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Applied Functionalanalysis

Script of “Applied Functionalanalysis” by Prof. Peter Kumlin

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foreword — cooperation

This document is a transcript of the lecture “Applied Functionalanalysis, WiSe 2016/2017, Term 1”, by Prof. Peter Kumlin. It mainly contains the written content of the lecture. I will not assume any responsibility for the correctness of the content! For questions, remarks and mistakes please write an email to `keil.menden@web.de`. I’m grateful for every email.

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1 Introduction

1.1 Introduction example

We have

$$\begin{cases} f'' + f = g, & \text{in } I = [0,1] \\ f(0) = 1,\ f'(0) = 1 \end{cases},$$

where g is a known continous function on I. We will now consider different cases:

1. $g = 0$

$$\Rightarrow\ f(x) = A\cos(x) + B\sin(x), x \in I,$$

   where $A, B \in \mathbb{R}$.

2. g arbitrary. We will now introduce the Method of variation of constants. Set

$$f(x) = A(x)\cos(x) + B(x)\sin(x).$$

   Differentiate

$$f'(x) = A'(x)\cos(x) + B'(x)\sin(x) - A(x)\sin(x) + B(x)\cos(x).$$

   Assume (this is part of the method)

$$A'(x)\cos(x) + B'(x)\sin(x) = 0, \qquad x \in I.$$

   Differentiate $f'(x)$ and get

$$f''(x) = \underbrace{-A(x)\cos(x) - B(x)\sin(x)}_{=-f(x)} - A'(x)\sin(x) + B'(x)\cos(x).$$

   We get

$$g(x) = f''(x) + f(x) = -A'(x)\sin(x) + B'(x)\cos(x).$$

   Now:

$$\begin{cases} A'(x)\cos(x) + B'(x)\sin(x) = 0, & x \in I \\ -A'(x)\sin(x) + B'(x)\cos(x) = g(x), & x \in I \\ A(0) = 1, \qquad B(0) = 0 \end{cases}.$$

   We get

$$\begin{aligned} A'(x) &= -g(x)\sin(x), \\ A(0) &= 1, \\ B'(x) &= g(x)\cos(x), \\ B(0) &= 0. \end{aligned}$$

This implies

$$\begin{aligned} A(x) &= A(0) + \int_0^x A'(t)\,\mathrm{d}t = 1 - \int_0^x g(t)\sin(t)\,\mathrm{d}t, \\ B(x) &= B(0) + \int_0^x B'(t)\,\mathrm{d}t = 0 + \int_0^x g(t)\cos(t)\,\mathrm{d}t. \end{aligned}$$

Hence

$$\begin{aligned} f(x) &= \cos(x) - \int_0^x g(t)\sin(t)\,\mathrm{d}t\cos(x) + \int_0^x g(t)\cos(t)\,\mathrm{d}t\sin(x) \\ &= \cos(x) + \int_0^x (\underbrace{\sin(x)\cos(t) - \sin(t)\cos(x)}_{=\sin(x-t)})g(t)\,\mathrm{d}t \\ &= \cos(x) + \int_0^x \sin(x-t)g(t)\,\mathrm{d}t \qquad (*). \end{aligned}$$

Check that $f(x)$ in $(*)$ satisfies the PDE.

special case:

Assume for $x \in I$

$$g(x) = k(x)f(x).$$

Here k is a known continous function on I. Insert this in $(*)$. We obtain

$$f(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f(t)\,\mathrm{d}t, \qquad x \in I \qquad (**).$$

Observe that f appears both in LHS and RHS. $(**)$ is a reformulation of the PDE with $g = kf$. Pick a $\underbrace{\text{continous function in } I}_{\in C(I)}$. call it f_0. Set

$$\begin{aligned} f_1(x) &= \cos(x) + \int_0^x \sin(x-t)k(t)f_0(t)\,\mathrm{d}t, \\ f_2(x) &= \cos(x) + \int_0^x \sin(x-t)k(t)f_1(t)\,\mathrm{d}t, \\ &\vdots \qquad \vdots \\ f_{n+1}(x) &= \cos(x) + \int_0^x \sin(x-t)k(t)f_n(t)\,\mathrm{d}t, \qquad n = 1,2,3,\ldots. \end{aligned}$$

Hope:

f_n tends to some continous function f on I, denoted $f_n \to f$. 'Tends to' has to be more precis!

$$\begin{array}{ccc} f_{n+1}(x) = & \cos(x) + \displaystyle\int_0^x \sin(x-t)k(t)f_n(t)\,\mathrm{d}t \\ \downarrow & \downarrow \\ f(x) = & \cos(x) + \displaystyle\int_0^x \sin(x-t)k(t)f(t)\,\mathrm{d}t \end{array}$$

for $x \in I$. Simplify notation set for $v \in C(I)$

$$\begin{cases} u(x) & = \cos(x) \\ kv(x) & = \int_0^x \sin(x-t)k(t)v(t)\,\mathrm{d}t \end{cases}.$$

We have $f_0 \in C(I)$, $f_{n+1} = u + kf_n$ for $n = 0, 1, 2, \ldots$ (!)
Facts from previous calculus classes:

Definition (Sequenze of continous functions)**.**

$$v_n \in C(I), \qquad n = 1, 2, \ldots.$$

We say that $(v_n)_{n=1}^{\infty}$ converges uniformly in I if

$$\max_{x\in I}|v_n(x) - v_m(x)| \to 0, \qquad n, m \to \infty,$$

i.e.

$$\forall\,\varepsilon > 0 \exists\,N : \forall\,n, m \geq N : \max_{x\in I}|v_n(x) - v_m(x)| < \varepsilon.$$

Lemma . Suppose that $(v_n)_{n=1}^{\infty}$ converges uniformly on I. then there exists $v \in C(I)$ such that

$$\max_{x\in I}|v_m(x) - v(x)| \to 0 \qquad \text{as } m \to \infty.$$

Back to (!):

More Notation:

$$k(kv) = k^2 v, \qquad v \in C(I)$$

and

$$k^{n+1}v = k(k^n v), \qquad n = 1, 2, \ldots.$$

We have

$$\begin{aligned} f_0 &\in C(I) \\ f_1 &= u + kf_0 \\ \text{and } f_2 &= u + kf_1 = u + k(u + kf_0) \end{aligned}$$

and so on. Note that

$$k(v+w) = kv + kw.$$

Then

$$\begin{aligned} f_2 &= u + k(u + kf_0) = k + ku + k(kf_0) = u + ku + k^2 f_0 \\ f_3 &= u + kf_2 = u + ku + k^2u + k^3 f_0 \end{aligned}$$

and in general for $n = 1, 2, \dots$

$$f_n = ku + \ldots + k^{n-1}u + k^n f_0, \qquad n = 1, 2, \ldots.$$

Assume $n > m$ then

$$f_n - f_m = k^m u + \ldots + k^{n-1}u + k^n f_0 - k^m f_0.$$

Set for $v \in C(I)$

$$\|v\| = \max_{x \in I} |v(x)|.$$

Note

$$\|v + w\| \le \|v\| + \|w\| \qquad \text{for } v, w \in C(I)$$

and

$$\|-v\| = \|v\|.$$

We have

$$\begin{aligned} \|f_n - f_m\| &= \left\|k^m u + \ldots + k^{n-1}u + k^n f_0 - k^m f_0\right\| \\ &\le \|k^m u\| + \ldots + \left\|k^{n-1}u\right\| + \|k^n f_0\| + \|-k^m f_0\|. \end{aligned}$$

Assumption:

$$\sum_{l=1}^{\infty} \left\|k^l v\right\| < \infty \qquad \text{for all } v \in C(I) \qquad (***).$$

Under this assumption

$$\|f_n - f_m\| \to 0 \qquad \text{as } n, m \to \infty$$

since

$$\begin{aligned} \sum_{l=1}^{\infty} \left\|k^l u\right\| &< \infty \qquad\qquad (u(x) = \cos(x)) \\ \sum_{l=1}^{\infty} \left\|k^l f_0\right\| &< \infty \qquad\qquad (f_0 \in C(I)). \end{aligned}$$

Conclusion: $(f_n)_{n=1}^{\infty}$ converges uniformly on I. By lemma above there exists $f \in C(I)$ such that

$$\max_{x\in I}|f_n(x) - f(x)| \to 0, \qquad n \to \infty,$$

i.e.

$$\|f_n - f\| \to 0, \qquad n \to \infty.$$

'Back hope': f_n tends to f, denoted $f_n \to f$ shall be interpretated as

$$\|f_n - f\| \to 0, \qquad n \to \infty.$$

Remember

$$f_{n+1}(x) = u(x) + kf_n(x) \to ?.$$

For $x \in I$ there is

$$\begin{aligned}
|kf_n(x) - kf(x)| &= |\int_0^x \sin(x-t)k(t)f_n(t)\,\mathrm{d}t - \int_0^x \sin(x-t)k(t)f(t)\,\mathrm{d}t| \\
&\le \int_0^x |\sin(x-t)k(t)|\underbrace{|f_n(t) - f(t)|}_{\le\|f_n - f\|}\,\mathrm{d}t \\
&\le \int_0^x |\sin(x-t)k(t)|\,\mathrm{d}t\|f_n - f\|.
\end{aligned}$$

In particular

$$\begin{aligned}
\|kf_n - kf\| &\le \max_{x\in I}\int_0^x \underbrace{|\sin(x-t)|}_{\le 1}\quad\underbrace{|k(t)|}_{\max_{t\in I}|k(t)|<\infty}\quad\mathrm{d}t\|f_n - f\| \\
&\le \|k\|\|f_n - f\|.
\end{aligned}$$

We have, provided $(***)$ holds, shown

$$\begin{aligned}
f_{n+1} &= u + kf_n \\
&\downarrow \\
f &= u + kf.
\end{aligned}$$

Let us try to prove $(***)$. For $v \in C(I)$ arbitrary and for $x \in I$

$$\begin{aligned}
\|kv(x)\| &= |\int_0^x \sin(x-t)k(t)v(t)\,\mathrm{d}t| \\
&\le \int_0^x \underbrace{|\sin(x-t)|}_{\le 1}\underbrace{|k(t)|}_{\le\|k\|}|v(t)|\,\mathrm{d}t \\
&\le \int_0^x \underbrace{|v(t)|}_{\le\|v\|}\,\mathrm{d}t\|k\| \\
&\le \|k\|\|v\|x.
\end{aligned}$$

In particular

$$\|kv\| \leq \|k\|\|v\|$$

and

$$\begin{aligned}|k^2 v(x)| &\leq \int_0^x |kv(t)| \, \mathrm{d}t \|k\| \\ &\leq \int_0^x \|k\|\|v\| t \, \mathrm{d}t \cdot \|k\| \\ &= \|k\|^2 \|v\| \frac{x^2}{2}.\end{aligned}$$

In particular

$$\left\|k^2 v\right\| \leq \|k\|^2 \|v\| \frac{1}{2}.$$

By induction we get

$$\begin{aligned}|k^n v(x)| &\leq \|k\|^n \|v\| \frac{x^m}{m!} \qquad x \in I \\ \|k^n v\| &\leq \|k\|^n \|v\| \frac{1}{n!}.\end{aligned}$$

So

$$\begin{aligned}\sum_{l=1}^{\infty} \left\|k^l v\right\| &\leq \sum_{l=1}^{\infty} \|k\|^l \|v\| \frac{1}{l!} \\ &= \|v\| \sum_{l=1}^{\infty} \frac{\|k\|^l}{l!} \\ &\leq \|v\| e^{\|k\|} < \infty.\end{aligned}$$

Consider Taylor expansion. $\Rightarrow (***)$ holds true.
We have now shown that $f = u + kf$ where $u(x) = \cos(x)$ and

$$kv = \int_0^x \sin(x-t) k(t) v(t) \, \mathrm{d}t.$$

$x \in I$ for $v \in C(I)$, has a solution $f \in C(I)$.

Question:

Is the solution unique?
Assume $f, \tilde{f} \in C(I)$ such that $f = u + kf$ and $\tilde{f} = u + k\tilde{f}$. Set

$$v = f - \tilde{f} \in C(I)$$

$$\begin{aligned}\Rightarrow v &= (u + kf) - (u + k\tilde{f}) \\ &= kf - k\tilde{f} \\ &= k(f - \tilde{f}) \\ &= kv.\end{aligned}$$

We have $v = kv$, implies that $kv = k(kv) = k^2 v$. So for $n = 1, 2, \ldots$

$$v = kv = k^2 v = \ldots = k^n v.$$

We know

$$\sum_{n=1}^{\infty} \|k^n \hat{v}\| < \infty \qquad \text{for all } \hat{v} \in C(I).$$

Apply this to $\hat{v} = v$:

$$\sum_{n=1}^{\infty} \underbrace{\|k^n v\|}_{=\|v\|} < \infty.$$

So $\|v\| = 0$ with implies $v(x) = 0$ for all $x \in I$. So we have $f(x) = \tilde{f}(x)$ for $x \in I$.
$\Rightarrow$ Answer to the question above: YES !

We have more or less proved the following theorem:

Theorem 1.1. Set $I = [0, 1]$. Suppose $u \in C(I)$ and $k \in C(I \times I)$. Consider

$$f(x) = u(x) + \int_0^x k(x,t) f(t) \, \mathrm{d}t, \qquad x \in I \qquad (1).$$

Then (1) has a unique solution $f \in C(I)$

With the same technology we can prove:

Theorem 1.2. Set $I = [0, 1]$. Suppose $u \in C(I)$, $k \in C(I \times I)$ and $\max_{(x,t) \in I \times I} |k(x,t)| < 1$. Consider

$$f(x) = u(x) + \int_0^1 k(x,t) f(t) \, \mathrm{d}t, \qquad x \in I \qquad (2).$$

Then (2) has a unique solution $f \in C(I)$.

Different notions: see introductional example.

2 Normed Spaces and Banach Spaces

Definition (vector space). $C(I)$ with the operations for $x \in I$:

addition $v, w \in C(I)$: $\qquad (v + w)(x) = v(x) + w(x)$,

mult. by scalar $v \in C(I)$, $\lambda \in \mathbb{R}$: $\qquad (\lambda v)(x) = \lambda v(x)$.

Note that $v + w, \lambda v \in C(I)$.

Definition (norm). Norm on $C(I)$ for instance

$$\|v\| = \max_{x \in I} |v(x)|$$

with norm given we can talk about convergence and continuity.

Definition (Cauchy sequence). In our example a sequence $(f_n)_{n=1}^{\infty}$ is called Cauchy sequence if $\|f_n - f_m\| \to 0$ for $n, m \to \infty$.

Definition . $C(I)$ with the max-norm. Lemma above says that every Cauchy sequence converges i.e.

$$\|v_n - v_m\| \to 0, \qquad n, m \to \infty.$$

This applies

$$\exists\, v \in C(I) : \|v_n - v\| \to 0, \qquad n \to \infty.$$

This is the defining property of a Banach space.
K linear mapping $C(I) \to C(I)$ with

$$K(v + w) = K(v) + K(w)$$
$$K(\lambda v) = \lambda K(v)$$

for $v, w \in C(I)$, $\lambda \in \mathbb{R}$.
K bounded linear:

$$\|Kv\| \leq M\|v\| \qquad \forall\, v \in C(I),$$

where $M > 0$ independent of v.

Definition (operator norm). Define

$$\|K\| := \inf\{M > 0 \,|\, \|Kv\| \leq M\|v\| \text{ for all } v \in C(I)\}.$$

fixed point results:
Our example: $f = u + kf =: T(f)$ and $f_0 \in C(I)$ fixed.
Form sequence of iterants $(f_n)_{n=1}^{\infty}$, $f_n = T(f_{n-1})$, $n = 1, 2, \ldots$ if

$$\|T(v) - T(w)\| \leq c\|v - w\|$$

for all $v, w \in C(I)$ for some $c < 1$. Then there is a unique $v \in C(I)$ such that $v = T(v)$.
This is Banach's fixed point theorem.

Definition (Green's function). Our example:

$$L = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + 1$$

differential operator. Boundary conditions

$$f(0) = f'(0) = 0.$$

Then

$$f(x) = \int_0^1 g(x,t)h(t)\,\mathrm{d}t$$

is a solution to

$$\begin{cases} f'' + f & = h, \\ f(0) = f'(0) & = 0. \end{cases}$$

Definition (real vector space). We say that E is a real vector space if it is a non-empty set with the operations

addition $E \times E \to E$, $\qquad (x,y) \mapsto x+y$

mult. with scalar $\mathbb{R} \times E \to E$, $\qquad (\lambda, x) \mapsto \lambda x$

satisfying the axioms:

(1) $x + y = y + x$, $\qquad$ for all $x, y \in E$,

(2) $x + (y + z) = (x + y) + z$, $\qquad$ for all $x, y, z \in E$,

(3) For all $x, y \in E$ there exists $z \in E$ such that $x + z = y$,

(4) $\alpha(\beta x) = (\alpha \cdot \beta)x$, $\qquad$ for all $\alpha, \beta \in \mathbb{R}, x \in E$,

(5) $\alpha(x + y) = \alpha x + \alpha y$, $\qquad$ for all $\alpha \in \mathbb{R}, x, y \in E$,

(6) $(\alpha + \beta)x = \alpha x + \beta x$, $\qquad$ for all $\alpha, \beta \in \mathbb{R}, x \in E$,

(7) $1 \cdot x = x$, $\qquad$ for all $x \in E$.

Remark. E is a complex vector space if all $\mathbb{R}$ in the definition above are replaced by $\mathbb{C}$.

Remark. (1)

$$\exists\,!0 \in E: \qquad x + 0 = x \qquad \text{for all } x \in E.$$

Since: Fix $x \in E$, by (3), $\exists\, 0_x$ such that $0_x + x = x$.
Fix $y \in E$. We want to show that $y + 0_y = y$. By (3), there exists $z \in E$ such that

$x + z = y$. So

$$\begin{aligned} y + 0_x &= (x+z) + 0_x \\ &\overset{(1)}{=} (z+x) + 0_x \\ &\overset{(2)}{=} z + (x + 0_x) \\ &= z + x \\ &\overset{(1)}{=} x + z \\ &= y. \end{aligned}$$

Assume $x + 0_1 = x$, $x + 0_2 = x$ for all $x \in E$. We want to show $0_1 = 0_2$:

$$0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2,$$

(2)

$$\forall\, x \in E : \exists! - x \in E : x + (-x) = 0.$$

proof: exercise.

(3)

$$\begin{aligned} 0x &= 0 \qquad \text{for all } x \in E \\ (-1)x &= -x \qquad \text{for all } x \in E. \end{aligned}$$

Examples (Examples of real vector spaces). 1) $\mathbb{R}$ with standard addition and mult. by scalar.

2) $\mathbb{R}^n$, $n = 2, 3, \ldots$.
addition $(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (x_1 + y_1, x_2 + y_2, \ldots)$
mult. $\lambda(x_1, x_2, \ldots) = (\lambda x_1, \lambda x_2, \ldots)$

3) $\mathbb{R}^\infty = \{(x_1, \ldots, x_n, \ldots) \,|\, x_n \in \mathbb{R}, n = 1, 2, \ldots\}$

4) $1 \le p < \infty$,

$$l^p = \left\{ (x_1, \ldots, x_n, \ldots) \in \mathbb{R}^\infty \,\middle|\, \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty \right\}$$

with the same addition and mult. by scalar as in $\mathbb{R}^\infty$. We have to check:

(1) $x, y \in l^p \quad \Rightarrow \quad x + y \in l^p$

(2) $x \in l^p, \lambda \in \mathbb{R} \quad \Rightarrow \quad \lambda x \in l^p$.

For (1) we assume $x = (x_1, \ldots, x_n, \ldots)$ and $y = (y_1, \ldots, y_n, \ldots)$.

$$\begin{aligned} x \in l^p \quad &\Rightarrow \quad \sum_{n=1}^{\infty} |x_n|^p < \infty \\ y \in l^p \quad &\Rightarrow \quad \sum_{n=1}^{\infty} |y_n|^p < \infty \end{aligned}$$

$$\Rightarrow \qquad x+y=(x_1+y_1,\dots)\overset{?}{\in} l^p?$$

$$\begin{aligned}\Rightarrow \sum_{n=1}^{\infty}|x_n+y_n|^p &\le \{|x_n+y_n|\le|x_n|+|y_n|\le 2\max\{|x_n|,|y_n|\}\}\\ &\{|x_n+y_n|^p\le 2^p\left(|x_n|^p+|y_n|^p\right)\}\\ &\le \sum_{n=1}^{\infty}2^p(|x_n|^p+|y_n|^p)\\ &= 2^p\underbrace{\sum|x_n|^p}_{<\infty}+2^p\underbrace{\sum|y_n|^p}_{<\infty}<\infty\end{aligned}$$

and

$$\sum_{n=1}^{\infty}|\lambda x_n|^p=\sum_{n=1}^{\infty}|\lambda|^p\cdot|x_n|^p=|\lambda|^p\sum_{n=1}^{\infty}|x_n|^p<\infty.$$

5) Function spaces, say real-valued functions on I.

   addition: $(f+g)(x)=f(x)+g(x), \qquad x\in I$

   mult. by scalar: $(\lambda f)(x)=\lambda f(x) \qquad$ for functions f and g

6) $C(I)$: addition and mult. by scalar as in (5).
   f,g continuous in I implies that $f+g$ is continuous in I.
   Also if f is continuous and $\lambda\in\mathbb{R}$ then (λf) is continuous in I.

7) $P(I)=$ polynomials in I.

8) $P_k(I)=$ polynomials of degree at most k in I.

Theorem 2.1 (Hölder's inequality). Assume $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$.
Let $(x_1,\dots,x_n,\dots)$ and $(y_1,y_2,\dots,y_n,\dots)$ be sequences of complex numbers. Then

$$\sum_{n=1}^{\infty}|x_ny_n|\le\left(\sum_{n=1}^{\infty}|x_n|^p\right)^{\frac{1}{p}}\cdot\left(\sum_{n=1}^{\infty}|y_n|^q\right)^{\frac{1}{q}}.$$

Remark there the LHS can be infinity, but the RHS can also be infinity.

proof. **Step 1** We're going to proof

$$ab\le\frac{a^p}{p}+\frac{b^q}{q}, \qquad \text{for all } a,b>0.$$

$$\int_0^a x^{p-1}\,\mathrm{d}x=\frac{a^p}{p}.$$

Note $y = x^{p-1}$ gives

$$x = y^{\frac{1}{p-1}} = y^{\frac{1}{\frac{1}{1-\frac{1}{q}}-1}} = y^{\frac{1}{\frac{q}{q-1}-1}} = y^{q-1}$$

so

$$\int_0^b y^{q-1}\,\mathrm{d}y = \frac{b^q}{q}.$$

We get

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

(You also get condition for $=$)

Step 2 It is enough to consider the cases LHS > 0 and RHS $< \infty$. There exists an integer N such that

$$0 < \sum_{n=1}^{N} |x_n|^p, \sum_{n=1}^{N} |y_n|^q < \infty.$$

Set

$$a = \frac{|x_k|}{\left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}}}, \qquad k = 1, 2, \ldots, N,$$
$$b = \frac{|y_k|}{\left(\sum_{n=1}^N |y_n|^q\right)^{\frac{1}{q}}}, \qquad k = 1, 2, \ldots, N.$$

Insert into

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

$$\frac{|x_k y_k|}{\left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^N |y_n|^q\right)^{\frac{1}{q}}} \leq \frac{|x_k|^p}{p\sum_{n=1}^N |x_n|^p} + \frac{|y_k|^q}{q\sum_{n=1}^N |y_n|^q}, \qquad k = 1, 2, \ldots, N.$$

We sum over k from 1 to N:

$$\sum_{k=1}^N |x_k y_k| \leq \left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{n=1}^N |y_n|^q\right)^{\frac{1}{q}}.$$

Let $N \to \infty$. First in RHS and then in LHS.

$\square$

Theorem 2.2 (Minkowski's inequality). Assume $1 \leq p < \infty$. and $X, Y \in l^p$. Then

$$\|X + Y\|_{l^p} \leq \|X\|_{l^p} + \|Y\|_{l^p}.$$

proof. $p = 1$:

$$\begin{aligned}
\|X+Y\|_{l^1} &= \|(x_1, x_2, \ldots, x_n, \ldots) + (y_1, y_2, \ldots, y_n, \ldots)\|_{l^1} \\
&= \|(x_1+y_1, \ldots, x_n+y_n, \ldots)\|_{l^1} \\
&= \sum_{n=1}^{\infty} |x_n + y_n| \\
&\leq \sum_{n=1}^{\infty} (|x_n| + |y_n|) \\
&= \sum_{n=1}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| \\
&= \|X\|_{l^1} + \|Y\|_{l^1}
\end{aligned}$$

$1 < p < \infty$:

$$\begin{aligned}
\|X+Y\|_{l^p}^p &= \sum_{n=1}^{\infty} |x_n + y_n|^p \\
&= \sum_{n=1}^{\infty} |x_n + y_n||x_n + y_n|^{p-1} \\
&\leq \sum_{n=1}^{\infty} |x_n||x_n + y_n|^{p-1} + \sum_{n=1}^{\infty} |y_n||x_n + y_n|^{p-1}.
\end{aligned}$$

Use Hölder to get

$$\begin{aligned}
\sum_{n=1}^{\infty} |x_n||x_n + y_n|^{p-1} &\leq \underbrace{\left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}}_{=\|X\|_{l^p}} \cdot \left(\sum_{n=1}^{\infty} |x_n + y_n|^{(p-1)q}\right)^{\frac{1}{q}} \\
&= \left\{ (p-1)q = (p-1)\frac{1}{1-\frac{1}{p}} = p \right\} \\
&= \|X\|_{l^p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{q}}.
\end{aligned}$$

We have

$$\|X+Y\|_{l^p}^p \leq \left(\|X\|_{l^p} + \|Y\|_{l^p}\right) \|X+Y\|_{l^p}^{\frac{p}{q}}.$$

If $\|X+Y\|_{l^p} \neq 0$ then

$$\|X+Y\|_{l^p}^{p-\frac{p}{q}} \leq \|X\|_{l^p} + \|Y\|_{l^p}$$

there

$$p - \frac{p}{q} = p(1 - \frac{1}{q}) = p\frac{1}{p} = 1.$$

$\square$

Remark. $f \in C([0,1])$ then for $1 \leq p < \infty$

$$\|f\|_{L^p} = \left(\int_0^1 |f(t)|^p \,\mathrm{d}t \right)^{\frac{1}{p}}.$$

Claim:

$$\|fq\|_{L^1} = \int_0^1 |f(t) \cdot g(t)| \,\mathrm{d}t \leq \|f\|_{L^p} \cdot \|g\|_{L^q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Also we have

$$\|f + q\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

This is proven with the same technique as we used for l^p. $\sum_{n=1}^{\infty}$ is replaced by $\int_0^1 \,\mathrm{d}t$.
E real/complex vector space. $x_1, \ldots, x_n \in E$, $\lambda_1, \ldots, \lambda_n$ scalar. We say that

$$\lambda_1 x_1, \ldots, \lambda_n x_n$$

is a linear combination of $x_1, \ldots, x_n$. We say that $x_1, \ldots, x_n$ are linear independent if

$$\alpha_1 x_1 + \ldots + \alpha_n x_n = 0 \qquad \Rightarrow \qquad \alpha_1 = \ldots = \alpha_n = 0.$$

If $A \subset E$, we say that A is linear independant if every linear combination of vectors in A is linear independent.

Examples. (1) Set $E = P([0,1])$ and $A = \left\{ p_k \,\middle|\, p_k(x) = x^k, x \in [0,1], k = 0, 1, \ldots \right\}$. A is linear independant since:
Consider

$$\alpha_0 p_0 + \alpha_1 p_1 + \ldots + \alpha_n p_n = 0,$$

i.e.

$$\alpha_0 p_0(x) + \alpha_1 p_1(x) + \ldots + \alpha_n p_n(x) = 0(x), \qquad x \in [0,1],$$

i.e.

$$\alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n = 0, \qquad x \in [0,1].$$

If $x = 0$ then $\alpha_0 = 0$

$$\alpha_1 x + \ldots + \alpha_n x^n = 0, \qquad x \in [0,1].$$

Differentiate

$$\alpha_1 + 2\alpha_2 x + \ldots + n\alpha_n x^{n-1} = 0$$

gives $\alpha_1 = 0$. Continue and get

$$\alpha_0 = \alpha_1 = \ldots = \alpha_n = 0.$$

Set $B \subset E$ where

$$\begin{aligned} \operatorname{span} B &= \{\text{set of all linear combinations of elements in B}\} \\ &= \left\{ \sum_{k=1}^{n} \lambda_k x_k \,\middle|\, x_k \in B, \lambda_k \in \mathbb{R}, k = 1, 2, \ldots, n \text{ where n is a positive integer} \right\}. \end{aligned}$$

Remark.

$$\sum_{k=1}^{n} \lambda_k x_k \in E,$$

$$\sum_{k=1}^{\infty} \lambda_k x_k \text{ has no meaning.}$$

$C \subset E$ is called a basis for E if

1) C linear independent,
2) $\operatorname{span} C = E$.

Continue of the example above:
Claim: A is a basis for E.

(2) Set $E = l^2$ and

$$A = \{X_k \,|\, k = 1, 2, \ldots\},$$

$$X_k = (0, 0, \ldots, 0, 1, 0, 0, \ldots).$$

Claim: A is linear independent since

$$\alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_n X_n = 0.$$

Here

$$\alpha_1 X_1 = (\alpha_1, 0, 0, \ldots), \qquad etc$$

and

$$0 = (0, 0, \ldots).$$

So

$$(\alpha_1, \alpha_2, \ldots, \alpha_n, 0, \ldots) = (0, 0, \ldots).$$

So $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.
Question: Is A a basis for l^2?
We note: If $X \in \operatorname{span} A$ then

$$X = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$$

for some positive integer n, i.e. X has only finitely many nonzero positions.
Consider:

$$X := (1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots),$$

$$\|X\|_{l^2} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{2}} < \infty.$$

So $X \in l^2 \setminus \operatorname{span} A$.

Remark. Every vector space has a basis (if we are allowed to use Axiom of Choice/ zorns lemma).
Basis = vector space basis = Hamel basis

Assume $x_1, \dots, x_n$ is a basis for E. Then every basis for E must contain n different elements.

$$n = \dim E$$

is well-defined. (System of linear equations, homogeneous with more unknowns than equations. Then there exists a nontrivial solution.)

Definition (norm). E vector space. We say that $\|.\| : E \to [0, \infty)$ is a norm on E if

1) $\|x\| = 0 \quad \Rightarrow x = 0$,
2) $\|\lambda x\| = |\lambda| \|x\| \quad$ for all $x \in E, \lambda \in \mathbb{R}$,
3) $\|x + y\| \leq \|x\| + \|y\| \quad$ for all $x, y \in E$.

Remark.

$$\|0\| = \|0 \cdot 0\| = \underbrace{|0|}_{=0} \|0\| = 0.$$

Examples. (1) $1 < p < \infty$ and

$$\|X\|_{l^p} = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

is a norm on l^p. Check $1)$,$2)$ and $3)$ above:

1)

$$0 = \|X\|_{l^p} = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}.$$

It follows

$$x_n = 0, \qquad n = 1, 2, \dots,$$

$$\Rightarrow \qquad X = (x_1, x_2, \dots) = (0, 0, \dots) = 0.$$

2)

$$\|\lambda X\|_{l^p} = \left(\sum_{n=1}^{\infty} |\lambda x_n|^p \right)^{\frac{1}{p}} = \left(|\lambda|^p \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} = |\lambda| \|X\|_{l^p}$$

3)
$$\|X+Y\|_{l^p} \leq \{\text{Minkowski's inequality}\} \leq \|X\|_{l^p} + \|Y\|_{l^p}$$

(2) $E = C([0,1])$ and $f \in E$
$$\|f\| = \max_{t\in[0,1]} |f(t)| \in [0,\infty).$$

Check the axioms above

1) If $\|f\| = 0$ it follows
$$|f(t)| = 0 \text{ for all } t \in [0,1], \qquad \Rightarrow \qquad f = 0$$

2)
$$\|\lambda f\| = \max_{t\in[0,1]} \underbrace{|\underbrace{(\lambda f)(t)}_{\lambda f(t)}|}_{|\lambda||f(t)|} = |\lambda| \max_{t\in[0,1]} |f(t)| = |\lambda|\|f\|$$

3)
$$\|f+g\| = \max_{t\in[0,1]} |\underbrace{(f+g)(t)}_{f(t)+g(t)}| = \max_{t\in[0,1]} (|f(t)|+|g(t)|) \leq \max_{t\in[0,1]} |f(t)| + \max_{t\in[0,1]} |g(t)| = \|f\| + \|g\| \quad \blacksquare$$

(3) $E = C([0,1])$ and $f \in E$.
$$\|f\|_{L^1} = \int_0^1 |f(t)|\,\mathrm{d}t$$

defines also a norm on E.

3)
$$\begin{aligned}\|f+g\|_{L^1} &= \int_0^1 |\underbrace{(f+g)(t)}_{f(t)+g(t)}|\,\mathrm{d}t \\ &\leq \int_0^1 (|f(t)|+|g(t)|)\,\mathrm{d}t \\ &= \int_0^1 |f(t)|\,\mathrm{d}t + \int_0^1 |g(t)|\,\mathrm{d}t \\ &= \|f\|_{L^1} + \|g\|_{L^1}\end{aligned}$$

2)
$$\|\lambda f\| = \int_0^1 \underbrace{|(\lambda f)(t)|}_{=|\lambda||f(t)|}\,\mathrm{d}t = |\lambda|\|f\|_{L^1}$$

1)
$$0 = \|f\|_{L^1} = \int_0^1 |f(t)|\,\mathrm{d}t$$

This implies $f(t) = 0$ for $t \in [0,1]$ since f is continuous! i.e. $f = 0$.

Theorem 2.3 (equivalent norm). E vector space with norms $\|.\|$ and $\|.\|_*$. We say that $\|.\|$ and $\|.\|_*$ are equivalent if there exists $\alpha, \beta > 0$ such that

$$\alpha\|x\|_* \leq \|x\| \leq \beta\|x\|_* \qquad \text{for all } x \in E.$$

Example.

$E = C([0,1])$. Choose $y = f(t)$ and $y = |f(t)|$

$$\|f\| = \max_{t\in[0,1]} |f(t)|, \qquad \|f\|_* = \|f\|_{L^1} = \text{area}.$$

Question: Are these norms equivalent?
Claim: $f \in C([0,1])$

$$\|f\|_* = \int_0^1 \underbrace{|f(t)|}_{\leq\|f\|} \,\mathrm{d}t \leq \|f\|.$$

Choose $f_n(t)$ such that

$$\|f_n\| = 1, \qquad \|f_n\|_* = \frac{1}{2n}.$$

So

$$\frac{\|f_n\|_*}{\|f_n\|} = \frac{1}{2n} \to 0 \qquad n \to \infty.$$

The norms are not equivalent! Answer: NO !

Theorem 2.4. E vector space with $\dim E < \infty$.
$\Rightarrow$ All norms on E are equivalent.

proof. Assume $n = \dim E$ with a positive integer n. Let $x_1, x_2, \ldots, x_n$ be a basis for E. For every $x \in E$

$$x = \alpha_1(x)x_1 + \ldots + \alpha_n(x)x_n,$$

where $\alpha_1(x), \ldots, \alpha_n(x)$ unique. Set

$$\|x\|_* = |\alpha_1(x)| + \ldots + |\alpha_n(x)|, \qquad x \in E$$

Claim: $\|.\|_*$ defines a norm on E (easy proof)
Fix an arbitrary norm $\|.\|$ on E.
Claim: $\|.\|_*$ and $\|.\|$ are equivalent.
Note for $x \in E$

$$\begin{aligned}\|x\| &= \|\alpha_1(x)x_1 + \ldots + \alpha_n(x)x_n\| \\ &\leq |\alpha_1(x)|\|x_1\| + \ldots + |\alpha_n(x)|\|x_n\| \\ &\leq \max_{k=1,2,\ldots,n} \|x_k\| \underbrace{(|\alpha_1(x)| + \ldots + |\alpha_n(x)|)}_{=\|x\|_*}.\end{aligned}$$

Set $\beta = \max_{k=1,2,\ldots,n} \|x_k\|$.
Then

$$\|x\| \leq \beta \|x\|_* \qquad \text{for all } x \in E.$$

Remains to prove: There exists $\alpha > 0$ such that

$$\alpha \|x\|_* \leq \|x\| \qquad \text{for all } x \in E \qquad (*).$$

Let E be a vector space with norm $\|.\|$ and $(v_m)_{m=1}^{\infty}$ a sequence in E. We say that $(v_m)_{m=1}^{\infty}$ converges in $(E, \|.\|)$ if there exists $v \in E$ such that $\|v_m - v\| \to 0$ for $n \to \infty$.
Notation: $v_m \to v$ in $(E, \|.\|)$.
Note: If we have $\|.\|$ and $\|.\|_*$ are equivalent, then

$$v_n \to v \text{ in } (E, \|.\|) \qquad \Leftrightarrow \qquad v_n \to v \text{ in } (E, \|.\|_*).$$

Back to $(*)$: Argue by contradiction.
Assume there is no $\alpha > 0$ such that

$$\alpha \|x\|_* \leq \|x\| \qquad \text{for all } x \in E.$$

For $k = 1, 2, 3, \ldots$ there are $y_k \in E$ such that

$$\frac{1}{k} \|y_k\|_* > \|y_k\|. \qquad (**).$$

We have

$$y_k = \alpha_1^{(k)} x_1 + \ldots + \alpha_n^{(k)} x_n,$$

where $\alpha_1^{(k)}, \ldots, \alpha_n^{(k)}$ are unique scalars and $k = 1, 2, \ldots$.
$(**)$ implies that

$$k\|y_k\| < |\alpha_1^{(k)}| + \ldots + |\alpha_n^{(k)}|,$$

WLOG we can assume $|\alpha_1^{(k)}| + \ldots + |\alpha_n^{(k)}| = 1$. (If not consider

$$\begin{aligned}
\lambda z &= \lambda(\alpha_1(z)x_1 + \ldots + \alpha_n(z)x_n) \\
&= (\lambda\alpha_1(z))x_1 + \ldots + (\lambda\alpha_n(z))x_n \\
&= \alpha_1(\lambda z)x_1 + \ldots + \alpha_n(\lambda z)x_n.
\end{aligned}$$

We have

$$\alpha_k(\lambda z) = \lambda \alpha_k(z), \qquad k = 1, 2, \ldots, n).$$

We have

$$k\|y_k\| < 1 \qquad k = 1, 2, \ldots$$

which implies $y_k \to 0$ in $(E, \|.\|)$.
<u>IF:</u>

$$\begin{aligned}
\alpha_1^{(k)} &\to \bar{\alpha_1} \\
\alpha_2^{(k)} &\to \bar{\alpha_2} \\
&\vdots \\
\alpha_n^{(k)} &\to \bar{\alpha_n}
\end{aligned}$$

for $k \to \infty$. Then set

$$\bar{y} = \bar{\alpha_1} x_1 + \ldots + \bar{\alpha_n} x_n$$

and get

$$\begin{aligned} \|y_k - \bar{y}\| &= \left\| (\alpha_1^{(k)} - \bar{\alpha_1}) x_1 + \ldots + (\alpha_n^{(k)} - \bar{\alpha_n}) x_n \right\| \\ &\le \underbrace{|\alpha_1^{(k)} - \bar{\alpha_1}|}_{\to 0} \underbrace{\|x_1\|}_{<\infty} + \ldots + \underbrace{|\alpha_n^{(k)} - \bar{\alpha_n}|}_{\to 0} \underbrace{\|x_n\|}_{<\infty} \to 0, \qquad k \to \infty \\ \|\bar{y}\| = \|\bar{y} - y_k + y_k\| &\le \underbrace{\bar{y} - y_k}_{\to 0} + \underbrace{\|y_k\|}_{\to 0} \to 0, \qquad k \to \infty. \end{aligned}$$

So $\|\bar{y}\| = 0$ hence $\bar{y} = 0$. But

$$|\bar{\alpha_1}| + |\bar{\alpha_2}| + \ldots + |\bar{\alpha_n}| = 1.$$

This contradicts $x_1, \ldots, x_n$ is a basis.
We have for $k = 1, 2, \ldots$ the vector $(\alpha_1^{(k)}, \alpha_2^{(k)}, \ldots, \alpha_n^{(k)})$ where

$$|\alpha_1^{(k)}| + \ldots + |\alpha_n^{(k)}| = 1.$$

We focus on the first one and we have

$$|\alpha_1^{(k)}| \le 1, \qquad k = 1, 2, \ldots.$$

By Bolzano-Weierstraß then there exists a converging subsequence $(\alpha_{1,1}^{(k)})_{k=1}^{\infty}$ of $(\alpha_1^{(k)})_{k=1}^{\infty}$. Set

$$\bar{\alpha_1} = \lim_{k \to \infty} \alpha_{1,1}^{(k)}$$

and consider

$$(\alpha_{1,1}^{(k)}, \alpha_{2,1}^{(k)}, \ldots, \alpha_{n,1}^{(k)}), \qquad k = 1, 2, \ldots.$$

We have

$$|\alpha_{2,1}^{(k)}| \le 1, \qquad k = 1, 2, \ldots.$$

Bolzano-Weierstraß implies that there exists a converging subsequenz $(\alpha_{2,2}^{(k)})_{k=1}^{\infty}$ of $(\alpha_{2,1}^{(k)})_{k=1}^{\infty}$. Set

$$\bar{\alpha_2} = \lim_{k \to \infty} \alpha_{2,2}^{(k)}.$$

□

Definition (normed space). Let E be a vector space over $\mathbb{R}$ or $\mathbb{C}$. $\|.\| : E \to \mathbb{R}$ a norm on E if

(i) $\|x\| > 0$ for any $x \in E \setminus \{0\}$,

(ii) $\|\lambda x\| = |\lambda x|$ for any $\lambda \in \mathbb{C}, x \in E$,

(iii) $\|x + y\| \le \|x\| + \|y\|$ for any $x, y \in E$.

Obs. $\|x\| = 0$ if $x = 0$. $(E, \|.\|)$ is called a normed space. A norm generates a distance

function (metric)
$$L(x, y) := \|x - y\| \qquad \text{for any } x, y \in E.$$

Examples.

- $\mathbb{R}^n$ with $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$ is the eukledian norm.
- $C([0, 1])$ continuous functions in $[0, 1]$ with
$$L(f, g) = \|f - g\|_\infty := \max_{x \in [0,1]} |f(x) - g(x)|$$

Definition (balls). Let $x \in E$, $r > 0$. Define

$$B(x, r) := \{y \in E \,|\, \|x - y\| < r\} \qquad \text{open ball,}$$
$$\bar{B}(x, r) := \{y \in E \,|\, \|x - y\| \leq r\} \qquad \text{closed ball.}$$

Definition (open/closed). A subset $A \subset E$ of a normed space $(E, \|.\|)$ is called open of any point x of A is inner, i.e

$$\exists\, r > 0 \,:\, B(x, r) \subset A.$$

It is called closed if the complement $E \setminus A$ is open.

Remark.

- open balls are open sets.
- closed balls are closed.
- $(C([0, 1]), \|.\|_\infty)$ with $\|f\|_\infty = \max_{x \in [0,1]} |f(x)|$.
$$A := \{g \in C([0, 1\,|\,)\}]f(x) < g(x), \,\forall\, x \in [0, 1]$$
is an open set $C([0, 1])$.
$$B := \{g \in C([0, 1\,|\,)\}]f(x) \leq g(x), \,\forall\, x \in [0, 1]$$
is a closed set.

Properties

- Any union of open sets is an open set.
- Any finite intersection of open sets is open.
- $\emptyset, E$ are both closed and open.
- Normed spaces are topological spaces.

Definition (convergence in normed spaces). Let $(E, \|.\|)$ be a normed space $\{x_n\}_n \subset E$. We say that x_n converges to $x \in E$ if

$$\|x_n - x\| \to 0, \qquad n \to \infty.$$

One can define open and closed using the definition of convergence:

Statement 2.5. $A \subseteq E$ is closed if any convergent sequence in A has a limit in A, i.e

$$\underbrace{x_n \to x}_{\substack{\text{for } n\to\infty \\ x_n \in A}} \qquad \Rightarrow \qquad x \in A.$$

proof. $\Rightarrow$: Assume that A is closed and $x_n \to x$. $x_n \in A$, but $x \notin A$. (try to get a contradiction).

A is closed $\Rightarrow E \setminus A$ is open and hence $\exists\, r > 0$ such that

$$B(x, r) \subset E \setminus A.$$

Hence $\|x_n - x\| \geq r$ for any n. This is a contradiction because in that case $x_n \not\to x$.

$\Leftarrow$: Assume that for any sequence $\{x_n\} \subset A$ such that $x_n \to x$ we have $x \in A$. We try to get a contradiction and assume that A is not closed. Hence $E \setminus A$ is not open and therefore $\exists\, x \in E \setminus A$ which is not inner.

$$\Rightarrow \qquad \forall\, B(x, \frac{1}{n}) \text{ containts points outside } E \setminus A,$$

i.e.

$$\exists\, x_n \in B(x, \frac{1}{n}),\ x_n \in A.$$

We get a sequence $\{x_n\} \subset A$ such that

$$\|x_n - x\| < \frac{1}{n} \qquad \Rightarrow \qquad x_n \to x.$$

This is a contradiction.

□

Definition (closure). $A \subset E$. The closure of A is the minimal closed subset containing A. We write $\bar{A}$.

Proposition 2.6. $\bar{A}$ is the set of all limit points of A which means

$$\bar{A} := \{x \in E \,|\, \text{there exists } \{x_n\} \subseteq A \text{ such that } x_n \to x\}.$$

proof. Exercise. □

Definition (dense). $A \subset E$ is dense in E if

$$\bar{A} = E.$$

Remark. This definition of dense is equivalent to the following definition:

$$\forall\, x \in E, \forall\, \varepsilon > 0\, \exists\, y \in A \text{ such that } \|x - y\| < \varepsilon.$$

Examples. 1) $\mathbb{Q} \subseteq \mathbb{R}$ with $|.|$ usual absolut value function. $\mathbb{Q}$ is dense in $\mathbb{R}$.

2) $C([a, b])$. The <u>Weierstraß-Theorem</u> says that the set of all polynomials are dense in $(C([a, b], \|.\|_\infty))$:

$$\forall\, f \in C([a, b]), \forall\, \varepsilon > 0\, \exists\, p - \text{polynomial such that } \max_{x \in [a,b]} |f(x) - p(x)| < \varepsilon.$$

Another example is $(C_0, \|.\|_\infty)$ where

$$C_0 = \{x = (x_1, x_2, \dots) \,|\, x_k \to 0 \text{ as } k \to \infty\},$$

$$\|x\|_\infty = \sup_i |x_i|.$$

$(C_0, \|.\|_\infty)$ is a normed space.

$$C_F = \{x = (x_1, x_2, \dots) \,|\, \text{only a finite number of } x_i \neq 0\} \subset C_0.$$

Statement 2.7. C_F is dense in C_0.

proof.

$$\forall\, x \in C_0\, \forall\, \varepsilon > 0 \text{ must find } y \in C_F \text{ such that } \|y - x\|_\infty < \varepsilon.$$

$$x \in C_0 \qquad \Rightarrow \qquad x_k \to 0 \text{ for } k \to \infty$$

$$\Rightarrow \qquad \forall\, \varepsilon > 0\, \exists\, K \text{ such that } |x_k| < \varepsilon\, \forall\, k \geq K.$$

Let now $y = (x_1, x_2, \dots, x_K, 0, \dots) \in C_F$. Then

$$\|x - y\|_\infty = \|(0, 0, \dots, 0, x_{K+1}, x_{K+2}, \dots)\|_\infty = \sup_{k > K} |x_k| < \varepsilon.$$

□

Definition (separable). A normed space $(E, \|.\|)$ is called separable if it contains a countable dense subset.

Examples.
- $(\mathbb{R}, |.|)$ is separable as $\mathbb{Q}$ is countable and dense in $\mathbb{R}$.
- $(\mathbb{R}^n, \|.\|_2)$ is separable, $\mathbb{Q}^n$ is countable and dense in $\mathbb{R}$.

Definition (compact set). For a normed space $(E, \|.\|)$ is $A \subset E$ a compact set if any sequence $\{x_n\} \subset A$ has a subsequence convergent to an element $x \in A$.

Example. Any bounded and closed subset in $\mathbb{R}, \mathbb{R}^n, \mathbb{C}^n$ is compact. A sequence $\{x_n\}$ of a bounded set is bounded. From real Analysis one knows it has a subsequence that is convergent. If the subset is closed then the limit point is inside the set.

Lemma . $S \subset E$ compact in $(E, \|.\|)$ implies that S is closed and bounded. (Bounded means that $S \subset B(0, R)$ for some $R > 0$).

proof. Let S be a compact subset of E. Assume that S is not bounded. Hence for any $n > 0$ there exists points in S which are outside $B(0, n)$, i.e.

$$\exists x_n \in S : \|x_n\| > n.$$

Then $\{x_n\}$ can not have a convergent subsequence as if $x_{n_k} \to x$ then

$$n_k < \|x_{n_k}\| = \|x_{n_k} - x + x\| \leq \|x_{n_k} - x\| + \|x\| \to \|x\|$$

but $n_k \to \infty$. This is a contradiction, hence S must be bounded.
S must be closed, because if $x_n \to x$ then any subsequence converges to x. From the definition of compactness and uniqueness of the limit we have $x \in S$.

$\square$

Remark. In general, S bounded and closed doesn't imply that S is compact.
For instance let $E = C([0, 1])$. Then $S = \{g \in C((0, 1)) \,|\, \|g\|_\infty \leq 1\}$ is closed and bounded, but not compact.
Take $x_n(t) := t^n$. Then $x_n \in S$. $\{x_n\}$ does not have a subsequence convergent to a continuous function.

Theorem 2.8. $(E, \|.\|)$ normed space and $\dim E < \infty$
iff

$$\forall A \subset E,\ A \text{ compact } \Leftrightarrow A \text{ is closed and bounded.}$$

proof. $\Rightarrow$: If $\dim E < \infty$ then A is compact iff A is bounded and closed (exsercise).

$\Leftarrow$: Enough to prove the following:
If $\dim E = \infty$ then the unit ball $S = \{x \in E \,|\, \|x\| \leq 1\}$ is not compact.

Lemma 2.9 (Riesz's lemma). If X is a proper closed subspace of a normed space $(E, \|.\|)$ then for every $\varepsilon \in (0,1)$ there exists an $x_\varepsilon \in E$ with $\|x_\varepsilon\| = 1$ such that

$$\|x_\varepsilon - x\| \geq \varepsilon \qquad \forall\, x \in X.$$

proof. Let $z \in E \setminus X$ (X proper and hence $E \setminus X \neq \emptyset$). Set

$$d := \inf_{x \in X} \|z - x\|.$$

As X is closed, $d > 0$, otherwise z is a limit point in $E \setminus X$. Fix $\varepsilon \in (0,1)$. Then there exists $x_0 \in X$ such that

$$d \leq \|z - x_0\| < \frac{d}{\varepsilon}.$$

Let $x_\varepsilon := \frac{z - x_0}{\|z - x_0\|}$; We have $\|x_\varepsilon\| = 1$ and

$$\begin{aligned}
\|x - x_\varepsilon\| &= \left\| x - \frac{z - x_0}{\|z - x_0\|} \right\| \\
&= \frac{\|x\|z - x_0\| - z + x_0\|}{\|z - x_0\|} \\
&= \frac{\left\| \overbrace{x\|z - x_0\| + x_0}^{\in X} - z \right\|}{\|z - x_0\|} \\
&\geq \frac{d}{d}\varepsilon = \varepsilon.
\end{aligned}$$

□

Continue now the proof of the theorem above:
Let $x_1 \in S$. Consider $X = \text{span}\{x_1\}$ which is a proper closed subspace of E. Hence by Riesz's lemma exists x_2 with $\|x_2\| = 1$ such that

$$\|x_2 - x_1\| \geq \frac{1}{2}$$

and

$$\|x_2 - x\| \geq \frac{1}{2} \qquad \forall\, x \in X.$$

Now consider span$\{x_1, x_2\}$ which is a proper closed subspace of E. By Riesz's lemma follows

$$\exists\, x_3 \in E,\ \|x_3\| = 1 :\ \|x_3 - x_1\| \geq \frac{1}{2}, \|x_3 - x_2\| \geq \frac{1}{2}.$$

Continuing in the same fashion we get $\{x_n\}$, $\|x_n\| = 1$ such that

$$\|x_n - x_m\| \geq \frac{1}{2} \qquad \forall\, n, m,\ n \neq m.$$

Clearly $\{x_n\} \subset S$ has no convergent subsequence. Hence S is not compact. □

Definition (Cauchy sequence). $(E, \|.\|)$ normed space. $\{x_n\} \subseteq E$ is called Cauchy if

$$\forall \varepsilon > 0 \, \exists N : \|x_n - x_m\| < \varepsilon \text{ for any } n, m \geq N.$$

Example. $(C_F, \|.\|_\infty)$, $\|x\|_\infty = \sup_{k\in\mathbb{N}}|x_k|$ where $x = (x_1, x_2, \ldots)$. Define

$$x_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, \ldots).$$

Then $\{x_n\}$ is Cauchy, as for $n > m$

$$\begin{aligned}\|x_n - x_m\|_\infty &= \left\|(0, \ldots, 0, \frac{1}{m+1}, \ldots, \frac{1}{n}, 0, \ldots)\right\|_\infty \\ &= \frac{1}{m+1}.\end{aligned}$$

Observe that x_n is convergent in $(C_0, \|.\|_\infty)$

$$\underbrace{x_n}_{\in C_F} \to (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots) \in C_0 \setminus C_F.$$

Statement 2.10. A convergent sequence is always a Cauchy sequence.

Definition (complete space). A normed vector space $(E, \|.\|)$ is called complete if any Cauchy sequence in E is convergent in E.

$(C_F, \|.\|_\infty)$ is not complete.

Definition (Banach space). A complete normed space is called Banach space.

Examples.

- $(\mathbb{R}, |.|)$ is a Banach space.
- $(\mathbb{C}, |.|)$ is a Banach space.
- $(l^2, \|.\|_2)$ where
$$l^2 = \left\{(x_1, x_2, \ldots) \,\middle|\, \sum_{i=1}^{\infty} |x_i|^2 < \infty, x_i \in \mathbb{C}\right\}$$
and
$$\|(x_1, x_2, \ldots)\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}.$$
$(l^2, \|.\|_2)$ is complete.

proof. Let $x_n = (x_1^n, x_2^n, \ldots)$ be a Cauchy sequence in l^2. We must show that it has a limit in l^2. We will do it in a few steps:

Step 1: Find a candidate for a limit a.

Step 2: Show that $a \in l^2$.

Step 3: $\|x_n - a\|_2 \to 0$ as $n \to \infty$.

Step 1: Let

$$
\begin{aligned}
x_1 &= (x_1^1, x_2^1, \ldots) \\
x_2 &= (x_1^2, x_2^2, \ldots) \\
\vdots \quad & \quad \vdots \\
x_n &= (x_1^n, x_2^n, \ldots).
\end{aligned}
$$

For each k consider sequence $\{x_k^n\} \subset \mathbb{C}$ (k-th coordinates in each x_n). Each sequence is Cauchy, as for all $n, m \geq N$

$$|x_k^n - x_k^m| < \left(\sum_{k=1}^{\infty} |x_k^n - x_k^m|^2\right)^{\frac{1}{2}} = \|x_n - x_m\|_2 < \varepsilon.$$

As $(\mathbb{C}, |.|)$ is complete, $\{x_k^n\}_n$ has a limit $a_k \in \mathbb{C}$. Candidate for limit of x_n is

$$a = (a_1, a_2, \ldots, a_k, \ldots).$$

Step 2: Write

$$a = \underbrace{x_n}_{\in l^2} - (x_n - a).$$

In order to show that $a \in l^2$ it is enough to see that $x_n - a \in l^2$ for some n. $\{x_n\}$ Cauchy implies

$$\forall\, \varepsilon > 0\, \exists\, N : \forall\, n, m \geq N : \|x_n - x_m\|_2 < \varepsilon.$$

Consider for some $u > 0$

$$\sum_{i=1}^{u} |x_i^n - x_i^m|^2 \leq \sum_{i=1}^{\infty} |x_i^n - x_i^m|^2 = \|x_n - x_m\|_2^2 < \varepsilon^2.$$

Let $m \to \infty$. We get

$$\sum_{i=1}^{m} |x_i^n - a_i|^2 \leq \varepsilon^2.$$

This holds for any $u \in \mathbb{N}$. Hence for any $n \geq \mathbb{N}$

$$\underbrace{\sum_{i=1}^{\infty} |x_i^n - a_i|^2}_{=\|x_n - a\|_2^2} \leq \varepsilon^2.$$

Hence $x_n - a \in l^2$ and moreover $\|x_n - a\| \to 0$ as $n \to \infty$.

□

- $(C([a,b]), \|.\|_\infty)$ is a Banach space.
- $(l^p, \|.\|_{l^p})$ for $1 \leq p < \infty$ are all Banach spaces.
- $(C([a,b]), \|.\|_2)$ with

$$\|f\|_2 = \left(\int |f(t)|^2 \, \mathrm{d}t\right)^{\frac{1}{2}}.$$

  One can prove that $(C([a,b]), \|.\|_2)$ is not a Banach space.

  Exercise:

  $[a,b] = [0,1]$ and

$$f_n(t) = \begin{cases} 0, & \text{falls } t < \frac{1}{2} - \frac{1}{n} \\ 1, & \text{falls } t > \frac{1}{2} \\ \text{continuous linear function} & \end{cases}.$$

  Show that $\{f_n\}$ is Cauchy in $C([0,1], \|.\|_2)$ but $f_n \not\to f \in C([0,1])$.

Definition (Convergent and absolutely convergent series). A series $\sum_{n=1}^\infty x_n$ in E is called convergent if $\{\sum_{n=1}^m x_n\}_m$, a sequence of partial sums, is convergent in E. If $\sum_{n=1}^\infty \|x_n\| < \infty$ then we say that $\sum_{n=1}^\infty x_n$ converges absolutely.

Theorem 2.11. A normed space E is complete iff every absolutely convergent series converges in E.

proof. $\Rightarrow$: Suppose E is complete and $\sum_{n=1}^\infty \|x_n\| < \infty$. Let

$$S_N := \sum_{n=1}^N x_n \in E.$$

For $M > N$:

$$\begin{aligned} \|S_N - S_M\| &= \left\| \sum_{n=N+1}^M x_n \right\| \\ &\leq \sum_{n=N+1}^M \|x_n\| \\ &\leq \sum_{n=N+1}^\infty \|x_n\| \to 0 \qquad \text{as } N \to \infty. \end{aligned}$$

Hence $\{S_N\}$ is Cauchy. As E is complete, S_N has a limit in E i.e. $\sum_{n=1}^\infty x_n$ converges in E.

$\Leftarrow$: Assume that every absolut convergent series is convergent in E. We want to see that E is complete.

Let $\{x_n\}$ be a Cauchy sequence. We want to prove that $\{x_n\}$ has a limit in E. We know that

$$\forall\, k \,\exists\, n_k : \; \|x_n - x_m\| < \frac{1}{2^k} \qquad \forall\, n, m \geq n_k.$$

We can assume that $\{n_k\}$ is an increasing sequence. Write

$$x_{n_k} = (x_{n_k} - x_{n_{k-1}}) + (x_{n_{k-1}} - x_{n_{k-2}}) + \ldots + (x_{n_1} - \underbrace{x_{n_0}}_{=0}) = \sum_{l=1}^{k} (x_{n_l} - x_{n_{l-1}}).$$

$$\sum_{l=1}^{\infty} \|x_{n_l} - x_{n_{l-1}}\| \leq \sum_{l=1}^{\infty} \frac{1}{2^l} < \infty.$$

Hence $\sum_{l=1}^{\infty} (x_{n_l} - x_{n_{l-1}})$ is absolutely convergent. By assumption

$$\sum_{l=1}^{\infty} (x_{n_l} - x_{n_{l-1}})$$

is convergent in E. Hence the partial sums is convergent. Subsequence is convergent. $\{x_{n_k}\}$ is convergent to some $x \in E$.

Exercise:

Show that the whole $\{x_n\} \to x$.

$\square$

Recall:
converging squences $(x_n)_{n=1}^{\infty}$ in $(E, \|.\|)$. $\|x_n - x\| \to 0$ for $n \to \infty$ for some $x \in E$. (Notation: $x_n \to x$ in $(E, \|.\|)$)

Remark. Assume $x_n \to x$ in $(E, \|.\|)$. Then

1) $\|x_n\| \to \|x\|$ in $(E, \|.\|)$.
2) $\sup_n \|x_n\| < \infty$.

because

1)
$$\|x_n\| \leq \|x_n - x\| + \|x\|,$$

so

$$\|x_n\| - \|x\| \leq \|x_n - x\|.$$

It follows

$$-(\|x_n\| - \|x\|) \leq \|x_n - x\|.$$

So

$$|\|x_n\| - \|x\|| \leq \|x_n - x\| \to 0, \qquad \text{for } n \to \infty.$$

Cauchy sequence in $(x_n)_{n=1}^{\infty}$ in $(E, \|.\|)$ if $\|x_n - x_m\| \to 0$ for $n, m \to \infty$.
We obtain: $(x_n)_{n=1}^{\infty}$ converges in $(E, \|.\|)$ $\Rightarrow$ $(x_n)_{n=1}^{\infty}$ Cauchy sequence in $(E, \|.\|)$. ($\not\Leftarrow$ in general). If $\Leftarrow$ then we call $(E, \|.\|)$ a Banach space.

$\sum_{n=1}^{\infty} x_m$ converges in $(E, \|.\|)$ if $\left(\sum_{n=1}^{k} x_n\right)_{k=1}^{\infty}$ converges in $(E, \|.\|)$.
$\sum_{n=1}^{\infty} x_m$ converges absolutely in $(E, \|.\|)$ if $\sum_{n=1}^{\infty} \|x_n\|$ converges $(\mathbb{R}, \|.\|)$.

2.1 Mappings between normed spaces

Definition . Let $(E_1, \|.\|_1)$, $(E_2, \|.\|_2)$ be normed spaces. $T : E_1 \to E_2$ (not necessarily linear) is called continuous at $x_0 \in E_1$, if

$$x_n \to x_0 \text{ in } (E_1, \|.\|_1) \qquad \Rightarrow \qquad T(x_n) \to T(x_0) \text{ in } (E_2, \|.\|_2).$$

T is called continuous if it is continuous at $x_0 \in E_1$ for all $x_0 \in E_1$. We say that $T : E_1 \to E_2$ is linear if

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$$

for all scalars λ_1, λ_2 and $x_1, x_2 \in E_1$.
$T : E_1 \to E_2$ linear is called bounded if there exists $M > 0$ such that

$$\|T(x)\|_2 \leq M\|x\|_1 \qquad \text{for all } x \in E_1.$$

If T is bounded linear $E_1 \to E_2$ define

$$\|T\| = \|T\|_{E_1 \to E_2} := \inf\{M \geq 0 \,|\, \|T(x)\|_2 \leq M\|x\|_1 \text{ for all } x \in E_1\}.$$

Lemma .

$$\|T\| = \sup_{\substack{x \in E_1 \\ x \neq 0}} \frac{\|T(x)\|_2}{\|x\|_1} = \sup_{\substack{x \in E_1 \\ \|x\|_1 = 1}} \|T(x)\|_2.$$

Proposition 2.12. Assume $T : E_1 \to E_2$ linear. Then all the following statements are equivalent:

(1) T continuous at $0 \in E_1$.

(2) T continuous at $x_0 \in E_1$ for some $x_0 \in E_1$.

(3) T continuous at $x_0 \in E_1$ for all $x_0 \in E_1$.

(4) T is bounded.

proof. $(1) \Rightarrow (4)$**:** Assume T is continuous at $0 \in E_1$, i.e.

$$x_n \to 0 \text{ in } (E_1, \|.\|_1) \qquad \Rightarrow \qquad T(x_n) \to T(\underbrace{0}_{\in E_1}) = \underbrace{0}_{\in E_2} \text{ in } (E_2, \|.\|_2).$$

We want to prove that T is bounded. We search a $M > 0$ such that

$$\|T(x)\|_2 \le M\|x\|_1.$$

We assume that this doesn't hold true.
For $n = 1, 2, \dots$ there exists $x_n \in E_1$ such that

$$\|T(x_n)\|_2 > n\|x_n\|_1.$$

Set for $n = 1, 2, \dots$

$$z_n := \frac{1}{n\|x_n\|_1} x_n.$$

(Note that $\|x_n\|_1 > 0$. Otherwise we would get a contradiction.)
Note

$$\|z_n\|_1 = \left\| \frac{1}{n\|x_n\|_1} \right\|_1 = \frac{1}{n\|x_n\|_1}\|x_n\|_1 = \frac{1}{n} \to 0, \qquad \text{for } n \to \infty.$$

We have $z_n \to 0$ in $(E_1, \|.\|_1)$. But

$$\|T(z_n)\|_2 = \left\| \frac{1}{n\|x_n\|_1} T(x_n) \right\|_2 = \frac{1}{n\|x_n\|_1}\|T(x_n)\|_2 > 1 \qquad \text{for all } n.$$

Hence

$$T(z_n) \not\to 0 \qquad \text{in } (E_2, \|.\|_2).$$

This is a contradiction.

$(1) \Leftarrow (4)$**:** Assume T is bounded. For some $M > 0$

$$\|T(x)\|_2 \le M\|x\|_1, \qquad \text{for all } x \in E_1.$$

We need to show that T is continuous at $0 \in E_1$, i.e.

$$x_n \to 0 \text{ in } (E_1, \|.\|_1) \qquad \Rightarrow \qquad T(x_n) \to T(0) = 0 \text{ in } (E_2, \|.\|_2).$$

From

$$\|T(x_n)\|_2 \le M\|x_n\|_1 \to 0$$

so

$$T(x_n) \to \underbrace{0}_{=T(0)} \text{ in } (E_2, \|.\|_2).$$

□

Examples. (A) $E_1 = E_2 = C([0,1])$, $\|.\|_1 = \|.\|_2 = \|.\|_\infty =: \|.\|$, i.e.

$$\|f\| := \max_{x\in[0,1]} |f(x)|.$$

$$T(f)(x) = \int_0^{1-x} \min(x,y) f(y)\,\mathrm{d}y, \qquad \text{for } f \in C([0,1]), x \in [0,1].$$

(1) $T(f) \in C([0,1])$ for $f \in C([0,1])$,

(2) T linear,

(3) T bounded,

(4) Calculate $\|T\|$.

proof. (1) Fix $f \in C([0,1])$ arbitrary and fix $x \in [0,1]$. Show that $T(f)$ is continuous at x. Consider a sequence $(x_n)_{n=1}^\infty$ in $[0,1]$ such that $x_n \to x$ in $(\mathbb{R}, |.|)$. To show $T(f)(x_n) \to T(f)(x)$ in $(\mathbb{R}, |.|)$.

$$\begin{aligned}
|T(f)(x_n) - T(f)(x)| &= \{\text{assume that } x_n \le x\} \\
&= \left|\int_0^{1-x_n} \min(x_n,y) f(y)\,\mathrm{d}y - \int_0^{1-x} \min(x,y) f(y)\,\mathrm{d}y\right| \\
&\le \left|\int_0^{1-x} (\min(x_n,y) - \min(x,y)) f(y)\,\mathrm{d}y\right| \\
&\quad + \left|\int_{1-x}^{1-x_n} \min(x_n,y) f(y)\,\mathrm{d}y\right| \\
&\le \underbrace{\int_0^{1-x} \underbrace{|\min(x_n,y) - \min(x,y)|}_{\le |x_n - x|} \underbrace{|f(y)|}_{\le \|f\|}\,\mathrm{d}y}_{\le |x_n - x| \|f\|} \\
&\quad + \underbrace{\int_{1-x}^{1-x_n} \underbrace{\min(x_n,y)}_{\le 1} \underbrace{|f(y)|}_{\le \|f\|}\,\mathrm{d}y}_{0 \le \ldots \le |x_n - x| \cdot \|f\|} \to 0, \qquad \text{as } n \to \infty
\end{aligned}$$

If $x_n > x$ we get a similar calculation. Conclusion:

$$T(f)(x_n) \to T(f)(x) \text{ in } (\mathbb{R}, |.|) \text{ as } n \to \infty.$$

(2) Fix $f_1, f_2 \in C([0,1])$ and λ_1, λ_2 scalars. Then

$$\begin{aligned}
T(\lambda_1 f_1 + \lambda_2 f_2)(x) &= \int_0^{1-x} \min(x,y) \underbrace{(\lambda_1 f_1 + \lambda_2 f_2)(y)}_{= \lambda_1 f_1(y) + \lambda_2 f_2(y)}\,\mathrm{d}y \\
&= \lambda_1 \int_0^{1-x} \min(x,y) f_1(y)\,\mathrm{d}y + \lambda_2 \int_0^{1-x} \min(x,y) f_2(y)\,\mathrm{d}y \\
&= \lambda_1 T(f_1)(x) + \lambda_2 T(f_2)(x) \qquad \text{for } x \in [0,1]
\end{aligned}$$

(3) Fix $f \in C([0,1])$. For $x \in [0,1]$

$$
\begin{aligned}
|T(f)(x)| &= |\int_0^{1-x} \underbrace{\min(x,y)}_{\geq 0} f(y)\,\mathrm{d}y| \\
&\overset{(*_1)}{\leq} \int_0^{1-x} \min(x,y) \underbrace{|f(y)|}_{\leq \|f\|} \mathrm{d}y \\
&\overset{(*_2)}{\leq} \int_0^{1-x} \min(x,y)\,\mathrm{d}y \|f\|.
\end{aligned}
$$

Clearly

$$\max_{x\in[0,1]} \int_0^{1-x} \min(x,y)\,\mathrm{d}y \leq 1.$$

This gives:

$$\|T(f)\| = \max_{x\in[0,1]} |T(f)(x)| \leq 1 \cdot \|f\|, \qquad \text{for all } f \in C([0,1]).$$

Conclusion: T is bounded with ($M = 1$)

(4) Consider the unequality above. $(*_1)$ is an equality if f has a constant sign. $(*_2)$ is an equality if f is a constant function. So we have to calculate

$$\int_0^{1-x} \min(x,y)\,\mathrm{d}y \qquad \text{for } x \in [0,1].$$

case 1: $1 - x \leq x$ i.e. $\frac{1}{2} \leq x$ and we get

$$
\begin{aligned}
\int_0^{1-x} \underbrace{\min(x,y)}_{=y} \mathrm{d}y &= \left[\frac{1}{2}y^2\right]_0^{1-x} \\
&= \frac{1}{2}(1-x)^2.
\end{aligned}
$$

case 2: $x < 1 - x$ i.e. $x < \frac{1}{2}$ and we get

$$
\begin{aligned}
\int_0^{1-x} \min(x,y)\,\mathrm{d}y &= \int_0^{x} y\,\mathrm{d}y + \int_x^{1-x} x\,\mathrm{d}y \\
&= \frac{1}{2}x^2 + x(1-2x) \\
&= x - \frac{3}{2}x^2.
\end{aligned}
$$

Claim:

$$\|T\| = \max\left(\max_{x\in[\frac{1}{2},1]} \frac{1}{2}(1-x)^2, \max_{x\in[0,\frac{1}{2}]} \left(x - \frac{3}{2}x^2\right)\right) = \ldots = \frac{1}{6}.$$

Note

- $\|T(f)\| \leq \|T\| \cdot \|f\|$ for all $f \in C([0,1])$,
- $\|T(1)\| = \|T\| \cdot \|1\|$ where $1(x) = 1$ for $x \in [0,1]$.

$\square$

(B) $E_1 = C([0,1])$ with maximumnorm, $E_2 = \mathbb{R}$ with absolut value. $T : E_1 \to E_2$ with

$$T(f) = \int_0^{\frac{1}{2}} f(y)\,\mathrm{d}y - \int_{\frac{1}{2}}^1 f(y)\,\mathrm{d}y \qquad \text{for } f \in E_1$$

$$\begin{aligned}
|T(f)| &= |\int_0^{\frac{1}{2}} f(y)\,\mathrm{d}y - \int_{\frac{1}{2}}^1 f(y)\,\mathrm{d}y| \\
&\leq |\int_0^{\frac{1}{2}} f(y)\,\mathrm{d}y| + |\int_{\frac{1}{2}}^1 f(y)\,\mathrm{d}y| \\
&\leq \int_0^{\frac{1}{2}} \underbrace{|f(y)|}_{\leq \|f\|}\,\mathrm{d}y + \int_{\frac{1}{2}}^1 \underbrace{|f(y)|}_{\leq \|f\|}\,\mathrm{d}y \\
&\leq 1\|f\|.
\end{aligned}$$

Hence T is bounded and $\|T\| \leq 1$.

$$T(f) = \int_0^1 k(y) f(y)\,\mathrm{d}y,$$

where

$$T(f_n) = \begin{cases} \text{to be completed}, & \text{falls } case \end{cases}.$$

$$T(f_n) \leq 1\left(\frac{1}{2} - \frac{1}{2n} + \frac{1}{2} - \frac{1}{2n}\right) = 1 - \frac{1}{n}, \qquad n = 1, 2, \ldots.$$

Note

$$k(y) f_n(y) \geq 0 \qquad \text{for } y \in [0,1].$$

Hence $\|T\| \leq 1 - \frac{1}{n}$ for $n = 1, 2, \ldots$. Note $\|f_n\| = 1$ for all n. Conclusion $\|T\| = 1$.
Here

$$|T(f)| \leq \underbrace{\|T\|}_{\leq 1}\|f\| \text{ for all } f \in C([0,1])$$

but

$$|T(f)| < \|T\|\|f\| \qquad \text{for all } f \in C([0,1]).$$

Statement 2.13. T_1, T_2 bounded linear mappings $(E_1, \|.\|_1) \to (E_2, \|.\|_2)$ and λ scalar. Set

$$\begin{aligned}
(T_1 + T_2)(x) &= T_1(x) + T_2(x) \qquad x \in E_1 \\
(\lambda T_1)(x) &= \lambda T_1(x) \qquad x \in E_1.
\end{aligned}$$

Claim:

(1) $T_1 + T_2$ and λT_1 are both linear mappings $(E_1, \|.\|_1) \to (E_2, \|.\|_2)$,

(2) $T_1 + T_2$ and λT_1 are both bounded mappings $(E_1, \|.\|_1) \to (E_2, \|.\|_2)$.
$B(E_1, E_2)$ denote the vector space of all bounded linear mappings $(E_1, \|.\|_1) \to (E_2, \|.\|_2)$.

(3)
$$\|T\|_{E_1 \to E_2} := \inf\{M > 0 \,|\, \|T(x)\|_2 \le M\|x\|_1 \text{ for all } x \in E_1\}$$
defines a norm in $B(E_1, E_2)$.

proof. (1) $\|T\| = 0$ implies that $\|T(x)\|_2 = 0$ for all $x \in E_1 \;\Rightarrow\; T(x) = 0 \in E_2$.

$$T = 0 \in B(E_1, E_2)$$

(2) $T \in B(E_1, E_2)$ and λ scalar.

$$\begin{aligned}
\|\lambda T\| &= \inf\{M > 0 \,|\, \|(\lambda T)(x)\|_2 \le M\|x\|_1 \text{ for all } x \in E_1\} \\
&= \inf\{M > 0 \,|\, |\lambda| \|T(x)\|_2 \le M\|x\|_1 \text{ for all } x \in E_1\} \\
&= \{\text{if } \lambda \neq 0\} \\
&= \inf\left\{ \underbrace{M}_{=|\lambda|\tilde{M}} > 0 \,\middle|\, \|T(x)\|_2 \le \underbrace{\frac{M}{|\lambda|}}_{=\tilde{M}} \|x\|_1 \text{ for all } x \in E_1 \right\} \\
&= |\lambda| \inf\left\{ \tilde{M} > 0 \,\middle|\, \|T(x)\|_2 \le \tilde{M}\|x\|_1 \text{ for all } x \in E_1 \right\} \\
&= |\lambda| \|T\|
\end{aligned}$$

(3) Set $T_1, T_2 \in B(E_1, E_2)$.

$$\begin{aligned}
\|T_1 + T_2\| &= \inf\{M > 0 \,|\, \|(T_1 + T_2)(x)\|_2 \le M\|x\|_1 \text{ for all } x \in E_1\} \\
&\le \inf\{M_1 + M_2 > 0 \,|\, \|T_1(x)\|_2 \le M_1\|x\|_1,\ \|T_2(x)\|_2 \le M_2\|x\|_1 \text{ for all } x \in E_1\} \\
&= \|T_1\| + \|T_2\|
\end{aligned}$$

□

Conclusion: $(B(E_1, B_2), \|.\|_{E_1 \to E_2})$ is a normed space.

Statement 2.14. $(B(E_1, B_2), \|.\|_{E_1 \to E_2})$ is a Banach space if $(E_2, \|.\|_2)$ is a Banach space.

proof. Assume $(T_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(B(E_1, B_2), \|.\|_{E_1\to E_2})$ where $(E_2, \|.\|_2)$ is a Banach space. Fix $x \in E_1$

$$\begin{aligned}\|T_n(x) - T_m(x)\|_2 &= \|(T_n - T_m)(x)\|_2 \\ &\leq \underbrace{\|T_n - T_m\|_{E_1\to E_2}}_{\substack{\to 0 \\ n,m\to\infty}} \cdot \|x\|_1 \to 0, \qquad n, m \to \infty.\end{aligned}$$

Hence $(T_n(x))_{n=1}^{\infty}$ is a Cauchy sequence in $(E_2, \|.\|_2)$. This is a Banach space which implies that $(T_n(x))_{n=1}^{\infty}$ converges in $(E_2, \|.\|_2)$. Call the limit $T(x) \in E_2$ for all $x \in E_1$. Show now

(1) $T : E_1 \to E_2$ is linear,

(2) T is bounded,

(3) $\|T_n - T\|_{E_1\to E_2} \to 0$ for $n \to \infty$.

(1) Observe

$$T(\lambda_1 x_1 + \lambda_2 x_2) \leftarrow T_n(\lambda_1 x_1 + \lambda_2 x_2) = \{T \text{ linear}\} = \underbrace{\underbrace{\lambda_1 \underbrace{T_n(x_1)}_{\to T(x_1)}}_{\to \lambda_1 T(x_1)} + \underbrace{\lambda_2 \underbrace{T_n(x_2)}_{\to T(x_2)}}_{\to \lambda_2 T(x_2)}}_{\to \lambda_1 T(x_1) + \lambda_2 T(x_2)}.$$

So for $n \to \infty$ it is

$$T(\lambda_1 x_1 + \lambda_2 + x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2) \qquad \text{in } (E_2, \|.\|_2).$$

(2) Fix $\varepsilon > 0$. Then there exists N such that:

$$\|T_n - T_m\|_{E_1\to E_2} < \varepsilon \qquad \text{for } n, m \geq N$$

So for $x \in E_1$

$$\|T_n(x) - T_m(x)\|_2 \leq \|T_n - T_m\|_{E_1\to E_2} \|x\|_1 < \varepsilon \|x\|_1 \qquad \text{for } n, m \geq N.$$

Let $m \to \infty$.

$$\|T_n(x) - T(x)\|_2 \leq \varepsilon \|x\|_1 \qquad \text{for } n \geq N$$

So

$$\begin{aligned}\|T(x)\|_2 &\leq \|T(x) - T_N(x)\|_2 + \|T_N(x)\|_2 \\ &\leq \varepsilon \|x\|_1 + \|T_N\|_{E_1\to E_2} \cdot \|x\|_1 \\ &= \left(\varepsilon + \|T_N\|_{E_1\to E_2}\right) \|x\|_1 \qquad \text{for } x \in E_1.\end{aligned}$$

(3) Look above and get

$$\|T_n - T\|_{E_1\to E_2} \to 0, \qquad n \to \infty.$$

□

Theorem 2.15 (Banach-Steinhaus Theorem (uniform boundedness principle)). Set $(E_1, \|.\|_1)$ Banach space, $(E_2, \|.\|_2)$ normed space and $\mathcal{F} \subset B(E_1, E_2)$. Assume

$$\sup_{T\in\mathcal{F}} \|T(x)\|_2 < \infty \qquad \text{for all } x \in E_1$$

then

$$\sup_{T\in\mathcal{F}} \|T\|_{E_1\to E_2} < \infty.$$

Remark. The implication $\Leftarrow$ is easy to prove. If $\mathcal{F}$ is a finite set, the theorem is trivial.

proof. Step 1: Assume

$$\exists\, x_0 \in E_1 \,\exists\, r > 0 \,\exists\, M > 0 : \forall x \in \overline{B(x_0, r)} \,\forall T \in \mathcal{F} : \|T(x)\|_2 \leq M.$$

We have to show that

$$\sup_{T\in\mathcal{F}} \|T\|_{E_1\to E_2} < \infty.$$

Fix $T \in \mathcal{F}$. For $\|x\|_1 \leq r$

$$\|T(x_0 + x)\|_2 \leq M.$$

Note that $x_0 + x \in \overline{B(x_0, r)}$.

$$\begin{aligned}
\|T(x)\|_2 &= \|T(x_0 + x - x_0)\|_2 \\
&= \{T \text{ linear}\} \\
&= \|T(x_0 + x) - T(x_0)\|_2 \\
&\leq \|T(x_0 + x)\|_2 + \|T(x_0)\|_2 \\
&\leq 2M.
\end{aligned}$$

For $0 \neq x \in E_1$

$$\left\| T\left(\frac{r}{\|x\|_1} x\right) \right\|_2 \leq 2M.$$

$\frac{r}{\|x\|_1} x$ has the $\|.\|_1$-norm equal to r. This implies, since T linear,

$$\frac{r}{\|x\|_1} \|T(x)\|_2 \leq 2M,$$

i.e.

$$\|T(x)\|_2 \leq \frac{2M}{r} \|x\|_1 \qquad \text{for all } 0 \neq x \in E_1.$$

We have

$$\|T\|_{E_1\to E_2} \leq \underbrace{\frac{2M}{r}}_{\substack{\text{independant} \\ \text{of } T}} < \infty$$

$$\sup_{T\in\mathcal{F}} \|T\|_{E_1\to E_2} \leq \frac{2M}{r} < \infty.$$

Step 2: Justify the assumption in step 1. This assumption is equivalent to

$$\exists\, x_0 \in E_1 \,\exists\, r > 0 \,\exists\, M > 0 : \forall\, x \in B(x_0, r) \,\forall\, T \in \mathcal{F} : \|T(x)\|_2 \leq M.$$

(Note $\overline{B(x_0, r_1)} \subset B(x_0, r) \subset B(x_0, r_2)$ for $0 < r_1 < r < r_2$).
Argue by contradiction. Assume that the assumption is false. Then it holds

$$\forall\, x_0 \in E_1 \,\forall\, r > 0 \,\forall\, M > 0 : \exists\, x \in B(x_0, r) \,\exists\, T \in \mathcal{F} : \|T(x)\|_2 > M.$$

Idea: Find a converging sequence $x_n \in E_1$, $x_n \to x$ in $(E_1, \|.\|_1)$ and a sequence $(T_n)_{n=1}^{\infty} \subset \mathcal{F}$ such that

$$\|T_n(x_n)\|_2 > n \qquad \text{for all } n, \qquad \text{and} \qquad \|T_n(x)\|_2 > n \qquad \text{for all } n.$$

We have from above $x_1 \in B(0, 1)$ and $T_1 \in \mathcal{F}$ such that

$$\|T_1(x_1)\|_2 > 1.$$

T_1 is bounded linear, hence continuous. This implies that there exists $0 < r_1 < \frac{1}{2}$ such that

$$\|T_1(x)\|_2 > 1 \qquad \text{for } x \in B(x_1, r_1)$$

and

$$\overline{B(x_1, r_1)} \subset B(0, 1).$$

□

2.2 Fixed point theory

Example. Consider

$$f(x) + 5\int_0^{1-x} \min(x, y) f(y)\,\mathrm{d}y = g(x), \qquad x \in [0, 1] \qquad (*)$$

where $g \in C([0, 1])$.
Claim: There exists an unique solution $f \in C([0, 1])$ that $(*)$.
Idea:

$$f(x) = f(x) - 5\int_0^{1-x} \min(x, y) f(y)\,\mathrm{d}y, \qquad x \in [0, 1].$$

Set for $x \in [0, 1]$

$$\tilde{T}(f)(x) = RHS(x).$$

To find a solution to $(*)$ is the same finding $f \in C([0, 1])$ such that

$$f = \tilde{T}(f).$$

Clearly $\tilde{T} : C([0, 1]) \to C([0, 1])$. (continual later).

Theorem 2.16 (Banach's fixed point theorem). $(E, \|.\|)$ Banach space. $T : E \to E$ (no assumption on linearity) is a contraction on E, i.e. there exists $c < 1$ such that

$$\|T(x) - T(\tilde{x})\| \le c\|x - \tilde{x}\| \qquad \text{for all } x, \tilde{x} \in E.$$

Then there exists a unique $\bar{x} \in E$ such that

$$\bar{x} = T(\bar{x}).$$

($\bar{x}$ is a fixed point)

proof. **Uniqueness:** Assume $T(\bar{x}) = \bar{x}$ and $T(\tilde{x}) = \tilde{x}$. Then

$$\underbrace{\|\bar{x} - \tilde{x}\|}_{\ge 0} = \|T(\bar{x}) - T(\tilde{x})\| \le \underbrace{c}_{<1}\|\bar{x} - \tilde{x}\|.$$

Thus $\|\bar{x} - \tilde{x}\| = 0$, i.e. $\bar{x} = \tilde{x}$.

Existence: Pick an arbitrary $x_0 \in E$. Set

$$x_{n+1} = T(x_n), \qquad n = 0, 1, 2, \dots.$$

Claim: $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(E, \|.\|)$. Note:

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|T(x_n) - T(x_{n-1})\| \\
&\le c\|x_n - x_{n-1}\| \\
&\le \dots \\
&\le c^n\|x_1 - x_0\|, \qquad n = 1, 2, \dots.
\end{aligned}$$

For $n > m$

$$\begin{aligned}
\|x_n - x_m\| &= \|x_n - x_{n-1} + x_{n-1} - \ldots + x_{m+1} - x_m\| \\
&\le \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \ldots + \|x_{m+1} - x_m\| \\
&\le (c^{n-1} + c^{n-2} + \ldots c^m)\|x_1 - x_0\| \\
&\le \frac{c^m}{1-c}\|x_1 - x_0\| \to 0 \qquad \text{as } n, m \to \infty.
\end{aligned}$$

Hence $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(E, \|.\|)$. $(E, \|.\|)$ is a Banach space. So $(x_n)_{n=1}^{\infty}$ converges in $(E, \|.\|)$. Call the limit $\bar{x}$.
Claim: $\bar{x}$ is a fixed point for T.

$$\begin{aligned}
\|\bar{x} - T(\bar{x})\| &= \|\bar{x} - x_{n+1} + x_{n+1} - T(\bar{x})\| \\
&\le \|\bar{x} - x_{n+1}\| + \left\|\underbrace{x_{n+1}}_{T(x_n)} - T(\bar{x})\right\| \\
&\le \underbrace{\|\bar{x} - x_{n+1}\|}_{\to 0} + c\underbrace{\|x_n - \bar{x}\|}_{\to 0} \to 0, \qquad n \to \infty
\end{aligned}$$

$\square$

Remark. (1) $x_n \to \bar{x}$ for $n \to \infty$ independend of the choice of x_0

(2) Fix $z \in E$

$$\begin{aligned} \|\bar{x} - z\| &= \|T(\bar{x}) - T(z) + T(z) - z\| \\ &\leq \|T(\bar{x}) - T(z)\| + \|T(z) - z\| \\ &\leq c\|\bar{x} - z\| + \|T(z) - z\|. \end{aligned}$$

Hence

$$\|\bar{x} - z\| \leq \frac{1}{1-c}\|T(z) - z\|.$$

Example. Consider now the example from above: $(C([0,1]), \|.\|)$ with $\|f\| = \max_{x\in[0,1]}|f(x)|$ is a Banach space! To apply Banach's fixed point theorem we need $\tilde{T}$ to be a contraction. Fix $f_1, f_2 \in C([0,1])$ and get for $x \in [0,1]$

$$\begin{aligned} |(\tilde{T}(f_1) - \tilde{T}(f_2))(x)| &= |5\int_0^{1-x} \min(x,y) f_2(y)\,\mathrm{d}y - 5\int_0^{1-x} \min(x,y) f_1(y)\,\mathrm{d}y| \\ &= |5\int_0^{1-x} \min(x,y)(f_2(y) - f_1(y))\,\mathrm{d}y| \\ &\leq 5\int_0^{1-x} \min(x,y)\underbrace{|f_2(y) - f_1(y)|}_{\leq\|f_2 - f_1\|}\,\mathrm{d}y \\ &\leq \underbrace{5\int_0^{1-x} \min(x,y)\,\mathrm{d}y}_{0\leq\ldots\leq\frac{1}{6}}\|f_2 - f_1\| \\ &\leq \frac{5}{6}\|f_2 - f_1\|. \end{aligned}$$

Hence

$$\left\|\tilde{T}(f_1) - \tilde{T}(f_2)\right\| \leq \frac{5}{6}\|f_1 - f_2\|.$$

We conclude that $\tilde{T}$ is a contraction. We can take $c = \frac{5}{6}$. By Banach's fixed point theorem $\tilde{T}$ has a unique fixed point. Finally $(*)$ has a unique solution $f \in C([0,1])$ which is the fixed point.

Theorem 2.17 (Banach's fixed point theorem (generalization)). $(E, \|.\|)$ Banach space. $T : F \to F$ where F is a closed set in E. N positive integer. Assume $T^N = \underbrace{T \circ T \circ \ldots \circ T}_{N-\text{times}}$ is a contraction on F, i.e. there exists $c > 1$ such that

$$\left\|T^N(x) - T^N(\tilde{x})\right\| \leq c\|x - \tilde{x}\|, \qquad \text{for all } x, \tilde{x} \in F.$$

Then T has unique fixed point $\bar{x}$, i.e.

$$\bar{x} = T(\bar{x}) \in F.$$

proof. $N = 1$: Fix $x_0 \in F$ and consider $(x_n)_{n=1}^{\infty}$ where $x_{n+1} = T(x_n)$ for $n = 0, 1, 2, \ldots$. There $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence and hence this converges in E since this is a Banach space. Call the limit $\bar{x}$. Note

$$\underbrace{x_n}_{\in F} \to \bar{x} \text{ in } E \text{ and } F \text{ is closed}$$

implies $\bar{x} \in F$. The rest of the argument is the same as before.

$N > 1$: By previous result we know that T^N has a unique fixed point $\bar{x} \in F$, i.e. $\bar{x} = T^N(\bar{x})$.
Claim: $\bar{x}$ is a fixed point for T.

$$\begin{aligned} \|T(\bar{x}) - \bar{x}\| &= \left\|T(T^N(\bar{x})) - T^N(\bar{x})\right\| \\ &= \left\|T^N(T(\bar{x})) - T^N(\bar{x})\right\| \\ &\leq c\|T(\bar{x}) - \bar{x}\|. \end{aligned}$$

This gives

$$\|T(\bar{x} - \bar{x})\| = 0, \qquad \text{i.e. } \bar{x} = T(\bar{x}).$$

Existence of a fixed point for T done. For the uniqueness assume $\bar{x} = T(\bar{x})$ and $\tilde{x} = T(\tilde{x})$. Then

$$\begin{aligned} \bar{x} &= T(\bar{x}) = T^2(\bar{x}) = \ldots = T^N(\bar{x}) \\ \tilde{x} &= T(\tilde{x}) = T^2(\tilde{x}) = \ldots = T^N(\tilde{x}). \end{aligned}$$

But T^N has a unique fixed point so

$$\bar{x} = \tilde{x}.$$

□

Remark. (1) $T : (0, 1] \to (0, 1]$ where $T(x) = \frac{x}{2}$. Clearly T is a contraction on $(0, 1]$ but has no fixed point. Note that $(0, 1]$ is not a closed intervall.

(2) $T : [0, \infty) \to [0, \infty)$, where $T(x) = x + \frac{1}{x}$. Clearly $[0, \infty)$ is a closed intervall in $\mathbb{R}$ but T has no fixed point.
Claim: T is not a contraction but 'close' to be a contraction.

$$|T(x) - T(\tilde{x})| < |x - \tilde{x}| \qquad \text{for } x, \tilde{x} \in [1, \infty), x \neq \tilde{x}$$

Note

$$|T(x) - T(\tilde{x})| = |\underbrace{T'(x)}_{\substack{(1-\frac{1}{t})\leq 1 \\ \text{for } t\in[1,\infty)}}||x - \tilde{x}|$$

for some t betweeen x and $\tilde{x}$.

Example. $(E, \|.\|)$ Banach space. K compact set in E and $T : K \to K$ where

$$\|T(x) - T(\bar{x})\| < \|x - \bar{x}\| \qquad \text{for all } x, \bar{x} \in K, x \neq \bar{x}.$$

Show: T has a unique fixed point in K.

Uniqueness: Assume $\bar{x} = T(\bar{x})$ and $\tilde{x} = T(\tilde{x})$ and $\bar{x} \neq \tilde{x}$ for $\bar{x}, \tilde{x} \in K$. Then

$$\|\bar{x} - \tilde{x}\| = \|T(\bar{x}) - \tilde{x}\| < \|\bar{x} - \tilde{x}\|.$$

Contradiction because then $\bar{x} = \tilde{x}$.

Existence: To show: There exists $x \in K$ such that $x = T(x)$, i.e.

$$\|T(x) - x\| = 0.$$

Set $d := \inf_{x \in K} \|T(x) - x\|$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in K such that

$$\|T(x_n) - x_n\| \to d, \qquad \text{as } n \to \infty.$$

K compact implies that there exists a subsequence $(\tilde{x}_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $(\tilde{x}_n)_{n=1}^{\infty}$ converges in K. Call the limit element $\bar{x} \in K$. We know

$$\tilde{x}_n \to \bar{x} \qquad \text{in } K$$

and

$$\|T(\tilde{x}_n) - \tilde{x}_n\| \to d.$$

Question:

$$T(\tilde{x}_n) \to T(\bar{x}) \qquad \text{in } K?$$

But since

$$\|T(x) - T(\tilde{x})\| \leq \|x - \tilde{x}\| \qquad \text{for all } x, \tilde{x} \in K$$

we have

$$\tilde{x}_n \to \bar{x} \qquad \text{in } K$$

which implies

$$T(\tilde{x}_n) \to T(\bar{x}) \text{ in } K.$$

Hence:

$$\|T(\bar{x}) - \bar{x}\| \leftarrow \|T(\tilde{x}_n) - \tilde{x}_n\| \to d, \qquad n \to \infty.$$

We obtain

$$\|T(\bar{x}) - \bar{x}\| = d.$$

Question: Is $d = 0$?
If $d > 0$ then $\bar{x} \neq T(\bar{x})$, $\bar{x}, T(\bar{x}) \in K$

$$\|T(\bar{x}) - T(T(\bar{x}))\| < \|\bar{x} - T(\bar{x})\| = d = \inf_{x \in K} \|x - T(x)\|.$$

This is a contradiction which gives $d = 0$ and so $\bar{x} = T(\bar{x})$.

Example. Consider

$$f(x) = \int_0^x k(x,y)h(y,f(y))\,\mathrm{d}y + g(x), \qquad x \in [0,1] \qquad (*),$$

where $g \in C([0,1])$, $k \in C([0,1] \times [0,1])$ and $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ continuous and satisfies:
There exists $M > 0$ such that

$$|h(x,z_1) - h(x,z_2)| \leq M|z_1 - z_2| \qquad \text{for all } x \in [0,1],\ z_1, z_2 \in \mathbb{R}.$$

Claim: $(*)$ has a unique solution $f \in C([0,1])$.
For $f \in C([0,1])$ set

$$T(f)(x) = \int_0^x k(x,y)h(y,f(y))\,\mathrm{d}y + g(x) \qquad x \in [0,1].$$

Here $T(f)(x) \in C([0,1])$.
Want to show: $T : C([0,1]) \to C([0,1])$ has a unique fixed point.
Start with the Banach space $(C([0,1]), \text{max-norm})$. Check if T is a contraction in $C([0,1])$.
Fix $f_1, f2 \in C([0,1])$

$$T(f_1)(x) - T(f_2)(x) = \int_0^x k(x,y)(h(y,f_1(y)) - h(y,f_2(y)))\,\mathrm{d}y.$$

k is continuous on the compact set $[0,1] \times [0,1]$ so

$$\sup_{(x,y)\in[0,1]\times[0,1]} |k(x,y)| =: N < \infty.$$

We obtain

$$\begin{aligned}
|(T(f_1) - T(f_2))(x)| &\leq \int_0^x \underbrace{|k(x,y)|}_{\leq N}\underbrace{h(y,f_1(y)) - h(y,f_2(y))}_{\leq M \underbrace{f_1(y) - f_2(y)}_{\leq \|f_1 - f_2\|}}\,\mathrm{d}y \\
&\leq \int_0^x NM\,\mathrm{d}y \|f_1 - f_2\| \\
&\leq NM\|f_1 - f_2\|.
\end{aligned}$$

This yields

$$\|T(f_1) - T(f_2)\| \leq NM\|f_1 - f_2\|.$$

<u>IF:</u> $NM < 1$ Then T is a contaction.
Trick: For $a > 0$ set

$$\|f\|_a = \max_{x\in[0,1]} e^{-ax}|f(x)|$$

for $f \in C([0,1])$.
Claim: $\|.\|_a$ defines a norm on $C([0,1])$. This is easy to check.

Claim: $\|.\|$ and $\|.\|_a$ are equivalent.
This follows from

$$e^{-a}\|f\| \leq \|f\|_a \leq \|f\|$$

for all $f \in C([0,1])$ (note that $\|.\|$ is the max-norm).
Claim: $(C([0,1]), \|.\|_a)$ is a Banach space.
This follows from the fact that $\|.\|$ und $\|.\|_a$ are equivalent and $(C([0,1]), \|.\|)$ is a Banach space.
Claim: T is a contraction on $(C([0,1]), \|.\|_a)$ for $a > 0$ large enough.
For $f_1, f_2 \in C([0,1])$ and $x \in [0,1]$ we have

$$\begin{aligned} |(T(f_1) - T(f_2))(x)| &\leq \int_0^x NM|(f_1 - f_2)(y)|\,\mathrm{d}y \\ &= \int_0^x NMe^{ay} \cdot \underbrace{e^{-ay}|(f_1 - f_2)(x)|}_{\leq \|f_1 - f_2\|_a}\,\mathrm{d}y \\ &\leq NM \underbrace{\int_0^x e^{ay}\,\mathrm{d}y}_{\frac{1}{a}(e^{ax}-1)} \|f_1 - f_2\|_a. \end{aligned}$$

So

$$e^{-ax}|(T(f_1) - T(f_2))(x)| \leq \frac{NM}{a}(1 - e^{-ax})\|f_1 - f_2\|_a$$

and

$$\|T(f_1) - T(f_2)\|_a \leq \frac{NM}{a}\|f_1 - f_2\|_a$$

For $a > NM$ is T a contraction on $(C([0,1]), \|.\|_a)$. Banach fixed point theorem implies that there is a unique $f \in C([0,1])$ that solves $(*)$.

Theorem 2.18. $(E, \|.\|)$ Banach space, $(Y, \|.\|)$ normed space. $T : E \times Y \to E$ where

(1) There exists a $C < 1$ such that

$$\|T(x,y) - T(\tilde{x}, y)\| \leq C\|x - \tilde{x}\| \qquad \text{for all } x, \tilde{x} \in E,\ y \in Y.$$

(2) $T_x : Y \to E$ where $T_x(y) = T(x,y)$ is continuous for all $x \in E$.

$\Rightarrow$ For every $y \in Y$ there exists a unique $g(y) \in E$ such that

$$g(y) = T(g(y), y)$$

and $g : Y \to E$ is continuous.

proof. The existence of a unique element $g(y) \in E$ for every $y \in Y$ follows from Banach's fixed point theorem.
Assume $y_n \to \tilde{y}$ in $(Y, \|.\|_*)$, i.e.

$$\|y_n - \tilde{y}\|_* \to 0, \qquad n \to \infty.$$

Remains to show

$$g(y_m) \to g(\tilde{y}) \qquad \text{in } (E(, \|.\|)).$$

$$\begin{aligned}\|g(y_n) - g(\tilde{y})\| &= \|T(g(y_n), y_n) - T(g(\tilde{y}), \tilde{y})\| \\ &\leq \underbrace{\|T(g(y_n), y_n) - T(g(\tilde{y}), y_n)\|}_{\overset{(1)}{\leq} c\|g(y_n) - g(\tilde{y})\|} + \underbrace{\|T(g(\tilde{y}), y_n) - T(g(\tilde{y}), \tilde{y})\|}_{(2) \to 0,\, n \to \infty}\end{aligned}$$

We obtain

$$\|g(y_n) - g(\tilde{y})\| \leq \frac{1}{1-c}\|T(g(\tilde{y}), y_n) - T(g(\tilde{y}), \tilde{y})\| \to 0, \qquad n \to \infty.$$

□

Theorem 2.19 (Brouwer's fixed point theorem). K compact ($=$ closed and bounded) convex subset of $\mathbb{R}^n$ and $T : K \to K$ continuous. Then T has a fixed point, i.e. there exists $\bar{x} \in K$ with

$$T(\bar{x}) = \bar{x}.$$

Remark.
- No uniqueness! Consider the case $T = \mathrm{id}_K$.
- Set $K \subseteq \mathbb{R}^n$ (in general) is convex if

$$x, \tilde{x} \in K \text{ and } \lambda \in [0,1] \qquad \Rightarrow \qquad \lambda x + (1-\lambda)\tilde{x} \in K.$$

Theorem 2.20 (Perron's theorem). A real-valued $n \times n$-Matrix with positive entries. $A = [a_{ij}]_{i,j=1,\ldots,n}$ all $a_{ij} > 0$.
$\Rightarrow$ The mapping for $x \in \mathbb{R}^n$

$$x \mapsto Ax$$

has an eigenvalue > 0 with an eigenvector with positive entries, i.e. there exists $\lambda > 0$ and $\tilde{x} \in \mathbb{R}^n$ with $A\tilde{x} = \lambda\tilde{x}$and all entries in $\tilde{x}$ are positive.

proof. We use Brouwer's fixed point theorem. Set

$$K := \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; x_k \geq 0, \sum_{i=1}^{n} x_i = 1 \right\}.$$

Claim: K is closed, bounded and a convex set in $\mathbb{R}^n$. Thus K is compact (since $K \subseteq \mathbb{R}^n$).
Set

$$T(x_1, \ldots, x_n) = \underbrace{\frac{1}{\|Ax\|_{l^1}} A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\in K} \qquad \text{for all } (x_1, \ldots, x_n) \in K$$

Claim: $T : K \to K$ is continuous.
Since

$$x_k \to x \qquad \text{in } K \text{ w.r.t. } l^1 - \text{norm}.$$

To show:

$$T(x_k) \to T(x) \qquad \text{in } K \text{ w.r.t. } l^1 - \text{norm}.$$

Set

$$\begin{aligned} x &= (x_1, x_2, \ldots, x_n) \\ x_k &= (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}) \qquad k = 1, 2, \ldots. \end{aligned}$$

Consider

$$\begin{aligned} \|T(x_k) - T(x)\|_{l^1} &= \left\| \frac{1}{\|Ax_k\|_{l^1}} Ax_k - \frac{1}{\|Ax\|_{l^1}} Ax \right\|_{l^1} \\ &\leq \left\| \frac{1}{\|Ax_k\|_{l^1}} Ax_k - \frac{1}{\|Ax\|_{l^1}} Ax_k \right\|_{l^1} + \left\| \frac{1}{\|Ax\|_{l^1}} Ax_k - \frac{1}{\|Ax\|_{l^1}} Ax \right\|_{l^1} \\ &= \left| \frac{1}{\|Ax_k\|_{l^1}} - \frac{1}{\|Ax\|_{l^1}} \right| \|Ax_k\|_{l^1} + \frac{1}{\|Ax\|_{l^1}} \|A(x - x_k)\|_{l^1} \end{aligned}$$

and

$$\begin{aligned} \|A(x - x_k)\|_{l^1} &= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij}(x_j - x_j^{(k)}) \right| \\ &\leq \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} |x_j - x_j^{(k)}| \\ &\leq \underbrace{n \cdot \max_{i,j} a_{ij}}_{<\infty} \underbrace{\|x - x_k\|_{l^1}}_{\to 0} \to 0, \qquad k \to \infty. \end{aligned}$$

So

$$Ax_k \to Ax \qquad \text{in } l^1.$$

This implies

$$\|Ax_k\|_{l^1} \to \|Ax\|_{l^1} \qquad \text{in } \mathbb{R}.$$

Brouwer's fixed point theorem implies that T has a fixed point $\bar{x} \in K$.

$$\begin{aligned} \bar{x} &= (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \\ \bar{x} &= T(\bar{x}) = \frac{1}{\|A\bar{x}\|_{l^1}} A\bar{x} \end{aligned}$$

Hence $A\bar{x} = \|A\bar{x}\|_{l^1} \bar{x}$ where $\|A\bar{x}\|_{l^1} > 0$ and $\bar{x}$ has all entries > 0. $\square$

Theorem 2.21 (Schauder's fixed point theorem). $(E, \|.\|)$ Banach space. K compact, convex set in E. $T : K \to K$ continuous.
$\Rightarrow T$ has a fixed point in K.

Example.

$$S = \left\{ f \in C((0,1)) \,\middle|\, f(0) = 0,\ f(1) = 1,\ \|f\| = \max_{x\in[0,1]} |f(x)| \leq 1 \right\}$$

$T : S \to S$ defined by

$$T(f)(x) = f(x^2), \qquad x \in [0,1].$$

$C([0,1])$ is equipped with the max-norm.
Claim:

- S is closed, bounded and convex in $C([0,1])$.
- $T : S \to S$ is continuous.
- T has no fixed point in S.

- S bounded: $f \in S$ implies $\|f\| \leq 1$.
- S closed: $f_n \to f$ in $(C([0,1]), \|.\|)$.
  To show: $f \in S$.
  Note
  $$\max_{x\in[0,1]} |f_n(x) - f(x)| \to 0, \qquad n \to \infty.$$
  This implies
  $$|f(0)| = |f_n(0) - f(0)| \to 0, \qquad n \to \infty.$$
  So $f(0) = 0$.
  $$|1 - f(1)| = \|f_n(1) - f(1)\| \to 0, \qquad n \to \infty.$$
  So $f(1) = 1$. For $x \in [0,1]$ we get
  $$\begin{aligned} |f(x)| &\leq \|f(x) - f_n(x)\| + |f_n(x)| \\ &\leq \underbrace{\|f - f_n\|}_{\to 0} + \underbrace{\|f_n\|}_{\leq 1}. \end{aligned}$$
  Conclusion $f \in S$
  $$\|f\| = \max_{x\in[0,1]} |f(x)| \leq 1.$$

- $f, \tilde{f} \in S$ and $\lambda \in [0,1]$.
  To show:
  $$\lambda f + (1-\lambda)\tilde{f} \in S.$$

Trivial since

$$(\lambda f + (1-\lambda)\tilde{f})(0) = 0$$
$$(\lambda f + (1-\lambda)\tilde{f})(1) = \lambda f(1) + (1-\lambda)\tilde{f}(1) = 1$$

and

$$\left\|\lambda f + (1-\lambda)\tilde{f}\right\| \leq |\lambda|\|f\| + |1-\lambda|\left\|\tilde{f}\right\| \leq 1.$$

We want to show that $T : S \to S$ is continuous. (obvious that $T(S) \subseteq S$)
Assume $f_n \to f$ in S in max-norm, i.e.

$$\max_{x\in[0,1]} |f_n(x) - f(x)| \to 0, \qquad n \to \infty.$$

To show: $T(f_n) \to T(f)$ in S in max-norm.

$$\begin{aligned}\|T(f_n) - T(f)\| &= \max_{x\in[0,1]} |T(f_n)(x) - T(f)(x)| \\ &= \max_{x\in[0,1]} |f_n(x^2) - f(x^2)| \\ &= \|f_n - f\| \to 0, \qquad n \to \infty.\end{aligned}$$

$T : S \to S$ has no fixed point.
If $f \in S$ is a fixed point for T then

$$f(x^2) = T(f)(x) = f(x), \qquad x \in [0,1].$$

To show: there can be no such $f \in S$.
Set $a = \inf\{x \in [0,1\,|\,]\}f(x) = \frac{1}{2} \neq \emptyset$ since f is continuous. $a \in (0,1)$ since if $a = 0$ then there exists a sequence

$$a_n \in \{x \in [0,1\,|\,]\}f(x) = \frac{1}{2}$$

such that $a_n \to a$ in $\mathbb{R}$ as $n \to \infty$. Contradiction since

$$\frac{1}{2} = f(a_n) \to f(a) = f(0) = 0$$

since f is continuous.
But $0 < a^2 < a$ and $f(a^2) = f(a) = \frac{1}{2}$. This is a contradiction.
If we believe in Schauder then we can conclude that $S \subseteq C([0,1])$ is not compact.

Theorem 2.22 (Arzela-Ascoli theorem). Assume K is a compact set in $\mathbb{R}^n$ (e.g. $K = [0,1]$ in $\mathbb{R}^n$ $n = 1$) and $S \subseteq C(K)$ where $C(K)$ is equipped with the max-norm.
$\Rightarrow$ S is relatively compact in $C(K)$ iff

(1) S uniformly bounded.

(2) S is equicontinuous.

Definition . (i) S is uniformly bounded if

$$\sup_{f \in S} \|f\| < \infty.$$

(ii) S is equicontinuous if: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - \tilde{x}| < \delta,\ x, \tilde{x} \in K \qquad \Rightarrow \qquad |f(x) - f(\tilde{x})| < \varepsilon.$$

$\delta = \delta(\varepsilon)$ must not depend on f.

S is relatively compact in $C(K)$ if for every sequence $(f_n)_{n=1}^{\infty}$ in S there exists a converging subsequence in $C(K)$.
To show: S is relatively compact in $C(K)$ iff the closure $\bar{S}$ is compact in $C(K)$.

Things to do:

(1) Proof of Schander's theorem.

(2) Proof of Arzela-Ascoli theorem.

(3) Application with Schander.

(4) Proof of Brouwer's thereom (special case).

(5) Completion of normed spaces.

For (4) wie consider the following lemma.

Lemma 2.23 (Sperner's lemma). Big triangle T

$$T = \bigcup_{a \in A} T_a.$$

$\{T_a\}_{a \in A}$ is triangle of T, i.e. for any pair T_a, $T_{\tilde{a}}$ in the triangulation

$$T_a \cup T_{\tilde{a}} = \{\emptyset \text{ or common vertrex or common side or } T_a = T_{\tilde{a}}\}.$$

$\Rightarrow$ There must exists a triangle T_a with all vertices colored differently. MISSING FIGURE!

Proof of Schander's fixed point theorem: To prove: $(E, \|.\|)$ Banach space, K compact convex set in E and $T : K \to K$ continuous.
Claim: T has a fixed point.

Lemma . Assume $(x_n)_{n=1}^{\infty}$ sequence in K such that

$$\|T(x_n) - x_n\| \to 0, \qquad n \to \infty.$$

T has a fixed point in K.

proof. Consider $(T(x_n))_{n=1}^{\infty}$ in K. K compact implies that there exists a $z \in K$ and a subsequence $(T(\tilde{x}_n))_{n=1}^{\infty}$ of $(T(x_n))_{n=1}^{\infty}$ such that

$$T(\tilde{x}_n) \to z \qquad \text{in } K \text{ as } n \to \infty.$$

Then

$$\Big\| \underbrace{T(\tilde{x}_n)}_{\to z} - \tilde{x}_n \Big\| \to 0, \qquad \text{as } n \to \infty.$$

So $\tilde{x}_n \to z$ for $n \to \infty$. But T continuous implies

$$z \leftarrow T(\tilde{x}_n) \to T(z), \qquad n \to \infty.$$

Conclusion: $z = T(z)$ so z is a fixed point. □

Lemma . K compact set in E. Let $\varepsilon > 0$. Then there exists a finite set $x_1, \dots, x_n \in K$ such that for all $x \in K$

$$\min_{k=1,\dots,N} \|x - x_k\| < \varepsilon.$$

proof. Assume there is no finite sequence $x_1, \dots, x_N$. Then there exists a sequence $(x_n)_{n=1}^{\infty}$ such that

$$\|x_k - x_l\| \geq \varepsilon, \qquad \text{for } k \neq l.$$

Clearly $(x_n)_{n=1}^{\infty}$ has no converging subsequence. This contradicts K beeing compact. □

Fix positive integer n. Apply previous lemma with $\varepsilon = \frac{1}{\varepsilon}$. then there exists a finite set $x_1, \dots, x_N$ such that

$$K \subset \bigcup_{k=1}^{N} B\left(x_k, \frac{1}{n}\right).$$

Set

$$\begin{aligned} K_n &= \{\text{set of all convex combinations of } x_1, \dots, x_N\} \\ &= \left\{ \sum_{k=1}^{N} \lambda_k x_k \,\middle|\, \lambda_k \geq 0 \text{ for all } k, \sum_{k=1}^{N} \lambda_k = 1 \right\}. \end{aligned}$$

This set is a closed and bounded set in span(K_n) finite dimensional. Also K_n is convex. (want $T_n : K_n \to K_n$ where T_n close to T).

Set $f_k(x) = \max\left(0, \frac{1}{n} - \|x - x_k\|\right)$ for $x \in K$ and $k = 1, 2, \ldots, N$.
For each $x \in K$ there exists a k such that $f_k(x) > 0$. Set

$$P_n(x) = \frac{f_1(x)x_1 + f_2(x_2) + \ldots + f_N(x_N)}{f_1(x) + f_2(x) + \ldots + f_N(x)}, \qquad x \in K.$$

P_n is a convex combination of $x_1, \ldots, x_N$ for every $x \in K$. So $P_n(x) \in K_n$ for every $x \in K$.
Claim: $\|P_n(x) - x\| < \frac{1}{n}$ for all $x \in K$.Set T_n to be defined like

$$T_n := P_n T : K_n \to K_n.$$

Here T_n is continuous since T and P_n are continuous. K_n is compact and convex in a finite dimensional space. Brouwer's fixed point theorem implies that T_n has a fixed point in K_n,i.e. there exists $x_n \in K_n$ such that

$$x_n = T_n(x_n) = P_n(x_n).$$

But then

$$\|x_n - T(x_n)\| \leq \underbrace{\Big\| x_n - \underbrace{P_n T(x_n)}_{=T_n} \Big\|}_{=0} + \underbrace{\|P_n T(x_n) - T(x_n)\|}_{<\frac{1}{n}}.$$

The first lemma above gives that T has a fixed point in K. $\square$

Example. Assume $k(x,y)$ continuous on $[0,1] \times [0,1]$ and $h(y,z)$ continuous on $[0,1] \times \mathbb{R}$ and

$$\sup_{(y,z) \in [0,1] \times \mathbb{R}} |h(y,z)| \equiv B < \infty.$$

Then there exists a solution $f \in C([0,1])$ to

$$f(x) = \int_0^1 k(x,y) h(y, f(y))\,\mathrm{d}y, \qquad x \in [0,1].$$

Method: Set $f \in C([0,1])$ and

$$T(f)(x) = \int_0^1 k(x,y) h(y, f(y))\,\mathrm{d}y, \qquad x \in [0,1] \qquad (*).$$

We want to apply (a generalized version of) Schauder's fixed point theorem. Assume $(E, \|.\|)$ is a Banach space and F closed convex subset of E. Moreover assume $T : E \to E$ continuos and $T(F)$ relatively compact in $(E, \|.\|)$. Then T has a fixed point in F.

Step 1: T as in $(*)$.
Claim: $T(C([0,1])) \subseteq C([0,1])$.
To proof this we note that k is continuous on $[0,1] \times [0,1]$ whicht is compact in $\mathbb{R}^2$.

This implies that k is uniformly continuous on $[0,1]\times[0,1]$. Fix now $\varepsilon>0$.
Then there exists $\delta=\delta(\varepsilon)>0$ such that

$$|k(x_1,y_1)-k(x_2,y_2)|<\frac{\varepsilon}{B}$$

for $|(x_1,y_1)-(x_2,y_2)|<\delta$.
Fix $f\in C([0,1])$

$$\begin{aligned}|T(f)(x_1)-T(f)(x_2)| &= |\int_0^1 (k(x_1,y)-k(x_2,y))h(y,f(y))\,\mathrm{d}y| \\ &\le \int_0^1 \underbrace{|k(x_1,y)-k(x_2,y)|}_{<\frac{\varepsilon}{B}\text{ if }|x_1-x_2|<\delta}\underbrace{|h(y,f(y))|}_{\le B}\,\mathrm{d}y<\varepsilon, \qquad \text{provided } |x_1-x_2|<\delta\end{aligned}$$

∎

Conclusion: $T(f)\in C([0,1])$ for $f\in C([0,1])$

Step 2: Choose F.
k is a continuous function on a compact set $[0,1]\times[0,1]$ implies

$$\sup_{(x,y)\in[0,1]\times[0,1]}|k(x,y)|\equiv A<\infty.$$

Hence

$$|T(f)(x)|\le AB \qquad \text{for all } f\in C([0,1]).$$

Set

$$F:=\{f\in C([0,1\,|\,)\}]\|f\|=\max_{x\in[0,1]}|f(x)|\le AB.$$

Clearly F is closed convex in $(C([0,1]),\|.\|)$ which is a Banach space.

Step 3: Claim: $T(F)$ is relatively compact.
To prove this we use the Arzela-Ascoli Theorem.

Let K be a compact set in $\mathbb{R}^n$. Let $\mathcal{S}\subset C(K)$ (realvalued continuous functions on K).
Then $\mathcal{S}$ is relatively compact in $(C(K),\|.\|_\infty)$ if

(1) $\mathcal{S}$ uniformly bounded, i.e.

$$\sup_{f\in\mathcal{S}}\|f\|<\infty.$$

(2) Equicontinuity of $f\in\mathcal{S}$, i.e.

$$\begin{aligned}&\forall\,\varepsilon>0\,\exists\,\delta=\delta(\varepsilon)>0:\ \forall\,f\in\mathcal{S}:\\ &|x_1-x_2|<\delta,\ x_1,x_2\in K \quad\Rightarrow\quad |f(x_2)-f(x_1)|<\varepsilon.\end{aligned}$$

In our example it is $\mathcal{S}=F$, $K=[0,1]$ in $\mathbb{R}$. Check that (1) and (2) in AA-Theorem are satisfied.

(1) F is uniformly bounded since

$$\sup_{f\in F}\|f\| \leq AB < \infty.$$

(2) Equicontinuity follows from calculations in Step 1.

Conclusion: $T(F)$ is relatively compact.

Step 4: Claim: $T : F \to F$ continuous
In step 1 we had $f \in F$ and $x_n \to x$ in $[0,1]$. We have shown that $T(f)(x_n) \to T(f)(x)$ in $\mathbb{R}$. So $T(f)$ is a continuous function.
Now we want to show that for $f_n \to f$ in F we've got $T(f_n) \to T(f)$ in $C([0,1])$.
Note that $h : [0,1] \times [-AB, AB] \to \mathbb{R}$ is continuous and $[0,1] \times [-AB, AB]$ is compact set in $\mathbb{R}^2$. So $h : [0,1] \times [-AB, AB] \to \mathbb{R}$ is uniformly continuous.
Fix $\varepsilon > 0$. Then there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$|h(y_1, z_1) - h(y_2, z_2)| < \frac{\varepsilon}{A}$$

for $|(y_1, z_1) - (y_2, z_2)| < \delta$. For $f_1, f_2 \in F$ with

$$\|f_1 - f_2\| < \delta.$$

We have

$$\begin{aligned}
|T(f_1)(x) - T(f_2)(x)| &= |\int_0^1 k(x,y)(h(y, f_1(y)) - h(y, f_2(y)))\,\mathrm{d}y| \\
&\leq \int_0^1 \underbrace{|k(x,y)|}_{\leq A}\underbrace{|h(y, f_1(y)) - h(y, f_2(y))|}_{<\frac{\varepsilon}{A}}\,\mathrm{d}y < \varepsilon.
\end{aligned}$$

Conclusion: $T : F \to F$ is continuous.

Step 5: Apply Schauder's fixed point theorem.

2.3 Completion of normed spaces

$(E, \|.\|)$ normed spaces. We say that $(\tilde{E}, \|.\|_*)$ is a completion of $(E, \|.\|)$ if $(\tilde{E}, \|.\|_*)$ is a normed space such that

(1) $\exists\, \Phi : E \to \tilde{E}$ injective and linear.

(2) $\|x\| = \|\Phi(x)\|_*$ for all $x \in E$.

(3) $\Phi(E)$ is dense in $\tilde{E}$.

(4) $(\tilde{E}, \|.\|_*)$ is a Banach space.

Construction:
Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be Cauchy sequences in $(E, \|.\|)$. We say that $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are equivalent, denoted by $(x_n) \sim (y_n)$, if

$$\|x_n - y_n\| \to 0, \qquad n \to \infty.$$

Set

$$\tilde{E} = \{((x_n))_N \,|\, (x_n)_{n=1}^{\infty} \text{ Cauchy sequence in } (E, \|.\|)\}.$$

Vector space structure:

$$\begin{cases} [(x_n)]_N + [(\tilde{x}_n)]_N & = [(x_n + \tilde{x}_n)]_N \\ \lambda[(x_n)]_N & = [(\lambda x)_n]_N. \end{cases}$$

Show that these definitions are well-defined, i.e. independent of the choice of representative norm

$$\|[(x_n)]_N\|_* = \lim_{n\to\infty} \|x_n\|.$$

Note

$$(x_n) \sim (y_n)$$

implies

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\|.$$

Since

$$|\|x_n\| - \|y_n\|| \le \|x_n - y_n\| \to 0, \qquad n \to \infty$$

Check that the axioms for being a norm are satisfied.
Now we have $(\tilde{E}, \|.\|_*)$ is a normed space.
Define Φ: For $x \in E$ set $\Phi(x) = [(x)_{n=1}^{\infty}]_N$ where

$$(x)_{n=1}^{\infty} = (x, x, x, \dots).$$

Claim 1 & 2: easy to prove.

Claim 3: $\Phi(E)$ dense in $(\tilde{E}, \|.\|_*)$. Fix $[(x_n)]_N \in \tilde{E}$. Consider $\Phi(x_k)$ where x_k is the element in the k-th position in the sequence $(x_1, x_2, \dots, x_n, \dots)$.

$$\|[(x_n)]_N - \Phi(x_k)\|_* = \lim_{n\to\infty} \|x_n - x_k\| \to 0 \qquad k \to \infty.$$

Since $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Claim 4: $(\tilde{E}, \|.\|_*)$ is a Banach space.
Consider a Cauchy sequence $z_n \in \tilde{E}$ such that $\|z_n - z\| \to 0$ as $n \to \infty$.
To show: There exists $z \in \tilde{E}$ such that

$$\|z_n - z\| \to 0, \qquad n \to \infty.$$

By 3 we have that $\Phi(E)$ is dense in $\tilde{E}$ so for $n = 1, 2, \ldots$ there exists $x_n \in E$, $n = 1, 2, \ldots$ such that

$$\|z_n - \Phi(z_n)\| < \frac{1}{n}, \qquad n = 1, 2, \ldots.$$

Set $z =: [(x_n)]_N$.
Need to show that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence

$$\begin{aligned}\|x_n - x_m\| &= \|\Phi(x_n) - \Phi(x_m)\|_* \\ &\leq \|\Phi(x_n) - z_n\|_* + \|z_n - z_m\|_* + \|z_m - \Phi(x_m)\|_* \\ &< \frac{1}{n} + \|z_n - z_m\| + \frac{1}{m} \to 0, \qquad n, m \to \infty.\end{aligned}$$

Conclusion: $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(E, \|.\|)$. Remains to show:

$$\|z_n - z\|_* \to 0, \qquad n \to \infty$$

$$\|z_n - z\|_* \leq \underbrace{\|z_n - \Phi(x_n)\|_*}_{<\frac{1}{n}} + \underbrace{\|\Phi(x_n) - z\|_*}_{=\lim_{n\to\infty}\|x_n - x_m\|} \to 0, \qquad n \to \infty.$$

Consider $f \in C([0, 1])$

- max-norm: $\|f\| = \max_{x\in[0,1]}|f(x)|$. Then $(C([0, 1]), \|.\|)$ is a Banach space.
- $p \geq 1$:

$$\|f\|_{L^p} = \left(\int_0^1 |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$$

defines a norm for $C([0, 1])$.

Remark. • Consider piecewise linear $f_n \in C([0, 1])$ for $n = 1, 2, \ldots$

$$f_n(x) = \begin{cases} 1, & \text{if } \frac{1}{2} \leq x \leq 1 \\ 0, & \text{if } x \leq \frac{1}{2} - \frac{1}{2n} \end{cases}$$

with

$$\|f_n - f_m\|_{L^1} \leq \frac{1}{2}\frac{1}{\min(m, n)} \to 0, \qquad n, m \to \infty.$$

So $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(C([0, 1]), \|.\|_{L^1})$ but $(f_n)_{n=1}^{\infty}$ does not converge in $(C([0, 1]), \|.\|_{L^1})$ since if $\|f_n - f\|_{L^1} \to 0$ as $n \to \infty$ and $f \in C([0, 1])$ then

$$f(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}) \\ 1, & \text{if } x \in [\frac{1}{2}, 1] \end{cases}.$$

Conclusion: $(C([0, 1]), \|.\|_{L^1})$ is not a Banach space.

- Consider:
$$f(x) = \begin{cases} 1, & \text{if } x = \frac{1}{2} \\ 0, & \text{if } x \in [0,1] \setminus \{\frac{1}{2}\} \end{cases}.$$
Then
$$\|f\|_{L^1} = 0 = \|0\|_{L^1}.$$
Compare this with the first axiom for a norm function.
- Replace $[0,1]$ with $\mathbb{R}$. For $f : \mathbb{R} \to \mathbb{R}$ set
$$\operatorname{supp}(f) = \{x \in \mathbb{R} \,|\, f(x) \neq 0\}.$$
Set
$$C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) \,|\, \operatorname{supp}(f) \text{ is compact in } \mathbb{R}\}.$$
Claim: $C_0(\mathbb{R})$ forms a vector space and for every $p \geq 1$ and $f \in C_0(\mathbb{R})$
$$\|f\|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}$$
defines a norm on $C_0(\mathbb{R})$.
Problem: $(C_0(\mathbb{R}), \|.\|_{L^p})$ for $p \geq 1$ are not Banach spaces.
$(L^1(\mathbb{R}), \|.\|_{L^1})$ is a completion of $(C_0(\mathbb{R}), \|.\|_{L^1})$.
Note $A \subset \mathbb{R}$ and A bounded. Define
$$f_A(x) \begin{cases} 1, & x \in A \\ 0, & \text{elsewhere} \end{cases}.$$
Lebesguesmeasure of $A = \|f_A\|_{L^1} = \mu(f_A)$. $A \subset \mathbb{R}$ and A unbounded
$$\mu(A) = \lim_{n \to \infty} \mu(A \cap [-n, n]).$$
We say that $A \subset \mathbb{R}$ is a 0- set if for all $\varepsilon > 0$ there exist open intervals I_n, $n = 1, 2, \ldots$ such that
  (1) $A \subseteq \bigcup_{n=1}^{\infty} I_n$,
  (2) $\sum_{n=1}^{\infty}$ lenghts of $I_m < \varepsilon$.
In particular
$$A = \mathbb{Q} = \{r_n \,|\, n = 1, 2, \ldots\} \qquad \text{is a } 0\text{-set.}$$

3 Hilbert spaces

Example. Consider $\mathbb{C}^n = \{(x_1, x_2, \dots, x_n) \,|\, x_i \in \mathbb{C}\}$ and $x, y \in \mathbb{C}^n$ with $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$. Define the inner product of x, y (scalar product)

$$\langle x\,,\, y\rangle = \sum_{i=1}^{n} x_i \bar{y}_i \in \mathbb{C}.$$

We have a map

$$\begin{aligned} \mathbb{C}^n \times \mathbb{C}^n &\to \mathbb{C} \\ (x, y) &\mapsto \langle x\,,\, y\rangle. \end{aligned}$$

This mapping has properties:

- $x \neq 0$ folgt $\langle x\,,\, x\rangle = \sum_{i=1}^{n} x_i \bar{x}_i = \sum_{i=1}^{n} |x_i|^2 > 0$
- $\langle \lambda x\,,\, y\rangle = \lambda \langle x\,,\, y\rangle$ for $x, y \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$.
- $\langle x\,,\, y\rangle = \sum_{i=1}^{n} x_i \bar{y}_i = \overline{\sum_{i=1}^{n} y_i \bar{x}_i}$ for $x, y \in \mathbb{C}^n$.
  In particular $\langle x\,,\, \lambda y\rangle = \bar{\lambda} \langle x\,,\, y\rangle$ for $\lambda \in \mathbb{C}$.
- $\langle x + y\,,\, z\rangle = \langle x\,,\, z\rangle + \langle y\,,\, z\rangle$ for $x, y, z \in \mathbb{C}^n$.

Definition . An inner product space V is a complex vector space with an inner product which is a map

$$\langle .\,,\, .\rangle : V \times V \to \mathbb{C}.$$

Satisfying

- $\langle \lambda x\,,\, y\rangle = \lambda \langle x\,,\, y\rangle$ for any $x, y \in V$, $\lambda \in \mathbb{C}$.
- $\langle x + y\,,\, z\rangle = \langle x\,,\, z\rangle + \langle y\,,\, z\rangle$ for any $x, y, z \in V$.
- $\langle x\,,\, y\rangle = \overline{\langle x\,,\, y\rangle}$ for any $x, y \in V$.
- $\langle x\,,\, x\rangle > 0$ for any $x \in V, x \neq 0$.

Can we generalize $\mathbb{C}^n$?

$$\mathbb{C}^{\mathbb{N}} \{(x_1, x_2, \dots) \,|\, x_i \in \mathbb{C}\}$$

with

$$\langle x\,,\, y\rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i.$$

This is not necessarily convergent.

Examples. (1)

$$l^2 = \left\{ (x_1, x_2, \dots) \,\middle|\, \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}.$$

We have with Cauchy Schwarz

$$\sum_{i=1}^{n} |x_i \bar{y}_i| \leq \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |y_i|^2 \right)^{\frac{1}{2}}$$

if $x \in l^2$ and $y \in l^2$ we get

$$\sum_{i=1}^{n} |x_i \bar{y}_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |y_i|^2 . \right)^{\frac{1}{2}} < \infty.$$

It follows that $\sum_{i=1}^{\infty} x_i \bar{y}_i$ converges absolutely and hence it is convergent. The following

$$\langle x \,,\, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i$$

is well-defined for vectors $x, y \in l^2$. Like for $\mathbb{C}^n$ one can easily check that $\langle . \,,\, . \rangle$ satisfies the axioms for inner products.
$(l^2, \langle . \,,\, . \rangle)$ is an inner product space.

(2) Consider $C([0,1])$ with the inner product

$$\langle f \,,\, g \rangle = \int_0^1 f(t)\overline{g(t)} \,\mathrm{d}t \qquad \forall\, f, g \in C([0,1]).$$

- $$\langle \lambda f \,,\, g \rangle = \int_0^1 \lambda f(t)\overline{g(t)} \,\mathrm{d}t = \lambda \int_0^1 f(t)\overline{g(t)} \,\mathrm{d}t = \lambda \langle f \,,\, g \rangle.$$
- $$\langle f \,,\, f \rangle = \int_0^1 f(t)\overline{f(t)} \,\mathrm{d}t = \int_0^1 |f(t)|^2 \,\mathrm{d}t > 0.$$
- ....

If we take $\mathbb{R}^3$ with the Eukledian norm on $\mathbb{R}^3$

$$\|(x_1, x_2, x_3)\| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \left(\sum_{i=1}^{3} |x_i|^2 \right)^{\frac{1}{2}} = \langle x \,,\, x \rangle^{\frac{1}{2}}.$$

Let V be an inner product space with $\langle . \,,\, . \rangle$ as the inner product. Let for $x \in V$

$$\|x\| := \langle x \,,\, x \rangle^{\frac{1}{2}}.$$

Statement 3.1. The $x \mapsto \|x\|$ with $\|.\|$ defined above is a norm.

We are going to prove the norm axioms but first we need another theorem.

Theorem 3.2 (Cauchy-Schwarz inequalitiy). For any $x, y \in V$ (inner product space)

$$|\langle x\,,\,y\rangle| \leq \langle x\,,\,x\rangle^{\frac{1}{2}}\langle y\,,\,y\rangle^{\frac{1}{2}}.$$

The equality holds iff x, y are linearly dependent.

proof. Assume x, y linearly dependent. We can assume that $x = \lambda y$ for some $\lambda \in \mathbb{C}$.

$$|\langle x\,,\,y\rangle| = |\langle \lambda y\,,\,y\rangle| = |\lambda|\langle y\,,\,y\rangle$$

and

$$\begin{aligned}\langle x\,,\,x\rangle^{\frac{1}{2}}\langle y\,,\,y\rangle^{\frac{1}{2}} &= \langle \lambda y\,,\,\lambda y\rangle^{\frac{1}{2}}\langle y\,,\,y\rangle^{\frac{1}{2}}\\ &= |\lambda|\langle y\,,\,y\rangle^{\frac{1}{2}}\langle y\,,\,y\rangle^{\frac{1}{2}}\\ &= |\lambda|\langle y\,,\,y\rangle.\end{aligned}$$

Hence

$$|\langle x\,,\,y\rangle| = \langle x\,,\,x\rangle^{\frac{1}{2}}\langle y\,,\,y\rangle^{\frac{1}{2}}.$$

Assume x, y are linearly independent. Hence $x + \lambda y \neq 0$ for any $\lambda \in \mathbb{C}$. By an axiom for inner product we get

$$0 < \langle x + \lambda y\,,\,x + \lambda y\rangle = \langle x\,,\,x\rangle + \lambda\langle y\,,\,x\rangle + \bar{\lambda}\langle x\,,\,y\rangle + |\lambda|^2\langle y\,,\,y\rangle.$$

Pick now

$$\lambda = -\frac{\langle x\,,\,y\rangle}{\langle y\,,\,y\rangle}.$$

(Note that $y \neq 0$ as x, y linearly independent.) We have

$$\begin{aligned}0 &< \langle x\,,\,x\rangle - \frac{\overbrace{\langle x\,,\,y\rangle\langle y\,,\,x\rangle}^{=|\langle x\,,\,y\rangle|^2}}{\langle y\,,\,y\rangle} - \frac{\overbrace{\overline{\langle x\,,\,y\rangle}\langle x\,,\,y\rangle}^{=|\langle x\,,\,y\rangle|^2}}{\langle y\,,\,y\rangle} + \frac{|\langle x\,,\,y\rangle|^2}{\langle y\,,\,y\rangle^2}\langle y\,,\,y\rangle\\ &= \langle x\,,\,x\rangle - \frac{|\langle x\,,\,y\rangle|^2}{\langle y\,,\,y\rangle}.\end{aligned}$$

This gives

$$\frac{|\langle x\,,\,y\rangle|^2}{\langle y\,,\,y\rangle} < \langle x\,,\,x\rangle$$

and it follows

$$|\langle x\,,\,y\rangle|^2 < \langle x\,,\,x\rangle\langle y\,,\,y\rangle.$$

$\square$

Now we can use this inequality to proof the statement above:

proof. (i) $\|x\| > 0$ for all $x \neq 0$ in V (Exercise).

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in V$, $\lambda \in \mathbb{C}$ (Exercise).

(iii) Let $x, y \in V$. Then

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y\,,\, x+y\rangle \\
&= \langle x\,,\, x\rangle + \langle x\,,\, y\rangle + \langle y\,,\, x\rangle + \langle y\,,\, y\rangle \\
&= \langle x\,,\, x\rangle + 2\operatorname{Re}(\langle x\,,\, y\rangle) + \langle y\,,\, y\rangle \\
&\leq \langle x\,,\, x\rangle + 2|\langle x\,,\, y\rangle| + \langle y\,,\, y\rangle \\
&\leq \langle x\,,\, x\rangle + 2\langle x\,,\, x\rangle^{\frac{1}{2}}\langle y\,,\, y\rangle^{\frac{1}{2}} + \langle y\,,\, y\rangle \\
&= \left(\langle x\,,\, x\rangle^{\frac{1}{2}} + \langle y\,,\, y\rangle^{\frac{1}{2}}\right)^2 .
\end{aligned}$$

So

$$\|x+y\|^2 \leq (\|x\| + \|y\|)^2 .$$

$\square$

Theorem 3.3 (The Parallelogram Law). Let $(V, \langle .\,,\, .\rangle)$ be an inner product space. Let $\|x\| = \langle x\,,\, x\rangle^{\frac{1}{2}}$. Then

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \qquad \forall\, x, y \in V.$$

Statement 3.4. l^p has inner product $\langle .\,,\, .\rangle_{l^p}$ such that

$$\|x\|_p = \sqrt{\langle x\,,\, x\rangle_{l^p}}$$

iff $p = 2$.

proof. Enough to show that $\|.\|_p$-norm does not satisfy the parallelogram law for some $x, y \in l^p$ if $p \neq 2$. Take for example $x = (1, 0, 0, \ldots)$ and $y = (0, 1, 0, \ldots)$. Note that $\|x\|_{l^p} = \|y\|_{l^p} = 1$

$$\begin{aligned}
\|x+y\|_{l^p}^2 &= \|(1,1,0,\ldots)\|_{l^p} = 2^{\frac{2}{p}} \\
\|x-y\|_{l^p}^2 &= \|(1,-1,0,\ldots)\|_{l^p} = 2^{\frac{2}{p}} \\
\|x+y\|_{l^p}^2 + \|x-y\|_{l^p}^2 &= 2\cdot 2^{\frac{2}{p}} = 2(\|x\|_{l^p}^2 + \|y\|_{l^p}^2) = 2\cdot 2.
\end{aligned}$$

$\square$

All l^p with $p \neq 2$ are not inner product spaces.

Exercise:
Show that $(C([0,1]), \|.\|_\infty)$ is not an inner product space.

Remark. Whenever a norm satisfies the parallelogram law then there exists an inner product on V such that

$$\|x\| = \langle x\,,\,x\rangle^{\frac{1}{2}}.$$

Theorem 3.5 (The Polarization Identity). Let $(V, \langle .\,,\,.\rangle)$ be an inner product space. Then

$$4\langle x\,,\,y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2.$$

Definition 3.6. Let $(V, \langle .\,,\,.\rangle)$ be an inner product space. We say that x, y in V are orthogonal if $\langle x\,,\,y\rangle = 0$ (We write $x \perp y$). Let $M \subseteq V$ Define the orthogonal complement

$$M^\perp = \{x \in V \,|\, x \perp y \text{ for any } y \in M\}.$$

Proposition 3.7. If $M \subseteq V$ then $M^\perp$ is a subspace of V.

Theorem 3.8 (Pythagorean formula). $x, y \in V$ (inner product space). Then

$$x \perp y \qquad \text{iff} \qquad \|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

3.1 Orthogonal Systems

Let $(V, \langle .\,,\,.\rangle)$ be an inner product space $\{u_n\} \subseteq V$ is called orthogonal system (with n finite or infinite) if $u_n \perp u_m$ for all $n \neq m$. It is an orthonormal system if in addition $\|u_n\| = 1$.

Examples. 1) $\{e_k\}_{k=1}^\infty \subseteq l^2$ with

$$\langle x\,,\,y\rangle = \sum_{i=1}^\infty x_i \bar{y}_i$$

with

$$e_k = (0, \ldots, 1, 0, \ldots).$$

$\Rightarrow \{e_k\}$ is an ON-system.

2) $C([-\pi, \pi])$ with

$$\langle f\,,\,g\rangle = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\,\mathrm{d}t.$$

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{-int} \,\middle|\, n \in \mathbb{Z} \right\}$$

is an orthonormal system.

Definition 3.9. Let $\{a_n \mid n \in \mathbb{N}\}$ be an orthonormal system in V. The formal series

$$\sum_{n=1}^{\infty} \langle x\,,\, a_n \rangle a_n$$

is called a fourier series of x corresponding $\{a_n \mid n \in \mathbb{N}\}$ and $\langle x\,,\, a_n \rangle$ are called fourier coefficients of x corresponding to $\{a_n \mid n \in \mathbb{N}\}$.

Theorem 3.10 (Bessel's Equality and Inequality). If $\{a_n\}$ orthonormal system in an inner product space V, then for all $x \in V$

$$\left\| x - \sum_{k=1}^{n} \langle x\,,\, a_k \rangle a_k \right\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x\,,\, a_k \rangle|^2$$

and

$$\sum_{k=1}^{\infty} |\langle x\,,\, a_k \rangle|^2 \leq \|x\|^2.$$

proof.

$$\begin{aligned}
\left\| x - \sum_{k=1}^{n} \langle x\,,\, a_k \rangle a_k \right\|^2 &= \langle x - \sum_{k=1}^{n} \langle x\,,\, a_k \rangle a_k\,,\, x - \sum_{k=1}^{n} \langle x\,,\, a_k \rangle a_k \rangle \\
&= \langle x\,,\, x \rangle - \sum_{k=1}^{n} \overline{\langle x\,,\, a_k \rangle} \langle x\,,\, a_k \rangle - \sum_{k=1}^{n} \langle x\,,\, a_k \rangle \langle a_k\,,\, x \rangle \\
&\qquad + \langle \sum_{k=1}^{n} \langle x\,,\, a_k \rangle a_k\,,\, \sum_{k=1}^{n} \langle x\,,\, a_k \rangle a_k \rangle \\
&= \|x\|^2 - \sum_{k=1}^{n} |\langle x\,,\, a_k \rangle|^2 - \sum_{k=1}^{n} |\langle x\,,\, a_k \rangle|^2 + \sum_{k=1}^{n} |\langle x\,,\, a_k \rangle|^2 \\
&= \|x\|^2 - \sum_{k=1}^{n} |\langle x\,,\, a_k \rangle|^2.
\end{aligned}$$

This gives also:

$$\sum_{k=1}^{n} |\langle x\,,\, a_k \rangle|^2 = \|x\|^2 - \left\| x - \sum_{k=1}^{n} \langle x\,,\, a_k \rangle a_k \right\| \leq \|x\|^2$$

for all $n \in \mathbb{N}$. Hence

$$\sum_{k=1}^{\infty} |\langle x\,,\, a_k \rangle|^2 \leq \|x\|^2.$$

□

Definition 3.11 (Hilbert space). A Hilbert space is an inner product space which is complete w.r.t. the norm is defined through the inner product.

Examples.
- $\mathbb{C}^n$ is an inner product space and complete w.r.t the Eukledean norm. Hence $\mathbb{C}^n$ is a Hilbert space.
- l^2 is a Banach space w.r.t.

$$\|x\|_{l^2} = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$$

  and

$$\|x\|_{l^2} = \langle x \, , \, x \rangle^{\frac{1}{2}},$$

  where

$$\langle x \, , \, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i.$$

- $(C([0,1]), \|.\|_\infty)$ is a Banach space but not an inner product space. Hence it is no Hilbert space.
- $(C([0,1]), \langle . \, , \, . \rangle)$ is an inner product space $f, g \in C([0,1])$ with

$$\langle f \, , \, g \rangle = \int_0^1 f(t)\overline{g(t)} \, \mathrm{d}t$$

  and the corresponding

$$\|f\|_2 = \langle f \, , \, f \rangle = \int_0^1 |f(t)|^2 \, \mathrm{d}t.$$

Remark. Other l^p spaces are not Hilbert spaces!!!! They are not inner product spaces.

Statement 3.12. $(C([0,1]), \langle . \, , \, . \rangle)$ is not a Hilbert space since $(C([0,1]), \|.\|_2)$ is not complete.

proof. Sketch: Show that $f_n(t)$, which is defined as a piecewise continuous function for example

$$f_n(x) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}] \\ 0, & \text{if } x \in [\frac{1}{2} + \frac{1}{n}] \\ \text{continuous}, & \text{elsewhere} \end{cases}$$

is a Cauchy sequence w.r.t $\|.\|_2$ but has no limit in $C([0,1])$. □

Consider

$$C_F = \{(x_1, x_2, \dots) \,|\, \text{only finite } x_i \neq 0\}$$

with

$$\langle x\,,\,y\rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i.$$

Show that $(C_F, \langle .\,,\,.\rangle)$ is not a Hilbert space.

Definition 3.13 (strongly and weakly convergent). A sequence $\{x_n\} \subseteq H$, where H is a Hilbert space, is called strongly convergent ($x_n \to x \in H$) if

$$\|x_n - x\| \to 0, \qquad n \to \infty.$$

(Norm induced by an inner product)
We say that x_n is weakly convergent ($x_n \rightharpoonup x$) if

$$\langle x_n\,,\,y\rangle \to \langle x\,,\,y\rangle, \qquad \forall\, y \in H.$$

Statement 3.14. $x_n \to x \qquad \Rightarrow \qquad x_n \rightharpoonup x.$

proof. Assume strong convergence for $(x_n)_{n\in\mathbb{N}}$. Then

$$\begin{aligned}
|\langle x_n\,,\,y\rangle - \langle x\,,\,y\rangle| &= |\langle x_n - x\,,\,y\rangle| \\
&\le \underbrace{\langle x_n - x\,,\,x_n - x\rangle^{\frac{1}{2}}}_{=\|x_n - x\|} \underbrace{\langle y\,,\,y\rangle^{\frac{1}{2}}}_{=\|y\|} \\
&= \underbrace{x_n - x}_{\to 0}\|y\| \to 0, \qquad n \to \infty.
\end{aligned}$$

Hence $\langle x_n\,,\,y\rangle \to \langle x\,,\,y\rangle$. □

Remark. The converse is not true in general:
Take $H = l^2$ and

$$\begin{aligned}
x_n = e_n &= (0, \ldots, 1, 0, \ldots) \\
y &= (y_1, y_2, \ldots) \in l^2.
\end{aligned}$$

We have for all $y \in H$

$$\langle e_n\,,\,y\rangle = y_n \to 0, \qquad n \to \infty$$

as

$$\|e_n - 0\|_{l^2} = \|e_n\|_{l^2} = 1.$$

Statement 3.15. $x_n \to x$ and $y_n \to y$ yields

$$\langle x_n\,,\,y_n\rangle \to \langle x\,,\,y\rangle.$$

In particular

$$x_n \to x \qquad \Rightarrow \qquad \|x_n\| \to \|x\|.$$

proof.

$$\begin{aligned}
|\langle x_n\,,\,y_n\rangle - \langle x\,,\,y\rangle| &= |\langle x_n\,,\,y_n\rangle - \langle x\,,\,y_n\rangle + \langle x\,,\,y_n\rangle - \langle x\,,\,y\rangle| \\
&= |\langle x_n - x\,,\,y_n\rangle + \langle x\,,\,y_n - y\rangle| \\
&\le |\langle x_n - x\,,\,y_n\rangle| + |\langle x\,,\,y_n - y\rangle| \\
&\le \underbrace{\|x_n - x\|}_{\to 0}\underbrace{\|y_n\|}_{<\infty} + \underbrace{\|x\|}_{<\infty}\,\underbrace{\|y_n - y\|}_{\to 0} \to 0, \qquad n \to \infty.
\end{aligned}$$

Check $\{\|y_n\|\}$ is bounded

$$\|y_n\| = \|y_n - y + y\| \le \underbrace{\|y_n - y\|}_{\to 0} + \underbrace{\|y\|}_{<\infty} \to 0, \qquad n \to \infty.$$

□

Statement 3.16. $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ yields

$$x_n \to x.$$

proof.

$$\begin{aligned}
\|x_n - x\|^2 &= \langle x_n - x\,,\,x_n - x\rangle \\
&= \underbrace{\langle x_n\,,\,x_n\rangle}_{=\|x_n\|^2} - \langle x\,,\,x_n\rangle - \langle x_n\,,\,x\rangle + \langle x\,,\,x\rangle \\
&= \|x_n\|^2 - \overline{\langle x_n\,,\,x\rangle} - \langle x_n\,,\,x\rangle + \|x\|^2 \\
&\to \|x\|^2 - \|x\|^2 - \|x\|^2 + \|x\|^2 = 0.
\end{aligned}$$

□

We have proved

$$x_n \to x \qquad \Rightarrow \qquad \{\|x_n\|\} \text{ is bounded.}$$

Theorem 3.17.

$$x_n \rightharpoonup x \qquad \Rightarrow \qquad \sup_{n\in\mathbb{N}} \|x_n\| < \infty.$$

proof. Let $x_n \rightharpoonup x$. Consider $f_n : H \to \mathbb{C}$ where

$$f_n(y) = \langle y\,,\,x_n\rangle, \qquad y \in H.$$

- f_n is a linear functional for every $n \in \mathbb{N}$.

- $\forall\, n \in \mathbb{N}$ f_n is a bounded ($\Leftrightarrow$ continuous) linear functional as if

$$y_k \overset{k\to\infty}{\to} y \qquad \Rightarrow \qquad f_n(y_k) = \langle y_k \,,\, x_n\rangle \to \langle y \,,\, x_n\rangle = f_n(y), \qquad k \to \infty.$$

- $f_n(y) \to \langle y \,,\, x\rangle$.
  $\{f_n(y)\}_n$ is a convergent sequence in $\mathbb{C}$ and hence bounded for all $y \in H$.
  Hence it exists M_y such that
  $$|f_n(y)| \leq M_y.$$
  By Banach-Steinhaus-Theorem it holds
  $$\|f_n\| \leq M \text{ for some } M > 0.$$
  We are done if we proof that $\|f_n\| = \|x_n\|$.
  $$|f_n(y)| = |\langle y \,,\, x_n\rangle| \leq \|y\|\|x_n\|, \qquad \forall\, y \in H.$$
  Hence
  $$\|f_n\| \leq \|x_n\| \qquad (1).$$
  On the other Hand we have
  $$f_n(x_n) = \langle x_n \,,\, x_n\rangle = \|x_n\|^2$$
  and thus
  $$\|f_n\| = \sup_{x\in H} \frac{|f_n(x)|}{\|x\|} \geq \frac{|f_n(x_n)|}{\|x_n\|} = \|x_n\| \qquad (2)$$
  With (1) and (2) we are finished.

□

3.2 Orthogonal decomposition in Hilbert spaces

Remember Linear Algebra. Take $\mathbb{R}^n$ and a subspace $M \subseteq \mathbb{R}^n$

$$\Rightarrow \qquad \forall\, x \in \mathbb{R}^n \qquad x = z + y, \qquad \text{where } z \in M, y \in M^\perp.$$

This can be done in a unique way

$$M = \operatorname{span}\{e_z\}$$
$$M^\perp = \operatorname{span}\{e_y\}$$

and

$$z = \operatorname{proj}_{M^\perp} x, \qquad \|x - \operatorname{proj}_M x\| = \min_{y\in M} \|x - y\|.$$

General Hilbert space case

Proposition 3.18. $M \subseteq H$, then $M^\perp$ is a closed subspace and

$$\left(M^\perp\right)^\perp = \overline{\operatorname{span} M}.$$

Statement 3.19. H Hilbert space and M-closed subspace of H and $x \in H$. Then there exists a unique $z \in M$ such that

$$\|x - z\| = \operatorname{dist}(x, M) := \inf_{y \in M} \|x - y\|.$$

(z analog of the proj$_M x$ in the other case).

Proposition 3.20. Taking $z \in M$ from the previous proposition. We have $x - z \in M^\perp$, i.e.

$$x = \underbrace{z}_{\in M} + \underbrace{(x - z)}_{\in M^\perp}.$$

Theorem 3.21 (Orthogonal Decompostion Theorem)**.** Let $(E, \langle .\,, .\rangle)$ be a Hilbert space and S be a closed subspace of E.

$$\Rightarrow \qquad E = S \oplus S^\perp$$

which means that for every $x \in E$ there exists an unique decomposition

$$x = y + z$$

with $y \in S$ and $z \in S^\perp$.

Example. Let $A \subseteq E$ where E is a Hilbert space. It follows

$$\overline{\operatorname{span} A} = \left(A^\perp\right)^\perp.$$

Note

$$A \subseteq \underbrace{\left(A^\perp\right)^\perp}_{\text{subspace of } E} \quad \Rightarrow \quad \operatorname{span} A \subseteq \underbrace{\left(A^\perp\right)^\perp}_{\text{closed}} \quad \Rightarrow \quad \overline{\operatorname{span} A} \subseteq \left(A^\perp\right)^\perp$$

$$A \subseteq \overline{\operatorname{span} A} \quad \Rightarrow \quad \overline{\operatorname{span} A}^\perp \subseteq A^\perp \quad \Rightarrow \quad \left(A^\perp\right)^\perp \subseteq \left(\overline{\operatorname{span} A}^\perp\right)^\perp.$$

Hence

$$\overline{\operatorname{span} A} \subseteq \left(A^\perp\right)^\perp \subseteq \left(\overline{\operatorname{span} A}^\perp\right)^\perp.$$

By the Orthogonal Decomposition Theorem we get

$$E = \overline{\operatorname{span} A} \oplus \overline{\operatorname{span} A}^{\perp} = \overline{\operatorname{span} A}^{\perp} \oplus \left(\overline{\operatorname{span} A}^{\perp}\right)^{\perp},$$

which implies

$$\overline{\operatorname{span} A} = \left(\overline{\operatorname{span} A}^{\perp}\right)^{\perp},$$

$$\Rightarrow \qquad \left(A^{\perp}\right)^{\perp} = \overline{\operatorname{span} A}.$$

Now we are going to prove the Orthogonal Decomposition Theorem.

proof. **Step 1:** S is a closed convex set in a Hilbert space E. This implies that

$$\forall\, x \in E\, \exists!\, y \in S: \qquad \|x-y\| \leq \|x-\tilde{y}\| \qquad \forall\, \tilde{y} \in S.$$

which means

$$\|x-y\| = \inf_{\tilde{y}\in S} \|x-\tilde{y}\|.$$

Fix $x \notin S$ with

$$\inf_{\tilde{y}\in S} \|x-\tilde{y}\| = d > 0.$$

Take a sequence $(y_n)_{n=1}^{\infty}$ in S such that

$$\|x-y_n\| \to d, \qquad n \to \infty.$$

Claim: This is a Cauchy sequence.
(use Parallelogram-law for $\|.\|$)

Step 2: S as in ODT.
Note: S must be convex.
Fix $x \in E$, choose $y \in S$ with

$$\|x-y\| \leq \|x-\tilde{y}\|, \qquad \forall\, \tilde{y} \in S.$$

Set

$$\underbrace{x}_{\in E} = \underbrace{y}_{\in S} + (x-y).$$

To show: $x-y \in S^{\perp}$. A variational argument of this is

$$\langle x-y\,,\, v\rangle = 0, \qquad \forall\, v \in S.$$

We know

$$\|x-y\|^2 \leq \|x-y+\alpha v\|^2 \qquad \forall\,\text{scalars } \alpha$$
$$\|x-y\|^2 \leq \langle x-y+\alpha v\,,\, x-y+\alpha v\rangle$$
$$= \|x-y\|^2 + \alpha\langle v\,,\, x-y\rangle + \bar{\alpha}\langle x-y\,,\, v\rangle + |\alpha|^2\|v\|^2$$

and
$$0 \le 2\,\mathrm{Re}(\alpha\langle x-y\,,\,v\rangle) + |\alpha|^2\|v\|^2.$$
Set
$$\alpha = t\overline{\langle x-y\,,\,v\rangle}, \qquad t \in \mathbb{R},$$
$$\Rightarrow \qquad 0 \le 2t|\langle x-y\,,\,v\rangle|^2 + t^2|\langle x-y\,,\,v\rangle|^2\|v\|^2.$$
Assume $\langle x-y\,,\,v\rangle \ne 0$:
We have
$$0 \le 2t + t^2\|v\|^2 \qquad \forall\, t \in \mathbb{R}$$
$$\Rightarrow \qquad -2t \le t^2\|v\|^2, \qquad \text{Let } t < 0$$
$$\Leftrightarrow \qquad 2 \le -t\|v\|^2, \qquad t < 0.$$
Let $t \to 0$, then
$$2 \le 0$$
which is a contradiction.

□

3.3 Bounded linear functionals on Hilbert spaces

Consider $(H, \langle .\,,\,.\rangle)$- Hilbert space (inner product space which is complete w.r.t. to a norm $\|x\| = \sqrt{\langle x\,,\,x\rangle}$).
Let M be a closed subspace of H.
$$\mathcal{M}^\perp = \{y \in H \mid \langle x\,,\,y\rangle = 0,\ \forall\, x \in M\}.$$
Then we know $H = M + M^\perp$, i.e. for any $x \in H$ there exists a unique $y \in M$ and $z \in M^\perp$ such that
$$x = y + z.$$

Theorem 3.22 (Riesz-Frechét represantation theorem)**.** Let $(H, \langle .\,,\,.\rangle)$ be a Hilbertspace. Let f be a bounded linear functional on H. Then there exists a unique $x_f \in H$ such that
$$f(x) = \langle x\,,\,x_f\rangle, \qquad \forall\, x \in H.$$
Moreover
$$\|f\| = \|x_f\|_H.$$

Remark. If $f : H \to \mathbb{C}$ is of the form
$$f(x) = \langle x\,,\,y\rangle, \qquad \text{for all } x \in H \text{ and some } y \in H.$$
Then f is bounded and linear (easy with Cauchy-Schwarz and properties of the scalarproduct).

proof. **Existence of x_f:** If f is a zero linear functional, i.e. $f(x) = 0$ for all $x \in H$ take $x_f = 0$. Assume now that f is not the zero functional. Consider

$$N(f) := \ker f = \{x \in H \,|\, f(x) = 0\}.$$

Then $N(f)$ is a closed subspace of H:
For $x_1, x_2 \in N(f)$, $\alpha, \beta \in \mathbb{C}$ it holds

$$f(\alpha x_1, \beta x_2) \overset{\text{lin}}{=} \alpha f(x_1) + \beta f(x_2).$$

Hence $\alpha x_1 + \beta x_2 \in N(f)$ and $N(f)$ is a subspace. $N(f)$ is closed since if $x_n \in N(f)$ with $x_n \to x$ strongly. Then

$$f(x_n) \to f(x)$$

because of bounded and hence continuous. But we know that $f(x_n) = 0$ so the limit has to be $f(x) = 0$, i.e $x \in N(f)$. $N(f)$ is a proper closed subspace. ($N(f) \neq H$). Consider now $N(f)^\perp$ which is non-zero.

- $\dim N(f)^\perp = 1$.
  Assume that $x_1 \neq 0, x_2 \neq 0 \in N(f)^\perp$. Then we have $f(x_1), f(x_2) \neq 0$. It exists $a \in \mathbb{C}$ such that
  $$f(x_1) + af(x_2) = 0.$$
  And also
  $$f(x_1 + ax_2) = 0$$
  which gives
  $$x_1 + ax_2 \in N(f) \cap N(f)^\perp = \{0\}.$$
  Hence
  $$x_1 + ax_2 = 0.$$
  Any two vectors are linearly dependent in $N(f)^\perp$ which gives
  $$\dim N(f)^\perp = 1.$$

Take $y' \in N(f)^\perp$ with $\|y'\| = 1$ and let

$$x_f = \overline{f(y')}y'.$$

We get

$$\langle x\,,\, x_f\rangle = \begin{cases} 0, & \text{if } x \in N(f) \\ \langle \lambda y'\,,\, \overline{f(y')}y'\rangle = f(y')\lambda\underbrace{\langle y'\,,\, y'\rangle}_{=1}, & \text{if } x = \lambda y' \end{cases}.$$

Furthermore

$$\langle x\,,\, x_f\rangle = \begin{cases} f(x), & \text{if } x \in N(f) \\ f(\lambda y') = f(x), & \text{if } x = \lambda y' \end{cases}.$$

Since every element in H is given by $x + \lambda y'$. For $x \in N(f)$ and $\lambda \in \mathbb{C}$. Using linearity we get

$$f(x + \lambda y') = f(x) + f(\lambda y') = \langle x\,,\, x_f\rangle + \langle \lambda y'\,,\, x_f\rangle = \langle x + \lambda y'\,,\, x_f\rangle$$

uniqueness: Assume there exists $x_1, x_2 \in H$ such that

$$f(x) = \langle x\,,\, x_1\rangle = \langle x\,,\, x_2\rangle, \qquad \forall\, x \in H.$$

We get

$$\langle x\,,\, x_1 - x_2\rangle = 0, \qquad \forall\, x \in H.$$

It holds in particular for $x = x_1 - x_2$ the following equality

$$\langle x_1 - x_2\,,\, x_1 - x_2\rangle = 0 \qquad \Rightarrow \qquad x_1 - x_2 = 0.$$

norm equality We must see that

$$\|f\| = \|x_f\|_H.$$

From remark we have

$$f(x) = \langle x\,,\, x_f\rangle \qquad \Rightarrow \qquad \|f\| \leq \|x_f\|.$$

We have for $x_f \neq 0$:

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} \geq \frac{|f(x_f)|}{\|x_f\|} = \frac{\|x_f\|^2}{\|x_f\|} = \|x_f\|.$$

This gives the desired result.

□

Example.

$$E = C_F = \{(x_1, x_2, \ldots) \,|\, \text{only finite number of } x_i \neq 0\} \subseteq l^2.$$

On C_F consider l^2-inner-product

$$\langle x\,,\, y\rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i \qquad \text{for } x, y \in C_F.$$

1. C_F is not a Hilbert space as it is not complete w.r.t

$$\|x\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}.$$

Find a Cauchy sequence that is not convergent to an element in C_F.
Find a proper closed subspace M such that $M^\perp = \{0\}$ (This would mean in particular that $C_F \neq M + M^\perp$)
Consider

$$M = \left\{(x_1, x_2, \ldots) \in C_F \,\middle|\, \sum_{k=1}^{\infty} x_k \frac{1}{k} = 0\right\},$$

$$x_f = (1, \frac{1}{2}, \frac{1}{3}, \ldots) \in l^2,$$

$$M = \ker f \cap C_F$$

where

$$f : l^2 \to \mathbb{C}$$
$$f(x) = \langle x \,,\, x_f \rangle = \sum_{k=1}^{\infty} x_k \frac{1}{k},$$

$$M^\perp = \text{all elements in } C_F \text{ which are in } (\ker f)^\perp.$$

From the proof of Riesz-Frechet theorem we have $(\ker f)^\perp$ is 1-dimensional and

$$x_f \in (\ker f)^\perp.$$

Hence

$$(\ker f)^\perp = \{\lambda x_f \,|\, \lambda \in \mathbb{C}\}.$$

We have

$$\underbrace{(\ker f)^\perp \cap C_F}_{=M^\perp} = \{0\}.$$

2. $(H, \langle . \,,\, . \rangle)$ Hilbert space and $\{u_i\} \subseteq H$ finite or infinite i. $\{u_i\}$ is an orthogonal system if

$$\langle u_i \,,\, u_j \rangle = 0, \qquad \forall\, i \neq j$$

and an orthonormal system if

$$\langle u_i \,,\, u_j \rangle = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}.$$

Proposition 3.23. Orthogonal system of non-zero vectors are linearly independent. (See linear algebra)

Having linearly independent family of vectors we can make it orthogonal with for example using Gram-Schmidt orthogonalization procedure.(See linear algebra for details).
Recall that we can write a Fourier series of x with $\langle x \,,\, u_i \rangle$ Fourier coefficients

$$x \in H \qquad \Rightarrow \qquad x = \sum_{i=1}^{\infty} \langle x \,,\, u_i \rangle u_i$$

with $\{u_i\}$-ON-system.
$C([-\pi, \pi])$ and $\{u_k\} = \left\{ \frac{1}{\sqrt{2\pi}} e^{ikt} \,\middle|\, k \in \mathbb{Z} \right\}$ equipped with the scalar product

$$\langle f \,,\, g \rangle = \int_{-\pi}^{\pi} f(t) \overline{g(t)} \,\mathrm{d}t.$$

It holds for the Fourier-series

$$\langle f\,,\,u_k\rangle = \hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(t)e^{-ikt}\,\mathrm{d}t.$$

We want to see when

$$\sum_{i=1}^{\infty}\langle x\,,\,u_i\rangle u_i$$

is convergent to x.

Definition 3.24. $\mathcal{A}_n$ ON-system is called an ON-basis for H if its span is dense in H. We say that an ON-system is complete if every $x \in H$ is

$$\sum_{i=1}^{\infty}\langle x\,,\,u_i\rangle u_i.$$

Theorem 3.25. $(H, \langle .\,,\,.\rangle)$- Hilbert space, $\{u_k\}$ is ON-system in H. The following statements are equivalent.

(1) $\{u_n\}$ is a complete ON-system.

(2) $\{u_n\}$ is an ON-basis for H.

(3) (Parsevals's Identity)

$$\|x\| = \left(\sum_{k=1}^{\infty}|\langle x\,,\,u_k\rangle|^2\right)^{\frac{1}{2}}, \qquad \forall\, x \in H.$$

(4) $\langle x\,,\,y\rangle = \sum_{k=1}^{\infty}\langle x\,,\,u_k\rangle\overline{\langle y\,,\,u_k\rangle}$ for all $x, y \in H$.

(5) $\langle x\,,\,u_k\rangle = 0$ for all $k \in \mathbb{N}$ follows $x = 0$.

proof. **(1) $\Rightarrow$ (2):** We have

$$x = \sum_{i=1}^{\infty}\langle x\,,\,u_i\rangle u_i$$

it means

$$x = \lim_{n\to\infty}\sum_{i=1}^{n}\langle x\,,\,u_i\rangle u_i \in \operatorname{span}\{u_i \,|\, i \geq 1\}.$$

This is implies that any $x \in H$ is in $\overline{\operatorname{span}\{u_i \,|\, i \geq 1\}}$, i.e. $\{u_i\}$ is ON-basis.

(2) $\Rightarrow$ (5): Let $\{u_i\}$ be a ON-basis. Assume

$$\langle x\,,\,u_k\rangle = 0, \qquad \forall\, k \in \mathbb{N}.$$

Then

$$\langle x\,,\,u\rangle = 0, \qquad \forall\, u \in \operatorname{span}\{u_k \,|\, k \geq 1\}.$$

By the property that strong convergence implies weak convergence we will have

$$\langle x\,,\,u\rangle = 0, \qquad \forall\, u \in \overline{\operatorname{span}\{u_k \,|\, k \geq 1\}} = H.$$

In particular

$$\langle x\,,\,u\rangle = 0, \qquad \text{for } u = x$$

which means

$$\langle x\,,\,x\rangle = 0 \qquad \Leftrightarrow \qquad x = 0.$$

(5) $\Rightarrow$ (1) Recall Bessel's equality. For $\{u_k\}$- ON-system then

$$\left\| x - \sum_{i=1}^{k} \langle x\,,\,u_k\rangle u_k \right\|^2 = \|x\|^2 - \sum_{i=1}^{k} |\langle x\,,\,u_k\rangle|^2$$

Assume (5), i.e.

$$\langle x\,,\,u_k\rangle = 0, \qquad \forall\, k \qquad \Rightarrow \qquad x = 0$$

We must see

$$x = \sum_{k=1}^{n} \langle x\,,\,u_k\rangle u_k \qquad \forall\, x \in H.$$

From Bessel's equality we have

$$\sum_{k=1}^{n} |\langle x\,,\,u_k\rangle|^2 = \|x\|^2 - \left\| x - \sum_{k=1}^{n} \langle x\,,\,u_k\rangle u_k \right\|^2 \leq \|x\|^2, \qquad \forall\, k \in \mathbb{N}$$

and hence $\sum_{k=1}^{n} |\langle x\,,\,u_k\rangle|^2$ is convergent. It implies that for $n > m$ we have

$$\begin{aligned} \left\| \sum_{k=1}^{n} \langle x\,,\,u_k\rangle u_k - \sum_{k=1}^{m} \langle x\,,\,u_k\rangle u_k \right\|^2 \quad &= \quad \left\| \sum_{k=m+1}^{n} \langle x\,,\,u_k\rangle u_k \right\|^2 \\ &\overset{\text{pythagorian thm}}{=} \sum_{k=m+1}^{n} |\langle x\,,\,u_k\rangle|^2 \|u_k\|^2 \\ &\to \quad 0, \qquad n, m \to 0 \qquad (*). \end{aligned}$$

Note that if $\{x_i\}$ are paarwise orthogonal it holds

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \|x\|^2.$$

From $(*)$ we know that the partial sum

$$S_n := \sum_{k=1}^{n} \langle x\,,\,u_k\rangle u_k$$

is a Cauchy sequence. As H is a Hilbert space, H is complete and hence S_n has a limit in H. Write

$$\sum_{i=1}^{\infty} \langle x , u_i \rangle u_i$$

for the limit. We must see that the limit is x. Consider

$$y := x - \sum_{i=1}^{\infty} \langle x , u_i \rangle u_i.$$

Then

$$\langle y , u_i \rangle = \langle x , u_i \rangle - \langle x , u_i \rangle = 0, \qquad \forall\, i.$$

By assumption (5) it follows

$$y = 0 \qquad \Leftrightarrow \qquad x = \sum_{i=1}^{\infty} \langle x , u_i \rangle u_i.$$

(1) $\Rightarrow$ **(3):** From Bessel's equality we have again

$$\left\| x - \sum_{i=1}^{n} \langle x , u_i \rangle u_i \right\|^2 = \|x\|^2 - \sum_{i=1}^{n} |\langle x , u_i \rangle|^2.$$

By assumption (1) the LHS tends to 0 as $n \to \infty$. On the other hand the RHS goes to

$$\to \|x\|^2 - \sum_{i=1}^{\infty} |\langle x , u_i \rangle|^2, \qquad n \to \infty.$$

This gives

$$\|x\|^2 - \sum_{i=1}^{\infty} |\langle x , u_i \rangle|^2 = 0.$$

(3) $\Rightarrow$ **(5)** trivial.

(4) $\Rightarrow$ **(5)** trivial (take $y = x$).

(1) $\Rightarrow$ **(4)** We have

$$x = \sum_{k=1}^{\infty} \langle x , u_k \rangle u_k.$$

Then

$$\langle x , y \rangle = \sum_{k=1}^{\infty} \langle x , u_k \rangle \langle u_k , y \rangle = \sum_{k=1}^{\infty} \langle x , u_k \rangle \overline{\langle y , u_k \rangle}.$$

□

Example. $L^2([-\pi,\pi])$ with

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{int} \,\middle|\, n \in \mathbb{Z} \right\}$$

is an ON-system in $L^2([-\pi,\pi])$ where

$$\langle f\,,\,g\rangle = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\,\mathrm{d}t.$$

Statement 3.26. The system above is an ON-basis for $L^2([-\pi,\pi])$. In particular, for any $f \in L^2([-\pi,\pi])$

$$f = \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{ikt}$$

convergent in the L^2-norm.

$$\|f\|_{L^2} = \left(\int_{-\pi}^{\pi} |f(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}$$

which is equivalent to

$$\left\| f - \sum_{k=-n}^{n} \hat{f}(k)e^{ikt} \right\|_{L^2}^2 \to 0.$$

Sketch of the proof:

(1) Stein-Weierstraß-Theorem. X compact set $C(X,\mathbb{C})$ continuous functions with complex values. Let $M \subseteq C(X,\mathbb{C})$ be a subspace that satisfies:

(a) it seperates points of X, i.e.

$$\forall\, x_1, x_2 \in X, x_1 \neq x_2 \,\exists\, f \in M: \qquad f(x_1) \neq f(x_2).$$

(b) M contains the constant function 1 ($f(x) = 1$ for all $x \in X$).

(c) It is closed under complex conjugation, i.e.

$$f \in M \qquad \Rightarrow \qquad \bar{f} \in M$$

and closed under product, i.e.

$$f_1, f_2 \in M \qquad \Rightarrow \qquad f_1 \cdot f_2 \in M.$$

Then M is dense in $C(X,\mathbb{C})$ w.r.t. $\|.\|_\infty$ (Continuous function by Polynomials) From this it follows

$$M = \{\text{all complex polynomials}\}$$

are dense in $C([a,b])$.

(2) $C([a,b])$ is dense in $L^2([a,b])$ w.r.t. $\|.\|_{L^2}$-norm.

We will use (1) and (2) to show that $\operatorname{span}\left\{\frac{1}{\sqrt{2\pi}}e^{int}\,\middle|\, n\in\mathbb{Z}\right\}$ is dense in $L^2([-\pi,\pi])$.

proof. Let

$$M := \operatorname{span}\left\{\frac{1}{\sqrt{2\pi}}e^{int}\,\middle|\, n\in\mathbb{Z}\right\} \subseteq \{f\in C([-\pi,\pi\,|\,)\}]f(\pi)=f(-\pi).$$

M seperates points, it contains the constant function 1 and it is closed under complex conjugation. Furthermore M is closed under taking products. With Stein-Weierstraß it follows that M is dense in

$$\{f\in C([-\pi,\pi\,|\,)\}]f(\pi)=f(-\pi).$$

By (2) we have $C([-\pi,\pi])$ is dense in $L^2([-\pi,\pi])$ w.r.t. the L^2-norm. From this one can see that even $\{f\in C([-\pi,\pi\,|\,)\}]f(\pi)=f(-\pi)$ is dense in $L^2([-\pi,\pi])$:

$$\forall\,\varepsilon>0,\,\forall\,f\in L^2\,\exists\,g\in C([-\pi,\pi]):\qquad \|f-g\|_{L^2}^2=\int_{-\pi}^{\pi}|f(t)-g(t)|^2\,\mathrm{d}t<\varepsilon.$$

Define g_ε such that it has a pike in $x=\pi-\varepsilon$ but it is continuous and is equal to g for $x<\pi-\varepsilon$. Then

$$g_\varepsilon\in C([-\pi,\pi]),\; g_\varepsilon(-\pi)=g_\varepsilon(\pi).$$

It holds

$$\begin{aligned}
\|f-g_\varepsilon\|_{L^2} &\le \underbrace{\|f-g\|_{L^2}}_{<\sqrt{\varepsilon}}+\|g-g_\varepsilon\|_{L^2}\\
&\le \sqrt{\varepsilon}+\left(\int_{\pi-\varepsilon}^{\pi}|g(t)-g_\varepsilon(t)|\,\mathrm{d}t\right)^{\frac12}\\
&\le \sqrt{\varepsilon}+\sqrt{\max_{x\in[-\pi-\varepsilon,\pi]}|g-g_\varepsilon|\varepsilon}\\
&=\sqrt{\varepsilon}+\sqrt{C}\sqrt{\varepsilon}.
\end{aligned}$$

We conclude: any $f=L^2-$limit g_n with $g_n\in C([-\pi,\pi])$ and $g_n(-\pi)=g_n(\pi)$. Each $g_n=$ $\|.\|_\infty$-norm limit of an element in $\operatorname{span}\left\{\frac{1}{\sqrt{2\pi}}e^{int}\,\middle|\, n\in\mathbb{Z}\right\}$ as

$$\|g-f\|_{L^2}\le\|g-f\|_\infty^{\frac12}(2\pi)^{\frac12}.$$

Note that

$$\left(\int_{-\pi}^{\pi}|g(t)-f(t)|^2\,\mathrm{d}t\right)^{\frac12}\le\max_{x\in[-\pi,\pi]}|g(t)-f(t)|\left(\int_{-\pi}^{\pi}\mathrm{d}t\right)^{\frac12}.$$

We get that each g_n can be approximated in the L^2-norm by elements in $\operatorname{span}\left\{\frac{1}{\sqrt{2\pi}}e^{int}\,\middle|\, n\in\mathbb{Z}\right\}$ hence

$$\operatorname{span}\left\{\frac{1}{\sqrt{2\pi}}e^{int}\,\middle|\, n\in\mathbb{Z}\right\}\subseteq L^2([-\pi,\pi]).$$

$\square$

3.4 Linear operators on Hilbert spaces

Set $(H_1, \langle . , . \rangle_1)$ and $(H_2, \langle . , . \rangle_2)$ Hilbert spaces. A bounded linear mapping $A : H_1 \to H_2$ is called bounded linear operator.
Bounded means in our case

$$\|Ax\|_2 \leq C\|x\|_1 \qquad \forall\, x \in H \text{ and some constant } C$$

Example. Set $H_1 = H_2 = L^2([0,1])$ and $K : [0,1] \times [0,1] \to \mathbb{C}$. Assume that K is continuous. Consider

$$(Af)(x) = \int_0^1 K(x,y) f(y) \,\mathrm{d}y.$$

A is linear (trivial). Show that A is bounded:

$$\begin{aligned}
\|Af\|_2 &= \int_0^1 |\int_0^1 K(x,y)f(y)\,\mathrm{d}y|^2 \,\mathrm{d}x \\
&\overset{\text{CS}}{\leq} \int_0^1 \left(\int_0^1 |K(x,y)|^2 \,\mathrm{d}y \cdot \int_0^1 |f(y)|^2 \,\mathrm{d}y \right) \mathrm{d}x \\
&= \underbrace{\int_0^1 \left(\int_0^1 |K(x,y)|^2 \,\mathrm{d}y \right) \mathrm{d}x}_{<\infty} \cdot \underbrace{\int_0^1 |f(y)|^2 \,\mathrm{d}y}_{=\|f\|_2^2}.
\end{aligned}$$

Hence

$$\|A\| \leq \left(\int_0^1 \int_0^1 |K(x,y)|^2 \,\mathrm{d}x \,\mathrm{d}y \right)^{\frac{1}{2}}.$$

Products $(A \cdot B)$ of operators $H \to H$ with $A : H \to H$ and $B : H \to H$ are defined by

$$(A \cdot B)(f) := A(Bf).$$

Statement 3.27. If A and B are bounded, then $A \cdot B$ is also bounded and

$$\|AB\| \leq \|A\|\|B\|.$$

In particular: for all $n \in \mathbb{N}$ A^n is bounded and

$$\|A^n\| \leq \|A\|^n.$$

Example. $E = L^2([0,1])$ and $f, g \in E$ with

$$\langle f\,,\, g \rangle_{L^2} = \int_0^1 f(x)\overline{g(x)} \,\mathrm{d}x, \qquad \|f\|_{L^2} = \left(\int_0^1 |f(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.$$

Set $h \in C([0,1] \times [0,1])$ and for $f \in L^2([0,1])$

$$A(f)(x) = \int_0^1 h(x,y) f(y) \,\mathrm{d}y, \qquad x \in [0,1].$$

Then

$$\|A\| \le \left(\int_0^1 \left(\int_0^1 |h(x,y)|^2 \,\mathrm{d}y \right) \mathrm{d}x \right)^{\frac{1}{2}} < \infty.$$

Example. Let $(E, \|.\|)$ be a normed space. Then there are no $A, B \in B(E,E)$ such that

$$AB - BA = I$$

where I is the identity ($I(x) = x$ for $x \in E$).

Remark. Consider $f \in E = C^\infty([0,1])$ and

$$A = \frac{\mathrm{d}}{\mathrm{d}x}, \qquad B = x.$$

Then

$$(AB - BA)(f)(x) = \frac{\mathrm{d}}{\mathrm{d}x}(x(f(x))) - x\frac{\mathrm{d}}{\mathrm{d}x} f(x) = f(x).$$

Argue by contradiction.
Assume $A, B \in B(E,E)$ with $AB - BA = I$.
Hint: Consider $A^n B - BA^n$ for $n = 1, 2, \ldots$. For $n = 2$ we have

$$\begin{aligned} A^2B - BA^2 &= A^2B - ABA + ABA - BA^2 \\ &= A(AB - BA) + (AB - BA)A \\ &= 2A. \end{aligned}$$

For $n = 3$ we have

$$\begin{aligned} A^3B - BA^3 &= A^3B - A^2BA + A^2BA - BA^3 \\ &= A^2(AB - BA) + (A^2B - BA^2)A \\ &= 3A^2. \end{aligned}$$

In general

$$A^nB - BA^n = nA^{n-1}, \qquad n = 2, 3, 4, \ldots \qquad (*)$$

Check using an induction argument. We obtain

$$n\big\|A^{n-1}\big\| = \|A^nB - BA^n\| \le \|A^nB\| + \|BA^n\| \le 2\big\|A^{n-1}\big\|\|A\|\|B\|$$

Hence

$$(2\|A\|\|B\| - n)\big\|A^{n-1}\big\| \ge 0, \qquad \forall\, n = 2, 3, \ldots.$$

We conclude that $\big\|A^{n-1}\big\| = 0$ for n large enough. Clearly the same for $\|A^n\|$. This yields $A^n = 0$ for n large enough. Repeated use of $(*)$ gives $A = 0$. This contradicts $AB - BA = I$ so the implication in the example is proven.

Recall a important theorem:

Theorem 3.28 (Riesz representation theorem). $(E, \langle . , . \rangle)$ Hilbert space $f \in B(E, \mathbb{C})$. f is bounded linear functional on E. This yields

$$\exists ! x_f \in E : \qquad f(x) = \langle x \, , \, x_f \rangle, \qquad \forall x \in E.$$

Also it holds

$$\underbrace{\|f\|}_{\text{operator norm of } f} = \underbrace{\|x_f\|}_{normofx_f \text{ in } E}.$$

Definition 3.29. $\varphi : E \times E \to \mathbb{C}$ is called:

- Bilinear, if for scalars α and β it holds

$$\varphi(\alpha x + \beta y, z) = \alpha\varphi(x, z) + \beta\varphi(y, z) \qquad \forall x, y, z \in E$$
$$\varphi(x, \alpha y + \beta z) = \bar{\alpha}\varphi(x, z) + \overline{\beta}\varphi(y, z) \qquad \forall x, y, z \in E.$$

- Bounded, if there exists $M > 0$ such that

$$|\varphi(x, y)| \leq M\|x\|\|y\|, \qquad \forall x, y \in E.$$

- Coercive, if there exists $K > 0$ such that

$$\varphi(x, x) \geq K\|x\|^2, \qquad \forall x \in E.$$

Clearly $\langle . , . \rangle$ in E is a bilinear, bounded and coercive functional in E (with $M = K = 1$). We will now introduce a Generalization of the Riesz representation theorem.

Theorem 3.30 (Lax-Milgram). $(E, \langle . , . \rangle)$ Hilbert space. Let $\varphi : E \times E \to \mathbb{C}$ be a bilinear, bounded and coercive functional. $f : E \to \mathbb{C}$ bounded linear functional in E. Then there exists an unique $x_f \in E$ such that

$$f(x) = \varphi(x, x_f), \qquad \forall x \in E.$$

proof. Step 1: $\exists !\ A \in B(E, E)$ with

$$\varphi(x, y) = \langle x \, , \, A(y) \rangle, \qquad \forall x, y \in E.$$

Step 2: A is injective and surjective.

Step 3: Apply RRT with $\tilde{x}_f = A^{-1}(x_f)$

$$\begin{aligned} f(x) &= \langle x \, , \, x_f \rangle \\ &= \langle x \, , \, A(A^{-1}(x_f)) \rangle \\ &= \varphi(x, \tilde{x}_f), \qquad \forall x \in E. \end{aligned}$$

Step 1: Fix $y \in E$ and consider for $x \in E$

$$x \overset{f_y}{\mapsto} \varphi(x, y) \in \mathbb{C}.$$

Claim: $f_y : E \to \mathbb{C}$ is a bounded linear functional.
For $x, y, z \in E$ and α, β scalars we have

$$\begin{aligned} f_y(\alpha x + \beta z) &= \varphi(\alpha x + \beta z, y) \\ &= \alpha\varphi(x, y) + \beta\varphi(z, y) \\ &= \alpha f_y(x) + \beta f_y(z). \end{aligned}$$

Hence f_y is linear. It is bounded because of

$$|f_y(x)| = |\varphi(x, y)| \leq (M\|y\|)\|x\|, \qquad \forall\, x \in E.$$

So f_y is bounded.
RRT implies $f_y(x) = \langle x\,,\, A(y)\rangle$ for all $x \in E$ for some $A(y) \in E$.
Now we have $A : E \to E$. **Claim:** $A \in B(E, E)$.
For $x, y, z \in E$ and scalars α, β we have

$$\begin{aligned} \langle x\,,\, A(\alpha y + \beta z)\rangle &= \varphi(x, \alpha y + \beta z) \\ &= \bar{\alpha}\varphi(x, y) + \bar{\beta}\varphi(x, z) \\ &= \bar{\alpha}\langle x\,,\, A(y)\rangle + \bar{\beta}\langle x\,,\, A(z)\rangle \\ &= \langle x\,,\, \alpha A(y)\rangle + \langle x\,,\, \beta A(z)\rangle. \end{aligned}$$

This is equivalent to

$$\langle x\,,\, A(\alpha y + \beta z) - \alpha A(y) - \beta A(z)\rangle = 0, \qquad x \in E.$$

This implies

$$\|A(\alpha y + \beta z) - \alpha A(y) - \beta A(z)\| = 0.$$

So

$$A(\alpha y + \beta z) = \alpha A(y) + \beta A(z) \qquad \forall\, y, z \in E \text{ and scalars } \beta, \alpha.$$

Hence, A is linear. We will now show that A is bounded:
We know because φ is continuous that for all $x, y \in E$

$$|\langle x\,,\, A(y)\rangle| = |\varphi(x, y)| \leq M\|x\|\|y\|.$$

Take $x = A(y)$ and get

$$\|A(y)\|^2 \leq M\|A(y)\|\|y\| \qquad \forall\, y \in E$$

which implies

$$\|A(y)\| \leq M\|y\| \qquad \forall\, y \in E.$$

Hence $\|A\| \leq M < \infty$.

Step 2: Note $\varphi(x,y) = \langle x\,,\,A(y)\rangle$ for alle $x, y \in E$.
Claim: A is injective, i.e.

$$A(x_1) = A(x_2) \qquad \Rightarrow \qquad x_1 = x_2.$$

φ is coercive so

$$\|x\|^2 \leq \frac{\varphi(x,x)}{K} = \frac{1}{K}\underbrace{|\langle x\,,\,A(x)\rangle|}_{>0} \leq \frac{1}{K}\|x\|\|A(x)\| \qquad \forall x \in E.$$

Hence

$$\|x\| \leq \frac{1}{K}\|A(x)\|, \qquad \forall x \in E.$$

If $A(x_1) = A(x_2)$ we have $A(x_1 - x_2) = 0 \in E$ then

$$\|x_1 - x_2\| \leq \frac{1}{K}\|A(x_1 - x_2)\| = 0.$$

We get $x_1 = x_2$.
Claim: A is surjective, i.e. the image of A is E:

$$\mathcal{R}(A) = \{A(x)\,|\,x \in E\} = E.$$

We first show that $\mathcal{R}(A)$ is a closed subspace of E.

- $\mathcal{R}(A)$ is a subspace in E since A is linear.
- $\mathcal{R}(A)$ is closed since

$$y_n \to y \qquad \text{in } (E, \|.\|) \qquad \Rightarrow y \in \mathcal{R}(A).$$

$\mathcal{R}(A)$ is linear. Take $y_1, y_2 \in \mathcal{R}(A)$ with preimages x_1, x_2 and yield

$$\alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 A(x_1) + \alpha_2 A(x_2) = A(\alpha_1 x_1 + \alpha_2 x_2).$$

So

$$\alpha_1 y_1 + \alpha_2 y_2 \in \mathcal{R}(A).$$

Assume

$$y_n \to y \qquad \text{in } (E, \|.\|).$$

For $n = 1, 2, \ldots$ there are $x_1, x_2, \ldots$ such that $y_n = A(x_n)$ for $n = 1, 2, \ldots$.
Claim: $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E since

$$\begin{aligned}
\|x_n - x_m\| &\leq \frac{1}{K}\|A(x_n - x_m)\| \\
&= \frac{1}{K}\|A(x_n) - A(x_m)\| \\
&= \frac{1}{K}\|y_n - y_m\| \to 0, \qquad n, m \to \infty
\end{aligned}$$

since $(y_n)_{n\in\mathbb{N}}$ converges.
Since $(E,\|.\|)$ is a Banach space $(x_n)_{n\in\mathbb{N}}$ converges in $(E,\|.\|)$. Call the limit $x \in E$. Hence

$$A(x_n) \to y$$

since A is bounded, continuos and linear. So $y = A(x)$ and we get $y \in \mathcal{R}(A)$.
Secondly A is surjective, i.e. $\mathcal{R}(A) = E$.
Assume that this is not true. The Orthogonal decomposition theorem gives

$$E = \mathcal{R}(A) \oplus \mathcal{R}(A)^{\perp}.$$

The first one is a closed subspace in E and the second one is not empty by assumption. Fix $z \in \mathcal{R}(A)^{\perp} \setminus \{0\}$. Note

$$\varphi(x,y) = \langle x\,,\, A(y)\rangle \qquad x,y \in E$$

With $x = y = z$ we get

$$\varphi(z,z) = \langle z\,,\, A(z)\rangle = 0$$

and

$$\varphi(z,z) \geq K\|z\|^2 \geq 0 \qquad \Rightarrow\ z = 0.$$

This is a contradiction.
The Conclusion is

$$\mathcal{R}(A)^{\perp} = \{0\} \qquad \Rightarrow \qquad \mathcal{R}(A) = E.$$

We have $\varphi(x,y) = \langle x\,,\, A(y)\rangle$ for all $x,y \in E$ and $A \in B(E,E)$ surjective.

Step 3: see above.

□

3.5 Adjoint operator

$(E, \langle .\,,\,.\rangle)$ Hilbert space and $A \in B(E,E)$ with adjoint A^*, i.e.

$$\langle A(x)\,,\, y\rangle = \langle x\,,\, A^*(y)\rangle, \qquad \forall\, x,y \in E.$$

Fix $y \in E$ and consider

$$x \overset{f_y}{\mapsto} \langle A(x)\,,\, y\rangle \in \mathbb{C}.$$

Claim: f_y is a bounded linear functional on E

- linear since A is linear.
- bounded since A is bounded with

$$|f_y(x)| \leq (\|A\|\|y\|)\|x\|, \qquad x \in E.$$

RRT implies

$$f_y(x) = \langle x\,,\,A^*(y)\rangle, \qquad x \in E.$$

We have $A^* : E \to E$ such that

$$\langle A(x)\,,\,y\rangle = \langle x\,,\,A^*(y)\rangle, \qquad \forall\, x, y \in E.$$

Proposition 3.31. $A \in B(E,E)$. Then $A^* \in B(E,E)$ and $\|A^*\| = \|A\|$.

proof. A^* linear:

$$\langle x\,,\,A^*(\alpha y + \beta z)\rangle = \langle x\,,\,\alpha A^*(y) + \beta A^*(z)\rangle \qquad \forall\, x, y \in E.$$

A^* bounded:
Take $x = A^*(y)$ and get

$$\begin{aligned} \|A^*(y)\|^2 &= |\langle A(A^*(y))\,,\,y\rangle| \\ &\leq \|A(A^*(y))\|\|y\| \\ &\leq \|A\|\|A^*(y)\|\|y\|, \qquad y \in E. \end{aligned}$$

We get

$$\|A^*(y)\| \leq \|A\|\|y\|, \qquad y \in E.$$

Conclucion: $A^* \in B(E,E)$. We also get

$$\|A^*\| \leq \|A\|.$$

But we also know that $A^{**} = A$ since

$$\begin{aligned} \langle x\,,\,A^{**}(y)\rangle &= \langle A^*(x)\,,\,y\rangle \\ &= \overline{\langle y\,,\,A^*(x)\rangle} \\ &= \overline{\langle A(y)\,,\,x\rangle} \\ &= \langle x\,,\,A(y)\rangle, \qquad x, y \in E. \end{aligned}$$

So

$$\|A\| = \|A^{**}\| \leq \|A^*\|$$

which impllies

$$\|A\| = \|A^*\|.$$

□

Remark. $A, B \in B(E,E)$ then

$$\begin{aligned} (A+B)^* &= A^* + B^* \\ (AB)^* &= B^*A^* \\ (\alpha A)^* &= \bar{\alpha}A^* \\ A^{**} &= A \\ I^* &= I. \end{aligned}$$

Example. Continuity of the example above: For $f \in L^2([0,1])$ consider

$$A(f)(x) = \int_0^1 h(x,y) f(y) \,\mathrm{d}y, \qquad x \in [0,1].$$

For $g \in L^2([0,1])$ it holds

$$\begin{aligned}
\langle A(f) \,,\, g \rangle_{L^2} &= \int_0^1 A(f)(x) \overline{g(x)} \,\mathrm{d}x \\
&= \int_0^1 \int_0^1 h(x,y) f(y) \,\mathrm{d}x \overline{g(x)} \,\mathrm{d}x \\
&= \int_0^1 f(y) \cdot \int_0^1 h(x,y) \overline{g(x)} \,\mathrm{d}x \,\mathrm{d}y \\
&= \int_0^1 f(y) \cdot \overline{\int_0^1 \overline{h(x,y)} g(x) \,\mathrm{d}x} \,\mathrm{d}y \\
&= \langle f \,,\, A^*(g) \rangle_{L^2}.
\end{aligned}$$

This gives us

$$A^*(f)(x) = \int_0^1 \overline{h(y,x)} f(y) \,\mathrm{d}y, \qquad x \in [0,1].$$

Example. $A \in B(E,E)$. It follows

$$\mathcal{R}(A)^\perp = N(A^*) = \{x \in E \mid A^*(x) = 0\}$$

since $x \in \mathcal{R}(A)^\perp$. It is equivalent that

$$\langle x \,,\, A(y) \rangle = 0, \qquad \forall\, y \in E$$

$$\Leftrightarrow \qquad \langle A^*(x) \,,\, y \rangle = 0, \qquad \forall\, y \in E$$

$$\Rightarrow \qquad A^*(x) = 0 \qquad \Leftrightarrow \qquad x \in N(A^*).$$

We get

$$N(A^*)^\perp = \overline{\mathcal{R}(A)}$$

since

$$N(A^*)^\perp = \left(R(A)^\perp\right)^\perp = \overline{\operatorname{span}(\mathcal{R}(A))} = \overline{\mathcal{R}(A)}.$$

Remark. $A \in B(E,E)$ is called self adjoint if $A^* = A$.

For $A \in B(E,E)$ we have

$$\|A\| = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |\langle A(x) \,,\, y \rangle|$$

since

$$\|\langle A(x) \,,\, y \rangle\| \leq \underbrace{\|A(x)\|}_{\leq \|A\|\|x\|} \leq \|A\|, \qquad \text{for } \|x\| = \|y\| = 1.$$

If $A(x) = 0$ for all $x \in E$ then $\|A\| = 0$ and also

$$\sup_{\substack{\|x\|=1\\ \|y\|=1}} |\langle A(x)\,,\, y\rangle| = 0.$$

For x with $A(x) \neq 0$ then it is

$$A\left(\frac{1}{\|x\|}x\right) \neq 0.$$

For such an x with $\|x\| = 1$ we have

$$|\langle A(x)\,,\, \frac{1}{\|A(x)\|}A(x)\rangle| = \frac{1}{\|A(x)\|}\|A(x)\|^2 = \|A(x)\|$$

and

$$\|A\| \leq \sup_{\|x\|=1} \|A(x)\| \leq \sup_{\substack{\|x\|=1\\ \|y\|=1}} |\langle A(x)\,,\, y\rangle| \leq \|A\|.$$

Proposition 3.32. Let $A \in B(E, E)$ be self-adjoint. Then

$$\|A\| = \sup_{\|x\|=1} |\langle A(x)\,,\, x\rangle|.$$

proof. Set

$$M = \sup_{\|x\|=1} |\langle A(x)\,,\, x\rangle|.$$

For $\|x\| = 1$ we have

$$|\langle A(x)\,,\, x\rangle| \leq \|A(x)\|\|x\| \leq \|A\|.$$

Furthermore

$$M \leq \|A\|.$$

It remains to prove: $\|A\| \leq M$.
For $x, z \in E$ consider:

$$\begin{aligned}\langle A(x+z)\,,\, x+z\rangle - \langle A(x-z)\,,\, x-z\rangle &= 2\langle A(x)\,,\, z\rangle + 2\langle A(z)\,,\, x\rangle \\ &= 2\left(\langle A(x)\,,\, z\rangle + \langle z\,,\, A^*(x)\rangle\right) \\ &= 2(\langle A(x)\,,\, z\rangle + \langle z\,,\, A(x)\rangle) \\ &= 4\,\mathrm{Re}(\langle A(x)\,,\, z\rangle).\end{aligned}$$

Assume now $A(x) \neq 0$ and set

$$z = \frac{1}{\|A(x)\|}A(x).$$

Hence

$$\|A(x)\| = \frac{1}{4}\left(\langle A(x + \frac{1}{\|A(x)\|}A(x))\,,\, x + \frac{1}{\|A(x)\|}A(x)\rangle - \langle A(x - \frac{1}{\|A(x)\|}A(x))\,,\, x - \frac{1}{\|A(x)\|}A(x)\rangle\right).$$

Note

$$|\langle A(y)\,,\,y\rangle| = \|y\|^2 |\langle A(\frac{1}{\|y\|}y)\,,\,\frac{1}{\|y\|}y\rangle| \leq M\|y\|^2.$$

We now obtain

$$\begin{aligned}
\|A(x)\| &\leq \frac{1}{4}\left(M\left\|x+\frac{1}{\|A(x)\|}A(x)\right\|^2 + M\left\|x-\frac{1}{\|A(x)\|}A(x)\right\|^2\right) \\
&= \frac{M}{4}2\left(\|x\|^2 + \left\|\frac{1}{\|A(x)\|}A(x)\right\|^2\right) \\
&= \frac{M}{2}(\|x\|^2+1).
\end{aligned}$$

So

$$\|A\| = \sup_{\|x\|=1}\|A(x)\| \leq M$$

and this yields

$$\|A\| = M.$$

□

Definition 3.33 (compact). If $A : E \to E$ is linear, then we say that A is compact if for all bounded sequences $(x_n)_{n=1}^\infty$ in E, $(A(x_n))_{n=1}^\infty$ has a convergent subsequence in E.

Lemma 3.34. A is compact and linear $\Rightarrow$ A is bounded.

proof. If A is not bounded then there exists a sequence $(y_n)_{n=1}^\infty$ in E such that

$$\|A(y_n)\| \geq n\|y_n\|, \qquad \text{for } n = 1, 2, \ldots.$$

Set $x_n = \frac{y_n}{\|y_n\|}$ for $n = 1, 2, \ldots$. Here $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\|A(x_n)\| = \left\|A\left(\frac{1}{\|y_n\|}y_n\right)\right\| = \frac{1}{\|y_n\|}\|A(y_n)\| > n, \qquad \forall\, n \in \mathbb{N}.$$

$(A(x_n))_{n=1}^\infty$ has no converging subsequence since $\|A(x_n)\| \to \infty$ for $n \to \infty$. □

Remark.

- $A \in B(E,E)$ and $F \subset E$ where F is bounded. Then

  $$A(F) = \{A(x)\,|\,x \in F\}$$

  is bounded.
- $A \in B(E,E)$ compact and $F \subset E$, F bounded. Then $\overline{A(F)}$ is compact.

Lemma 3.35. A, B compact linear operators $E \to E$ and α and β scalars. Then $\alpha A + \beta B$ is compact.

proof. Fix an arbitrary bounded sequence $(x_n)_{n=1}^{\infty}$ in E. Since A is compact there exists a converging subsequence $(A(x_{n_k}))_{k=1}^{\infty}$ of $(A(x_{n_k}))_{k=1}^{\infty}$.
Clearly $(\alpha A(x_n))_{n=1}^{\infty}$ converges in E.
Since B is compact there exists a converging subsequence $(B(x_{n_k}))_{k=1}^{\infty}$ of $(B(x_n))_{n=1}^{\infty}$.
Clearly $(\beta B(x_{n_k}))_{k=1}^{\infty}$ converges in E. Hence

$$(\alpha A(x_{n_k}) + \beta B(x_{n_k}))_{k=1}^{\infty} = ((\alpha A + \beta B)(x_{n_k}))_{k=1}^{\infty}$$

converges in E. □

Set

$$K(E, E) := \text{set of all compact linear mappings } E \to E.$$

We have $K(E, E)$ is a subspace in $(B(E, E), \|.\|_{E \to E})$.

Proposition 3.36. $K(E, E)$ is a <u>closed</u> subspace in $(B(E, E), \|.\|_{E \to E})$.

Before the proof we note:

1. Assume $(E, \langle . \, , \, . \rangle)$ to be a Hilbert space and $A \in B(E, E)$.

$$\begin{aligned} x_n \to x \text{ in } E \quad &\Rightarrow \quad A(x_n) \to A(x) \text{ in } E \\ x_n \rightharpoonup x \text{ in } E \quad &\Rightarrow \quad A(x_n) \rightharpoonup A(x) \text{ in } E \end{aligned}$$

   since for $y \in E$ we have

$$\langle A(x_n) \, , \, y \rangle = \langle x_n \, , \, A^*(y) \rangle \overset{n \to \infty}{\to} \langle x \, , \, A^*(y) \rangle = \langle A(x) \, , \, y \rangle.$$

2. $A \in K(E, E)$ and $x_n \rightharpoonup x$ in E

$$\Rightarrow A(x_n) \to A(x) \qquad \text{in } E.$$

3. $A \in B(E, E)$ finite-rank operator, i.e.

$$\dim \mathcal{R}(A) < \infty \qquad \Rightarrow \qquad A \in K(E, E)$$

   since: Let $e_1, e_2, \dots, e_N$ be an ON-basis for $\mathcal{R}(A)$ with $N = \dim(\mathcal{R}(A))$. We have

$$A(x) = \langle A(x) \, , \, e_1 \rangle e_1 + \ldots + \langle A(x) \, , \, e_N \rangle e_N.$$

   Fix an arbitrary bounded sequence $(x_n)_{n=1}^{\infty}$ in E. A is bounded which implies that $(A(x_n))_{n=1}^{\infty}$ is a bounded sequence. Furthermore

$$(\langle A(x_n) \, , \, e_1 \rangle)_{n=1}^{\infty}$$

is a bounded sequence in $\mathbb{C}$. Bolzano Weierstrass theorem implies that $(\langle A(x_n)\,,\,e_1\rangle)_{n=1}^{\infty}$ has a converging subsequence $(\langle A(x_{n_k})\,,\,e_1\rangle)_{k=1}^{\infty}$. Clearly $(\langle A(x_{n_k})\,,\,e_1\rangle e_1)_{k=1}^{\infty}$ converges in E.
Hence

$$A(x) = \langle A(x)\,,\,e_1\rangle e_1 + \ldots + \langle A(x)\,,\,e_N\rangle e_N$$

is a compact mapping since $K(E,E)$ is a subspace of $B(E,E)$.

proof. Assume $(A_n)_{n=1}^{\infty} \subseteq K(E,E)$ such that $A_n \to A$ in $(B(E,E), \|.\|_{E\to E})$.
We have to show: $A \in K(E,E)$
Fix an arbitrary bounded sequence $(x_n)_{n=1}^{\infty}$ in E. We want to show that $(A(x_n))_{n=1}^{\infty}$ has a converging subsequence in E.
Set

$$M = \sup_n \|x_n\| < \infty.$$

$$\begin{aligned} A_1 \in K(E,E) \quad &\Rightarrow \quad (A_1(x_n))_{n=1}^{\infty} \text{ has a converging subsequence } (A_1(x_{n_k}))_{k=1}^{\infty} \\ A_2 \in K(E,E) \quad &\Rightarrow \quad (A_2(x_n))_{n=1}^{\infty} \text{ has a converging subsequence } (A_2(x_{n_k}))_{k=1}^{\infty} \end{aligned}$$

proceed inductively:

$$A_k \in K(E,E) \quad \Rightarrow \quad (A_k(x_n))_{n=1}^{\infty} \text{ has a converging subsequence } (A_k(x_{n_l}))_{l=1}^{\infty}$$

Also: $(A_l(x_{n,k})_{n=1}^{\infty}$ converges in E for $l = 1, 2, \ldots, k$.
Here $(A_k(y_n))_{n=1}^{\infty}$ converges for $k = 1, 2, \ldots$.
So since $(E, \|.\|)$ is a Banach space it is enough to show that $(A(y_n))_{n=1}^{\infty}$ is a Cauchy sequence in $(E, \|.\|)$.
Fix an arbitrary $\varepsilon > 0$. We have

$$\|A(y_n) - A(y_m)\| \le \underbrace{\|A(y_n) - A_k(y_n)\|}_{\le \|A - A_k\|_{E\to E}\|y_n\|} + \|A_k(y_n) - A_k(y_m)\| + \|A_k(y_m) - A(y_m)\|.$$

Fix k large enough such that

$$\|A_k - A\| < \frac{\varepsilon}{3M}.$$

Then

$$\|A(y_n) - A(y_m)\| < \frac{2}{3}\varepsilon + \|A_k(y_n) - A_k(y_m)\|$$

$(A_k(y_n))_{n=1}^{\infty}$ converges in E. This implies the existence of N such that

$$\forall\, n, m \ge N: \qquad \|A_k(y_n) - A_k(y_n)\| < \varepsilon$$

$$\Rightarrow \|A(y_n) - A(y_m)\| < \varepsilon, \qquad \forall\, n, m \ge N$$

and thus $(A(y_n))_{n=1}^{\infty}$ is a Cauchy sequence. $\square$

Proposition 3.37. Let $(E, \langle . , . \rangle)$ be a seperable Hilbert space and $A \in K(E, E)$. then there exist finite-ranked operators $A_n \in K(E, E)$ such that

$$\|A - A_n\|_{E \to E} \to 0, \qquad n \to \infty.$$

proof. Let $(x_n)_{n=1}^{\infty}$ be an ON-basis for E. For

$$x = \sum_{k=1}^{\infty} \langle x , x_k \rangle x_k, \qquad x \in E.$$

Set

$$A_n(x) = A\left(\sum_{k=1}^{n} \langle x , x_k \rangle x_k\right) = \sum_{k=1}^{n} \langle x , x_k \rangle A(x_k), \qquad x \in E, \qquad n = 1, 2, \dots.$$

Here $\dim(\mathcal{R}(A_n)) \leq n$ for $n = 1, 2, \dots$.
So A_n is a finite ranked operator in E for $n = 1, 2, \dots$.
Fix $x \in E$ with $\|x\| = 1$ and consider:

$$\|(A - A_n)(x)\|^2 = \left\| A(\sum_{k=n+1}^{\infty} \langle x , x_k \rangle x_k) \right\|^2 \leq \sup_{\substack{\|y\|=1, \\ y \in \{x_1, \dots, x_n\}^{\perp}}} \|A(y)\|^2$$

and thus

$$\|A - A_n\|_{E \to E}^2 \leq \sup_{\substack{\|y\|=1, \\ y \in \{x_1, \dots, x_n\}^{\perp}}} \|A(y)\|^2.$$

Set

$$u_n := \sup_{\substack{\|y\|=1, \\ y \in \{x_1, \dots, x_n\}^{\perp}}} \|A(y)\|^2 < \infty, \qquad n = 1, 2, \dots.$$

Here $a_n \geq a_{n+1} \geq 0$ for $n = 1, 2, \dots$.
Clearly $(a_n)_{n=1}^{\infty}$ converges in $\mathbb{R}$. Set $a = \lim_{n \to \infty} a_n$. It remains to prove $a = 0$. Assume $a > 0$. Then there exists $(y_n)_{n=1}^{\infty}$ in E such that

1. $\|y_n\| = 1$,
2. $y \in \{x_1, \dots, x_n\}^{\perp}$,
3. $\|A(y_n)\|^2 \geq \frac{1}{2} a$.

Claim: $y_n \rightharpoonup 0$ in $(E, \langle . , . \rangle)$ since:

Fix an arbitrary $x \in E$ and

$$\begin{aligned}
|\langle y_n \,,\, x\rangle| &= |\langle y_n \,,\, \sum_{k=1}^{\infty} \langle x \,,\, x_k\rangle x_k\rangle| \\
&= |\langle y_n \,,\, \sum_{k=n+1}^{\infty} \langle x \,,\, x_k\rangle x_k\rangle| \\
&\leq \|y_n\| \cdot \left\| \sum_{k=n+1}^{\infty} \langle x \,,\, x_k\rangle x_k \right\| \\
&= \sqrt{\sum_{k=n+1}^{\infty} |\langle x \,,\, x_k\rangle|^2} \to 0, \qquad n \to \infty.
\end{aligned}$$

(Note that $\sum_{k=1}^{\infty} |\langle x \,,\, x_k\rangle|^2 = \|x\|^2 < \infty$)
We have $y_n \rightharpoonup 0$ in $(E, \langle . \,,\, .\rangle)$ and

$$A \in B(E, E) \qquad \Rightarrow \qquad A(y_n) \to A(0) = 0.$$

Contradiction to (3) above which gives us $a = 0$. $\square$

Proposition 3.38. $(E, \langle . \,,\, .\rangle)$ Hilbert space and $A \in K(E, E)$. Then

$$x_n \rightharpoonup x \text{ in } (E, \langle . \,,\, .\rangle) \qquad \Rightarrow \qquad A(x_n) \to A(x) \text{ in } (E, \langle . \,,\, .\rangle).$$

proof. $x_n \rightharpoonup x$ in $(E, \langle . \,,\, .\rangle)$ implies that $\sup_n \|x_n\| < \infty$ (according to important theorem). Since $A \in K(E, E)$, we know that $(A(x_n))_{n=1}^{\infty}$ has a converging subsequence $(A(x_{n_k})_{k=1}^{\infty}$ since $(x_n)_{n=1}^{\infty}$ is bounded.
Say $A(x_{n_k}) \to y$ in E. $A \in K(E, E) \subset B(E, E)$ and $x_n \rightharpoonup x$ in $(E, \langle . \,,\, .\rangle)$.
This implies

$$A(x_n) \rightharpoonup A(x) \qquad \text{in } (E, \langle . \,,\, .\rangle).$$

We get that $y = A(x)$. We have $A(x_{n_k}) \to A(x)$ in E.
Assume that $A(x_n) \not\to A(x)$ in E.
Then there exists an $\varepsilon > 0$ and a subsequence $(A(\tilde{x}_n))_{n=1}^{\infty}$ of $(A(x_n))_{n=1}^{\infty}$ such that

$$\|A(\tilde{x}_n) - A(x)\| \geq \varepsilon, \qquad \forall\, n.$$

But $\tilde{x}_n \rightharpoonup x$ in $(E, \langle . \,,\, .\rangle)$ and to be compact implies that $(A(\tilde{x}_n))_{n=1}^{\infty}$ has a converging subsequence $(A(\tilde{x}_{n_k})_{k=1}^{\infty}$ that converges to $A(x)$ (same argument as before) Conclusion: $A(x_n) \to A(x)$ in $(E, \langle . \,,\, .\rangle)$. $\square$

Proposition 3.39. $A \in K(E, E)$ and $(E, \langle . \,,\, .\rangle)$ Hilbert space $\qquad \Rightarrow A^* \in K(E, E)$.

proof. Fix any bounded sequence $(x_n)_{n=1}^{\infty}$ in E.

$$\begin{aligned} \|A^*(x_n) - A^*(x_m)\| &= \langle A^*(x_n) - A^*(x_m)\,,\, A^*(x_n) - A^*(x_m)\rangle \\ &= \langle x_n - x_m\,,\, A(A^*(x_n)) - A(A^*(x_m))\rangle \end{aligned}$$

then use $A \in K(E, E)$. □

Proposition 3.40. $A \in K(E, E)$, $B \in B(E, E)$ $\quad \Rightarrow AB, BA \in K(E, E)$.

Example. We already know this example: $k \in C([0,1] \times [0,1])$ with

$$A(f)(x) = \int_0^1 k(x,y) f(y)\, \mathrm{d}y, \qquad x \in [0,1], \qquad f \in L^2([0,1]).$$

We know that $A \in B(L^2([0,1]), L^2([0,1]))$

$$\|A\|_{L^2 \to L^2} \leq \|k\|_{L^2([0,1]\times[0,1])}.$$

Claim: $A \in K(L^2([0,1]), L^2([0,1]))$.
Approximate A by finite-ranked operators.
Note: set $A = A_k$ and $B = A_{k_n}$ where k_n is a nice function on $[0,1] \times [0,1]$ and

$$A - B = A_k - A_{k_n} = A_{k-k_n}.$$

So

$$\|A - B\|_{L^2 \to L^2} \leq \|k - k_n\|.$$

Set

$$\begin{aligned} I_f &= [x_j - \frac{1}{N}, x_j], \qquad & j &= 1, \dots, N, \qquad & x_j &= \frac{j}{N} \\ \tilde{I}_l &= [y_l - \frac{1}{N}, y_j], \qquad & l &= 1, \dots, N, \qquad & y_l &= \frac{l}{N}. \end{aligned}$$

Set

$$k_n(x,y) = \sum_{j=1}^{N} \sum_{l=1}^{N} k(x_j, y_l) \chi_{I_j}(x) \chi_{\tilde{I}_l}(y)$$

where

$$\chi_{I_j}(x) = \begin{cases} 1, & \text{if } x \in I_j \\ 0, & \text{elsewhere.} \end{cases}$$

Since $k \in C([0,1] \times [0,1])$ and $[0,1] \times [0,1]$ compact in $\mathbb{R}^2$ then k is uniformly continous on $[0,1] \times [0,1]$. We fix $\varepsilon > 0$.
Claim: It exists an N such that

$$\sup_{\substack{(x,y)\in \\ [0,1]\times[0,1]}} |k(x,y) - k_n(x,y)| < \infty,$$

$$A_{k_N}(f)(x) = \int_0^1 k_N(x,y)f(y)\,\mathrm{d}y = \sum_{j=1}^{N} \underbrace{\sum_{l=1}^{N} k(x_i,y_l)\int_0^1 \chi_{\tilde{I}_l}(y)f(y)\,\mathrm{d}y\chi_{I_j}(x)}_{\text{scalar}}.$$

$$\dim(\mathcal{R}(A_{k_N})) = N < \infty.$$

Hence $A_{k_N} \in K(L^2([0,1]), L^2([0,1]))$ for all N.
Moreover

$$\|A - A_{k_N}\|_{L^2\to L^2} \le \|k - k_N\|_{L^2([0,1]\times[0,1])} < \varepsilon$$

for N large enough. $K(E,E)$ is a closed set in $(B(E,E), \|.\|_{L^2\to L^2})$ so $A \in K(L^2, L^2)$.

Example. $(E, \langle .\,,\,. \rangle)$ Hilbert space, $(x_n)_{n=1}^{\infty}$ ON-basis and $(\lambda_n)_{n=1}^{\infty}$ sequence of scalars. Set

$$T(x) = \sum_{n=1}^{\infty} \lambda_n \langle x\,,\,x_n\rangle x_n, \qquad x \in E.$$

Claim:

1) $T \in B(E,E) \qquad \Leftrightarrow \qquad (\lambda_n)_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{C}$.
2) $T \in K(E,E) \qquad \Leftrightarrow \qquad \lambda_n \to 0$ for $n \to \infty$.

Note $x \in E$ and the Parseval's formula

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x\,,\,x_n\rangle|^2.$$

For $T(x) \in E$ we have

$$\|T(x)\|^2 = \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle x\,,\,x_n\rangle|^2.$$

If $(\lambda_n)_{n=1}^{\infty}$ bounded sequence in $\mathbb{C}$. Then $\sup|\lambda_n| \equiv M < \infty$ and

$$\|T(x)\|^2 \le \sum_{n=1}^{\infty} M^2 |\langle x\,,\,x_n\rangle|^2 = M^2\|x\|^2.$$

If $(\lambda_n)_{n=1}^{\infty}$ is not bounded then there exists a sequence $(\lambda_{n_k})_{k=1}^{\infty}$ such that $|\lambda_{n_k}| \to \infty$ as $k \to \infty$. But

$$\|T(x_{n_k})\| = |\lambda_{n_k}|\|x_{n_k}\| = |\lambda_{n_k}| \to \infty, \qquad k \to \infty$$

$$\sup_{\|x\|=1} \|T(x)\| = \infty.$$

So 1) is done. For 2) we assume $\lambda_n \to 0$ for $n \to \infty$. Set

$$T_k(x) = \sum_{n=1}^{k} \lambda_n \langle x\,,\,x_n\rangle x_n, \qquad x \in E$$

T_k is a finite rank operator for $k = 1, 2, \ldots$. SO $T_k \in K(E, E)$ for all k.

$$\begin{aligned}
\|T - T_k\|_{E\to E} &= \sup_{\|x\|=1} \|(T - T_k)(x)\| \\
&= \sup_{\|x\|=1} \left\| \sum_{k=n+1}^{\infty} \lambda_n \langle x\,,\, x_n\rangle x_n \right\| \\
&\leq \sup_{n=k+1,k+2,\ldots} |\lambda_n| \to 0, \qquad k \to \infty
\end{aligned}$$

Assume $\lambda_n \not\to 0$ for $n \to \infty$. Then there exists $\varepsilon > 0$ and a sequence $(\lambda_{n_k})_{k=1}^{\infty}$ such that

$$|\lambda_{n_k}| \geq \varepsilon.$$

Note

$$\begin{aligned}
T(x_{n_k}) &= \lambda_{n_k} x_{n_k}, \qquad k = 1, 2, \ldots \\
\|T(x_{n_k})\| &= |\lambda_{n_k}|\|x_{n_k}\| = |\lambda_{n_k}| \geq \varepsilon, \qquad k = 1, 2, \ldots
\end{aligned}$$

$x_{n_k} \overset{\mathrm{w}}{\to} 0$ in $(E, \langle .\,,\,.\rangle)$ since for $y \in E$

$$\langle x_{n_k}\,,\, y\rangle = \langle x_{n_k}\,,\, \sum_{n=1}^{\infty} \langle y\,,\, x_n\rangle x_n\rangle = \overline{\langle y\,,\, x_{n_k}\rangle} \to 0$$

since

$$\sum_{n=1}^{\infty} |\langle y\,,\, x_n\rangle|^2 = \|y\|^2 < \infty.$$

If $T \in K(E, E)$ then $T(x_{n_k}) \to T(0) = 0$ but

$$\|T(x_{n_k})\| \geq \varepsilon, \qquad \text{for all } k.$$

Hence

$$T \notin K(E, E).$$

Example. $(E, \langle .\,,\,.\rangle)$ Hilbert space, $A \in K(E, E)$ and $I(x) = x$ for all $x \in E$. It follows

$$\Rightarrow \qquad \mathcal{R}(I - A) \text{ closed in } E.$$

Remark.

$$\begin{aligned}
\mathcal{R}(I - A)^{\perp} &= \mathcal{N}((I - A)^*) = \mathcal{N}(I - A^*) \\
\overline{\mathcal{R}(I - A)} &= \left(\mathcal{R}(I - A)^{\perp}\right)^{\perp} = \mathcal{N}(I - A^*)^{\perp}.
\end{aligned}$$

If $A \in K(E, E)$ then

$$\overline{\mathcal{R}(I - A)} = \mathcal{R}(I - A).$$

Solve

$$x = A(x) + y \qquad \Leftrightarrow \qquad (I - A)(x) = y$$

Compare 'Fredholm alternative'.

proof. Take a sequence $(y_n)_{n\in\mathbb{N}} \subseteq R(I-A)$ such that $y_n \to y$ in $(E, \|.\|)$.
To show: $y \in \mathcal{R}(I-A)$, i.e. $y = (I-A)(x)$ for some $x \in E$ and $y_n = (I-A)(x_n)$ for some $x_n \in E$.

$$x_n \in E = \mathcal{N}(I-A) + \mathcal{N}(I-A)^{\perp}$$

such that

$$x_n = \tilde{x}_n + \hat{x}_n$$

with

$$\|x_n\|^2 = \|\tilde{x}_n\|^2 + \|\hat{x}_n\|^2.$$

Step 1: Show $(\hat{x}_n)_{n=1}^{\infty}$ bounded in E.
Step 2: $y_n = (I-A)(\hat{x}_n) = \hat{x}_n - A(\hat{x}_n)$.

$\square$

recall:
$(E, \langle . \, , \, . \rangle)$ Hilbert space and $(x_n)_{n=1}^{\infty}$ ON-basis and $(\lambda_n)_{n=1}^{\infty}$ sequence of complex numbers. Set

$$A(x) = \sum_{n=1}^{\infty} \lambda_n \langle x \, , \, x_n \rangle x_n.$$

We have:

- $A : E \to E$ if $(\lambda_n)_{n=1}^{\infty} \in l^{\infty}$
  if $(\lambda_n)_{n=1}^{\infty}$ is not bounded, there exists a subsequence $(\lambda_{n_k})_{k\in\mathbb{N}}$ such that

  $$|\lambda_{n_k}| \geq k, \qquad k = 1, 2, \ldots.$$

  Set

  $$x = \sum_{k=1}^{\infty} \frac{1}{k} x_{n_k}.$$

  Clearly $x \in E$ since $\left(\frac{1}{k}\right)_{k=1}^{\infty} \in l^{\infty}$. But

  $$T(x) = \sum_{k=1}^{\infty} \lambda_{n_k} \frac{1}{k} x_{n_k} \notin E$$

  since $\left(\lambda_{n_k} \cdot \frac{1}{k}\right)_{k=1}^{\infty} \notin l^2$.
  Note

  $$A \in B(E,E) \qquad \Leftrightarrow \qquad (\lambda_n)_{n=1}^{\infty} \in l^{\infty}$$

  and $\|A\| = \sup_n |\lambda_n|$.

- $A \in K(E,E)$ iff $\lambda_n \to 0$ for $n \to \infty$.

- A is self adjoint iff $\lambda_n \in \mathbb{R}$ for all $n \in \mathbb{N}$.

Basis facts:
Set $A \in B(E,E)$ where $(E, \langle . \, , \, . \rangle)$ is a Hilbert space. Then:

- If A is self-adjoint we have
$$\|A\| = \sup_{\|x\|=1} |\langle A(x) \, , \, x \rangle|.$$
- If A is self-adjoint it follows
$$\langle A(x) \, , \, x \rangle \in \mathbb{R}, \qquad \forall \, x \in E$$
since
$$\langle A(x) \, , \, x \rangle = \langle x \, , \, A^*(x) \rangle \overset{\text{self-adjoint}}{=} \langle x \, , \, A(x) \rangle = \overline{\langle A(x) \, , \, x \rangle}.$$
- $K(E,E)$ (Set of all compact linear operators) closed subspace in $(B(E,E), \|.\|_{E \to E})$.
- $A \in K(E,E)$ and $x_n \rightharpoonup x$ in E. Then
$$A(x_n) \to A(x), \qquad \text{in } E.$$
- $A \in K(E,E)$ and $B \in B(E,E)$. Then
  - $AB, BA \in K(E,E)$,
  - $A^* \in K(E,E)$,
  - $\mathcal{R}(B)^{\perp} = \mathcal{N}(B^*)$
    $\overline{\mathcal{R}(B)} = \mathcal{N}(B^*)^{\perp}$,
  - $\mathcal{R}(I-A)$ is a closed subspace in E.
- $E = \mathcal{R}(I-A) \oplus \mathcal{R}(I-A)^{\perp} = \mathcal{R}(I-A) \oplus \mathcal{N}(I-A^*)$.
- For any $A \in K(E,E)$
$$\dim(\mathcal{N}(I-A)) = \dim(\{x \in E \,|\, x - A(x) = 0\}) < \infty$$
since: if $\dim(\mathcal{N}(I-A)) = \infty$ then there exists an ON- sequence $(x_n)_{n=1}^{\infty}$ in $\mathcal{N}(I-A)$. Then
$$x_n \rightharpoonup E, \qquad \text{since } \langle x_n \, , \, y \rangle \to 0, n \to \infty$$
since for $y \in \overline{\operatorname{span}\{x_n \,|\, n = 1, 2, \dots\}}$ then
$$\|y\|^2 = \sum_{n=1}^{\infty} |\langle x_n \, , \, y \rangle|^2 < \infty.$$
$A \in K(E,E)$ implies that $A(x_n) \to A(0) = 0$ in E. But
$$x_n = A(x_n) \to 0 \qquad \text{in } E, \qquad \|x_n\| = 1 \text{ for all } n$$
This is a contradiction.
Conclusion: $\dim(I-A) < \infty$.

From above we have for $A \in K(E,E)$

$$E = \mathcal{R}(I - A) \oplus \mathcal{N}(I - A^*).$$

Consider the equation

$$x = A(x) + y \qquad\qquad (1).$$

(1) has a solution provided by $y \in \mathcal{R}(I - A)$. That is the case if $y \perp z$ for all $z \in \mathcal{N}(I - A^*)$. Since $\dim(\mathcal{N}(I - A^*)) < \infty$, this is just finitely many conditions.

Theorem 3.41 (Fredholm alternativ). $A \in K(E,E)$ where E is a Hilbert space. then exactly one of the statements below holds:

1. $x = A(x) + y$ is solvable for every $y \in E$.
2. $x = A(x)$ has a non trivial solution $x \in E$, i.e. $x \neq 0$.

(No assumption on A being self-adjoint.)

Remark. The statement in Fredholm Alternativ also holds if $(E, \|.\|)$ is a Banach space.

proof. **(1) $\Rightarrow \neg$ (2):** We want to show that there are no non-trivial solutions for $x = A(x)$. Assume that there exists a non-trivial solution $x_1 \in E$ to $x = A(x)$, i.e.

$$(I - A)(x_1) = 0, \qquad \text{with } x_1 \neq 0.$$

If (1) holds true there exists a $x_2 \in E$ such that

$$(I - A)(x_2) = x_1 \neq 0.$$

But

$$(I - A)(x_1) = (I - A)^2(x_2) = 0.$$

With (1) there exists $x_3 \in E$ such that

$$(I - A)(x_3) = x_2$$

which implies

$$(I - A)^2(x_3) = (I - A)(x_2) = x_1 \neq 0.$$

But once again

$$(I - A)^3(x_3) = 0.$$

Proceed inductively gives us a sequence $(x_k)_{k=1}^{\infty}$ such that

$$(I - A)^k(x_k) = 0, \qquad \text{but } (I - A)^{k-1}(x_k) \neq 0.$$

We obtain

$$\mathcal{N}(I - A) \subsetneq \mathcal{N}((I - A)^2) \subsetneq \mathcal{N}((I - A)^3) \subsetneq \dots.$$

This is a sequence of proper closed subspaces.
Apply now Riesz-Lemma:
There exists a sequence $(y_k)_{k=1}^{\infty}$ with $\|y_k\| = 1$ and $\|y_k - x\| \geq \frac{1}{2}$ for all $x \in \mathcal{N}((I - A)^{k-1})$ and $y_k \in \mathcal{N}((I-A)^k)$.
Claim: $\|A(y_n) - A(y_m)\| \geq \frac{1}{2}$ for all $n > m$.

$$\begin{aligned}\|A(y_m) - A(y_n)\| &= \Bigg\| \underbrace{(I-A)(y_n)}_{\in \mathcal{N}((I-A)^{n-1})} - y_n + \underbrace{A(y_m)}_{\in \mathcal{N}((I-A)^{n-1})} \Bigg\| \\ &= \Bigg\| y_n - \underbrace{((I-A)(y_n) + A(y_m))}_{\in \mathcal{N}((I-A)^{n-1})} \Bigg\| \geq \frac{1}{2}.\end{aligned}$$

So $(A(y_n))_{n=1}^{\infty}$ can not converge in E. But A is compact and $\|y_n\| = 1$ for all n. This is a contradiction.
Conclusion: There is no non-trivial solution of $A(x) = x$.

$\neg$ **(2)** $\Rightarrow$ **(1)** Assume that $x = A(x)$ has no non-trivial solution $x \in E$. We want to show that (1) holds.

$$E = \mathcal{R}(I - A^*) \oplus \mathcal{N}(I - A), \qquad \text{with } \mathcal{N}(I-A) = \{0\}.$$

Hence

$$x = A^*(x) + y$$

is solvable for every $y \in E$. From the first part of the proof it follows that

$$\mathcal{N}(I - A^*) = \{0\}.$$

But then

$$E = \mathcal{R}(I - A) \oplus \mathcal{N}(I - A^*) = \mathcal{R}(I - A).$$

Conclusion: $x = A(x) + y$ is solvable for all $y \in E$.

□

Example. $L^2([0,1])$, $k \in C([0,1] \times [0,1])$ and

$$A(f)(x) = \int_0^1 k(x,y) f(y)\,\mathrm{d}y, \qquad x \in [0,1].$$

Then

- $A \in B(L^2, L^2)$ with $\|A\|_{L^2 \to L^2} \leq \|k\|_{L^2([0,1]\times[0,1])}$,
- A self-adjoint if $k(x,y) = \overline{k(y,x)}$ for all $x, y \in [0,1]$,
- $A \in K(E,E)$ (by approximation by finite rank operators).

Theorem 3.42 (Hilbert-Schmidt-Theorem). $(E, \langle . , . \rangle)$ Hilbert spaces and $A \in K(E, E)$ self adjoint. Then there exists a sequence of non-zero eigenvalues of A denoted $(\lambda_n)_{n=1}^N$ for N finite or infinite, corresponding to eigenvectors $(u_n)_{n=1}^N$. Respectively where $(u_n)_{n=1}^N$ is an ON-sequence, and

$$|\lambda_1| \geq |\lambda_2| \geq \dots$$

with

$$\lim_{n\to\infty} \lambda_n = 0, \qquad \text{if } N = \infty$$

such that for $x \in E$

$$x = \sum_{n=1}^{N} \langle x , u_n \rangle u_n + v, \qquad v \in \mathcal{N}(A).$$

Moreover

$$A(x) = \sum_{n=1}^{N} \lambda_n \langle x , u_n \rangle u_n.$$

Remark. With notation from the theorem above we have

1. $$A^k(x) = \sum_{n=1}^{N} \lambda_n^k \langle x , u_n \rangle u_n, \qquad k = 1, 2, \dots.$$

2. If A is injective, i.e. $\mathcal{N}(A) = \{0\}$ then the Eigenvectors $(u_n)_{n=1}^N$ form an ON-basis for E.

Definition (Eigenvalues and Eigenvectors for $A \in B(E, E)$). $\lambda \in \mathbb{C}$ is called an eigenvalue of A if there exists an $0 \neq x \in E$ such that

$$A(x) = \lambda x.$$

Remark (properties for Eigenvalues and Eigenvectors). 1. $|\lambda| \leq \|A\|$ since

$$|\lambda| \|x\| = \|\lambda x\| = \|A(x)\| \leq \|A\| \cdot \|x\|.$$

2. A self-adjoint and λ eigenvalue. Then

$$\Rightarrow \lambda \in \mathbb{R}$$

since

$$\begin{aligned}\lambda\langle x\,,\,x\rangle &= \langle \lambda x\,,\,x\rangle \\ &= \langle A(x)\,,\,x\rangle \\ &= \langle x\,,\,A^*(x)\rangle \\ &= \langle x\,,\,A(x)\rangle \\ &= \langle x\,,\,\lambda x\rangle \\ &= \bar{\lambda}\langle x\,,\,x\rangle.\end{aligned}$$

So

$$\lambda = \bar{\lambda}, \qquad \Rightarrow \lambda \in \mathbb{R}.$$

3. A self-adjoint, $A(x) = \lambda x$ and $A(y) = \mu y$, where $x, y \neq 0$ and $\lambda \neq \mu$.

$$\Rightarrow\ x \perp y$$

since

$$\lambda\langle x\,,\,y\rangle = \ldots = \bar{\mu}\langle x\,,\,y\rangle.$$

So

$$\underbrace{(\lambda - \mu)}_{\neq 0}\langle x\,,\,y\rangle = 0.$$

4. $A \in K(E, E)$ and $\lambda \neq 0$ eigenvalue of A. Then

$$\dim E_\lambda = \dim\{x \in E \,|\, A(x) = \lambda x\} < \infty.$$

Proposition 3.43. $(E, \langle .\,,\,.\rangle)$ Hilbert space and $A \in K(E, E)$ self-adjoint. Then

$$\Rightarrow \|A\| \qquad \text{or} \qquad -\|A\|$$

is an eigenvalue of A.

proof. $A = 0$ then the statement is trivial.
Assume $A \neq 0$.
A self-adjoint implies that

$$\|A\| = \sup_{\|x\|=1} |\langle A(x)\,,\,x\rangle|.$$

Also self-adjoint implies that for all $x \in E$ we have

$$\langle A(x)\,,\,x\rangle \in \mathbb{R}.$$

Hence there exists a sequence $(x_n)_{n=1}^\infty$ in E with $\|x_n\| = 1$ for all n such that

$$\langle A(x_n)\,,\,x_n\rangle \to \lambda, \qquad n \to \infty.$$

where $\lambda \in \mathbb{R}$ and $|\lambda| = \|A\|$.
Claim: $A(x_n) - \lambda x_n \to 0$ in E.

$$\begin{aligned}\|A(x_n) - \lambda x_n\|^2 &= \langle A(x_n) - \lambda x_n \, , \, A(x_n) - \lambda x_n\rangle \\ &= \underbrace{\langle A(x_n) \, , \, A(x_n)\rangle}_{\substack{=\|A(x_n)\|^2 \\ \leq \|A\|^2\|x_n\|^2 \\ =\|A\|^2}} - \overbrace{\bar{\lambda}\underbrace{\langle A(x_n) \, , \, x_n\rangle}_{\to\lambda}}^{\to|\lambda|^2=\|A\|^2} - \overbrace{\lambda\underbrace{\langle x_n \, , \, A(x_n)\rangle}_{\to\lambda}}^{\to\lambda^2=\|A\|^2} + \underbrace{|\lambda|^2}_{=\|A\|^2}\underbrace{\langle x_n \, , \, x_n\rangle}_{=1} \\ &\to 0, \qquad n \to \infty.\end{aligned}$$

$A \in K(E, E)$ and $\|x_n\| = 1$ for all n we get that

$$(A(x_n))_{n=1}^{\infty}$$

has a converging subsequence $(A(x_{n_k})_{k=1}^{\infty}$ in E.
Call the limit element $y \in E$ so

$$A(x_{n_k}) \to y \qquad \text{in } E.$$

$$\begin{cases} A(x_n) - \lambda x_n & \to 0 \\ A(x_{n_k}) & \to y \end{cases} \qquad \text{in } E$$

implies

$$x_{n_k} \to \frac{1}{\lambda} y \qquad \text{in } E$$

(note $|\lambda| > 0$ since $A \neq 0$).
Set $x = \frac{1}{\lambda}y$. So $x_{n_k} \to x$ in E. Consider

$$\|A(x) - \lambda x\| \leq \|A(x) - A(x_{n_k})\| + \|A(x_{n_k}) - y\| \to 0, \qquad k \to \infty$$

Conclusion:

$$A(x) = \lambda x.$$

where $\|x\| = 1$ since $1 = \|x_{n_k}\| \to \|x\|$ as $k \to \infty$. □

We are now going to prove the Hilbert-Schmidt theorem:
proof. If $A = 0$ the theorem is trivial.
Assume $A \neq 0$.
By the proposition above there exists an eigenvalue λ_1 of A with $|\lambda_1| = \|A\|$ and an eigenvector u_1 with $\|u_1\| = 1$ corresponding to the eigenvalue λ_1.
Set $Q_1 = \{u_1\}^{\perp}$. Q_1 is a closed subspace of E and hence Q_1 is a Hilbert space.
For $x \in Q_1$ we have $A(x) \in Q_1$ since for $x \in Q_1$ we have

$$\begin{aligned}\langle A(x) \, , \, u_1\rangle &= \langle x \, , \, A^*(u_1)\rangle \\ &= \langle x \, , \, A(u_1)\rangle \\ &= \langle x \, , \, \underbrace{\lambda_1}_{\in\mathbb{R}} u_1\rangle \\ &= \lambda_1\langle x \, , \, u_1\rangle = 0.\end{aligned}$$

Now

$$A\big|_{Q_1} : Q_1 \to Q_1$$

is compact and also self-adjoint. By proposition above there exists an eigenvalue λ_2 of $A\big|_{Q_1}$ and a corresponding eigenvector u_2 with $\|u_2\| = 1$ where

$$|\lambda_2| = \left\| A\big|_{Q_1} \right\| \leq \|A\| = |\lambda_1|.$$

Here $A(u_2) = \lambda_2 u_2$ so λ_2 is an eigenvalue of A. Set $Q_2 = \{u_1, u_2\}^\perp$. Q_2 is a closed subspace of E and we have

$$x \in Q_2 \qquad \Rightarrow \qquad A(x) \in Q_2$$

since $x \in Q_2$ we have

$$\langle A(x) \, , \, u_1 \rangle = \langle x \, , \, A(u_1) \rangle = \langle x \, , \, \lambda_1 u_1 \rangle = 0$$
$$\langle A(x) \, , \, u_2 \rangle = \langle x \, , \, A(u_2) \rangle = \langle x \, , \, \lambda_2 u_2 \rangle = 0.$$

Proceed inductively.

Case 1: For a positive integer k we have

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_k| > 0$$

with corresponding eigenvectors $u_1, u_2, \ldots, u_k$ but $A\big|_{Q_k}$ with $Q_k = \{u_1, u_2, \ldots, u_k\}^\perp$, then is the zero-mapping $Q_k \to Q_k$. This corresponds to $N = k$ and

$$x = \sum_{n=1}^{k} \langle x \, , \, u_n \rangle u_n + v, \qquad \text{where } v \in \mathcal{N}(A).$$

Case 2: The process never terminates. We get

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n| \geq \ldots$$

with corresponding eigenvectors $u_1, u_2, \ldots, u_n, \ldots$.
We have $(u_n)_{n=1}^\infty$ ON-sequence in E corresponding to the non-zero eigenvalue $(\lambda_n)_{n=1}^\infty$.
$A \in K(E, E)$ and $u_n \to 0$ in E since $(u_n)_{n=1}^\infty$ is ON-sequence.
Then this implies $A(u_n) \to 0$ in E. So

$$|\lambda_n| = \|\lambda_n u_n\| = \|A(u_n)\| \to 0, \qquad n \to \infty.$$

Hence

$$\lim_{n\to\infty} \lambda_n = 0.$$

Set

$$S := \overline{\operatorname{span}\{u_1, \ldots, u_n, \ldots\}} = \left\{ \sum_{k=1}^{\infty} a_n u_n \,\middle|\, (a_n)_{n=1}^\infty \in l^\infty \right\}.$$

S is a closed subspace of E.
We have $E = S \oplus S^{\perp}$ where $S^{\perp} \subseteq Q_k = \{u_1, \ldots, u_k\}^{\perp}$ for all $k \in \mathbb{N}$. For $x \in E$ we have

$$\underbrace{\sum_{k=1}^{\infty} \langle x\,,\, u_k \rangle u_k}_{\in S} + \underbrace{v}_{\in S^{\perp}}$$

since $(\langle x\,,\, u_k\rangle)_{k=1}^{\infty} \in l^{\infty}$ by Bessel's inequality. To show: $A(v) = 0$. Clearly, $v \in Q_k$ for all k. If $v = 0$ there is nothing to prove. For $v \neq 0$ set $w = \frac{1}{\|v\|} v$ and get

$$\begin{aligned} |\langle A(v)\,,\, v\rangle| &= \|v\|^2 |\langle A(w)\,,\, w\rangle| \\ &\leq \|v\|^2 \underbrace{\sup_{\substack{\|z\|=1 \\ z \in Q_k}} |\langle A(z)\,,\, z\rangle|}_{= \left\| A\big|_{Q_k} \right\| = |\lambda_{k+1}| \to 0} \end{aligned}$$

Claim: $A\big|_{S^{\perp}} = 0$ and hence $v \in S^{\perp}$ implies $A(v) = 0$.

□

Theorem 3.44 (Spectral mapping theorem). $(E, \langle .\,,\, . \rangle)$ seperable Hilbert space and ∞-dimensional $A \in K(E, E)$ self-adjoint. Then there exists a ON-basis of eigenvectors $(\tilde{u}_n)_{n=1}^{\infty}$ corresponding to the eigenvalues $(\tilde{\lambda}_n)_{n=1}^{\infty}$ if A where $\lim_{n\to\infty} \tilde{\lambda}_n = 0$.

proof (consequence of HS-theorem). We have by HS-theorem an ON-sequence $(u_n)_{n=1}^{\infty}$ of eigenvectors corresponding to the non-zero eigenvalues $(\lambda_n)_{n=1}^{N}$.
Set

$$S = \overline{\operatorname{span}\{u_1, \ldots, u_n, \ldots\}}.$$

E is seperable implies E has an ON-basis $(v_n)_{n=1}^{\infty}$. By Gram-Schmidt Orthoganlization procedure we can obtain an ON-basis $(w_n)_{n=1}^{M}$ for $S^{\perp}$. Have M finite or infinite.

$$\begin{aligned} S &: u_1, u_2, \ldots && \text{ON-basis finite or infinite} \\ S^{\perp} &: w_1, w_2, \ldots && \text{ON-basis finite or infinite} \end{aligned}$$

Consider the ON-sequence $u_1, w_1, u_2, w_2, \ldots = \tilde{u}_1, \tilde{u}_2, \ldots$. This gives an ON-basis for E consisting of eigenvectors to A. Also

$$\lim_{n\to\infty} \tilde{\lambda}_n = 0.$$

□

- $(E, \langle .\,,\, . \rangle)$ complex Hilbert space.
- $A \in \mathcal{B}(\mathcal{E}, \mathcal{E})$.

- Consider the equation
$$x = A(x) + y, \qquad y \in E.$$
$$(I - A)(x) = y.$$
- Consider this problem for $\lambda \in \mathbb{C}$.
- Set
$$\rho(A) := \left\{\lambda \in \mathbb{C} \,\middle|\, (A - \lambda I)^{-1} \in \mathcal{B}(E, E)\right\}$$
- $\rho(A)$ is called the resolvent set for A.
- Set
$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$
- $\sigma(A)$ is called the spectrum of A.
- Clearly, a necessary condition for $(A - \lambda I)^{-1} \in \mathcal{B}(E, E)$ is that
$$A - \lambda I : E \to E$$
is a bijection.
- Linearity for $(A - \lambda I)^{-1}$ follows from the linearity of $A - \lambda I$.

Theorem 3.45 (Banachs's inverse mapping theorem). $(E, \|.\|)$ Banach space, $A \in \mathcal{B}(E, E)$. $A - \lambda I : E \to E$ bijection. Then
$$\Rightarrow (A - \lambda I)^{-1} \in \mathcal{B}(E, E)$$

proof. based on the Open mapping theorem. Proof is omitted. Assume $\lambda \in \sigma(A)$. Then $A - \lambda I : E \to E$ is not a bijection.

- If $A - \lambda I : E \to E$ is not injective then there exists $0 \neq x \in E$ such that
$$(A - \lambda I)(x) = 0,$$
i.e. λ is an eigenvalue of A. Set
$$\sigma_p(A) = \{\lambda \in \mathbb{C} \,|\, \lambda \text{eigenvalue of } A\}.$$
- If $A - \lambda I$ is injective, densely defined but not bounded then $\lambda \in \sigma(A)$. The set of such λ's is called the continuous spectrum of A, denoted $\sigma_c(A)$
- If $A - \lambda I$ is not surjective then the set of such λ's is called the residual spectrum, denoted $\sigma_r(A)$.

$\square$

Lemma . $(E, \|.\|)$ Banach space, $A \in \mathcal{B}(E,E)$ with $\|A\| < 1$. Then

$$(I-A)^{-1} \in \mathcal{B}(E,E)$$

and

$$(I-A)^{-1} = I + \sum_{n=1}^{\infty} A^n.$$

This series is called a Neumannseries.

proof. Observe

$$\|A^n\| = \|A \cdot A \cdots A\| \leq \|A\|^n, \qquad n = 1, 2, \dots$$

and

$$\sum_{n=1}^{\infty} \|A^n\| < \infty.$$

Since E is a Banach space we have

$$\sum_{n=1}^{\infty} A^n$$

converges in $\mathcal{B}(\mathcal{E},\mathcal{E})$. Since E Banach space implies $\mathcal{B}(E,E)$ is a Banach space.
Note

$$(I-A)\left(I + \sum_{n=1}^{N} A^n\right) = I - A^{N+1} \to I, \qquad \text{in } \mathcal{B}(E,E).$$

$$\left(I + \sum_{n=1}^{N} A^n\right)(I-A) = I - A^{N+1} \to I, \qquad \text{in } \mathcal{B}(E,E).$$

We get

$$\left(I + \sum_{n=1}^{\infty}\right)(I-A) = I = (I-A)(I + \sum_{n=1}^{\infty} A^n).$$

We have $(I-A)^{-1}$ exists and is equal to $I + \sum_{n=1}^{A^n}$. □

Lemma . $(E, \|.\|)$ Banach space and $A \in \mathcal{B}(E,E)$. Then

1. $\sigma(A) \neq \emptyset$.
2. $\sigma(A)$ closed set in $\mathbb{C}$.
3. $\sigma(A) \subseteq \overline{B(0, \|A\|)}$

proof. 1. omitted.

2. Enough to prove that $\rho(A)$ is an open set in $\mathbb{C}$.
   Fix $\lambda_0 \in \rho(A)$. So $(A - \lambda_0 I)^{-1} \in \mathcal{B}(E, E)$.
   Note:

$$\begin{aligned} A - \lambda I &= A - \lambda_0 I - (\lambda - \lambda_0) I \\ &= \underbrace{(A - \lambda_0 I)}_{\substack{\text{invertible} \\ \text{since } \lambda_0 \in \rho(A)}} \underbrace{\left(I - (\lambda - \lambda_0)(A - \lambda_0 I)^{-1}\right)}_{\substack{\text{invertible if} \\ \left\|(\lambda - \lambda_0)(A - \lambda_0 I)^{-1}\right\| < 1 \\ \text{by previous lemma, i.e.} \\ |\lambda - \lambda_0| < \frac{1}{\|(A - \lambda_0 I)^{-1}\|}}}. \end{aligned}$$

   Clearly, $A - \lambda I$ is invertible if

$$|\lambda - \lambda_0| < \frac{1}{\|(A - \lambda_0 I)^{-1}\|}.$$

3. It is enough to show that $\lambda \in \rho(A)$ if

$$|\lambda| > \|A\|.$$

   Note

$$A - \lambda I = -\lambda (I - \frac{1}{\lambda} A).$$

   Here

$$\left\| -\frac{1}{\lambda} A \right\| = \frac{1}{|\lambda|} \|A\| < 1.$$

   $I - \frac{1}{\lambda} A$ is invertible by previous lemma. So $\rho(A)$.

$\square$

Now assume $(E, \langle ., .\rangle)$ is a complex Hilbert space with infinite dimension. $A \in \mathcal{K}(E, E)$ (We don't assume A is self-adjoint). Then

1. $\lambda \in \sigma(A) \setminus \{0\} \quad \Rightarrow$ is an eigenvalue of A.
2. $\lambda \in \sigma(A) \setminus \{0\} \quad \Rightarrow \dim\{x \in E \,|\, A(x) = \lambda x\} < \infty$.
3. O is the only cluster point for $\sigma(A)$
4. $0 \in \sigma(A)$ since if $0 \notin \sigma(A)$ then $A^{-1} \in \mathcal{B}(E, E)$ and

$$\underbrace{\underbrace{A}_{\in \mathcal{K}(E,E)} \underbrace{A^{-1}}_{\in \mathcal{B}(E,E)}}_{\in \mathcal{K}(E,E)} = I.$$

   But $I \notin \mathcal{K}(E, E)$ since E ∞-dimensional. Just take an ON-sequence $(x_n)_{n=1}^{\infty}$ in E. Then

$$x_n \rightharpoonup 0, \qquad \text{in } E$$

but $\|x_n\| = 1$ for all n and if $I \in \mathcal{K}(E, E)$ then

$$x_n = I(x_n) \to I(0) = 0, \qquad \text{in } E$$

which implies that $\|x_n\| \to 0$ for $n \to \infty$. Moreover (by Hilbert-Schmidt theorem) $(E, \langle . \, , \, . \rangle)$ complex Hilbert space, seperable and ∞-dim. $A \in \mathcal{K}(E, E)$ and self-adjoint it follows

$$\Rightarrow \qquad (u_n)_{n=1}^{\infty} \text{ ON-basis for } E \text{ where}$$

$$A(u_n) = \lambda_n u_n, \qquad n = 1, 2, \ldots.$$

(λ_n eigenvalue of A with normalised eigenvector u_n) with

$$\lim_{n \to \infty} \lambda_n = 0.$$

For $x \in E$

$$x = \sum_{n=1}^{\infty} \langle x \, , \, u_n \rangle u_n$$

and

$$A(x) = \sum_{n=1}^{\infty} \lambda \langle x \, , \, \lambda_n \rangle u_n$$

Fredholm Alternativ:
E, A as above. Then

1. $x = A(x) + y$ is seperable for all $y \in E$.
   iff

2. $x = A(x)$ has no non-trivial solution $x \in E$.

Exactly one of the statements hold:

1. (1) from above

2. (2) has a non-trivial solution $x \in E$.

In general (1) is seperable for $y \in E$ iff

$$y \in \{x \in E \,|\, A(x) = x\}^{\perp}.$$

If so: If x is a solution to (1) then also $x + \tilde{x}$ is a solution to (1) where

$$\tilde{x} \in \{x \in E \,|\, A(x) = x\}$$

proof. Look at (1). Let $(u_n)_{n=1}^{\infty}$ be the ON-basis from the previous theorem.

$$x = \sum_{n=1}^{\infty} \langle x , u_n \rangle u_n, \qquad y = \sum_{n=1}^{\infty} \langle y , u_n \rangle u_n.$$

$$A(x) = \sum_{n=1}^{\infty} \lambda_n \langle x , u_n \rangle u_n.$$

(1) taked the form

$$\sum_{n=1}^{\infty} \left(\langle x , u_n \rangle - \lambda_n \langle x , u_n \rangle - \langle y , u_n \rangle \right) u_n = 0.$$

This implies

$$(I - \lambda) \langle x , u_n \rangle - \langle y , u_n \rangle = 0, \qquad n = 1, 2, \dots .$$

If $\lambda_n \neq 1$ then

$$\langle x , u_n \rangle = \frac{\langle y , u_n \rangle}{1 - \lambda_n}.$$

If $\lambda_n = 1$ then $-y$ must be orthogonal to every u_n corresponding by the eigenvalue 1.

$$\sum_{n=1}^{\infty} \frac{\langle y , u_n \rangle}{1 - \lambda_n} u_n \in E$$

since

$$\left(\frac{\langle y , u_n \rangle}{1 - \lambda_n} \right)_{n=1}^{\infty} \in l^2$$

since

$$\sup_{\substack{n \\ \lambda_n \neq 1}} \left| \frac{1}{1 - \lambda_n} \right| < \infty$$

since

$$\lim_{n \to \infty} \lambda_n = 0$$

and

$$(\langle y , u_n \rangle)_{n=1}^{\infty} \in l^2.$$

□

4 Boundary Value Problems for ODE's

Consider

$$(*) \qquad \begin{cases} Lu &= f \in C([0,1]) \\ R_j u &= 0 \qquad j = 1,2,\dots,n \end{cases}$$

(homogenuous boundary conditions), where

$$Lu := u^{(n)} + C_{n-1}(x)u^{(n-1)} + \ldots + c_1(x)u' + c_0(x)u, \qquad u \in C^n([0,1])$$

with

$$c_0(x), c_1(x), \dots, c_{n-1}(x) \in C([0,1]) -$$

$$R_j = \sum_{k=0}^{n-1} \left(\alpha_{jk} u^{(k)}(0) + \beta_{jk} u^{(k)}(1) \right), \qquad j = 1,2,\dots,n$$

with

$$\alpha_{jk}, \beta_{jk} \in \mathbb{C}, \qquad j = 1,\dots,n, \qquad k = 0,\dots,n-1$$

Reformulate (*).

$$u(x) = \int_0^1 \underbrace{g(x,y)}_{\substack{\text{Green's function} \\ \text{for } L \text{ and } R_j \\ j=1,\dots,n}} \; f(y)\,\mathrm{d}y \qquad \in C^n([0,1])$$

and satisfies the boundary conditions $R_j = 0$ for $j = 1,2,\dots,n$.
Consider the problem

$$(**) \qquad \begin{cases} Lu &= f(x,u), \qquad x \in [0,1) \\ R_j u &= 0, \qquad j = 1,2,\dots,n. \end{cases}$$

The reformulation above gives

$$u(x) = \int_0^1 g(x,y) f(y,u(y))\,\mathrm{d}y, \qquad x \in [0,1].$$

To find a solution set

$$T(u)(x) = \int_0^1 g(x,y) f(y,u(y))\,\mathrm{d}y, \qquad x \in [0,1].$$

$$T : C([0,1]) \to C([0,1])$$

A fixed point to T gives a solution to $(**)$. Note that if $u \in C([0,1])$ then

$$T(u) \in C^n([0,1])$$

and satisfies $R_j = 0$ for $j = 1,2,\dots$.
Given L and R_j for $j = 1,2,\dots,n$ find the corresponding Green's function.

Example.

$$\begin{cases} Lu &= u'' - u, \qquad \text{on } [0,1] \\ R_1 u &= u(0) = 0 \\ R_2 u &= u(1) = 0 \end{cases}$$

Theorem 4.1. $Lu = f \in C([0,1])$, where

$$Lu := u^{(n)} + c_{n-1}(x)u^{(n-1)} + \ldots + c_1(x)u' + c_0(x)u$$

and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$. Then for $x_0 \in [0,1]$

$$\Rightarrow \qquad \exists!\ u \in C^n([0,1]) \text{ with } Lu = f.$$

and

$$(u, u', \ldots, u^{(n-1)})\big|_{x_0} = \xi.$$

proof. Reformulate the problem as a system of first order differential equations.

$$\begin{cases} Lu &= f \\ (u, u', \ldots, u^{(n-1)})\big|_{x_0} &= \xi \end{cases}$$

corresponds to

$$\begin{cases} \tilde{u}' = \tilde{f} \\ \tilde{u}(x_0) = \xi \end{cases}$$

and is equivalent to

$$\tilde{u}(x) = \xi + \int_{x_0}^{x} \tilde{f}(s)\,\mathrm{d}s.$$

$\tilde{f}$ contains $\tilde{u}$ implicitly. The statement of the proof follows from an application of Banach's fixed point theorem. (See course homepage and proof of picard's existence theorem.) $\square$

Set

$$\mathcal{N}(L) = \{u \in C^n((0,1)) \mid Lu = 0\}$$

Claim: $\dim \mathcal{N}(L) = n$
Set

$$C^n_R([0,1]) = \{u \in C^n((0,1)) \mid R_j u = 0, j = 1, 2, \ldots, n\}$$

and $L_0 = L\big|_{C^n_R([0,1])}$. Let $u_1, \ldots, u_m \in \mathcal{N}(L)$

Theorem 4.2. The following statements are equivalent. Let $u_1, \ldots, u_n \in \mathcal{N}(L)$

1. $W(x) \neq 0$ for all $x \in [0,1]$.
2. $W(x) \neq 0$ for some $x \in [0,1]$.

3. $u_1, u_2, \dots, u_n$ is a basis for $\mathcal{N}(L)$.

where

$$W(x) = \det\left(\begin{pmatrix} u_1(x) & \dots & u_n(x) \\ u_1'(x) & \dots & u_n'(x) \\ \vdots & & \vdots \\ u_1^{(n-1)} & \dots & u_n^{(n-1)(x)} \end{pmatrix}\right), \qquad x \in [0,1].$$

Theorem 4.3. With the notation from above the following statements are equivalent.

1. $L_0 : C_R^n([0,1]) \to C([0,1])$ is a bijection.
2. $\det(R_j u_k)_{1 \le j,k \le n} \neq 0$.

Example (continue). From the example above we get

$$u_1(x) = e^x, \qquad u_2(x) = e^{-x}.$$

$$u(x) = Ae^x + Be^{-x}$$

and

$$\begin{aligned} R_1u_1 &= u_1(0) = e^0 = 1 \\ R_1u_2 &= u_2(0) = e^0 = 1 \\ R_2u_1 &= u_1(1) = e \\ R_2u_2 &= u_2(1) = \frac{1}{e} \end{aligned}$$

and

$$\det(R_ju_k) = \det(\begin{pmatrix} 1 & 1 \\ e & \frac{1}{e} \end{pmatrix}) = \frac{1}{e} - e \neq 0.$$

Theorem 4.4. Assume $u_1, \dots, u_n$ basis for $\mathcal{N}(L)$ and $\det(R_ju_k) \neq 0$. Set $G = L_0^{-1}$.

$$\Rightarrow \qquad \exists!\,\text{continuous } g \in C([0,1] \times [0,1])$$

such that

$$G(f) = \int_0^1 g(x,y) f(y)\, \mathrm{d}y$$

is a solution of

$$\begin{cases} Lu &= f \\ R_ju &= 0, \qquad j = 1, \dots, n \end{cases}.$$

Here

$$g(x,y) = \underbrace{\left(\sum_{k=1}^{n} a_k(y)u_k(x)\right)}_{\equiv e(x,y)}\theta(x-y) + \sum_{k=1}^{n} b_k(y)u_k(x).$$

where

$$\begin{aligned} e_x^{(k)}(y,y) &= 0, \qquad k = 0,1,\ldots,n-2 \\ e_x^{(n-1)}(y,y) &= 1 \end{aligned}$$

Note

$$Lu = 1u^{(n)} + c_{n-1}u^{(n-1)} + \ldots + c_0 u.$$

and

$$R_j(g(.,y)) = 0, \qquad 0 < y < 1, \qquad j = 1,2,\ldots,n$$

Note

$$\begin{aligned} \int_0^1 g(x,y)f(y)\,\mathrm{d}y &= \int_0^1 e(x,y)\theta(x-y)f(y)\,\mathrm{d}y + \int_0^1 \sum_{k=1}^{n} b_k(y)u_k(x)f(y)\,\mathrm{d}y \\ &= \underbrace{\underbrace{\int_0^x \sum_{k=1}^{\infty} a_k(y)u_k(x)f(y)\,\mathrm{d}y}_{=L[\ldots]=f} + \underbrace{\sum_{k=1}^{N}\int_0^1 b_k(y)f(y)\,\mathrm{d}y u_k(x)}_{L[\ldots]=0}}_{L[\ldots]=f} \end{aligned}$$

Calculate $g(x,y)$ for $n=2$:
Set

$$e(x,y) = a_1(y)u_1(x) + a_2(y)u_2(x)$$

$$\begin{cases} e(y,y) &= a_1(y)e^y + a_2(y)e^{-y} = 0 \\ e_x'(y,y) &= a_1(y)e^y - a_2(y)e^{-y} = 1 \end{cases}$$

So we get

$$\begin{aligned} a_1(y) &= \frac{1}{2}e^{-y} \\ a_2(y) &= -\frac{1}{2}e^{-y} \end{aligned}$$

and

$$\begin{aligned} e(x,y) &= \frac{1}{2}e^{-y}e^x - \frac{1}{2}e^y e^{-x} \\ &= \frac{1}{2}(e^{x-y} - e^{y-x}), \qquad (x,y) \in [0,1]\times[0,1] \end{aligned}$$

Set

$$g(x,y) = e(x,y)\theta(x-y) + b_1(y)u_1(x) + b_2(y)u_2(x)$$

For $0 < y < 1$

$$\begin{aligned} R_1 g(.,y) = 0, \text{ i.e. } g(0,y) = 0, \qquad & \text{for } y \in (0,1), \\ & \text{i.e. } b_1(y)u_1(0) + b_2 u_2(0) = 0 \text{ for } y \in (0,1), \\ & \text{So } b_1(y) + b_2(y) = 0. \end{aligned}$$

$$\begin{aligned} R_2 g(.,y) = 0, \text{ i.e. } g(1,y) = 0, \qquad & \text{for } y \in (0,1), \\ & \text{i.e. } e(1,y) + b_1(y)u_1(1) + b_2(y)u_2(1) = 0 \text{ for } y \in (0,1), \\ & \text{So } \frac{1}{2}\left(e^{1-y} - e^{y-1}\right) + b_1(y)e + b_2(y)e^{-1} = 0 \text{ for } y \in (0,1). \end{aligned}$$

So we have in total

$$\begin{cases} b_1(y) + b_2(y) & = 0 \\ \frac{1}{2}\left(e^{1-y} - e^{y-1}\right) + b_1(y)e + b_2(y)e^{-1} & = 0 \end{cases}.$$

We obtain

$$\begin{cases} b_1(y) & = -b_2(y) \\ b_2(y)\left(e^{-1} - e\right) & = \frac{1}{2}\left(e^{y-1} - e^{1-y}\right) \end{cases}.$$

So

$$b_2(y) = \frac{\frac{1}{2}(e^{y-1} - e^{1-y})}{(e^{-1} - e)} = \frac{1}{2}\frac{e^{1-y} - e^{y}}{e^2 - 1}$$

and

$$b_1(y) = \frac{1}{2}\frac{e^{y} - e^{2-y}}{e^2 - 1}.$$

We obtain

$$g(x,y) = \frac{1}{2}(e^{x-y} - e^{y-x})\theta(x-t) + \frac{1}{2}\frac{e^{x+y - e^{x+2-y}}}{e^2 - 1} + \frac{1}{2}\frac{e^{2-y-x} - e^{y-x}}{e^2 - 1}.$$

Question: $g(x,y) = g(y,x)$ for all $x, y \in [0,1]$?
In general, we say that $L_0 = L\big|_{C^n_R([0,1])}$ is <u>symmetrie</u> if

$$\langle L_0(u)\,,\, v\rangle_{L^2} = \langle u\,,\, L_0(v)\rangle_{L^2}, \qquad \forall\, u, v \in C^n_R([0,1])$$

Example (continue). As above we have

$$L(u) = u'' - u$$

with boundary conditions

$$u(0) = u(1) = 0$$

Set $u, v \in C^2_R([0,1])$

$$\begin{aligned}
\langle L_0(u)\,,\,v\rangle_{L^2} &= \int_0^1 L_0(u)\bar{v}\,\mathrm{d}x \\
&= \int_0^1 u''\bar{v} - u\bar{v}\,\mathrm{d}x \\
&= -\int_0^1 u'\bar{v} + u\bar{v}\,\mathrm{d}x + \underbrace{u'\bar{v}\Big|_0^1}_{=u'(1)\underbrace{\bar{v}(1)}_{=0}-u'(0)\underbrace{\bar{v}(0)}_{=0}} \\
&= -\int_0^1 \left(u'\bar{v}' + u\bar{v}\right)\,\mathrm{d}x \\
&= \int_0^1 u(\bar{v}'' - \bar{v})\,\mathrm{d}x \\
&= \int_0^1 u\overline{L_0 v}\,\mathrm{d}x \\
&= \langle u\,,\,L_0 v\rangle_{L^2}
\end{aligned}$$

Recall:
We have a bounded value problem with

$$\begin{cases} Lu & \equiv u^{(n)} + c_{n-1}(x)u^{(n-1)} + \ldots + c_1(x)u' + c_0(x)u = f \in C([0,1]) \\ R_j u & = \sum_{k=0}^{n-1}\left(\alpha_{kj}u^{(k)}(0) + \beta_{kj}u^{(k)}(1)\right) = 0, \qquad j = 1,2,\ldots,n \end{cases}$$

with $\alpha_{kj}, \beta_{kj} \in \mathbb{C}$ for $k = 0,\ldots,n-1$ and $j = 1,\ldots,n$.
We have

$$\mathcal{N}(L) = \{u \in C^n((0,1))\,|\,Lu = 0\}.$$

Set $u_1, u_2, \ldots, u_n$ as a basis for $\mathcal{N}(L)$ and set

$$L_0 = L\big|_{C^n_R([0,1])}$$

where

$$C^n_R([0,1]) = \{u \in C^n((0,1))\,|\,R_j u = 0,\; j = 1,\ldots,n\}.$$

Then $L_0 u \in C([0,1])$ for $u \in C^n_R([0,1])$. Remember Theorem 4.3 and Theorem 4.4:

Theorem . Assume

$$\det(R_j u_k) \neq 0.$$

Then

1. $L_0 : C^n_R([0,1]) \to C([0,1])$ is a bijection.
2. Set $G = L_0^{-1}$. Then there exists a continuous function $g(x,y)$ in $[0,1]\times[0,1]$ such

that

$$G(f) = \int_0^1 g(x,y)f(y)\,\mathrm{d}y, \qquad x \in [0,1].$$

with $f \in C([0,1])$ and g is called the Green's function for L and the boundary conditions R_j, $j = 1,\dots,n$ can be given by

$$g(x,y) = e(x,y)\theta(x-y) + \sum_{k=1}^{n} b_k(y)u_k(x)$$

where

$$e(x,y) = \sum_{k=1}^{n} a_k(y)u_k(x)$$

with a_k and b_k are defined through

$$\begin{cases} \left(\frac{\partial}{\partial x}\right)^l e(x,y)\big|_{x=y} & = 0, \qquad l = 0,1,\dots,n-2. \\ \left(\frac{\partial}{\partial x}\right)^{n-1} e(x,y)\big|_{x=y} & = 1 \end{cases}$$

and

$$R_j(g(.,y)) = 0, \qquad 0 < y < 1, \qquad j = 1,2,\dots,n.$$

Remark. Consider the problem

$$\begin{cases} Lu & = f(x,u), \qquad x \in [0,1] \\ R_j u & = c_j, \qquad j = 1,2,\dots,n \end{cases}.$$

Pick any $\tilde{u} \in C^n([0,1])$ such that

$$R_j\tilde{u} = c_j, \qquad j = 1,\dots,n.$$

Set $U = \tilde{u} + v$. Note that

$$R_j v = R_j(u - \tilde{u}) = R_j u - R_j \tilde{u} = 0, \qquad j = 1,2,\dots,n$$

and

$$L(\tilde{u} + v) = f(x, \tilde{u} + v)$$

gives

$$Lv = f(x, \tilde{u} + v) - L\tilde{u} = \hat{f}(x,v).$$

Solve

$$\begin{cases} Lv & = \hat{f}(x,v) \\ R_j v & = 0, \qquad j = 1,\dots,n \end{cases}$$

Moreover set

$$T(v)(x) = \int_0^1 g(x,y)\hat{f}(y,v(y))\,\mathrm{d}y, \qquad x \in [0,1].$$

Apply a fixed point theorem.
Warning: don't take for example the space $C^2([0,1])$ because in this case we need another norm to use the fixed point theorem (no Banach space). Take

$$T : C([0,1]) \to C([0,1])$$

instead where $(C([0,1]), \|.\|)$ is a Banach space.
If $u \in C([0,1])$ is a fixed point then

$$u(x) = \int_0^1 g(x,y) f(y, u(y))\, \mathrm{d}y, \qquad x \in [0,1].$$

WE ACTUALLY have $u \in C_R^n([0,1])$!!

Definition . Call L_0 symmetric if

$$\langle L_0 u \,,\, v \rangle_{L^2} = \langle u \,,\, L_0 v \rangle_{L^2}, \qquad \forall\, u, v \in C_R^n([0,1])$$

where

$$\langle f \,,\, h \rangle_{L^2} = \int_0^1 f(x) \overline{h(x)}\, \mathrm{d}x$$

Theorem 4.5. Assume that $L_0 : C_R^n([0,1]) \to C([0,1])$ is a bijection. The following statements are equivalent:

1. L_0 is symmetric.
2. $\tilde{G}$ self-adjoint.
3. $g(x,y) = \overline{g(y,x)}$ for all $(x,y) \in [0,1] \times [0,1]$.

Here

$$G(f) = \int_0^1 g(x,y) f(y)\, \mathrm{d}y, \qquad f \in C([0,1])$$
$$\tilde{G}(f) = \int_0^1 g(x,y) f(y)\, \mathrm{d}y, \qquad f \in L^2([0,1])$$

$\tilde{G} \in \mathcal{B}(L^2([0,1]), L^2([0,1]))$ and $\overline{C([0,1])}^{L^2([0,1])} = L^2([0,1])$.

proof. **(1) $\Rightarrow$ (2):** Assume that L_0 is symmetric. Then we have

$$\langle L_0(G(f)) \,,\, G(h) \rangle_{L^2} = \langle G(f) \,,\, L_0(G(h)) \rangle_{L^2}$$

for all $f, h \in C([0,1])$. Hence

$$\langle f \,,\, G(h) \rangle_{L^2} = \langle G(f) \,,\, h \rangle_{L^2}, \qquad \forall\, f, h \in C([0,1]).$$

Change

$$\langle f\,,\,\tilde{G}(h)\rangle_{L^2} = \langle \tilde{G}(f)\,,\,h\rangle_{L^2}, \qquad \forall\, f, h \in L^2([0,1]) \qquad (*)$$

Hence $\tilde{G}$ is self-adjoint in $\mathcal{B}(L^2([0,1]), L^2([0,1]))$. This yields (2).

(2) $\Rightarrow$ (3): Given (2). From $(*)$ we get

$$\begin{aligned}
\int_0^1 f(x) \int_0^1 g(x,y)h(y)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^1 \int_0^1 g(x,y)f(y)\,\mathrm{d}y\overline{h(x)}\,\mathrm{d}x \\
&= \int_0^1 f(y) \int_0^1 g(x,y)\overline{h(x)}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 f(x)\overline{\int_0^1 \overline{g(y,x)}h(y)\,\mathrm{d}y}\,\mathrm{d}x
\end{aligned}$$

We get

$$\int_0^1 f(x)\overline{\int_0^1 (g(x,y) - \overline{g(y,x)})h(y)\,\mathrm{d}y}\,\mathrm{d}x = 0, \qquad f, g, h\text{cont.}$$

This implies that

$$\int_0^1 (g(x,y) - \overline{g(y,x)})h(y)\,\mathrm{d}y = 0, \qquad \forall\, x \in [0,1].$$

This implies

$$g(x,y) = \overline{g(y,x)}, \qquad \forall\, x, y \in [0,1]$$

□

Theorem 4.6. Assume L_0 symmetric and bijection. Then

1. - 0 is not an eigenvalue of L_0.
   - 0 is not an eigenvalue of $\tilde{G}$.
2. f is an eigenfunction for L_0 with the eigenvalue μ if and only if f is an eigenfunction for $\tilde{G}$ with the eigenvalue $\frac{1}{\mu}$.

proof. 1. $\mathcal{N}(L_0) = \{0\}$. So 0 is not an eigenvalue of L_0. IF f is an eigenfunction for $\tilde{G}$ with eigenvalue 0 then for $u \in C_R^n([0,1])$ we have

$$\begin{aligned}
\langle f\,,\,u\rangle &= \langle f\,,\,G\,\underbrace{L_0(u)}_{\in C([0,1])}\,\rangle \\
&= \langle f\,,\,\tilde{G}(L_0 u)\rangle \\
&= \langle \tilde{G}(f)\,,\,L_0 u\rangle \\
&= \langle 0\,,\,L_0 u\rangle \\
&= 0
\end{aligned}$$

So we have

$$\langle f\,,\,u\rangle = 0, \qquad \forall\, u \in C^n_R([0,1]).$$

Claim: $C^n_R([0,1])$ is dense in $L^2([0,1])$.
If so, we get

$$f \equiv 0.$$

2. Assume

$$L_0(f) = \mu f, \qquad f \in C^n_R([0,1]) \setminus \{0\}.$$

We have

$$0 \neq f = G(L_0(f)) = G(\mu f) = \tilde{G}(\mu f) = \mu \tilde{G}(f)$$

So $\tilde{G}(f) = \frac{1}{\mu} f$.
Assume $\tilde{G}(f) = \frac{1}{\mu} f$ for $f \in L^2([0,1]) \setminus \{0\}$.
We have

$$\tilde{G}(f)(x) = \frac{1}{\mu} f(x), \qquad \text{for all } x \in [0,1] \text{ except for } x \text{ in a zero set.}$$

Consider

$$\underbrace{\mu \underbrace{\tilde{G}(f)(x)}_{\substack{\text{continuous}\\ \text{function}}}}_{\in C([0,1])}.$$

Set

$$h(x) := \mu \tilde{G}(f)(x), \qquad x \in [0,1].$$

We get

$$h(x) = \mu \tilde{G}(h)(x) = \mu \underbrace{G(h)}_{\in C^n_R([0,1])}(x).$$

and

$$L_0 h = L_0(\mu G(h)) = \mu L_0 G(h) = \mu h.$$

So

$$L_0(h) = \mu h.$$

□

Theorem 4.7. Assume L_0 to be symmetric and a bijection. Let $(\mu_n)_{n=1}^{\infty}$ be the eigenvalues of L_0 counted with multiplicity and $(e_n)_{n=1}^{\infty}$ corresponding ON-sequence of eigenfunctions. Then

1. $(e_n)_{n=1}^{\infty}$ is an ON-basis for $L^2([0,1])$.
2. The solution u of

$$\begin{cases} Lu &= f \in C([0,1]) \\ R_j u &= 0, \qquad j = 1, 2, \dots, n \end{cases}$$

is given by

$$u = \sum_{n=1}^{\infty} \frac{1}{\mu_n} \langle f\,,\, e_n \rangle e_n$$

in $L^2([0,1])$.
Note $Le_n = \mu e_n$.

4.1 Method of continuity

Theorem 4.8. Assume $(E, \|.\|)$ Banach space. $A_t \in \mathcal{B}(E,E)$ for $t \in [0,1]$. Assume that there exists $c > 0$ such that

1. $\|x\| \leq c\|A_t(x)\|$ for all $x \in E$ and $t \in [0,1]$.
2. $\|A_t(x) - A_s(x)\| \leq C|t-s|\|x\|$ for all $x \in E$ and $s, t \in [0,1]$.
3. A_0 is invertible.

Then A_1 is invertible.

proof. Assume A_t is invertible.

$$A_s = \underbrace{A_t}_{\text{invertible}} \underbrace{\big(I + A_t^{-1}(A_s - A_t)\big)}_{\text{invertible if}\left\|A_t^{-1}(A_s-A_t)\right\|<1\text{by 'Neumann serieslemma}}$$

But

$$\left\|A_t^{-1}(A_s - A_t)\right\| \leq \underbrace{\left\|A_t^{-1}\right\|}_{\leq c} \cdot \underbrace{\|A_s - A_t\|}_{\leq C|t-s|}$$

So if $|t-s| < \frac{1}{c^2}$ then A_s is invertible. Pick a sequence t_k for $k = 1, \ldots, N$ such that

$$\max_{k=1,\ldots,N-1} |t_{k+1} - t_k| < \frac{1}{c^2}$$

where $0 = t_1 < t_2 < \ldots < t_N = 1$. Finally we get if A_0 is invertible that A_{t_1} is invertible.
$\Rightarrow A_{t_2}$ is invertible. $\Rightarrow \ldots \Rightarrow A_1$ is invertible. □

4.2 Orthogonal projections

Attention: We should have done this earlier.
$(E, \langle .\,,\,. \rangle)$ Hilbert space and S closed subspace. We know that

$$E = S \oplus S^\perp$$

For every $x \in E$ there are unique $y \in S$, $z \in S^\perp$ such that

$$x = y + z$$

Define $P_S : E \to E$ with

$$P_s(x) = y, \qquad \forall\, x \in E.$$

We note:

- P_S is linear:
  For $y_1, y_2 \in S$ we find $z_1, z_2 \in S^\perp$

  $$\begin{aligned} x_1 &= y_1 + z_1 \\ x_2 &= y_2 + z_2 \end{aligned}$$

  For scalars α_1, α_2 we have

  $$\alpha_1 x_1 + \alpha_2 x_2 = \underbrace{\alpha_1 y_1 + \alpha_2 y_2}_{\in S} + \underbrace{\alpha_1 z_1 + \alpha_2 z_2}_{\in S^\perp}$$

  Then we have

  $$P_S(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 P_S(x_1) + \alpha_2 P_S(x_2)$$

  and

  $$P_S(x_1) = y_1, \qquad P_S(x_2) = y_2$$

- P_S is bounded:

  $$\begin{aligned} \|P_S(x)\|^2 &= \|y\|^2 \\ &\overset{y \perp z}{\leq} \|y\|^2 + \|z\|^2 \\ &= \|x\|^2 \end{aligned}$$

  So $\|P_S\| \leq 1$. But

  $$\|P_S(y)\| = \|y\|, \qquad y \in S.$$

  So $\|P_S\| = 1$.

- P_s is self-adjoint:

  $$\begin{aligned} \langle P_S(x_1)\,,\, x_2\rangle &= \langle y_1\,,\, x_2\rangle \\ &= \langle y_1\,,\, y_2\rangle \\ &= \langle x_1\,,\, y_2\rangle \\ &= \langle x_1\,,\, P_S(x_2)\rangle, \qquad \forall\, x_1, x_2 \in E. \end{aligned}$$

- $P_S^2 = P_S$:

  $$\begin{aligned} P_S^2(x) &= P_S(P_S(x)) \\ &= P_S(y) \\ &= y \\ &= P_S(x) \end{aligned}$$

Proposition 4.9. Assume $P \in \mathcal{B}(E, E)$. With $P^2 = P$ and P self-adjoint. Then there exists a closed subspace S in E such that $P = P_S$

proof. What is S? Set

$$S = \{x \in E \,|\, P(x) = x\}$$

Claim: S is a closed subspace in E.
This is obvious.
Claim: S is closed.
Assume $x_n \in S$ with $x_n \to x$ in E.

$$P(x_n) \to P(x) \qquad \text{in } E \text{ since } P \text{ continuous.}$$

So $x \in S$. By the Orthogonal Decomposition Theorem we have

$$E = S \oplus S^{\perp}.$$

It remains to show $P = P_S$. Fix $x \in E$.

$$x = P(x) + x - P(x)$$

show that $P(x) \in S$ and $x - P(x) \in S^{\perp}$.

1. $P(x) \in S$ since
$$P(P(x)) = P^2(x) = P(x), \qquad \text{by assumption}$$

2. $x - P(x) \in S^{\perp}$ since for $y \in S$ we have

$$\begin{aligned}
\langle y \,,\, x - P(x)\rangle &= \langle P(y) \,,\, x - P(x)\rangle \\
&= \langle y \,,\, P^*(x - P(x))\rangle \\
&= \langle y \,,\, P(x) - P(x)\rangle \\
&= 0
\end{aligned}$$

□

Example 4.1 [last example]
Set $(E, \langle . \,,\, . \rangle)$ Hilbert space. $(e_n)_{n=1}^{\infty}$ ON-basis, $(f_n)_{n=1}^{\infty}$ ON-sequence.
Assume

$$\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \infty$$

Then

$$\Rightarrow \qquad (f_n)_{n=1}^{\infty} \text{ is an ON-basis.}$$

proof. **Step 1:** Assume

$$\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1.$$

Assume that $(f_n)_{n=1}^{\infty}$ is not an ON-basis. Then there exists $0 \neq x \in E$ such that

$$\langle x\,,\, f_k\rangle = 0, \qquad k = 1, 2, \dots .$$

But $(e_n)_{n=1}^{\infty}$ is an ON-basis.

$$x = \sum_{n=1}^{\infty} \langle x\,,\, e_n\rangle e_n$$

Parseval's formula gives

$$\begin{aligned}
0 &< \|x\|^2 \\
&= \sum_{n=1}^{\infty} |\langle x\,,\, e_n\rangle|^2 \\
&= \sum_{n=1}^{\infty} |\langle x\,,\, e_n - f_n\rangle|^2 \\
&\leq \sum_{n=1}^{\infty} \|x\|^2 \|e_n - f_n\|^2 \\
&= \|x\|^2 \underbrace{\sum_{n=1}^{\infty} \|e_n - f_n\|^2}_{<1} \\
&< \|x\|^2
\end{aligned}$$

which is a contradiction.
Conslusion: $(f_n)_{n=1}^{\infty}$ is an ON-basis.

Step 2: Assume

$$\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \infty.$$

Assume once more that $(f_n)_{n=1}^{\infty}$ is not an ON-basis.
There exists $0 \neq x \in E$ such that

$$\langle x\,,\, f_k\rangle = 0, \qquad k = 1, 2, \dots .$$

Since

$$\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \infty$$

there exists a positive interger $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} \|e_n - f_n\|^2 < 1.$$

Note that $\operatorname{span}\{x, f_1, f_2, \dots, f_n\}$ has the dimension $N+1$.
Claim: There exists a non-trivial solution to

$$\alpha_0 x + \alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_N f_N \perp e_k, \qquad k = 1, 2, \dots, N.$$

N equations, $N+1$ unknowns and it is a homogenous linear system. There exists

$$y = \alpha_0 x + \alpha_1 f_1 + \ldots + \alpha_N f_N$$

where not all α_k's are 0 such that

$$y \perp e_k, \qquad \text{for } k = 1, 2, \dots, N.$$

Note that $y \neq 0$.
$(e_n)_{n=1}^{\infty}$ is an ON-basis for E. So

$$\begin{aligned} y &= \sum_{n=1}^{\infty} \langle y \,,\, e_n \rangle e_n \\ &= \sum_{n=N+1}^{\infty} \langle y \,,\, e_n \rangle e_n \end{aligned}$$

Parseval's formular gives

$$\begin{aligned} 0 &< \|y\|^2 \\ &= \sum_{n=N+1}^{\infty} |\langle y \,,\, e_n \rangle|^2 \\ &= \sum_{n=1=N+1}^{\infty} |\langle y \,,\, e_n - f_n \rangle|^2 \\ &\leq \|y\|^2 \cdot \sum_{n=N+1}^{\infty} \|e_n - f_n\|^2 < \|y\|^2 \end{aligned}$$

This is a contradiction and we are done.

$\square$