fix $y \in E$ and consider for $x \in E$

$$x \mapsto f_y(x) \in \mathbb{C}.$$

Claim: $f_y : E \rightarrow \mathbb{C}$ is a bounded linear functional.

For $x, y, z \in E$ and α, β scalars we have

$$\begin{aligned} f_y(\alpha x + \beta z) &= \varphi(\alpha x + \beta z, y) \\ &= \alpha \varphi(x, y) + \beta \varphi(z, y) \\ &= \alpha f_y(x) + \beta f_y(z). \end{aligned}$$

Hence f_y is linear. It is bounded because of

$$|f_y(x)| = |\varphi(x, y)| \leq (M\|y\|)\|x\|, \quad \forall x \in E.$$

So f_y is bounded.

RRT implies $f_y(x) = \langle x, A(y) \rangle$ for all $x \in E$ for some $A(y) \in E$.

Now we have $A : E \rightarrow E$. **Claim:** $A \in B(E, E)$.

For $x, y, z \in E$ and scalars α, β we have

$$\begin{aligned} \langle x, A(\alpha y + \beta z) \rangle &= \varphi(x, \alpha y + \beta z) \\ &= \bar{\alpha} \varphi(x, y) + \bar{\beta} \varphi(x, z) \\ &= \bar{\alpha} \langle x, A(y) \rangle + \bar{\beta} \langle x, A(z) \rangle \\ &= \langle x, \alpha A(y) \rangle + \langle x, \beta A(z) \rangle. \end{aligned}$$

This is equivalent to

$$\langle x, A(\alpha y + \beta z) - \alpha A(y) - \beta A(z) \rangle = 0, \quad x \in E.$$

This implies

$$\|A(\alpha y + \beta z) - \alpha A(y) - \beta A(z)\| = 0.$$

So

$$A(\alpha y + \beta z) = \alpha A(y) + \beta A(z) \quad \forall y, z \in E \text{ and scalars } \beta, \alpha.$$

Hence, A is linear. We will now show that A is bounded:

We know because φ is continuous that for all $x, y \in E$

$$|\langle x, A(y) \rangle| = |\varphi(x, y)| \leq M\|x\|\|y\|.$$

Take $x = A(y)$ and get

$$\|A(y)\|^2 \leq M\|A(y)\|\|y\| \quad \forall y \in E$$

which implies

$$\|A(y)\| \leq M\|y\| \quad \forall y \in E.$$

Hence $\|A\| \leq M < \infty$.

Step 2: Note $\varphi(x, y) = \langle x, A(y) \rangle$ for alle $x, y \in E$.

Claim: A is injective, i.e.

$$A(x_1) = A(x_2) \quad \Rightarrow \quad x_1 = x_2.$$

φ is coercive so

$$\|x\|^2 \leq \frac{\varphi(x, x)}{K} = \frac{1}{K} \underbrace{\geq 0}_{|\langle x, A(x) \rangle|} \leq \frac{1}{K} \|x\| \|A(x)\| \quad \forall x \in E.$$

Hence

$$\|x\| \leq \frac{1}{K} \|A(x)\|, \quad \forall x \in E.$$

If $A(x_1) = A(x_2)$ we have $A(x_1 - x_2) = 0 \in E$ then

$$\|x_1 - x_2\| \leq \frac{1}{K} \|A(x_1 - x_2)\| = 0.$$

We get $x_1 = x_2$.

Claim: A is surjective, i.e. the image of A is E :

$$\mathfrak{R}(A) = \{A(x) \mid x \in E\} = E.$$

We first show that $\mathfrak{R}(A)$ is a closed subspace of E .

- $\mathfrak{R}(A)$ is a subspace in E since A is linear.
- $\mathfrak{R}(A)$ is closed since

$$y_n \rightarrow y \quad \text{in } (E, \|\cdot\|) \quad \Rightarrow \quad y \in \mathfrak{R}(A).$$

$\mathfrak{R}(A)$ is linear. Take $y_1, y_2 \in \mathfrak{R}(A)$ with preimages x_1, x_2 and yield

$$\alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 A(x_1) + \alpha_2 A(x_2) = A(\alpha_1 x_1 + \alpha_2 x_2).$$

So

$$\alpha_1 y_1 + \alpha_2 y_2 \in \mathfrak{R}(A).$$

Assume

$$y_n \rightarrow y \quad \text{in } (E, \|\cdot\|).$$

For $n = 1, 2, \dots$ there are x_1, x_2, \dots such that $y_n = A(x_n)$ for $n = 1, 2, \dots$

Claim: $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in E since

$$\begin{aligned} \|x_n - x_m\| &\leq \frac{1}{K} \|A(x_n - x_m)\| \\ &= \frac{1}{K} \|A(x_n) - A(x_m)\| \\ &= \frac{1}{K} \|y_n - y_m\| \rightarrow 0, \quad n, m \rightarrow \infty \end{aligned}$$

since $(y_n)_{n \in \mathbb{N}}$ converges.

Since $(E, \|\cdot\|)$ is a Banach space $(x_n)_{n \in \mathbb{N}}$ converges in $(E, \|\cdot\|)$. Call the limit $x \in E$. Hence

$$A(x_n) \rightarrow y$$

since A is bounded, continuous and linear. So $y = A(x)$ and we get $y \in \mathfrak{R}(A)$.

Secondly A is surjective, i.e. $\mathfrak{R}(A) = E$.

Assume that this is not true. The Orthogonal decomposition theorem gives

$$E = \mathfrak{R}(A) \oplus \mathfrak{R}(A)^\perp.$$

The first one is a closed subspace in E and the second one is not empty by assumption.

Fix $z \in \mathfrak{R}(A)^\perp \setminus \{0\}$. Note

$$\varphi(x, y) = \langle x, A(y) \rangle \quad x, y \in E$$

With $x = y = z$ we get

$$\varphi(z, z) = \langle z, A(z) \rangle = 0$$

and

$$\varphi(z, z) \geq K\|z\|^2 \geq 0 \quad \Rightarrow \quad z = 0.$$

This is a contradiction.

The Conclusion is

$$\mathfrak{R}(A)^\perp = \{0\} \quad \Rightarrow \quad \mathfrak{R}(A) = E.$$

We have $\varphi(x, y) = \langle x, A(y) \rangle$ for all $x, y \in E$ and $A \in B(E, E)$ surjective.

Step 3: see above.

□

3.5 Adjoint operator

$(E, \langle \cdot, \cdot \rangle)$ Hilbert space and $A \in B(E, E)$ with adjoint A^* , i.e.

$$\langle A(x), y \rangle = \langle x, A^*(y) \rangle, \quad \forall x, y \in E.$$

Fix $y \in E$ and consider

$$x \mapsto \langle A(x), y \rangle \in \mathbb{C}.$$

Claim: f_y is a bounded linear functional on E

- linear since A is linear.
- bounded since A is bounded with

$$|f_y(x)| \leq (\|A\| \|y\|) \|x\|, \quad x \in E.$$

RRT implies

$$f_y(x) = \langle x, A^*(y) \rangle, \quad x \in E.$$

We have $A^* : E \rightarrow E$ such that

$$\langle A(x), y \rangle = \langle x, A^*(y) \rangle, \quad \forall x, y \in E.$$

Proposition 3.31. $A \in B(E, E)$. Then $A^* \in B(E, E)$ and $\|A^*\| = \|A\|$.

proof. A^* linear:

$$\langle x, A^*(\alpha y + \beta z) \rangle = \langle x, \alpha A^*(y) + \beta A^*(z) \rangle \quad \forall x, y \in E.$$

A^* bounded:

Take $x = A^*(y)$ and get

$$\begin{aligned} \|A^*(y)\|^2 &= |\langle A^*(y), A^*(y) \rangle| \\ &\leq \|A^*(y)\| \|y\| \\ &\leq \|A\| \|A^*(y)\| \|y\|, \quad y \in E. \end{aligned}$$

We get

$$\|A^*(y)\| \leq \|A\| \|y\|, \quad y \in E.$$

Conclusion: $A^* \in B(E, E)$. We also get

$$\|A^*\| \leq \|A\|.$$

But we also know that $A^{**} = A$ since

$$\begin{aligned} \langle x, A^{**}(y) \rangle &= \langle A^*(x), y \rangle \\ &= \overline{\langle y, A^*(x) \rangle} \\ &= \overline{\langle A(y), x \rangle} \\ &= \langle x, A(y) \rangle, \quad x, y \in E. \end{aligned}$$

So

$$\|A\| = \|A^{**}\| \leq \|A^*\|$$

which implies

$$\|A\| = \|A^*\|.$$

□

Remark. $A, B \in B(E, E)$ then

$$\begin{aligned} (A + B)^* &= A^* + B^* \\ (AB)^* &= B^* A^* \\ (\alpha A)^* &= \bar{\alpha} A^* \\ A^{**} &= A \\ I^* &= I. \end{aligned}$$

Example. Continuity of the example above: For $f \in L^2([0, 1])$ consider

$$A(f)(x) = \int_0^1 h(x, y)f(y) \, dy, \quad x \in [0, 1].$$

For $g \in L^2([0, 1])$ it holds

$$\begin{aligned} \langle A(f), g \rangle_{L^2} &= \int_0^1 A(f)(x) \overline{g(x)} \, dx \\ &= \int_0^1 \int_0^1 h(x, y)f(y) \, dx \overline{g(x)} \, dx \\ &= \int_0^1 f(y) \cdot \int_0^1 h(x, y) \overline{g(x)} \, dx \, dy \\ &= \int_0^1 f(y) \cdot \overline{\int_0^1 h(x, y)g(x) \, dx} \, dy \\ &= \langle f, A^*(g) \rangle_{L^2}. \end{aligned}$$

This gives us

$$A^*(f)(x) = \int_0^1 \overline{h(y, x)}f(y) \, dy, \quad x \in [0, 1].$$

Example. $A \in B(E, E)$. It follows

$$\Re(A)^\perp = N(A^*) = \{x \in E \mid A^*(x) = 0\}$$

since $x \in \Re(A)^\perp$. It is equivalent that

$$\begin{aligned} \langle x, A(y) \rangle &= 0, \quad \forall y \in E \\ \Leftrightarrow \quad \langle A^*(x), y \rangle &= 0, \quad \forall y \in E \\ \Rightarrow \quad A^*(x) &= 0 \quad \Leftrightarrow \quad x \in N(A^*). \end{aligned}$$

We get

$$N(A^*)^\perp = \overline{\Re(A)}$$

since

$$N(A^*)^\perp = \left(R(A)^\perp \right)^\perp = \overline{\text{span}(\Re(A))} = \overline{\Re(A)}.$$

Remark. $A \in B(E, E)$ is called self adjoint if $A^* = A$.

For $A \in B(E, E)$ we have

$$\|A\| = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |\langle A(x), y \rangle|$$

since

$$\|\langle A(x), y \rangle\| \leq \underbrace{\|A(x)\|}_{\leq \|A\|\|x\|} \leq \|A\|, \quad \text{for } \|x\| = \|y\| = 1.$$

If $A(x) = 0$ for all $x \in E$ then $\|A\| = 0$ and also

$$\sup_{\substack{\|x\|=1 \\ \|y\|=1}} |\langle A(x), y \rangle| = 0.$$

For x with $A(x) \neq 0$ then it is

$$A\left(\frac{1}{\|x\|}x\right) \neq 0.$$

For such an x with $\|x\| = 1$ we have

$$|\langle A(x), \frac{1}{\|A(x)\|}A(x) \rangle| = \frac{1}{\|A(x)\|} \|A(x)\|^2 = \|A(x)\|$$

and

$$\|A\| \leq \sup_{\|x\|=1} \|A(x)\| \leq \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |\langle A(x), y \rangle| \leq \|A\|.$$

Proposition 3.32. Let $A \in B(E, E)$ be self-adjoint. Then

$$\|A\| = \sup_{\|x\|=1} |\langle A(x), x \rangle|.$$

proof. Set

$$M = \sup_{\|x\|=1} |\langle A(x), x \rangle|.$$

For $\|x\| = 1$ we have

$$|\langle A(x), x \rangle| \leq \|A(x)\| \|x\| \leq \|A\|.$$

Furthermore

$$M \leq \|A\|.$$

It remains to prove: $\|A\| \leq M$.

For $x, z \in E$ consider:

$$\begin{aligned} \langle A(x+z), x+z \rangle - \langle A(x-z), x-z \rangle &= 2\langle A(x), z \rangle + 2\langle A(z), x \rangle \\ &= 2(\langle A(x), z \rangle + \langle z, A^*(x) \rangle) \\ &= 2(\langle A(x), z \rangle + \langle z, A(x) \rangle) \\ &= 4 \operatorname{Re}(\langle A(x), z \rangle). \end{aligned}$$

Assume now $A(x) \neq 0$ and set

$$z = \frac{1}{\|A(x)\|} A(x).$$

Hence

$$\|A(x)\| = \frac{1}{4} \left(\langle A(x + \frac{1}{\|A(x)\|} A(x)), x + \frac{1}{\|A(x)\|} A(x) \rangle - \langle A(x - \frac{1}{\|A(x)\|} A(x)), x - \frac{1}{\|A(x)\|} A(x) \rangle \right).$$

Note

$$|\langle A(y), y \rangle| = \|y\|^2 |\langle A(\frac{1}{\|y\|}y), \frac{1}{\|y\|}y \rangle| \leq M\|y\|^2.$$

We now obtain

$$\begin{aligned} \|A(x)\| &\leq \frac{1}{4} \left(M \left\| x + \frac{1}{\|A(x)\|} A(x) \right\|^2 + M \left\| x - \frac{1}{\|A(x)\|} A(x) \right\|^2 \right) \\ &= \frac{M}{4} 2 \left(\|x\|^2 + \left\| \frac{1}{\|A(x)\|} A(x) \right\|^2 \right) \\ &= \frac{M}{2} (\|x\|^2 + 1). \end{aligned}$$

So

$$\|A\| = \sup_{\|x\|=1} \|A(x)\| \leq M$$

and this yields

$$\|A\| = M.$$

□

Definition 3.33 (compact). If $A : E \rightarrow E$ is linear, then we say that A is compact if for all bounded sequences $(x_n)_{n=1}^\infty$ in E , $(A(x_n))_{n=1}^\infty$ has a bounded subsequence in E .

Lemma 3.34. A is compact and linear $\Rightarrow A$ is bounded.

proof. If A is not bounded then there exists a sequence $(y_n)_{n=1}^\infty$ in E such that

$$\|A(y_n)\| \geq n\|y_n\|, \quad \text{for } n = 1, 2, \dots$$

Set $x_n = \frac{y_n}{\|y_n\|}$ for $n = 1, 2, \dots$. Here $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\|A(x_n)\| = \left\| A \left(\frac{1}{\|y_n\|} y_n \right) \right\| = \frac{1}{\|y_n\|} \|A(y_n)\| > n, \quad \forall n \in \mathbb{N}.$$

$(A(x_n))_{n=1}^\infty$ has no converging subsequence since $\|A(x_n)\| \rightarrow \infty$ for $n \rightarrow \infty$. □

Remark. • $A \in B(E, E)$ and $F \subset E$ where F is bounded. Then

$$A(F) = \{A(x) \mid x \in F\}$$

is bounded.

• $A \in B(E, E)$ compact and $F \subset E$, F bounded. Then $\overline{A(F)}$ is compact.

Lemma 3.35. A, B compact linear operators $E \rightarrow E$ and α and β scalars. Then $\alpha A + \beta B$ is compact.

proof. Fix an arbitrary bounded sequence $(x_n)_{n=1}^\infty$ in E . Since A is compact there exists a converging subsequence $(A(x_{n_k}))_{k=1}^\infty$ of $(A(x_n))_{n=1}^\infty$.

Clearly $(\alpha A(x_n))_{n=1}^\infty$ converges in E .

Since B is compact there exists a converging subsequence $(B(x_{n_k}))_{k=1}^\infty$ of $(B(x_n))_{n=1}^\infty$.

Clearly $(\beta B(x_{n_k}))_{k=1}^\infty$ converges in E . Hence

$$(\alpha A(x_{n_k}) + \beta B(x_{n_k}))_{k=1}^\infty = ((\alpha A + \beta B)(x_{n_k}))_{k=1}^\infty$$

converges in E . □

Set

$$K(E, E) := \text{set of all compact linear mappings } E \rightarrow E.$$

We have $K(E, E)$ is a subspace in $(B(E, E), \|\cdot\|_{E \rightarrow E})$.

Proposition 3.36. $K(E, E)$ is a closed subspace in $(B(E, E), \|\cdot\|_{E \rightarrow E})$.

Before the proof we note:

1. Assume $(E, \langle \cdot, \cdot \rangle)$ to be a Hilbert space and $A \in B(E, E)$.

$$\begin{aligned} x_n \rightarrow x \text{ in } E &\Rightarrow A(x_n) \rightarrow A(x) \text{ in } E \\ x_n \rightharpoonup x \text{ in } E &\Rightarrow A(x_n) \rightharpoonup A(x) \text{ in } E \end{aligned}$$

since for $y \in E$ we have

$$\langle A(x_n), y \rangle = \langle x_n, A^*(y) \rangle \xrightarrow{n \rightarrow \infty} \langle x, A^*(y) \rangle = \langle A(x), y \rangle.$$

2. $A \in K(E, E)$ and $x_n \rightharpoonup x$ in E

$$\Rightarrow A(x_n) \rightarrow A(x) \quad \text{in } E.$$

3. $A \in B(E, E)$ finite-rank operator, i.e.

$$\dim \mathcal{R}(A) < \infty \quad \Rightarrow \quad A \in K(E, E)$$

since: Let e_1, e_2, \dots, e_N be an ON-basis for $\mathcal{R}(A)$ with $N = \dim(\mathcal{R}(A))$. We have

$$A(x) = \langle A(x), e_1 \rangle e_1 + \dots + \langle A(x), e_N \rangle e_N.$$

Fix an arbitrary bounded sequence $(x_n)_{n=1}^\infty$ in E . A is bounded which implies that $(A(x_n))_{n=1}^\infty$ is a bounded sequence. Furthermore

$$(\langle A(x_n), e_1 \rangle)_{n=1}^\infty$$

is a bounded sequence in \mathbb{C} . Bolzano Weierstrass theorem implies that $(\langle A(x_n), e_1 \rangle)_{n=1}^\infty$ has a converging subsequence $(\langle A(x_{n_k}), e_1 \rangle)_{k=1}^\infty$. Clearly $(\langle A(x_{n_k}), e_1 \rangle)_{k=1}^\infty$ converges in E .

Hence

$$A(x) = \langle A(x), e_1 \rangle e_1 + \dots + \langle A(x), e_N \rangle e_N$$

is a compact mapping since $K(E, E)$ is a subspace of $B(E, E)$.

proof. Assume $(A_n)_{n=1}^\infty \subseteq K(E, E)$ such that $A_n \rightarrow A$ in $(B(E, E), \|\cdot\|_{E \rightarrow E})$.

We have to show: $A \in K(E, E)$

Fix an arbitrary bounded sequence $(x_n)_{n=1}^\infty$ in E . We want to show that $(A(x_n))_{n=1}^\infty$ has a converging subsequence in E .

Set

$$M = \sup_n \|x_n\| < \infty.$$

$$\begin{aligned} A_1 \in K(E, E) &\Rightarrow (A_1(x_n))_{n=1}^\infty \text{ has a converging subsequence } (A_1(x_{n_k}))_{k=1}^\infty \\ A_2 \in K(E, E) &\Rightarrow (A_2(x_n))_{n=1}^\infty \text{ has a converging subsequence } (A_2(x_{n_k}))_{k=1}^\infty \end{aligned}$$

proceed inductively:

$$A_k \in K(E, E) \Rightarrow (A_k(x_n))_{n=1}^\infty \text{ has a converging subsequence } (A_k(x_{n_l}))_{l=1}^\infty$$

Also: $(A_l(x_{n,k}))_{n=1}^\infty$ converges in E for $l = 1, 2, \dots, k$.

Here $(A_k(y_n))_{n=1}^\infty$ converges for $k = 1, 2, \dots$

So since $(E, \|\cdot\|)$ is a Banach space it is enough to show that $(A(y_n))_{n=1}^\infty$ is a Cauchy sequence in $(E, \|\cdot\|)$.

Fix an arbitrary $\varepsilon > 0$. We have

$$\|A(y_n) - A(y_m)\| \leq \underbrace{\|A(y_n) - A_k(y_n)\|}_{\leq \|A - A_k\|_{E \rightarrow E} \|y_n\|} + \|A_k(y_n) - A_k(y_m)\| + \|A_k(y_m) - A(y_m)\|.$$

Fix k large enough such that

$$\|A_k - A\| < \frac{\varepsilon}{3M}.$$

Then

$$\|A(y_n) - A(y_m)\| < \frac{2}{3}\varepsilon + \|A_k(y_n) - A_k(y_m)\|$$

$(A_k(y_n))_{n=1}^\infty$ converges in E . This implies the existence of N such that

$$\forall n, m \geq N : \|A_k(y_n) - A_k(y_m)\| < \varepsilon$$

$$\Rightarrow \|A(y_n) - A(y_m)\| < \varepsilon, \quad \forall n, m \geq N$$

and thus $(A(y_n))_{n=1}^\infty$ is a Cauchy sequence. □

Proposition 3.37. Let $(E, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space and $A \in K(E, E)$. then there exist finite-ranked operators $A_n \in K(E, E)$ such that

$$\|A - A_n\|_{E \rightarrow E} \rightarrow 0, \quad n \rightarrow \infty.$$

proof. Let $(x_n)_{n=1}^\infty$ be an ON-basis for E . For

$$x = \sum_{k=1}^{\infty} \langle x, x_k \rangle x_k, \quad x \in E.$$

Set

$$A_n(x) = A \left(\sum_{k=1}^n \langle x, x_k \rangle x_k \right) = \sum_{k=1}^n \langle x, x_k \rangle A(x_k), \quad x \in E, \quad n = 1, 2, \dots$$

Here $\dim(\mathcal{R}(A_n)) \leq n$ for $n = 1, 2, \dots$

So A_n is a finite ranked operator in E for $n = 1, 2, \dots$

Fix $x \in E$ with $\|x\| = 1$ and consider:

$$\|(A - A_n)(x)\|^2 = \left\| A \left(\sum_{k=n+1}^{\infty} \langle x, x_k \rangle x_k \right) \right\|^2 \leq \sup_{\substack{\|y\|=1, \\ y \in \{x_1, \dots, x_n\}^\perp}} \|A(y)\|^2$$

and thus

$$\|A - A_n\|_{E \rightarrow E}^2 \leq \sup_{\substack{\|y\|=1, \\ y \in \{x_1, \dots, x_n\}^\perp}} \|A(y)\|^2.$$

Set

$$u_n := \sup_{\substack{\|y\|=1, \\ y \in \{x_1, \dots, x_n\}^\perp}} \|A(y)\|^2 < \infty, \quad n = 1, 2, \dots$$

Here $a_n \geq a_{n+1} \geq 0$ for $n = 1, 2, \dots$

Clearly $(a_n)_{n=1}^\infty$ converges in \mathbb{R} . Set $a = \lim_{n \rightarrow \infty} a_n$. It remains to prove $a = 0$. Assume $a > 0$. Then there exists $(y_n)_{n=1}^\infty$ in E such that

1. $\|y_n\| = 1$,
2. $y \in \{x_1, \dots, x_n\}^\perp$,
3. $\|A(y_n)\|^2 \geq \frac{1}{2}a$.

Claim: $y_n \rightharpoonup 0$ in $(E, \langle \cdot, \cdot \rangle)$ since:

Fix an arbitrary $x \in E$ and

$$\begin{aligned}
 |\langle y_n, x \rangle| &= |\langle y_n, \sum_{k=1}^{\infty} \langle x, x_k \rangle x_k \rangle| \\
 &= |\langle y_n, \sum_{k=n+1}^{\infty} \langle x, x_k \rangle x_k \rangle| \\
 &\leq \|y_n\| \cdot \left\| \sum_{k=n+1}^{\infty} \langle x, x_k \rangle x_k \right\| \\
 &= \sqrt{\sum_{k=n+1}^{\infty} |\langle x, x_k \rangle|^2} \rightarrow 0, \quad n \rightarrow \infty.
 \end{aligned}$$

(Note that $\sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 = \|x\|^2 < \infty$)

We have $y_n \rightarrow 0$ in $(E, \langle \cdot, \cdot \rangle)$ and

$$A \in B(E, E) \quad \Rightarrow \quad A(y_n) \rightarrow A(0) = 0.$$

Contradiction to (3) above which gives us $a = 0$. □

Proposition 3.38. $(E, \langle \cdot, \cdot \rangle)$ Hilbert space and $A \in K(E, E)$. Then

$$x_n \rightharpoonup x \text{ in } (E, \langle \cdot, \cdot \rangle) \quad \Rightarrow \quad A(x_n) \rightarrow A(x) \text{ in } (E, \langle \cdot, \cdot \rangle).$$

proof. $x_n \rightharpoonup x$ in $(E, \langle \cdot, \cdot \rangle)$ implies that $\sup_n \|x_n\| < \infty$ (according to important theorem). Since $A \in K(E, E)$, we know that $(A(x_n))_{n=1}^{\infty}$ has a converging subsequence $(A(x_{n_k}))_{k=1}^{\infty}$ since $(x_n)_{n=1}^{\infty}$ is bounded.

Say $A(x_{n_k}) \rightarrow y$ in E . $A \in K(E, E) \subset B(E, E)$ and $x_n \rightharpoonup x$ in $(E, \langle \cdot, \cdot \rangle)$.

This implies

$$A(x_n) \rightharpoonup A(x) \quad \text{in } (E, \langle \cdot, \cdot \rangle).$$

We get that $y = A(x)$. We have $A(x_{n_k}) \rightarrow A(x)$ in E .

Assume that $A(x_n) \not\rightarrow A(x)$ in E .

Then there exists an $\varepsilon > 0$ and a subsequence $(A(\tilde{x}_n))_{n=1}^{\infty}$ of $(A(x_n))_{n=1}^{\infty}$ such that

$$\|A(\tilde{x}_n) - A(x)\| \geq \varepsilon, \quad \forall n.$$

But $\tilde{x}_n \rightharpoonup x$ in $(E, \langle \cdot, \cdot \rangle)$ and to be compact implies that $(A(\tilde{x}_n))_{n=1}^{\infty}$ has a converging subsequence $(A(\tilde{x}_{n_k}))_{k=1}^{\infty}$ that converges to $A(x)$ (same argument as before) Conclusion: $A(x_n) \rightarrow A(x)$ in $(E, \langle \cdot, \cdot \rangle)$. □

Proposition 3.39. $A \in K(E, E)$ and $(E, \langle \cdot, \cdot \rangle)$ Hilbert space $\Rightarrow A^* \in K(E, E)$.

proof. Fix any bounded sequence $(x_n)_{n=1}^\infty$ in E .

$$\begin{aligned}\|A^*(x_n) - A^*(x_m)\| &= \langle A^*(x_n) - A^*(x_m), A^*(x_n) - A^*(x_m) \rangle \\ &= \langle x_n - x_m, A(A^*(x_n)) - A(A^*(x_m)) \rangle\end{aligned}$$

then use $A \in K(E, E)$. □

Proposition 3.40. $A \in K(E, E), B \in B(E, E) \Rightarrow AB, BA \in K(E, E)$.

Example. We already know this example: $k \in C([0, 1] \times [0, 1])$ with

$$A(f)(x) = \int_0^1 k(x, y)f(y) dy, \quad x \in [0, 1], \quad f \in L^2([0, 1]).$$

We know that $A \in B(L^2([0, 1]), L^2([0, 1]))$

$$\|A\|_{L^2 \rightarrow L^2} \leq \|k\|_{L^2([0, 1] \times [0, 1])}.$$

Claim: $A \in K(L^2([0, 1]), L^2([0, 1]))$.

Approximate A by finite-ranked operators.

Note: set $A = A_k$ and $B = A_{k_n}$ where k_n is a nice function on $[0, 1] \times [0, 1]$ and

$$A - B = A_k - A_{k_n} = A_{k-k_n}.$$

So

$$\|A - B\|_{L^2 \rightarrow L^2} \leq \|k - k_n\|.$$

Set

$$\begin{aligned}I_j &= [x_j - \frac{1}{N}, x_j], \quad j = 1, \dots, N, \quad x_j = \frac{j}{N} \\ \tilde{I}_l &= [y_l - \frac{1}{N}, y_l], \quad l = 1, \dots, N, \quad y_l = \frac{l}{N}.\end{aligned}$$

Set

$$k_n(x, y) = \sum_{j=1}^N \sum_{l=1}^N k(x_j, y_l) \chi_{I_j}(x) \chi_{\tilde{I}_l}(y)$$

where

$$\chi_{I_j}(x) = \begin{cases} 1, & \text{if } x \in I_j \\ 0, & \text{elsewhere.} \end{cases}$$

Since $k \in C([0, 1] \times [0, 1])$ and $[0, 1] \times [0, 1]$ compact in \mathbb{R}^2 then k is uniformly continuous on $[0, 1] \times [0, 1]$. We fix $\varepsilon > 0$.

Claim: It exists an N such that

$$\sup_{\substack{(x, y) \in \\ [0, 1] \times [0, 1]}} |k(x, y) - k_n(x, y)| < \varepsilon,$$

$$A_{k_N}(f)(x) = \int_0^1 k_N(x, y) f(y) dy = \sum_{j=1}^N \underbrace{\sum_{l=1}^N k(x_i, y_l) \int_0^1 \chi_{\tilde{I}_l}(y) f(y) dy \chi_{I_j}(x)}_{\text{scalar}}.$$

$$\dim(\mathcal{R}(A_{k_N})) = N < \infty.$$

Hence $A_{k_N} \in K(L^2([0, 1]), L^2([0, 1]))$ for all N .

Moreover

$$\|A - A_{k_N}\|_{L^2 \rightarrow L^2} \leq \|k - k_N\|_{L^2([0,1] \times [0,1])} < \varepsilon$$

for N large enough. $K(E, E)$ is a closed set in $(B(E, E), \|\cdot\|_{L^2 \rightarrow L^2})$ so $A \in K(L^2, L^2)$.

Example. $(E, \langle \cdot, \cdot \rangle)$ Hilbert space, $(x_n)_{n=1}^\infty$ ON-basis and $(\lambda_n)_{n=1}^\infty$ sequence of scalars. Set

$$T(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, x_n \rangle x_n, \quad x \in E.$$

Claim:

$$1) \quad T \in B(E, E) \quad \Leftrightarrow \quad (\lambda_n)_{n=1}^\infty \text{ is a bounded sequence in } \mathbb{C}.$$

$$2) \quad T \in K(E, E) \quad \Leftrightarrow \quad \lambda_n \rightarrow 0 \text{ for } n \rightarrow \infty.$$

Note $x \in E$ and the Parseval's formula

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2.$$

For $T(x) \in E$ we have

$$\|T(x)\|^2 = \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle x, x_n \rangle|^2.$$

If $(\lambda_n)_{n=1}^\infty$ bounded sequence in \mathbb{C} . Then $\sup |\lambda_n| \equiv M < \infty$ and

$$\|T(x)\|^2 \leq \sum_{n=1}^{\infty} M^2 |\langle x, x_n \rangle|^2 = M^2 \|x\|^2.$$

If $(\lambda_n)_{n=1}^\infty$ is not bounded then there exists a sequence $(\lambda_{n_k})_{k=1}^\infty$ such that $|\lambda_{n_k}| \rightarrow \infty$ as $k \rightarrow \infty$. But

$$\|T(x_{n_k})\| = |\lambda_{n_k}| \|x_{n_k}\| = |\lambda_{n_k}| \rightarrow \infty, \quad k \rightarrow \infty$$

$$\sup_{\|x\|=1} \|T(x)\| = \infty.$$

So 1) is done. For 2) we assume $\lambda_n \rightarrow 0$ for $n \rightarrow \infty$. Set

$$T_k(x) = \sum_{n=1}^k \lambda_n \langle x, x_n \rangle x_n, \quad x \in E$$

T_k is a finite rank operator for $k = 1, 2, \dots$ SO $T_k \in K(E, E)$ for all k .

$$\begin{aligned} \|T - T_k\|_{E \rightarrow E} &= \sup_{\|x\|=1} \|(T - T_k)(x)\| \\ &= \sup_{\|x\|=1} \left\| \sum_{n=k+1}^{\infty} \lambda_n \langle x, x_n \rangle x_n \right\| \\ &\leq \sup_{n=k+1, k+2, \dots} |\lambda_n| \rightarrow 0, \quad k \rightarrow \infty \end{aligned}$$

Assume $\lambda_n \not\rightarrow 0$ for $n \rightarrow \infty$. Then there exists $\varepsilon > 0$ and a sequence $(\lambda_{n_k})_{k=1}^{\infty}$ such that

$$|\lambda_{n_k}| \geq \varepsilon.$$

Note

$$T(x_{n_k}) = \lambda_{n_k} x_{n_k}, \quad k = 1, 2, \dots$$

$$\|T(x_{n_k})\| = |\lambda_{n_k}| \|x_{n_k}\| = |\lambda_{n_k}| \geq \varepsilon, \quad k = 1, 2, \dots$$

$x_{n_k} \xrightarrow{w} 0$ in $(E, \langle \cdot, \cdot \rangle)$ since for $y \in E$

$$\langle x_{n_k}, y \rangle = \langle x_{n_k}, \sum_{n=1}^{\infty} \langle y, x_n \rangle x_n \rangle = \overline{\langle y, x_{n_k} \rangle} \rightarrow 0$$

since

$$\sum_{n=1}^{\infty} |\langle y, x_n \rangle|^2 = \|y\|^2 < \infty.$$

If $T \in K(E, E)$ then $T(x_{n_k}) \rightarrow T(0) = 0$ but

$$\|T(x_{n_k})\| \geq \varepsilon, \quad \text{for all } k.$$

Hence

$$T \notin K(E, E).$$

Example. $(E, \langle \cdot, \cdot \rangle)$ Hilbert space, $A \in K(E, E)$ and $I(x) = x$ for all $x \in E$. It follows

$$\Rightarrow R(I - A) \text{ closed in } E.$$

Remark.

$$\begin{aligned} R(I - A)^{\perp} &= \mathcal{N}((I - A)^*) = \mathcal{N}(I - A^*) \\ \overline{R(I - A)} &= R(I - A)^{\perp\perp} = \mathcal{N}(I - A^*)^{\perp}. \end{aligned}$$

If $A \in K(E, E)$ then

$$\overline{R(I - A)} = R(I - A).$$

Solve

$$x = A(x) + y \quad \Leftrightarrow \quad (I - A)(x) = y$$

Compare 'Fredholm alternative'

proof. Take a sequence $(y_n)_{n \in \mathbb{N}} \subseteq R(I - A)$ such that $y_n \rightarrow y$ in $(E, \|\cdot\|)$.

To show: $y \in R(I - A)$, i.e. $y = (I - A)(x)$ for some $x \in E$ and $y_n = (I - A)(x_n)$ for some $x_n \in E$.

$$x_n \in E = \mathcal{N}(I - A) + \mathcal{N}(I - A)^\perp$$

such that

$$x_n = \tilde{x}_n + \hat{x}_n$$

with

$$\|x_n\|^2 = \|\tilde{x}_n\|^2 + \|\hat{x}_n\|^2.$$

Step 1: Show $(\hat{x}_n)_{n=1}^\infty$ bounded in E .

Step 2: $y_n = (I - A)(\hat{x}_n) = \hat{x}_n - A(\hat{x}_n)$.

□

recall:

$(E, \langle \cdot, \cdot \rangle)$ Hilbert space and $(x_n)_{n=1}^\infty$ ON-basis and $(\lambda_n)_{n=1}^\infty$ sequence of complex numbers. Set

$$A(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, x_n \rangle x_n.$$

We have:

- $A : E \rightarrow E$ if $(\lambda_n)_{n=1}^\infty \in l^\infty$
if $(\lambda_n)_{n=1}^\infty$ is not bounded, there exists a subsequence $(\lambda_{n_k})_{k \in \mathbb{N}}$ such that

$$|\lambda_{n_k}| \geq k, \quad k = 1, 2, \dots$$

Set

$$x = \sum_{k=1}^{\infty} \frac{1}{k} x_{n_k}.$$

Clearly $x \in E$ since $(\frac{1}{k})_{k=1}^\infty \in l^\infty$. But

$$T(x) = \sum_{k=1}^{\infty} \lambda_{n_k} \frac{1}{k} x_{n_k} \notin E$$

since $(\lambda_{n_k} \cdot \frac{1}{k})_{k=1}^\infty \notin l^2$.

Note

$$A \in B(E, E) \quad \Leftrightarrow \quad (\lambda_n)_{n=1}^\infty \in l^\infty$$

and $\|A\| = \sup_n |\lambda_n|$.

- $A \in K(E, E)$ iff $\lambda_n \rightarrow 0$ for $n \rightarrow \infty$.
- A is self adjoint iff $\lambda_n \in \mathbb{R}$ for all $n \in \mathbb{N}$.

Basis facts:

Set $A \in B(E, E)$ where $(E, \langle \cdot, \cdot \rangle)$ is a Hilbert space. Then:

- If A is self-adjoint we have

$$\|A\| = \sup_{\|x\|=1} |\langle A(x), x \rangle|.$$

- If A is self-adjoint it follows

$$\langle A(x), x \rangle \in \mathbb{R}, \quad \forall x \in E$$

since

$$\langle A(x), x \rangle = \langle x, A^*(x) \rangle \stackrel{\text{self-adjoint}}{=} \langle x, A(x) \rangle = \overline{\langle A(x), x \rangle}.$$

- $K(E, E)$ (Set of all compact linear operators) closed subspace in $(B(E, E), \|\cdot\|_{E \rightarrow E})$.
- $A \in K(E, E)$ and $x_n \rightharpoonup x$ in E . Then

$$A(x_n) \rightarrow A(x), \quad \text{in } E.$$

- $A \in K(E, E)$ and $B \in B(E, E)$. Then
 - $AB, BA \in K(E, E)$,
 - $A^* \in K(E, E)$,
 - $\mathcal{R}(B)^\perp = \mathcal{N}(B^*)$
 $\mathcal{R}(B) = \mathcal{N}(B^*)^\perp$,
 - $\mathcal{R}(I - A)$ is a closed subspace in E .
- $E = \mathcal{R}(I - A) \oplus \mathcal{R}(I - A)^\perp = \mathcal{R}(I - A) \oplus \mathcal{N}(I - A^*)$.
- For any $A \in K(E, E)$

$$\dim(\mathcal{N}(I - A)) = \dim(\{x \in E \mid x - A(x) = 0\}) < \infty$$

since: if $\dim(\mathcal{N}(I - A)) = \infty$ then there exists an ON- sequence $(x_n)_{n=1}^\infty$ in $\mathcal{N}(I - A)$.
Then

$$x_n \rightharpoonup 0, \quad \text{since } \langle x_n, y \rangle \rightarrow 0, n \rightarrow \infty$$

since for $y \in \overline{\text{span}\{x_n \mid n = 1, 2, \dots\}}$ then

$$\|y\|^2 = \sum_{n=1}^{\infty} |\langle x_n, y \rangle|^2 < \infty.$$

$A \in K(E, E)$ implies that $A(x_n) \rightarrow A(0) = 0$ in E . But

$$x_n = A(x_n) \rightarrow 0 \quad \text{in } E, \quad \|x_n\| = 1 \text{ for all } n$$

This is a contradiction.

Conclusion: $\dim(I - A) < \infty$.

From above we have for $A \in K(E, E)$

$$E = \mathcal{R}(I - A) \oplus \mathcal{N}(I - A^*).$$

Consider the equation

$$x = A(x) + y \quad (1).$$

(1) has a solution provided by $y \in \mathcal{R}(I - A)$. That is the case if $y \perp z$ for all $z \in \mathcal{N}(I - A^*)$. Since $\dim(\mathcal{N}(I - A^*)) < \infty$, this is just finitely many conditions.

Theorem 3.41 (Fredholm alternativ). $A \in K(E, E)$ where E is a Hilbert space. then exactly one of the statements below holds:

1. $x = A(x) + y$ is solvable for every $y \in E$.
2. $x = A(x)$ has a non trivial solution $x \in E$, i.e. $x \neq 0$.

(No assumption on A being self-adjoint.)

Remark. The statement in Fredholm Alternativ also holds if $(E, \|\cdot\|)$ is a Banach space.

proof. (1) $\Rightarrow \neg$ (2): We want to show that there are no non-trivial solutions for $x = A(x)$. Assume that there exists a non-trivial solution $x_1 \in E$ to $x = A(x)$, i.e.

$$(I - A)(x_1) = 0, \quad \text{with } x_1 \neq 0.$$

If (1) holds true there exists a $x_2 \in E$ such that

$$(I - A)(x_2) = x_1 \neq 0.$$

But

$$(I - A)(x_1) = (I - A)^2(x_2) = 0.$$

With (1) there exists $x_3 \in E$ such that

$$(I - A)(x_3) = x_2$$

which implies

$$(I - A)^2(x_3) = (I - A)(x_2) = x_1 \neq 0.$$

But once again

$$(I - A)^3(x_3) = 0.$$

Proceed inductively gives us a sequence $(x_k)_{k=1}^{\infty}$ such that

$$(I - A)^k(x_k) = 0, \quad \text{but } (I - A)^{k-1}(x_k) \neq 0.$$

We obtain

$$\mathcal{N}(I - A) \subsetneq \mathcal{N}((I - A)^2) \subsetneq \mathcal{N}((I - A)^3) \subsetneq \dots$$

This is a sequence of proper closed subspaces.

Apply now Riesz-Lemma:

There exists a sequence $(y_k)_{k=1}^\infty$ with $\|y_k\| = 1$ and $\|y_k - x\| \geq \frac{1}{2}$ for all $x \in \mathcal{N}((I - A)^{k-1})$ and $y_k \in \mathcal{N}((I - A)^k)$.

Claim: $\|A(y_n) - A(y_m)\| \geq \frac{1}{2}$ for all $n > m$.

$$\begin{aligned} \|A(y_m) - A(y_n)\| &= \left\| \underbrace{(I - A)(y_n)}_{\in \mathcal{N}((I - A)^{n-1})} - y_n + \underbrace{A(y_m)}_{\in \mathcal{N}((I - A)^{n-1})} \right\| \\ &= \left\| y_n - \underbrace{((I - A)(y_n) + A(y_m))}_{\in \mathcal{N}((I - A)^{n-1})} \right\| \geq \frac{1}{2}. \end{aligned}$$

So $(A(y_n))_{n=1}^\infty$ can not converge in E . But A is compact and $\|y_n\| = 1$ for all n . This is a contradiction.

Conclusion: There is no non-trivial solution of $A(x) = x$.

\neg (2) \Rightarrow (1) Assume that $x = A(x)$ has a no non-trivial solution $x \in E$. We want to show that (1) holds.

$$E = \mathcal{R}(I - A^*) \oplus \mathcal{N}(I - A), \quad \text{with } \mathcal{N}(I - A) = \{0\}.$$

Hence

$$x = A^*(x) + y$$

is solvable for every $y \in E$. From the first part of the proof it follows that

$$\mathcal{N}(I - A^*) = \{0\}.$$

But then

$$E = \mathcal{R}(I - A) \oplus \mathcal{N}((I - A)^*) = \mathcal{R}(I - A).$$

Conclusion: $x = A(x) + y$ is solvable for all $y \in E$.

□

Example. $L^2([0, 1])$, $k \in C([0, 1] \times [0, 1])$ and

$$A(f)(x) = \int_0^1 k(x, y) f(y) dy, \quad x \in [0, 1].$$

Then

- $A \in B(L^2, L^2)$ with $\|A\|_{L^2 \rightarrow L^2} \leq \|k\|_{L^2([0, 1] \times [0, 1])}$,
- A self-adjoint if $k(x, y) = \overline{k(y, x)}$ for all $x, y \in [0, 1]$,
- $A \in K(E, E)$ (by approximation by finite rank operators).

Theorem 3.42 (Hilbert-Schmidt-Theorem). $(E, \langle \cdot, \cdot \rangle)$ Hilbert spaces and $A \in K(E, E)$ self adjoint. Then there exists a sequence of non-zero eigenvalues of A denoted $(\lambda_n)_{n=1}^N$ for N finite or infinite, corresponding to Eigenvectors $(u_n)_{n=1}^N$. Respectively where $(u_n)_{n=1}^N$ is an ON-sequence, and

$$|\lambda_1| \geq |\lambda_2| \geq \dots$$

with

$$\lim_{n \rightarrow \infty} \lambda_n = 0, \quad \text{if } N = \infty$$

such that for $x \in E$

$$x = \sum_{n=1}^N \langle x, u_n \rangle u_n + v, \quad v \in \mathcal{N}(A).$$

Moreover

$$A(x) = \sum_{n=1}^N \lambda_n \langle x, u_n \rangle u_n.$$

Remark. With notation from the theorem above we have

1.

$$A^k(x) = \sum_{n=1}^N \lambda_n^k \langle x, u_n \rangle u_n, \quad k = 1, 2, \dots$$

2. If A is injective, i.e. $\mathcal{N}(A) = \{0\}$ then the Eigenvectors $(u_n)_{n=1}^N$ form an ON-basis for E .

Definition (Eigenvalues and Eigenvectors for $A \in B(E, E)$). $\lambda \in \mathbb{C}$ is called an eigenvalue of A if there exists an $0 \neq x \in E$ such that

$$A(x) = \lambda x.$$

Remark (properties for Eigenvalues and Eigenvectors). 1. $|\lambda| \leq \|A\|$ since

$$|\lambda| \|x\| = \|\lambda x\| = \|A(x)\| \leq \|A\| \cdot \|x\|.$$

2. A self-adjoint and λ eigenvalue. Then

$$\Rightarrow \lambda \in \mathbb{R}$$

since

$$\begin{aligned}\lambda \langle x, x \rangle &= \langle \lambda x, x \rangle \\ &= \langle A(x), x \rangle \\ &= \langle x, A^*(x) \rangle \\ &= \langle x, A(x) \rangle \\ &= \langle x, \lambda x \rangle \\ &= \bar{\lambda} \langle x, x \rangle.\end{aligned}$$

So

$$\lambda = \bar{\lambda}, \quad \Rightarrow \lambda \in \mathbb{R}.$$

3. A self-adjoint, $A(x) = \lambda x$ and $A(y) = \mu y$, where $x, y \neq 0$ and $\lambda \neq \mu$.

$$\Rightarrow x \perp y$$

since

$$\lambda \langle x, y \rangle = \dots = \bar{\mu} \langle x, y \rangle.$$

So

$$\underbrace{(\lambda - \mu)}_{\neq 0} \langle x, y \rangle = 0.$$

4. $A \in K(E, E)$ and $\lambda \neq 0$ eigenvalue of A . Then

$$\dim E_\lambda = \dim \{x \in E \mid A(x) = \lambda x\} < \infty.$$

Proposition 3.43. $(E, \langle \cdot, \cdot \rangle)$ Hilbert space and $A \in K(E, E)$ self-adjoint. Then

$$\Rightarrow \|A\| \quad \text{or} \quad -\|A\|$$

is an eigenvalue of A .

proof. $A = 0$ then the statement is trivial.

Assume $A \neq 0$.

A self-adjoint implies that

$$\|A\| = \sum_{\|x\|=1} |\langle A(x), x \rangle|.$$

Also self-adjoint implies that for all $x \in E$ we have

$$\langle A(x), x \rangle \in \mathbb{R}.$$

Hence there exists a sequence $(x_n)_{n=1}^\infty$ in E with $\|x_n\| = 1$ for all n such that

$$\langle A(x_n), x_n \rangle \rightarrow \lambda, \quad n \rightarrow \infty.$$

where $\lambda \in \mathbb{R}$ and $|\lambda| = \|A\|$.

Claim: $A(x_n) - \lambda x_n \rightarrow 0$ in E .

$$\begin{aligned}
 \|A(x_n) - \lambda x_n\|^2 &= \langle A(x_n) - \lambda x_n, A(x_n) - \lambda x_n \rangle \\
 &= \underbrace{\langle A(x_n), A(x_n) \rangle}_{=\|A(x_n)\|^2} - \underbrace{\overline{\lambda} \langle A(x_n), x_n \rangle}_{\rightarrow \lambda} - \underbrace{\lambda \langle x_n, A(x_n) \rangle}_{\rightarrow \lambda} + \underbrace{|\lambda| \langle x_n, x_n \rangle}_{=\|A\|^2} \\
 &\leq \|A\|^2 \|x_n\|^2 \\
 &= \|A\|^2 \\
 &\rightarrow 0, \quad n \rightarrow \infty.
 \end{aligned}$$

$A \in K(E, E)$ and $\|x_n\| = 1$ for all n we get that

$$(A(x_n))_{n=1}^\infty$$

has a converging subsequence $(A(x_{n_k}))_{k=1}^\infty$ in E .

Call the limit element $y \in E$ so

$$\begin{aligned}
 A(x_{n_k}) &\rightarrow y \quad \text{in } E. \\
 \begin{cases} A(x_n) - \lambda x_n \rightarrow 0 \\ A(x_{n_k}) \rightarrow y \end{cases} &\quad \text{in } E
 \end{aligned}$$

implies

$$x_{n_k} \rightarrow \frac{1}{\lambda} y \quad \text{in } E$$

(note $|\lambda| > 0$ since $A \neq 0$).

Set $x = \frac{1}{\lambda} y$. So $x_{n_k} \rightarrow x$ in E . Consider

$$\|A(x) - \lambda x\| \leq \|A(x) - A(x_{n_k})\| + \|A(x_{n_k}) - y\| \rightarrow 0, \quad k \rightarrow \infty$$

Conclusion:

$$A(x) = \lambda x.$$

where $\|x\| = 1$ since $1 = \|x_{n_k}\| \rightarrow \|x\|$ as $k \rightarrow \infty$. □

We are now going to prove the Hilbert-Schmidt theorem:

proof. If $A = 0$ the theorem is trivial.

Assume $A \neq 0$.

By the proposition above there exists an eigenvalue λ_1 of A with $|\lambda_1| = \|A\|$ and an eigenvector u_1 with $\|u_1\| = 1$ corresponding to the eigenvalue λ_1 .

Set $Q_1 = \{u_1\}^\perp$. Q_1 is a closed subspace of E and hence Q_1 is a Hilbert space.

For $x \in Q_1$ we have $A(x) \in Q_1$ since for $x \in Q_1$ we have

$$\begin{aligned}
 \langle A(x), u_1 \rangle &= \langle x, A^*(u_1) \rangle \\
 &= \langle x, A(u_1) \rangle \\
 &= \langle x, \underbrace{\lambda_1}_{\in \mathbb{R}} u_1 \rangle \\
 &= \lambda_1 \langle x, u_1 \rangle = 0.
 \end{aligned}$$

Now

$$A|_{Q_1} : Q_1 \rightarrow Q_1$$

is compact and also self-adjoint. By proposition above there exists an eigenvalue λ_2 of $A|_{Q_1}$ and a corresponding eigenvector u_2 with $\|u_2\| = 1$ where

$$|\lambda_2| = \|A|_{Q_1}\| \leq \|A\| = |\lambda_1|.$$

Here $A(u_2) = \lambda_2 u_2$ so λ_2 is an eigenvalue of A . Set $Q_2 = \{u_1, u_2\}^\perp$. Q_2 is a closed subspace of E and we have

$$x \in Q_2 \quad \Rightarrow \quad A(x) \in Q_2$$

since $x \in Q_2$ we have

$$\begin{aligned} \langle A(x), u_1 \rangle &= \langle x, A(u_1) \rangle = \langle x, \lambda_1 u_1 \rangle = 0 \\ \langle A(x), u_2 \rangle &= \langle x, A(u_2) \rangle = \langle x, \lambda_2 u_2 \rangle = 0. \end{aligned}$$

Proceed inductively.

Case 1: For a positive integer k we have

$$|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_k| > 0$$

with corresponding eigenvectors u_1, u_2, \dots, u_k but $A|_{Q_k}$ with $Q_k = \{u_1, u_2, \dots, u_k\}^\perp$, then is the zero-mapping $Q_k \rightarrow Q_k$. This corresponds to $N = k$ and

$$x = \sum_{n=1}^k \langle x, u_n \rangle u_n + v, \quad \text{where } v \in \mathcal{N}(A).$$

Case 2: The process never terminates. We get

$$|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n| \geq \dots$$

with corresponding eigenvectors $u_1, u_2, \dots, u_n, \dots$

We have $(u_n)_{n=1}^\infty$ ON-sequence in E corresponding to the non-zero EW $(\lambda_n)_{n=1}^\infty$. $A \in K(E, E)$ und $w_n \rightarrow 0$ in E since $(u_n)_{n=1}^\infty$ is ON-sequence.

Then this implies $A(u_n) \rightarrow 0$ in E . So

$$|\lambda_n| = \|\lambda_n u_n\| = \|A(u_n)\| \rightarrow 0, \quad n \rightarrow \infty.$$

Hence

$$\lim_{n \rightarrow \infty} \lambda_n = 0.$$

Set

$$S := \overline{\text{span}\{u_1, \dots, u_n, \dots\}} = \left\{ \sum_{k=1}^n \alpha u_k \mid (a_n)_{n=1}^\infty \in l^\infty \right\}.$$

S is a closed subspace of E .

We have $E = S \oplus S^\perp$ where $S^\perp \subseteq Q_k = \{u_1, \dots, u_k\}^\perp$ for all $k \in \mathbb{N}$. For $x \in E$ we have

$$\underbrace{\sum_{k=1}^{\infty} \langle x, u_k \rangle u_k}_{\in S} + \underbrace{v}_{\in S^\perp}$$

since $(\langle x, u_k \rangle)_{k=1}^{\infty} \in l^\infty$ by Bessel's inequality. To show: $A(u) = 0$. Clearly, $v \in Q_k$ for all k . If $v = 0$ there is nothing to prove. For $v \neq 0$ set $w = \frac{1}{\|v\|}v$ and get

$$\begin{aligned} |\langle A(v), v \rangle| &= \|v\|^2 |\langle A(w), w \rangle| \\ &\leq \|w\|^2 \sup_{\substack{\|z\|=1 \\ z \in Q_k}} |\langle A(z), z \rangle| \\ &= \|A|_{Q_k}\| = |\lambda_{k+1}| \rightarrow 0 \end{aligned}$$

Claim: $A|_{S^\perp} = 0$ and hence $v \in S^\perp$ implies $A(v) = 0$.

□

Theorem 3.44 (Spectral mapping theorem). $(E, \langle \cdot, \cdot \rangle)$ separable Hilbert space and ∞ -dimensional $A \in K(E, E)$ self-adjoint. Then there exists a ON-basis of eigenvectors $(\tilde{u}_n)_{n=1}^{\infty}$ corresponding to the eigenvalues $(\tilde{\lambda}_n)_{n=1}^{\infty}$ if A where $\lim_{n \rightarrow \infty} \tilde{\lambda}_n = 0$.

proof (consequence of HS-theorem). We have by HS-theorem an ON-sequence $(u_n)_{n=1}^{\infty}$ of eigenvectors corresponding to the non-zero eigenvalues $(\lambda_n)_{n=1}^N$.

Set

$$S = \overline{\text{span}\{u_1, \dots, u_n, \dots\}}.$$

E is separable implies E has an ON-basis $(v_n)_{n=1}^{\infty}$. By Gram-Schmidt Orthogonalization procedure we can obtain an ON-basis $(w_n)_{n=1}^M$ for S^\perp . Have M finite or infinite.

$$\begin{array}{ll} S : u_1, u_2, \dots & \text{ON-basis finite or infinite} \\ S^\perp : w_1, w_2, \dots & \text{ON-basis finite or infinite} \end{array}$$

Consider the ON-sequence $u_1, w_1, u_2, w_2, \dots = \tilde{u}_1, \tilde{u}_2, \dots$. This gives an ON-basis for E consisting of eigenvectors to A . Also

$$\lim_{n \rightarrow \infty} \tilde{\lambda}_n = 0.$$

□

