# Applied Functionalanalysis

**Script of “Applied Functionalanalysis” by Prof. Peter Kumlin**

Tim Keil

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## foreword — cooperation

This document is a transcript of the lecture "Applied Functionalanalysis, WiSe 2016/2017, Term 1", by Prof. Peter Kumlin. It mainly contains the written content of the lecture. I will not assume any responsibility for the correctness of the content! For questions, remarks and mistakes please write an email to `keil.menden@web.de`. I'm grateful for every email.

# Contents

**1 Introduction** **1**
1.1 Introduction example . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1
1.2 Mappings between normal spaces . . . . . . . . . . . . . . . . . . . . . . . . . 30
1.3 Fixed point theory . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 38

# 1 Introduction

## 1.1 Introduction example

We have

$$\begin{cases} f'' + f = g, & \text{in } I = [0,1] \\ f(0) = 1,\ f'(0) = 1 \end{cases}$$

where $g$ is a known continous function in $I$. We will now consider different cases:

1. $g = 0$

   $$\Rightarrow\ f(x) = A\cos(x) + B\sin(x), x \in I$$

   where $A, B \in \mathbb{R}$.

2. $g$ arbitrary. We will now introduce the Method of variation of constants. Set

   $$f(x) = A(x)\cos(x) + B(x)\sin(x)$$

   Differentiate

   $$f'(x) = A'(x)\cos(x) + B'(x)\sin(x) - A(x)\sin(x) + B(x)\cos(x)$$

   Aussume (This is part of the method)

   $$A'(x)\cos(x) + B'(x)\sin(x) = 0, \qquad x \in I$$

   Differentiate $f'(x)$ and get

   $$f''(x) = \underbrace{-A(x)\cos(x) - B(x)\sin(x)}_{=-f(x)} - A'(x)\sin(x) + B'(x)\cos(x)$$

   We get

   $$g(x) = f''(x) + f(x) = -A'(x)\sin(x) + B'(x)\cos(x).$$

   Now:

   $$\begin{cases} A'(x)\cos(x) + B'(x)\sin(x) = 0, & x \in I \\ -A'(x)\sin(x) + B'(x)\cos(x) = g(x), & x \in I \\ A(0) = 1, \qquad B(0) = 0 \end{cases}$$

   We get

   $$\begin{aligned} A'(x) &= -g(x)\sin(x) \\ A(0) &= 1 \\ B'(x) &= g(x)\cos(x) \\ B(0) &= 0 \end{aligned}$$

This implies

$$
\begin{aligned}
A(x) &= A(0) + \int_0^x A'(t)\,\mathrm{d}t = 1 - \int_0^x g(t)\sin(t)\,\mathrm{d}t \\
B(x) &= B(0) + \int_0^x B'(t)\,\mathrm{d}t = 0 + \int_0^x g(t)\cos(t)\,\mathrm{d}t
\end{aligned}
$$

Hence

$$
\begin{aligned}
f(x) &= \cos(x) - \int_0^x g(t)\sin(t)\,\mathrm{d}t\cos(x) + \int_0^x g(t)\cos(t)\,\mathrm{d}t\sin(x) \\
&= \cos(x) + \int_0^x (\underbrace{\sin(x)\cos(t) - \sin(t)\cos(x)}_{=\sin(x-t)})g(t)\,\mathrm{d}t \\
&= \cos(x) + \int_0^x \sin(x-t)g(t)\,\mathrm{d}t \qquad (*)
\end{aligned}
$$

Check that $f(x)$ in $(*)$ satisfies the PDE.

**special case:**

Assume for $x \in I$

$$g(x) = k(x)f(x)$$

Here $k$ is a known continous function in $I$. Insert this in $(*)$. We obtain

$$f(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f(t)\,\mathrm{d}t, \qquad x \in I \qquad (**)$$

Observe that $f$ appears both in LHS and RHS. $(**)$ is a reformulation of the PDE with $g = kf$. Pick a $\underbrace{\text{continous function in } I}_{\in C(I)}$. call it $f_0$. Set

$$
\begin{aligned}
f_1(x) &= \cos(x) + \int_0^x \sin(x-t)k(t)f_0(t)\,\mathrm{d}t \\
f_2(x) &= \cos(x) + \int_0^x \sin(x-t)k(t)f_1(t)\,\mathrm{d}t \\
\vdots \qquad & \qquad \vdots \\
f_{n+1}(x) &= \cos(x) + \int_0^x \sin(x-t)k(t)f_n(t)\,\mathrm{d}t, \qquad n = 1,2,3,\dots
\end{aligned}
$$

**Hope:**

$f_n$ tends to some continous function $f$ on $I$, denoted $f_n \to f$. 'Tends to' has to be more precis!

$$\begin{array}{ccc} f_{n+1}(x) = \cos(x) + & \displaystyle\int_0^x \sin(x-t)k(t)f_n(t)\,\mathrm{d}t \\ \downarrow & \downarrow \\ f(x) = \cos(x) + & \displaystyle\int_0^x \sin(x-t)k(t)f(t)\,\mathrm{d}t \end{array}$$

for $x \in I$. Simplify notation set for $v \in C(I)$

$$\begin{cases} u(x) & = \cos(x) \\ kv(x) & = \int_0^x \sin(x-t)k(t)v(t)\,\mathrm{d}t \end{cases}$$

We have $f_0 \in C(I)$, $f_{n+1} = u + kf_n$ for $n = 0, 1, 2, \ldots$ (!)
Facts from previous calculus classes:

**Definition** (Sequenze of continous functions)**.**

$$v_n \in C(I), \qquad n = 1, 2, \ldots$$

We say that $(v_n)_{n=1}^{\infty}$ converges uniformly in $I$ if

$$\max_{x\in I} |v_n(x) - v_m(x)| \to 0, \qquad n, m \to \infty$$

i.e.

$$\forall \varepsilon > 0 \exists N : \forall n, m \geq N : \max_{x\in I} |v_n(x) - v_m(x)| < \varepsilon$$

**Lemma .** Suppose that $(v_n)_{n=1}^{\infty}$ converges uniformly on $I$. then there exists $v \in C(I)$ such that

$$\max_{x\in I} |v_m(x) - v(x)| \to 0 \qquad \text{as } m \to \infty$$

Back to (!):
More Notation:

$$k(kv) = k^2 v, \qquad v \in C(I)$$

and

$$k^{n+1} v = k(k^n v), \qquad n = 1, 2, \ldots$$

We have $f_0 \in C(I)$, $f_1 = u + kf_0$ and

$$f_2 = u + kf_1 = u + k(u + kf_0)$$

and so on. Note that

$$k(v + w) = kv + kw$$

Then

$$
\begin{aligned}
f_2 &= u + k(u + kf_0) = k + ku + k(kf_0) = u + ku + k^2 f_0 \\
f_3 &= u + kf_2 = u + ku + k^2 u + k^3 f_0
\end{aligned}
$$

and in general for $n = 1, 2, \dots$

$$
f_n = ku + \dots + k^{n-1}u + k^n f_0, \qquad n = 1, 2, \dots
$$

Assume $n > m$ then

$$
f_n - f_m = k^m u + \dots + k^{n-1}u + k^n f_0 - k^m f_0
$$

Set for $v \in C(I)$

$$
\|v\| = \max_{x \in I} |v(x)|
$$

Note

$$
\|v + w\| \leq \|v\| + \|w\| \qquad \text{for } v, w \in C(I)
$$

and

$$
\|-v\| = \|v\|.
$$

We have

$$
\begin{aligned}
\|f_n - f_m\| &= \left\|k^m u + \dots + k^{n-1}u + k^n f_0 - k^m f_0\right\| \\
&\leq \|k^m u\| + \dots + \left\|k^{n-1}u\right\| + \|k^n f_0\| + \|-k^m f_0\|.
\end{aligned}
$$

Assumption

$$
\sum_{l=1}^{\infty} \left\|k^l v\right\| < \infty \qquad \text{for all } v \in C(I) \qquad (***)
$$

Under this assumption

$$
\|f_n - f_m\| \to 0 \qquad \text{as } n, m \to \infty
$$

since

$$
\begin{aligned}
\sum_{l=1}^{\infty} \left\|k^l u\right\| &< \infty \qquad\qquad (u(x) = \cos(x)) \\
\sum_{l=1}^{\infty} \left\|k^l f_0\right\| &< \infty \qquad\qquad (f_0 \in C(I))
\end{aligned}
$$

conclusion: $(f_n)_{n=1}^{\infty}$ converges uniformly on $I$. By lemma above there exists $f \in C(I)$ such that

$$
\max_{x \in I} |f_n(x) - f(x)| \to 0, \qquad n \to \infty
$$

i.e.

$$
\|f_n - f\| \to 0, \qquad n \to \infty
$$

'Back hope': $f_n$ tends to $f$, denoted $f_n \to f$ shall be interpretated as

$$\|f_n - f\| \to 0, \qquad n \to \infty$$

Remember

$$f_{n+1}(x) = u(x) + kf_n(x) \to ?$$

For $x \in I$ there is

$$\begin{aligned}
|kf_n(x) - kf(x)| &= |\int_0^x \sin(x-t)k(t)f_n(t)\,\mathrm{d}t - \int_0^x \sin(x-t)k(t)f(t)\,\mathrm{d}t| \\
&\leq \int_0^x |\sin(x-t)k(t)|\underbrace{|f_n(t) - f(t)|}_{\leq \|f_n - f\|}\,\mathrm{d}t \\
&\leq \int_0^x |\sin(x-t)k(t)|\,\mathrm{d}t\|f_n - f\|
\end{aligned}$$

In particular

$$\begin{aligned}
\|kf_n - kf\| &\leq \max_{x\in I} \int_0^x \underbrace{|\sin(x-t)|}_{\leq 1} \underbrace{|k(t)|}_{\max_{t\in I}|k(t)|<\infty} \,\mathrm{d}t\|f_n - f\| \\
&\leq \|k\|\|f_n - f\|
\end{aligned}$$

We have, provided $(***)$ holds, shown

$$\begin{aligned}
f_{n+1} &= u + kf_n \\
&\downarrow \\
f &= u + kf
\end{aligned}$$

Let us try to prove $(***)$. For $v \in C(I)$ arbitrary and for $x \in I$

$$\begin{aligned}
\|kv(x)\| &= |\int_0^x \sin(x-t)k(t)v(t)\,\mathrm{d}t| \\
&\leq \int_0^x \underbrace{|\sin(x-t)|}_{\leq 1}\underbrace{|k(t)|}_{\leq \|k\|}|v(t)|\,\mathrm{d}t \\
&\leq \int_0^x \underbrace{|v(t)|}_{\leq \|v\|}\,\mathrm{d}t\|k\| \\
&\leq \|k\|\|v\|x
\end{aligned}$$

In particular

$$\|kv\| \leq \|k\|\|v\|$$

and

$$\begin{aligned}
|k^2 v(x)| &\leq \int_0^x |kv(t)|\,\mathrm{d}t\|k\| \\
&\leq \int_0^x \|k\|\|v\|t\,\mathrm{d}t \cdot \|k\| \\
&= \|k\|^2\|v\|\frac{x^2}{2}
\end{aligned}$$

In particular

$$\left\|k^2 v\right\| \leq \|k\|^2 \|v\| \frac{1}{2}$$

By induction we get

$$|k^n v(x)| \leq \|k\|^n \|v\| \frac{x^m}{m!} \qquad x \in I$$
$$\|k^n v\| \leq \|k\|^n \|v\| \frac{1}{n!}$$

So

$$\begin{aligned}\sum_{l=1}^{\infty} \left\|k^l v\right\| &\leq \sum_{l=1}^{\infty} \|k\|^l \|v\| \frac{1}{l!} \\ &= \|v\| \sum_{l=1}^{\infty} \frac{\|k\|^l}{l!} \\ &\leq \|v\| e^{\|k\|} < \infty\end{aligned}$$

consider Taylor expansion. $\Rightarrow (***)$ holds true.
We have now shown that $f = u + kf$ where $u(x) = \cos(x)$ and

$$kv = \int_0^x \sin(x-t) k(t) v(t) \, \mathrm{d}t$$

$x \in I$ for $v \in C(I)$, has a solution $f \in C(I)$.
Question: Is the solution unique?
Assume $f, \tilde{f} \in C(I)$ such that $f = u + kf$ and $\tilde{f} = u + k\tilde{f}$. Set

$$v = f - \tilde{f} \in C(I)$$

$$\begin{aligned}\Rightarrow v &= (u + kf) - (u + k\tilde{f}) \\ &= kf - k\tilde{f} \\ &= k(f - \tilde{f}) \\ &= kv\end{aligned}$$

We have $v = kv$, implies that $kv = k(kv) = k^2 v$. So for $n = 1, 2, \ldots$

$$v = kv = k^2 v = \ldots = k^n v$$

We know

$$\sum_{n=1}^{\infty} \|k^n \hat{v}\| < \infty \qquad \text{for all } \hat{v} \in C(I).$$

Apply this to $\hat{v} = v$:

$$\sum_{n=1}^{\infty} \underbrace{\|k^n v\|}_{=\|v\|} < \infty$$

So $\|v\| = 0$ with implies $v(x) = 0$ for all $x \in I$. So we have $f(x) = \tilde{f}(x)$ for $x \in I$.
$\Rightarrow$ Answer to the question above: YES !

We have more or less proved the following theorem:

**Theorem 1.1.** Set $I = [0,1]$. Suppose $u \in C(I)$ and $k \in C(I \times I)$. Consider

$$f(x) = u(x) + \int_0^x k(x,t) f(t)\,\mathrm{d}t, \qquad x \in I \qquad (1)$$

Then $(1)$ has a unique solution $f \in C(I)$

With the same technology we can prove:

**Theorem 1.2.** Set $I = [0,1]$. Suppose $u \in C(I)$, $k \in C(I \times I)$ and $\max\limits_{(x,t)\in I\times I} |k(x,t)| < 1$. Consider

$$f(x) = u(x) + \int_0^1 k(x,t) f(t)\,\mathrm{d}t, \qquad x \in I \qquad (2).$$

Then $(2)$ has a unique solution $f \in C(I)$.

Different notions: see intoductory example.

**Definition** (vector space). $C(I)$ with the operations for $x \in I$

**addition** $v, w \in C(I)$: $\qquad (v + w)(x) = v(x) + w(x)$

**mult. by scalar** $v \in C(I)$, $\lambda \in \mathbb{R}$: $\qquad (\lambda v)(x) = \lambda v(x)$

Note that $v + w, \lambda v \in C(I)$.

**Definition** (norm). norm on $C(I)$ for instance

$$\|v\| = \max_{x\in I} |v(x)|$$

with norm given we can talk about convergence and confirmity

**Definition** (Cauchy sequence). In our example a sequence $(f_n)_{n=1}^{\infty}$ is called Cauchy sequence if $\|f_n - f_m\| \to 0$ for $n, m \to \infty$.

**Definition .** $C(I)$ with the max-norm. Lemma above says that every Cauchy sequence converges i.e.

$$\|v_n - v_m\| \to 0, \qquad n, m \to \infty$$

This applies

$$\exists\, v \in C(I) : \|v_n - v\| \to 0, \qquad n \to \infty$$

This is the defining property of a Banach space.

$K$ linear mapping $C(I) \to C(I)$ with

$$\begin{aligned} K(v+w) &= K(v) + K(w) \\ K(\lambda v) &= \lambda K(v) \end{aligned}$$

for $v, w \in C(I)$, $\lambda \in \mathbb{R}$.
$K$ bounded linear:

$$\|Kv\| \leq M\|v\| \qquad \forall v \in C(I)$$

where $M > 0$ independent of $v$.

**Definition** (operator norm). Define

$$\|K\| = \inf\{M > 0 \,|\, \|Kv\| \leq M\|v\| \text{ for all } v \in C(I)\}.$$

**fixed point results:**
Our example: $f = u + kf =: T(f)$ and $f_0 \in C(I)$ fixed.
Form sequence of iterants $(f_n)_{n=1}^{\infty}$, $f_n = T(f_{n-1})$, $n = 1, 2, \ldots$ if

$$\|T(v) - T(w)\| \leq c\|v - w\|$$

for all $v, w \in C(I)$ for some $c < 1$. Then there is a unique $v \in C(I)$ such that $v = T(v)$.
This is Banach's fixed point theorem.

**Definition** (Green's function). Our example:

$$L = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + 1$$

differential operator. Boundary conditions

$$f(0) = f'(0) = 0.$$

Then

$$f(x) = \int_0^1 g(x,t)h(t)\,\mathrm{d}t$$

is a solution to

$$\begin{cases} f'' + f &= h, \\ f(0) = f'(0) &= 0 \end{cases}$$

**Definition** (real vector space). We say that $E$ is a real vector space if it is a non-empty set with the operations

**addition** $E \times E \to E, \qquad (x, y) \mapsto x + y$

**mult. with scalar** $\mathbb{R} \times E \to E$, $\qquad (\lambda, x) \mapsto \lambda x$

satisfying the axioms:

(1) $x + y = y + x$, $\qquad$ for all $x, y \in E$

(2) $x + (y + z) = (x + y) + z$, $\qquad$ for all $x, y, z \in E$

(3) For all $x, y \in E$ there exists $z \in E$ such that $x + z = y$

(4) $\alpha(\beta x) = (\alpha \cdot \beta)x$, $\qquad$ for all $\alpha, \beta \in \mathbb{R}, x \in E$

(5) $\alpha(x + y) = \alpha x + \alpha y$, $\qquad$ for all $\alpha \in \mathbb{R}, x, y \in E$

(6) $(\alpha + \beta)x = \alpha x + \beta x$, $\qquad$ for all $\alpha, \beta \in \mathbb{R}, x \in E$

(7) $1 \cdot x = x$, $\qquad$ for all $x \in E$.

**Remark.** $E$ is a complec vector space if all $\mathbb{R}$ in the definition above are replaced by $\mathbb{C}$.

**Remark.** (1)

$$\exists !0 \in E : \qquad x + 0 = x \qquad \text{for all } x \in E.$$

since: Fix $x \in E$, by $(3)$, $\exists\, 0_x$ such that $0_x + x = x$.
Fix $y \in E$. We want to show that $y + 0_y = y$. By $(3)$, there exists $z \in E$ such that $x + z = y$. So

$$\begin{aligned} y + 0_x &= (x + z) + 0_x \\ &\overset{(1)}{=} (z + x) + 0_x \\ &\overset{(2)}{=} z + (x + 0_x) \\ &= z + x \\ &\overset{(1)}{=} x + z \\ &= y. \end{aligned}$$

Assume $x + 0_1 = x$, $x + 0_2 = x$ for all $x \in E$. We want to show $0_1 = 0_2$:

$$0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2$$

(2)

$$\forall x \in E : \exists ! -x \in E : x + (-x) = 0$$

proof: exercise.

(3)

$$\begin{aligned} 0x &= 0 \qquad \text{for all } x \in E \\ (-1)x &= -x \qquad \text{for all } x \in E \end{aligned}$$

**Examples** (Examples of real vector spaces). 1) $\mathbb{R}$ with standard addition and mult. by scalar.

2) $\mathbb{R}^n$, $n = 2, 3, \dots$

**addition** $(x_1, x_2, \dots) + (y_1, y_2, \dots) = (x_1 + y_1, x_2 + y_2, \dots)$

**mult.** $\lambda(x_1, x_2, \dots) = (\lambda x_1, \lambda x_2, \dots)$

3) $\mathbb{R}^\infty = \{(x_1, \dots, x_n, \dots) \,|\, x_n \in \mathbb{R}, n = 1, 2, \dots\}$

4) $1 \le p < \infty$,

$$l^p = \left\{ (x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty \,\middle|\, \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} < \infty \right\}$$

with the same addition and mult. by scalar as in $\mathbb{R}^\infty$. We have to check:

(1) $x, y \in l^p \quad \Rightarrow \quad x + y \in l^p$

(2) $x \in l^p, \lambda \in \mathbb{R} \quad \Rightarrow \quad \lambda x \in l^p$

For $(1)$ we assume $x = (x_1, \dots, x_n, \dots)$ and $y = (y_1, \dots, y_n, \dots)$.

$$x \in l^p \quad \Rightarrow \quad \sum_{n=1}^{\infty} |x_n|^p < \infty$$

$$y \in l^p \quad \Rightarrow \quad \sum_{n=1}^{\infty} |y_n|^p < \infty$$

$$\Rightarrow \quad x + y = (x_1 + y_1, \dots) \overset{?}{\in} l^p?$$

$$\begin{aligned}
\Rightarrow \sum_{n=1}^{\infty} |x_n + y_n|^p &\le \{|x_n + y_n| \le |x_n| + |y_n| \le 2\max\{|x_n|, |y_n|\}\} \\
&\quad \{|x_n + y_n|^p \le 2^p \left(|x_n|^p + |y_n|^p\right)\} \\
&\le \sum_{n=1}^{\infty} 2^p (|x_n|^p + |y_n|^p) \\
&= 2^p \underbrace{\sum |x_n|^p}_{<\infty} + 2^p \underbrace{\sum |y_n|^p}_{<\infty} < \infty
\end{aligned}$$

and

$$\sum_{n=1}^{\infty} |\lambda x_n|^p = \sum_{n=1}^{\infty} |\lambda|^p \cdot |x_n|^p = |\lambda|^p \sum_{n=1}^{\infty} |x_n|^p < \infty$$

5) function spaces, say real-valued functions on $I$.

**addition:** $(f + g)(x) = f(x) + g(x), \qquad x \in I$

**mult. by scalar:** $(\lambda f)(x) = \lambda f(x) \qquad$ for functions $f$ and $g$

6) $C(I)$ : addition and mult. by scalar as in $(5)$.
   $f, g$ continuous in $I$ implies that $f + g$ is continuous in $I$.
   Also if $f$ is continuous and $\lambda \in \mathbb{R}$ then $(\lambda f)$ is continuous in $I$.

7) $P(I) =$ polynomials in $I$.

8) $P_k(I) =$ polynomials of degree at most $k$ in $I$.

**Theorem** (Hölder's inequality). Assume $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $(x_1, \dots, x_n, \dots)$ and $(y_1, y_2, \dots, y_n, \dots)$ be sequences of complex numbers. Then

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{n=1}^{\infty} |y_n|^q \right)^{\frac{1}{q}}$$

Remark there the LHS can be infinity, but the RHS can also be infinity.

**proof.** **Step 1** We're going to proof

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \qquad \text{for all } a, b > 0$$

$$\int_0^a x^{p-1} \, \mathrm{d}x = \frac{a^p}{p}$$

Note $y = x^{p-1}$ gives

$$x = y^{\frac{1}{p-1}} = y^{\frac{1}{\frac{1}{1-\frac{1}{q}} - 1}} = y^{\frac{1}{\frac{q}{q-1} - 1}} = y^{q-1}$$

so

$$\int_0^b y^{q-1} \, \mathrm{d}y = \frac{b^q}{q}$$

We get

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

(You also get condition for $=$)

**Step 2** It is enough to consider the cases LHS $> 0$ and RHS $< \infty$. There consists integer $N$ such that

$$0 < \sum_{n=1}^{N} |x_n|^p, \sum_{n=1}^{N} |y_n|^q < \infty$$

Set

$$a = \frac{|x_k|}{\left( \sum_{n=1}^{N} |x_n|^p \right)^{\frac{1}{p}}}, \qquad k = 1, 2, \dots, N,$$

$$b = \frac{|y_k|}{\left( \sum_{n=1}^{N} |y_n|^q \right)^{\frac{1}{q}}}, \qquad k = 1, 2, \dots, N.$$

Insert into

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

$$\frac{|x_k y_k|}{\left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^N |y_n|^q\right)^{\frac{1}{q}}} \leq \frac{|x_k|^p}{p \sum_{n=1}^N |x_n|^p} + \frac{|y_k|^q}{q \sum_{n=1}^N |y_n|^q}, \qquad k = 1, 2, \ldots, N.$$

We sum over $k$ from $1$ to $N$.

$$\sum_{k=1}^N |x_k y_k| \leq \left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{n=1}^N |y_n|^q\right)^{\frac{1}{q}}$$

Let $N \to \infty$. First in RHS and then in LHS.

□

**Theorem** (Minkowski's inequality). Assume $1 \leq p < \infty$. and $X, Y \in l^p$. Then

$$\|X + Y\|_{l^p} \leq \|X\|_{l^p} + \|Y\|_{l^p}$$

**proof.** $p = 1$

$$\begin{aligned}
\|X + Y\|_{l^1} &= \|(x_1, x_2, \ldots, x_n, \ldots) + (y_1, y_2, \ldots, y_n, \ldots)\|_{l^1} \\
&= \|(x_1 + y_1, \ldots, x_n + y_n, \ldots)\|_{l^1} \\
&= \sum_{n=1}^\infty |x_n + y_n| \\
&\leq \sum_{n=1}^\infty (|x_n| + |y_n|) \\
&= \sum_{n=1}^\infty |x_n| + \sum_{n=1}^\infty |y_n| \\
&= \|X\|_{l^1} + \|Y\|_{l^1}
\end{aligned}$$

$1 < p < \infty$

$$\begin{aligned}
\|X + Y\|_{l^p}^p &= \sum_{n=1}^\infty |x_n + y_n|^p \\
&= \sum_{n=1}^\infty |x_n + y_n| |x_n + y_n|^{p-1} \\
&\leq \sum_{n=1}^\infty |x_n| |x_n + y_n|^{p-1} + \sum_{n=1}^\infty |y_n| |x_n + y_n|^{p-1}.
\end{aligned}$$

Use Hölder to get

$$\sum_{n=1}^{\infty}|x_n||x_n+y_n|^{p-1} \leq \underbrace{\left(\sum_{n=1}^{\infty}|x_n|^p\right)^{\frac{1}{p}}}_{=\|X\|_{l^p}} \cdot \left(\sum_{n=1}^{\infty}|x_n+y_n|^{(p-1)q}\right)^{\frac{1}{q}}$$
$$= \left\{(p-1)q = (p-1)\frac{1}{1-\frac{1}{p}} = p\right\}$$
$$= \|X\|_{l^p}\left(\sum_{n=1}^{\infty}|x_n+y_n|^p\right)^{\frac{1}{q}}.$$

We have

$$\|X+Y\|_{l^p}^p \leq (\|X\|_{l^p} + \|Y\|_{l^p})\,\|X+Y\|_{l^p}^{\frac{p}{q}}$$

If $\|X+Y\|_{l^p} \neq 0$ then

$$\|X+Y\|_{l^p}^{p-\frac{p}{q}} \leq \|X\|_{l^p} + \|Y\|_{l^p}$$

there

$$p - \frac{p}{q} = p(1-\frac{1}{q}) = p\frac{1}{p} = 1.$$

□

**Remark.** $f \in C([0,1])$ then for $1 \leq p < \infty$

$$\|f\|_{L^p} = \left(\int_0^1 |f(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}}.$$

Claim:

$$\|fq\|_{L^1} = \int_0^1 |f(t)\cdot g(t)|\,\mathrm{d}t \leq \|f\|_{L^p}\cdot\|g\|_{L^q}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Also we have

$$\|f+q\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

This is proven with the same technique as we used for $l^p$. $\sum_{n=1}^{\infty}$ is replaced by $\int_0^1\,\mathrm{d}t$.
$E$ real/complex vector space. $x_1,\ldots,x_n \in E$, $\lambda_1,\ldots,\lambda_n$ scalar. We say that

$$\lambda_1 x_1,\ldots,\lambda_n x_n$$

is a linear combination of $x_1,\ldots,x_n$. We say that $x_1,\ldots,x_n$ are linear independent if

$$\alpha_1 x_1 + \ldots + \alpha_n x_n = 0 \qquad \Rightarrow \qquad \alpha_1 = \ldots = \alpha_n = 0.$$

If $A \subset E$, we say that $A$ is linear independant if every linear combination of vectors in $A$ is linear independant.

**Examples.** (1) Set $E = P([0,1])$ and $A = \{p_k \,|\, p_k(x) = x^k, x \in [0,1], k = 0,1,\ldots\}$. A is linear independant since:
consider

$$\alpha_0 p_0 + \alpha_1 p_1 + \ldots + \alpha_n p_n = 0$$

i.e.

$$\alpha_0 p_0(x) + \alpha_1 p_1(x) + \ldots + \alpha_n p_n(x) = 0(x), \qquad x \in [0,1]$$

i.e.

$$\alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n = 0, \qquad x \in [0,1]$$

If $x = 0$ then $\alpha_0 = 0$

$$\alpha_1 x + \ldots + \alpha_n x^n = 0, \qquad x \in [0,1].$$

Differentiate

$$\alpha_1 + 2\alpha_2 x + \ldots + n\alpha_n x^{n-1} = 0$$

gives $\alpha_1 = 0$. Continue and get

$$\alpha_0 = \alpha_1 = \ldots = \alpha_n = 0.$$

Set $B \subset E$ where

$$\begin{aligned} \text{span } B &= \{\text{set of all linear combinations of elements in B}\} \\ &= \left\{ \sum_{k=1}^{n} \lambda_k x_k \,\middle|\, x_k \in B, \lambda_k \in \mathbb{R}, k = 1,2,\ldots,n \text{ where n is a positive integer} \right\} \end{aligned}$$

**Remark.**

$$\sum_{k=1}^{n} \lambda_k x_k \in E$$

$$\sum_{k=1}^{\infty} \lambda_k x_k \text{ has no meaning}$$

$C \subset E$ is called a basis for E if

1) $C$ linear independant.
2) span $C = E$

continue of the example above:
Claim: $A$ is a basis for E.

(2) Set $E = l^2$ and

$$A = \{X_k \,|\, k = 1,2,\ldots\}$$

$$X_k = (0,0,\ldots,0,1,0,0,\ldots)$$

Claim: A is linear independant since

$$\alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_n X_n = 0$$

Here

$$\alpha_1 X_1 = (\alpha_1, 0, 0, \ldots), \qquad etc$$

and

$$0 = (0, 0, \ldots)$$

So

$$(\alpha_1, \alpha_2, \ldots, \alpha_n, 0, \ldots) = (0, 0, \ldots)$$

So $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.
Question: Is $A$ a basis for $l^2$?
We note: If $X \in$ span $A$ then

$$X = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$$

for some positive integer $n$, i.e. $X$ has only finitely many nonzero positions.
Cosider:

$$X := (1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots)$$

$$\|X\|_{l^2} = \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} < \infty$$

So $X \in l^2 \setminus$ span $A$.

**Remark.** Every vector space has a basis (if we are allowed to use Axiom of Choice/ zorns lemma).
Basis = vector space basis = Hamel basis

Assume $x_1, \ldots, x_n$ is a basis for $E$. Then every basis for $E$ must contain $n$ different elements.

$$n = \dim E$$

is well-defined. (System of linear equations, homogeneous with more unknowns than equations. Then there exists a nontrivial solution.)

**Definition** (norm). $E$ vector space. We say that $\|.\| : E \to [0, \infty)$ is a norm on $E$ if

1) $\|x\| = 0 \qquad \Rightarrow x = 0$
2) $\|\lambda x\| = |\lambda| \|x\| \qquad$ for all $x \in E, \lambda \in \mathbb{R}$
3) $\|x + y\| \leq \|x\| + \|y\| \qquad$ for all $x, y \in E$

**Remark.**

$$\|0\| = \|0 \cdot 0\| = \underbrace{|0|}_{=0} \|0\| = 0$$

**Examples.** (1) $1 < p < \infty$ and

$$\|X\|_{l^p} = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

is a norm on $l^p$. Check $1)$,$2)$ and $3)$ above:

1)

$$0 = \|X\|_{l^p} = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

It follows

$$\begin{aligned} & x_n = 0, \qquad n = 1, 2, \ldots \\ \Rightarrow \qquad & X = (x_1, x_2, \ldots) = (0, 0, \ldots) = 0 \end{aligned}$$

2)

$$\|\lambda X\|_{l^p} = \left( \sum_{n=1}^{\infty} |\lambda x_n|^p \right)^{\frac{1}{p}} = \left( |\lambda|^p \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} = |\lambda| \|X\|_{l^p}$$

3)

$$\|X + Y\|_{l^p} \leq \{\text{Minkowski's inequality}\} \leq \|X\|_{l^p} + \|Y\|_{l^p}$$

(2) $E = C([0,1])$ and $f \in E$

$$\|f\| = \max_{t \in [0,1]} |f(t)| \in [0, \infty)$$

Check the axioms above

1) If $\|f\| = 0$ it follows

$$|f(t)| = 0 \text{ for all } t \in [0,1], \qquad \Rightarrow \qquad f = 0$$

2)

$$\|\lambda f\| = \max_{t \in [0,1]} \underbrace{|\underbrace{(\lambda f)(t)}_{\lambda f(t)}|}_{|\lambda||f(t)|} = |\lambda| \max_{t \in [0,1]} |f(t)| = |\lambda| \|f\|$$

3)

$$\|f + g\| = \max_{t \in [0,1]} |\underbrace{(f + g)(t)}_{f(t)+g(t)}| = \max_{t \in [0,1]} (|f(t)| + |g(t)|) \leq \max_{t \in [0,1]} |f(t)| + \max_{t \in [0,1]} |g(t)| = \|f\| + \|g\| \blacksquare$$

(3) $E = C([0,1])$ and $f \in E$.

$$\|f\|_{L^1} = \int_0^1 |f(t)|\,\mathrm{d}t$$

defines also a norm on $E$.

**3)**

$$\begin{aligned}
\|f+g\|_{L^1} &= \int_0^1 |\underbrace{(f+g)(t)}_{f(t)+g(t)}|\,\mathrm{d}t \\
&\leq \int_0^1 (|f(t)| + |g(t)|)\,\mathrm{d}t \\
&= \int_0^1 |f(t)|\,\mathrm{d}t + \int_0^1 |g(t)|\,\mathrm{d}t \\
&= \|f\|_{L^1} + \|g\|_{L^1}
\end{aligned}$$

**2)**

$$\|\lambda f\| = \int_0^1 \underbrace{|(\lambda f)(t)|}_{=|\lambda||f(t)|}\,\mathrm{d}t = |\lambda| \|f\|_{L^1}$$

**1)**

$$0 = \|f\|_{L^1} = \int_0^1 |f(t)|\,\mathrm{d}t$$

This implies $f(t) = 0$ for $t \in [0,1]$ since f is continuous! i.e. $f = 0$

**Theorem** (equivalent norm). $E$ vector space with norms $\|.\|$ and $\|.\|_*$. We say that $\|.\|$ and $\|.\|_*$ are equivalent if there exists $\alpha, \beta > 0$ such that

$$\alpha \|x\|_* \leq \|x\| \leq \beta \|x\|_* \qquad \text{for all } x \in E.$$

**Example.**

$E = C([0,1])$. Choose $y = f(t)$ and $y = |f(t)|$

$$\|f\| = \max_{t \in [0,1]} |f(t)|, \qquad \|f\|_* = \|f\|_{L^1} = \text{area}.$$

Question: Are these norms equivalent?
Claim $f \in C([0,1])$

$$\|f\|_* = \int_0^1 \underbrace{|f(t)|}_{\leq \|f\|}\,\mathrm{d}t \leq \|f\|$$

Choose $f_n(t)$ such that

$$\|f_n\| = 1, \qquad \|f_n\|_* = \frac{1}{2n}$$

So

$$\frac{\|f_n\|_*}{\|f_n\|} = \frac{1}{2n} \to 0 \qquad n \to \infty$$

The norms are not equivalent! Answer: NO !

**Theorem .** $E$ vector space with $\dim E < \infty$.
$\Rightarrow$ All norms on $E$ are equivalent.

**proof.** Assume $n = \dim E$ with a positive integer $n$. Let $x_1, x_2, \dots, x_n$ be a basis for $E$. For every $x \in E$

$$x = \alpha_1(x)x_1 + \ldots + \alpha_n(x)x_n$$

where $\alpha_1(x), \dots, \alpha_n(x)$ unique. Set

$$\|x\|_* = |\alpha_1(x)| + \ldots + |\alpha_n(x)|, \qquad x \in E$$

Claim: $\|.\|_*$ defines a norm on $E$ (easy proof)
Fix an arbitrary norm $\|.\|$ on $E$.
Claim: $\|.\|_*$ and $\|.\|$ are equivalent.
Note for $x \in E$

$$\begin{aligned} \|x\| &= \|\alpha_1(x)x_1 + \ldots + \alpha_n(x)x_n\| \\ &\le |\alpha_1(x)|\|x_1\| + \ldots + |\alpha_n(x)|\|x_n\| \\ &\le \max_{k=1,2,\dots,n} \|x_k\| (\underbrace{|\alpha_1(x)| + \ldots + |\alpha_n(x)|}_{=\|x\|_*}) \end{aligned}$$

Set $\beta = \max_{k=1,2,\dots,n} \|x_k\|$. Then

$$\|x\| \le \beta \|x\|_* \qquad \text{for all } x \in E.$$

Remains to prove: There exists $\alpha > 0$ such that

$$\alpha \|x\|_* \le \|x\| \qquad \text{for all } x \in E \qquad (*)$$

Let $E$ be a vector space with norm $\|.\|$ and $(v_m)_{m=1}^\infty$ a sequence in $E$. We say that $(v_m)_{m=1}^\infty$ converges in $(E, \|.\|)$ if there exists $v \in E$ such that $\|v_m - v\| \to 0$ for $n \to \infty$.
Notation: $v_m \to v$ in $(E, \|.\|)$.
Note: If we have $\|.\|$ and $\|.\|_*$ are equivalent, then

$$v_n \to v \text{ in } (E, \|.\|) \qquad \Leftrightarrow \qquad v_n \to v \text{ in } (E, \|.\|_*)$$

Back to $(*)$: Argue by contradiction.
Assume there is no $\alpha > 0$ such that

$$\alpha \|x\|_* \le \|x\| \qquad \text{for all } x \in E$$

For $k = 1, 2, 3, \ldots$ there are $y_k \in E$ such that

$$\frac{1}{k}\|y_k\|_* > \|y_k\|. \qquad (**)$$

We have

$$y_k = \alpha_1^{(k)} x_1 + \ldots + \alpha_n^{(k)} x_n$$

where $\alpha_1^{(k)}, \ldots, \alpha_n^{(k)}$ are unique scalars and $k = 1, 2, \ldots$.
$(**)$ implies that

$$k\|y_k\| < |\alpha_1^{(k)}| + \ldots + |\alpha_n^{(k)}|$$

WLOG we can assume $|\alpha_1^{(k)}| + \ldots + |\alpha_n^{(k)}| = 1$. ( If not consider

$$\begin{aligned}\lambda z &= \lambda(\alpha_1(z)x_1 + \ldots + \alpha_n(z)x_n) \\ &= (\lambda\alpha_1(z))x_1 + \ldots + (\lambda\alpha_n(z))x_n \\ &= \alpha_1(\lambda z)x_1 + \ldots + \alpha_n(\lambda z)x_n\end{aligned}$$

We have

$$\alpha_k(\lambda z) = \lambda\alpha_k(z), \qquad k = 1, 2, \ldots, n)$$

We have

$$k\|y_k\| < 1 \qquad k = 1, 2, \ldots$$

which implies $y_k \to 0$ in $(E, \|.\|)$.

<u>IF:</u>

$$\begin{aligned}\alpha_1^{(k)} &\to \bar{\alpha_1} \\ \alpha_2^{(k)} &\to \bar{\alpha_2} \\ &\vdots \\ \alpha_n^{(k)} &\to \bar{\alpha_n}\end{aligned}$$

for $k \to \infty$. Then set

$$\bar{y} = \bar{\alpha_1}x_1 + \ldots + \bar{\alpha_n}x_n$$

and get

$$\begin{aligned}\|y_k - \bar{y}\| &= \left\|(\alpha_1^{(k)} - \bar{\alpha_1})x_1 + \ldots + (\alpha_n^{(k)} - \bar{\alpha_n})x_n\right\| \\ &\le \underbrace{|\alpha_1^{(k)} - \bar{\alpha_1}|}_{\to 0}\underbrace{\|x_1\|}_{<\infty} + \ldots + \underbrace{|\alpha_n^{(k)} - \bar{\alpha_n}|}_{\to 0}\underbrace{\|x_n\|}_{<\infty} \to 0, \qquad k \to \infty\end{aligned}$$

$$\|\bar{y}\| = \|\bar{y} - y_k + y_k\| \le \underbrace{\bar{y} - y_k}_{\to 0} + \underbrace{\|y_k\|}_{\to 0} \to 0, \qquad k \to \infty$$

So $\|\bar{y}\| = 0$ hence $\bar{y} = 0$. But

$$|\bar{\alpha_1}| + |\bar{\alpha_2}| + \ldots + |\bar{\alpha_n}| = 1.$$

This contradicts $x_1, \dots, x_n$ is a basis.
We have for $k = 1, 2, \dots$ the vector $(\alpha_1^{(k)}, \alpha_2^{(k)}, \dots, \alpha_n^{(k)})$ where

$$|\alpha_1^{(k)}| + \dots + |\alpha_n^{(k)}| = 1$$

We focus on the first one and we have

$$|\alpha_1^{(k)}| \leq 1, \qquad k = 1, 2, \dots$$

By Bolzano-Weierstraß then there exists a converging subsequence $(\alpha_{1,1}^{(k)})_{k=1}^{\infty}$ of $(\alpha_1^{(k)})_{k=1}^{\infty}$. Set

$$\bar{\alpha_1} = \lim_{k \to \infty} \alpha_{1,1}^{(k)}$$

consider

$$(\alpha_{1,1}^{(k)}, \alpha_{2,1}^{(k)}, \dots, \alpha_{n,1}^{(k)}), \qquad k = 1, 2, \dots$$

We have

$$|\alpha_{2,1}^{(k)}| \leq 1, \qquad k = 1, 2, \dots$$

Bolzano-Weierstraß implies that there exists a converging subsequenz $(\alpha_{2,2}^{(k)})_{k=1}^{\infty}$ of $(\alpha_{2,1}^{(k)})_{k=1}^{\infty}$. Set

$$\bar{\alpha_2} = \lim_{k \to \infty} \alpha_{2,2}^{(k)}$$

□

**Definition** (normed space). Let $E$ be a vector space over $\mathbb{R}$ or $\mathbb{C}$. $\|.\| : E \to \mathbb{R}$ a norm on $E$ if

(i) $\|.\| > 0$ for any $x \in E \setminus \{0\}$

(ii) $\|\lambda x\| = |\lambda x|$ for any $\lambda \in \mathbb{C}, x \in E$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in E$.

Obs. $\|x\| = 0$ if $x = 0$. $(E, \|.\|)$ is called a normed space. A norm generates a distance function (metric)

$$L(x, y) := \|x - y\| \qquad \text{for any } x, y \in E.$$

**Examples.**

- $\mathbb{R}^n$ with $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$ is the eucledian norm.
- $C([0,1])$ continuous functions in $[0,1]$ with

$$L(f, g) = \|f - g\|_\infty := \max_{x \in [0,1]} |f(x) - g(x)|$$

**Definition** (balls). Let $x \in E$, $r > 0$. Define

$$\begin{aligned} B(x,r) &:= \{y \in E \,|\, \|x-y\| < r\} && \text{open ball} \\ \bar{B}(x,r) &:= \{y \in E \,|\, \|x-y\| \le r\} && \text{closed ball} \end{aligned}$$

**Definition** (open/closed). A subset $A \subset E$ of a normed space $(E, \|.\|)$ is called open of any point $x$ of $A$ is inner, i.e

$$\exists\, r > 0 \,:\, B(x,r) \subset A.$$

It is called closed if the complement $E \setminus A$ is open.

**Remark.**

- open balls are open sets.
- closed balls are closed.
- $(C([0,1]), \|.\|_\infty)$ with $\|f\|_\infty = \max_{x\in[0,1]}|f(x)|$.

  $$A := \{g \in C([0,1\,|\,)\}]f(x) < g(x),\, \forall\, x \in [0,1]$$

  is an open set $C([0,1])$.

  $$B := \{g \in C([0,1\,|\,)\}]f(x) \le g(x),\, \forall\, x \in [0,1]$$

  is a closed set.

**Properties**

- Any union of open sets is an open set.
- Any finite intersection of open sets is open.
- $\emptyset, E$ are both closed and open.
- Normed spaces are topological spaces.

**Definition** (convergence in normed spaces). Let $(E, \|.\|)$ be a normed space $\{x_n\}_n \subset E$. We say that $x_n$ converges to $x \in E$ if

$$\|x_n - x\| \to 0, \qquad n \to \infty$$

One can define open and closed using the definition of convergence:

**Satz.** $A \subseteq E$ is closed if any convergent sequence in $A$ has a limit in $A$, i.e

$$\substack{x_n \to x \\ \text{for } n\to\infty \\ x_n \in A} \Rightarrow x \in A$$

**proof.** $\Rightarrow$: Assume that $A$ is closed and $x_n \to x$. $x_n \in A$, but $x_n \notin A$. (try to get a contradiction).
$A$ is closed $\Rightarrow E \setminus A$ is open and hence $\exists\, r > 0$ such that

$$B(x, r) \subset E \setminus A.$$

Hence $\|x_n - x\| \geq r$ for any $n$. This is a contradiction because in that case $x_n \not\to x$

$\Leftarrow$: Assume that for any sequence $\{x_n\} \subset A$ such that $x_n \to x$ we have $x \in A$. We try to get a contradiction and assume that $A$ is not closed. Hence $E \setminus A$ is not open and therefore $\exists\, x \in E \setminus A$ which is not inner.

$$\Rightarrow \qquad \forall\, B(x, \frac{1}{n}) \text{ containts points outside } E \setminus A$$

i.e.

$$\exists\, x_n \in B(x, \frac{1}{n}),\ x_n \in A.$$

We get a sequence $\{x_n\} \subset A$ such that

$$\|x_n - x\| < \frac{1}{n} \qquad \Rightarrow \qquad x_n \to x$$

This is a contradiction

□

**Definition** (closure). $A \subset E$. The closure of $A$ is the minimal closed subset containing $A$. We write $\bar{A}$.

**Proposition .** $\bar{A}$ is the set of all limit points of $A$ which means

$$\bar{A} := \{x \in E \,|\, \text{there exists } \{x_n\} \subseteq A \text{ such that } x_n \to x\}$$

**proof.** exercise. □

**Definition** (dense). $A \subset E$ is dense in $E$ if

$$\bar{A} = E.$$

**Remark.** This definition of dense is equivalent to the following definition:

$$\forall\, x \in E,\ \forall\, \varepsilon > 0\, \exists\, y \in A \text{ such that } \|x - y\| < \varepsilon.$$

**Examples.** 1) $\mathbb{Q} \subseteq \mathbb{R}$ with $|.|$ usual absolut value function. $\mathbb{Q}$ is dense in $\mathbb{R}$.

2) $C([a,b])$. The Weirestrasstheorem says that the set of all polynomials are dense in $(C([a,b],\|.\|_\infty))$:

$$\forall\, f \in C([a,b]),\, \forall\, \varepsilon > 0\, \exists\, p - \text{polynomial such that } \max_{x\in[a,b]} |f(x) - p(x)| < \varepsilon.$$

Another example is $(C_0, \|.\|_\infty)$ where

$$C_0 = \{x = (x_1, x_2, \ldots) \,|\, x_k \to 0 \text{ as } k \to \infty\}$$

$$\|x\|_\infty = \sup_i |x_i|$$

$(C_0, \|.\|_\infty)$ is a normed space.

$$C_F = \{x = (x_1, x_2, \ldots) \,|\, \text{only a finite number of } x_i \neq 0\} \subset C_0$$

**Satz.** $C_F$ is dense in $C_0$

**proof.**

$$\forall\, x \in C_0\, \forall\, \varepsilon > 0 \text{ must find } y \in C_F \text{ such that } \|y - x\|_\infty < \varepsilon.$$

$$x \in C_0 \qquad \Rightarrow \qquad x_k \to 0 \text{ for } k \to \infty$$

$$\Rightarrow \qquad \forall\, \varepsilon > 0\, \exists\, K \text{ such that } |x_k| < \varepsilon\, \forall\, k \geq K$$

Let now $y = (x_1, x_2, \ldots, x_K, 0, \ldots) \in C_F$. Then

$$\|x - y\|_\infty = \|(0, 0, \ldots, 0, x_{K+1}, x_{K+2}, \ldots)\|_\infty = \sup_{k>K} |x_k| < \varepsilon$$

□

**Definition** (separable). A normed space $(E, \|.\|)$ is called separable if it contains a countable dense subset.

**Examples.**

- $(\mathbb{R}, |.|)$ is separable as $\mathbb{Q}$ is countable and dense in $\mathbb{R}$.
- $(\mathbb{R}^n, \|.\|_2)$ is separable, $\mathbb{Q}^n$ is countable and dense in $\mathbb{R}$.

**Definition** (compact set). For a normed space $(E, \|.\|)$ is $A \subset E$ a compact set if any sequence $\{x_n\} \subset A$ has a subsequence convergent to an element $x \in A$.

**Example.** Any bounded and closed subset in $\mathbb{R}, \mathbb{R}^n, \mathbb{C}^n$ is compact. A sequence $\{x_n\}$ of a bounded set is bounded. From real Analysis one knows it has a subsequence that is convergent. If the subset is closed then the limit point is inside the set.

**Lemma .** $S \subset$ compact in $(E, \|.\|)$ implies that $S$ is closed and bounded.(Bounded means that $S \subset B(0,R)$ for some $R > 0$)

**proof.** Let $S$ be a compact subset of $E$. Assume that $S$ is not bounded. Hence for any $n > 0$ there exists points in $S$ which are outside $B(0,n)$, i.e.

$$\exists x_n \in S : \|x_n\| > n.$$

Then $\{x_n\}$ can not have a convergent subsequence as if $x_{n_k} \to x$ then

$$n_k < \|x_{n_k}\| = \|x_{n_k} - x + x\| \leq \|x_{n_k} - x\| + \|x\| \to \|x\|$$

but $n_k \to \infty$. This is a contradiction, hence $S$ must be bounded.
$S$ must be closed, because if $x_n \to x$ then any subsequence converges to $x$. From the definition of compactness and uniqueness of the limit we have $x \in S$.

$\square$

**Remark.** In general, $S$ bounded and closed doesn't imply that $S$ is compact.
For instance let $E = C([0,1])$. Then $S = \{g \in C([0,1\,|\,)\}]\|g\|_\infty \leq 1$ is closed and bounded, but not compact.
Take $x_n(t) := t^n$. Then $x_n \in S$. $\{x_n\}$ does not have a subsequence convergent to a continuous function.

**Theorem .** $(E, \|.\|)$ normed space and $\dim E < \infty$
iff $\{\forall\, A \subset E,\ A$compact $\Leftrightarrow A$ is closed and bounded$\}$

**proof.** $\Rightarrow$: If $\dim E < \infty$ then $A$ is compact iff $A$ is bounded and closed (exsercise)

$\Leftarrow$: Enough to prove the following:
If $\dim E = \infty$ then the unit ball $S = \{x \in E \,|\, \|x\| \leq 1\}$ is not compact.

**Lemma** (Riesz's lemma)**.** If $X$ is a proper closed subspace of a normed space $(E, \|.\|)$ then for every $\varepsilon \in (0,1)$ there exists an $x_\varepsilon \in E$ with $\|x_\varepsilon\| = 1$ such that

$$\|x_\varepsilon - x\| \geq \varepsilon \qquad \forall x \in X.$$

**proof.** Let $z \in E \setminus X$ ($X$ proper and hence $E \setminus X \neq \emptyset$). Set

$$d := \inf_{x \in X} \|z - x\|$$

As $X$ is closed, $d > 0$, otherwise $z$ is a limit point in $E \setminus X$. Fix $\varepsilon \in (0,1)$. Then there exists $x_0 \in X$ such that

$$d \leq \|z - x_0\| < \frac{d}{\varepsilon}.$$

Let $x_\varepsilon := \frac{z-x_0}{\|z-x_0\|}$; We have $\|x_\varepsilon\| = 1$ and

$$\begin{aligned}\|x - x_\varepsilon\| &= \left\| x - \frac{z-x_0}{\|z-x_0\|} \right\| \\ &= \frac{\|x\|z-x_0\| - z + x_0\|}{\|z-x_0\|} \\ &= \frac{\left\| \overbrace{x\|z-x_0\| + x_0}^{\in X} - z \right\|}{\|z-x_0\|} \\ &\geq \frac{d}{d}\varepsilon = \varepsilon\end{aligned}$$

□

Continue now proof of the theorem above:
Let $x_1 \in S$. Consider $X = \text{span}\{x_1\}$ which is a proper closed subspace of $E$. Hence by Riesz's lemma exists $x_2$ with $\|x_2\| = 1$ such that

$$\|x_2 - x_1\| \geq \frac{1}{2}$$

and

$$\|x_2 - x\| \geq \frac{1}{2} \qquad \forall\, x \in X.$$

Now consider span$\{x_1, x_2\}$ which is a proper closed subspace of $E$. By Riesz's lemma follows

$$\exists\, x_3 \in E,\ \|x_3\| = 1 :\ \|x_3 - x_1\| \geq \frac{1}{2}, \|x_3 - x_2\| \geq \frac{1}{2}.$$

Continuing in the same fashion we get $\{x_n\}$, $\|x_n\| = 1$ such that

$$\|x_n - x_m\| \geq \frac{1}{2} \qquad \forall\, n, m,\ n \neq m.$$

Clearly $\{x_n\} \subset S$ has no convergent subsequence. Hence $S$ is not compact. □

**Definition** (Cauchy sequence). $(E, \|.\|)$ normed space. $\{x_n\} \subseteq E$ is called Cauchy if

$$\forall\, \varepsilon > 0\, \exists\, N :\ \|x_n - x_m\| < \varepsilon \text{ for any } n, m \geq N.$$

**Example.** $(C_F, \|.\|_\infty)$, $\|x\|_\infty = \sup_{k\in\mathbb{N}}|x_k|$ where $x = (x_1, x_2, \ldots)$. Define

$$x_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, \ldots)$$

Then $\{x_n\}$ is Cauchy, as for $n > m$

$$\begin{aligned}\|x_n - x_m\|_\infty &= \left\|(0,\dots,0,\frac{1}{m+1},\dots,\frac{1}{n},0,\dots)\right\|_\infty \\ &= \frac{1}{m+1}\end{aligned}$$

Observe that $x_n$ is convergent in $(C_0, \|.\|_\infty)$

$$\underbrace{x_n}_{\in C_F} \to (1,\frac{1}{2},\frac{1}{3},\dots,\frac{1}{n},\dots) \in C_0 \setminus C_F$$

**Statement 1.3.** A convergent sequence is always a Cauchy sequence.

**Definition** (complete space). A normed vector space $(E, \|.\|)$ is called complete if any Cauchy sequence in $E$ is convergent in $E$.

**Definition** (Banach space). A complete normed space is called Banach space.

**Examples.**

- $(\mathbb{R}, |.|)$ is a Banach space.
- $(\mathbb{C}, |.|)$ as well.
- $(l^2, \|.\|_2)$ where

$$l^2 = \left\{(x_1, x_2, \dots) \,\middle|\, \sum_{i=1}^{\infty} |x_i|^2 < \infty, x_i \in \mathbb{C}\right\}$$

and

$$\|(x_1, x_2, \dots)\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$$

$(l^2, \|.\|_2)$ is complete.
**proof.** Let $x_n = (x_1^n, x_2^n, \dots)$ be a Cauchy sequence in $l^2$. We must show that it has a limit in $l^2$. We will do it in a few steps:

Step 1: Find a candidate for a limit $a$

Step 2: Show that $a \in l^2$.

Step 3: $\|x_n - a\|_2 \to 0$ as $n \to \infty$.

Step 1: Let

$$\begin{aligned} x_1 &= (x_1^1, x_2^1, \dots) \\ x_2 &= (x_1^2, x_2^2, \dots) \\ &\vdots \\ x_n &= (x_1^n, x_2^n, \dots) \end{aligned}$$

For each $k$ consider sequence $\{x_k^n\} \subset \mathbb{C}$ ($k$-th coordinates in each $x_n$).
Each sequence is Cauchy, as for all $n, m \geq N$

$$|x_k^n - x_k^m| < \left( \sum_{k=1}^{\infty} |x_k^n - x_k^m|^2 \right)^{\frac{1}{2}} = \|x_n - x_m\|_2 < \varepsilon$$

As $(\mathbb{C}, |.|)$ is complete, $\{x_k^n\}_n$ has a limit $a_k \in \mathbb{C}$. Candidate for limit of $x_n$ is

$$a = (a_1, a_2, \dots, a_k, \dots).$$

Step 2: Write

$$a = \underbrace{x_n}_{\in l^2} - (x_n - a)$$

In order to show that $a \in l^2$ it is enough to see that $x_n - a \in l^2$ for some $n$.
$\{x_n\}$ Cauchy implies

$$\forall \varepsilon > 0 \, \exists N : \forall n, m \geq N : \|x_n - x_m\|_2 < \varepsilon.$$

Consider for some $u > 0$

$$\sum_{i=1}^{u} |x_i^n - x_i^m|^2 \leq \sum_{i=1}^{\infty} |x_i^n - x_i^m|^2 = \|x_n - x_m\|_2^2 < \varepsilon^2$$

Let $m \to \infty$. We get

$$\sum_{i=1}^{m} |x_i^n - a_i|^2 \leq \varepsilon^2$$

This holds for any $u \in \mathbb{N}$. Hence for any $n \geq \mathbb{N}$

$$\underbrace{\sum_{i=1}^{\infty} |x_i^n - a_i|^2}_{= \|x_n - a\|_2^2} \leq \varepsilon^2.$$

Hence $x_n - a \in l^2$ and moreover $\|x_n - a\| \to 0$ as $n \to \infty$.

$\square$

- $(C([a, b]), \|.\|_\infty)$ is a Banach space.

- $(l^p, \|.\|_{l^p})$ for $1 \leq p < \infty$ are all Banach spaces.
- $(C([a,b]), \|.\|_2)$ with
$$\|f\|_2 = \left( \int |f(t)|^2 \, \mathrm{d}t \right)^{\frac{1}{2}}$$
One can prove that $(C([a,b]), \|.\|_2)$ is not a Banach space.

**Exercise:**

$[a,b] = [0,1]$ and

$$f_n(t) = \begin{cases} 0, & \text{falls } t < \frac{1}{2} - \frac{1}{n} \\ 1, & \text{falls } t > \frac{1}{2} \\ \text{continuous linear function} & \end{cases}.$$

Show that $\{f_n\}$ is Cauchy in $C([0,1], \|.\|_2)$ but $f_n \not\to f \in C([0,1])$.

**Definition** (Convergent and absolutely convergent series). A series $\sum_{n=1}^{\infty} x_n$ in $E$ is called convergent if $\{\sum_{n=1}^{m} x_n\}_m$, a sequence of partial sums, is convergent in $E$. If $\sum_{n=1}^{\infty} \|x_n\| < \infty$ then we say that $\sum_{n=1}^{\infty} x_n$ converges absolutely.

**Theorem .** A normed space $E$ is complete iff every absolutely convergent series converges in $E$.

**proof.** $\Rightarrow$: Suppose $X$ is complete and $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Let

$$S_N := \sum_{n=1}^{N} x_n \in E.$$

For $M > N$:

$$\begin{aligned} \|S_N - S_M\| &= \left\| \sum_{n=N+1}^{M} x_n \right\| \\ &\leq \sum_{n=N+1}^{M} \|x_n\| \\ &\leq \sum_{n=N+1}^{\infty} \|x_n\| \to 0 \qquad \text{as } N \to \infty \end{aligned}$$

Hence $\{S_N\}$ is Cauchy. As $E$ is complete, $S_N$ has a limit in $E$ i.e. $\sum_{n=1}^{\infty} x_n$ converges in $E$.

**⇐:** Assume that every absolut convergent series is convergent in $E$. We want to see that $E$ is complete.
Let $\{x_n\}$ be a Cauchy sequence. We want to prove that $\{x_n\}$ has a limit in $E$. We know that

$$\forall\, k\, \exists\, n_k :\ \|x_n - x_m\| < \frac{1}{2^k} \qquad \forall\, n, m \geq n_k.$$

We can assume that $\{n_k\}$ is an increasing sequence. Write

$$x_{n_k} = (x_{n_k} - x_{n_{k-1}}) + (x_{n_{k-1}} - x_{n_{k-2}}) + \ldots + (x_{n_1} - \underbrace{x_{n0}}_{=0}) = \sum_{l=1}^{k}(x_{n_l} - x_{n_{l-1}}).$$

$$\sum_{l=1}^{\infty}\|x_{n_l} - x_{n_{l-1}}\| \leq \sum_{l=1}^{\infty}\frac{1}{2^l} < \infty$$

Hence $\sum_{l=1}^{\infty}(x_{n_l} - x_{n_{l-1}})$ is absolutely convergent. By assumption

$$\sum_{l=1}^{\infty}(x_{n_l} - x_{n_{l-1}})$$

is convergent in $E$. Hence the partial sums is convergent. Subsequence is convergent. $\{x_{n_k}\}$ is convergent to some $x \in E$.

**Exercise:**

Show that the whole $\{x_n\} \to x$.

□

**Recall:**
converging squences $(x_n)_{n=1}^{\infty}$ in $(E, \|.\|)$. $\|x_n - x\| \to 0$ for $n \to \infty$ for some $x \in E$. (Notation: $x_n \to x$ in $(E, \|.\|)$)

**Remark.** Assume $x_n \to x$ in $(E, \|.\|)$ Then

1) $\|x_n\| \to \|x\|$ in $(E, \|.\|)$.

2) $\sup_n \|x_n\| < \infty$.

because

1)

$$\|x_n\| \leq \|x_n - x\| + \|x\|$$

so

$$\|x_n\| - \|x\| \leq \|x_n - x\|$$

it follows

$$-(\|x_n\| - \|x\|) \leq \|x_n - x\|$$

So

$$|\|x_n\| - \|x\|| \le \|x_n - x\| \to 0, \qquad \text{for } n \to \infty$$

Cauchy sequence in $(x_n)_{n=1}^{\infty}$ in $(E, \|.\|)$ if $\|x_n - x_m\| \to 0$ for $n, m \to \infty$.
We obtain: $(x_n)_{n=1}^{\infty}$ converges in $(E, \|.\|)$ $\Rightarrow$ $(x_n)_{n=1}^{\infty}$ Cauchy sequence in $(E, \|.\|)$. ($\not\Leftarrow$ in general). If $\Leftarrow$ then we call $(E, \|.\|)$ a Banach space.

$\sum_{n=1}^{\infty} x_m$ converges in $(E, \|.\|)$ if $\left(\sum_{n=1}^{k} x_n\right)_{k=1}^{\infty}$ converges in $(E, \|.\|)$.
$\sum_{n=1}^{\infty} x_m$ converges absolutely in $(E, \|.\|)$ if $\sum_{n=1}^{\infty} \|x_n\|$ converges $(\mathbb{R}, \|.\|)$.

## 1.2 Mappings between normal spaces

**Definition .** Let $(E_1, \|.\|_1)$, $(E_2, \|.\|_2)$ be normal spaces. $T : E_1 \to E_2$ (not necessarily linear) is called continuous at $x_0 \in E_1$, if

$$x_n \to x_0 \text{ in } (E_1, \|.\|_1) \qquad \Rightarrow \qquad T(x_n) \to T(x_0) \text{ in } (E_2, \|.\|_2)$$

$T$ is called <u>continuous</u> if it is continuous at $x_0 \in E_1$ for all $x_0 \in E_1$. We say that $T : E_1 \to E_2$ is <u>linear</u> if

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$$

for all scalars $\lambda_1$, $\lambda_2$ and $x_1, x_2 \in E_1$.
$T : E_1 \to E_2$ linear is called <u>bounded</u> if there exists $M > 0$ such that

$$\|T(x)\|_2 \le M \|x\|_1 \qquad \text{for all } x \in E_1.$$

If $T$ is bounded linear $E_1 \to E_2$ define

$$\|T\| = \|T\|_{E_1 \to E_2} := \inf\{M \ge 0 \,|\, \|T(x)\|_2 \le M\|x\|_1 \text{ for all } x \in E_1\}$$

**Lemma .**

$$\|T\| = \sup_{\substack{x \in E_1 \\ x \neq 0}} \frac{\|T(x)\|_2}{\|x\|_1} = \sup_{\substack{x \in E_1 \\ \|x\|_1 = 1}} \|T(x)\|_2$$

**Proposition .** Assume $T : E_1 \to E_2$ linear. Then all the following statements are equivalent:

(1) $T$ continuous at $0 \in E_1$.

(2) $T$ continuous at $x_0 \in E_1$ for some $x_0 \in E_1$.

(3) $T$ continuous at $x_0 \in E_1$ for all $x_0 \in E_1$.

(4) $T$ is bounded.

**proof.** $(1) \Rightarrow (4)$**:** Assume $T$ is continuous at $0 \in E_1$. i.e.

$$x_n \to 0 \text{ in } (E_1, \|.\|_1) \qquad \Rightarrow \qquad T(x_n) \to T(\underbrace{0}_{\in E_1}) = \underbrace{0}_{\in E_2} \text{ in } (E_2, \|.\|_2)$$

We want to prove that $T$ is bounded. We search a $M > 0$ such that

$$\|T(x)\|_2 \leq M\|x\|_1$$

We assume that this doesn't hold true.
For $n = 1, 2, \ldots$ there exists $x_n \in E_1$ such that

$$\|T(x_n)\|_2 > n\|x_n\|_1.$$

Set for $n = 1, 2, \ldots$

$$z_n := \frac{1}{n\|x_n\|_1} x_n$$

(Note that $\|x_n\|_1 > 0$. Otherwise we would get a contradiction.)
Note

$$\|z_n\|_1 = \left\|\frac{1}{n\|x_n\|_1}\right\|_1 = \frac{1}{n\|x_n\|_1}\|x_n\|_1 = \frac{1}{n} \to 0, \qquad \text{for } n \to \infty$$

We have $z_n \to 0$ in $(E_1, \|.\|_1)$. But

$$\|T(z_n)\|_2 = \left\|\frac{1}{n\|x_n\|_1} T(x_n)_2\right\| = \frac{1}{n\|x_n\|_1}\|T(x_n)\|_2 > 1 \qquad \text{for all } n.$$

Hence

$$T(z_n) \not\to 0 \qquad \text{in } (E_2, \|.\|_2).$$

This is a contradiction.

$(1) \Leftarrow (4)$**:** Assume $T$ is bounded. For some $M > 0$

$$\|T(x)\|_2 \leq M\|x\|_1, \qquad \text{for all } x \in E_1.$$

We need to show that $T$ is continuous at $0 \in E_1$, i.e.

$$x_n \to 0 \text{ in } (E_1, \|.\|_1) \qquad \Rightarrow \qquad T(x_n) \to T(0) = 0 \text{ in } (E_2, \|.\|_2)$$

From

$$\|T(x_n)\|_2 \leq M\|x_n\|_1 \to 0$$

so

$$T(x_n) \to \underbrace{0}_{=T(0)} \text{ in } (E_2, \|.\|_2).$$

$\square$

**Examples.** (A) $E_1 = E_2 = C([0,1])$, $\|.\|_1 = \|.\|_2 = \|.\|_\infty =: \|.\|$, i.e.

$$\|f\| := \max_{x\in[0,1]} |f(x)|.$$

$$T(f)(x) = \int_0^{1-x} \min(x,y) f(y)\,\mathrm{d}y, \qquad \text{for } f \in C([0,1]), x \in [0,1].$$

(1) $T(f) \in C([0,1])$ for $f \in C([0,1])$,

(2) $T$ linear,

(3) $T$ bounded,

(4) Calculate $\|T\|$.

**proof.** (1) Fix $f \in C([0,1])$ arbitrary and fix $x \in [0,1]$. Show that $T(f)$ is continuous at $x$. Consider a sequence $(x_n)_{n=1}^\infty$ in $[0,1]$ such that $x_n \to x$ in $(\mathbb{R}, |.|)$.
To show $T(f)(x_n) \to T(f)(x)$ in $(\mathbb{R}, |.|)$

$$\begin{aligned}
|T(f)(x_n) - T(f)(x)| &= \{\text{assume that } x_n \le x\} \\
&= \left|\int_0^{1-x_n} \min(x_n,y) f(y)\,\mathrm{d}y - \int_0^{1-x} \min(x,y) f(y)\,\mathrm{d}y\right| \\
&\le \left|\int_0^{1-x} (\min(x_n,y) - \min(x,y)) f(y)\,\mathrm{d}y\right| \\
&\quad + \left|\int_{1-x}^{1-x_n} \min(x_n,y) f(y)\,\mathrm{d}y\right| \\
&\le \underbrace{\int_0^{1-x} \underbrace{|\min(x_n,y) - \min(x,y)|}_{\le |x_n - x|} \underbrace{|f(y)|}_{\le \|f\|}\,\mathrm{d}y}_{\le |x_n - x| \|f\|} \\
&\quad + \underbrace{\int_{1-x}^{1-x_n} \underbrace{\min(x_n,y)}_{\le 1} \underbrace{|f(y)|}_{\le \|f\|}\,\mathrm{d}y}_{0 \le \ldots \le |x_n - x| \cdot \|f\|} \to 0, \qquad \text{as } n \to \infty
\end{aligned}$$

If $x_n > x$ we get a similar calculation. Conclusion:

$$T(f)(x_n) \to T(f)(x) \text{ in } (\mathbb{R}, |.|) \text{ as } n \to \infty.$$

(2) Fix $f_1, f_2 \in C([0,1])$ and $\lambda_1, \lambda_2$ scalars. Then

$$\begin{aligned}
T(\lambda_1 f_1 + \lambda_2 f_2)(x) &= \int_0^{1-x} \min(x,y) \underbrace{(\lambda_1 f_1 + \lambda_2 f_2)(y)}_{= \lambda_1 f_1(y) + \lambda_2 f_2(y)}\,\mathrm{d}y \\
&= \lambda_1 \int_0^{1-x} \min(x,y) f_1(y)\,\mathrm{d}y + \lambda_2 \int_0^{1-x} \min(x,y) f_2(y)\,\mathrm{d}y \\
&= \lambda_1 T(f_1)(x) + \lambda_2 T(f_2)(x) \qquad \text{for } x \in [0,1]
\end{aligned}$$

(3) Fix $f \in C([0,1])$. For $x \in [0,1]$

$$\begin{aligned} |T(f)(x)| &= |\int_0^{1-x} \underbrace{\min(x,y)f(y)}_{\geq 0} \,\mathrm{d}y| \\ &\overset{(*_1)}{\leq} \int_0^{1-x} \min(x,y) \underbrace{|f(y)|}_{\leq \|f\|} \,\mathrm{d}y \\ &\overset{(*_2)}{\leq} \int_0^{1-x} \min(x,y) \,\mathrm{d}y \|f\| \end{aligned}$$

Clearly

$$\max_{x \in [0,1]} \int_0^{1-x} \min(x,y) \,\mathrm{d}y \leq 1$$

This gives:

$$\|T(f)\| = \max_{x \in [0,1]} |T(f)(x)| \leq 1 \cdot \|f\|, \qquad \text{for all } f \in C([0,1]).$$

Conclusion: $T$ is bounded with ($M = 1$)

(4) Consider the unequality above. $(*_1)$ is an equality if $f$ has a constant sign. $(*_2)$ is an equality if $f$ is a constant function. So we have to calculate

$$\int_0^{1-x} \min(x,y) \,\mathrm{d}y \qquad \text{for } x \in [0,1].$$

**case 1:** $1 - x \leq x$ i.e. $\frac{1}{2} \leq x$ and we get

$$\begin{aligned} \int_0^{1-x} \underbrace{\min(x,y)}_{=y} \,\mathrm{d}y &= \left[\frac{1}{2} y^2\right]_0^{1-x} \\ &= \frac{1}{2}(1-x)^2 \end{aligned}$$

**case 2:** $x < 1 - x$ i.e. $x < \frac{1}{2}$ and we get

$$\begin{aligned} \int_0^{1-x} \min(x,y) \,\mathrm{d}y &= \int_0^x y \,\mathrm{d}y + \int_x^{1-x} x \,\mathrm{d}y \\ &= \frac{1}{2}x^2 + x(1-2x) \\ &= x - \frac{3}{2}x^2 \end{aligned}$$

Claim

$$\|T\| = \max\left( \max_{x \in [\frac{1}{2},1]} \frac{1}{2}(1-x)^2, \max_{x \in [0,\frac{1}{2}]} \left(x - \frac{3}{2}x^2\right) \right) = \ldots = \frac{1}{6}$$

Note

- $\|T(f)\| \leq \|T\| \cdot \|f\|$ for all $f \in C([0,1])$,
- $\|T(1)\| = \|T\| \cdot \|1\|$ where $1(x) = 1$ for $x \in [0,1]$.

$\square$

(B) $E_1 = C([0,1])$ with maximumnorm, $E_2 = \mathbb{R}$ with absolut value. $T : E_1 \to E_2$ with

$$T(f) = \int_0^{\frac{1}{2}} f(y)\,\mathrm{d}y - \int_{\frac{1}{2}}^1 f(y)\,\mathrm{d}y \qquad \text{for } f \in E_1$$

$$\begin{aligned}
|T(f)| &= |\int_0^{\frac{1}{2}} f(y)\,\mathrm{d}y - \int_{\frac{1}{2}}^1 f(y)\,\mathrm{d}y| \\
&\leq |\int_0^{\frac{1}{2}} f(y)\,\mathrm{d}y| + |\int_{\frac{1}{2}}^1 f(y)\,\mathrm{d}y| \\
&\leq \int_0^{\frac{1}{2}} \underbrace{|f(y)|}_{\leq\|f\|}\,\mathrm{d}y + \int_{\frac{1}{2}}^1 \underbrace{|f(y)|}_{\leq\|f\|}\,\mathrm{d}y \\
&\leq 1\|f\|
\end{aligned}$$

Hence $T$ is bounded and $\|T\| \leq 1$.

$$T(f) = \int_0^1 k(y) f(y)\,\mathrm{d}y$$

where

$$T(f_n) = \begin{cases} nachholen, & \text{falls } case \end{cases}$$

$$T(f_n) \leq 1\left(\frac{1}{2} - \frac{1}{2n} + \frac{1}{2} - \frac{1}{2n}\right) = 1 - \frac{1}{n}, \qquad n = 1, 2, \ldots$$

note

$$k(y) f_n(y) \geq 0 \qquad \text{for } y \in [0,1].$$

Hence $\|T\| \leq 1 - \frac{1}{n}$ for $n = 1, 2, \ldots$. Note $\|f_n\| = 1$ for all $n$. Conclusion $\|T\| = 1$.
Here

$$|T(f)| \leq \underbrace{\|T\|}_{\leq 1}\|f\| \text{ for all } f \in C([0,1])$$

but

$$|T(f)| < \|T\|\|f\| \qquad \text{for all } f \in C([0,1]).$$

**Satz.** $T_1, T_2$ bounded linear mappings $(E_1, \|.\|_1) \to (E_2, \|.\|_2)$ and $\lambda$ scalar. Set

$$\begin{aligned}
(T_1 + T_2)(x) &= T_1(x) + T_2(x) \qquad x \in E_1 \\
(\lambda T_1)(x) &= \lambda T_1(x) \qquad x \in E_1
\end{aligned}$$

Claim:

(1) $T_1 + T_2$ and $\lambda T_1$ are both linear mappings $(E_1, \|.\|_1) \to (E_2, \|.\|_2)$,

(2) $T_1 + T_2$ and $\lambda T_1$ are both bounded mappings $(E_1, \|.\|_1) \to (E_2, \|.\|_2)$.
$B(E_1, E_2)$ denote the vector space of all bounded linear mappings $(E_1, \|.\|_1) \to (E_2, \|.\|_2)$.

(3)
$$\|T\|_{E_1 \to E_2} := \inf\{M > 0 \,|\, \|T(x)\|_2 \le M\|x\|_1 \text{ for all } x \in E_1\}$$
defines a norm in $B(E_1, E_2)$.

**proof.** (1) $\|T\| = 0$ implies that $\|T(x)\|_2 = 0$ for all $x \in E_1 \;\Rightarrow\; T(x) = 0 \in E_2$.
$$T = 0 \in B(E_1, E_2)$$

(2) $T \in B(E_1, E_2)$ and $\lambda$ scalar.
$$\begin{aligned}
\|\lambda T\| &= \inf\{M > 0 \,|\, \|(\lambda T)(x)\|_2 \le M\|x\|_1 \text{ for all } x \in E_1\} \\
&= \inf\{M > 0 \,|\, |\lambda|\|T(x)\|_2 \le M\|x\|_1 \text{ for all } x \in E_1\} \\
&= \{\text{if } \lambda \ne 0\} \\
&= \inf\left\{ \underbrace{M}_{=|\lambda|\tilde{M}} > 0 \,\middle|\, \|T(x)\|_2 \le \underbrace{\frac{M}{|\lambda|}}_{=\tilde{M}} \|x\|_1 \text{ for all } x \in E_1 \right\} \\
&= |\lambda| \inf\left\{ \tilde{M} > 0 \,\middle|\, \|T(x)\|_2 \le \tilde{M}\|x\|_1 \text{ for all } x \in E_1 \right\} \\
&= |\lambda|\|T\|
\end{aligned}$$

(3) Set $T_1, T_2 \in B(E_1, E_2)$.
$$\begin{aligned}
\|T_1 + T_2\| &= \inf\{M > 0 \,|\, \|(T_1 + T_2)(x)\|_2 \le M\|x\|_1 \text{ for all } x \in E_1\} \\
&\le \inf\{M_1 + M_2 > 0 \,|\, \|T_1(x)\|_2 \le M_1\|x\|_1,\ \|T_2(x)\|_2 \le M_2\|x\|_1 \text{ for all } x \in E_1\} \\
&= \|T_1\| + \|T_2\|
\end{aligned}$$

□

Conclusion: $(B(E_1, B_2), \|.\|_{E_1 \to E_2})$ is a normal space.

**Satz.** $(B(E_1, B_2), \|.\|_{E_1 \to E_2})$ is a Banach space if $(E_2, \|.\|_2)$ is a Banach space.

**proof.** Assume $(T_n)_{n=1}^\infty$ is a Cauchy sequence in $(B(E_1, B_2), \|.\|_{E_1 \to E_2})$ where $(E_2, \|.\|_2)$ is a Banach space. Fix $x \in E_1$
$$\begin{aligned}
\|T_n(x) - T_m(x)\|_2 &= \|(T_n - T_m)(x)\|_2 \\
&\le \underbrace{\|T_n - T_m\|_{E_1 \to E_2}}_{\substack{\to 0 \\ n,m \to \infty}} \cdot \|x\|_1 \to 0, \qquad n, m \to \infty
\end{aligned}$$

Hence $(T_n(x))_{n=1}^\infty$ is a Cauchy sequence in $(E_2, \|.\|_2)$. This is a Banach space which implies that $(T_n(x))_{n=1}^\infty$ converges in $(E_2, \|.\|_2)$. Call the limit $T(x) \in E_2$ for all $x \in E_1$. Show now

(1) $T : E_1 \to E_2$ is linear,

(2) $T$ is bounded,

(3) $\|T_n - T\|_{E_1 \to E_2} \to 0$ for $n \to \infty$.

(1) Observe

$$T(\lambda_1 x_1 + \lambda_2 + x_2) \leftarrow T_n(\lambda_1 x_1 + \lambda_2 x_2) = \{T \text{ linear}\} = \underbrace{\underbrace{\lambda_1 \underbrace{T_n(x_1)}_{\to T(x_1)}}_{\to \lambda_1 T(x_1)} + \underbrace{\lambda_2 \underbrace{T_n(x_2)}_{\to T(x_2)}}_{\to \lambda_2 T(x_2)}}_{\to \lambda_1 T(x_1) + \lambda_2 T(x_2)}$$

So for $n \to \infty$ it is

$$T(\lambda_1 x_1 + \lambda_2 + x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2) \qquad \text{in } (E_2, \|.\|_2).$$

(2) Fix $\varepsilon > 0$. Then there exists $N$ such that:

$$\|T_n - T_m\|_{E_1 \to E_2} < \varepsilon \qquad \text{for } n, m \geq N$$

So for $x \in E_1$

$$\|T_n(x) - T_m(x)\|_2 \leq \|T_n - T_m\|_{E_1 \to E_2} \|x\|_1 < \varepsilon \|x\|_1 \qquad \text{for } n, m \geq N$$

Let $m \to \infty$.

$$\|T_n(x) - T(x)\|_2 \leq \varepsilon \|x\|_1 \qquad \text{for } n \geq N$$

So

$$\begin{aligned} \|T(x)\|_2 &\leq \|T(x) - T_N(x)\|_2 + \|T_N(x)\|_2 \\ &\leq \varepsilon \|x\|_1 + \|T_N\|_{E_1 \to E_2} \cdot \|x\|_1 \\ &= \left(\varepsilon + \|T_N\|_{E_1 \to E_2}\right) \|x\|_1 \qquad \text{for } x \in E_1 \end{aligned}$$

(3) Look above and get

$$\|T_n - T\|_{E_1 \to E_2} \to 0, \qquad n \to \infty.$$

□

**Theorem** (Banach-Steinhaus theorem (uniform boundedness principle)). $(E_1, \|.\|_1)$ Banach space, $(E_2, \|.\|_2)$ normal space and $\mathcal{F} \subset B(E_1, E_2)$. Assume

$$\sup_{T \in \mathcal{F}} \|T(x)\|_2 < \infty \qquad \text{for all } x \in E_1$$

then

$$\sup_{T \in \mathcal{F}} \|T\|_{E_1 \to E_2} < \infty.$$

**Remark.** The implication $\Leftarrow$ is easy to prove. If $\mathcal{F}$ is a finite set, the theorem is trivial.

**proof.** step 1: Assume

$$\exists\, x_0 \in E_1 \,\exists\, r > 0 \,\exists\, M > 0 : \forall\, x \in \overline{B(x_0, r)} \,\forall\, T \in \mathcal{F} : \|T(x)\|_2 \le M$$

We have to show that

$$\sup_{T \in \mathcal{F}} \|T\|_{E_1 \to E_2} < \infty.$$

Fix $T \in \mathcal{F}$. For $\|x\|_1 \le r$

$$\|T(x_0 + x)\|_2 \le M$$

Note that $x_0 + x \in \overline{B(x_0, r)}$.

$$\begin{aligned}
\|T(x)\|_2 &= \|T(x_0 + x - x_0)\|_2 \\
&= \{T \text{ linear}\} \\
&= \|T(x_0 + x) - T(x_0)\|_2 \\
&\le \|T(x_0 + x)\|_2 + \|T(x_0)\|_2 \\
&\le 2M
\end{aligned}$$

For $0 \ne x \in E_1$

$$\left\| T\left( \frac{r}{\|x\|_1} x \right) \right\|_2 \le 2M$$

$\frac{r}{\|x\|_1}$ has the $\|.\|_1$-norm equal to $r$. This implies , since T linear,

$$\frac{r}{\|x\|_1} \|T(x)\|_2 \le 2M$$

i.e.

$$\|T(x)\|_2 \le \frac{2M}{r} \|x\|_1 \qquad \text{for all } 0 \ne x \in E_1.$$

We have

$$\|t\|_{E_1 \to E_2} \le \underbrace{\frac{2M}{r}}_{\substack{\text{independant} \\ \text{of } T}} < \infty$$

$$\sup_{T \in \mathcal{F}} \|T\|_{E_1 \to E_2} \le \frac{2M}{r} < \infty$$

step 2: Justify the assumption in step 1. This assumption is equivalent to

$$\exists\, x_0 \in E_1 \,\exists\, r > 0 \,\exists\, M > 0 : \forall\, x \in B(x_0, r) \,\forall\, T \in \mathcal{F} : \|T(x)\|_2 \le M$$

(Note $\overline{B(x_0, r_1)} \subset B(x_0, r) \subset B(x_0, r_2)$ for $0 < r_1 < r < r_2$).
Argue by contradiction. Assume that the assumption is false. Then it holds

$$\forall\, x_0 \in E_1 \,\forall\, r > 0 \,\forall\, M > 0 : \exists\, x \in B(x_0, r) \,\exists\, T \in \mathcal{F} : \|T(x)\|_2 > M.$$

Idea: Find a converging sequence $x_n \in E_1$, $x_n \to x$ in $(E_1, \|.\|_1)$ and a sequence $(T_n)_{n=1}^{\infty} \subset \mathcal{F}$ such that

$$\|T_n(x_n)\|_2 > n \qquad \text{for all } n, \qquad \text{and} \qquad \|T_n(x)\|_2 > n \qquad \text{for all } n.$$

We have from above $x_1 \in B(0,1)$ and $T_1 \in \mathcal{F}$ such that

$$\|T_1(x_1)\|_2 > 1.$$

$T_1$ is bounded linear, hence continuous. This implies that there exists $0 < r_1 < \frac{1}{2}$ such that

$$\|T_1(x)\|_2 > 1 \qquad \text{for } x \in B(x_1, r_1)$$

and

$$\overline{B(x_1, r_1)} \subset B(0,1).$$

□

## 1.3 Fixed point theory

**Example.** Consider

$$f(x) + 5\int_0^{1-x} \min(x,y) f(y)\,\mathrm{d}y = g(x), \qquad x \in [0,1] \qquad (*)$$

where $g \in C([0,1])$.
Claim: There exists an unique solution $f \in C([0,1])$ that $(*)$.
Idea:

$$f(x) = f(x) - 5\int_0^{1-x} \min(x,y) f(y)\,\mathrm{d}y, \qquad x \in [0,1]$$

Set für $x \in [0,1]$

$$\tilde{T}(f)(x) = RHS(x)$$

To find a solution to $(*)$ is the same finding $f \in C([0,1])$ such that

$$f = \tilde{T}(f)$$

Clearly $\tilde{T} : C([0,1]) \to C([0,1])$. (continual later).

**Theorem** (Banach's fixed point theorem). $(E, \|.\|)$ Banach space. $T : E \to E$ (no assumption on linearity) is a contraction on $E$, i.e. there exists $c > 1$ such that

$$\|T(x) - T(\tilde{x})\| \leq c\|x - \tilde{x}\| \qquad \text{for all } x, \tilde{x} \in E.$$

Then there exists a unique $\bar{x} \in E$ such that

$$\bar{x} = T(\bar{x})$$

($\bar{x}$ is a fixed point)

**proof.** **Uniqueness:** Assume $T(\bar{x}) = \bar{x}$ and $T(\tilde{x}) = \tilde{x}$. Then

$$\underbrace{\|\bar{x} - \tilde{x}\|}_{\geq 0} = \|T(\bar{x}) - T(\tilde{x})\| \leq \underbrace{c}_{<1}\|\bar{x} - \tilde{x}\|$$

Thus $\|\bar{x} - \tilde{x}\| = 0$, i.e. $\bar{x} = \tilde{x}$.

**Existence** Pick an arbitrary $x_0 \in E$. Set

$$x_{n+1} = T(x_n), \qquad n = 0, 1, 2, \dots$$

Claim: $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(E, \|.\|)$. Note:

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|T(x_n) - T(x_{n-1})\| \\ &\leq c\|x_n - x_{n-1}\| \\ &\leq \dots \\ &\leq c^n\|x_1 - x_0\|, \qquad n = 1, 2, \dots \end{aligned}$$

For $n > m$

$$\begin{aligned} \|x_n - x_m\| &= \|x_n - x_{n-1} + x_{n-1} - \dots + x_{m+1} - x_m\| \\ &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \dots + \|x_{m+1} - x_m\| \\ &\leq (c^{n-1} + c^{n-2} + \dots c^m)\|x_1 - x_0\| \\ &\leq \frac{c^m}{1-c}\|x_1 - x_0\| \to 0 \qquad \text{as } n, m \to \infty \end{aligned}$$

Hence $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(E, \|.\|)$. $(E, \|.\|)$ is a Banach space. So $(x_n)_{n=1}^{\infty}$ converges in $(E, \|.\|)$. Call the limit $\bar{x}$.
Claim: $\bar{x}$ is a fixed point for $T$.

$$\begin{aligned} \|\bar{x} - T(\bar{x})\| &= \|\bar{x} - x_{n+1} + x_{n+1} - T(\bar{x})\| \\ &\leq \|\bar{x} - x_{n+1}\| + \left\|\underbrace{x_{n+1}}_{T(x_n)} - T(\bar{x})\right\| \\ &\leq \underbrace{\|\bar{x} - x_{n+1}\|}_{\to 0} + c\underbrace{\|x_n - \bar{x}\|}_{\to 0} \to 0, \qquad n \to \infty \end{aligned}$$

$\square$

**Remark.** (1) $x_n \to \bar{x}$ for $n \to \infty$ independend of the choice of $x_0$

(2) Fix $z \in E$

$$\begin{aligned} \|\bar{x} - z\| &= \|T(\bar{x}) - T(z) + T(z) - z\| \\ &\leq \|T(\bar{x}) - T(z)\| + \|T(z) - z\| \\ &\leq c\|\bar{x} - z\| + \|T(z) - z\| \end{aligned}$$

Hence

$$\|\bar{x} - z\| \leq \frac{1}{1-c}\|T(z) - z\|$$

**Example.** Consider now the example from above: $(C([0,1]), \|.\|)$ with $\|f\| = \max_{x\in[0,1]}|f(x)|$ is a Banach space! To apply Banach's fixed point theorem we need $\tilde{T}$ to be a contraction. Fix $f_1, f_2 \in C([0,1])$ and get for $x \in [0,1]$

$$\begin{aligned}
|(\tilde{T}(f_1) - \tilde{T}(f_2))(x)| &= |5\int_0^{1-x} \min(x,y) f_2(y)\,\mathrm{d}y - 5\int_0^{1-x} \min(x,y) f(y)\,\mathrm{d}y| \\
&= |5\int_0^{1-x} \min(x,y)(f_2(y) - f_1(y))\,\mathrm{d}y| \\
&\leq 5\int_0^{1-x} \min(x,y)\underbrace{|f_2(y) - f_1(y)|}_{\leq\|f_2 - f_1\|}\,\mathrm{d}y \\
&\leq \underbrace{5\int_0^{1-x} \min(x,y)\,\mathrm{d}y}_{0\leq\ldots\leq\frac{1}{6}}\|f_2 - f_1\| \\
&\leq \frac{5}{6}\|f_2 - f_1\|
\end{aligned}$$

Hence

$$\left\|\tilde{T}(f_1) - \tilde{T}(f_2)\right\| \leq \frac{5}{6}\|f_1 - f_2\|$$

We conclude that $\tilde{T}$ is a contraction. We can take $c = \frac{5}{6}$. By Banach's fixed point theorem $\tilde{T}$ has a unique fixed point. Finally $(*)$ has a unique solution $f \in C([0,1])$ which is the fixed point.

**Theorem** (Banach's fixed point theorem (generalization)). $(E, \|.\|)$ Banach space. $T : F \to F$ where $F$ is a closed set in $E$. $N$ positive integer. Assume $T^N = \underbrace{T \circ T \circ \ldots \circ T}_{N-\text{times}}$ is a contraction on $F$, i.e. there exists $c > 1$ such that

$$\left\|T^N(x) - T^N(\tilde{x})\right\| \leq c\|x - \tilde{x}\|, \qquad \text{for all } x, \tilde{x} \in F.$$

Then $T$ has unique fixed point $\bar{x}$, i.e.

$$\bar{x} = T(\bar{x}) \in F$$

**proof.** $N = 1$: Fix $x_0 \in F$ and consider $(x_n)_{n=1}^{\infty}$ where $x_{n+1} = T(x_n)$ for $n = 0, 1, 2, \ldots$. There $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence and hence this converges in $E$ since this is a Banach space. Call the limit $\bar{x}$. Note

$$\underbrace{x_n}_{\in F} \to \bar{x} \text{ in } E \text{ and } F \text{ is closed}$$

implies $\bar{x} \in F$. The rest of the argument is the same as before.

$N > 1$: By previous result we know that $T^N$ has a unique fixpoint $\bar{x} \in F$, i.e. $\bar{x} = T^N(\bar{x})$.
Claim: $\bar{x}$ is a fixed point for $T$.

$$\begin{aligned}\|T(\bar{x}) - \bar{x}\| &= \left\|T(T^N(\bar{x})) - T^N(\bar{x})\right\| \\ &= \left\|T^N(T(\bar{x})) - T^N(\bar{x})\right\| \\ &\leq c\|T(\bar{x}) - \bar{x}\|\end{aligned}$$

This gives

$$\|T(\bar{x} - \bar{x})\| = 0, \qquad \text{i.e. } \bar{x} = T(\bar{x}).$$

Existence of a fixed point for $T$ done. For the uniqueness assume $\bar{x} = T(\bar{x})$ and $\tilde{x} = T(\tilde{x})$. Then

$$\begin{aligned}\bar{x} &= T(\bar{x}) = T^2(\bar{x}) = \ldots = T^N(\bar{x}) \\ \tilde{x} &= T(\tilde{x}) = T^2(\tilde{x}) = \ldots = T^N(\tilde{x})\end{aligned}$$

But $T^N$ has a unique fixed point so

$$\bar{x} = \tilde{x}$$

□

**Remark.** (1) $T : (0,1] \to (0,1]$ where $T(x) = \frac{x}{2}$. Clearly $T$ is a contraction on $(0,1]$ but has no fixed point. Note that $(0,1]$ is not a closed intervall.

(2) $T : [0,\infty) \to [0,\infty)$, where $T(x) = x + \frac{1}{x}$. Clearly $[0,\infty)$ is a closed intervall in $\mathbb{R}$ but $T$ has no fixed point.
Claim: $T$ is not a contraction but 'close' to be a contraction.

$$|T(x) - T(\tilde{x})| < |x - \tilde{x}| \qquad \text{for } x, \tilde{x} \in [1,\infty), x \neq \tilde{x}$$

Note

$$|T(x) - T(\tilde{x})| = |\underbrace{T'(x)}_{\substack{(1-\frac{1}{t})\leq 1 \\ \text{for } t\in[1,\infty)}}||x - \tilde{x}|$$

for some $t$ betweeen $x$ and $\tilde{x}$.

**Example.** $(E, \|.\|)$ Banach space. $K$ compact set in $E$ and $T : K \to K$ where

$$\|T(x) - T(\bar{x})\| < \|x - \bar{x}\| \qquad \text{for all } x, \bar{x} \in K, x \neq \bar{x}.$$

Show: $T$ has a unique fixed point in $K$.

**Uniqueness:** Assume $\bar{x} = T(\bar{x})$ and $\tilde{x} = T(\tilde{x})$ and $\bar{x} \neq \tilde{x}$ for $\bar{x}, \tilde{x} \in K$. Then

$$\|\bar{x} - \tilde{x}\| = \|T(\bar{x}) - \tilde{x}\| < \|\bar{x} - \tilde{x}\|$$

Contradiction because then $\bar{x} = \tilde{x}$.

**Existence:** To show: There exists $x \in K$ such that $x = T(x)$, i.e.

$$\|T(x) - x\| = 0.$$

Set $d := \inf_{x \in K} \|T(x) - x\|$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $K$ such that

$$\|T(x_n) - x_n\| \to d, \qquad \text{as } n \to \infty.$$

$K$ compact implies that there exists a subsequence $(\tilde{x}_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $(\tilde{x}_n)_{n=1}^{\infty}$ converges in $K$. Call the limit element $\bar{x} \in K$. We know

$$\tilde{x}_n \to \bar{x} \qquad \text{in } K$$

and

$$\|T(\tilde{x}_n) - \tilde{x}_n\| \to d.$$

Question:

$$T(\tilde{x}_n) \to T(\bar{x}) \qquad \text{in } K?$$

But since

$$\|T(x) - T(\tilde{x})\| \leq \|x - \tilde{x}\| \qquad \text{for all } x, \tilde{x} \in K$$

we have

$$\tilde{x}_n \to \bar{x} \qquad \text{in } K$$

which implies

$$T(\tilde{x}_n) \to T(\bar{x}) \text{ in } K.$$

Hence:

$$\|T(\bar{x}) - \bar{x}\| \leftarrow \|T(\tilde{x}_n) - \tilde{x}_n\| \to d, \qquad n \to \infty.$$

We obtain

$$\|T(\bar{x}) - \bar{x}\| = d.$$

Question: Is $d = 0$?
If $d > 0$ then $\bar{x} \neq T(\bar{x})$, $\bar{x}, T(\bar{x}) \in K$

$$\|T(\bar{x}) - T(T(\bar{x}))\| < \|\bar{x} - T(\bar{x})\| = d = \inf_{x \in K} \|x - T(x)\|.$$

This is a contradiction which gives $d = 0$ and so $\bar{x} = T(\bar{x})$.

**Example.** Consider

$$f(x) = \int_0^x k(x,y) h(y, f(y)) \, \mathrm{d}y + g(x), \qquad x \in [0,1] \qquad (*)$$

where $g \in C([0,1])$, $k \in C([0,1] \times [0,1])$ and $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ continuous and satisfies:
There exists $M > 0$ such that

$$|h(x, z_1) - h(x, z_2)| \leq M |z_1 - z_2| \qquad \text{for all } x \in [0,1],\ z_1, z_2 \in \mathbb{R}$$

Claim: $(*)$ has a unique solution $f \in C([0,1])$. For $f \in C([0,1])$ set

$$T(f)(x) = \int_0^x k(x,y)h(y,f(y))\,\mathrm{d}y + g(x) \qquad x \in [0,1].$$

Here $T(f)(x) \in C([0,1])$.
Want to show: $T : C([0,1]) \to C([0,1])$ has a unique fixed point.
Start with the Banach space $(C([0,1]), \text{max-norm})$. Check if $T$ is a contraction in $C([0,1])$.
Fix $f_1, f2 \in C([0,1])$

$$T(f_1)(x) - T(f_2)(x) = \int_0^x k(x,y)(h(y,f_1(y)) - h(y,f_2(y)))\,\mathrm{d}y$$

$k$ is continuous on the compact set $[0,1]\times[0,1]$ so

$$\sup_{(x,y)\in[0,1]\times[0,1]} |k(x,y)| =: N < \infty.$$

We obtain

$$\begin{aligned}
|(T(f_1) - T(f_2))(x)| &\leq \int_0^x \underbrace{|k(x,y)|}_{\leq N}\underbrace{h(y,f_1(y)) - h(y,f_2(y))}_{\leq M\underbrace{f_1(y) - f_2(y)}_{\leq \|f_1 - f_2\|}}\,\mathrm{d}y \\
&\leq \int_0^x NM\,\mathrm{d}y \|f_1 - f_2\| \\
&\leq NM\|f_1 - f_2\|
\end{aligned}$$

this yields

$$\|T(f_1) - T(f_2)\| \leq NM\|f_1 - f_2\|.$$

<u>IF:</u> $NM < 1$ Then $T$ is a contaction.
Trick: For $a > 0$ set

$$\|f\|_a = \max_{x\in[0,1]} e^{-ax}|f(x)|$$

for $f \in C([0,1])$.
Claim: $\|.\|_a$ defines a norm on $C([0,1])$. This is easy to check.
Claim: $\|.\|$ and $\|.\|_a$ are equivalent.
This follows from

$$e^{-a}\|f\| \leq \|f\|_a \leq \|f\|$$

for all $f \in C([0,1])$ (note that $\|.\|$ is the max-norm).
Claim: $(C([0,1]), \|.\|_a)$ is a Banach space.
This follows from the fact that $\|.\|$ und $\|.\|_a$ are equivalent and $(C([0,1]), \|.\|)$ is a Banach space.
Claim: $T$ is a contraction on $(C([0,1]), \|.\|_a)$ for $a > 0$ large enough.

For $f_1, f_2 \in C([0,1])$ and $x \in [0,1]$ we have

$$\begin{aligned}
|(T(f_1) - T(f_2))(x)| &\leq \int_0^x NM|(f_1 - f_2)(y)|\,\mathrm{d}y \\
&= \int_0^x NMe^{ay} \cdot \underbrace{e^{-ay}|(f_1 - f_2)(x)|}_{\leq \|f_1 - f_2\|_a}\,\mathrm{d}y \\
&\leq NM \underbrace{\int_0^x e^{ay}\,\mathrm{d}y}_{\frac{1}{a}(e^{ax}-1)} \|f_1 - f_2\|_a
\end{aligned}$$

So

$$e^{-ax}|(T(f_1) - T(f_2))(x)| \leq \frac{NM}{a}(1 - e^{-ax})\|f_1 - f_2\|_a$$

and

$$\|T(f_1) - T(f_2)\|_a \leq \frac{NM}{a}\|f_1 - f_2\|_a$$

For $a > NM$ is $T$ a contraction on $(C([0,1]), \|.\|_a)$. Banach fixed point theorem implies that there is a unique $f \in C([0,1])$ that solves $(*)$.