



GÖTEBORGS UNIVERSITET



# Applied Functionalanalysis

Script of “Applied Functionalanalysis” by Prof. Peter Kumlin

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## foreword — cooperation

This document is a transcript of the lecture “Applied Functionalanalysis, WiSe 2016/2017, Term 1”, by Prof. Peter Kumlin. It mainly contains the written content of the lecture. I will not assume any responsibility for the correctness of the content! For questions, remarks and mistakes please write an email to [keil.menden@web.de](mailto:keil.menden@web.de). I’m grateful for every email.



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# 1 Introduction

## 1.1 Introduction example

We have

$$\begin{cases} f'' + f = g, & \text{in } I = [0, 1] \\ f(0) = 1, f'(0) = 1 \end{cases},$$

where  $g$  is a known continuous function on  $I$ . We will now consider different cases:

1.  $g = 0$

$$\Rightarrow f(x) = A \cos(x) + B \sin(x), x \in I,$$

where  $A, B \in \mathbb{R}$ .

2.  $g$  arbitrary. We will now introduce the Method of variation of constants. Set

$$f(x) = A(x) \cos(x) + B(x) \sin(x).$$

Differentiate

$$f'(x) = A'(x) \cos(x) + B'(x) \sin(x) - A(x) \sin(x) + B(x) \cos(x).$$

Assume (this is part of the method)

$$A'(x) \cos(x) + B'(x) \sin(x) = 0, \quad x \in I.$$

Differentiate  $f'(x)$  and get

$$f''(x) = \underbrace{-A(x) \cos(x) - B(x) \sin(x)}_{=-f(x)} - A'(x) \sin(x) + B'(x) \cos(x).$$

We get

$$g(x) = f''(x) + f(x) = -A'(x) \sin(x) + B'(x) \cos(x).$$

Now:

$$\begin{cases} A'(x) \cos(x) + B'(x) \sin(x) = 0, & x \in I \\ -A'(x) \sin(x) + B'(x) \cos(x) = g(x), & x \in I. \\ A(0) = 1, & B(0) = 0 \end{cases}$$

We get

$$\begin{aligned} A'(x) &= -g(x) \sin(x), \\ A(0) &= 1, \\ B'(x) &= g(x) \cos(x), \\ B(0) &= 0. \end{aligned}$$

This implies

$$\begin{aligned}A(x) &= A(0) + \int_0^x A'(t) dt = 1 - \int_0^x g(t) \sin(t) dt, \\B(x) &= B(0) + \int_0^x B'(t) dt = 0 + \int_0^x g(t) \cos(t) dt.\end{aligned}$$

Hence

$$\begin{aligned}f(x) &= \cos(x) - \int_0^x g(t) \sin(t) dt \cos(x) + \int_0^x g(t) \cos(t) dt \sin(x) \\&= \cos(x) + \int_0^x \underbrace{(\sin(x) \cos(t) - \sin(t) \cos(x))}_{=\sin(x-t)} g(t) dt \\&= \cos(x) + \int_0^x \sin(x-t) g(t) dt \quad (*).\end{aligned}$$

Check that  $f(x)$  in  $(*)$  satisfies the PDE.

**special case:**

Assume for  $x \in I$

$$g(x) = k(x)f(x).$$

Here  $k$  is a known continuous function on  $I$ . Insert this in  $(*)$ . We obtain

$$f(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f(t) dt, \quad x \in I \quad (**).$$

Observe that  $f$  appears both in LHS and RHS.  $(**)$  is a reformulation of the PDE with  $g = kf$ . Pick a continuous function in  $I$ . call it  $f_0$ . Set  $\in C(I)$

$$\begin{aligned}f_1(x) &= \cos(x) + \int_0^x \sin(x-t)k(t)f_0(t) dt, \\f_2(x) &= \cos(x) + \int_0^x \sin(x-t)k(t)f_1(t) dt, \\&\vdots \\f_{n+1}(x) &= \cos(x) + \int_0^x \sin(x-t)k(t)f_n(t) dt, \quad n = 1, 2, 3, \dots\end{aligned}$$

**Hope:**

$f_n$  tends to some continuous function  $f$  on  $I$ , denoted  $f_n \rightarrow f$ . 'Tends to' has to be more precis!

$$\begin{array}{ccc}
 f_{n+1}(x) & = & \cos(x) + \int_0^x \sin(x-t)k(t)f_n(t) dt \\
 \downarrow & & \downarrow \\
 f(x) & = & \cos(x) + \int_0^x \sin(x-t)k(t)f(t) dt
 \end{array}$$

for  $x \in I$ . Simplify notation set for  $v \in C(I)$

$$\begin{cases} u(x) & = \cos(x) \\ kv(x) & = \int_0^x \sin(x-t)k(t)v(t) dt \end{cases} .$$

We have  $f_0 \in C(I)$ ,  $f_{n+1} = u + kf_n$  for  $n = 0, 1, 2, \dots$  (!)

Facts from previous calculus classes:

**Definition** (Sequence of continuous functions).

$$v_n \in C(I), \quad n = 1, 2, \dots$$

We say that  $(v_n)_{n=1}^\infty$  converges uniformly in  $I$  if

$$\max_{x \in I} |v_n(x) - v_m(x)| \rightarrow 0, \quad n, m \rightarrow \infty,$$

i.e.

$$\forall \varepsilon > 0 \exists N : \forall n, m \geq N : \max_{x \in I} |v_n(x) - v_m(x)| < \varepsilon.$$

**Lemma .** Suppose that  $(v_n)_{n=1}^\infty$  converges uniformly on  $I$ . then there exists  $v \in C(I)$  such that

$$\max_{x \in I} |v_m(x) - v(x)| \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Back to (!):

**More Notation:**

$$k(kv) = k^2v, \quad v \in C(I)$$

and

$$k^{n+1}v = k(k^n v), \quad n = 1, 2, \dots$$

We have

$$\begin{aligned}f_0 &\in C(I) \\f_1 &= u + kf_0 \\ \text{and } f_2 &= u + kf_1 = u + k(u + kf_0)\end{aligned}$$

and so on. Note that

$$k(v + w) = kv + kw.$$

Then

$$\begin{aligned}f_2 &= u + k(u + kf_0) = u + ku + k(kf_0) = u + ku + k^2f_0 \\f_3 &= u + kf_2 = u + ku + k^2u + k^3f_0\end{aligned}$$

and in general for  $n = 1, 2, \dots$

$$f_n = ku + \dots + k^{n-1}u + k^n f_0, \quad n = 1, 2, \dots$$

Assume  $n > m$  then

$$f_n - f_m = k^m u + \dots + k^{n-1}u + k^n f_0 - k^m f_0.$$

Set for  $v \in C(I)$

$$\|v\| = \max_{x \in I} |v(x)|.$$

Note

$$\|v + w\| \leq \|v\| + \|w\| \quad \text{for } v, w \in C(I)$$

and

$$\|-v\| = \|v\|.$$

We have

$$\begin{aligned}\|f_n - f_m\| &= \|k^m u + \dots + k^{n-1}u + k^n f_0 - k^m f_0\| \\&\leq \|k^m u\| + \dots + \|k^{n-1}u\| + \|k^n f_0\| + \|-k^m f_0\|.\end{aligned}$$

Assumption:

$$\sum_{l=1}^{\infty} \|k^l v\| < \infty \quad \text{for all } v \in C(I) \quad (**).$$

Under this assumption

$$\|f_n - f_m\| \rightarrow 0 \quad \text{as } n, m \rightarrow \infty$$

since

$$\begin{aligned}\sum_{l=1}^{\infty} \|k^l u\| &< \infty & (u(x) = \cos(x)) \\ \sum_{l=1}^{\infty} \|k^l f_0\| &< \infty & (f_0 \in C(I)).\end{aligned}$$



Conclusion:  $(f_n)_{n=1}^{\infty}$  converges uniformly on  $I$ . By lemma above there exists  $f \in C(I)$  such that

$$\max_{x \in I} |f_n(x) - f(x)| \rightarrow 0, \quad n \rightarrow \infty,$$

i.e.

$$\|f_n - f\| \rightarrow 0, \quad n \rightarrow \infty.$$

'Back hope':  $f_n$  tends to  $f$ , denoted  $f_n \rightarrow f$  shall be interpreted as

$$\|f_n - f\| \rightarrow 0, \quad n \rightarrow \infty.$$

Remember

$$f_{n+1}(x) = u(x) + k f_n(x) \rightarrow ?.$$

For  $x \in I$  there is

$$\begin{aligned} |k f_n(x) - k f(x)| &= \left| \int_0^x \sin(x-t) k(t) f_n(t) dt - \int_0^x \sin(x-t) k(t) f(t) dt \right| \\ &\leq \int_0^x |\sin(x-t) k(t)| \underbrace{|f_n(t) - f(t)|}_{\leq \|f_n - f\|} dt \\ &\leq \int_0^x |\sin(x-t) k(t)| dt \|f_n - f\|. \end{aligned}$$

In particular

$$\begin{aligned} \|k f_n - k f\| &\leq \max_{x \in I} \int_0^x \underbrace{|\sin(x-t)|}_{\leq 1} \underbrace{|k(t)|}_{\max_{t \in I} |k(t)| < \infty} dt \|f_n - f\| \\ &\leq \|k\| \|f_n - f\|. \end{aligned}$$

We have, provided  $(***)$  holds, shown

$$\begin{aligned} f_{n+1} &= u + k f_n \\ \downarrow \\ f &= u + k f. \end{aligned}$$

Let us try to prove  $(***)$ . For  $v \in C(I)$  arbitrary and for  $x \in I$

$$\begin{aligned} \|k v(x)\| &= \left| \int_0^x \sin(x-t) k(t) v(t) dt \right| \\ &\leq \int_0^x \underbrace{|\sin(x-t)|}_{\leq 1} \underbrace{|k(t)|}_{\leq \|k\|} |v(t)| dt \\ &\leq \int_0^x \underbrace{|v(t)|}_{\leq \|v\|} dt \|k\| \\ &\leq \|k\| \|v\| x. \end{aligned}$$

In particular

$$\|kv\| \leq \|k\|\|v\|$$

and

$$\begin{aligned} |k^2v(x)| &\leq \int_0^x |kv(t)| \, dt \|k\| \\ &\leq \int_0^x \|k\|\|v\|t \, dt \cdot \|k\| \\ &= \|k\|^2\|v\|\frac{x^2}{2}. \end{aligned}$$

In particular

$$\|k^2v\| \leq \|k\|^2\|v\|\frac{1}{2}.$$

By induction we get

$$\begin{aligned} |k^nv(x)| &\leq \|k\|^n\|v\|\frac{x^n}{n!} \quad x \in I \\ \|k^nv\| &\leq \|k\|^n\|v\|\frac{1}{n!}. \end{aligned}$$

So

$$\begin{aligned} \sum_{l=1}^{\infty} \|k^lv\| &\leq \sum_{l=1}^{\infty} \|k\|^l\|v\|\frac{1}{l!} \\ &= \|v\| \sum_{l=1}^{\infty} \frac{\|k\|^l}{l!} \\ &\leq \|v\|e^{\|k\|} < \infty. \end{aligned}$$

Consider Taylor expansion.  $\Rightarrow (***)$  holds true.

We have now shown that  $f = u + kf$  where  $u(x) = \cos(x)$  and

$$kv = \int_0^x \sin(x-t)k(t)v(t) \, dt.$$

$x \in I$  for  $v \in C(I)$ , has a solution  $f \in C(I)$ .

**Question:**

Is the solution unique?

Assume  $f, \tilde{f} \in C(I)$  such that  $f = u + kf$  and  $\tilde{f} = u + k\tilde{f}$ . Set

$$v = f - \tilde{f} \in C(I)$$

$$\begin{aligned} \Rightarrow v &= (u + kf) - (u + k\tilde{f}) \\ &= kf - k\tilde{f} \\ &= k(f - \tilde{f}) \\ &= kv. \end{aligned}$$

We have  $v = kv$ , implies that  $kv = k(kv) = k^2v$ . So for  $n = 1, 2, \dots$

$$v = kv = k^2v = \dots = k^nv.$$

We know

$$\sum_{n=1}^{\infty} \|k^n \hat{v}\| < \infty \quad \text{for all } \hat{v} \in C(I).$$

Apply this to  $\hat{v} = v$ :

$$\sum_{n=1}^{\infty} \underbrace{\|k^n v\|}_{=\|v\|} < \infty.$$

So  $\|v\| = 0$  with implies  $v(x) = 0$  for all  $x \in I$ . So we have  $f(x) = \tilde{f}(x)$  for  $x \in I$ .  
 $\Rightarrow$  Answer to the question above: YES !

We have more or less proved the following theorem:

**Theorem 1.1.** Set  $I = [0, 1]$ . Suppose  $u \in C(I)$  and  $k \in C(I \times I)$ . Consider

$$f(x) = u(x) + \int_0^x k(x, t)f(t) dt, \quad x \in I \quad (1).$$

Then (1) has a unique solution  $f \in C(I)$

With the same technology we can prove:

**Theorem 1.2.** Set  $I = [0, 1]$ . Suppose  $u \in C(I)$ ,  $k \in C(I \times I)$  and  $\max_{(x,t) \in I \times I} |k(x, t)| < 1$ .

Consider

$$f(x) = u(x) + \int_0^1 k(x, t)f(t) dt, \quad x \in I \quad (2).$$

Then (2) has a unique solution  $f \in C(I)$ .

Different notions: see introductory example.

## 2 Normed Spaces and Banach Spaces

**Definition** (vector space).  $C(I)$  with the operations for  $x \in I$ :

**addition**  $v, w \in C(I)$ :  $(v + w)(x) = v(x) + w(x)$ ,

**mult. by scalar**  $v \in C(I)$ ,  $\lambda \in \mathbb{R}$ :  $(\lambda v)(x) = \lambda v(x)$ .

Note that  $v + w, \lambda v \in C(I)$ .

**Definition (norm).** Norm on  $C(I)$  for instance

$$\|v\| = \max_{x \in I} |v(x)|$$

with norm given we can talk about convergence and continuity.

**Definition (Cauchy sequence).** In our example a sequence  $(f_n)_{n=1}^\infty$  is called Cauchy sequence if  $\|f_n - f_m\| \rightarrow 0$  for  $n, m \rightarrow \infty$ .

**Definition .**  $C(I)$  with the max-norm. Lemma above says that every Cauchy sequence converges i.e.

$$\|v_n - v_m\| \rightarrow 0, \quad n, m \rightarrow \infty.$$

This applies

$$\exists v \in C(I) : \|v_n - v\| \rightarrow 0, \quad n \rightarrow \infty.$$

This is the defining property of a Banach space.

$K$  linear mapping  $C(I) \rightarrow C(I)$  with

$$K(v + w) = K(v) + K(w)$$

$$K(\lambda v) = \lambda K(v)$$

for  $v, w \in C(I)$ ,  $\lambda \in \mathbb{R}$ .

$K$  bounded linear:

$$\|Kv\| \leq M\|v\| \quad \forall v \in C(I),$$

where  $M > 0$  independent of  $v$ .

**Definition (operator norm).** Define

$$\|K\| := \inf\{M > 0 \mid \|Kv\| \leq M\|v\| \text{ for all } v \in C(I)\}.$$

**fixed point results:**

Our example:  $f = u + kf =: T(f)$  and  $f_0 \in C(I)$  fixed.

Form sequence of iterants  $(f_n)_{n=1}^\infty$ ,  $f_n = T(f_{n-1})$ ,  $n = 1, 2, \dots$  if

$$\|T(v) - T(w)\| \leq c\|v - w\|$$

for all  $v, w \in C(I)$  for some  $c < 1$ . Then there is a unique  $v \in C(I)$  such that  $v = T(v)$ .

This is Banach's fixed point theorem.

**Definition** (Green's function). Our example:

$$L = \left( \frac{d}{dx} \right)^2 + 1$$

differential operator. Boundary conditions

$$f(0) = f'(0) = 0.$$

Then

$$f(x) = \int_0^1 g(x, t) h(t) dt$$

is a solution to

$$\begin{cases} f'' + f &= h, \\ f(0) = f'(0) &= 0. \end{cases}$$

**Definition** (real vector space). We say that  $E$  is a real vector space if it is a non-empty set with the operations

**addition**  $E \times E \rightarrow E$ ,  $(x, y) \mapsto x + y$

**mult. with scalar**  $\mathbb{R} \times E \rightarrow E$ ,  $(\lambda, x) \mapsto \lambda x$

satisfying the axioms:

- (1)  $x + y = y + x$ , for all  $x, y \in E$ ,
- (2)  $x + (y + z) = (x + y) + z$ , for all  $x, y, z \in E$ ,
- (3) For all  $x, y \in E$  there exists  $z \in E$  such that  $x + z = y$ ,
- (4)  $\alpha(\beta x) = (\alpha \cdot \beta)x$ , for all  $\alpha, \beta \in \mathbb{R}, x \in E$ ,
- (5)  $\alpha(x + y) = \alpha x + \alpha y$ , for all  $\alpha \in \mathbb{R}, x, y \in E$ ,
- (6)  $(\alpha + \beta)x = \alpha x + \beta x$ , for all  $\alpha, \beta \in \mathbb{R}, x \in E$ ,
- (7)  $1 \cdot x = x$ , for all  $x \in E$ .

**Remark.**  $E$  is a complex vector space if all  $\mathbb{R}$  in the definition above are replaced by  $\mathbb{C}$ .

**Remark.** (1)

$$\exists ! 0 \in E : \quad x + 0 = x \quad \text{for all } x \in E.$$

Since: Fix  $x \in E$ , by (3),  $\exists 0_x$  such that  $0_x + x = x$ .

Fix  $y \in E$ . We want to show that  $y + 0_y = y$ . By (3), there exists  $z \in E$  such that

$x + z = y$ . So

$$\begin{aligned} y + 0_x &= (x + z) + 0_x \\ &\stackrel{(1)}{=} (z + x) + 0_x \\ &\stackrel{(2)}{=} z + (x + 0_x) \\ &= z + x \\ &\stackrel{(1)}{=} x + z \\ &= y. \end{aligned}$$

Assume  $x + 0_1 = x$ ,  $x + 0_2 = x$  for all  $x \in E$ . We want to show  $0_1 = 0_2$ :

$$0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2,$$

(2)

$$\forall x \in E : \exists! -x \in E : x + (-x) = 0.$$

proof: exercise.

(3)

$$\begin{aligned} 0x &= 0 && \text{for all } x \in E \\ (-1)x &= -x && \text{for all } x \in E. \end{aligned}$$

**Examples** (Examples of real vector spaces). 1)  $\mathbb{R}$  with standard addition and mult. by scalar.

2)  $\mathbb{R}^n$ ,  $n = 2, 3, \dots$

**addition**  $(x_1, x_2, \dots) + (y_1, y_2, \dots) = (x_1 + y_1, x_2 + y_2, \dots)$

**mult.**  $\lambda(x_1, x_2, \dots) = (\lambda x_1, \lambda x_2, \dots)$

3)  $\mathbb{R}^\infty = \{(x_1, \dots, x_n, \dots) \mid x_n \in \mathbb{R}, n = 1, 2, \dots\}$

4)  $1 \leq p < \infty$ ,

$$l^p = \left\{ (x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty \mid \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty \right\}$$

with the same addition and mult. by scalar as in  $\mathbb{R}^\infty$ . We have to check:

$$(1) \quad x, y \in l^p \quad \Rightarrow \quad x + y \in l^p$$

$$(2) \quad x \in l^p, \lambda \in \mathbb{R} \quad \Rightarrow \quad \lambda x \in l^p.$$

For (1) we assume  $x = (x_1, \dots, x_n, \dots)$  and  $y = (y_1, \dots, y_n, \dots)$ .

$$\begin{aligned} x \in l^p &\Rightarrow \sum_{n=1}^{\infty} |x_n|^p < \infty \\ y \in l^p &\Rightarrow \sum_{n=1}^{\infty} |y_n|^p < \infty \end{aligned}$$

$$\Rightarrow \quad x + y = (x_1 + y_1, \dots) \stackrel{?}{\in} l^p?$$

$$\begin{aligned} \Rightarrow \sum_{n=1}^{\infty} |x_n + y_n|^p &\leq \{ |x_n + y_n| \leq |x_n| + |y_n| \leq 2 \max\{|x_n|, |y_n|\} \} \\ &\{ |x_n + y_n|^p \leq 2^p (|x_n|^p + |y_n|^p) \} \\ &\leq \sum_{n=1}^{\infty} 2^p (|x_n|^p + |y_n|^p) \\ &= 2^p \underbrace{\sum_{n=1}^{\infty} |x_n|^p}_{< \infty} + 2^p \underbrace{\sum_{n=1}^{\infty} |y_n|^p}_{< \infty} < \infty \end{aligned}$$

and

$$\sum_{n=1}^{\infty} |\lambda x_n|^p = \sum_{n=1}^{\infty} |\lambda|^p \cdot |x_n|^p = |\lambda|^p \sum_{n=1}^{\infty} |x_n|^p < \infty.$$

5) Function spaces, say real-valued functions on  $I$ .

**addition:**  $(f + g)(x) = f(x) + g(x), \quad x \in I$

**mult. by scalar:**  $(\lambda f)(x) = \lambda f(x) \quad \text{for functions } f \text{ and } g$

6)  $C(I)$  : addition and mult. by scalar as in (5).

$f, g$  continuous in  $I$  implies that  $f + g$  is continuous in  $I$ .

Also if  $f$  is continuous and  $\lambda \in \mathbb{R}$  then  $(\lambda f)$  is continuous in  $I$ .

7)  $P(I)$  = polynomials in  $I$ .

8)  $P_k(I)$  = polynomials of degree at most  $k$  in  $I$ .

**Theorem 2.1** (Hölder's inequality). Assume  $1 < p < \infty$  and  $\frac{1}{p} + \frac{1}{q} = 1$ .

Let  $(x_1, \dots, x_n, \dots)$  and  $(y_1, y_2, \dots, y_n, \dots)$  be sequences of complex numbers. Then

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{n=1}^{\infty} |y_n|^q \right)^{\frac{1}{q}}.$$

Remark there the LHS can be infinity, but the RHS can also be infinity.

**proof. Step 1** We're going to proof

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \text{for all } a, b > 0.$$

$$\int_0^a x^{p-1} dx = \frac{a^p}{p}.$$

Note  $y = x^{p-1}$  gives

$$x = y^{\frac{1}{p-1}} = y^{\frac{\frac{1}{1-\frac{1}{q}}}{1-\frac{1}{q}-1}} = y^{\frac{\frac{1}{q}}{\frac{q}{q-1}-1}} = y^{q-1}$$

so

$$\int_0^b y^{q-1} dy = \frac{b^q}{q}.$$

We get

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

(You also get condition for =)

**Step 2** It is enough to consider the cases  $\text{LHS} > 0$  and  $\text{RHS} < \infty$ . There exists an integer  $N$  such that

$$0 < \sum_{n=1}^N |x_n|^p, \sum_{n=1}^N |y_n|^q < \infty.$$

Set

$$a = \frac{|x_k|}{\left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}}}, \quad k = 1, 2, \dots, N,$$

$$b = \frac{|y_k|}{\left(\sum_{n=1}^N |y_n|^q\right)^{\frac{1}{q}}}, \quad k = 1, 2, \dots, N.$$

Insert into

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

$$\frac{|x_k y_k|}{\left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^N |y_n|^q\right)^{\frac{1}{q}}} \leq \frac{|x_k|^p}{p \sum_{n=1}^N |x_n|^p} + \frac{|y_k|^q}{q \sum_{n=1}^N |y_n|^q}, \quad k = 1, 2, \dots, N.$$

We sum over  $k$  from 1 to  $N$ :

$$\sum_{k=1}^N |x_k y_k| \leq \left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{n=1}^N |y_n|^q\right)^{\frac{1}{q}}.$$

Let  $N \rightarrow \infty$ . First in RHS and then in LHS.

□

**Theorem 2.2 (Minkowski's inequality).** Assume  $1 \leq p < \infty$ . and  $X, Y \in l^p$ . Then

$$\|X + Y\|_{l^p} \leq \|X\|_{l^p} + \|Y\|_{l^p}.$$



**proof.**  $p = 1$ :

$$\begin{aligned}
 \|X + Y\|_{l^1} &= \|(x_1, x_2, \dots, x_n, \dots) + (y_1, y_2, \dots, y_n, \dots)\|_{l^1} \\
 &= \|(x_1 + y_1, \dots, x_n + y_n, \dots)\|_{l^1} \\
 &= \sum_{n=1}^{\infty} |x_n + y_n| \\
 &\leq \sum_{n=1}^{\infty} (|x_n| + |y_n|) \\
 &= \sum_{n=1}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| \\
 &= \|X\|_{l^1} + \|Y\|_{l^1}
 \end{aligned}$$

$1 < p < \infty$ :

$$\begin{aligned}
 \|X + Y\|_{l^p}^p &= \sum_{n=1}^{\infty} |x_n + y_n|^p \\
 &= \sum_{n=1}^{\infty} |x_n + y_n| |x_n + y_n|^{p-1} \\
 &\leq \sum_{n=1}^{\infty} |x_n| |x_n + y_n|^{p-1} + \sum_{n=1}^{\infty} |y_n| |x_n + y_n|^{p-1}.
 \end{aligned}$$

Use Hölder to get

$$\begin{aligned}
 \sum_{n=1}^{\infty} |x_n| |x_n + y_n|^{p-1} &\leq \underbrace{\left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}}_{=\|X\|_{l^p}} \cdot \left( \sum_{n=1}^{\infty} |x_n + y_n|^{(p-1)q} \right)^{\frac{1}{q}} \\
 &= \left\{ (p-1)q = (p-1) \frac{1}{1 - \frac{1}{p}} = p \right\} \\
 &= \|X\|_{l^p} \left( \sum_{n=1}^{\infty} |x_n + y_n|^p \right)^{\frac{1}{q}}.
 \end{aligned}$$

We have

$$\|X + Y\|_{l^p}^p \leq (\|X\|_{l^p} + \|Y\|_{l^p}) \|X + Y\|_{l^p}^{\frac{p}{q}}.$$

If  $\|X + Y\|_{l^p} \neq 0$  then

$$\|X + Y\|_{l^p}^{p - \frac{p}{q}} \leq \|X\|_{l^p} + \|Y\|_{l^p}$$

there

$$p - \frac{p}{q} = p(1 - \frac{1}{q}) = p \frac{1}{p} = 1.$$

□

**Remark.**  $f \in C([0, 1])$  then for  $1 \leq p < \infty$

$$\|f\|_{L^p} = \left( \int_0^1 |f(t)|^p dt \right)^{\frac{1}{p}}.$$

**Claim:**

$$\|fg\|_{L^1} = \int_0^1 |f(t) \cdot g(t)| dt \leq \|f\|_{L^p} \cdot \|g\|_{L^q},$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ . Also we have

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

This is proven with the same technique as we used for  $l^p$ .  $\sum_{n=1}^{\infty}$  is replaced by  $\int_0^1 dt$ .  
 $E$  real/complex vector space.  $x_1, \dots, x_n \in E$ ,  $\lambda_1, \dots, \lambda_n$  scalar. We say that

$$\lambda_1 x_1, \dots, \lambda_n x_n$$

is a linear combination of  $x_1, \dots, x_n$ . We say that  $x_1, \dots, x_n$  are linear independent if

$$\alpha_1 x_1 + \dots + \alpha_n x_n = 0 \quad \Rightarrow \quad \alpha_1 = \dots = \alpha_n = 0.$$

If  $A \subset E$ , we say that  $A$  is linear independent if every linear combination of vectors in  $A$  is linear independent.

**Examples.** (1) Set  $E = P([0, 1])$  and  $A = \{p_k \mid p_k(x) = x^k, x \in [0, 1], k = 0, 1, \dots\}$ .  $A$  is linear independent since:

Consider

$$\alpha_0 p_0 + \alpha_1 p_1 + \dots + \alpha_n p_n = 0,$$

i.e.

$$\alpha_0 p_0(x) + \alpha_1 p_1(x) + \dots + \alpha_n p_n(x) = 0(x), \quad x \in [0, 1],$$

i.e.

$$\alpha_0 + \alpha_1 x + \dots + \alpha_n x^n = 0, \quad x \in [0, 1].$$

If  $x = 0$  then  $\alpha_0 = 0$

$$\alpha_1 x + \dots + \alpha_n x^n = 0, \quad x \in [0, 1].$$

Differentiate

$$\alpha_1 + 2\alpha_2 x + \dots + n\alpha_n x^{n-1} = 0$$

gives  $\alpha_1 = 0$ . Continue and get

$$\alpha_0 = \alpha_1 = \dots = \alpha_n = 0.$$

Set  $B \subset E$  where

$\text{span } B = \{\text{set of all linear combinations of elements in } B\}$

$$= \left\{ \sum_{k=1}^n \lambda_k x_k \mid x_k \in B, \lambda_k \in \mathbb{R}, k = 1, 2, \dots, n \text{ where } n \text{ is a positive integer} \right\}.$$

**Remark.**

$$\sum_{k=1}^n \lambda_k x_k \in E,$$

$$\sum_{k=1}^{\infty} \lambda_k x_k \text{ has no meaning.}$$

$C \subset E$  is called a basis for  $E$  if

- 1)  $C$  linear independent,
- 2)  $\text{span } C = E$ .

Continue of the example above:

**Claim:**  $A$  is a basis for  $E$ .

(2) Set  $E = l^2$  and

$$A = \{X_k \mid k = 1, 2, \dots\},$$

$$X_k = (0, 0, \dots, 0, 1, 0, 0, \dots).$$

**Claim:**  $A$  is linear independent since

$$\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_n X_n = 0.$$

Here

$$\alpha_1 X_1 = (\alpha_1, 0, 0, \dots), \quad \text{etc}$$

and

$$0 = (0, 0, \dots).$$

So

$$(\alpha_1, \alpha_2, \dots, \alpha_n, 0, \dots) = (0, 0, \dots).$$

So  $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ .

Question: Is  $A$  a basis for  $l^2$ ?

We note: If  $X \in \text{span } A$  then

$$X = (x_1, x_2, \dots, x_n, 0, 0, \dots)$$

for some positive integer  $n$ , i.e.  $X$  has only finitely many nonzero positions.

Consider:

$$X := (1, \frac{1}{2}, \dots, \frac{1}{n}, \dots),$$

$$\|X\|_{l^2} = \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} < \infty.$$

So  $X \in l^2 \setminus \text{span } A$ .

**Remark.** Every vector space has a basis (if we are allowed to use Axiom of Choice/ Zorn's lemma).

Basis = vector space basis = Hamel basis

Assume  $x_1, \dots, x_n$  is a basis for  $E$ . Then every basis for  $E$  must contain  $n$  different elements.

$$n = \dim E$$

is well-defined. (System of linear equations, homogeneous with more unknowns than equations. Then there exists a nontrivial solution.)

**Definition (norm).**  $E$  vector space. We say that  $\|\cdot\| : E \rightarrow [0, \infty)$  is a norm on  $E$  if

- 1)  $\|x\| = 0 \Rightarrow x = 0$ ,
- 2)  $\|\lambda x\| = |\lambda| \|x\|$  for all  $x \in E, \lambda \in \mathbb{R}$ ,
- 3)  $\|x + y\| \leq \|x\| + \|y\|$  for all  $x, y \in E$ .

**Remark.**

$$\|0\| = \|0 \cdot 0\| = \underbrace{|0|}_{=0} \|0\| = 0.$$

**Examples.** (1)  $1 < p < \infty$  and

$$\|X\|_{l^p} = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

is a norm on  $l^p$ . Check 1), 2) and 3) above:

1)

$$0 = \|X\|_{l^p} = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}.$$

It follows

$$\begin{aligned} x_n &= 0, \quad n = 1, 2, \dots, \\ \Rightarrow X &= (x_1, x_2, \dots) = (0, 0, \dots) = 0. \end{aligned}$$

2)

$$\|\lambda X\|_{l^p} = \left( \sum_{n=1}^{\infty} |\lambda x_n|^p \right)^{\frac{1}{p}} = \left( |\lambda|^p \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} = |\lambda| \|X\|_{l^p}$$

3)

$$\|X + Y\|_{l^p} \leq \{\text{Minkowski's inequality}\} \leq \|X\|_{l^p} + \|Y\|_{l^p}$$

(2)  $E = C([0, 1])$  and  $f \in E$ 

$$\|f\| = \max_{t \in [0, 1]} |f(t)| \in [0, \infty).$$

Check the axioms above

1) If  $\|f\| = 0$  it follows

$$|f(t)| = 0 \text{ for all } t \in [0, 1], \quad \Rightarrow \quad f = 0$$

2)

$$\|\lambda f\| = \max_{t \in [0, 1]} \underbrace{|(\lambda f)(t)|}_{\substack{\lambda f(t) \\ |\lambda| |f(t)|}} = |\lambda| \max_{t \in [0, 1]} |f(t)| = |\lambda| \|f\|$$

3)

$$\|f + g\| = \max_{t \in [0, 1]} \underbrace{|(f + g)(t)|}_{f(t)+g(t)} = \max_{t \in [0, 1]} (|f(t)| + |g(t)|) \leq \max_{t \in [0, 1]} |f(t)| + \max_{t \in [0, 1]} |g(t)| = \|f\| + \|g\|$$

(3)  $E = C([0, 1])$  and  $f \in E$ .

$$\|f\|_{L^1} = \int_0^1 |f(t)| dt$$

defines also a norm on  $E$ .

3)

$$\begin{aligned} \|f + g\|_{L^1} &= \int_0^1 \underbrace{|(f + g)(t)|}_{f(t)+g(t)} dt \\ &\leq \int_0^1 (|f(t)| + |g(t)|) dt \\ &= \int_0^1 |f(t)| dt + \int_0^1 |g(t)| dt \\ &= \|f\|_{L^1} + \|g\|_{L^1} \end{aligned}$$

2)

$$\|\lambda f\| = \int_0^1 \underbrace{|(\lambda f)(t)|}_{=|\lambda| |f(t)|} dt = |\lambda| \|f\|_{L^1}$$

1)

$$0 = \|f\|_{L^1} = \int_0^1 |f(t)| dt$$

This implies  $f(t) = 0$  for  $t \in [0, 1]$  since  $f$  is continuous! i.e.  $f = 0$ .

**Theorem 2.3 (equivalent norm).**  $E$  vector space with norms  $\|\cdot\|$  and  $\|\cdot\|_*$ . We say that  $\|\cdot\|$  and  $\|\cdot\|_*$  are equivalent if there exists  $\alpha, \beta > 0$  such that

$$\alpha\|x\|_* \leq \|x\| \leq \beta\|x\|_* \quad \text{for all } x \in E.$$

**Example.**

$E = C([0, 1])$ . Choose  $y = f(t)$  and  $y = |f(t)|$

$$\|f\| = \max_{t \in [0, 1]} |f(t)|, \quad \|f\|_* = \|f\|_{L^1} = \text{area}.$$

Question: Are these norms equivalent?

**Claim:**  $f \in C([0, 1])$

$$\|f\|_* = \int_0^1 \underbrace{|f(t)|}_{\leq \|f\|} dt \leq \|f\|.$$

Choose  $f_n(t)$  such that

$$\|f_n\| = 1, \quad \|f_n\|_* = \frac{1}{2n}.$$

So

$$\frac{\|f_n\|_*}{\|f_n\|} = \frac{1}{2n} \rightarrow 0 \quad n \rightarrow \infty.$$

The norms are not equivalent! Answer: NO !

**Theorem 2.4.**  $E$  vector space with  $\dim E < \infty$ .

$\Rightarrow$  All norms on  $E$  are equivalent.

**proof.** Assume  $n = \dim E$  with a positive integer  $n$ . Let  $x_1, x_2, \dots, x_n$  be a basis for  $E$ . For every  $x \in E$

$$x = \alpha_1(x)x_1 + \dots + \alpha_n(x)x_n,$$

where  $\alpha_1(x), \dots, \alpha_n(x)$  unique. Set

$$\|x\|_* = |\alpha_1(x)| + \dots + |\alpha_n(x)|, \quad x \in E$$

**Claim:**  $\|\cdot\|_*$  defines a norm on  $E$  (easy proof)

Fix an arbitrary norm  $\|\cdot\|$  on  $E$ .

**Claim:**  $\|\cdot\|_*$  and  $\|\cdot\|$  are equivalent.

Note for  $x \in E$

$$\begin{aligned} \|x\| &= \|\alpha_1(x)x_1 + \dots + \alpha_n(x)x_n\| \\ &\leq |\alpha_1(x)|\|x_1\| + \dots + |\alpha_n(x)|\|x_n\| \\ &\leq \max_{k=1,2,\dots,n} \|x_k\| \underbrace{(|\alpha_1(x)| + \dots + |\alpha_n(x)|)}_{=\|x\|_*}. \end{aligned}$$

Set  $\beta = \max_{k=1,2,\dots,n} \|x_k\|$ .

Then

$$\|x\| \leq \beta \|x\|_* \quad \text{for all } x \in E.$$

Remains to prove: There exists  $\alpha > 0$  such that

$$\alpha \|x\|_* \leq \|x\| \quad \text{for all } x \in E \quad (*).$$

Let  $E$  be a vector space with norm  $\|\cdot\|$  and  $(v_m)_{m=1}^\infty$  a sequence in  $E$ . We say that  $(v_m)_{m=1}^\infty$  converges in  $(E, \|\cdot\|)$  if there exists  $v \in E$  such that  $\|v_m - v\| \rightarrow 0$  for  $n \rightarrow \infty$ .

Notation:  $v_m \rightarrow v$  in  $(E, \|\cdot\|)$ .

Note: If we have  $\|\cdot\|$  and  $\|\cdot\|_*$  are equivalent, then

$$v_n \rightarrow v \text{ in } (E, \|\cdot\|) \Leftrightarrow v_n \rightarrow v \text{ in } (E, \|\cdot\|_*).$$

Back to (\*): Argue by contradiction.

Assume there is no  $\alpha > 0$  such that

$$\alpha \|x\|_* \leq \|x\| \quad \text{for all } x \in E.$$

For  $k = 1, 2, 3, \dots$  there are  $y_k \in E$  such that

$$\frac{1}{k} \|y_k\|_* > \|y_k\|. \quad (**).$$

We have

$$y_k = \alpha_1^{(k)} x_1 + \dots + \alpha_n^{(k)} x_n,$$

where  $\alpha_1^{(k)}, \dots, \alpha_n^{(k)}$  are unique scalars and  $k = 1, 2, \dots$

(\*\*) implies that

$$k \|y_k\| < |\alpha_1^{(k)}| + \dots + |\alpha_n^{(k)}|,$$

WLOG we can assume  $|\alpha_1^{(k)}| + \dots + |\alpha_n^{(k)}| = 1$ . ( If not consider

$$\begin{aligned} \lambda z &= \lambda(\alpha_1(z)x_1 + \dots + \alpha_n(z)x_n) \\ &= (\lambda\alpha_1(z))x_1 + \dots + (\lambda\alpha_n(z))x_n \\ &= \alpha_1(\lambda z)x_1 + \dots + \alpha_n(\lambda z)x_n. \end{aligned}$$

We have

$$\alpha_k(\lambda z) = \lambda \alpha_k(z), \quad k = 1, 2, \dots, n).$$

We have

$$k \|y_k\| < 1 \quad k = 1, 2, \dots$$

which implies  $y_k \rightarrow 0$  in  $(E, \|\cdot\|)$ .

IF:

$$\begin{aligned} \alpha_1^{(k)} &\rightarrow \bar{\alpha}_1 \\ \alpha_2^{(k)} &\rightarrow \bar{\alpha}_2 \\ &\vdots \\ \alpha_n^{(k)} &\rightarrow \bar{\alpha}_n \end{aligned}$$

for  $k \rightarrow \infty$ . Then set

$$\bar{y} = \bar{\alpha}_1 x_1 + \dots + \bar{\alpha}_n x_n$$

and get

$$\begin{aligned} \|y_k - \bar{y}\| &= \left\| (\alpha_1^{(k)} - \bar{\alpha}_1)x_1 + \dots + (\alpha_n^{(k)} - \bar{\alpha}_n)x_n \right\| \\ &\leq \underbrace{|\alpha_1^{(k)} - \bar{\alpha}_1|}_{\rightarrow 0} \underbrace{\|x_1\|}_{< \infty} + \dots + \underbrace{|\alpha_n^{(k)} - \bar{\alpha}_n|}_{\rightarrow 0} \underbrace{\|x_n\|}_{< \infty} \rightarrow 0, \quad k \rightarrow \infty \\ \|\bar{y}\| &= \|\bar{y} - y_k + y_k\| \leq \underbrace{\|\bar{y} - y_k\|}_{\rightarrow 0} + \underbrace{\|y_k\|}_{\rightarrow 0} \rightarrow 0, \quad k \rightarrow \infty. \end{aligned}$$

So  $\|\bar{y}\| = 0$  hence  $\bar{y} = 0$ . But

$$|\bar{\alpha}_1| + |\bar{\alpha}_2| + \dots + |\bar{\alpha}_n| = 1.$$

This contradicts  $x_1, \dots, x_n$  is a basis.

We have for  $k = 1, 2, \dots$  the vector  $(\alpha_1^{(k)}, \alpha_2^{(k)}, \dots, \alpha_n^{(k)})$  where

$$|\alpha_1^{(k)}| + \dots + |\alpha_n^{(k)}| = 1.$$

We focus on the first one and we have

$$|\alpha_1^{(k)}| \leq 1, \quad k = 1, 2, \dots$$

By Bolzano-Weierstraß then there exists a converging subsequence  $(\alpha_{1,1}^{(k)})_{k=1}^\infty$  of  $(\alpha_1^{(k)})_{k=1}^\infty$ . Set

$$\bar{\alpha}_1 = \lim_{k \rightarrow \infty} \alpha_{1,1}^{(k)}$$

and consider

$$(\alpha_{1,1}^{(k)}, \alpha_{2,1}^{(k)}, \dots, \alpha_{n,1}^{(k)}), \quad k = 1, 2, \dots$$

We have

$$|\alpha_{2,1}^{(k)}| \leq 1, \quad k = 1, 2, \dots$$

Bolzano-Weierstraß implies that there exists a converging subsequence  $(\alpha_{2,2}^{(k)})_{k=1}^\infty$  of  $(\alpha_{2,1}^{(k)})_{k=1}^\infty$ . Set

$$\bar{\alpha}_2 = \lim_{k \rightarrow \infty} \alpha_{2,2}^{(k)}.$$

□

**Definition** (normed space). Let  $E$  be a vector space over  $\mathbb{R}$  or  $\mathbb{C}$ .  $\|\cdot\| : E \rightarrow \mathbb{R}$  a norm on  $E$  if

- (i)  $\|x\| > 0$  for any  $x \in E \setminus \{0\}$ ,
- (ii)  $\|\lambda x\| = |\lambda| \|x\|$  for any  $\lambda \in \mathbb{C}, x \in E$ ,
- (iii)  $\|x + y\| \leq \|x\| + \|y\|$  for any  $x, y \in E$ .

Obs.  $\|x\| = 0$  if  $x = 0$ .  $(E, \|\cdot\|)$  is called a normed space. A norm generates a distance



function (metric)

$$L(x, y) := \|x - y\| \quad \text{for any } x, y \in E.$$

**Examples.** •  $\mathbb{R}^n$  with  $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$  is the eukledian norm.

- $C([0, 1])$  continuous functions in  $[0, 1]$  with

$$L(f, g) = \|f - g\|_\infty := \max_{x \in [0, 1]} |f(x) - g(x)|$$

**Definition (balls).** Let  $x \in E, r > 0$ . Define

$$\begin{aligned} B(x, r) &:= \{y \in E \mid \|x - y\| < r\} && \text{open ball,} \\ \bar{B}(x, r) &:= \{y \in E \mid \|x - y\| \leq r\} && \text{closed ball.} \end{aligned}$$

**Definition (open/closed).** A subset  $A \subset E$  of a normed space  $(E, \|\cdot\|)$  is called open if any point  $x$  of  $A$  is inner, i.e

$$\exists r > 0 : B(x, r) \subset A.$$

It is called closed if the complement  $E \setminus A$  is open.

**Remark.** • open balls are open sets.

- closed balls are closed.
- $(C([0, 1]), \|\cdot\|_\infty)$  with  $\|f\|_\infty = \max_{x \in [0, 1]} |f(x)|$ .

$$A := \{g \in C([0, 1]) \mid f(x) < g(x), \forall x \in [0, 1]\}$$

is an open set  $C([0, 1])$ .

$$B := \{g \in C([0, 1]) \mid f(x) \leq g(x), \forall x \in [0, 1]\}$$

is a closed set.

### Properties

- Any union of open sets is an open set.
- Any finite intersection of open sets is open.
- $\emptyset, E$  are both closed and open.
- Normed spaces are topological spaces.

**Definition** (convergence in normed spaces). Let  $(E, \|\cdot\|)$  be a normed space  $\{x_n\}_n \subset E$ . We say that  $x_n$  converges to  $x \in E$  if

$$\|x_n - x\| \rightarrow 0, \quad n \rightarrow \infty.$$

One can define open and closed using the definition of convergence:

**Statement 2.5.**  $A \subseteq E$  is closed if any convergent sequence in  $A$  has a limit in  $A$ , i.e

$$\underbrace{x_n \rightarrow x}_{\substack{\text{for } n \rightarrow \infty \\ x_n \in A}} \Rightarrow x \in A.$$

**proof.**  $\Rightarrow$ : Assume that  $A$  is closed and  $x_n \rightarrow x$ .  $x_n \in A$ , but  $x \notin A$ . (try to get a contradiction).

$A$  is closed  $\Rightarrow E \setminus A$  is open and hence  $\exists r > 0$  such that

$$B(x, r) \subset E \setminus A.$$

Hence  $\|x_n - x\| \geq r$  for any  $n$ . This is a contradiction because in that case  $x_n \not\rightarrow x$ .

$\Leftarrow$ : Assume that for any sequence  $\{x_n\} \subset A$  such that  $x_n \rightarrow x$  we have  $x \in A$ . We try to get a contradiction and assume that  $A$  is not closed. Hence  $E \setminus A$  is not open and therefore  $\exists x \in E \setminus A$  which is not inner.

$$\Rightarrow \forall B(x, \frac{1}{n}) \text{ contains points outside } E \setminus A,$$

i.e.

$$\exists x_n \in B(x, \frac{1}{n}), x_n \in A.$$

We get a sequence  $\{x_n\} \subset A$  such that

$$\|x_n - x\| < \frac{1}{n} \Rightarrow x_n \rightarrow x.$$

This is a contradiction. □

**Definition** (closure).  $A \subset E$ . The closure of  $A$  is the minimal closed subset containing  $A$ . We write  $\bar{A}$ .

**Proposition 2.6.**  $\bar{A}$  is the set of all limit points of  $A$  which means

$$\bar{A} := \{x \in E \mid \text{there exists } \{x_n\} \subseteq A \text{ such that } x_n \rightarrow x\}.$$

**proof.** Exercise. □

**Definition (dense).**  $A \subset E$  is dense in  $E$  if

$$\bar{A} = E.$$

**Remark.** This definition of dense is equivalent to the following definition:

$$\forall x \in E, \forall \varepsilon > 0 \exists y \in A \text{ such that } \|x - y\| < \varepsilon.$$

**Examples.** 1)  $\mathbb{Q} \subseteq \mathbb{R}$  with  $|\cdot|$  usual absolute value function.  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .

2)  $C([a, b])$ . The Weierstraß-Theorem says that the set of all polynomials are dense in  $(C([a, b], \|\cdot\|_\infty))$ :

$$\forall f \in C([a, b]), \forall \varepsilon > 0 \exists p - \text{polynomial such that } \max_{x \in [a, b]} |f(x) - p(x)| < \varepsilon.$$

Another example is  $(C_0, \|\cdot\|_\infty)$  where

$$C_0 = \{x = (x_1, x_2, \dots) \mid x_k \rightarrow 0 \text{ as } k \rightarrow \infty\},$$

$$\|x\|_\infty = \sup_i |x_i|.$$

$(C_0, \|\cdot\|_\infty)$  is a normed space.

$$C_F = \{x = (x_1, x_2, \dots) \mid \text{only a finite number of } x_i \neq 0\} \subset C_0.$$

**Statement 2.7.**  $C_F$  is dense in  $C_0$ .

**proof.**

$$\forall x \in C_0 \forall \varepsilon > 0 \text{ must find } y \in C_F \text{ such that } \|y - x\|_\infty < \varepsilon.$$

$$x \in C_0 \quad \Rightarrow \quad x_k \rightarrow 0 \text{ for } k \rightarrow \infty$$

$$\Rightarrow \quad \forall \varepsilon > 0 \exists K \text{ such that } |x_k| < \varepsilon \forall k \geq K.$$

Let now  $y = (x_1, x_2, \dots, x_K, 0, \dots) \in C_F$ . Then

$$\|x - y\|_\infty = \|(0, 0, \dots, 0, x_{K+1}, x_{K+2}, \dots)\|_\infty = \sup_{k > K} |x_k| < \varepsilon.$$

□

**Definition (separable).** A normed space  $(E, \|\cdot\|)$  is called separable if it contains a countable dense subset.

- Examples.**
- $(\mathbb{R}, |\cdot|)$  is separable as  $\mathbb{Q}$  is countable and dense in  $\mathbb{R}$ .
  - $(\mathbb{R}^n, \|\cdot\|_2)$  is separable,  $\mathbb{Q}^n$  is countable and dense in  $\mathbb{R}^n$ .

**Definition (compact set).** For a normed space  $(E, \|\cdot\|)$  is  $A \subset E$  a compact set if any sequence  $\{x_n\} \subset A$  has a subsequence convergent to an element  $x \in A$ .

**Example.** Any bounded and closed subset in  $\mathbb{R}, \mathbb{R}^n, \mathbb{C}^n$  is compact. A sequence  $\{x_n\}$  of a bounded set is bounded. From real Analysis one knows it has a subsequence that is convergent. If the subset is closed then the limit point is inside the set.

**Lemma .**  $S \subset E$  compact in  $(E, \|\cdot\|)$  implies that  $S$  is closed and bounded. (Bounded means that  $S \subset B(0, R)$  for some  $R > 0$ ).

**proof.** Let  $S$  be a compact subset of  $E$ . Assume that  $S$  is not bounded. Hence for any  $n > 0$  there exists points in  $S$  which are outside  $B(0, n)$ , i.e.

$$\exists x_n \in S : \|x_n\| > n.$$

Then  $\{x_n\}$  can not have a convergent subsequence as if  $x_{n_k} \rightarrow x$  then

$$n_k < \|x_{n_k}\| = \|x_{n_k} - x + x\| \leq \|x_{n_k} - x\| + \|x\| \rightarrow \|x\|$$

but  $n_k \rightarrow \infty$ . This is a contradiction, hence  $S$  must be bounded.

$S$  must be closed, because if  $x_n \rightarrow x$  then any subsequence converges to  $x$ . From the definition of compactness and uniqueness of the limit we have  $x \in S$ .

□

**Remark.** In general,  $S$  bounded and closed doesn't imply that  $S$  is compact.

For instance let  $E = C([0, 1])$ . Then  $S = \{g \in C([0, 1]) \mid \|g\|_\infty \leq 1\}$  is closed and bounded, but not compact.

Take  $x_n(t) := t^n$ . Then  $x_n \in S$ .  $\{x_n\}$  does not have a subsequence convergent to a continuous function.

**Theorem 2.8.**  $(E, \|\cdot\|)$  normed space and  $\dim E < \infty$   
iff

$$\forall A \subset E, A \text{ compact} \Leftrightarrow A \text{ is closed and bounded.}$$

**proof.**  $\Rightarrow$ : If  $\dim E < \infty$  then  $A$  is compact iff  $A$  is bounded and closed (exercise).

$\Leftarrow$ : Enough to prove the following:

If  $\dim E = \infty$  then the unit ball  $S = \{x \in E \mid \|x\| \leq 1\}$  is not compact.

**Lemma 2.9 (Riesz's lemma).** If  $X$  is a proper closed subspace of a normed space  $(E, \|\cdot\|)$  then for every  $\varepsilon \in (0, 1)$  there exists an  $x_\varepsilon \in E$  with  $\|x_\varepsilon\| = 1$  such that

$$\|x_\varepsilon - x\| \geq \varepsilon \quad \forall x \in X.$$

**proof.** Let  $z \in E \setminus X$  ( $X$  proper and hence  $E \setminus X \neq \emptyset$ ). Set

$$d := \inf_{x \in X} \|z - x\|.$$

As  $X$  is closed,  $d > 0$ , otherwise  $z$  is a limit point in  $E \setminus X$ . Fix  $\varepsilon \in (0, 1)$ . Then there exists  $x_0 \in X$  such that

$$d \leq \|z - x_0\| < \frac{d}{\varepsilon}.$$

Let  $x_\varepsilon := \frac{z - x_0}{\|z - x_0\|}$ ; We have  $\|x_\varepsilon\| = 1$  and

$$\begin{aligned} \|x - x_\varepsilon\| &= \left\| x - \frac{z - x_0}{\|z - x_0\|} \right\| \\ &= \frac{\|x\|z - x_0\| - z + x_0\|}{\|z - x_0\|} \\ &= \frac{\left\| \overbrace{x\|z - x_0\| + x_0 - z}^{\in X} \right\|}{\|z - x_0\|} \\ &\geq \frac{d}{d/\varepsilon} = \varepsilon. \end{aligned}$$

□

Continue now the proof of the theorem above:

Let  $x_1 \in S$ . Consider  $X = \text{span}\{x_1\}$  which is a proper closed subspace of  $E$ . Hence by Riesz's lemma exists  $x_2$  with  $\|x_2\| = 1$  such that

$$\|x_2 - x_1\| \geq \frac{1}{2}$$

and

$$\|x_2 - x\| \geq \frac{1}{2} \quad \forall x \in X.$$

Now consider  $\text{span}\{x_1, x_2\}$  which is a proper closed subspace of  $E$ . By Riesz's lemma follows

$$\exists x_3 \in E, \|x_3\| = 1 : \|x_3 - x_1\| \geq \frac{1}{2}, \|x_3 - x_2\| \geq \frac{1}{2}.$$

Continuing in the same fashion we get  $\{x_n\}$ ,  $\|x_n\| = 1$  such that

$$\|x_n - x_m\| \geq \frac{1}{2} \quad \forall n, m, n \neq m.$$

Clearly  $\{x_n\} \subset S$  has no convergent subsequence. Hence  $S$  is not compact. □

**Definition** (Cauchy sequence).  $(E, \|\cdot\|)$  normed space.  $\{x_n\} \subseteq E$  is called Cauchy if

$$\forall \varepsilon > 0 \exists N : \|x_n - x_m\| < \varepsilon \text{ for any } n, m \geq N.$$

**Example.**  $(C_F, \|\cdot\|_\infty)$ ,  $\|x\|_\infty = \sup_{k \in \mathbb{N}} |x_k|$  where  $x = (x_1, x_2, \dots)$ . Define

$$x_n = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, \dots).$$

Then  $\{x_n\}$  is Cauchy, as for  $n > m$

$$\begin{aligned} \|x_n - x_m\|_\infty &= \left\| (0, \dots, 0, \frac{1}{m+1}, \dots, \frac{1}{n}, 0, \dots) \right\|_\infty \\ &= \frac{1}{m+1}. \end{aligned}$$

Observe that  $x_n$  is convergent in  $(C_0, \|\cdot\|_\infty)$

$$\underbrace{x_n}_{\in C_F} \rightarrow (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots) \in C_0 \setminus C_F.$$

**Statement 2.10.** A convergent sequence is always a Cauchy sequence.

**Definition** (complete space). A normed vector space  $(E, \|\cdot\|)$  is called complete if any Cauchy sequence in  $E$  is convergent in  $E$ .

$(C_F, \|\cdot\|_\infty)$  is not complete.

**Definition** (Banach space). A complete normed space is called Banach space.

**Examples.** •  $(\mathbb{R}, |\cdot|)$  is a Banach space.

•  $(\mathbb{C}, |\cdot|)$  is a Banach space.

•  $(l^2, \|\cdot\|_2)$  where

$$l^2 = \left\{ (x_1, x_2, \dots) \left| \sum_{i=1}^{\infty} |x_i|^2 < \infty, x_i \in \mathbb{C} \right. \right\}$$

and

$$\|(x_1, x_2, \dots)\|_2 = \left( \sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}.$$

$(l^2, \|\cdot\|_2)$  is complete.

**proof.** Let  $x_n = (x_1^n, x_2^n, \dots)$  be a Cauchy sequence in  $l^2$ . We must show that it has a limit in  $l^2$ . We will do it in a few steps:

Step 1: Find a candidate for a limit  $a$ .

Step 2: Show that  $a \in l^2$ .

Step 3:  $\|x_n - a\|_2 \rightarrow 0$  as  $n \rightarrow \infty$ .

**Step 1:** Let

$$\begin{aligned} x_1 &= (x_1^1, x_2^1, \dots) \\ x_2 &= (x_1^2, x_2^2, \dots) \\ &\vdots \\ x_n &= (x_1^n, x_2^n, \dots). \end{aligned}$$

For each  $k$  consider sequence  $\{x_k^n\} \subset \mathbb{C}$  ( $k$ -th coordinates in each  $x_n$ ).  
Each sequence is Cauchy, as for all  $n, m \geq N$

$$|x_k^n - x_k^m| < \left( \sum_{k=1}^{\infty} |x_k^n - x_k^m|^2 \right)^{\frac{1}{2}} = \|x_n - x_m\|_2 < \varepsilon.$$

As  $(\mathbb{C}, |\cdot|)$  is complete,  $\{x_k^n\}_n$  has a limit  $a_k \in \mathbb{C}$ . Candidate for limit of  $x_n$  is

$$a = (a_1, a_2, \dots, a_k, \dots).$$

**Step 2:** Write

$$a = \underbrace{x_n}_{\in l^2} - (x_n - a).$$

In order to show that  $a \in l^2$  it is enough to see that  $x_n - a \in l^2$  for some  $n$ .  
 $\{x_n\}$  Cauchy implies

$$\forall \varepsilon > 0 \exists N : \forall n, m \geq N : \|x_n - x_m\|_2 < \varepsilon.$$

Consider for some  $u > 0$

$$\sum_{i=1}^u |x_i^n - x_i^m|^2 \leq \sum_{i=1}^{\infty} |x_i^n - x_i^m|^2 = \|x_n - x_m\|_2^2 < \varepsilon^2.$$

Let  $m \rightarrow \infty$ . We get

$$\sum_{i=1}^m |x_i^n - a_i|^2 \leq \varepsilon^2.$$

This holds for any  $u \in \mathbb{N}$ . Hence for any  $n \geq N$

$$\underbrace{\sum_{i=1}^{\infty} |x_i^n - a_i|^2}_{=\|x_n - a\|_2^2} \leq \varepsilon^2.$$

Hence  $x_n - a \in l^2$  and moreover  $\|x_n - a\| \rightarrow 0$  as  $n \rightarrow \infty$ .



- $(C([a, b]), \|\cdot\|_\infty)$  is a Banach space.
- $(l^p, \|\cdot\|_{l^p})$  for  $1 \leq p < \infty$  are all Banach spaces.
- $(C([a, b]), \|\cdot\|_2)$  with

$$\|f\|_2 = \left( \int |f(t)|^2 dt \right)^{\frac{1}{2}}.$$

One can prove that  $(C([a, b]), \|\cdot\|_2)$  is not a Banach space.

**Exercise:**

$[a, b] = [0, 1]$  and

$$f_n(t) = \begin{cases} 0, & \text{falls } t < \frac{1}{2} - \frac{1}{n} \\ 1, & \text{falls } t > \frac{1}{2} \\ \text{continuous linear function} & \end{cases}$$

Show that  $\{f_n\}$  is Cauchy in  $C([0, 1], \|\cdot\|_2)$  but  $f_n \not\rightarrow f \in C([0, 1])$ .

**Definition** (Convergent and absolutely convergent series). A series  $\sum_{n=1}^{\infty} x_n$  in  $E$  is called convergent if  $\{\sum_{n=1}^m x_n\}_m$ , a sequence of partial sums, is convergent in  $E$ . If  $\sum_{n=1}^{\infty} \|x_n\| < \infty$  then we say that  $\sum_{n=1}^{\infty} x_n$  converges absolutely.

**Theorem 2.11.** A normed space  $E$  is complete iff every absolutely convergent series converges in  $E$ .

**proof.**  $\Rightarrow$ : Suppose  $E$  is complete and  $\sum_{n=1}^{\infty} \|x_n\| < \infty$ . Let

$$S_N := \sum_{n=1}^N x_n \in E.$$

For  $M > N$ :

$$\begin{aligned} \|S_N - S_M\| &= \left\| \sum_{n=N+1}^M x_n \right\| \\ &\leq \sum_{n=N+1}^M \|x_n\| \\ &\leq \sum_{n=N+1}^{\infty} \|x_n\| \rightarrow 0 \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Hence  $\{S_N\}$  is Cauchy. As  $E$  is complete,  $S_N$  has a limit in  $E$  i.e.  $\sum_{n=1}^{\infty} x_n$  converges in  $E$ .



$\Leftarrow$ : Assume that every absolutely convergent series is convergent in  $E$ . We want to see that  $E$  is complete.

Let  $\{x_n\}$  be a Cauchy sequence. We want to prove that  $\{x_n\}$  has a limit in  $E$ . We know that

$$\forall k \exists n_k : \|x_n - x_m\| < \frac{1}{2^k} \quad \forall n, m \geq n_k.$$

We can assume that  $\{n_k\}$  is an increasing sequence. Write

$$x_{n_k} = (x_{n_k} - x_{n_{k-1}}) + (x_{n_{k-1}} - x_{n_{k-2}}) + \dots + (x_{n_1} - \underbrace{x_{n_0}}_{=0}) = \sum_{l=1}^k (x_{n_l} - x_{n_{l-1}}).$$

$$\sum_{l=1}^{\infty} \|x_{n_l} - x_{n_{l-1}}\| \leq \sum_{l=1}^{\infty} \frac{1}{2^l} < \infty.$$

Hence  $\sum_{l=1}^{\infty} (x_{n_l} - x_{n_{l-1}})$  is absolutely convergent. By assumption

$$\sum_{l=1}^{\infty} (x_{n_l} - x_{n_{l-1}})$$

is convergent in  $E$ . Hence the partial sums are convergent. Subsequence is convergent.  $\{x_{n_k}\}$  is convergent to some  $x \in E$ .

**Exercise:**

Show that the whole  $\{x_n\} \rightarrow x$ .

□

**Recall:**

converging sequences  $(x_n)_{n=1}^{\infty}$  in  $(E, \|\cdot\|)$ .  $\|x_n - x\| \rightarrow 0$  for  $n \rightarrow \infty$  for some  $x \in E$ . (Notation:  $x_n \rightarrow x$  in  $(E, \|\cdot\|)$ )

**Remark.** Assume  $x_n \rightarrow x$  in  $(E, \|\cdot\|)$ . Then

$$1) \|x_n\| \rightarrow \|x\| \text{ in } (E, \|\cdot\|).$$

$$2) \sup_n \|x_n\| < \infty.$$

because

1)

$$\|x_n\| \leq \|x_n - x\| + \|x\|,$$

so

$$\|x_n\| - \|x\| \leq \|x_n - x\|.$$

It follows

$$-(\|x_n\| - \|x\|) \leq \|x_n - x\|.$$

So

$$\|x_n\| - \|x\| \leq \|x_n - x\| \rightarrow 0, \quad \text{for } n \rightarrow \infty.$$

Cauchy sequence in  $(x_n)_{n=1}^\infty$  in  $(E, \|\cdot\|)$  if  $\|x_n - x_m\| \rightarrow 0$  for  $n, m \rightarrow \infty$ .

We obtain:  $(x_n)_{n=1}^\infty$  converges in  $(E, \|\cdot\|)$   $\Rightarrow$   $(x_n)_{n=1}^\infty$  Cauchy sequence in  $(E, \|\cdot\|)$ . ( $\Leftarrow$  in general). If  $\Leftarrow$  then we call  $(E, \|\cdot\|)$  a Banach space.

$\sum_{n=1}^\infty x_n$  converges in  $(E, \|\cdot\|)$  if  $\left(\sum_{n=1}^k x_n\right)_{k=1}^\infty$  converges in  $(E, \|\cdot\|)$ .

$\sum_{n=1}^\infty x_n$  converges absolutely in  $(E, \|\cdot\|)$  if  $\sum_{n=1}^\infty \|x_n\|$  converges  $(\mathbb{R}, \|\cdot\|)$ .

## 2.1 Mappings between normed spaces

**Definition .** Let  $(E_1, \|\cdot\|_1)$ ,  $(E_2, \|\cdot\|_2)$  be normed spaces.  $T : E_1 \rightarrow E_2$  (not necessarily linear) is called continuous at  $x_0 \in E_1$ , if

$$x_n \rightarrow x_0 \text{ in } (E_1, \|\cdot\|_1) \quad \Rightarrow \quad T(x_n) \rightarrow T(x_0) \text{ in } (E_2, \|\cdot\|_2).$$

$T$  is called continuous if it is continuous at  $x_0 \in E_1$  for all  $x_0 \in E_1$ . We say that  $T : E_1 \rightarrow E_2$  is linear if

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$$

for all scalars  $\lambda_1, \lambda_2$  and  $x_1, x_2 \in E_1$ .

$T : E_1 \rightarrow E_2$  linear is called bounded if there exists  $M > 0$  such that

$$\|T(x)\|_2 \leq M\|x\|_1 \quad \text{for all } x \in E_1.$$

If  $T$  is bounded linear  $E_1 \rightarrow E_2$  define

$$\|T\| = \|T\|_{E_1 \rightarrow E_2} := \inf\{M \geq 0 \mid \|T(x)\|_2 \leq M\|x\|_1 \text{ for all } x \in E_1\}.$$

**Lemma .**

$$\|T\| = \sup_{\substack{x \in E_1 \\ x \neq 0}} \frac{\|T(x)\|_2}{\|x\|_1} = \sup_{\substack{x \in E_1 \\ \|x\|_1 = 1}} \|T(x)\|_2.$$

**Proposition 2.12.** Assume  $T : E_1 \rightarrow E_2$  linear. Then all the following statements are equivalent:

- (1)  $T$  continuous at  $0 \in E_1$ .
- (2)  $T$  continuous at  $x_0 \in E_1$  for some  $x_0 \in E_1$ .
- (3)  $T$  continuous at  $x_0 \in E_1$  for all  $x_0 \in E_1$ .

**(4)  $T$  is bounded.**

**proof.** (1)  $\Rightarrow$  (4): Assume  $T$  is continuous at  $0 \in E_1$ , i.e.

$$x_n \rightarrow 0 \text{ in } (E_1, \|\cdot\|_1) \quad \Rightarrow \quad T(x_n) \rightarrow T(\underbrace{0}_{\in E_1}) = \underbrace{0}_{\in E_2} \text{ in } (E_2, \|\cdot\|_2).$$

We want to prove that  $T$  is bounded. We search a  $M > 0$  such that

$$\|T(x)\|_2 \leq M\|x\|_1.$$

We assume that this doesn't hold true.

For  $n = 1, 2, \dots$  there exists  $x_n \in E_1$  such that

$$\|T(x_n)\|_2 > n\|x_n\|_1.$$

Set for  $n = 1, 2, \dots$

$$z_n := \frac{1}{n\|x_n\|_1} x_n.$$

(Note that  $\|x_n\|_1 > 0$ . Otherwise we would get a contradiction.)

Note

$$\|z_n\|_1 = \left\| \frac{1}{n\|x_n\|_1} \right\|_1 = \frac{1}{n\|x_n\|_1} \|x_n\|_1 = \frac{1}{n} \rightarrow 0, \quad \text{for } n \rightarrow \infty.$$

We have  $z_n \rightarrow 0$  in  $(E_1, \|\cdot\|_1)$ . But

$$\|T(z_n)\|_2 = \left\| \frac{1}{n\|x_n\|_1} T(x_n) \right\|_2 = \frac{1}{n\|x_n\|_1} \|T(x_n)\|_2 > 1 \quad \text{for all } n.$$

Hence

$$T(z_n) \not\rightarrow 0 \quad \text{in } (E_2, \|\cdot\|_2).$$

This is a contradiction.

(1)  $\Leftarrow$  (4): Assume  $T$  is bounded. For some  $M > 0$

$$\|T(x)\|_2 \leq M\|x\|_1, \quad \text{for all } x \in E_1.$$

We need to show that  $T$  is continuous at  $0 \in E_1$ , i.e.

$$x_n \rightarrow 0 \text{ in } (E_1, \|\cdot\|_1) \quad \Rightarrow \quad T(x_n) \rightarrow T(0) = 0 \text{ in } (E_2, \|\cdot\|_2).$$

From

$$\|T(x_n)\|_2 \leq M\|x_n\|_1 \rightarrow 0$$

so

$$T(x_n) \rightarrow \underbrace{0}_{=T(0)} \text{ in } (E_2, \|\cdot\|_2).$$

□

**Examples.** (A)  $E_1 = E_2 = C([0, 1])$ ,  $\|\cdot\|_1 = \|\cdot\|_2 = \|\cdot\|_\infty =: \|\cdot\|$ , i.e.

$$\|f\| := \max_{x \in [0, 1]} |f(x)|.$$

$$T(f)(x) = \int_0^{1-x} \min(x, y) f(y) dy, \quad \text{for } f \in C([0, 1]), x \in [0, 1].$$

(1)  $T(f) \in C([0, 1])$  for  $f \in C([0, 1])$ ,

(2)  $T$  linear,

(3)  $T$  bounded,

(4) Calculate  $\|T\|$ .

**proof.** (1) Fix  $f \in C([0, 1])$  arbitrary and fix  $x \in [0, 1]$ . Show that  $T(f)$  is continuous at  $x$ . Consider a sequence  $(x_n)_{n=1}^\infty$  in  $[0, 1]$  such that  $x_n \rightarrow x$  in  $(\mathbb{R}, |\cdot|)$ .

To show  $T(f)(x_n) \rightarrow T(f)(x)$  in  $(\mathbb{R}, |\cdot|)$ .

$$\begin{aligned} |T(f)(x_n) - T(f)(x)| &= \{\text{assume that } x_n \leq x\} \\ &= \left| \int_0^{1-x_n} \min(x_n, y) f(y) dy - \int_0^{1-x} \min(x, y) f(y) dy \right| \\ &\leq \left| \int_0^{1-x} (\min(x_n, y) - \min(x, y)) f(y) dy \right| \\ &\quad + \left| \int_{1-x}^{1-x_n} \min(x_n, y) f(y) dy \right| \\ &\leq \underbrace{\int_0^{1-x} \underbrace{|\min(x_n, y) - \min(x, y)|}_{\leq |x_n - x|} \underbrace{|f(y)|}_{\leq \|f\|} dy}_{\leq |x_n - x| \|f\|} \\ &\quad + \underbrace{\int_{1-x}^{1-x_n} \underbrace{\min(x_n, y)}_{\leq 1} \underbrace{|f(y)|}_{\leq \|f\|} dy}_{0 \leq \dots \leq |x_n - x| \cdot \|f\|} \rightarrow 0, \quad \text{as } n \rightarrow \infty \end{aligned}$$

If  $x_n > x$  we get a similar calculation. Conclusion:

$$T(f)(x_n) \rightarrow T(f)(x) \text{ in } (\mathbb{R}, |\cdot|) \text{ as } n \rightarrow \infty.$$

(2) Fix  $f_1, f_2 \in C([0, 1])$  and  $\lambda_1, \lambda_2$  scalars. Then

$$\begin{aligned} T(\lambda_1 f_1 + \lambda_2 f_2)(x) &= \int_0^{1-x} \min(x, y) \underbrace{(\lambda_1 f_1 + \lambda_2 f_2)(y)}_{=\lambda_1 f_1(y) + \lambda_2 f_2(y)} dy \\ &= \lambda_1 \int_0^{1-x} \min(x, y) f_1(y) dy + \lambda_2 \int_0^{1-x} \min(x, y) f_2(y) dy \\ &= \lambda_1 T(f_1)(x) + \lambda_2 T(f_2)(x) \quad \text{for } x \in [0, 1] \end{aligned}$$

(3) Fix  $f \in C([0, 1])$ . For  $x \in [0, 1]$

$$\begin{aligned}
 |T(f)(x)| &= \left| \int_0^{1-x} \underbrace{\min(x, y)}_{\geq 0} f(y) \, dy \right| \\
 &\stackrel{(*_1)}{\leq} \int_0^{1-x} \min(x, y) \underbrace{|f(y)|}_{\leq \|f\|} \, dy \\
 &\stackrel{(*_2)}{\leq} \int_0^{1-x} \min(x, y) \, dy \|f\|.
 \end{aligned}$$

Clearly

$$\max_{x \in [0, 1]} \int_0^{1-x} \min(x, y) \, dy \leq 1.$$

This gives:

$$\|T(f)\| = \max_{x \in [0, 1]} |T(f)(x)| \leq 1 \cdot \|f\|, \quad \text{for all } f \in C([0, 1]).$$

Conclusion:  $T$  is bounded with  $(M = 1)$

- (4) Consider the inequality above.  $(*_1)$  is an equality if  $f$  has a constant sign.  $(*_2)$  is an equality if  $f$  is a constant function. So we have to calculate

$$\int_0^{1-x} \min(x, y) \, dy \quad \text{for } x \in [0, 1].$$

**case 1:**  $1 - x \leq x$  i.e.  $\frac{1}{2} \leq x$  and we get

$$\begin{aligned}
 \int_0^{1-x} \underbrace{\min(x, y)}_{=y} \, dy &= \left[ \frac{1}{2} y^2 \right]_0^{1-x} \\
 &= \frac{1}{2} (1-x)^2.
 \end{aligned}$$

**case 2:**  $x < 1 - x$  i.e.  $x < \frac{1}{2}$  and we get

$$\begin{aligned}
 \int_0^{1-x} \min(x, y) \, dy &= \int_0^x y \, dy + \int_x^{1-x} x \, dy \\
 &= \frac{1}{2} x^2 + x(1-2x) \\
 &= x - \frac{3}{2} x^2.
 \end{aligned}$$

**Claim:**

$$\|T\| = \max \left( \max_{x \in [\frac{1}{2}, 1]} \frac{1}{2} (1-x)^2, \max_{x \in [0, \frac{1}{2}]} \left( x - \frac{3}{2} x^2 \right) \right) = \dots = \frac{1}{6}.$$

Note

- $\|T(f)\| \leq \|T\| \cdot \|f\|$  for all  $f \in C([0, 1])$ ,
- $\|T(1)\| = \|T\| \cdot \|1\|$  where  $1(x) = 1$  for  $x \in [0, 1]$ .

□

(B)  $E_1 = C([0, 1])$  with maximumnorm,  $E_2 = \mathbb{R}$  with absolut value.  $T : E_1 \rightarrow E_2$  with

$$T(f) = \int_0^{\frac{1}{2}} f(y) dy - \int_{\frac{1}{2}}^1 f(y) dy \quad \text{for } f \in E_1$$

$$\begin{aligned} |T(f)| &= \left| \int_0^{\frac{1}{2}} f(y) dy - \int_{\frac{1}{2}}^1 f(y) dy \right| \\ &\leq \left| \int_0^{\frac{1}{2}} f(y) dy \right| + \left| \int_{\frac{1}{2}}^1 f(y) dy \right| \\ &\leq \int_0^{\frac{1}{2}} \underbrace{|f(y)|}_{\leq \|f\|} dy + \int_{\frac{1}{2}}^1 \underbrace{|f(y)|}_{\leq \|f\|} dy \\ &\leq 1 \|f\|. \end{aligned}$$

Hence  $T$  is bounded and  $\|T\| \leq 1$ .

$$T(f) = \int_0^1 k(y) f(y) dy,$$

where

$$T(f_n) = \left\{ \begin{array}{l} \text{to be completed,} \\ \text{falls case .} \end{array} \right.$$

$$T(f_n) \leq 1 \left( \frac{1}{2} - \frac{1}{2n} + \frac{1}{2} - \frac{1}{2n} \right) = 1 - \frac{1}{n}, \quad n = 1, 2, \dots$$

Note

$$k(y) f_n(y) \geq 0 \quad \text{for } y \in [0, 1].$$

Hence  $\|T\| \leq 1 - \frac{1}{n}$  for  $n = 1, 2, \dots$ . Note  $\|f_n\| = 1$  for all  $n$ . Conclusion  $\|T\| = 1$ . Here

$$|T(f)| \leq \underbrace{\|T\|}_{\leq 1} \|f\| \quad \text{for all } f \in C([0, 1])$$

but

$$|T(f)| < \|T\| \|f\| \quad \text{for all } f \in C([0, 1]).$$

**Statement 2.13.**  $T_1, T_2$  bounded linear mappings  $(E_1, \|\cdot\|_1) \rightarrow (E_2, \|\cdot\|_2)$  and  $\lambda$  scalar. Set

$$\begin{aligned} (T_1 + T_2)(x) &= T_1(x) + T_2(x) & x \in E_1 \\ (\lambda T_1)(x) &= \lambda T_1(x) & x \in E_1. \end{aligned}$$

**Claim:**

- (1)  $T_1 + T_2$  and  $\lambda T_1$  are both linear mappings  $(E_1, \|\cdot\|_1) \rightarrow (E_2, \|\cdot\|_2)$ ,
- (2)  $T_1 + T_2$  and  $\lambda T_1$  are both bounded mappings  $(E_1, \|\cdot\|_1) \rightarrow (E_2, \|\cdot\|_2)$ .  
 $B(E_1, E_2)$  denote the vector space of all bounded linear mappings  $(E_1, \|\cdot\|_1) \rightarrow (E_2, \|\cdot\|_2)$ .
- (3)
- $$\|T\|_{E_1 \rightarrow E_2} := \inf\{M > 0 \mid \|T(x)\|_2 \leq M\|x\|_1 \text{ for all } x \in E_1\}$$
- defines a norm in  $B(E_1, E_2)$ .

**proof.** (1)  $\|T\| = 0$  implies that  $\|T(x)\|_2 = 0$  for all  $x \in E_1 \Rightarrow T(x) = 0 \in E_2$ .

$$T = 0 \in B(E_1, E_2)$$

(2)  $T \in B(E_1, E_2)$  and  $\lambda$  scalar.

$$\begin{aligned} \|\lambda T\| &= \inf\{M > 0 \mid \|(\lambda T)(x)\|_2 \leq M\|x\|_1 \text{ for all } x \in E_1\} \\ &= \inf\{M > 0 \mid |\lambda| \|T(x)\|_2 \leq M\|x\|_1 \text{ for all } x \in E_1\} \\ &= \{\text{if } \lambda \neq 0\} \\ &= \inf\left\{ \underbrace{M}_{=|\lambda|\tilde{M}} > 0 \mid \|T(x)\|_2 \leq \underbrace{\frac{M}{|\lambda|}}_{=\tilde{M}} \|x\|_1 \text{ for all } x \in E_1 \right\} \\ &= |\lambda| \inf\left\{ \tilde{M} > 0 \mid \|T(x)\|_2 \leq \tilde{M}\|x\|_1 \text{ for all } x \in E_1 \right\} \\ &= |\lambda| \|T\| \end{aligned}$$

(3) Set  $T_1, T_2 \in B(E_1, E_2)$ .

$$\begin{aligned} \|T_1 + T_2\| &= \inf\{M > 0 \mid \|(T_1 + T_2)(x)\|_2 \leq M\|x\|_1 \text{ for all } x \in E_1\} \\ &\leq \inf\{M_1 + M_2 > 0 \mid \|T_1(x)\|_2 \leq M_1\|x\|_1, \|T_2(x)\|_2 \leq M_2\|x\|_1 \text{ for all } x \in E_1\} \\ &= \|T_1\| + \|T_2\| \end{aligned}$$

□

**Conclusion:**  $(B(E_1, E_2), \|\cdot\|_{E_1 \rightarrow E_2})$  is a normed space.

**Statement 2.14.**  $(B(E_1, E_2), \|\cdot\|_{E_1 \rightarrow E_2})$  is a Banach space if  $(E_2, \|\cdot\|_2)$  is a Banach space.

**proof.** Assume  $(T_n)_{n=1}^\infty$  is a Cauchy sequence in  $(B(E_1, E_2), \|\cdot\|_{E_1 \rightarrow E_2})$  where  $(E_2, \|\cdot\|_2)$  is a Banach space. Fix  $x \in E_1$

$$\begin{aligned} \|T_n(x) - T_m(x)\|_2 &= \|(T_n - T_m)(x)\|_2 \\ &\leq \underbrace{\|T_n - T_m\|_{E_1 \rightarrow E_2}}_{\substack{\rightarrow 0 \\ n, m \rightarrow \infty}} \cdot \|x\|_1 \rightarrow 0, \quad n, m \rightarrow \infty. \end{aligned}$$

Hence  $(T_n(x))_{n=1}^\infty$  is a Cauchy sequence in  $(E_2, \|\cdot\|_2)$ . This is a Banach space which implies that  $(T_n(x))_{n=1}^\infty$  converges in  $(E_2, \|\cdot\|_2)$ . Call the limit  $T(x) \in E_2$  for all  $x \in E_1$ . Show now

- (1)  $T : E_1 \rightarrow E_2$  is linear,
- (2)  $T$  is bounded,
- (3)  $\|T_n - T\|_{E_1 \rightarrow E_2} \rightarrow 0$  for  $n \rightarrow \infty$ .

(1) Observe

$$\begin{aligned} T(\lambda_1 x_1 + \lambda_2 x_2) &\leftarrow T_n(\lambda_1 x_1 + \lambda_2 x_2) = \{T \text{ linear}\} = \lambda_1 \underbrace{T_n(x_1)}_{\rightarrow T(x_1)} + \lambda_2 \underbrace{T_n(x_2)}_{\rightarrow T(x_2)} \\ &\quad \underbrace{\rightarrow \lambda_1 T(x_1)}_{\rightarrow \lambda_1 T(x_1)} + \underbrace{\rightarrow \lambda_2 T(x_2)}_{\rightarrow \lambda_2 T(x_2)} \\ &\quad \rightarrow \lambda_1 T(x_1) + \lambda_2 T(x_2) \end{aligned}$$

So for  $n \rightarrow \infty$  it is

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2) \quad \text{in } (E_2, \|\cdot\|_2).$$

(2) Fix  $\varepsilon > 0$ . Then there exists  $N$  such that:

$$\|T_n - T_m\|_{E_1 \rightarrow E_2} < \varepsilon \quad \text{for } n, m \geq N$$

So for  $x \in E_1$

$$\|T_n(x) - T_m(x)\|_2 \leq \|T_n - T_m\|_{E_1 \rightarrow E_2} \|x\|_1 < \varepsilon \|x\|_1 \quad \text{for } n, m \geq N.$$

Let  $m \rightarrow \infty$ .

$$\|T_n(x) - T(x)\|_2 \leq \varepsilon \|x\|_1 \quad \text{for } n \geq N$$

So

$$\begin{aligned} \|T(x)\|_2 &\leq \|T(x) - T_N(x)\|_2 + \|T_N(x)\|_2 \\ &\leq \varepsilon \|x\|_1 + \|T_N\|_{E_1 \rightarrow E_2} \cdot \|x\|_1 \\ &= (\varepsilon + \|T_N\|_{E_1 \rightarrow E_2}) \|x\|_1 \quad \text{for } x \in E_1. \end{aligned}$$

(3) Look above and get

$$\|T_n - T\|_{E_1 \rightarrow E_2} \rightarrow 0, \quad n \rightarrow \infty.$$

□



**Theorem 2.15** (Banach-Steinhaus Theorem (uniform boundedness principle)). Set  $(E_1, \|\cdot\|_1)$  Banach space,  $(E_2, \|\cdot\|_2)$  normed space and  $\mathcal{F} \subset B(E_1, E_2)$ . Assume

$$\sup_{T \in \mathcal{F}} \|T(x)\|_2 < \infty \quad \text{for all } x \in E_1$$

then

$$\sup_{T \in \mathcal{F}} \|T\|_{E_1 \rightarrow E_2} < \infty.$$

**Remark.** The implication  $\Leftarrow$  is easy to prove. If  $\mathcal{F}$  is a finite set, the theorem is trivial.

**proof.** Step 1: Assume

$$\exists x_0 \in E_1 \exists r > 0 \exists M > 0 : \forall x \in \overline{B(x_0, r)} \forall T \in \mathcal{F} : \|T(x)\|_2 \leq M.$$

We have to show that

$$\sup_{T \in \mathcal{F}} \|T\|_{E_1 \rightarrow E_2} < \infty.$$

Fix  $T \in \mathcal{F}$ . For  $\|x\|_1 \leq r$

$$\|T(x_0 + x)\|_2 \leq M.$$

Note that  $x_0 + x \in \overline{B(x_0, r)}$ .

$$\begin{aligned} \|T(x)\|_2 &= \|T(x_0 + x - x_0)\|_2 \\ &= \{T \text{ linear}\} \\ &= \|T(x_0 + x) - T(x_0)\|_2 \\ &\leq \|T(x_0 + x)\|_2 + \|T(x_0)\|_2 \\ &\leq 2M. \end{aligned}$$

For  $0 \neq x \in E_1$

$$\left\| T \left( \frac{r}{\|x\|_1} x \right) \right\|_2 \leq 2M.$$

$\frac{r}{\|x\|_1} x$  has the  $\|\cdot\|_1$ -norm equal to  $r$ . This implies, since  $T$  linear,

$$\frac{r}{\|x\|_1} \|T(x)\|_2 \leq 2M,$$

i.e.

$$\|T(x)\|_2 \leq \frac{2M}{r} \|x\|_1 \quad \text{for all } 0 \neq x \in E_1.$$

We have

$$\begin{aligned} \|T\|_{E_1 \rightarrow E_2} &\leq \underbrace{\frac{2M}{r}}_{\text{independent of } T} < \infty \\ \sup_{T \in \mathcal{F}} \|T\|_{E_1 \rightarrow E_2} &\leq \frac{2M}{r} < \infty. \end{aligned}$$

Step 2: Justify the assumption in step 1. This assumption is equivalent to

$$\exists x_0 \in E_1 \exists r > 0 \exists M > 0 : \forall x \in B(x_0, r) \forall T \in \mathcal{F} : \|T(x)\|_2 \leq M.$$

(Note  $\overline{B(x_0, r_1)} \subset B(x_0, r) \subset B(x_0, r_2)$  for  $0 < r_1 < r < r_2$ ).

Argue by contradiction. Assume that the assumption is false. Then it holds

$$\forall x_0 \in E_1 \forall r > 0 \forall M > 0 : \exists x \in B(x_0, r) \exists T \in \mathcal{F} : \|T(x)\|_2 > M.$$

Idea: Find a converging sequence  $x_n \in E_1$ ,  $x_n \rightarrow x$  in  $(E_1, \|\cdot\|_1)$  and a sequence  $(T_n)_{n=1}^\infty \subset \mathcal{F}$  such that

$$\|T_n(x_n)\|_2 > n \quad \text{for all } n, \quad \text{and} \quad \|T_n(x)\|_2 > n \quad \text{for all } n.$$

We have from above  $x_1 \in B(0, 1)$  and  $T_1 \in \mathcal{F}$  such that

$$\|T_1(x_1)\|_2 > 1.$$

$T_1$  is bounded linear, hence continuous. This implies that there exists  $0 < r_1 < \frac{1}{2}$  such that

$$\|T_1(x)\|_2 > 1 \quad \text{for } x \in B(x_1, r_1)$$

and

$$\overline{B(x_1, r_1)} \subset B(0, 1).$$

□

## 2.2 Fixed point theory

**Example.** Consider

$$f(x) + 5 \int_0^{1-x} \min(x, y) f(y) dy = g(x), \quad x \in [0, 1] \quad (*)$$

where  $g \in C([0, 1])$ .

**Claim:** There exists an unique solution  $f \in C([0, 1])$  that (\*).

Idea:

$$f(x) = f(x) - 5 \int_0^{1-x} \min(x, y) f(y) dy, \quad x \in [0, 1].$$

Set for  $x \in [0, 1]$

$$\tilde{T}(f)(x) = RHS(x).$$

To find a solution to (\*) is the same finding  $f \in C([0, 1])$  such that

$$f = \tilde{T}(f).$$

Clearly  $\tilde{T} : C([0, 1]) \rightarrow C([0, 1])$ . (continual later).

**Theorem 2.16** (Banach's fixed point theorem).  $(E, \|\cdot\|)$  Banach space.  $T : E \rightarrow E$  (no assumption on linearity) is a contraction on  $E$ , i.e. there exists  $c < 1$  such that

$$\|T(x) - T(\tilde{x})\| \leq c\|x - \tilde{x}\| \quad \text{for all } x, \tilde{x} \in E.$$

Then there exists a unique  $\bar{x} \in E$  such that

$$\bar{x} = T(\bar{x}).$$

( $\bar{x}$  is a fixed point)

**proof. Uniqueness:** Assume  $T(\bar{x}) = \bar{x}$  and  $T(\tilde{x}) = \tilde{x}$ . Then

$$\underbrace{\|\bar{x} - \tilde{x}\|}_{\geq 0} = \|T(\bar{x}) - T(\tilde{x})\| \leq \underbrace{c}_{< 1} \|\bar{x} - \tilde{x}\|.$$

Thus  $\|\bar{x} - \tilde{x}\| = 0$ , i.e.  $\bar{x} = \tilde{x}$ .

**Existence:** Pick an arbitrary  $x_0 \in E$ . Set

$$x_{n+1} = T(x_n), \quad n = 0, 1, 2, \dots$$

**Claim:**  $(x_n)_{n=1}^{\infty}$  is a Cauchy sequence in  $(E, \|\cdot\|)$ . Note:

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|T(x_n) - T(x_{n-1})\| \\ &\leq c\|x_n - x_{n-1}\| \\ &\leq \dots \\ &\leq c^n\|x_1 - x_0\|, \quad n = 1, 2, \dots \end{aligned}$$

For  $n > m$

$$\begin{aligned} \|x_n - x_m\| &= \|x_n - x_{n-1} + x_{n-1} - \dots + x_{m+1} - x_m\| \\ &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \dots + \|x_{m+1} - x_m\| \\ &\leq (c^{n-1} + c^{n-2} + \dots + c^m)\|x_1 - x_0\| \\ &\leq \frac{c^m}{1-c}\|x_1 - x_0\| \rightarrow 0 \quad \text{as } n, m \rightarrow \infty. \end{aligned}$$

Hence  $(x_n)_{n=1}^{\infty}$  is a Cauchy sequence in  $(E, \|\cdot\|)$ .  $(E, \|\cdot\|)$  is a Banach space. So  $(x_n)_{n=1}^{\infty}$  converges in  $(E, \|\cdot\|)$ . Call the limit  $\bar{x}$ .

**Claim:**  $\bar{x}$  is a fixed point for  $T$ .

$$\begin{aligned} \|\bar{x} - T(\bar{x})\| &= \|\bar{x} - x_{n+1} + x_{n+1} - T(\bar{x})\| \\ &\leq \|\bar{x} - x_{n+1}\| + \left\| \underbrace{x_{n+1}}_{T(x_n)} - T(\bar{x}) \right\| \\ &\leq \underbrace{\|\bar{x} - x_{n+1}\|}_{\rightarrow 0} + c \underbrace{\|x_n - \bar{x}\|}_{\rightarrow 0} \rightarrow 0, \quad n \rightarrow \infty \end{aligned}$$

□

**Remark.** (1)  $x_n \rightarrow \bar{x}$  for  $n \rightarrow \infty$  independent of the choice of  $x_0$

(2) Fix  $z \in E$

$$\begin{aligned}\|\bar{x} - z\| &= \|T(\bar{x}) - T(z) + T(z) - z\| \\ &\leq \|T(\bar{x}) - T(z)\| + \|T(z) - z\| \\ &\leq c\|\bar{x} - z\| + \|T(z) - z\|.\end{aligned}$$

Hence

$$\|\bar{x} - z\| \leq \frac{1}{1-c}\|T(z) - z\|.$$

**Example.** Consider now the example from above:  $(C([0, 1]), \|\cdot\|)$  with  $\|f\| = \max_{x \in [0, 1]} |f(x)|$  is a Banach space! To apply Banach's fixed point theorem we need  $\tilde{T}$  to be a contraction. Fix  $f_1, f_2 \in C([0, 1])$  and get for  $x \in [0, 1]$

$$\begin{aligned}|(\tilde{T}(f_1) - \tilde{T}(f_2))(x)| &= |5 \int_0^{1-x} \min(x, y) f_2(y) dy - 5 \int_0^{1-x} \min(x, y) f_1(y) dy| \\ &= |5 \int_0^{1-x} \min(x, y) (f_2(y) - f_1(y)) dy| \\ &\leq 5 \int_0^{1-x} \min(x, y) \underbrace{|f_2(y) - f_1(y)|}_{\leq \|f_2 - f_1\|} dy \\ &\leq 5 \underbrace{\int_0^{1-x} \min(x, y) dy}_{0 \leq \dots \leq \frac{1}{6}} \|f_2 - f_1\| \\ &\leq \frac{5}{6} \|f_2 - f_1\|.\end{aligned}$$

Hence

$$\|\tilde{T}(f_1) - \tilde{T}(f_2)\| \leq \frac{5}{6} \|f_1 - f_2\|.$$

We conclude that  $\tilde{T}$  is a contraction. We can take  $c = \frac{5}{6}$ . By Banach's fixed point theorem  $\tilde{T}$  has a unique fixed point. Finally (\*) has a unique solution  $f \in C([0, 1])$  which is the fixed point.

**Theorem 2.17** (Banach's fixed point theorem (generalization)).  $(E, \|\cdot\|)$  Banach space.  $T : F \rightarrow F$  where  $F$  is a closed set in  $E$ .  $N$  positive integer. Assume  $T^N = \underbrace{T \circ T \circ \dots \circ T}_{N\text{-times}}$

is a contraction on  $F$ , i.e. there exists  $c > 1$  such that

$$\|T^N(x) - T^N(\tilde{x})\| \leq c\|x - \tilde{x}\|, \quad \text{for all } x, \tilde{x} \in F.$$

Then  $T$  has unique fixed point  $\bar{x}$ , i.e.

$$\bar{x} = T(\bar{x}) \in F.$$

**proof.**  $N = 1$ : Fix  $x_0 \in F$  and consider  $(x_n)_{n=1}^\infty$  where  $x_{n+1} = T(x_n)$  for  $n = 0, 1, 2, \dots$ . There  $(x_n)_{n=1}^\infty$  is a Cauchy sequence and hence this converges in  $E$  since this is a Banach space. Call the limit  $\bar{x}$ . Note

$$\underbrace{x_n}_{\in F} \rightarrow \bar{x} \text{ in } E \text{ and } F \text{ is closed}$$

implies  $\bar{x} \in F$ . The rest of the argument is the same as before.

$N > 1$ : By previous result we know that  $T^N$  has a unique fixed point  $\bar{x} \in F$ , i.e.  $\bar{x} = T^N(\bar{x})$ .

**Claim:**  $\bar{x}$  is a fixed point for  $T$ .

$$\begin{aligned} \|T(\bar{x}) - \bar{x}\| &= \|T(T^N(\bar{x})) - T^N(\bar{x})\| \\ &= \|T^N(T(\bar{x})) - T^N(\bar{x})\| \\ &\leq c\|T(\bar{x}) - \bar{x}\|. \end{aligned}$$

This gives

$$\|T(\bar{x} - \bar{x})\| = 0, \quad \text{i.e. } \bar{x} = T(\bar{x}).$$

Existence of a fixed point for  $T$  done. For the uniqueness assume  $\bar{x} = T(\bar{x})$  and  $\tilde{x} = T(\tilde{x})$ . Then

$$\begin{aligned} \bar{x} &= T(\bar{x}) = T^2(\bar{x}) = \dots = T^N(\bar{x}) \\ \tilde{x} &= T(\tilde{x}) = T^2(\tilde{x}) = \dots = T^N(\tilde{x}). \end{aligned}$$

But  $T^N$  has a unique fixed point so

$$\bar{x} = \tilde{x}.$$

□

**Remark.** (1)  $T : (0, 1] \rightarrow (0, 1]$  where  $T(x) = \frac{x}{2}$ . Clearly  $T$  is a contraction on  $(0, 1]$  but has no fixed point. Note that  $(0, 1]$  is not a closed interval.

(2)  $T : [0, \infty) \rightarrow [0, \infty)$ , where  $T(x) = x + \frac{1}{x}$ . Clearly  $[0, \infty)$  is a closed interval in  $\mathbb{R}$  but  $T$  has no fixed point.

**Claim:**  $T$  is not a contraction but 'close' to be a contraction.

$$|T(x) - T(\tilde{x})| < |x - \tilde{x}| \quad \text{for } x, \tilde{x} \in [1, \infty), x \neq \tilde{x}$$

Note

$$|T(x) - T(\tilde{x})| = \underbrace{|T'(t)|}_{(1-\frac{1}{t}) \leq 1 \text{ for } t \in [1, \infty)} |x - \tilde{x}|$$

for some  $t$  between  $x$  and  $\tilde{x}$ .

**Example.**  $(E, \|\cdot\|)$  Banach space.  $K$  compact set in  $E$  and  $T : K \rightarrow K$  where

$$\|T(x) - T(\bar{x})\| < \|x - \bar{x}\| \quad \text{for all } x, \bar{x} \in K, x \neq \bar{x}.$$

Show:  $T$  has a unique fixed point in  $K$ .

**Uniqueness:** Assume  $\bar{x} = T(\bar{x})$  and  $\tilde{x} = T(\tilde{x})$  and  $\bar{x} \neq \tilde{x}$  for  $\bar{x}, \tilde{x} \in K$ . Then

$$\|\bar{x} - \tilde{x}\| = \|T(\bar{x}) - T(\tilde{x})\| < \|\bar{x} - \tilde{x}\|.$$

Contradiction because then  $\bar{x} = \tilde{x}$ .

**Existence:** To show: There exists  $x \in K$  such that  $x = T(x)$ , i.e.

$$\|T(x) - x\| = 0.$$

Set  $d := \inf_{x \in K} \|T(x) - x\|$ . Let  $(x_n)_{n=1}^\infty$  be a sequence in  $K$  such that

$$\|T(x_n) - x_n\| \rightarrow d, \quad \text{as } n \rightarrow \infty.$$

$K$  compact implies that there exists a subsequence  $(\tilde{x}_n)_{n=1}^\infty$  of  $(x_n)_{n=1}^\infty$  such that  $(\tilde{x}_n)_{n=1}^\infty$  converges in  $K$ . Call the limit element  $\bar{x} \in K$ . We know

$$\tilde{x}_n \rightarrow \bar{x} \quad \text{in } K$$

and

$$\|T(\tilde{x}_n) - \tilde{x}_n\| \rightarrow d.$$

Question:

$$T(\tilde{x}_n) \rightarrow T(\bar{x}) \quad \text{in } K?$$

But since

$$\|T(x) - T(\tilde{x})\| \leq \|x - \tilde{x}\| \quad \text{for all } x, \tilde{x} \in K$$

we have

$$\tilde{x}_n \rightarrow \bar{x} \quad \text{in } K$$

which implies

$$T(\tilde{x}_n) \rightarrow T(\bar{x}) \text{ in } K.$$

Hence:

$$\|T(\bar{x}) - \bar{x}\| \leftarrow \|T(\tilde{x}_n) - \tilde{x}_n\| \rightarrow d, \quad n \rightarrow \infty.$$

We obtain

$$\|T(\bar{x}) - \bar{x}\| = d.$$

Question: Is  $d = 0$ ?

If  $d > 0$  then  $\bar{x} \neq T(\bar{x})$ ,  $\bar{x}, T(\bar{x}) \in K$

$$\|T(\bar{x}) - T(T(\bar{x}))\| < \|\bar{x} - T(\bar{x})\| = d = \inf_{x \in K} \|x - T(x)\|.$$

This is a contradiction which gives  $d = 0$  and so  $\bar{x} = T(\bar{x})$ .

**Example.** Consider

$$f(x) = \int_0^x k(x, y)h(y, f(y)) \, dy + g(x), \quad x \in [0, 1] \quad (*),$$

where  $g \in C([0, 1])$ ,  $k \in C([0, 1] \times [0, 1])$  and  $h : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  continuous and satisfies:  
There exists  $M > 0$  such that

$$|h(x, z_1) - h(x, z_2)| \leq M|z_1 - z_2| \quad \text{for all } x \in [0, 1], z_1, z_2 \in \mathbb{R}.$$

**Claim:** (\*) has a unique solution  $f \in C([0, 1])$ .

For  $f \in C([0, 1])$  set

$$T(f)(x) = \int_0^x k(x, y)h(y, f(y)) \, dy + g(x) \quad x \in [0, 1].$$

Here  $T(f)(x) \in C([0, 1])$ .

Want to show:  $T : C([0, 1]) \rightarrow C([0, 1])$  has a unique fixed point.

Start with the Banach space  $(C([0, 1]), \text{max-norm})$ . Check if  $T$  is a contraction in  $C([0, 1])$ .

Fix  $f_1, f_2 \in C([0, 1])$

$$T(f_1)(x) - T(f_2)(x) = \int_0^x k(x, y)(h(y, f_1(y)) - h(y, f_2(y))) \, dy.$$

$k$  is continuous on the compact set  $[0, 1] \times [0, 1]$  so

$$\sup_{(x,y) \in [0,1] \times [0,1]} |k(x, y)| =: N < \infty.$$

We obtain

$$\begin{aligned} |(T(f_1) - T(f_2))(x)| &\leq \int_0^x \underbrace{|k(x, y)|}_{\leq N} \underbrace{|h(y, f_1(y)) - h(y, f_2(y))|}_{\leq M|f_1(y) - f_2(y)|} \, dy \\ &\leq \int_0^x NM \, dy \|f_1 - f_2\| \\ &\leq NM \|f_1 - f_2\|. \end{aligned}$$

This yields

$$\|T(f_1) - T(f_2)\| \leq NM \|f_1 - f_2\|.$$

**IF:**  $NM < 1$  Then  $T$  is a contraction.

Trick: For  $a > 0$  set

$$\|f\|_a = \max_{x \in [0,1]} e^{-ax} |f(x)|$$

for  $f \in C([0, 1])$ .

**Claim:**  $\|\cdot\|_a$  defines a norm on  $C([0, 1])$ . This is easy to check.

**Claim:**  $\|\cdot\|$  and  $\|\cdot\|_a$  are equivalent.

This follows from

$$e^{-a}\|f\| \leq \|f\|_a \leq \|f\|$$

for all  $f \in C([0, 1])$  (note that  $\|\cdot\|$  is the max-norm).

**Claim:**  $(C([0, 1]), \|\cdot\|_a)$  is a Banach space.

This follows from the fact that  $\|\cdot\|$  and  $\|\cdot\|_a$  are equivalent and  $(C([0, 1]), \|\cdot\|)$  is a Banach space.

**Claim:**  $T$  is a contraction on  $(C([0, 1]), \|\cdot\|_a)$  for  $a > 0$  large enough.

For  $f_1, f_2 \in C([0, 1])$  and  $x \in [0, 1]$  we have

$$\begin{aligned} |(T(f_1) - T(f_2))(x)| &\leq \int_0^x NM|(f_1 - f_2)(y)| dy \\ &= \int_0^x NM e^{ay} \cdot \underbrace{e^{-ay}|(f_1 - f_2)(x)|}_{\leq \|f_1 - f_2\|_a} dy \\ &\leq NM \underbrace{\int_0^x e^{ay} dy}_{\frac{1}{a}(e^{ax} - 1)} \|f_1 - f_2\|_a. \end{aligned}$$

So

$$e^{-ax}|(T(f_1) - T(f_2))(x)| \leq \frac{NM}{a}(1 - e^{-ax})\|f_1 - f_2\|_a$$

and

$$\|T(f_1) - T(f_2)\|_a \leq \frac{NM}{a}\|f_1 - f_2\|_a$$

For  $a > NM$  is  $T$  a contraction on  $(C([0, 1]), \|\cdot\|_a)$ . Banach fixed point theorem implies that there is a unique  $f \in C([0, 1])$  that solves (\*).

**Theorem 2.18.**  $(E, \|\cdot\|)$  Banach space,  $(Y, \|\cdot\|)$  normed space.  $T : E \times Y \rightarrow E$  where

(1) There exists a  $C < 1$  such that

$$\|T(x, y) - T(\tilde{x}, y)\| \leq C\|x - \tilde{x}\| \quad \text{for all } x, \tilde{x} \in E, y \in Y.$$

(2)  $T_x : Y \rightarrow E$  where  $T_x(y) = T(x, y)$  is continuous for all  $x \in E$ .

$\Rightarrow$  For every  $y \in Y$  there exists a unique  $g(y) \in E$  such that

$$g(y) = T(g(y), y)$$

and  $g : Y \rightarrow E$  is continuous.

**proof.** The existence of a unique element  $g(y) \in E$  for every  $y \in Y$  follows from Banach's fixed point theorem.

Assume  $y_n \rightarrow \tilde{y}$  in  $(Y, \|\cdot\|_*)$ , i.e.

$$\|y_n - \tilde{y}\|_* \rightarrow 0, \quad n \rightarrow \infty.$$



Remains to show

$$g(y_m) \rightarrow g(\tilde{y}) \quad \text{in } (E, \|\cdot\|).$$

$$\begin{aligned} \|g(y_n) - g(\tilde{y})\| &= \|T(g(y_n), y_n) - T(g(\tilde{y}), \tilde{y})\| \\ &\leq \underbrace{\|T(g(y_n), y_n) - T(g(\tilde{y}), y_n)\|}_{\stackrel{(1)}{\leq c\|g(y_n) - g(\tilde{y})\|}} + \underbrace{\|T(g(\tilde{y}), y_n) - T(g(\tilde{y}), \tilde{y})\|}_{\stackrel{(2)}{\rightarrow 0, n \rightarrow \infty}} \end{aligned}$$

We obtain

$$\|g(y_n) - g(\tilde{y})\| \leq \frac{1}{1-c} \|T(g(\tilde{y}), y_n) - T(g(\tilde{y}), \tilde{y})\| \rightarrow 0, \quad n \rightarrow \infty.$$

□

**Theorem 2.19** (Brouwer's fixed point theorem).  $K$  compact (= closed and bounded) convex subset of  $\mathbb{R}^n$  and  $T : K \rightarrow K$  continuous. Then  $T$  has a fixed point, i.e. there exists  $\bar{x} \in K$  with

$$T(\bar{x}) = \bar{x}.$$

**Remark.** • No uniqueness! Consider the case  $T = \text{id}_K$ .

- Set  $K \subseteq \mathbb{R}^n$  (in general) is convex if

$$x, \tilde{x} \in K \text{ and } \lambda \in [0, 1] \quad \Rightarrow \quad \lambda x + (1 - \lambda)\tilde{x} \in K.$$

**Theorem 2.20** (Perron's theorem).  $A$  real-valued  $n \times n$ -Matrix with positive entries.

$A = [a_{ij}]_{i,j=1,\dots,n}$  all  $a_{ij} > 0$ .

$\Rightarrow$  The mapping for  $x \in \mathbb{R}^n$

$$x \mapsto Ax$$

has an eigenvalue  $> 0$  with an eigenvector with positive entries, i.e. there exists  $\lambda > 0$  and  $\tilde{x} \in \mathbb{R}^n$  with  $A\tilde{x} = \lambda\tilde{x}$  and all entries in  $\tilde{x}$  are positive.

**proof.** We use Brouwer's fixed point theorem. Set

$$K := \left\{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_k \geq 0, \sum_{i=1}^n x_i = 1 \right\}.$$

**Claim:**  $K$  is closed, bounded and a convex set in  $\mathbb{R}^n$ . Thus  $K$  is compact (since  $K \subseteq \mathbb{R}^n$ ). Set

$$T(x_1, \dots, x_n) = \underbrace{\frac{1}{\|Ax\|_{l^1}} A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\in K} \quad \text{for all } (x_1, \dots, x_n) \in K$$

**Claim:**  $T : K \rightarrow K$  is continuous.

Since

$$x_k \rightarrow x \quad \text{in } K \text{ w.r.t. } l^1\text{-norm.}$$

To show:

$$T(x_k) \rightarrow T(x) \quad \text{in } K \text{ w.r.t. } l^1\text{-norm.}$$

Set

$$\begin{aligned} x &= (x_1, x_2, \dots, x_n) \\ x_k &= (x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}) \quad k = 1, 2, \dots \end{aligned}$$

Consider

$$\begin{aligned} \|T(x_k) - T(x)\|_{l^1} &= \left\| \frac{1}{\|Ax_k\|_{l^1}} Ax_k - \frac{1}{\|Ax\|_{l^1}} Ax \right\|_{l^1} \\ &\leq \left\| \frac{1}{\|Ax_k\|_{l^1}} Ax_k - \frac{1}{\|Ax\|_{l^1}} Ax_k \right\|_{l^1} + \left\| \frac{1}{\|Ax\|_{l^1}} Ax_k - \frac{1}{\|Ax\|_{l^1}} Ax \right\|_{l^1} \\ &= \left| \frac{1}{\|Ax_k\|_{l^1}} - \frac{1}{\|Ax\|_{l^1}} \right| \|Ax_k\|_{l^1} + \frac{1}{\|Ax\|_{l^1}} \|A(x - x_k)\|_{l^1} \end{aligned}$$

and

$$\begin{aligned} \|A(x - x_k)\|_{l^1} &= \sum_{i=1}^n \left| \sum_{j=1}^n a_{ij} (x_j - x_j^{(k)}) \right| \\ &\leq \sum_{i=1}^n \sum_{j=1}^n a_{ij} |x_j - x_j^{(k)}| \\ &\leq \underbrace{n \cdot \max_{i,j} a_{ij}}_{< \infty} \underbrace{\|x - x_k\|_{l^1}}_{\rightarrow 0} \rightarrow 0, \quad k \rightarrow \infty. \end{aligned}$$

So

$$Ax_k \rightarrow Ax \quad \text{in } l^1.$$

This implies

$$\|Ax_k\|_{l^1} \rightarrow \|Ax\|_{l^1} \quad \text{in } \mathbb{R}.$$

Brouwer's fixed point theorem implies that  $T$  has a fixed point  $\bar{x} \in K$ .

$$\begin{aligned} \bar{x} &= (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \\ \bar{x} &= T(\bar{x}) = \frac{1}{\|A\bar{x}\|_{l^1}} A\bar{x} \end{aligned}$$

Hence  $A\bar{x} = \|A\bar{x}\|_{l^1} \bar{x}$  where  $\|A\bar{x}\|_{l^1} > 0$  and  $\bar{x}$  has all entries  $> 0$ . □

**Theorem 2.21** (Schauder's fixed point theorem).  $(E, \|\cdot\|)$  Banach space.  $K$  compact, convex set in  $E$ .  $T : K \rightarrow K$  continuous.  
 $\Rightarrow T$  has a fixed point in  $K$ .

**Example.**

$$S = \left\{ f \in C([0, 1]) \mid f(0) = 0, f(1) = 1, \|f\| = \max_{x \in [0, 1]} |f(x)| \leq 1 \right\}$$

$T : S \rightarrow S$  defined by

$$T(f)(x) = f(x^2), \quad x \in [0, 1].$$

$C([0, 1])$  is equipped with the max-norm.

**Claim:**

- $S$  is closed, bounded and convex in  $C([0, 1])$ .
- $T : S \rightarrow S$  is continuous.
- $T$  has no fixed point in  $S$ .
- $S$  bounded:  $f \in S$  implies  $\|f\| \leq 1$ .
- $S$  closed:  $f_n \rightarrow f$  in  $(C([0, 1]), \|\cdot\|)$ .

To show:  $f \in S$ .

Note

$$\max_{x \in [0, 1]} |f_n(x) - f(x)| \rightarrow 0, \quad n \rightarrow \infty.$$

This implies

$$|f(0)| = |f_n(0) - f(0)| \rightarrow 0, \quad n \rightarrow \infty.$$

So  $f(0) = 0$ .

$$|1 - f(1)| = \|f_n(1) - f(1)\| \rightarrow 0, \quad n \rightarrow \infty.$$

So  $f(1) = 1$ . For  $x \in [0, 1]$  we get

$$\begin{aligned} |f(x)| &\leq \|f(x) - f_n(x)\| + |f_n(x)| \\ &\leq \underbrace{\|f - f_n\|}_{\rightarrow 0} + \underbrace{\|f_n\|}_{\leq 1}. \end{aligned}$$

Conclusion  $f \in S$

$$\|f\| = \max_{x \in [0, 1]} |f(x)| \leq 1.$$

- $f, \tilde{f} \in S$  and  $\lambda \in [0, 1]$ .

To show:

$$\lambda f + (1 - \lambda)\tilde{f} \in S.$$

Trivial since

$$\begin{aligned}(\lambda f + (1 - \lambda)\tilde{f})(0) &= 0 \\ (\lambda f + (1 - \lambda)\tilde{f})(1) &= \lambda f(1) + (1 - \lambda)\tilde{f}(1) = 1\end{aligned}$$

and

$$\|\lambda f + (1 - \lambda)\tilde{f}\| \leq |\lambda|\|f\| + |1 - \lambda|\|\tilde{f}\| \leq 1.$$

We want to show that  $T : S \rightarrow S$  is continuous. (obvious that  $T(S) \subseteq S$ )  
Assume  $f_n \rightarrow f$  in  $S$  in max-norm, i.e.

$$\max_{x \in [0,1]} |f_n(x) - f(x)| \rightarrow 0, \quad n \rightarrow \infty.$$

To show:  $T(f_n) \rightarrow T(f)$  in  $S$  in max-norm.

$$\begin{aligned}\|T(f_n) - T(f)\| &= \max_{x \in [0,1]} |T(f_n)(x) - T(f)(x)| \\ &= \max_{x \in [0,1]} |f_n(x^2) - f(x^2)| \\ &= \|f_n - f\| \rightarrow 0, \quad n \rightarrow \infty.\end{aligned}$$

$T : S \rightarrow S$  has no fixed point.

If  $f \in S$  is a fixed point for  $T$  then

$$f(x^2) = T(f)(x) = f(x), \quad x \in [0, 1].$$

To show: there can be no such  $f \in S$ .

Set  $a = \inf\{x \in [0, 1] \mid f(x) = \frac{1}{2}\} \neq \emptyset$  since  $f$  is continuous.  $a \in (0, 1)$  since if  $a = 0$  then there exists a sequence

$$a_n \in \{x \in [0, 1] \mid f(x) = \frac{1}{2}\}$$

such that  $a_n \rightarrow a$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ . Contradiction since

$$\frac{1}{2} = f(a_n) \rightarrow f(a) = f(0) = 0$$

since  $f$  is continuous.

But  $0 < a^2 < a$  and  $f(a^2) = f(a) = \frac{1}{2}$ . This is a contradiction.

If we believe in Schauder then we can conclude that  $S \subseteq C([0, 1])$  is not compact.

**Theorem 2.22** (Arzela-Ascoli theorem). Assume  $K$  is a compact set in  $\mathbb{R}^n$  (e.g.  $K = [0, 1]$  in  $\mathbb{R}^n$   $n = 1$ ) and  $S \subseteq C(K)$  where  $C(K)$  is equipped with the max-norm.  
 $\Rightarrow S$  is relatively compact in  $C(K)$  iff

- (1)  $S$  uniformly bounded.
- (2)  $S$  is equicontinuous.

**Definition .** (i)  $S$  is uniformly bounded if

$$\sup_{f \in S} \|f\| < \infty.$$

(ii)  $S$  is equicontinuous if: for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$|x - \tilde{x}| < \delta, x, \tilde{x} \in K \quad \Rightarrow \quad |f(x) - f(\tilde{x})| < \varepsilon.$$

$\delta = \delta(\varepsilon)$  must not depend on  $f$ .

$S$  is relatively compact in  $C(K)$  if for every sequence  $(f_n)_{n=1}^\infty$  in  $S$  there exists a converging subsequence in  $C(K)$ .

To show:  $S$  is relatively compact in  $C(K)$  iff the closure  $\bar{S}$  is compact in  $C(K)$ .

**Things to do:**

- (1) Proof of Schander's theorem.
- (2) Proof of Arzela-Ascoli theorem.
- (3) Application with Schander.
- (4) Proof of Brouwer's theorem (special case).
- (5) Completion of normed spaces.

For (4) we consider the following lemma.

**Lemma 2.23** (Sperner's lemma). Big triangle  $T$

$$T = \bigcup_{a \in A} T_a.$$

$\{T_a\}_{a \in A}$  is triangle of  $T$ , i.e. for any pair  $T_a, T_{\tilde{a}}$  in the triangulation

$$T_a \cup T_{\tilde{a}} = \{\emptyset \text{ or common vertex or common side or } T_a = T_{\tilde{a}}\}.$$

$\Rightarrow$  There must exist a triangle  $T_a$  with all vertices colored differently. MISSING FIGURE!

**Proof of Schander's fixed point theorem:** To prove:  $(E, \|\cdot\|)$  Banach space,  $K$  compact convex set in  $E$  and  $T : K \rightarrow K$  continuous.

**Claim:**  $T$  has a fixed point.

**Lemma .** Assume  $(x_n)_{n=1}^\infty$  sequence in  $K$  such that

$$\|T(x_n) - x_n\| \rightarrow 0, \quad n \rightarrow \infty.$$

$T$  has a fixed point in  $K$ .

**proof.** Consider  $(T(x_n))_{n=1}^\infty$  in  $K$ .  $K$  compact implies that there exists a  $z \in K$  and a subsequence  $(T(\tilde{x}_n))_{n=1}^\infty$  of  $(T(x_n))_{n=1}^\infty$  such that

$$T(\tilde{x}_n) \rightarrow z \quad \text{in } K \text{ as } n \rightarrow \infty.$$

Then

$$\left\| \underbrace{T(\tilde{x}_n)}_{\rightarrow z} - \tilde{x}_n \right\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

So  $\tilde{x}_n \rightarrow z$  for  $n \rightarrow \infty$ . But  $T$  continuous implies

$$z \leftarrow T(\tilde{x}_n) \rightarrow T(z), \quad n \rightarrow \infty.$$

Conclusion:  $z = T(z)$  so  $z$  is a fixed point. □

**Lemma .**  $K$  compact set in  $E$ . Let  $\varepsilon > 0$ . Then there exists a finite set  $x_1, \dots, x_N \in K$  such that for all  $x \in K$

$$\min_{k=1, \dots, N} \|x - x_k\| < \varepsilon.$$

**proof.** Assume there is no finite sequence  $x_1, \dots, x_N$ . Then there exists a sequence  $(x_n)_{n=1}^\infty$  such that

$$\|x_k - x_l\| \geq \varepsilon, \quad \text{for } k \neq l.$$

Clearly  $(x_n)_{n=1}^\infty$  has no converging subsequence. This contradicts  $K$  being compact. □

Fix positive integer  $n$ . Apply previous lemma with  $\varepsilon = \frac{1}{n}$ . then there exists a finite set  $x_1, \dots, x_N$  such that

$$K \subset \bigcup_{k=1}^N B\left(x_k, \frac{1}{n}\right).$$

Set

$$\begin{aligned} K_n &= \{\text{set of all convex combinations of } x_1, \dots, x_N\} \\ &= \left\{ \sum_{k=1}^N \lambda_k x_k \mid \lambda_k \geq 0 \text{ for all } k, \sum_{k=1}^N \lambda_k = 1 \right\}. \end{aligned}$$

This set is a closed and bounded set in  $\text{span}(K_n)$  finite dimensional. Also  $K_n$  is convex. (want  $T_n : K_n \rightarrow K_n$  where  $T_n$  close to  $T$ ).

Set  $f_k(x) = \max(0, \frac{1}{n} - \|x - x_k\|)$  for  $x \in K$  and  $k = 1, 2, \dots, N$ .  
For each  $x \in K$  there exists a  $k$  such that  $f_k(x) > 0$ . Set

$$P_n(x) = \frac{f_1(x)x_1 + f_2(x)x_2 + \dots + f_N(x)x_N}{f_1(x) + f_2(x) + \dots + f_N(x)}, \quad x \in K.$$

$P_n$  is a convex combination of  $x_1, \dots, x_N$  for every  $x \in K$ . So  $P_n(x) \in K_n$  for every  $x \in K$ .

**Claim:**  $\|P_n(x) - x\| < \frac{1}{n}$  for all  $x \in K$ . Set  $T_n$  to be defined like

$$T_n := P_n T : K_n \rightarrow K_n.$$

Here  $T_n$  is continuous since  $T$  and  $P_n$  are continuous.  $K_n$  is compact and convex in a finite dimensional space. Brouwer's fixed point theorem implies that  $T_n$  has a fixed point in  $K_n$ , i.e. there exists  $x_n \in K_n$  such that

$$x_n = T_n(x_n) = P_n(x_n).$$

But then

$$\|x_n - T(x_n)\| \leq \underbrace{\left\| x_n - \underbrace{P_n T(x_n)}_{=T_n} \right\|}_{=0} + \underbrace{\|P_n T(x_n) - T(x_n)\|}_{< \frac{1}{n}}.$$

The first lemma above gives that  $T$  has a fixed point in  $K$ . □

**Example.** Assume  $k(x, y)$  continuous on  $[0, 1] \times [0, 1]$  and  $h(y, z)$  continuous on  $[0, 1] \times \mathbb{R}$  and

$$\sup_{(y,z) \in [0,1] \times \mathbb{R}} |h(y, z)| \equiv B < \infty.$$

Then there exists a solution  $f \in C([0, 1])$  to

$$f(x) = \int_0^1 k(x, y) h(y, f(y)) dy, \quad x \in [0, 1].$$

Method: Set  $f \in C([0, 1])$  and

$$T(f)(x) = \int_0^1 k(x, y) h(y, f(y)) dy, \quad x \in [0, 1] \quad (*).$$

We want to apply (a generalized version of) Schauder's fixed point theorem. Assume  $(E, \|\cdot\|)$  is a Banach space and  $F$  closed convex subset of  $E$ . Moreover assume  $T : E \rightarrow E$  continuous and  $T(F)$  relatively compact in  $(E, \|\cdot\|)$ . Then  $T$  has a fixed point in  $F$ .

**Step 1:**  $T$  as in  $(*)$ .

**Claim:**  $T(C([0, 1])) \subseteq C([0, 1])$ .

To prove this we note that  $k$  is continuous on  $[0, 1] \times [0, 1]$  which is compact in  $\mathbb{R}^2$ .

This implies that  $k$  is uniformly continuous on  $[0, 1] \times [0, 1]$ . Fix now  $\varepsilon > 0$ . Then there exists  $\delta = \delta(\varepsilon) > 0$  such that

$$|k(x_1, y_1) - k(x_2, y_2)| < \frac{\varepsilon}{B}$$

for  $|(x_1, y_1) - (x_2, y_2)| < \delta$ .  
Fix  $f \in C([0, 1])$

$$\begin{aligned} |T(f)(x_1) - T(f)(x_2)| &= \left| \int_0^1 (k(x_1, y) - k(x_2, y)) h(y, f(y)) \, dy \right| \\ &\leq \int_0^1 \underbrace{|k(x_1, y) - k(x_2, y)|}_{< \frac{\varepsilon}{B} \text{ if } |x_1 - x_2| < \delta} \underbrace{|h(y, f(y))|}_{\leq B} \, dy < \varepsilon, \quad \text{provided } |x_1 - x_2| < \delta \end{aligned}$$

Conclusion:  $T(f) \in C([0, 1])$  for  $f \in C([0, 1])$

**Step 2:** Choose  $F$ .

$k$  is a continuous function on a compact set  $[0, 1] \times [0, 1]$  implies

$$\sup_{(x,y) \in [0,1] \times [0,1]} |k(x, y)| \equiv A < \infty.$$

Hence

$$|T(f)(x)| \leq AB \quad \text{for all } f \in C([0, 1]).$$

Set

$$F := \{f \in C([0, 1]) \mid \|f\| = \max_{x \in [0,1]} |f(x)| \leq AB\}.$$

Clearly  $F$  is closed convex in  $(C([0, 1]), \|\cdot\|)$  which is a Banach space.

**Step 3: Claim:**  $T(F)$  is relatively compact.

To prove this we use the Arzela-Ascoli Theorem.

Let  $K$  be a compact set in  $\mathbb{R}^n$ . Let  $\mathcal{S} \subset C(K)$  (realvalued continuous functions on  $K$ ). Then  $\mathcal{S}$  is relatively compact in  $(C(K), \|\cdot\|_\infty)$  if

(1)  $\mathcal{S}$  uniformly bounded, i.e.

$$\sup_{f \in \mathcal{S}} \|f\| < \infty.$$

(2) Equicontinuity of  $f \in \mathcal{S}$ , i.e.

$$\begin{aligned} \forall \varepsilon > 0 \exists \delta = \delta(\varepsilon) > 0 : \forall f \in \mathcal{S} : \\ |x_1 - x_2| < \delta, x_1, x_2 \in K \quad \Rightarrow \quad |f(x_2) - f(x_1)| < \varepsilon. \end{aligned}$$

In our example it is  $\mathcal{S} = F$ ,  $K = [0, 1]$  in  $\mathbb{R}$ . Check that (1) and (2) in AA-Theorem are satisfied.



(1)  $F$  is uniformly bounded since

$$\sup_{f \in F} \|f\| \leq AB < \infty.$$

(2) Equicontinuity follows from calculations in Step 1.

Conclusion:  $T(F)$  is relatively compact.

**Step 4: Claim:**  $T : F \rightarrow F$  continuous

In step 1 we had  $f \in F$  and  $x_n \rightarrow x$  in  $[0, 1]$ . We have shown that  $T(f)(x_n) \rightarrow T(f)(x)$  in  $\mathbb{R}$ . So  $T(f)$  is a continuous function.

Now we want to show that for  $f_n \rightarrow f$  in  $F$  we've got  $T(f_n) \rightarrow T(f)$  in  $C([0, 1])$ .

Note that  $h : [0, 1] \times [-AB, AB] \rightarrow \mathbb{R}$  is continuous and  $[0, 1] \times [-AB, AB]$  is compact set in  $\mathbb{R}^2$ . So  $h : [0, 1] \times [-AB, AB] \rightarrow \mathbb{R}$  is uniformly continuous.

Fix  $\varepsilon > 0$ . Then there exists a  $\delta = \delta(\varepsilon) > 0$  such that

$$|h(y_1, z_1) - h(y_2, z_2)| < \frac{\varepsilon}{A}$$

for  $|(y_1, z_1) - (y_2, z_2)| < \delta$ . For  $f_1, f_2 \in F$  with

$$\|f_1 - f_2\| < \delta.$$

We have

$$\begin{aligned} |T(f_1)(x) - T(f_2)(x)| &= \left| \int_0^1 k(x, y) (h(y, f_1(y)) - h(y, f_2(y))) \, dy \right| \\ &\leq \int_0^1 \underbrace{|k(x, y)|}_{\leq A} \underbrace{|h(y, f_1(y)) - h(y, f_2(y))|}_{< \frac{\varepsilon}{A}} \, dy < \varepsilon. \end{aligned}$$

Conclusion:  $T : F \rightarrow F$  is continuous.

**Step 5:** Apply Schauder's fixed point theorem.

## 2.3 Completion of normed spaces

$(E, \|\cdot\|)$  normed spaces. We say that  $(\tilde{E}, \|\cdot\|_*)$  is a completion of  $(E, \|\cdot\|)$  if  $(\tilde{E}, \|\cdot\|_*)$  is a normed space such that

- (1)  $\exists \Phi : E \rightarrow \tilde{E}$  injective and linear.
- (2)  $\|x\| = \|\Phi(x)\|_*$  for all  $x \in E$ .
- (3)  $\Phi(E)$  is dense in  $\tilde{E}$ .
- (4)  $(\tilde{E}, \|\cdot\|_*)$  is a Banach space.

**Construction:**

Let  $(x_n)_{n=1}^\infty$  and  $(y_n)_{n=1}^\infty$  be Cauchy sequences in  $(E, \|\cdot\|)$ . We say that  $(x_n)_{n=1}^\infty$  and  $(y_n)_{n=1}^\infty$  are equivalent, denoted by  $(x_n) \sim (y_n)$ , if

$$\|x_n - y_n\| \rightarrow 0, \quad n \rightarrow \infty.$$

Set

$$\tilde{E} = \{((x_n))_N \mid (x_n)_{n=1}^\infty \text{ Cauchy sequence in } (E, \|\cdot\|)\}.$$

Vector space structure:

$$\begin{cases} [(x_n)]_N + [(\tilde{x}_n)]_N &= [(x_n + \tilde{x}_n)]_N \\ \lambda[(x_n)]_N &= [(\lambda x_n)]_N. \end{cases}$$

Show that these definitions are well-defined, i.e. independent of the choice of representative norm

$$\|[(x_n)]_N\|_* = \lim_{n \rightarrow \infty} \|x_n\|.$$

Note

$$(x_n) \sim (y_n)$$

implies

$$\lim_{n \rightarrow \infty} \|x_n\| = \lim_{n \rightarrow \infty} \|y_n\|.$$

Since

$$\| \|x_n\| - \|y_n\| \| \leq \|x_n - y_n\| \rightarrow 0, \quad n \rightarrow \infty$$

Check that the axioms for being a norm are satisfied.

Now we have  $(\tilde{E}, \|\cdot\|_*)$  is a normed space.

Define  $\Phi$ : For  $x \in E$  set  $\Phi(x) = [(x)_{n=1}^\infty]_N$  where

$$(x)_{n=1}^\infty = (x, x, x, \dots).$$

**Claim 1 & 2:** easy to prove.

**Claim 3:**  $\Phi(E)$  dense in  $(\tilde{E}, \|\cdot\|_*)$ . Fix  $[(x_n)]_N \in \tilde{E}$ . Consider  $\Phi(x_k)$  where  $x_k$  is the element in the  $k$ -th position in the sequence  $(x_1, x_2, \dots, x_n, \dots)$ .

$$\|[(x_n)]_N - \Phi(x_k)\|_* = \lim_{n \rightarrow \infty} \|x_n - x_k\| \rightarrow 0 \quad k \rightarrow \infty.$$

Since  $(x_n)_{n=1}^\infty$  is a Cauchy sequence.

**Claim 4:**  $(\tilde{E}, \|\cdot\|_*)$  is a Banach space.

Consider a Cauchy sequence  $z_n \in \tilde{E}$  such that  $\|z_n - z\| \rightarrow 0$  as  $n \rightarrow \infty$ .

To show: There exists  $z \in \tilde{E}$  such that

$$\|z_n - z\| \rightarrow 0, \quad n \rightarrow \infty.$$

By 3 we have that  $\Phi(E)$  is dense in  $\tilde{E}$  so for  $n = 1, 2, \dots$  there exists  $x_n \in E$ ,  $n = 1, 2, \dots$  such that

$$\|z_n - \Phi(z_n)\| < \frac{1}{n}, \quad n = 1, 2, \dots$$

Set  $z =: [(x_n)]_N$ .

Need to show that  $(x_n)_{n=1}^\infty$  is a Cauchy sequence

$$\begin{aligned} \|x_n - x_m\| &= \|\Phi(x_n) - \Phi(x_m)\|_* \\ &\leq \|\Phi(x_n) - z_n\|_* + \|z_n - z_m\|_* + \|z_m - \Phi(x_m)\|_* \\ &< \frac{1}{n} + \|z_n - z_m\| + \frac{1}{m} \rightarrow 0, \quad n, m \rightarrow \infty. \end{aligned}$$

Conclusion:  $(x_n)_{n=1}^\infty$  is a Cauchy sequence in  $(E, \|\cdot\|)$ . Remains to show:

$$\begin{aligned} \|z_n - z\|_* &\rightarrow 0, \quad n \rightarrow \infty \\ \|z_n - z\|_* &\leq \underbrace{\|z_n - \Phi(x_n)\|_*}_{< \frac{1}{n}} + \underbrace{\|\Phi(x_n) - z\|_*}_{=\lim_{n \rightarrow \infty} \|x_n - x_m\|} \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Consider  $f \in C([0, 1])$

- max-norm:  $\|f\| = \max_{x \in [0, 1]} |f(x)|$ . Then  $(C([0, 1]), \|\cdot\|)$  is a Banach space.
- $p \geq 1$ :

$$\|f\|_{L^p} = \left( \int_0^1 |f(x)|^p dx \right)^{\frac{1}{p}}$$

defines a norm for  $C([0, 1])$ .

**Remark.** • Consider piecewise linear  $f_n \in C([0, 1])$  for  $n = 1, 2, \dots$

$$f_n(x) = \begin{cases} 1, & \text{if } \frac{1}{2} \leq x \leq 1 \\ 0, & \text{if } x \leq \frac{1}{2} - \frac{1}{2n} \end{cases}$$

with

$$\|f_n - f_m\|_{L^1} \leq \frac{1}{2 \min(m, n)} \rightarrow 0, \quad n, m \rightarrow \infty.$$

So  $(f_n)_{n=1}^\infty$  is a Cauchy sequence in  $(C([0, 1]), \|\cdot\|_{L^1})$  but  $(f_n)_{n=1}^\infty$  does not converge in  $(C([0, 1]), \|\cdot\|_{L^1})$  since if  $\|f_n - f\|_{L^1} \rightarrow 0$  as  $n \rightarrow \infty$  and  $f \in C([0, 1])$  then

$$f(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}) \\ 1, & \text{if } x \in [\frac{1}{2}, 1] \end{cases}.$$

Conclusion:  $(C([0, 1]), \|\cdot\|_{L^1})$  is not a Banach space.

- Consider:

$$f(x) = \begin{cases} 1, & \text{if } x = \frac{1}{2} \\ 0, & \text{if } x \in [0, 1] \setminus \{\frac{1}{2}\} \end{cases}.$$

Then

$$\|f\|_{L^1} = 0 = \|0\|_{L^1}.$$

Compare this with the first axiom for a norm function.

- Replace  $[0, 1]$  with  $\mathbb{R}$ . For  $f : \mathbb{R} \rightarrow \mathbb{R}$  set

$$\text{supp}(f) = \{x \in \mathbb{R} \mid f(x) \neq 0\}.$$

Set

$$C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) \mid \text{supp}(f) \text{ is compact in } \mathbb{R}\}.$$

**Claim:**  $C_0(\mathbb{R})$  forms a vector space and for every  $p \geq 1$  and  $f \in C_0(\mathbb{R})$

$$\|f\|_{L^p} = \left( \int_{\mathbb{R}} |f(x)|^p dx \right)^{\frac{1}{p}}$$

defines a norm on  $C_0(\mathbb{R})$ .

**Problem:**  $(C_0(\mathbb{R}), \|\cdot\|_{L^p})$  for  $p \geq 1$  are not Banach spaces.

$(L^1(\mathbb{R}), \|\cdot\|_{L^1})$  is a completion of  $(C_0(\mathbb{R}), \|\cdot\|_{L^1})$ .

Note  $A \subset \mathbb{R}$  and  $A$  bounded. Define

$$f_A(x) = \begin{cases} 1, & x \in A \\ 0, & \text{elsewhere} \end{cases}.$$

Lebesguesmeasure of  $A = \|f_A\|_{L^1} = \mu(f_A)$ .  $A \subset \mathbb{R}$  and  $A$  unbounded

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A \cap [-n, n]).$$

We say that  $A \subset \mathbb{R}$  is a 0- set if for all  $\varepsilon > 0$  there exist open intervals  $I_n$ ,  $n = 1, 2, \dots$  such that

- (1)  $A \subseteq \bigcup_{n=1}^{\infty} I_n$ ,
- (2)  $\sum_{n=1}^{\infty} \text{lengths of } I_n < \varepsilon$ .

In particular

$$A = \mathbb{Q} = \{r_n \mid n = 1, 2, \dots\} \quad \text{is a 0-set.}$$

### 3 Hilbert spaces

**Example.** Consider  $\mathbb{C}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{C}\}$  and  $x, y \in \mathbb{C}^n$  with  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n)$ . Define the inner product of  $x, y$  (scalar product)

$$\langle x, y \rangle = \sum_{i=1}^n x_i \bar{y}_i \in \mathbb{C}.$$

We have a map

$$\begin{aligned} \mathbb{C}^n \times \mathbb{C}^n &\rightarrow \mathbb{C} \\ (x, y) &\mapsto \langle x, y \rangle. \end{aligned}$$

This mapping has properties:

- $x \neq 0$  folgt  $\langle x, x \rangle = \sum_{i=1}^n x_i \bar{x}_i = \sum_{i=1}^n |x_i|^2 > 0$
- $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$  for  $x, y \in \mathbb{C}^n, \lambda \in \mathbb{C}$ .
- $\langle x, y \rangle = \sum_{i=1}^n x_i \bar{y}_i = \overline{\sum_{i=1}^n y_i \bar{x}_i}$  for  $x, y \in \mathbb{C}^n$ .  
In particular  $\langle x, \lambda y \rangle = \bar{\lambda} \langle x, y \rangle$  for  $\lambda \in \mathbb{C}$ .
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$  for  $x, y, z \in \mathbb{C}^n$ .

**Definition .** An inner product space  $V$  is a complex vector space with an inner product which is a map

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}.$$

Satisfying

- $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$  for any  $x, y \in V, \lambda \in \mathbb{C}$ .
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$  for any  $x, y, z \in V$ .
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$  for any  $x, y \in V$ .
- $\langle x, x \rangle > 0$  for any  $x \in V, x \neq 0$ .

Can we generalize  $\mathbb{C}^n$ ?

$$\mathbb{C}^{\mathbb{N}} \{(x_1, x_2, \dots) \mid x_i \in \mathbb{C}\}$$

with

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i.$$

This is not necessarily convergent.

**Examples. (1)**

$$l^2 = \left\{ (x_1, x_2, \dots) \mid \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}.$$

We have with Cauchy Schwarz

$$\sum_{i=1}^n |x_i \bar{y}_i| \leq \left( \sum_{i=1}^n |x_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n |y_i|^2 \right)^{\frac{1}{2}}$$

if  $x \in l^2$  and  $y \in l^2$  we get

$$\sum_{i=1}^n |x_i \bar{y}_i| \leq \left( \sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{\infty} |y_i|^2 \right)^{\frac{1}{2}} < \infty.$$

It follows that  $\sum_{i=1}^{\infty} x_i \bar{y}_i$  converges absolutely and hence it is convergent. The following

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i$$

is well-defined for vectors  $x, y \in l^2$ . Like for  $\mathbb{C}^n$  one can easily check that  $\langle \cdot, \cdot \rangle$  satisfies the axioms for inner products.

$(l^2, \langle \cdot, \cdot \rangle)$  is an inner product space.

**(2) Consider  $C([0, 1])$  with the inner product**

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} dt \quad \forall f, g \in C([0, 1]).$$

•

$$\langle \lambda f, g \rangle = \int_0^1 \lambda f(t) \overline{g(t)} dt = \lambda \int_0^1 f(t) \overline{g(t)} dt = \lambda \langle f, g \rangle.$$

•

$$\langle f, f \rangle = \int_0^1 f(t) \overline{f(t)} dt = \int_0^1 |f(t)|^2 dt > 0.$$

• ...

If we take  $\mathbb{R}^3$  with the Eukledian norm on  $\mathbb{R}^3$

$$\|(x_1, x_2, x_3)\| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \left( \sum_{i=1}^3 |x_i|^2 \right)^{\frac{1}{2}} = \langle x, x \rangle^{\frac{1}{2}}.$$

Let  $V$  be an inner product space with  $\langle \cdot, \cdot \rangle$  as the inner product. Let for  $x \in V$

$$\|x\| := \langle x, x \rangle^{\frac{1}{2}}.$$

**Statement 3.1.** The  $x \mapsto \|x\|$  with  $\|\cdot\|$  defined above is a norm.

We are going to prove the norm axioms but first we need another theorem.

**Theorem 3.2** (Cauchy-Schwarz inequality). For any  $x, y \in V$  (inner product space)

$$|\langle x, y \rangle| \leq \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}}.$$

The equality holds iff  $x, y$  are linearly dependent.

**proof.** Assume  $x, y$  linearly dependent. We can assume that  $x = \lambda y$  for some  $\lambda \in \mathbb{C}$ .

$$|\langle x, y \rangle| = |\langle \lambda y, y \rangle| = |\lambda| \langle y, y \rangle$$

and

$$\begin{aligned} \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}} &= \langle \lambda y, \lambda y \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}} \\ &= |\lambda| \langle y, y \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}} \\ &= |\lambda| \langle y, y \rangle. \end{aligned}$$

Hence

$$|\langle x, y \rangle| = \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}}.$$

Assume  $x, y$  are linearly independent. Hence  $x + \lambda y \neq 0$  for any  $\lambda \in \mathbb{C}$ . By an axiom for inner product we get

$$0 < \langle x + \lambda y, x + \lambda y \rangle = \langle x, x \rangle + \lambda \langle y, x \rangle + \bar{\lambda} \langle x, y \rangle + |\lambda|^2 \langle y, y \rangle.$$

Pick now

$$\lambda = -\frac{\langle x, y \rangle}{\langle y, y \rangle}.$$

(Note that  $y \neq 0$  as  $x, y$  linearly independent.) We have

$$\begin{aligned} 0 &< \langle x, x \rangle - \frac{\overbrace{\langle x, y \rangle \langle y, x \rangle}^{=|\langle x, y \rangle|^2}}{\langle y, y \rangle} - \frac{\overbrace{\langle x, y \rangle \langle x, y \rangle}^{=|\langle x, y \rangle|^2}}{\langle y, y \rangle} + \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle^2} \langle y, y \rangle \\ &= \langle x, x \rangle - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle}. \end{aligned}$$

This gives

$$\frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} < \langle x, x \rangle$$

and it follows

$$|\langle x, y \rangle|^2 < \langle x, x \rangle \langle y, y \rangle.$$

□

Now we can use this inequality to proof the statement above:

**proof.** (i)  $\|x\| > 0$  for all  $x \neq 0$  in  $V$  (Exercise).

(ii)  $\|\lambda x\| = |\lambda| \|x\|$  for all  $x \in V, \lambda \in \mathbb{C}$  (Exercise).

(iii) Let  $x, y \in V$ . Then

$$\begin{aligned}\|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \langle x, x \rangle + 2\operatorname{Re}(\langle x, y \rangle) + \langle y, y \rangle \\ &\leq \langle x, x \rangle + 2|\langle x, y \rangle| + \langle y, y \rangle \\ &\leq \langle x, x \rangle + 2\langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}} + \langle y, y \rangle \\ &= \left( \langle x, x \rangle^{\frac{1}{2}} + \langle y, y \rangle^{\frac{1}{2}} \right)^2.\end{aligned}$$

So

$$\|x + y\|^2 \leq (\|x\| + \|y\|)^2.$$

□

**Theorem 3.3 (The Parallelogram Law).** Let  $(V, \langle \cdot, \cdot \rangle)$  be an inner product space. Let  $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$ . Then

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \forall x, y \in V.$$

**Statement 3.4.**  $l^p$  has inner product  $\langle \cdot, \cdot \rangle_{l^p}$  such that

$$\|x\|_p = \sqrt{\langle x, x \rangle_{l^p}}$$

iff  $p = 2$ .

**proof.** Enough to show that  $\|\cdot\|_p$ -norm does not satisfy the parallelogram law for some  $x, y \in l^p$  if  $p \neq 2$ . Take for example  $x = (1, 0, 0, \dots)$  and  $y = (0, 1, 0, \dots)$ . Note that  $\|x\|_{l^p} = \|y\|_{l^p} = 1$

$$\begin{aligned}\|x + y\|_{l^p}^2 &= \|(1, 1, 0, \dots)\|_{l^p}^2 = 2^{\frac{2}{p}} \\ \|x - y\|_{l^p}^2 &= \|(1, -1, 0, \dots)\|_{l^p}^2 = 2^{\frac{2}{p}} \\ \|x + y\|_{l^p}^2 + \|x - y\|_{l^p}^2 &= 2 \cdot 2^{\frac{2}{p}} = 2(\|x\|_{l^p}^2 + \|y\|_{l^p}^2) = 2 \cdot 2.\end{aligned}$$

□

All  $l^p$  with  $p \neq 2$  are not inner product spaces.



**Exercise:**

Show that  $(C([0, 1]), \|\cdot\|_\infty)$  is not an inner product space.

**Remark.** Whenever a norm satisfies the parallelogram law then there exists an inner product on  $V$  such that

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}.$$

**Theorem 3.5 (The Polarization Identity).** Let  $(V, \langle \cdot, \cdot \rangle)$  be an inner product space. Then

$$4\langle x, y \rangle = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2.$$

**Definition 3.6.** Let  $(V, \langle \cdot, \cdot \rangle)$  be an inner product space. We say that  $x, y$  in  $V$  are orthogonal if  $\langle x, y \rangle = 0$  (We write  $x \perp y$ ). Let  $M \subseteq V$ . Define the orthogonal complement

$$M^\perp = \{x \in V \mid x \perp y \text{ for any } y \in M\}.$$

**Proposition 3.7.** If  $M \subseteq V$  then  $M^\perp$  is a subspace of  $V$ .

**Theorem 3.8 (Pythagorean formula).**  $x, y \in V$  (inner product space). Then

$$x \perp y \quad \text{iff} \quad \|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

**3.1 Orthogonal Systems**

Let  $(V, \langle \cdot, \cdot \rangle)$  be an inner product space  $\{u_n\} \subseteq V$  is called orthogonal system (with  $n$  finite or infinite) if  $u_n \perp u_m$  for all  $n \neq m$ . It is an orthonormal system if in addition  $\|u_n\| = 1$ .

**Examples.** 1)  $\{e_k\}_{k=1}^\infty \subseteq \ell^2$  with

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i$$

with

$$e_k = (0, \dots, 1, 0, \dots).$$

$\Rightarrow \{e_k\}$  is an ON-system.

2)  $C([-\pi, \pi])$  with

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt.$$

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{-int} \mid n \in \mathbb{Z} \right\}$$

is an orthonormal system.

**Definition 3.9.** Let  $\{a_n \mid n \in \mathbb{N}\}$  be an orthonormal system in  $V$ . The formal series

$$\sum_{n=1}^{\infty} \langle x, a_n \rangle a_n$$

is called a fourier series of  $x$  corresponding  $\{a_n \mid n \in \mathbb{N}\}$  and  $\langle x, a_n \rangle$  are called fourier coefficients of  $x$  corresponding to  $\{a_n \mid n \in \mathbb{N}\}$ .

**Theorem 3.10** (Bessel's Equality and Inequality). If  $\{a_n\}$  orthonormal system in an inner product space  $V$ , then for all  $x \in V$

$$\left\| x - \sum_{k=1}^n \langle x, a_k \rangle a_k \right\|^2 = \|x\|^2 - \sum_{k=1}^n |\langle x, a_k \rangle|^2$$

and

$$\sum_{k=1}^{\infty} |\langle x, a_k \rangle|^2 \leq \|x\|^2.$$

**proof.**

$$\begin{aligned} \left\| x - \sum_{k=1}^n \langle x, a_k \rangle a_k \right\|^2 &= \left\langle x - \sum_{k=1}^n \langle x, a_k \rangle a_k, x - \sum_{k=1}^n \langle x, a_k \rangle a_k \right\rangle \\ &= \langle x, x \rangle - \sum_{k=1}^n \overline{\langle x, a_k \rangle} \langle x, a_k \rangle - \sum_{k=1}^n \langle x, a_k \rangle \langle a_k, x \rangle \\ &\quad + \left\langle \sum_{k=1}^n \langle x, a_k \rangle a_k, \sum_{k=1}^n \langle x, a_k \rangle a_k \right\rangle \\ &= \|x\|^2 - \sum_{k=1}^n |\langle x, a_k \rangle|^2 - \sum_{k=1}^n |\langle x, a_k \rangle|^2 + \sum_{k=1}^n |\langle x, a_k \rangle|^2 \\ &= \|x\|^2 - \sum_{k=1}^n |\langle x, a_k \rangle|^2. \end{aligned}$$

This gives also:

$$\sum_{k=1}^n |\langle x, a_k \rangle|^2 = \|x\|^2 - \left\| x - \sum_{k=1}^n \langle x, a_k \rangle a_k \right\|^2 \leq \|x\|^2$$

for all  $n \in \mathbb{N}$ . Hence

$$\sum_{k=1}^{\infty} |\langle x, a_k \rangle|^2 \leq \|x\|^2.$$

□

**Definition 3.11** (Hilbert space). A Hilbert space is an inner product space which is complete w.r.t. the norm is defined through the inner product.

**Examples.** •  $\mathbb{C}^n$  is an inner product space and complete w.r.t the Eukledean norm. Hence  $\mathbb{C}^n$  is a Hilbert space.

- $l^2$  is a Banach space w.r.t.

$$\|x\|_{l^2} = \left( \sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}$$

and

$$\|x\|_{l^2} = \langle x, x \rangle^{\frac{1}{2}},$$

where

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i.$$

- $(C([0, 1]), \|\cdot\|_{\infty})$  is a Banach space but not an inner product space. Hence it is no Hilbert space.
- $(C([0, 1]), \langle \cdot, \cdot \rangle)$  is an inner product space  $f, g \in C([0, 1])$  with

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} dt$$

and the corresponding

$$\|f\|_2 = \langle f, f \rangle = \int_0^1 |f(t)|^2 dt.$$

**Remark.** Other  $l^p$  spaces are not Hilbert spaces!!!! They are not inner product spaces.

**Statement 3.12.**  $(C([0, 1]), \langle \cdot, \cdot \rangle)$  is not a Hilbert space since  $(C([0, 1]), \|\cdot\|_2)$  is not complete.

**proof.** Sketch: Show that  $f_n(t)$ , which is defined as a piecewise continuous function for example

$$f_n(x) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}] \\ 0, & \text{if } x \in [\frac{1}{2} + \frac{1}{n}, 1] \\ \text{continuous,} & \text{elsewhere} \end{cases}$$

is a Cauchy sequence w.r.t  $\|\cdot\|_2$  but has no limit in  $C([0, 1])$ . □

Consider

$$C_F = \{(x_1, x_2, \dots) \mid \text{only finite } x_i \neq 0\}$$

with

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i.$$

Show that  $(C_F, \langle \cdot, \cdot \rangle)$  is not a Hilbert space.

**Definition 3.13** (strongly and weakly convergent). A sequence  $\{x_n\} \subseteq H$ , where  $H$  is a Hilbert space, is called strongly convergent ( $x_n \rightarrow x \in H$ ) if

$$\|x_n - x\| \rightarrow 0, \quad n \rightarrow \infty.$$

(Norm induced by an inner product)

We say that  $x_n$  is weakly convergent ( $x_n \rightharpoonup x$ ) if

$$\langle x_n, y \rangle \rightarrow \langle x, y \rangle, \quad \forall y \in H.$$

**Statement 3.14.**  $x_n \rightarrow x \Rightarrow x_n \rightharpoonup x.$

**proof.** Assume strong convergence for  $(x_n)_{n \in \mathbb{N}}$ . Then

$$\begin{aligned} |\langle x_n, y \rangle - \langle x, y \rangle| &= |\langle x_n - x, y \rangle| \\ &\leq \underbrace{\langle x_n - x, x_n - x \rangle^{\frac{1}{2}}}_{=\|x_n - x\|} \underbrace{\langle y, y \rangle^{\frac{1}{2}}}_{=\|y\|} \\ &= \underbrace{\|x_n - x\|}_{\rightarrow 0} \|y\| \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Hence  $\langle x_n, y \rangle \rightarrow \langle x, y \rangle$ . □

**Remark.** The converse is not true in general:

Take  $H = l^2$  and

$$\begin{aligned} x_n &= e_n = (0, \dots, 1, 0, \dots) \\ y &= (y_1, y_2, \dots) \in l^2. \end{aligned}$$

We have for all  $y \in H$

$$\langle e_n, y \rangle = y_n \rightarrow 0, \quad n \rightarrow \infty$$

as

$$\|e_n - 0\|_{l^2} = \|e_n\|_{l^2} = 1.$$

**Statement 3.15.**  $x_n \rightarrow x$  and  $y_n \rightarrow y$  yields

$$\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle.$$

In particular

$$x_n \rightarrow x \quad \Rightarrow \quad \|x_n\| \rightarrow \|x\|.$$

**proof.**

$$\begin{aligned}
 |\langle x_n, y_n \rangle - \langle x, y \rangle| &= |\langle x_n, y_n \rangle - \langle x, y_n \rangle + \langle x, y_n \rangle - \langle x, y \rangle| \\
 &= |\langle x_n - x, y_n \rangle + \langle x, y_n - y \rangle| \\
 &\leq |\langle x_n - x, y_n \rangle| + |\langle x, y_n - y \rangle| \\
 &\leq \underbrace{\|x_n - x\|}_{\rightarrow 0} \underbrace{\|y_n\|}_{< \infty} + \underbrace{\|x\|}_{< \infty} \underbrace{\|y_n - y\|}_{\rightarrow 0} \rightarrow 0, \quad n \rightarrow \infty.
 \end{aligned}$$

Check  $\{\|y_n\|\}$  is bounded

$$\|y_n\| = \|y_n - y + y\| \leq \underbrace{\|y_n - y\|}_{\rightarrow 0} + \underbrace{\|y\|}_{< \infty} \rightarrow 0, \quad n \rightarrow \infty.$$

□

**Statement 3.16.**  $x_n \rightarrow x$  and  $\|x_n\| \rightarrow \|x\|$  yields

$$x_n \rightarrow x.$$

**proof.**

$$\begin{aligned}
 \|x_n - x\|^2 &= \langle x_n - x, x_n - x \rangle \\
 &= \underbrace{\langle x_n, x_n \rangle}_{=\|x_n\|^2} - \langle x, x_n \rangle - \langle x_n, x \rangle + \langle x, x \rangle \\
 &= \|x_n\|^2 - \overline{\langle x_n, x \rangle} - \langle x_n, x \rangle + \|x\|^2 \\
 &\rightarrow \|x\|^2 - \|x\|^2 - \|x\|^2 + \|x\|^2 = 0.
 \end{aligned}$$

□

We have proved

$$x_n \rightarrow x \quad \Rightarrow \quad \{\|x_n\|\} \text{ is bounded.}$$

**Theorem 3.17.**

$$x_n \rightarrow x \quad \Rightarrow \quad \sup_{n \in \mathbb{N}} \|x_n\| < \infty.$$

**proof.** Let  $x_n \rightarrow x$ . Consider  $f_n : H \rightarrow \mathbb{C}$  where

$$f_n(y) = \langle y, x_n \rangle, \quad y \in H.$$

- $f_n$  is a linear functional for every  $n \in \mathbb{N}$ .

- $\forall n \in \mathbb{N}$   $f_n$  is a bounded ( $\Leftrightarrow$  continuous) linear functional as if

$$y_k \xrightarrow{k \rightarrow \infty} y \quad \Rightarrow \quad f_n(y_k) = \langle y_k, x_n \rangle \rightarrow \langle y, x_n \rangle = f_n(y), \quad k \rightarrow \infty.$$

- $f_n(y) \rightarrow \langle y, x \rangle$ .  
 $\{f_n(y)\}_n$  is a convergent sequence in  $\mathbb{C}$  and hence bounded for all  $y \in H$ .  
Hence it exists  $M_y$  such that

$$|f_n(y)| \leq M_y.$$

By Banach-Steinhaus-Theorem it holds

$$\|f_n\| \leq M \text{ for some } M > 0.$$

We are done if we proof that  $\|f_n\| = \|x_n\|$ .

$$|f_n(y)| = |\langle y, x_n \rangle| \leq \|y\| \|x_n\|, \quad \forall y \in H.$$

Hence

$$\|f_n\| \leq \|x_n\| \quad (1).$$

On the other Hand we have

$$f_n(x_n) = \langle x_n, x_n \rangle = \|x_n\|^2$$

and thus

$$\|f_n\| = \sup_{x \in H} \frac{|f_n(x)|}{\|x\|} \geq \frac{|f_n(x_n)|}{\|x_n\|} = \|x_n\| \quad (2)$$

With (1) and (2) we are finished.

□

### 3.2 Orthogonal decomposition in Hilbert spaces

Remember Linear Algebra. Take  $\mathbb{R}^n$  and a subspace  $M \subseteq \mathbb{R}^n$

$$\Rightarrow \quad \forall x \in \mathbb{R}^n \quad x = z + y, \quad \text{where } z \in M, y \in M^\perp.$$

This can be done in a unique way

$$\begin{aligned} M &= \text{span}\{e_z\} \\ M^\perp &= \text{span}\{e_y\} \end{aligned}$$

and

$$z = \text{proj}_{M^\perp} x, \quad \|x - \text{proj}_M x\| = \min_{y \in M} \|x - y\|.$$

**General Hilbert space case**

**Proposition 3.18.**  $M \subseteq H$ , then  $M^\perp$  is a closed subspace and

$$(M^\perp)^\perp = \overline{\text{span } M}.$$

**Statement 3.19.**  $H$  Hilbert space and  $M$ -closed subspace of  $H$  and  $x \in H$ . Then there exists a unique  $z \in M$  such that

$$\|x - z\| = \text{dist}(x, M) := \inf_{y \in M} \|x - y\|.$$

( $z$  analog of the  $\text{proj}_M x$  in the other case).

**Proposition 3.20.** Taking  $z \in M$  from the previous proposition. We have  $x - z \in M^\perp$ , i.e.

$$x = \underbrace{z}_{\in M} + \underbrace{(x - z)}_{\in M^\perp}.$$

**Theorem 3.21** (Orthogonal Decomposition Theorem). Let  $(E, \langle \cdot, \cdot \rangle)$  be a Hilbert space and  $S$  be a closed subspace of  $E$ .

$$\Rightarrow E = S \oplus S^\perp$$

which means that for every  $x \in E$  there exists a unique decomposition

$$x = y + z$$

with  $y \in S$  and  $z \in S^\perp$ .

**Example.** Let  $A \subseteq E$  where  $E$  is a Hilbert space. It follows

$$\overline{\text{span } A} = (A^\perp)^\perp.$$

Note

$$A \subseteq \underbrace{(A^\perp)^\perp}_{\text{subspace of } E} \Rightarrow \text{span } A \subseteq \underbrace{(A^\perp)^\perp}_{\text{closed}} \Rightarrow \overline{\text{span } A} \subseteq (A^\perp)^\perp$$

$$A \subseteq \overline{\text{span } A} \Rightarrow \overline{\text{span } A}^\perp \subseteq A^\perp \Rightarrow (A^\perp)^\perp \subseteq (\overline{\text{span } A}^\perp)^\perp.$$

Hence

$$\overline{\text{span } A} \subseteq (A^\perp)^\perp \subseteq (\overline{\text{span } A}^\perp)^\perp.$$

By the Orthogonal Decomposition Theorem we get

$$E = \overline{\text{span } A} \oplus \overline{\text{span } A}^\perp = \overline{\text{span } A}^\perp \oplus \left( \overline{\text{span } A}^\perp \right)^\perp,$$

which implies

$$\begin{aligned} \overline{\text{span } A} &= \left( \overline{\text{span } A}^\perp \right)^\perp, \\ \Rightarrow \quad \left( A^\perp \right)^\perp &= \overline{\text{span } A}. \end{aligned}$$

Now we are going to prove the Orthogonal Decomposition Theorem.

**proof. Step 1:**  $S$  is a closed convex set in a Hilbert space  $E$ . This implies that

$$\forall x \in E \exists! y \in S : \quad \|x - y\| \leq \|x - \tilde{y}\| \quad \forall \tilde{y} \in S.$$

which means

$$\|x - y\| = \inf_{\tilde{y} \in S} \|x - \tilde{y}\|.$$

Fix  $x \notin S$  with

$$\inf_{\tilde{y} \in S} \|x - \tilde{y}\| = d > 0.$$

Take a sequence  $(y_n)_{n=1}^\infty$  in  $S$  such that

$$\|x - y_n\| \rightarrow d, \quad n \rightarrow \infty.$$

**Claim:** This is a Cauchy sequence.

(use Parallelogram-law for  $\|\cdot\|$ )

**Step 2:**  $S$  as in ODT.

Note:  $S$  must be convex.

Fix  $x \in E$ , choose  $y \in S$  with

$$\|x - y\| \leq \|x - \tilde{y}\|, \quad \forall \tilde{y} \in S.$$

Set

$$\underbrace{x}_{\in E} = \underbrace{y}_{\in S} + (x - y).$$

To show:  $x - y \in S^\perp$ . A variational argument of this is

$$\langle x - y, v \rangle = 0, \quad \forall v \in S.$$

We know

$$\begin{aligned} \|x - y\|^2 &\leq \|x - y + \alpha v\|^2 \quad \forall \text{ scalars } \alpha \\ \|x - y\|^2 &\leq \langle x - y + \alpha v, x - y + \alpha v \rangle \\ &= \|x - y\|^2 + \alpha \langle v, x - y \rangle + \bar{\alpha} \langle x - y, v \rangle + |\alpha|^2 \|v\|^2 \end{aligned}$$



and

$$0 \leq 2 \operatorname{Re}(\alpha \langle x - y, v \rangle) + |\alpha|^2 \|v\|^2.$$

Set

$$\begin{aligned} \alpha &= t \overline{\langle x - y, v \rangle}, \quad t \in \mathbb{R}, \\ \Rightarrow \quad 0 &\leq 2t |\langle x - y, v \rangle|^2 + t^2 |\langle x - y, v \rangle|^2 \|v\|^2. \end{aligned}$$

Assume  $\langle x - y, v \rangle \neq 0$ :

We have

$$\begin{aligned} 0 &\leq 2t + t^2 \|v\|^2 \quad \forall t \in \mathbb{R} \\ \Rightarrow \quad -2t &\leq t^2 \|v\|^2, \quad \text{Let } t < 0 \\ \Leftrightarrow \quad 2 &\leq -t \|v\|^2, \quad t < 0. \end{aligned}$$

Let  $t \rightarrow 0$ , then

$$2 \leq 0$$

which is a contradiction. □

### 3.3 Bounded linear functionals on Hilbert spaces

Consider  $(H, \langle \cdot, \cdot \rangle)$ - Hilbert space (inner product space which is complete w.r.t. to a norm  $\|x\| = \sqrt{\langle x, x \rangle}$ ).

Let  $M$  be a closed subspace of  $H$ .

$$M^\perp = \{y \in H \mid \langle x, y \rangle = 0, \forall x \in M\}.$$

Then we know  $H = M + M^\perp$ , i.e. for any  $x \in H$  there exists a unique  $y \in M$  and  $z \in M^\perp$  such that

$$x = y + z.$$

**Theorem 3.22** (Riesz-Freché representation theorem). Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbertspace. Let  $f$  be a bounded linear functional on  $H$ . Then there exists a unique  $x_f \in H$  such that

$$f(x) = \langle x, x_f \rangle, \quad \forall x \in H.$$

Moreover

$$\|f\| = \|x_f\|_H.$$

**Remark.** If  $f : H \rightarrow \mathbb{C}$  is of the form

$$f(x) = \langle x, y \rangle, \quad \text{for all } x \in H \text{ and some } y \in H.$$

Then  $f$  is bounded and linear (easy with Cauchy-Schwarz and properties of the scalar product).

**proof. Existence of  $x_f$ :** If  $f$  is a zero linear functional, i.e.  $f(x) = 0$  for all  $x \in H$  take  $x_f = 0$ . Assume now that  $f$  is not the zero functional. Consider

$$N(f) := \ker f = \{x \in H \mid f(x) = 0\}.$$

Then  $N(f)$  is a closed subspace of  $H$ :

For  $x_1, x_2 \in N(f)$ ,  $\alpha, \beta \in \mathbb{C}$  it holds

$$f(\alpha x_1 + \beta x_2) \stackrel{\text{lin}}{=} \alpha f(x_1) + \beta f(x_2).$$

Hence  $\alpha x_1 + \beta x_2 \in N(f)$  and  $N(f)$  is a subspace.  $N(f)$  is closed since if  $x_n \in N(f)$  with  $x_n \rightarrow x$  strongly. Then

$$f(x_n) \rightarrow f(x)$$

because of bounded and hence continuous. But we know that  $f(x_n) = 0$  so the limit has to be  $f(x) = 0$ , i.e.  $x \in N(f)$ .  $N(f)$  is a proper closed subspace. ( $N(f) \neq H$ ). Consider now  $N(f)^\perp$  which is non-zero.

- $\dim N(f)^\perp = 1$ .

Assume that  $x_1 \neq 0, x_2 \neq 0 \in N(f)^\perp$ . Then we have  $f(x_1), f(x_2) \neq 0$ . It exists  $a \in \mathbb{C}$  such that

$$f(x_1) + a f(x_2) = 0.$$

And also

$$f(x_1 + a x_2) = 0$$

which gives

$$x_1 + a x_2 \in N(f) \cap N(f)^\perp = \{0\}.$$

Hence

$$x_1 + a x_2 = 0.$$

Any two vectors are linearly dependent in  $N(f)^\perp$  which gives

$$\dim N(f)^\perp = 1.$$

Take  $y' \in N(f)^\perp$  with  $\|y'\| = 1$  and let

$$x_f = \overline{f(y')} y'.$$

We get

$$\langle x, x_f \rangle = \begin{cases} 0, & \text{if } x \in N(f) \\ \langle \lambda y', \overline{f(y')} y' \rangle = f(y') \lambda \underbrace{\langle y', y' \rangle}_{=1}, & \text{if } x = \lambda y' \end{cases}.$$

Furthermore

$$\langle x, x_f \rangle = \begin{cases} f(x), & \text{if } x \in N(f) \\ f(\lambda y') = f(x), & \text{if } x = \lambda y' \end{cases}.$$

Since every element in  $H$  is given by  $x + \lambda y'$ . For  $x \in N(f)$  and  $\lambda \in \mathbb{C}$ . Using linearity we get

$$f(x + \lambda y') = f(x) + f(\lambda y') = \langle x, x_f \rangle + \langle \lambda y', x_f \rangle = \langle x + \lambda y', x_f \rangle$$

**uniqueness:** Assume there exists  $x_1, x_2 \in H$  such that

$$f(x) = \langle x, x_1 \rangle = \langle x, x_2 \rangle, \quad \forall x \in H.$$

We get

$$\langle x, x_1 - x_2 \rangle = 0, \quad \forall x \in H.$$

It holds in particular for  $x = x_1 - x_2$  the following equality

$$\langle x_1 - x_2, x_1 - x_2 \rangle = 0 \quad \Rightarrow \quad x_1 - x_2 = 0.$$

**norm equality** We must see that

$$\|f\| = \|x_f\|_H.$$

From remark we have

$$f(x) = \langle x, x_f \rangle \quad \Rightarrow \quad \|f\| \leq \|x_f\|.$$

We have for  $x_f \neq 0$ :

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} \geq \frac{|f(x_f)|}{\|x_f\|} = \frac{\|x_f\|^2}{\|x_f\|} = \|x_f\|.$$

This gives the desired result. □

**Example.**

$$E = C_F = \{(x_1, x_2, \dots) \mid \text{only finite number of } x_i \neq 0\} \subseteq l^2.$$

On  $C_F$  consider  $l^2$ -inner-product

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i \quad \text{for } x, y \in C_F.$$

1.  $C_F$  is not a Hilbert space as it is not complete w.r.t

$$\|x\|_2 = \left( \sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}.$$

Find a Cauchy sequence that is not convergent to an element in  $C_F$ .

Find a proper closed subspace  $M$  such that  $M^\perp = \{0\}$  (This would mean in particular that  $C_F \neq M + M^\perp$ )

Consider

$$M = \left\{ (x_1, x_2, \dots) \in C_F \mid \sum_{k=1}^{\infty} x_k \frac{1}{k} = 0 \right\},$$

$$x_f = (1, \frac{1}{2}, \frac{1}{3}, \dots) \in l^2,$$

$$M = \ker f \cap C_F$$

where

$$f : l^2 \rightarrow \mathbb{C}$$

$$f(x) = \langle x, x_f \rangle = \sum_{k=1}^{\infty} x_k \frac{1}{k},$$

$$M^{\perp} = \text{all elements in } C_F \text{ which are in } (\ker f)^{\perp}.$$

From the proof of Riesz-Frechet theorem we have  $(\ker f)^{\perp}$  is 1-dimensional and

$$x_f \in (\ker f)^{\perp}.$$

Hence

$$(\ker f)^{\perp} = \{\lambda x_f \mid \lambda \in \mathbb{C}\}.$$

We have

$$\underbrace{(\ker f)^{\perp} \cap C_F}_{=M^{\perp}} = \{0\}.$$

2.  $(H, \langle \cdot, \cdot \rangle)$  Hilbert space and  $\{u_i\} \subseteq H$  finite or infinite  $i$ .  $\{u_i\}$  is an orthogonal system if

$$\langle u_i, u_j \rangle = 0, \quad \forall i \neq j$$

and an orthonormal system if

$$\langle u_i, u_j \rangle = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}.$$

**Proposition 3.23.** Orthogonal system of non-zero vectors are linearly independent. (See linear algebra)

Having linearly independent family of vectors we can make it orthogonal with for example using Gram-Schmidt orthogonalization procedure. (See linear algebra for details).

Recall that we can write a Fourier series of  $x$  with  $\langle x, u_i \rangle$  Fourier coefficients

$$x \in H \quad \Rightarrow \quad x = \sum_{i=1}^{\infty} \langle x, u_i \rangle u_i$$

with  $\{u_i\}$ -ON-system.

$C([-\pi, \pi])$  and  $\{u_k\} = \left\{ \frac{1}{\sqrt{2\pi}} e^{ikt} \mid k \in \mathbb{Z} \right\}$  equipped with the scalar product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt.$$

It holds for the Fourier-series

$$\langle f, u_k \rangle = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt.$$

We want to see when

$$\sum_{i=1}^{\infty} \langle x, u_i \rangle u_i$$

is convergent to  $x$ .

**Definition 3.24.**  $\mathcal{A}_n$  ON-system is called an ON-basis for  $H$  if its span is dense in  $H$ . We say that an ON-system is complete if every  $x \in H$  is

$$\sum_{i=1}^{\infty} \langle x, u_i \rangle u_i.$$

**Theorem 3.25.**  $(H, \langle \cdot, \cdot \rangle)$ - Hilbert space,  $\{u_k\}$  is ON-system in  $H$ . The following statements are equivalent.

- (1)  $\{u_n\}$  is a complete ON-system.
- (2)  $\{u_n\}$  is an ON-basis for  $H$ .
- (3) (Parseval's Identity)

$$\|x\| = \left( \sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2 \right)^{\frac{1}{2}}, \quad \forall x \in H.$$

- (4)  $\langle x, y \rangle = \sum_{k=1}^{\infty} \langle x, u_k \rangle \overline{\langle y, u_k \rangle}$  for all  $x, y \in H$ .
- (5)  $\langle x, u_k \rangle = 0$  for all  $k \in \mathbb{N}$  follows  $x = 0$ .

**proof.** (1)  $\Rightarrow$  (2): We have

$$x = \sum_{i=1}^{\infty} \langle x, u_i \rangle u_i$$

it means

$$x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \langle x, u_i \rangle u_i \in \overline{\text{span}\{u_i \mid i \geq 1\}}.$$

This implies that any  $x \in H$  is in  $\overline{\text{span}\{u_i \mid i \geq 1\}}$ , i.e.  $\{u_i\}$  is ON-basis.

(2)  $\Rightarrow$  (5): Let  $\{u_i\}$  be a ON-basis. Assume

$$\langle x, u_k \rangle = 0, \quad \forall k \in \mathbb{N}.$$

Then

$$\langle x, u \rangle = 0, \quad \forall u \in \text{span}\{u_k \mid k \geq 1\}.$$

By the property that strong convergence implies weak convergence we will have

$$\langle x, u \rangle = 0, \quad \forall u \in \text{span}\{u_k \mid k \geq 1\} = H.$$

In particular

$$\langle x, u \rangle = 0, \quad \text{for } u = x$$

which means

$$\langle x, x \rangle = 0 \quad \Leftrightarrow \quad x = 0.$$

(5)  $\Rightarrow$  (1) Recall Bessel's equality. For  $\{u_k\}$ - ON-system then

$$\left\| x - \sum_{i=1}^k \langle x, u_i \rangle u_i \right\|^2 = \|x\|^2 - \sum_{i=1}^k |\langle x, u_i \rangle|^2$$

Assume (5), i.e.

$$\langle x, u_k \rangle = 0, \quad \forall k \quad \Rightarrow \quad x = 0$$

We must see

$$x = \sum_{k=1}^n \langle x, u_k \rangle u_k \quad \forall x \in H.$$

From Bessel's equality we have

$$\sum_{k=1}^n |\langle x, u_k \rangle|^2 = \|x\|^2 - \left\| x - \sum_{k=1}^n \langle x, u_k \rangle u_k \right\|^2 \leq \|x\|^2, \quad \forall k \in \mathbb{N}$$

and hence  $\sum_{k=1}^n |\langle x, u_k \rangle|^2$  is convergent. It implies that for  $n > m$  we have

$$\begin{aligned} \left\| \sum_{k=1}^n \langle x, u_k \rangle u_k - \sum_{k=1}^m \langle x, u_k \rangle u_k \right\|^2 &= \left\| \sum_{k=m+1}^n \langle x, u_k \rangle u_k \right\|^2 \\ &\stackrel{\text{pythagorean thm}}{=} \sum_{k=m+1}^n |\langle x, u_k \rangle|^2 \|u_k\|^2 \\ &\rightarrow 0, \quad n, m \rightarrow \infty \quad (*). \end{aligned}$$

Note that if  $\{x_i\}$  are pairwise orthogonal it holds

$$\left\| \sum_{i=1}^n x_i \right\|^2 = \sum_{i=1}^n \|x_i\|^2.$$

From (\*) we know that the partial sum

$$S_n := \sum_{k=1}^n \langle x, u_k \rangle u_k$$

is a Cauchy sequence. As  $H$  is a Hilbert space,  $H$  is complete and hence  $S_n$  has a limit in  $H$ . Write

$$\sum_{i=1}^{\infty} \langle x, u_i \rangle u_i$$

for the limit. We must see that the limit is  $x$ . Consider

$$y := x - \sum_{i=1}^{\infty} \langle x, u_i \rangle u_i.$$

Then

$$\langle y, u_i \rangle = \langle x, u_i \rangle - \langle x, u_i \rangle = 0, \quad \forall i.$$

By assumption (5) it follows

$$y = 0 \quad \Leftrightarrow \quad x = \sum_{i=1}^{\infty} \langle x, u_i \rangle u_i.$$

**(1)  $\Rightarrow$  (3):** From Bessel's equality we have again

$$\left\| x - \sum_{i=1}^n \langle x, u_i \rangle u_i \right\|^2 = \|x\|^2 - \sum_{i=1}^n |\langle x, u_i \rangle|^2.$$

By assumption (1) the LHS tends to 0 as  $n \rightarrow \infty$ . On the other hand the RHS goes to

$$\rightarrow \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, u_i \rangle|^2, \quad n \rightarrow \infty.$$

This gives

$$\|x\|^2 - \sum_{i=1}^{\infty} |\langle x, u_i \rangle|^2 = 0.$$

**(3)  $\Rightarrow$  (5)** trivial.

**(4)  $\Rightarrow$  (5)** trivial (take  $y = x$ ).

**(1)  $\Rightarrow$  (4)** We have

$$x = \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k.$$

Then

$$\langle x, y \rangle = \sum_{k=1}^{\infty} \langle x, u_k \rangle \langle u_k, y \rangle = \sum_{k=1}^{\infty} \langle x, u_k \rangle \overline{\langle y, u_k \rangle}.$$

□

**Example.**  $L^2([-\pi, \pi])$  with

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{int} \mid n \in \mathbb{Z} \right\}$$

is an ON-system in  $L^2([-\pi, \pi])$  where

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt.$$

**Statement 3.26.** The system above is an ON-basis for  $L^2([-\pi, \pi])$ . In particular, for any  $f \in L^2([-\pi, \pi])$

$$f = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikt}$$

convergent in the  $L^2$ -norm.

$$\|f\|_{L^2} = \left( \int_{-\pi}^{\pi} |f(t)|^2 dt \right)^{\frac{1}{2}}$$

which is equivalent to

$$\left\| f - \sum_{k=-n}^n \hat{f}(k) e^{ikt} \right\|_{L^2}^2 \rightarrow 0.$$

**Sketch of the proof:**

(1) Stein-Weierstraß-Theorem.  $X$  compact set  $C(X, \mathbb{C})$  continuous functions with complex values. Let  $M \subseteq C(X, \mathbb{C})$  be a subspace that satisfies:

(a) it separates points of  $X$ , i.e.

$$\forall x_1, x_2 \in X, x_1 \neq x_2 \exists f \in M : f(x_1) \neq f(x_2).$$

(b)  $M$  contains the constant function 1 ( $f(x) = 1$  for all  $x \in X$ ).

(c) It is closed under complex conjugation, i.e.

$$f \in M \Rightarrow \bar{f} \in M$$

and closed under product, i.e.

$$f_1, f_2 \in M \Rightarrow f_1 \cdot f_2 \in M.$$

Then  $M$  is dense in  $C(X, \mathbb{C})$  w.r.t.  $\|\cdot\|_{\infty}$  (Continuous function by Polynomials) From this it follows

$$M = \{\text{all complex polynomials}\}$$

are dense in  $C([a, b])$ .

(2)  $C([a, b])$  is dense in  $L^2([a, b])$  w.r.t.  $\|\cdot\|_{L^2}$ -norm.



We will use (1) and (2) to show that  $\text{span}\left\{\frac{1}{\sqrt{2\pi}}e^{int} \mid n \in \mathbb{Z}\right\}$  is dense in  $L^2([-\pi, \pi])$ .

**proof.** Let

$$M := \text{span}\left\{\frac{1}{\sqrt{2\pi}}e^{int} \mid n \in \mathbb{Z}\right\} \subseteq \{f \in C([-\pi, \pi]) \mid f(\pi) = f(-\pi)\}.$$

$M$  separates points, it contains the constant function 1 and it is closed under complex conjugation. Furthermore  $M$  is closed under taking products. With Stein-Weierstraß it follows that  $M$  is dense in

$$\{f \in C([-\pi, \pi]) \mid f(\pi) = f(-\pi)\}.$$

By (2) we have  $C([-\pi, \pi])$  is dense in  $L^2([-\pi, \pi])$  w.r.t. the  $L^2$ -norm. From this one can see that even  $\{f \in C([-\pi, \pi]) \mid f(\pi) = f(-\pi)\}$  is dense in  $L^2([-\pi, \pi])$ :

$$\forall \varepsilon > 0, \forall f \in L^2 \exists g \in C([-\pi, \pi]) : \quad \|f - g\|_{L^2}^2 = \int_{-\pi}^{\pi} |f(t) - g(t)|^2 dt < \varepsilon.$$

Define  $g_\varepsilon$  such that it has a peak in  $x = \pi - \varepsilon$  but it is continuous and is equal to  $g$  for  $x < \pi - \varepsilon$ . Then

$$g_\varepsilon \in C([-\pi, \pi]), \quad g_\varepsilon(-\pi) = g_\varepsilon(\pi).$$

It holds

$$\begin{aligned} \|f - g_\varepsilon\|_{L^2} &\leq \underbrace{\|f - g\|_{L^2}}_{< \sqrt{\varepsilon}} + \|g - g_\varepsilon\|_{L^2} \\ &\leq \sqrt{\varepsilon} + \left( \int_{\pi-\varepsilon}^{\pi} |g(t) - g_\varepsilon(t)|^2 dt \right)^{\frac{1}{2}} \\ &\leq \sqrt{\varepsilon} + \sqrt{\max_{x \in [-\pi-\varepsilon, \pi]} |g - g_\varepsilon| \varepsilon} \\ &= \sqrt{\varepsilon} + \sqrt{C} \sqrt{\varepsilon}. \end{aligned}$$

We conclude: any  $f = L^2$ -limit  $g_n$  with  $g_n \in C([-\pi, \pi])$  and  $g_n(-\pi) = g_n(\pi)$ . Each  $g_n = \|\cdot\|_\infty$ -norm limit of an element in  $\text{span}\left\{\frac{1}{\sqrt{2\pi}}e^{int} \mid n \in \mathbb{Z}\right\}$  as

$$\|g - f\|_{L^2} \leq \|g - f\|_\infty^{\frac{1}{2}} (2\pi)^{\frac{1}{2}}.$$

Note that

$$\left( \int_{-\pi}^{\pi} |g(t) - f(t)|^2 dt \right)^{\frac{1}{2}} \leq \max_{x \in [-\pi, \pi]} |g(t) - f(t)| \left( \int_{-\pi}^{\pi} dt \right)^{\frac{1}{2}}.$$

We get that each  $g_n$  can be approximated in the  $L^2$ -norm by elements in  $\text{span}\left\{\frac{1}{\sqrt{2\pi}}e^{int} \mid n \in \mathbb{Z}\right\}$  hence

$$\text{span}\left\{\frac{1}{\sqrt{2\pi}}e^{int} \mid n \in \mathbb{Z}\right\} \subseteq L^2([-\pi, \pi]).$$

□

### 3.4 Linear operators on Hilbert spaces

Set  $(H_1, \langle \cdot, \cdot \rangle_1)$  and  $(H_2, \langle \cdot, \cdot \rangle_2)$  Hilbert spaces. A bounded linear mapping  $A : H_1 \rightarrow H_2$  is called bounded linear operator.

Bounded means in our case

$$\|Ax\|_2 \leq C\|x\|_1 \quad \forall x \in H \text{ and some constant } C$$

**Example.** Set  $H_1 = H_2 = L^2([0, 1])$  and  $K : [0, 1] \times [0, 1] \rightarrow \mathbb{C}$ . Assume that  $K$  is continuous. Consider

$$(Af)(x) = \int_0^1 K(x, y)f(y) dy.$$

$A$  is linear (trivial). Show that  $A$  is bounded:

$$\begin{aligned} \|Af\|_2 &= \int_0^1 \left| \int_0^1 K(x, y)f(y) dy \right|^2 dx \\ &\stackrel{\text{CS}}{\leq} \int_0^1 \left( \int_0^1 |K(x, y)|^2 dy \cdot \int_0^1 |f(y)|^2 dy \right) dx \\ &= \underbrace{\int_0^1 \left( \int_0^1 |K(x, y)|^2 dy \right) dx}_{< \infty} \cdot \underbrace{\int_0^1 |f(y)|^2 dy}_{=\|f\|_2^2}. \end{aligned}$$

Hence

$$\|A\| \leq \left( \int_0^1 \int_0^1 |K(x, y)|^2 dx dy \right)^{\frac{1}{2}}.$$

Products  $(A \cdot B)$  of operators  $H \rightarrow H$  with  $A : H \rightarrow H$  and  $B : H \rightarrow H$  are defined by

$$(A \cdot B)(f) := A(Bf).$$

**Statement 3.27.** If  $A$  and  $B$  are bounded, then  $A \cdot B$  is also bounded and

$$\|AB\| \leq \|A\|\|B\|.$$

In particular: for all  $n \in \mathbb{N}$   $A^n$  is bounded and

$$\|A^n\| \leq \|A\|^n.$$

**Example.**  $E = L^2([0, 1])$  and  $f, g \in E$  with

$$\langle f, g \rangle_{L^2} = \int_0^1 f(x) \overline{g(x)} dx, \quad \|f\|_{L^2} = \left( \int_0^1 |f(x)|^2 dx \right)^{\frac{1}{2}}.$$

Set  $h \in C([0, 1] \times [0, 1])$  and for  $f \in L^2([0, 1])$

$$A(f)(x) = \int_0^1 h(x, y)f(y) \, dy, \quad x \in [0, 1].$$

Then

$$\|A\| \leq \left( \int_0^1 \left( \int_0^1 |h(x, y)|^2 \, dy \right) \, dx \right)^{\frac{1}{2}} < \infty.$$

**Example.** Let  $(E, \|\cdot\|)$  be a normed space. Then there are no  $A, B \in B(E, E)$  such that

$$AB - BA = I$$

where  $I$  is the identity ( $I(x) = x$  for  $x \in E$ ).

**Remark.** Consider  $f \in E = C^\infty([0, 1])$  and

$$A = \frac{d}{dx}, \quad B = x.$$

Then

$$(AB - BA)(f)(x) = \frac{d}{dx}(x(f(x))) - x \frac{d}{dx}f(x) = f(x).$$

Argue by contradiction.

Assume  $A, B \in B(E, E)$  with  $AB - BA = I$ .

Hint: Consider  $A^n B - BA^n$  for  $n = 1, 2, \dots$ . For  $n = 2$  we have

$$\begin{aligned} A^2 B - BA^2 &= A^2 B - ABA + ABA - BA^2 \\ &= A(AB - BA) + (AB - BA)A \\ &= 2A. \end{aligned}$$

For  $n = 3$  we have

$$\begin{aligned} A^3 B - BA^3 &= A^3 B - A^2 BA + A^2 BA - BA^3 \\ &= A^2(AB - BA) + (A^2 B - BA^2)A \\ &= 3A^2. \end{aligned}$$

In general

$$A^n B - BA^n = nA^{n-1}, \quad n = 2, 3, 4, \dots \quad (*)$$

Check using an induction argument. We obtain

$$n\|A^{n-1}\| = \|A^n B - BA^n\| \leq \|A^n B\| + \|BA^n\| \leq 2\|A^{n-1}\|\|A\|\|B\|$$

Hence

$$(2\|A\|\|B\| - n)\|A^{n-1}\| \geq 0, \quad \forall n = 2, 3, \dots$$

We conclude that  $\|A^{n-1}\| = 0$  for  $n$  large enough. Clearly the same for  $\|A^n\|$ . This yields  $A^n = 0$  for  $n$  large enough. Repeated use of  $(*)$  gives  $A = 0$ . This contradicts  $AB - BA = I$  so the implication in the example is proven.

Recall a important theorem:

**Theorem 3.28** (Riesz representation theorem).  $(E, \langle \cdot, \cdot \rangle)$  Hilbert space  $f \in B(E, \mathbb{C})$ .  $f$  is bounded linear functional on  $E$ . This yields

$$\exists ! x_f \in E : \quad f(x) = \langle x, x_f \rangle, \quad \forall x \in E.$$

Also it holds

$$\underbrace{\|f\|}_{\text{operator norm of } f} = \underbrace{\|x_f\|}_{\text{norm of } x_f \text{ in } E}.$$

**Definition 3.29.**  $\varphi : E \times E \rightarrow \mathbb{C}$  is called:

- Bilinear, if for scalars  $\alpha$  and  $\beta$  it holds

$$\begin{aligned} \varphi(\alpha x + \beta y, z) &= \alpha \varphi(x, z) + \beta \varphi(y, z) & \forall x, y, z \in E \\ \varphi(x, \alpha y + \beta z) &= \bar{\alpha} \varphi(x, y) + \bar{\beta} \varphi(x, z) & \forall x, y, z \in E. \end{aligned}$$

- Bounded, if there exists  $M > 0$  such that

$$|\varphi(x, y)| \leq M \|x\| \|y\|, \quad \forall x, y \in E.$$

- Coercive, if there exists  $K > 0$  such that

$$\varphi(x, x) \geq K \|x\|^2, \quad \forall x \in E.$$

Clearly  $\langle \cdot, \cdot \rangle$  in  $E$  is a bilinear, bounded and coercive functional in  $E$  (with  $M = K = 1$ ). We will now introduce a Generalization of the Riesz representation theorem.

**Theorem 3.30** (Lax-Milgram).  $(E, \langle \cdot, \cdot \rangle)$  Hilbert space. Let  $\varphi : E \times E \rightarrow \mathbb{C}$  be a bilinear, bounded and coercive functional.  $f : E \rightarrow \mathbb{C}$  bounded linear functional in  $E$ . Then there exists an unique  $x_f \in E$  such that

$$f(x) = \varphi(x, x_f), \quad \forall x \in E.$$

**proof.** Step 1:  $\exists ! A \in B(E, E)$  with

$$\varphi(x, y) = \langle x, A(y) \rangle, \quad \forall x, y \in E.$$

Step 2:  $A$  is injective and surjective.

Step 3: Apply RRT with  $\tilde{x}_f = A^{-1}(x_f)$

$$\begin{aligned} f(x) &= \langle x, x_f \rangle \\ &= \langle x, A(A^{-1}(x_f)) \rangle \\ &= \varphi(x, \tilde{x}_f), \quad \forall x \in E. \end{aligned}$$

**Step 1:** Fix  $y \in E$  and consider for  $x \in E$

$$x \mapsto f_y(x) \in \mathbb{C}.$$

**Claim:**  $f_y : E \rightarrow \mathbb{C}$  is a bounded linear functional.

For  $x, y, z \in E$  and  $\alpha, \beta$  scalars we have

$$\begin{aligned} f_y(\alpha x + \beta z) &= \varphi(\alpha x + \beta z, y) \\ &= \alpha \varphi(x, y) + \beta \varphi(z, y) \\ &= \alpha f_y(x) + \beta f_y(z). \end{aligned}$$

Hence  $f_y$  is linear. It is bounded because of

$$|f_y(x)| = |\varphi(x, y)| \leq (M\|y\|)\|x\|, \quad \forall x \in E.$$

So  $f_y$  is bounded.

RRT implies  $f_y(x) = \langle x, A(y) \rangle$  for all  $x \in E$  for some  $A(y) \in E$ .

Now we have  $A : E \rightarrow E$ . **Claim:**  $A \in B(E, E)$ .

For  $x, y, z \in E$  and scalars  $\alpha, \beta$  we have

$$\begin{aligned} \langle x, A(\alpha y + \beta z) \rangle &= \varphi(x, \alpha y + \beta z) \\ &= \bar{\alpha} \varphi(x, y) + \bar{\beta} \varphi(x, z) \\ &= \bar{\alpha} \langle x, A(y) \rangle + \bar{\beta} \langle x, A(z) \rangle \\ &= \langle x, \alpha A(y) \rangle + \langle x, \beta A(z) \rangle. \end{aligned}$$

This is equivalent to

$$\langle x, A(\alpha y + \beta z) - \alpha A(y) - \beta A(z) \rangle = 0, \quad x \in E.$$

This implies

$$\|A(\alpha y + \beta z) - \alpha A(y) - \beta A(z)\| = 0.$$

So

$$A(\alpha y + \beta z) = \alpha A(y) + \beta A(z) \quad \forall y, z \in E \text{ and scalars } \beta, \alpha.$$

Hence,  $A$  is linear. We will now show that  $A$  is bounded:

We know because  $\varphi$  is continuous that for all  $x, y \in E$

$$|\langle x, A(y) \rangle| = |\varphi(x, y)| \leq M\|x\|\|y\|.$$

Take  $x = A(y)$  and get

$$\|A(y)\|^2 \leq M\|A(y)\|\|y\| \quad \forall y \in E$$

which implies

$$\|A(y)\| \leq M\|y\| \quad \forall y \in E.$$

Hence  $\|A\| \leq M < \infty$ .

**Step 2:** Note  $\varphi(x, y) = \langle x, A(y) \rangle$  for alle  $x, y \in E$ .

**Claim:**  $A$  is injective, i.e.

$$A(x_1) = A(x_2) \quad \Rightarrow \quad x_1 = x_2.$$

$\varphi$  is coercive so

$$\|x\|^2 \leq \frac{\varphi(x, x)}{K} = \frac{1}{K} \underbrace{|\langle x, A(x) \rangle|}_{>0} \leq \frac{1}{K} \|x\| \|A(x)\| \quad \forall x \in E.$$

Hence

$$\|x\| \leq \frac{1}{K} \|A(x)\|, \quad \forall x \in E.$$

If  $A(x_1) = A(x_2)$  we have  $A(x_1 - x_2) = 0 \in E$  then

$$\|x_1 - x_2\| \leq \frac{1}{K} \|A(x_1 - x_2)\| = 0.$$

We get  $x_1 = x_2$ .

**Claim:**  $A$  is surjective, i.e. the image of  $A$  is  $E$ :

$$\mathcal{R}(A) = \{A(x) \mid x \in E\} = E.$$

We first show that  $\mathcal{R}(A)$  is a closed subspace of  $E$ .

- $\mathcal{R}(A)$  is a subspace in  $E$  since  $A$  is linear.
- $\mathcal{R}(A)$  is closed

$$y_n \rightarrow y \quad \text{in } (E, \|\cdot\|) \quad \Rightarrow \quad y \in \mathcal{R}(A).$$

$\mathcal{R}(A)$  is linear. Take  $y_1, y_2 \in \mathcal{R}(A)$  with preimages  $x_1, x_2$  and yield

$$\alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 A(x_1) + \alpha_2 A(x_2) = A(\alpha_1 x_1 + \alpha_2 x_2).$$

So

$$\alpha_1 y_1 + \alpha_2 y_2 \in \mathcal{R}(A).$$

Assume

$$y_n \rightarrow y \quad \text{in } (E, \|\cdot\|).$$

For  $n = 1, 2, \dots$  there are  $x_1, x_2, \dots$  such that  $y_n = A(x_n)$  for  $n = 1, 2, \dots$

**Claim:**  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $E$  since

$$\begin{aligned} \|x_n - x_m\| &\leq \frac{1}{K} \|A(x_n - x_m)\| \\ &= \frac{1}{K} \|A(x_n) - A(x_m)\| \\ &= \frac{1}{K} \|y_n - y_m\| \rightarrow 0, \quad n, m \rightarrow \infty \end{aligned}$$

since  $(y_n)_{n \in \mathbb{N}}$  converges.

Since  $(E, \|\cdot\|)$  is a Banach space  $(x_n)_{n \in \mathbb{N}}$  converges in  $(E, \|\cdot\|)$ . Call the limit  $x \in E$ . Hence

$$A(x_n) \rightarrow y$$

since  $A$  is bounded, continuous and linear. So  $y = A(x)$  and we get  $y \in \mathcal{R}(A)$ .

Secondly  $A$  is surjective, i.e.  $\mathcal{R}(A) = E$ .

Assume that this is not true. The Orthogonal decomposition theorem gives

$$E = \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp.$$

The first one is a closed subspace in  $E$  and the second one is not empty by assumption.

Fix  $z \in \mathcal{R}(A)^\perp \setminus \{0\}$ . Note

$$\varphi(x, y) = \langle x, A(y) \rangle \quad x, y \in E$$

With  $x = y = z$  we get

$$\varphi(z, z) = \langle z, A(z) \rangle = 0$$

and

$$\varphi(z, z) \geq K\|z\|^2 \geq 0 \quad \Rightarrow \quad z = 0.$$

This is a contradiction.

The Conclusion is

$$\mathcal{R}(A)^\perp = \{0\} \quad \Rightarrow \quad \mathcal{R}(A) = E.$$

We have  $\varphi(x, y) = \langle x, A(y) \rangle$  for all  $x, y \in E$  and  $A \in B(E, E)$  surjective.

**Step 3:** see above.

□

### 3.5 Adjoint operator

$(E, \langle \cdot, \cdot \rangle)$  Hilbert space and  $A \in B(E, E)$  with adjoint  $A^*$ , i.e.

$$\langle A(x), y \rangle = \langle x, A^*(y) \rangle, \quad \forall x, y \in E.$$

Fix  $y \in E$  and consider

$$x \mapsto \langle A(x), y \rangle \in \mathbb{C}.$$

**Claim:**  $f_y$  is a bounded linear functional on  $E$

- linear since  $A$  is linear.
- bounded since  $A$  is bounded with

$$|f_y(x)| \leq (\|A\| \|y\|) \|x\|, \quad x \in E.$$

RRT implies

$$f_y(x) = \langle x, A^*(y) \rangle, \quad x \in E.$$

We have  $A^* : E \rightarrow E$  such that

$$\langle A(x), y \rangle = \langle x, A^*(y) \rangle, \quad \forall x, y \in E.$$

**Proposition 3.31.**  $A \in B(E, E)$ . Then  $A^* \in B(E, E)$  and  $\|A^*\| = \|A\|$ .

**proof.**  $A^*$  linear:

$$\langle x, A^*(\alpha y + \beta z) \rangle = \langle x, \alpha A^*(y) + \beta A^*(z) \rangle \quad \forall x, y \in E.$$

$A^*$  bounded:

Take  $x = A^*(y)$  and get

$$\begin{aligned} \|A^*(y)\|^2 &= |\langle A^*(y), A^*(y) \rangle| \\ &\leq \|A^*(y)\| \|y\| \\ &\leq \|A\| \|A^*(y)\| \|y\|, \quad y \in E. \end{aligned}$$

We get

$$\|A^*(y)\| \leq \|A\| \|y\|, \quad y \in E.$$

Conclusion:  $A^* \in B(E, E)$ . We also get

$$\|A^*\| \leq \|A\|.$$

But we also know that  $A^{**} = A$  since

$$\begin{aligned} \langle x, A^{**}(y) \rangle &= \langle A^*(x), y \rangle \\ &= \overline{\langle y, A^*(x) \rangle} \\ &= \overline{\langle A(y), x \rangle} \\ &= \langle x, A(y) \rangle, \quad x, y \in E. \end{aligned}$$

So

$$\|A\| = \|A^{**}\| \leq \|A^*\|$$

which implies

$$\|A\| = \|A^*\|.$$

□

**Remark.**  $A, B \in B(E, E)$  then

$$\begin{aligned} (A + B)^* &= A^* + B^* \\ (AB)^* &= B^* A^* \\ (\alpha A)^* &= \bar{\alpha} A^* \\ A^{**} &= A \\ I^* &= I. \end{aligned}$$



**Example.** Continuity of the example above: For  $f \in L^2([0, 1])$  consider

$$A(f)(x) = \int_0^1 h(x, y)f(y) \, dy, \quad x \in [0, 1].$$

For  $g \in L^2([0, 1])$  it holds

$$\begin{aligned} \langle A(f), g \rangle_{L^2} &= \int_0^1 A(f)(x) \overline{g(x)} \, dx \\ &= \int_0^1 \int_0^1 h(x, y)f(y) \, dx \overline{g(x)} \, dx \\ &= \int_0^1 f(y) \cdot \int_0^1 h(x, y) \overline{g(x)} \, dx \, dy \\ &= \int_0^1 f(y) \cdot \overline{\int_0^1 h(x, y)g(x) \, dx} \, dy \\ &= \langle f, A^*(g) \rangle_{L^2}. \end{aligned}$$

This gives us

$$A^*(f)(x) = \int_0^1 \overline{h(y, x)}f(y) \, dy, \quad x \in [0, 1].$$

**Example.**  $A \in B(E, E)$ . It follows

$$\mathcal{R}(A)^\perp = N(A^*) = \{x \in E \mid A^*(x) = 0\}$$

since  $x \in \mathcal{R}(A)^\perp$ . It is equivalent that

$$\begin{aligned} \langle x, A(y) \rangle &= 0, \quad \forall y \in E \\ \Leftrightarrow \quad \langle A^*(x), y \rangle &= 0, \quad \forall y \in E \\ \Rightarrow \quad A^*(x) &= 0 \quad \Leftrightarrow \quad x \in N(A^*). \end{aligned}$$

We get

$$N(A^*)^\perp = \overline{\mathcal{R}(A)}$$

since

$$N(A^*)^\perp = \left( \mathcal{R}(A)^\perp \right)^\perp = \overline{\text{span}(\mathcal{R}(A))} = \overline{\mathcal{R}(A)}.$$

**Remark.**  $A \in B(E, E)$  is called self adjoint if  $A^* = A$ .

For  $A \in B(E, E)$  we have

$$\|A\| = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |\langle A(x), y \rangle|$$

since

$$\|\langle A(x), y \rangle\| \leq \underbrace{\|A(x)\|}_{\leq \|A\|\|x\|} \leq \|A\|, \quad \text{for } \|x\| = \|y\| = 1.$$

If  $A(x) = 0$  for all  $x \in E$  then  $\|A\| = 0$  and also

$$\sup_{\substack{\|x\|=1 \\ \|y\|=1}} |\langle A(x), y \rangle| = 0.$$

For  $x$  with  $A(x) \neq 0$  then it is

$$A\left(\frac{1}{\|x\|}x\right) \neq 0.$$

For such an  $x$  with  $\|x\| = 1$  we have

$$|\langle A(x), \frac{1}{\|A(x)\|}A(x) \rangle| = \frac{1}{\|A(x)\|} \|A(x)\|^2 = \|A(x)\|$$

and

$$\|A\| \leq \sup_{\|x\|=1} \|A(x)\| \leq \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |\langle A(x), y \rangle| \leq \|A\|.$$

**Proposition 3.32.** Let  $A \in B(E, E)$  be self-adjoint. Then

$$\|A\| = \sup_{\|x\|=1} |\langle A(x), x \rangle|.$$

**proof.** Set

$$M = \sup_{\|x\|=1} |\langle A(x), x \rangle|.$$

For  $\|x\| = 1$  we have

$$|\langle A(x), x \rangle| \leq \|A(x)\| \|x\| \leq \|A\|.$$

Furthermore

$$M \leq \|A\|.$$

It remains to prove:  $\|A\| \leq M$ .

For  $x, z \in E$  consider:

$$\begin{aligned} \langle A(x+z), x+z \rangle - \langle A(x-z), x-z \rangle &= 2\langle A(x), z \rangle + 2\langle A(z), x \rangle \\ &= 2(\langle A(x), z \rangle + \langle z, A^*(x) \rangle) \\ &= 2(\langle A(x), z \rangle + \langle z, A(x) \rangle) \\ &= 4 \operatorname{Re}(\langle A(x), z \rangle). \end{aligned}$$

Assume now  $A(x) \neq 0$  and set

$$z = \frac{1}{\|A(x)\|} A(x).$$

Hence

$$\|A(x)\| = \frac{1}{4} \left( \langle A(x + \frac{1}{\|A(x)\|} A(x)), x + \frac{1}{\|A(x)\|} A(x) \rangle - \langle A(x - \frac{1}{\|A(x)\|} A(x)), x - \frac{1}{\|A(x)\|} A(x) \rangle \right).$$

Note

$$|\langle A(y), y \rangle| = \|y\|^2 |\langle A(\frac{1}{\|y\|}y), \frac{1}{\|y\|}y \rangle| \leq M\|y\|^2.$$

We now obtain

$$\begin{aligned} \|A(x)\| &\leq \frac{1}{4} \left( M \left\| x + \frac{1}{\|A(x)\|} A(x) \right\|^2 + M \left\| x - \frac{1}{\|A(x)\|} A(x) \right\|^2 \right) \\ &= \frac{M}{4} 2 \left( \|x\|^2 + \left\| \frac{1}{\|A(x)\|} A(x) \right\|^2 \right) \\ &= \frac{M}{2} (\|x\|^2 + 1). \end{aligned}$$

So

$$\|A\| = \sup_{\|x\|=1} \|A(x)\| \leq M$$

and this yields

$$\|A\| = M.$$

□

**Definition 3.33 (compact).** If  $A : E \rightarrow E$  is linear, then we say that  $A$  is compact if for all bounded sequences  $(x_n)_{n=1}^\infty$  in  $E$ ,  $(A(x_n))_{n=1}^\infty$  has a convergent subsequence in  $E$ .

**Lemma 3.34.**  $A$  is compact and linear  $\Rightarrow A$  is bounded.

**proof.** If  $A$  is not bounded then there exists a sequence  $(y_n)_{n=1}^\infty$  in  $E$  such that

$$\|A(y_n)\| \geq n\|y_n\|, \quad \text{for } n = 1, 2, \dots$$

Set  $x_n = \frac{y_n}{\|y_n\|}$  for  $n = 1, 2, \dots$ . Here  $\|x_n\| = 1$  for all  $n \in \mathbb{N}$  and

$$\|A(x_n)\| = \left\| A \left( \frac{1}{\|y_n\|} y_n \right) \right\| = \frac{1}{\|y_n\|} \|A(y_n)\| > n, \quad \forall n \in \mathbb{N}.$$

$(A(x_n))_{n=1}^\infty$  has no converging subsequence since  $\|A(x_n)\| \rightarrow \infty$  for  $n \rightarrow \infty$ . □

**Remark.** •  $A \in B(E, E)$  and  $F \subset E$  where  $F$  is bounded. Then

$$A(F) = \{A(x) \mid x \in F\}$$

is bounded.

•  $A \in B(E, E)$  compact and  $F \subset E$ ,  $F$  bounded. Then  $\overline{A(F)}$  is compact.

**Lemma 3.35.**  $A, B$  compact linear operators  $E \rightarrow E$  and  $\alpha$  and  $\beta$  scalars. Then  $\alpha A + \beta B$  is compact.

**proof.** Fix an arbitrary bounded sequence  $(x_n)_{n=1}^\infty$  in  $E$ . Since  $A$  is compact there exists a converging subsequence  $(A(x_{n_k}))_{k=1}^\infty$  of  $(A(x_n))_{n=1}^\infty$ .

Clearly  $(\alpha A(x_n))_{n=1}^\infty$  converges in  $E$ .

Since  $B$  is compact there exists a converging subsequence  $(B(x_{n_k}))_{k=1}^\infty$  of  $(B(x_n))_{n=1}^\infty$ .

Clearly  $(\beta B(x_{n_k}))_{k=1}^\infty$  converges in  $E$ . Hence

$$(\alpha A(x_{n_k}) + \beta B(x_{n_k}))_{k=1}^\infty = ((\alpha A + \beta B)(x_{n_k}))_{k=1}^\infty$$

converges in  $E$ . □

Set

$$K(E, E) := \text{set of all compact linear mappings } E \rightarrow E.$$

We have  $K(E, E)$  is a subspace in  $(B(E, E), \|\cdot\|_{E \rightarrow E})$ .

**Proposition 3.36.**  $K(E, E)$  is a closed subspace in  $(B(E, E), \|\cdot\|_{E \rightarrow E})$ .

Before the proof we note:

1. Assume  $(E, \langle \cdot, \cdot \rangle)$  to be a Hilbert space and  $A \in B(E, E)$ .

$$\begin{aligned} x_n \rightarrow x \text{ in } E &\Rightarrow A(x_n) \rightarrow A(x) \text{ in } E \\ x_n \rightharpoonup x \text{ in } E &\Rightarrow A(x_n) \rightharpoonup A(x) \text{ in } E \end{aligned}$$

since for  $y \in E$  we have

$$\langle A(x_n), y \rangle = \langle x_n, A^*(y) \rangle \xrightarrow{n \rightarrow \infty} \langle x, A^*(y) \rangle = \langle A(x), y \rangle.$$

2.  $A \in K(E, E)$  and  $x_n \rightharpoonup x$  in  $E$

$$\Rightarrow A(x_n) \rightarrow A(x) \text{ in } E.$$

3.  $A \in B(E, E)$  finite-rank operator, i.e.

$$\dim \mathcal{R}(A) < \infty \Rightarrow A \in K(E, E)$$

since: Let  $e_1, e_2, \dots, e_N$  be an ON-basis for  $\mathcal{R}(A)$  with  $N = \dim(\mathcal{R}(A))$ . We have

$$A(x) = \langle A(x), e_1 \rangle e_1 + \dots + \langle A(x), e_N \rangle e_N.$$

Fix an arbitrary bounded sequence  $(x_n)_{n=1}^\infty$  in  $E$ .  $A$  is bounded which implies that  $(A(x_n))_{n=1}^\infty$  is a bounded sequence. Furthermore

$$(\langle A(x_n), e_1 \rangle)_{n=1}^\infty$$

is a bounded sequence in  $\mathbb{C}$ . Bolzano Weierstrass theorem implies that  $(\langle A(x_n), e_1 \rangle)_{n=1}^\infty$  has a converging subsequence  $(\langle A(x_{n_k}), e_1 \rangle)_{k=1}^\infty$ . Clearly  $(\langle A(x_{n_k}), e_1 \rangle)_{k=1}^\infty$  converges in  $E$ .

Hence

$$A(x) = \langle A(x), e_1 \rangle e_1 + \dots + \langle A(x), e_N \rangle e_N$$

is a compact mapping since  $K(E, E)$  is a subspace of  $B(E, E)$ .

**proof.** Assume  $(A_n)_{n=1}^\infty \subseteq K(E, E)$  such that  $A_n \rightarrow A$  in  $(B(E, E), \|\cdot\|_{E \rightarrow E})$ .

We have to show:  $A \in K(E, E)$

Fix an arbitrary bounded sequence  $(x_n)_{n=1}^\infty$  in  $E$ . We want to show that  $(A(x_n))_{n=1}^\infty$  has a converging subsequence in  $E$ .

Set

$$M = \sup_n \|x_n\| < \infty.$$

$$\begin{aligned} A_1 \in K(E, E) &\Rightarrow (A_1(x_n))_{n=1}^\infty \text{ has a converging subsequence } (A_1(x_{n_1}))_{n=1}^\infty \\ A_2 \in K(E, E) &\Rightarrow (A_2(x_{n_1}))_{n=1}^\infty \text{ has a converging subsequence } (A_2(x_{n_2}))_{n=1}^\infty \end{aligned}$$

proceed inductively:

$$A_k \in K(E, E) \Rightarrow (A_k(x_{n_{k-1}}))_{n=1}^\infty \text{ has a converging subsequence } (A_k(x_{n_k}))_{n=1}^\infty$$

Also:  $(A_l(x_{n_k}))_{n=1}^\infty$  converges in  $E$  for  $l = 1, 2, \dots, k$ .

Here  $(A_k(y_n))_{n=1}^\infty$  converges for  $k = 1, 2, \dots$

So since  $(E, \|\cdot\|)$  is a Banach space it is enough to show that  $(A(y_n))_{n=1}^\infty$  is a Cauchy sequence in  $(E, \|\cdot\|)$ .

Fix an arbitrary  $\varepsilon > 0$ . We have

$$\|A(y_n) - A(y_m)\| \leq \underbrace{\|A(y_n) - A_k(y_n)\|}_{\leq \|A - A_k\|_{E \rightarrow E} \|y_n\|} + \|A_k(y_n) - A_k(y_m)\| + \|A_k(y_m) - A(y_m)\|.$$

Fix  $k$  large enough such that

$$\|A_k - A\| < \frac{\varepsilon}{3M}.$$

Then

$$\|A(y_n) - A(y_m)\| < \frac{2}{3}\varepsilon + \|A_k(y_n) - A_k(y_m)\|$$

$(A_k(y_n))_{n=1}^\infty$  converges in  $E$ . This implies the existence of  $N$  such that

$$\forall n, m \geq N : \|A_k(y_n) - A_k(y_m)\| < \varepsilon$$

$$\Rightarrow \|A(y_n) - A(y_m)\| < \varepsilon, \quad \forall n, m \geq N$$

and thus  $(A(y_n))_{n=1}^\infty$  is a Cauchy sequence. □

**Proposition 3.37.** Let  $(E, \langle \cdot, \cdot \rangle)$  be a separable Hilbert space and  $A \in K(E, E)$ . then there exist finite-ranked operators  $A_n \in K(E, E)$  such that

$$\|A - A_n\|_{E \rightarrow E} \rightarrow 0, \quad n \rightarrow \infty.$$

**proof.** Let  $(x_n)_{n=1}^\infty$  be an ON-basis for  $E$ . For

$$x = \sum_{k=1}^{\infty} \langle x, x_k \rangle x_k, \quad x \in E.$$

Set

$$A_n(x) = A \left( \sum_{k=1}^n \langle x, x_k \rangle x_k \right) = \sum_{k=1}^n \langle x, x_k \rangle A(x_k), \quad x \in E, \quad n = 1, 2, \dots$$

Here  $\dim(\mathcal{R}(A_n)) \leq n$  for  $n = 1, 2, \dots$

So  $A_n$  is a finite ranked operator in  $E$  for  $n = 1, 2, \dots$

Fix  $x \in E$  with  $\|x\| = 1$  and consider:

$$\|(A - A_n)(x)\|^2 = \left\| A \left( \sum_{k=n+1}^{\infty} \langle x, x_k \rangle x_k \right) \right\|^2 \leq \sup_{\substack{\|y\|=1, \\ y \in \{x_1, \dots, x_n\}^\perp}} \|A(y)\|^2$$

and thus

$$\|A - A_n\|_{E \rightarrow E}^2 \leq \sup_{\substack{\|y\|=1, \\ y \in \{x_1, \dots, x_n\}^\perp}} \|A(y)\|^2.$$

Set

$$u_n := \sup_{\substack{\|y\|=1, \\ y \in \{x_1, \dots, x_n\}^\perp}} \|A(y)\|^2 < \infty, \quad n = 1, 2, \dots$$

Here  $a_n \geq a_{n+1} \geq 0$  for  $n = 1, 2, \dots$

Clearly  $(a_n)_{n=1}^\infty$  converges in  $\mathbb{R}$ . Set  $a = \lim_{n \rightarrow \infty} a_n$ . It remains to prove  $a = 0$ . Assume  $a > 0$ . Then there exists  $(y_n)_{n=1}^\infty$  in  $E$  such that

1.  $\|y_n\| = 1$ ,
2.  $y \in \{x_1, \dots, x_n\}^\perp$ ,
3.  $\|A(y_n)\|^2 \geq \frac{1}{2}a$ .

**Claim:**  $y_n \rightharpoonup 0$  in  $(E, \langle \cdot, \cdot \rangle)$  since:

Fix an arbitrary  $x \in E$  and

$$\begin{aligned}
 |\langle y_n, x \rangle| &= |\langle y_n, \sum_{k=1}^{\infty} \langle x, x_k \rangle x_k \rangle| \\
 &= |\langle y_n, \sum_{k=n+1}^{\infty} \langle x, x_k \rangle x_k \rangle| \\
 &\leq \|y_n\| \cdot \left\| \sum_{k=n+1}^{\infty} \langle x, x_k \rangle x_k \right\| \\
 &= \sqrt{\sum_{k=n+1}^{\infty} |\langle x, x_k \rangle|^2} \rightarrow 0, \quad n \rightarrow \infty.
 \end{aligned}$$

(Note that  $\sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 = \|x\|^2 < \infty$ )

We have  $y_n \rightarrow 0$  in  $(E, \langle \cdot, \cdot \rangle)$  and

$$A \in B(E, E) \quad \Rightarrow \quad A(y_n) \rightarrow A(0) = 0.$$

Contradiction to (3) above which gives us  $a = 0$ . □

**Proposition 3.38.**  $(E, \langle \cdot, \cdot \rangle)$  Hilbert space and  $A \in K(E, E)$ . Then

$$x_n \rightharpoonup x \text{ in } (E, \langle \cdot, \cdot \rangle) \quad \Rightarrow \quad A(x_n) \rightarrow A(x) \text{ in } (E, \langle \cdot, \cdot \rangle).$$

**proof.**  $x_n \rightharpoonup x$  in  $(E, \langle \cdot, \cdot \rangle)$  implies that  $\sup_n \|x_n\| < \infty$  (according to important theorem). Since  $A \in K(E, E)$ , we know that  $(A(x_n))_{n=1}^{\infty}$  has a converging subsequence  $(A(x_{n_k}))_{k=1}^{\infty}$  since  $(x_n)_{n=1}^{\infty}$  is bounded.

Say  $A(x_{n_k}) \rightarrow y$  in  $E$ .  $A \in K(E, E) \subset B(E, E)$  and  $x_n \rightharpoonup x$  in  $(E, \langle \cdot, \cdot \rangle)$ .

This implies

$$A(x_n) \rightharpoonup A(x) \quad \text{in } (E, \langle \cdot, \cdot \rangle).$$

We get that  $y = A(x)$ . We have  $A(x_{n_k}) \rightarrow A(x)$  in  $E$ .

Assume that  $A(x_n) \not\rightarrow A(x)$  in  $E$ .

Then there exists an  $\varepsilon > 0$  and a subsequence  $(A(\tilde{x}_n))_{n=1}^{\infty}$  of  $(A(x_n))_{n=1}^{\infty}$  such that

$$\|A(\tilde{x}_n) - A(x)\| \geq \varepsilon, \quad \forall n.$$

But  $\tilde{x}_n \rightharpoonup x$  in  $(E, \langle \cdot, \cdot \rangle)$  and to be compact implies that  $(A(\tilde{x}_n))_{n=1}^{\infty}$  has a converging subsequence  $(A(\tilde{x}_{n_k}))_{k=1}^{\infty}$  that converges to  $A(x)$  (same argument as before) Conclusion:  $A(x_n) \rightarrow A(x)$  in  $(E, \langle \cdot, \cdot \rangle)$ . □

**Proposition 3.39.**  $A \in K(E, E)$  and  $(E, \langle \cdot, \cdot \rangle)$  Hilbert space  $\Rightarrow A^* \in K(E, E)$ .

**proof.** Fix any bounded sequence  $(x_n)_{n=1}^\infty$  in  $E$ .

$$\begin{aligned}\|A^*(x_n) - A^*(x_m)\| &= \langle A^*(x_n) - A^*(x_m), A^*(x_n) - A^*(x_m) \rangle \\ &= \langle x_n - x_m, A(A^*(x_n)) - A(A^*(x_m)) \rangle\end{aligned}$$

then use  $A \in K(E, E)$ . □

**Proposition 3.40.**  $A \in K(E, E), B \in B(E, E) \Rightarrow AB, BA \in K(E, E)$ .

**Example.** We already know this example:  $k \in C([0, 1] \times [0, 1])$  with

$$A(f)(x) = \int_0^1 k(x, y)f(y) dy, \quad x \in [0, 1], \quad f \in L^2([0, 1]).$$

We know that  $A \in B(L^2([0, 1]), L^2([0, 1]))$

$$\|A\|_{L^2 \rightarrow L^2} \leq \|k\|_{L^2([0, 1] \times [0, 1])}.$$

**Claim:**  $A \in K(L^2([0, 1]), L^2([0, 1]))$ .

Approximate  $A$  by finite-ranked operators.

Note: set  $A = A_k$  and  $B = A_{k_n}$  where  $k_n$  is a nice function on  $[0, 1] \times [0, 1]$  and

$$A - B = A_k - A_{k_n} = A_{k - k_n}.$$

So

$$\|A - B\|_{L^2 \rightarrow L^2} \leq \|k - k_n\|.$$

Set

$$\begin{aligned}I_j &= [x_j - \frac{1}{N}, x_j], & j &= 1, \dots, N, & x_j &= \frac{j}{N} \\ \tilde{I}_l &= [y_l - \frac{1}{N}, y_l], & l &= 1, \dots, N, & y_l &= \frac{l}{N}.\end{aligned}$$

Set

$$k_n(x, y) = \sum_{j=1}^N \sum_{l=1}^N k(x_j, y_l) \chi_{I_j}(x) \chi_{\tilde{I}_l}(y)$$

where

$$\chi_{I_j}(x) = \begin{cases} 1, & \text{if } x \in I_j \\ 0, & \text{elsewhere.} \end{cases}$$

Since  $k \in C([0, 1] \times [0, 1])$  and  $[0, 1] \times [0, 1]$  compact in  $\mathbb{R}^2$  then  $k$  is uniformly continuous on  $[0, 1] \times [0, 1]$ . We fix  $\varepsilon > 0$ .

**Claim:** It exists an  $N$  such that

$$\sup_{\substack{(x, y) \in \\ [0, 1] \times [0, 1]}} |k(x, y) - k_n(x, y)| < \varepsilon,$$



$$A_{k_N}(f)(x) = \int_0^1 k_N(x, y) f(y) dy = \sum_{j=1}^N \underbrace{\sum_{l=1}^N k(x_i, y_l) \int_0^1 \chi_{\tilde{I}_l}(y) f(y) dy \chi_{I_j}(x)}_{\text{scalar}}.$$

$$\dim(\mathcal{R}(A_{k_N})) = N < \infty.$$

Hence  $A_{k_N} \in K(L^2([0, 1]), L^2([0, 1]))$  for all  $N$ .

Moreover

$$\|A - A_{k_N}\|_{L^2 \rightarrow L^2} \leq \|k - k_N\|_{L^2([0,1] \times [0,1])} < \varepsilon$$

for  $N$  large enough.  $K(E, E)$  is a closed set in  $(B(E, E), \|\cdot\|_{L^2 \rightarrow L^2})$  so  $A \in K(L^2, L^2)$ .

**Example.**  $(E, \langle \cdot, \cdot \rangle)$  Hilbert space,  $(x_n)_{n=1}^\infty$  ON-basis and  $(\lambda_n)_{n=1}^\infty$  sequence of scalars. Set

$$T(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, x_n \rangle x_n, \quad x \in E.$$

**Claim:**

- 1)  $T \in B(E, E) \Leftrightarrow (\lambda_n)_{n=1}^\infty$  is a bounded sequence in  $\mathbb{C}$ .
- 2)  $T \in K(E, E) \Leftrightarrow \lambda_n \rightarrow 0$  for  $n \rightarrow \infty$ .

Note  $x \in E$  and the Parseval's formula

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2.$$

For  $T(x) \in E$  we have

$$\|T(x)\|^2 = \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle x, x_n \rangle|^2.$$

If  $(\lambda_n)_{n=1}^\infty$  bounded sequence in  $\mathbb{C}$ . Then  $\sup |\lambda_n| \equiv M < \infty$  and

$$\|T(x)\|^2 \leq \sum_{n=1}^{\infty} M^2 |\langle x, x_n \rangle|^2 = M^2 \|x\|^2.$$

If  $(\lambda_n)_{n=1}^\infty$  is not bounded then there exists a sequence  $(\lambda_{n_k})_{k=1}^\infty$  such that  $|\lambda_{n_k}| \rightarrow \infty$  as  $k \rightarrow \infty$ . But

$$\begin{aligned} \|T(x_{n_k})\| &= |\lambda_{n_k}| \|x_{n_k}\| = |\lambda_{n_k}| \rightarrow \infty, \quad k \rightarrow \infty \\ \sup_{\|x\|=1} \|T(x)\| &= \infty. \end{aligned}$$

So 1) is done. For 2) we assume  $\lambda_n \rightarrow 0$  for  $n \rightarrow \infty$ . Set

$$T_k(x) = \sum_{n=1}^k \lambda_n \langle x, x_n \rangle x_n, \quad x \in E$$

$T_k$  is a finite rank operator for  $k = 1, 2, \dots$  SO  $T_k \in K(E, E)$  for all  $k$ .

$$\begin{aligned} \|T - T_k\|_{E \rightarrow E} &= \sup_{\|x\|=1} \|(T - T_k)(x)\| \\ &= \sup_{\|x\|=1} \left\| \sum_{n=k+1}^{\infty} \lambda_n \langle x, x_n \rangle x_n \right\| \\ &\leq \sup_{n=k+1, k+2, \dots} |\lambda_n| \rightarrow 0, \quad k \rightarrow \infty \end{aligned}$$

Assume  $\lambda_n \not\rightarrow 0$  for  $n \rightarrow \infty$ . Then there exists  $\varepsilon > 0$  and a sequence  $(\lambda_{n_k})_{k=1}^{\infty}$  such that

$$|\lambda_{n_k}| \geq \varepsilon.$$

Note

$$T(x_{n_k}) = \lambda_{n_k} x_{n_k}, \quad k = 1, 2, \dots$$

$$\|T(x_{n_k})\| = |\lambda_{n_k}| \|x_{n_k}\| = |\lambda_{n_k}| \geq \varepsilon, \quad k = 1, 2, \dots$$

$x_{n_k} \xrightarrow{w} 0$  in  $(E, \langle \cdot, \cdot \rangle)$  since for  $y \in E$

$$\langle x_{n_k}, y \rangle = \langle x_{n_k}, \sum_{n=1}^{\infty} \langle y, x_n \rangle x_n \rangle = \overline{\langle y, x_{n_k} \rangle} \rightarrow 0$$

since

$$\sum_{n=1}^{\infty} |\langle y, x_n \rangle|^2 = \|y\|^2 < \infty.$$

If  $T \in K(E, E)$  then  $T(x_{n_k}) \rightarrow T(0) = 0$  but

$$\|T(x_{n_k})\| \geq \varepsilon, \quad \text{for all } k.$$

Hence

$$T \notin K(E, E).$$

**Example.**  $(E, \langle \cdot, \cdot \rangle)$  Hilbert space,  $A \in K(E, E)$  and  $I(x) = x$  for all  $x \in E$ . It follows

$$\Rightarrow \mathcal{R}(I - A) \text{ closed in } E.$$

**Remark.**

$$\begin{aligned} \mathcal{R}(I - A)^{\perp} &= \mathcal{N}((I - A)^*) = \mathcal{N}(I - A^*) \\ \overline{\mathcal{R}(I - A)} &= \left( \mathcal{R}(I - A)^{\perp} \right)^{\perp} = \mathcal{N}(I - A^*)^{\perp}. \end{aligned}$$

If  $A \in K(E, E)$  then

$$\overline{\mathcal{R}(I - A)} = \mathcal{R}(I - A).$$

Solve

$$x = A(x) + y \quad \Leftrightarrow \quad (I - A)(x) = y$$

Compare 'Fredholm alternative'

**proof.** Take a sequence  $(y_n)_{n \in \mathbb{N}} \subseteq R(I - A)$  such that  $y_n \rightarrow y$  in  $(E, \|\cdot\|)$ .

To show:  $y \in \mathcal{R}(I - A)$ , i.e.  $y = (I - A)(x)$  for some  $x \in E$  and  $y_n = (I - A)(x_n)$  for some  $x_n \in E$ .

$$x_n \in E = \mathcal{N}(I - A) + \mathcal{N}(I - A)^\perp$$

such that

$$x_n = \tilde{x}_n + \hat{x}_n$$

with

$$\|x_n\|^2 = \|\tilde{x}_n\|^2 + \|\hat{x}_n\|^2.$$

Step 1: Show  $(\hat{x}_n)_{n=1}^\infty$  bounded in  $E$ .

Step 2:  $y_n = (I - A)(\hat{x}_n) = \hat{x}_n - A(\hat{x}_n)$ .

□

**recall:**

$(E, \langle \cdot, \cdot \rangle)$  Hilbert space and  $(x_n)_{n=1}^\infty$  ON-basis and  $(\lambda_n)_{n=1}^\infty$  sequence of complex numbers. Set

$$A(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, x_n \rangle x_n.$$

We have:

- $A : E \rightarrow E$  if  $(\lambda_n)_{n=1}^\infty \in l^\infty$   
if  $(\lambda_n)_{n=1}^\infty$  is not bounded, there exists a subsequence  $(\lambda_{n_k})_{k \in \mathbb{N}}$  such that

$$|\lambda_{n_k}| \geq k, \quad k = 1, 2, \dots$$

Set

$$x = \sum_{k=1}^{\infty} \frac{1}{k} x_{n_k}.$$

Clearly  $x \in E$  since  $(\frac{1}{k})_{k=1}^\infty \in l^\infty$ . But

$$T(x) = \sum_{k=1}^{\infty} \lambda_{n_k} \frac{1}{k} x_{n_k} \notin E$$

since  $(\lambda_{n_k} \cdot \frac{1}{k})_{k=1}^\infty \notin l^2$ .

Note

$$A \in B(E, E) \quad \Leftrightarrow \quad (\lambda_n)_{n=1}^\infty \in l^\infty$$

and  $\|A\| = \sup_n |\lambda_n|$ .

- $A \in K(E, E)$  iff  $\lambda_n \rightarrow 0$  for  $n \rightarrow \infty$ .
- $A$  is self adjoint iff  $\lambda_n \in \mathbb{R}$  for all  $n \in \mathbb{N}$ .

**Basis facts:**

Set  $A \in B(E, E)$  where  $(E, \langle \cdot, \cdot \rangle)$  is a Hilbert space. Then:

- If  $A$  is self-adjoint we have

$$\|A\| = \sup_{\|x\|=1} |\langle A(x), x \rangle|.$$

- If  $A$  is self-adjoint it follows

$$\langle A(x), x \rangle \in \mathbb{R}, \quad \forall x \in E$$

since

$$\langle A(x), x \rangle = \langle x, A^*(x) \rangle \stackrel{\text{self-adjoint}}{=} \langle x, A(x) \rangle = \overline{\langle A(x), x \rangle}.$$

- $K(E, E)$  (Set of all compact linear operators) closed subspace in  $(B(E, E), \|\cdot\|_{E \rightarrow E})$ .
- $A \in K(E, E)$  and  $x_n \rightharpoonup x$  in  $E$ . Then

$$A(x_n) \rightarrow A(x), \quad \text{in } E.$$

- $A \in K(E, E)$  and  $B \in B(E, E)$ . Then
  - $AB, BA \in K(E, E)$ ,
  - $A^* \in K(E, E)$ ,
  - $\mathcal{R}(B)^\perp = \mathcal{N}(B^*)$   
 $\mathcal{R}(B) = \mathcal{N}(B^*)^\perp$ ,
  - $\mathcal{R}(I - A)$  is a closed subspace in  $E$ .
- $E = \mathcal{R}(I - A) \oplus \mathcal{R}(I - A)^\perp = \mathcal{R}(I - A) \oplus \mathcal{N}(I - A^*)$ .
- For any  $A \in K(E, E)$

$$\dim(\mathcal{N}(I - A)) = \dim(\{x \in E \mid x - A(x) = 0\}) < \infty$$

since: if  $\dim(\mathcal{N}(I - A)) = \infty$  then there exists an ON- sequence  $(x_n)_{n=1}^\infty$  in  $\mathcal{N}(I - A)$ .  
Then

$$x_n \rightharpoonup 0, \quad \text{since } \langle x_n, y \rangle \rightarrow 0, n \rightarrow \infty$$

since for  $y \in \overline{\text{span}\{x_n \mid n = 1, 2, \dots\}}$  then

$$\|y\|^2 = \sum_{n=1}^{\infty} |\langle x_n, y \rangle|^2 < \infty.$$

$A \in K(E, E)$  implies that  $A(x_n) \rightarrow A(0) = 0$  in  $E$ . But

$$x_n = A(x_n) \rightarrow 0 \quad \text{in } E, \quad \|x_n\| = 1 \text{ for all } n$$

This is a contradiction.

Conclusion:  $\dim(I - A) < \infty$ .

From above we have for  $A \in K(E, E)$

$$E = \mathcal{R}(I - A) \oplus \mathcal{N}(I - A^*).$$

Consider the equation

$$x = A(x) + y \quad (1).$$

(1) has a solution provided by  $y \in \mathcal{R}(I - A)$ . That is the case if  $y \perp z$  for all  $z \in \mathcal{N}(I - A^*)$ . Since  $\dim(\mathcal{N}(I - A^*)) < \infty$ , this is just finitely many conditions.

**Theorem 3.41** (Fredholm alternativ).  $A \in K(E, E)$  where  $E$  is a Hilbert space. then exactly one of the statements below holds:

1.  $x = A(x) + y$  is solvable for every  $y \in E$ .
2.  $x = A(x)$  has a non trivial solution  $x \in E$ , i.e.  $x \neq 0$ .

(No assumption on  $A$  being self-adjoint.)

**Remark.** The statement in Fredholm Alternativ also holds if  $(E, \|\cdot\|)$  is a Banach space.

**proof.** (1)  $\Rightarrow$   $\neg$  (2): We want to show that there are no non-trivial solutions for  $x = A(x)$ . Assume that there exists a non-trivial solution  $x_1 \in E$  to  $x = A(x)$ , i.e.

$$(I - A)(x_1) = 0, \quad \text{with } x_1 \neq 0.$$

If (1) holds true there exists a  $x_2 \in E$  such that

$$(I - A)(x_2) = x_1 \neq 0.$$

But

$$(I - A)(x_1) = (I - A)^2(x_2) = 0.$$

With (1) there exists  $x_3 \in E$  such that

$$(I - A)(x_3) = x_2$$

which implies

$$(I - A)^2(x_3) = (I - A)(x_2) = x_1 \neq 0.$$

But once again

$$(I - A)^3(x_3) = 0.$$

Proceed inductively gives us a sequence  $(x_k)_{k=1}^{\infty}$  such that

$$(I - A)^k(x_k) = 0, \quad \text{but } (I - A)^{k-1}(x_k) \neq 0.$$

We obtain

$$\mathcal{N}(I - A) \subsetneq \mathcal{N}((I - A)^2) \subsetneq \mathcal{N}((I - A)^3) \subsetneq \dots$$

This is a sequence of proper closed subspaces.

Apply now Riesz-Lemma:

There exists a sequence  $(y_k)_{k=1}^\infty$  with  $\|y_k\| = 1$  and  $\|y_k - x\| \geq \frac{1}{2}$  for all  $x \in \mathcal{N}((I - A)^{k-1})$  and  $y_k \in \mathcal{N}((I - A)^k)$ .

**Claim:**  $\|A(y_n) - A(y_m)\| \geq \frac{1}{2}$  for all  $n > m$ .

$$\begin{aligned} \|A(y_m) - A(y_n)\| &= \left\| \underbrace{(I - A)(y_n)}_{\in \mathcal{N}((I-A)^{n-1})} - y_n + \underbrace{A(y_m)}_{\in \mathcal{N}((I-A)^{n-1})} \right\| \\ &= \left\| y_n - \underbrace{((I - A)(y_n) + A(y_m))}_{\in \mathcal{N}((I-A)^{n-1})} \right\| \geq \frac{1}{2}. \end{aligned}$$

So  $(A(y_n))_{n=1}^\infty$  can not converge in  $E$ . But  $A$  is compact and  $\|y_n\| = 1$  for all  $n$ . This is a contradiction.

Conclusion: There is no non-trivial solution of  $A(x) = x$ .

¬ (2) ⇒ (1) Assume that  $x = A(x)$  has no non-trivial solution  $x \in E$ . We want to show that (1) holds.

$$E = \mathcal{R}(I - A^*) \oplus \mathcal{N}(I - A), \quad \text{with } \mathcal{N}(I - A) = \{0\}.$$

Hence

$$x = A^*(x) + y$$

is solvable for every  $y \in E$ . From the first part of the proof it follows that

$$\mathcal{N}(I - A^*) = \{0\}.$$

But then

$$E = \mathcal{R}(I - A) \oplus \mathcal{N}(I - A^*) = \mathcal{R}(I - A).$$

Conclusion:  $x = A(x) + y$  is solvable for all  $y \in E$ .

□

**Example.**  $L^2([0, 1])$ ,  $k \in C([0, 1] \times [0, 1])$  and

$$A(f)(x) = \int_0^1 k(x, y) f(y) dy, \quad x \in [0, 1].$$

Then

- $A \in B(L^2, L^2)$  with  $\|A\|_{L^2 \rightarrow L^2} \leq \|k\|_{L^2([0,1] \times [0,1])}$ ,
- $A$  self-adjoint if  $k(x, y) = \overline{k(y, x)}$  for all  $x, y \in [0, 1]$ ,
- $A \in K(E, E)$  (by approximation by finite rank operators).

**Theorem 3.42** (Hilbert-Schmidt-Theorem).  $(E, \langle \cdot, \cdot \rangle)$  Hilbert spaces and  $A \in K(E, E)$  self adjoint. Then there exists a sequence of non-zero eigenvalues of  $A$  denoted  $(\lambda_n)_{n=1}^N$  for  $N$  finite or infinite, corresponding to eigenvectors  $(u_n)_{n=1}^N$ . Respectively where  $(u_n)_{n=1}^N$  is an ON-sequence, and

$$|\lambda_1| \geq |\lambda_2| \geq \dots$$

with

$$\lim_{n \rightarrow \infty} \lambda_n = 0, \quad \text{if } N = \infty$$

such that for  $x \in E$

$$x = \sum_{n=1}^N \langle x, u_n \rangle u_n + v, \quad v \in \mathcal{N}(A).$$

Moreover

$$A(x) = \sum_{n=1}^N \lambda_n \langle x, u_n \rangle u_n.$$

**Remark.** With notation from the theorem above we have

1.

$$A^k(x) = \sum_{n=1}^N \lambda_n^k \langle x, u_n \rangle u_n, \quad k = 1, 2, \dots$$

2. If  $A$  is injective, i.e.  $\mathcal{N}(A) = \{0\}$  then the Eigenvectors  $(u_n)_{n=1}^N$  form an ON-basis for  $E$ .

**Definition** (Eigenvalues and Eigenvectors for  $A \in B(E, E)$ ).  $\lambda \in \mathbb{C}$  is called an eigenvalue of  $A$  if there exists an  $0 \neq x \in E$  such that

$$A(x) = \lambda x.$$

**Remark** (properties for Eigenvalues and Eigenvectors). 1.  $|\lambda| \leq \|A\|$  since

$$|\lambda| \|x\| = \|\lambda x\| = \|A(x)\| \leq \|A\| \cdot \|x\|.$$

2.  $A$  self-adjoint and  $\lambda$  eigenvalue. Then

$$\Rightarrow \lambda \in \mathbb{R}$$

since

$$\begin{aligned}\lambda \langle x, x \rangle &= \langle \lambda x, x \rangle \\ &= \langle A(x), x \rangle \\ &= \langle x, A^*(x) \rangle \\ &= \langle x, A(x) \rangle \\ &= \langle x, \lambda x \rangle \\ &= \bar{\lambda} \langle x, x \rangle.\end{aligned}$$

So

$$\lambda = \bar{\lambda}, \quad \Rightarrow \lambda \in \mathbb{R}.$$

3.  $A$  self-adjoint,  $A(x) = \lambda x$  and  $A(y) = \mu y$ , where  $x, y \neq 0$  and  $\lambda \neq \mu$ .

$$\Rightarrow x \perp y$$

since

$$\lambda \langle x, y \rangle = \dots = \bar{\mu} \langle x, y \rangle.$$

So

$$\underbrace{(\lambda - \mu)}_{\neq 0} \langle x, y \rangle = 0.$$

4.  $A \in K(E, E)$  and  $\lambda \neq 0$  eigenvalue of  $A$ . Then

$$\dim E_\lambda = \dim \{x \in E \mid A(x) = \lambda x\} < \infty.$$

**Proposition 3.43.**  $(E, \langle \cdot, \cdot \rangle)$  Hilbert space and  $A \in K(E, E)$  self-adjoint. Then

$$\Rightarrow \|A\| \quad \text{or} \quad -\|A\|$$

is an eigenvalue of  $A$ .

**proof.**  $A = 0$  then the statement is trivial.

Assume  $A \neq 0$ .

$A$  self-adjoint implies that

$$\|A\| = \sup_{\|x\|=1} |\langle A(x), x \rangle|.$$

Also self-adjoint implies that for all  $x \in E$  we have

$$\langle A(x), x \rangle \in \mathbb{R}.$$

Hence there exists a sequence  $(x_n)_{n=1}^\infty$  in  $E$  with  $\|x_n\| = 1$  for all  $n$  such that

$$\langle A(x_n), x_n \rangle \rightarrow \lambda, \quad n \rightarrow \infty.$$



where  $\lambda \in \mathbb{R}$  and  $|\lambda| = \|A\|$ .

**Claim:**  $A(x_n) - \lambda x_n \rightarrow 0$  in  $E$ .

$$\begin{aligned}
 \|A(x_n) - \lambda x_n\|^2 &= \langle A(x_n) - \lambda x_n, A(x_n) - \lambda x_n \rangle \\
 &= \underbrace{\langle A(x_n), A(x_n) \rangle}_{=\|A(x_n)\|^2} - \underbrace{\overline{\lambda} \langle A(x_n), x_n \rangle}_{\rightarrow \lambda} - \underbrace{\lambda \langle x_n, A(x_n) \rangle}_{\rightarrow \lambda} + \underbrace{|\lambda|^2 \langle x_n, x_n \rangle}_{=\|A\|^2} \\
 &\leq \|A\|^2 \|x_n\|^2 \\
 &= \|A\|^2 \\
 &\rightarrow 0, \quad n \rightarrow \infty.
 \end{aligned}$$

$A \in K(E, E)$  and  $\|x_n\| = 1$  for all  $n$  we get that

$$(A(x_n))_{n=1}^\infty$$

has a converging subsequence  $(A(x_{n_k}))_{k=1}^\infty$  in  $E$ .

Call the limit element  $y \in E$  so

$$\begin{aligned}
 A(x_{n_k}) &\rightarrow y \quad \text{in } E. \\
 \begin{cases} A(x_n) - \lambda x_n \rightarrow 0 \\ A(x_{n_k}) \rightarrow y \end{cases} &\quad \text{in } E
 \end{aligned}$$

implies

$$x_{n_k} \rightarrow \frac{1}{\lambda} y \quad \text{in } E$$

(note  $|\lambda| > 0$  since  $A \neq 0$ ).

Set  $x = \frac{1}{\lambda} y$ . So  $x_{n_k} \rightarrow x$  in  $E$ . Consider

$$\|A(x) - \lambda x\| \leq \|A(x) - A(x_{n_k})\| + \|A(x_{n_k}) - y\| \rightarrow 0, \quad k \rightarrow \infty$$

**Conclusion:**

$$A(x) = \lambda x.$$

where  $\|x\| = 1$  since  $1 = \|x_{n_k}\| \rightarrow \|x\|$  as  $k \rightarrow \infty$ . □

We are now going to prove the Hilbert-Schmidt theorem:

**proof.** If  $A = 0$  the theorem is trivial.

Assume  $A \neq 0$ .

By the proposition above there exists an eigenvalue  $\lambda_1$  of  $A$  with  $|\lambda_1| = \|A\|$  and an eigenvector  $u_1$  with  $\|u_1\| = 1$  corresponding to the eigenvalue  $\lambda_1$ .

Set  $Q_1 = \{u_1\}^\perp$ .  $Q_1$  is a closed subspace of  $E$  and hence  $Q_1$  is a Hilbert space.

For  $x \in Q_1$  we have  $A(x) \in Q_1$  since for  $x \in Q_1$  we have

$$\begin{aligned}
 \langle A(x), u_1 \rangle &= \langle x, A^*(u_1) \rangle \\
 &= \langle x, A(u_1) \rangle \\
 &= \langle x, \underbrace{\lambda_1}_{\in \mathbb{R}} u_1 \rangle \\
 &= \lambda_1 \langle x, u_1 \rangle = 0.
 \end{aligned}$$

Now

$$A|_{Q_1} : Q_1 \rightarrow Q_1$$

is compact and also self-adjoint. By proposition above there exists an eigenvalue  $\lambda_2$  of  $A|_{Q_1}$  and a corresponding eigenvector  $u_2$  with  $\|u_2\| = 1$  where

$$|\lambda_2| = \|A|_{Q_1}\| \leq \|A\| = |\lambda_1|.$$

Here  $A(u_2) = \lambda_2 u_2$  so  $\lambda_2$  is an eigenvalue of  $A$ . Set  $Q_2 = \{u_1, u_2\}^\perp$ .  $Q_2$  is a closed subspace of  $E$  and we have

$$x \in Q_2 \quad \Rightarrow \quad A(x) \in Q_2$$

since  $x \in Q_2$  we have

$$\begin{aligned} \langle A(x), u_1 \rangle &= \langle x, A(u_1) \rangle = \langle x, \lambda_1 u_1 \rangle = 0 \\ \langle A(x), u_2 \rangle &= \langle x, A(u_2) \rangle = \langle x, \lambda_2 u_2 \rangle = 0. \end{aligned}$$

Proceed inductively.

**Case 1:** For a positive integer  $k$  we have

$$|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_k| > 0$$

with corresponding eigenvectors  $u_1, u_2, \dots, u_k$  but  $A|_{Q_k}$  with  $Q_k = \{u_1, u_2, \dots, u_k\}^\perp$ , then is the zero-mapping  $Q_k \rightarrow Q_k$ . This corresponds to  $N = k$  and

$$x = \sum_{n=1}^k \langle x, u_n \rangle u_n + v, \quad \text{where } v \in \mathcal{N}(A).$$

**Case 2:** The process never terminates. We get

$$|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n| \geq \dots$$

with corresponding eigenvectors  $u_1, u_2, \dots, u_n, \dots$

We have  $(u_n)_{n=1}^\infty$  ON-sequence in  $E$  corresponding to the non-zero eigenvalue  $(\lambda_n)_{n=1}^\infty$ .

$A \in K(E, E)$  and  $u_n \rightarrow 0$  in  $E$  since  $(u_n)_{n=1}^\infty$  is ON-sequence.

Then this implies  $A(u_n) \rightarrow 0$  in  $E$ . So

$$|\lambda_n| = \|\lambda_n u_n\| = \|A(u_n)\| \rightarrow 0, \quad n \rightarrow \infty.$$

Hence

$$\lim_{n \rightarrow \infty} \lambda_n = 0.$$

Set

$$S := \overline{\text{span}\{u_1, \dots, u_n, \dots\}} = \left\{ \sum_{k=1}^{\infty} a_k u_k \mid (a_n)_{n=1}^\infty \in l^\infty \right\}.$$

$S$  is a closed subspace of  $E$ .

We have  $E = S \oplus S^\perp$  where  $S^\perp \subseteq Q_k = \{u_1, \dots, u_k\}^\perp$  for all  $k \in \mathbb{N}$ . For  $x \in E$  we have

$$\underbrace{\sum_{k=1}^{\infty} \langle x, u_k \rangle u_k}_{\in S} + \underbrace{v}_{\in S^\perp}$$

since  $(\langle x, u_k \rangle)_{k=1}^{\infty} \in l^\infty$  by Bessel's inequality. To show:  $A(v) = 0$ . Clearly,  $v \in Q_k$  for all  $k$ . If  $v = 0$  there is nothing to prove. For  $v \neq 0$  set  $w = \frac{1}{\|v\|}v$  and get

$$\begin{aligned} |\langle A(v), v \rangle| &= \|v\|^2 |\langle A(w), w \rangle| \\ &\leq \|v\|^2 \sup_{\substack{\|z\|=1 \\ z \in Q_k}} |\langle A(z), z \rangle| \\ &= \|A|_{Q_k}\| = |\lambda_{k+1}| \rightarrow 0 \end{aligned}$$

**Claim:**  $A|_{S^\perp} = 0$  and hence  $v \in S^\perp$  implies  $A(v) = 0$ .

□

**Theorem 3.44** (Spectral mapping theorem).  $(E, \langle \cdot, \cdot \rangle)$  separable Hilbert space and  $\infty$ -dimensional  $A \in K(E, E)$  self-adjoint. Then there exists a ON-basis of eigenvectors  $(\tilde{u}_n)_{n=1}^{\infty}$  corresponding to the eigenvalues  $(\tilde{\lambda}_n)_{n=1}^{\infty}$  of  $A$  where  $\lim_{n \rightarrow \infty} \tilde{\lambda}_n = 0$ .

**proof** (consequence of HS-theorem). We have by HS-theorem an ON-sequence  $(u_n)_{n=1}^{\infty}$  of eigenvectors corresponding to the non-zero eigenvalues  $(\lambda_n)_{n=1}^N$ .

Set

$$S = \overline{\text{span}\{u_1, \dots, u_n, \dots\}}.$$

$E$  is separable implies  $E$  has an ON-basis  $(v_n)_{n=1}^{\infty}$ . By Gram-Schmidt Orthogonalization procedure we can obtain an ON-basis  $(w_n)_{n=1}^M$  for  $S^\perp$ . Have  $M$  finite or infinite.

$$\begin{array}{ll} S : u_1, u_2, \dots & \text{ON-basis finite or infinite} \\ S^\perp : w_1, w_2, \dots & \text{ON-basis finite or infinite} \end{array}$$

Consider the ON-sequence  $u_1, w_1, u_2, w_2, \dots = \tilde{u}_1, \tilde{u}_2, \dots$ . This gives an ON-basis for  $E$  consisting of eigenvectors to  $A$ . Also

$$\lim_{n \rightarrow \infty} \tilde{\lambda}_n = 0.$$

□

- $(E, \langle \cdot, \cdot \rangle)$  complex Hilbert space.
- $A \in \mathcal{B}(\mathcal{E}, \mathcal{E})$ .

- Consider the equation

$$\begin{aligned}x &= A(x) + y, & y &\in E. \\(I - A)(x) &= y.\end{aligned}$$

- Consider this problem for  $\lambda \in \mathbb{C}$ .

- Set

$$\rho(A) := \{\lambda \in \mathbb{C} \mid (A - \lambda I)^{-1} \in \mathcal{B}(E, E)\}$$

- $\rho(A)$  is called the resolvent set for  $A$ .

- Set

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

- $\sigma(A)$  is called the spectrum of  $A$ .

- Clearly, a necessary condition for  $(A - \lambda I)^{-1} \in \mathcal{B}(E, E)$  is that

$$A - \lambda I : E \rightarrow E$$

is a bijection.

- Linearity for  $(A - \lambda I)^{-1}$  follows from the linearity of  $A - \lambda I$ .

**Theorem 3.45** (Banach's inverse mapping theorem).  $(E, \|\cdot\|)$  Banach space,  $A \in \mathcal{B}(E, E)$ .  $A - \lambda I : E \rightarrow E$  bijection. Then

$$\Rightarrow (A - \lambda I)^{-1} \in \mathcal{B}(E, E)$$

**proof.** based on the Open mapping theorem. Proof is omitted. Assume  $\lambda \in \sigma(A)$ . Then  $A - \lambda I : E \rightarrow E$  is not a bijection.

- If  $A - \lambda I : E \rightarrow E$  is not injective then there exists  $0 \neq x \in E$  such that

$$(A - \lambda I)(x) = 0,$$

i.e.  $\lambda$  is an eigenvalue of  $A$ . Set

$$\sigma_p(A) = \{\lambda \in \mathbb{C} \mid \lambda \text{ eigenvalue of } A\}.$$

- If  $A - \lambda I$  is injective, densely defined but not bounded then  $\lambda \in \sigma(A)$ . The set of such  $\lambda$ 's is called the continuous spectrum of  $A$ , denoted  $\sigma_c(A)$
- If  $A - \lambda I$  is not surjective then the set of such  $\lambda$ 's is called the residual spectrum, denoted  $\sigma_r(A)$ .

□

**Lemma .**  $(E, \|\cdot\|)$  Banach space,  $A \in \mathcal{B}(E, E)$  with  $\|A\| < 1$ . Then

$$(I - A)^{-1} \in \mathcal{B}(E, E)$$

and

$$(I - A)^{-1} = I + \sum_{n=1}^{\infty} A^n.$$

This series is called a Neumannseries.

**proof.** Observe

$$\|A^n\| = \|A \cdot A \cdots A\| \leq \|A\|^n, \quad n = 1, 2, \dots$$

and

$$\sum_{n=1}^{\infty} \|A^n\| < \infty.$$

Since  $E$  is a Banach space we have

$$\sum_{n=1}^{\infty} A^n$$

converges in  $\mathcal{B}(E, E)$ . Since  $E$  Banach space implies  $\mathcal{B}(E, E)$  is a Banach space. Note

$$(I - A) \left( I + \sum_{n=1}^N A^n \right) = I - A^{N+1} \rightarrow I, \quad \text{in } \mathcal{B}(E, E).$$

$$\left( I + \sum_{n=1}^N A^n \right) (I - A) = I - A^{N+1} \rightarrow I, \quad \text{in } \mathcal{B}(E, E).$$

We get

$$\left( I + \sum_{n=1}^{\infty} A^n \right) (I - A) = I = (I - A) \left( I + \sum_{n=1}^{\infty} A^n \right).$$

We have  $(I - A)^{-1}$  exists and is equal to  $I + \sum_{n=1}^{\infty} A^n$ . □

**Lemma .**  $(E, \|\cdot\|)$  Banach space and  $A \in \mathcal{B}(E, E)$ . Then

1.  $\sigma(A) \neq \emptyset$ .
2.  $\sigma(A)$  closed set in  $\mathbb{C}$ .
3.  $\sigma(A) \subseteq \overline{B(0, \|A\|)}$

**proof.** 1. omitted.

2. Enough to prove that  $\rho(A)$  is an open set in  $\mathbb{C}$ .

Fix  $\lambda_0 \in \rho(A)$ . So  $(A - \lambda_0 I)^{-1} \in \mathcal{B}(E, E)$ .

Note:

$$\begin{aligned} A - \lambda I &= A - \lambda_0 I - (\lambda - \lambda_0)I \\ &= \underbrace{(A - \lambda_0 I)}_{\substack{\text{invertible} \\ \text{since } \lambda_0 \in \rho(A)}} \underbrace{(I - (\lambda - \lambda_0)(A - \lambda_0 I)^{-1})}_{\substack{\text{invertible if} \\ \|( \lambda - \lambda_0 )(A - \lambda_0 I)^{-1} \| < 1 \\ \text{by previous lemma, i.e.} \\ |\lambda - \lambda_0| < \frac{1}{\|(A - \lambda_0 I)^{-1}\|}}} \end{aligned}$$

Clearly,  $A - \lambda I$  is invertible if

$$|\lambda - \lambda_0| < \frac{1}{\|(A - \lambda_0 I)^{-1}\|}.$$

3. It is enough to show that  $\lambda \in \rho(A)$  if

$$|\lambda| > \|A\|.$$

Note

$$A - \lambda I = -\lambda(I - \frac{1}{\lambda}A).$$

Here

$$\left\| -\frac{1}{\lambda}A \right\| = \frac{1}{|\lambda|} \|A\| < 1.$$

$I - \frac{1}{\lambda}A$  is invertible by previous lemma. So  $\rho(A)$ .

□

Now assume  $(E, \langle \cdot, \cdot \rangle)$  is a complex Hilbert space with infinite dimension.  $A \in \mathcal{K}(E, E)$  (We don't assume  $A$  is self-adjoint). Then

1.  $\lambda \in \sigma(A) \setminus \{0\} \Rightarrow$  is an eigenvalue of  $A$ .
2.  $\lambda \in \sigma(A) \setminus \{0\} \Rightarrow \dim\{x \in E \mid A(x) = \lambda x\} < \infty$ .
3.  $0$  is the only cluster point for  $\sigma(A)$
4.  $0 \in \sigma(A)$  since if  $0 \notin \sigma(A)$  then  $A^{-1} \in \mathcal{B}(E, E)$  and

$$\underbrace{\underbrace{A}_{\in \mathcal{K}(E, E)} \underbrace{A^{-1}}_{\in \mathcal{B}(E, E)}}_{\in \mathcal{K}(E, E)} = I.$$

But  $I \notin \mathcal{K}(E, E)$  since  $E$   $\infty$ -dimensional. Just take an ON-sequence  $(x_n)_{n=1}^\infty$  in  $E$ . Then

$$x_n \rightharpoonup 0, \quad \text{in } E$$

but  $\|x_n\| = 1$  for all  $n$  and if  $I \in \mathcal{K}(E, E)$  then

$$x_n = I(x_n) \rightarrow I(0) = 0, \quad \text{in } E$$

which implies that  $\|x_n\| \rightarrow 0$  for  $n \rightarrow \infty$ . Moreover (by Hilbert-Schmidt theorem)  $(E, \langle \cdot, \cdot \rangle)$  complex Hilbert space, separable and  $\infty$ -dim.  $A \in \mathcal{K}(E, E)$  and self-adjoint it follows

$$\Rightarrow (u_n)_{n=1}^{\infty} \text{ ON-basis for } E \text{ where}$$

$$A(u_n) = \lambda_n u_n, \quad n = 1, 2, \dots$$

( $\lambda_n$  eigenvalue of  $A$  with normalised eigenvector  $u_n$ ) with

$$\lim_{n \rightarrow \infty} \lambda_n = 0.$$

For  $x \in E$

$$x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$$

and

$$A(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, u_n \rangle u_n$$

### Fredholm Alternativ:

$E, A$  as above. Then

1.  $x = A(x) + y$  is separable for all  $y \in E$ .  
iff
2.  $x = A(x)$  has no non-trivial solution  $x \in E$ .

Exactly one of the statements hold:

1. (1) from above
2. (2) has a non-trivial solution  $x \in E$ .

In general (1) is separable for  $y \in E$  iff

$$y \in \{x \in E \mid A(x) = x\}^{\perp}.$$

If so: If  $x$  is a solution to (1) then also  $x + \tilde{x}$  is a solution to (1) where

$$\tilde{x} \in \{x \in E \mid A(x) = x\}$$

**proof.** Look at (1). Let  $(u_n)_{n=1}^{\infty}$  be the ON-basis from the previous theorem.

$$x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n, \quad y = \sum_{n=1}^{\infty} \langle y, u_n \rangle u_n.$$

$$A(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, u_n \rangle u_n.$$

(1) taked the form

$$\sum_{n=1}^{\infty} (\langle x, u_n \rangle - \lambda_n \langle x, u_n \rangle - \langle y, u_n \rangle) u_n = 0.$$

This implies

$$(I - \lambda) \langle x, u_n \rangle - \langle y, u_n \rangle = 0, \quad n = 1, 2, \dots$$

If  $\lambda_n \neq 1$  then

$$\langle x, u_n \rangle = \frac{\langle y, u_n \rangle}{1 - \lambda_n}.$$

If  $\lambda_n = 1$  then  $-y$  must be orthogonal to every  $u_n$  corresponding by the eigenvalue 1.

$$\sum_{n=1}^{\infty} \frac{\langle y, u_n \rangle}{1 - \lambda_n} u_n \in E$$

since

$$\left( \frac{\langle y, u_n \rangle}{1 - \lambda_n} \right)_{n=1}^{\infty} \in l^2$$

since

$$\sup_{\lambda_n \neq 1} \left| \frac{1}{1 - \lambda_n} \right| < \infty$$

since

$$\lim_{n \rightarrow \infty} \lambda_n = 0$$

and

$$(\langle y, u_n \rangle)_{n=1}^{\infty} \in l^2.$$

□



## 4 Boundary Value Problems for ODE's

Consider

$$(*) \quad \begin{cases} Lu &= f \in C([0, 1]) \\ R_j u &= 0 \quad j = 1, 2, \dots, n \end{cases}$$

(homogeneous boundary conditions), where

$$Lu := u^{(n)} + C_{n-1}(x)u^{(n-1)} + \dots + c_1(x)u' + c_0(x)u, \quad u \in C^n([0, 1])$$

with

$$c_0(x), c_1(x), \dots, c_{n-1}(x) \in C([0, 1]) -$$

$$R_j = \sum_{k=0}^{n-1} \left( \alpha_{jk} u^{(k)}(0) + \beta_{jk} u^{(k)}(1) \right), \quad j = 1, 2, \dots, n$$

with

$$\alpha_{jk}, \beta_{jk} \in \mathbb{C}, \quad j = 1, \dots, n, \quad k = 0, \dots, n-1$$

Reformulate (\*).

$$u(x) = \int_0^1 \underbrace{g(x, y)}_{\substack{\text{Green's function} \\ \text{for } L \text{ and } R_j \\ j=1, \dots, n}} f(y) dy \in C^n([0, 1])$$

and satisfies the boundary conditions  $R_j = 0$  for  $j = 1, 2, \dots, n$ .

Consider the problem

$$(**) \quad \begin{cases} Lu &= f(x, u), \quad x \in [0, 1] \\ R_j u &= 0, \quad j = 1, 2, \dots, n. \end{cases}$$

The reformulation above gives

$$u(x) = \int_0^1 g(x, y) f(y, u(y)) dy, \quad x \in [0, 1].$$

To find a solution set

$$T(u)(x) = \int_0^1 g(x, y) f(y, u(y)) dy, \quad x \in [0, 1].$$

$$T : C([0, 1]) \rightarrow C([0, 1])$$

A fixed point to  $T$  gives a solution to (\*\*). Note that if  $u \in C([0, 1])$  then

$$T(u) \in C^n([0, 1])$$

and satisfies  $R_j = 0$  for  $j = 1, 2, \dots$

Given  $L$  and  $R_j$  for  $j = 1, 2, \dots, n$  find the corresponding Green's function.

**Example.**

$$\begin{cases} Lu &= u'' - u, & \text{on } [0, 1] \\ R_1 u &= u(0) = 0 \\ R_2 u &= u(1) = 0 \end{cases}$$

**Theorem 4.1.**  $Lu = f \in C([0, 1])$ , where

$$Lu := u^{(n)} + c_{n-1}(x)u^{(n-1)} + \dots + c_1(x)u' + c_0(x)u$$

and  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ . Then for  $x_0 \in [0, 1]$

$$\Rightarrow \quad \exists! u \in C^n([0, 1]) \text{ with } Lu = f.$$

and

$$(u, u', \dots, u^{(n-1)})|_{x_0} = \xi.$$

**proof.** Reformulate the problem as a system of first order differential equations.

$$\begin{cases} Lu &= f \\ (u, u', \dots, u^{(n-1)})|_{x_0} &= \xi \end{cases}$$

corresponds to

$$\begin{cases} \tilde{u}' = \tilde{f} \\ \tilde{u}(x_0) = \xi \end{cases}$$

and is equivalent to

$$\tilde{u}(x) = \xi + \int_{x_0}^x \tilde{f}(s) \, ds.$$

$\tilde{f}$  contains  $\tilde{u}$  implicitly. The statement of the proof follows from an application of Banach's fixed point theorem. (See course homepage and proof of picard's existence theorem.)  $\square$

Set

$$\mathcal{N}(L) = \{u \in C^n((0, 1)) \mid Lu = 0\}$$

**Claim:**  $\dim \mathcal{N}(L) = n$

Set

$$C_R^n([0, 1]) = \{u \in C^n((0, 1)) \mid R_j u = 0, j = 1, 2, \dots, n\}$$

and  $L_0 = L|_{C_R^n([0, 1])}$ . Let  $u_1, \dots, u_m \in \mathcal{N}(L)$

**Theorem 4.2.** The following statements are equivalent. Let  $u_1, \dots, u_n \in \mathcal{N}(L)$

1.  $W(x) \neq 0$  for all  $x \in [0, 1]$ .
2.  $W(x) \neq 0$  for some  $x \in [0, 1]$ .

3.  $u_1, u_2, \dots, u_n$  is a basis for  $\mathcal{N}(L)$ .

where

$$W(x) = \det \begin{pmatrix} u_1(x) & \dots & u_n(x) \\ u_1'(x) & \dots & u_n'(x) \\ \vdots & & \vdots \\ u_1^{(n-1)} & \dots & u_n^{(n-1)}(x) \end{pmatrix}, \quad x \in [0, 1].$$

**Theorem 4.3.** With the notation from above the following statements are equivalent.

1.  $L_0 : C_R^n([0, 1]) \rightarrow C([0, 1])$  is a bijection.
2.  $\det(R_j u_k)_{1 \leq j, k \leq n} \neq 0$ .

**Example (continue).** From the example above we get

$$u_1(x) = e^x, \quad u_2(x) = e^{-x}.$$

$$u(x) = Ae^x + Be^{-x}$$

and

$$R_1 u_1 = u_1(0) = e^0 = 1$$

$$R_1 u_2 = u_2(0) = e^0 = 1$$

$$R_2 u_1 = u_1(1) = e$$

$$R_2 u_2 = u_2(1) = \frac{1}{e}$$

and

$$\det(R_j u_k) = \det \begin{pmatrix} 1 & 1 \\ e & \frac{1}{e} \end{pmatrix} = \frac{1}{e} - e \neq 0.$$

**Theorem 4.4.** Assume  $u_1, \dots, u_n$  basis for  $\mathcal{N}(L)$  and  $\det(R_j u_k) \neq 0$ . Set  $G = L_0^{-1}$ .

$$\Rightarrow \quad \exists ! \text{continuous } g \in C([0, 1] \times [0, 1])$$

such that

$$G(f) = \int_0^1 g(x, y) f(y) dy$$

is a solution of

$$\begin{cases} Lu &= f \\ R_j u &= 0, \quad j = 1, \dots, n \end{cases}.$$

Here

$$g(x, y) = \underbrace{\left( \sum_{k=1}^n a_k(y) u_k(x) \right)}_{\equiv e(x, y)} \theta(x - y) + \sum_{k=1}^n b_k(y) u_k(x).$$

where

$$\begin{aligned} e_x^{(k)}(y, y) &= 0, & k &= 0, 1, \dots, n-2 \\ e_x^{(n-1)}(y, y) &= 1 \end{aligned}$$

Note

$$Lu = 1u^{(n)} + c_{n-1}u^{(n-1)} + \dots + c_0u.$$

and

$$R_j(g(\cdot, y)) = 0, \quad 0 < y < 1, \quad j = 1, 2, \dots, n$$

Note

$$\begin{aligned} \int_0^1 g(x, y) f(y) dy &= \int_0^1 e(x, y) \theta(x - y) f(y) dy + \int_0^1 \sum_{k=1}^n b_k(y) u_k(x) f(y) dy \\ &= \underbrace{\int_0^x \sum_{k=1}^{\infty} a_k(y) u_k(x) f(y) dy}_{=L[\dots]=f} + \underbrace{\sum_{k=1}^N \int_0^1 b_k(y) f(y) dy u_k(x)}_{L[\dots]=0} \\ &\quad \underbrace{\hspace{10em}}_{L[\dots]=f} \end{aligned}$$

Calculate  $g(x, y)$  for  $n = 2$ :

Set

$$e(x, y) = a_1(y) u_1(x) + a_2(y) u_2(x)$$

$$\begin{cases} e(y, y) &= a_1(y) e^y + a_2(y) e^{-y} = 0 \\ e'_x(y, y) &= a_1(y) e^y - a_2(y) e^{-y} = 1 \end{cases}$$

So we get

$$\begin{aligned} a_1(y) &= \frac{1}{2} e^{-y} \\ a_2(y) &= -\frac{1}{2} e^{-y} \end{aligned}$$

and

$$\begin{aligned} e(x, y) &= \frac{1}{2} e^{-y} e^x - \frac{1}{2} e^y e^{-x} \\ &= \frac{1}{2} (e^{x-y} - e^{y-x}), \quad (x, y) \in [0, 1] \times [0, 1] \end{aligned}$$

Set

$$g(x, y) = e(x, y)\theta(x - y) + b_1(y)u_1(x) + b_2(y)u_2(x)$$

For  $0 < y < 1$

$$\begin{aligned} R_1 g(., y) &= 0, \text{ i.e. } g(0, y) = 0, & \text{for } y \in (0, 1), \\ &\text{i.e. } b_1(y)u_1(0) + b_2(y)u_2(0) = 0 & \text{for } y \in (0, 1), \\ &\text{So } b_1(y) + b_2(y) = 0. \end{aligned}$$

$$\begin{aligned} R_2 g(., y) &= 0, \text{ i.e. } g(1, y) = 0, & \text{for } y \in (0, 1), \\ &\text{i.e. } e(1, y) + b_1(y)u_1(1) + b_2(y)u_2(1) = 0 & \text{for } y \in (0, 1), \\ &\text{So } \frac{1}{2}(e^{1-y} - e^{y-1}) + b_1(y)e + b_2(y)e^{-1} = 0 & \text{for } y \in (0, 1). \end{aligned}$$

So we have in total

$$\begin{cases} b_1(y) + b_2(y) &= 0 \\ \frac{1}{2}(e^{1-y} - e^{y-1}) + b_1(y)e + b_2(y)e^{-1} &= 0 \end{cases}.$$

We obtain

$$\begin{cases} b_1(y) &= -b_2(y) \\ b_2(y)(e^{-1} - e) &= \frac{1}{2}(e^{y-1} - e^{1-y}) \end{cases}.$$

So

$$b_2(y) = \frac{\frac{1}{2}(e^{y-1} - e^{1-y})}{(e^{-1} - e)} = \frac{1}{2} \frac{e^{1-y} - e^y}{e^2 - 1}$$

and

$$b_1(y) = \frac{1}{2} \frac{e^y - e^{2-y}}{e^2 - 1}.$$

We obtain

$$g(x, y) = \frac{1}{2}(e^{x-y} - e^{y-x})\theta(x - y) + \frac{1}{2} \frac{e^{x+y} - e^{x+2-y}}{e^2 - 1} + \frac{1}{2} \frac{e^{2-y-x} - e^{y-x}}{e^2 - 1}.$$

Question:  $g(x, y) = g(y, x)$  for all  $x, y \in [0, 1]$ ?

In general, we say that  $L_0 = L|_{C_R^n([0,1])}$  is symmetric if

$$\langle L_0(u), v \rangle_{L^2} = \langle u, L_0(v) \rangle_{L^2}, \quad \forall u, v \in C_R^n([0, 1])$$

**Example (continue).** As above we have

$$L(u) = u'' - u$$

with boundary conditions

$$u(0) = u(1) = 0$$

Set  $u, v \in C_R^2([0, 1])$

$$\begin{aligned}
 \langle L_0(u), v \rangle_{L^2} &= \int_0^1 L_0(u) \bar{v} \, dx \\
 &= \int_0^1 u'' \bar{v} - u \bar{v} \, dx \\
 &= - \int_0^1 u' \bar{v}' + u \bar{v} \, dx + \underbrace{u' \bar{v}}_{=0} \Big|_0^1 \\
 &= - \int_0^1 (u' \bar{v}' + u \bar{v}) \, dx \\
 &= \int_0^1 u (\bar{v}'' - \bar{v}) \, dx \\
 &= \int_0^1 u \overline{L_0 v} \, dx \\
 &= \langle u, L_0 v \rangle_{L^2}
 \end{aligned}$$

**Recall:**

We have a bounded value problem with

$$\begin{cases} Lu & \equiv u^{(n)} + c_{n-1}(x)u^{(n-1)} + \dots + c_1(x)u' + c_0(x)u = f \in C([0, 1]) \\ R_j u & = \sum_{k=0}^{n-1} (\alpha_{kj} u^{(k)}(0) + \beta_{kj} u^{(k)}(1)) = 0, \quad j = 1, 2, \dots, n \end{cases}$$

with  $\alpha_{kj}, \beta_{kj} \in \mathbb{C}$  for  $k = 0, \dots, n-1$  and  $j = 1, \dots, n$ .

We have

$$\mathcal{N}(L) = \{u \in C^n((0, 1)) \mid Lu = 0\}.$$

Set  $u_1, u_2, \dots, u_n$  as a basis for  $\mathcal{N}(L)$  and set

$$L_0 = L|_{C_R^n([0, 1])}$$

where

$$C_R^n([0, 1]) = \{u \in C^n((0, 1)) \mid R_j u = 0, j = 1, \dots, n\}.$$

Then  $L_0 u \in C([0, 1])$  for  $u \in C_R^n([0, 1])$ . Remember Theorem 4.3 and Theorem 4.4:

**Theorem .** Assume

$$\det(R_j u_k) \neq 0.$$

Then

1.  $L_0 : C_R^n([0, 1]) \rightarrow C([0, 1])$  is a bijection.
2. Set  $G = L_0^{-1}$ . Then there exists a continuous function  $g(x, y)$  in  $[0, 1] \times [0, 1]$  such

that

$$G(f) = \int_0^1 g(x, y) f(y) \, dy, \quad x \in [0, 1].$$

with  $f \in C([0, 1])$  and  $g$  is called the Green's function for  $L$  and the boundary conditions  $R_j$ ,  $j = 1, \dots, n$  can be given by

$$g(x, y) = e(x, y)\theta(x - y) + \sum_{k=1}^n b_k(y)u_k(x)$$

where

$$e(x, y) = \sum_{k=1}^n a_k(y)u_k(x)$$

with  $a_k$  and  $b_k$  are defined through

$$\begin{cases} \left(\frac{\partial}{\partial x}\right)^l e(x, y)\big|_{x=y} = 0, & l = 0, 1, \dots, n-2. \\ \left(\frac{\partial}{\partial x}\right)^{n-1} e(x, y)\big|_{x=y} = 1 \end{cases}$$

and

$$R_j(g(\cdot, y)) = 0, \quad 0 < y < 1, \quad j = 1, 2, \dots, n.$$

**Remark.** Consider the problem

$$\begin{cases} Lu = f(x, u), & x \in [0, 1] \\ R_j u = c_j, & j = 1, 2, \dots, n \end{cases}.$$

Pick any  $\tilde{u} \in C^n([0, 1])$  such that

$$R_j \tilde{u} = c_j, \quad j = 1, \dots, n.$$

Set  $U = \tilde{u} + v$ . Note that

$$R_j v = R_j(u - \tilde{u}) = R_j u - R_j \tilde{u} = 0, \quad j = 1, 2, \dots, n$$

and

$$L(\tilde{u} + v) = f(x, \tilde{u} + v)$$

gives

$$Lv = f(x, \tilde{u} + v) - L\tilde{u} = \hat{f}(x, v).$$

Solve

$$\begin{cases} Lv = \hat{f}(x, v) \\ R_j v = 0, & j = 1, \dots, n \end{cases}$$

Moreover set

$$T(v)(x) = \int_0^1 g(x, y) \hat{f}(y, v(y)) \, dy, \quad x \in [0, 1].$$

Apply a fixed point theorem.

**Warning:** don't take for example the space  $C^2([0, 1])$  because in this case we need another norm to use the fixed point theorem (no Banach space). Take

$$T : C([0, 1]) \rightarrow C([0, 1])$$

instead where  $(C([0, 1]), \|\cdot\|)$  is a Banach space.

If  $u \in C([0, 1])$  is a fixed point then

$$u(x) = \int_0^1 g(x, y) f(y, u(y)) \, dy, \quad x \in [0, 1].$$

WE ACTUALLY have  $u \in C_R^n([0, 1])!!$

**Definition .** Call  $L_0$  symmetric if

$$\langle L_0 u, v \rangle_{L^2} = \langle u, L_0 v \rangle_{L^2}, \quad \forall u, v \in C_R^n([0, 1])$$

where

$$\langle f, h \rangle_{L^2} = \int_0^1 f(x) \overline{h(x)} \, dx$$

**Theorem 4.5.** Assume that  $L_0 : C_R^n([0, 1]) \rightarrow C([0, 1])$  is a bijection. The following statements are equivalent:

1.  $L_0$  is symmetric.
2.  $\tilde{G}$  self-adjoint.
3.  $g(x, y) = \overline{g(y, x)}$  for all  $(x, y) \in [0, 1] \times [0, 1]$ .

Here

$$G(f) = \int_0^1 g(x, y) f(y) \, dy, \quad f \in C([0, 1])$$

$$\tilde{G}(f) = \int_0^1 g(x, y) f(y) \, dy, \quad f \in L^2([0, 1])$$

$$\tilde{G} \in \mathcal{B}(L^2([0, 1]), L^2([0, 1])) \text{ and } \overline{C([0, 1])}^{L^2([0, 1])} = L^2([0, 1]).$$

**proof. (1)  $\Rightarrow$  (2):** Assume that  $L_0$  is symmetric. Then we have

$$\langle L_0(G(f)), G(h) \rangle_{L^2} = \langle G(f), L_0(G(h)) \rangle_{L^2}$$

for all  $f, h \in C([0, 1])$ . Hence

$$\langle f, G(h) \rangle_{L^2} = \langle G(f), h \rangle_{L^2}, \quad \forall f, h \in C([0, 1]).$$



Change

$$\langle f, \tilde{G}(h) \rangle_{L^2} = \langle \tilde{G}(f), h \rangle_{L^2}, \quad \forall f, h \in L^2([0, 1]) \quad (*)$$

Hence  $\tilde{G}$  is self-adjoint in  $\mathcal{B}(L^2([0, 1]), L^2([0, 1]))$ . This yields (2).

(2)  $\Rightarrow$  (3): Given (2). From (\*) we get

$$\begin{aligned} \int_0^1 f(x) \int_0^1 g(x, y) h(y) dy dx &= \int_0^1 \int_0^1 g(x, y) f(y) dy \overline{h(x)} dx \\ &= \int_0^1 f(y) \int_0^1 g(x, y) \overline{h(x)} dx dy \\ &= \int_0^1 f(x) \overline{\int_0^1 g(y, x) h(y) dy} dx \end{aligned}$$

We get

$$\int_0^1 f(x) \overline{\int_0^1 (g(x, y) - \overline{g(y, x)}) h(y) dy} dx = 0, \quad f, g, h \text{ cont.}$$

This implies that

$$\int_0^1 (g(x, y) - \overline{g(y, x)}) h(y) dy = 0, \quad \forall x \in [0, 1].$$

This implies

$$g(x, y) = \overline{g(y, x)}, \quad \forall x, y \in [0, 1]$$

□

**Theorem 4.6.** Assume  $L_0$  symmetric and bijection. Then

1. • 0 is not an eigenvalue of  $L_0$ .  
• 0 is not an eigenvalue of  $\tilde{G}$ .
2.  $f$  is an eigenfunction for  $L_0$  with the eigenvalue  $\mu$  if and only if  $f$  is an eigenfunction for  $\tilde{G}$  with the eigenvalue  $\frac{1}{\mu}$ .

**proof.** 1.  $\mathcal{N}(L_0) = \{0\}$ . So 0 is not an eigenvalue of  $L_0$ . IF  $f$  is an eigenfunction for  $\tilde{G}$  with eigenvalue 0 then for  $u \in C_R^n([0, 1])$  we have

$$\begin{aligned} \langle f, u \rangle &= \langle f, \underbrace{G L_0(u)}_{\in C([0, 1])} \rangle \\ &= \langle f, \tilde{G}(L_0 u) \rangle \\ &= \langle \tilde{G}(f), L_0 u \rangle \\ &= \langle 0, L_0 u \rangle \\ &= 0 \end{aligned}$$

So we have

$$\langle f, u \rangle = 0, \quad \forall u \in C_R^n([0, 1]).$$

**Claim:**  $C_R^n([0, 1])$  is dense in  $L^2([0, 1])$ .

If so, we get

$$f \equiv 0.$$

2. Assume

$$L_0(f) = \mu f, \quad f \in C_R^n([0, 1]) \setminus \{0\}.$$

We have

$$0 \neq f = G(L_0(f)) = G(\mu f) = \tilde{G}(\mu f) = \mu \tilde{G}(f)$$

$$\text{So } \tilde{G}(f) = \frac{1}{\mu} f.$$

$$\text{Assume } \tilde{G}(f) = \frac{1}{\mu} f \text{ for } f \in L^2([0, 1]) \setminus \{0\}.$$

We have

$$\tilde{G}(f)(x) = \frac{1}{\mu} f(x), \quad \text{for all } x \in [0, 1] \text{ except for } x \text{ in a zero set.}$$

Consider

$$\underbrace{\mu \tilde{G}(f)(x)}_{\substack{\text{continuous} \\ \text{function}}} \in C([0, 1])$$

Set

$$h(x) := \mu \tilde{G}(f)(x), \quad x \in [0, 1].$$

We get

$$h(x) = \mu \tilde{G}(h)(x) = \mu \underbrace{G(h)}_{\in C_R^n([0, 1])}(x).$$

and

$$L_0 h = L_0(\mu G(h)) = \mu L_0 G(h) = \mu h.$$

So

$$L_0(h) = \mu h.$$

□

**Theorem 4.7.** Assume  $L_0$  to be symmetric and a bijection. Let  $(\mu_n)_{n=1}^\infty$  be the eigenvalues of  $L_0$  counted with multiplicity and  $(e_n)_{n=1}^\infty$  corresponding ON-sequence of eigenfunctions. Then

1.  $(e_n)_{n=1}^\infty$  is an ON-basis for  $L^2([0, 1])$ .

2. The solution  $u$  of

$$\begin{cases} Lu &= f \in C([0, 1]) \\ R_j u &= 0, \quad j = 1, 2, \dots, n \end{cases}$$

is given by

$$u = \sum_{n=1}^{\infty} \frac{1}{\mu_n} \langle f, e_n \rangle e_n$$

in  $L^2([0, 1])$ .

Note  $Le_n = \mu e_n$ .

## 4.1 Method of continuity

**Theorem 4.8.** Assume  $(E, \|\cdot\|)$  Banach space.  $A_t \in \mathcal{B}(E, E)$  for  $t \in [0, 1]$ . Assume that there exists  $c > 0$  such that

1.  $\|x\| \leq c\|A_t(x)\|$  for all  $x \in E$  and  $t \in [0, 1]$ .
2.  $\|A_t(x) - A_s(x)\| \leq C|t - s|\|x\|$  for all  $x \in E$  and  $s, t \in [0, 1]$ .
3.  $A_0$  is invertible.

Then  $A_1$  is invertible.

**proof.** Assume  $A_t$  is invertible.

$$A_s = \underbrace{A_t}_{\text{invertible}} \underbrace{(I + A_t^{-1}(A_s - A_t))}_{\text{invertible if } \|A_t^{-1}(A_s - A_t)\| < 1 \text{ by Neumann series lemma}}$$

But

$$\|A_t^{-1}(A_s - A_t)\| \leq \underbrace{\|A_t^{-1}\|}_{\leq c} \cdot \underbrace{\|A_s - A_t\|}_{\leq C|t-s|}$$

So if  $|t - s| < \frac{1}{c^2}$  then  $A_s$  is invertible. Pick a sequence  $t_k$  for  $k = 1, \dots, N$  such that

$$\max_{k=1, \dots, N-1} |t_{k+1} - t_k| < \frac{1}{c^2}$$

where  $0 = t_1 < t_2 < \dots < t_N = 1$ . Finally we get if  $A_0$  is invertible that  $A_{t_1}$  is invertible.  $\Rightarrow A_{t_2}$  is invertible.  $\Rightarrow \dots \Rightarrow A_1$  is invertible.  $\square$

## 4.2 Orthogonal projections

Attention: We should have done this earlier.

$(E, \langle \cdot, \cdot \rangle)$  Hilbert space and  $S$  closed subspace. We know that

$$E = S \oplus S^\perp$$

For every  $x \in E$  there are unique  $y \in S, z \in S^\perp$  such that

$$x = y + z$$

Define  $P_S : E \rightarrow E$  with

$$P_S(x) = y, \quad \forall x \in E.$$

We note:

- $P_S$  is linear:  
For  $y_1, y_2 \in S$  we find  $z_1, z_2 \in S^\perp$

$$x_1 = y_1 + z_1$$

$$x_2 = y_2 + z_2$$

For scalars  $\alpha_1, \alpha_2$  we have

$$\alpha_1 x_1 + \alpha_2 x_2 = \underbrace{\alpha_1 y_1 + \alpha_2 y_2}_{\in S} + \underbrace{\alpha_1 z_1 + \alpha_2 z_2}_{\in S^\perp}$$

Then we have

$$P_S(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 P_S(x_1) + \alpha_2 P_S(x_2)$$

and

$$P_S(x_1) = y_1, \quad P_S(x_2) = y_2$$

- $P_S$  is bounded:

$$\begin{aligned} \|P_S(x)\|^2 &= \|y\|^2 \\ &\stackrel{y \perp z}{\leq} \|y\|^2 + \|z\|^2 \\ &= \|x\|^2 \end{aligned}$$

So  $\|P_S\| \leq 1$ . But

$$\|P_S(y)\| = \|y\|, \quad y \in S.$$

So  $\|P_S\| = 1$ .

- $P_S$  is self-adjoint:

$$\begin{aligned} \langle P_S(x_1), x_2 \rangle &= \langle y_1, x_2 \rangle \\ &= \langle y_1, y_2 \rangle \\ &= \langle x_1, y_2 \rangle \\ &= \langle x_1, P_S(x_2) \rangle, \quad \forall x_1, x_2 \in E. \end{aligned}$$

- $P_S^2 = P_S$ :

$$\begin{aligned} P_S^2(x) &= P_S(P_S(x)) \\ &= P_S(y) \\ &= y \\ &= P_S(x) \end{aligned}$$

**Proposition 4.9.** Assume  $P \in \mathcal{B}(E, E)$ . With  $P^2 = P$  and  $P$  self-adjoint. Then there exists a closed subspace  $S$  in  $E$  such that  $P = P_S$

**proof.** What is  $S$ ? Set

$$S = \{x \in E \mid P(x) = x\}$$

**Claim:**  $S$  is a closed subspace in  $E$ .

This is obvious.

**Claim:**  $S$  is closed.

Assume  $x_n \in S$  with  $x_n \rightarrow x$  in  $E$ .

$$P(x_n) \rightarrow P(x) \quad \text{in } E \text{ since } P \text{ continuous.}$$

So  $x \in S$ . By the Orthogonal Decomposition Theorem we have

$$E = S \oplus S^\perp.$$

It remains to show  $P = P_S$ . Fix  $x \in E$ .

$$x = P(x) + x - P(x)$$

show that  $P(x) \in S$  and  $x - P(x) \in S^\perp$ .

1.  $P(x) \in S$  since

$$P(P(x)) = P^2(x) = P(x), \quad \text{by assumption}$$

2.  $x - P(x) \in S^\perp$  since for  $y \in S$  we have

$$\begin{aligned} \langle y, x - P(x) \rangle &= \langle P(y), x - P(x) \rangle \\ &= \langle y, P^*(x - P(x)) \rangle \\ &= \langle y, P(x) - P(x) \rangle \\ &= 0 \end{aligned}$$

□

#### Example 4.1 [last example]

Set  $(E, \langle \cdot, \cdot \rangle)$  Hilbert space.  $(e_n)_{n=1}^\infty$  ON-basis,  $(f_n)_{n=1}^\infty$  ON-sequence.  
Assume

$$\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \infty$$

Then

$$\Rightarrow (f_n)_{n=1}^\infty \text{ is an ON-basis.}$$

**proof. Step 1:** Assume

$$\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1.$$

Assume that  $(f_n)_{n=1}^{\infty}$  is not an ON-basis. Then there exists  $0 \neq x \in E$  such that

$$\langle x, f_k \rangle = 0, \quad k = 1, 2, \dots$$

But  $(e_n)_{n=1}^{\infty}$  is an ON-basis.

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$$

Parseval's formula gives

$$\begin{aligned} 0 &< \|x\|^2 \\ &= \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \\ &= \sum_{n=1}^{\infty} |\langle x, e_n - f_n \rangle|^2 \\ &\leq \sum_{n=1}^{\infty} \|x\|^2 \|e_n - f_n\|^2 \\ &= \|x\|^2 \underbrace{\sum_{n=1}^{\infty} \|e_n - f_n\|^2}_{<1} \\ &< \|x\|^2 \end{aligned}$$

which is a contradiction.

Conclusion:  $(f_n)_{n=1}^{\infty}$  is an ON-basis.

**Step 2:** Assume

$$\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \infty.$$

Assume once more that  $(f_n)_{n=1}^{\infty}$  is not an ON-basis.

There exists  $0 \neq x \in E$  such that

$$\langle x, f_k \rangle = 0, \quad k = 1, 2, \dots$$

Since

$$\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \infty$$

there exists a positive integer  $N \in \mathbb{N}$  such that

$$\sum_{n=N+1}^{\infty} \|e_n - f_n\|^2 < 1.$$

Note that  $\text{span}\{x, f_1, f_2, \dots, f_N\}$  has the dimension  $N + 1$ .

**Claim:** There exists a non-trivial solution to

$$\alpha_0 x + \alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_N f_N \perp e_k, \quad k = 1, 2, \dots, N.$$

$N$  equations,  $N + 1$  unknowns and it is a homogenous linear system. There exists

$$y = \alpha_0 x + \alpha_1 f_1 + \dots + \alpha_N f_N$$

where not all  $\alpha_k$ 's are 0 such that

$$y \perp e_k, \quad \text{for } k = 1, 2, \dots, N.$$

Note that  $y \neq 0$ .

$(e_n)_{n=1}^{\infty}$  is an ON-basis for  $E$ . So

$$\begin{aligned} y &= \sum_{n=1}^{\infty} \langle y, e_n \rangle e_n \\ &= \sum_{n=N+1}^{\infty} \langle y, e_n \rangle e_n \end{aligned}$$

Parseval's formular gives

$$\begin{aligned} 0 &< \|y\|^2 \\ &= \sum_{n=N+1}^{\infty} |\langle y, e_n \rangle|^2 \\ &= \sum_{n=1=N+1}^{\infty} |\langle y, e_n - f_n \rangle|^2 \\ &\leq \|y\|^2 \cdot \sum_{n=N+1}^{\infty} \|e_n - f_n\|^2 < \|y\|^2 \end{aligned}$$

This is a contradiction and we are done. □

