parameters: α, β, p, a: risk-aversion factor, r: risk-free

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Assume that $V_t(W_t) = -b_t \cdot e^{-c_t W_t}$ $Y_t \begin{cases} \alpha & \text{prob} = p \\ \beta & \text{prob} = 1-p \end{cases}$

we have $W_{t+1} = x_t \cdot (Y_t - r) + W_t \cdot (1+r)$

$$V_t^*(W_t) = \max_{x_t} \left\{ E_{Y \sim \text{Bernoli}(\alpha, \beta, p)} -b_{t+1} e^{-c_{t+1} W_{t+1}} \right\}$$

$$= \max_{x_t} \left\{ E_{Y \sim \text{Bernoli}(\alpha, \beta, p)} -b_{t+1} \cdot e^{-c_{t+1} \cdot [x_t \cdot (Y_t - r) + W_t \cdot (1+r)]} \right\}$$

since $V_t^*(W_t) = \max_{x_t} Q_t(W_t, x_t)$

$$\Rightarrow Q_t(W_t, x_t) = -b_{t+1} \cdot \left\{ \underbrace{e^{-c_{t+1} [x_t \cdot (r-\beta) - W_t (1+r)]}}_{\text{define as } J} \cdot \underbrace{[1-p + p \cdot e^{-c_{t+1} x_t (\alpha - \beta)}]}_{\text{define as } K} \right\}$$

Find optimal x_t^* s.t. it maximize $Q_t(W_t, x_t)$

$$\frac{\partial Q_t}{\partial x_t} = -b_{t+1} \cdot \left\{ \underbrace{e^{-c_{t+1} [x_t \cdot (r-b) - W_t (1+r)]}}_{J} \cdot c_{t+1} (r-b) \cdot K + J \cdot p \cdot e^{-c_{t+1} x_t (\alpha - \beta)} - c_{t+1} (\alpha - \beta) \right\} = 0$$

$$\Rightarrow (r-b) p \cdot K - (\alpha - \beta) \cdot p \cdot e^{-c_{t+1} [x_t (\alpha - \beta)]} = 0$$

$$\Rightarrow e^{-c_{t+1} (x_t (\alpha - \beta))} = -\frac{r-b}{r-a} \frac{(1-p)}{p} \Rightarrow x_t^* = \frac{1}{-c_{t+1} (\alpha - \beta)} \ln \left(\frac{r-b}{r-a} \cdot \frac{p-1}{p} \right)$$

$\alpha > r > \beta$

check 2nd order condition:

$$\cancel{\frac{\partial^2 Q_t}{\partial x_t^2} = -b_{t+1} [J \cdot c_{t+1} (r-b)^2 \cdot k + J \cdot c_{t+1} \cdot (r-b) \cdot (-c_{t+1}) \cdot (\alpha - \beta) \cdot p \cdot e^{-c_{t+1} x_t (\alpha - \beta)}}$$

$\Rightarrow$

$$Q_t = -b_{t+1} \cdot e^{-c_{t+1} W_t (1+r)} \cdot E[e^{-c_{t+1} x_t (Y_t - r)}] = -b_{t+1} \cdot e^{-c_{t+1} W_t (1+r)} \cdot [p \cdot e^{-c_{t+1} x_t (\alpha - r)} + (1-p) \cdot e^{-c_{t+1} x_t (\beta - r)}]$$

$$\frac{\partial Q_t}{\partial x_t} = -b_{t+1} \cdot e^{-c_{t+1} W_t (1+r)} \cdot [p \cdot e^{-c_{t+1} x_t (\alpha - r)} \cdot (-c_{t+1} (\alpha - r)) + (1-p) \cdot e^{-c_{t+1} x_t (\beta - r)} \cdot (-c_{t+1}) (\beta - r)]$$

$$\frac{\partial^2 Q_t}{\partial x_t^2} = -b_{t+1} \cdot e^{-c_{t+1} W_t (1+r)} \left[p \cdot e^{-c_{t+1} x_t (\alpha - r)} \cdot [c_{t+1}^2 (\alpha - r)] + (1-p) \cdot e^{-c_{t+1} x_t (\beta - r)} \cdot [(-c_{t+1})(\beta - r)]^2 \right\}$$

$< 0 \Rightarrow x_t^*$ is global maximum

$$x_t^* = \frac{1}{-c_{t+1}\cdot(\alpha-\beta)} \ln\left(\frac{r-b}{r-a}\cdot\frac{p-1}{p}\right)$$

we know $V_t^*(W_t) = \max_{x_t} Q_t(x_t, W_t)$

$$= -b_{t+1}\cdot e^{-c_{t+1}W_t(1+r)}\cdot\left[p\cdot e^{-c_{t+1}x_t^*(\alpha-r)} + (1-p)\cdot e^{-c_t x_t^*(\beta-r)}\right]$$

plug in x^*

$$= -b_{t+1}\cdot e^{-c_{t+1}W_t(1+r)}\cdot \underbrace{\left[p\cdot e^{\frac{\alpha-r}{\alpha-\beta}\ln\left(\frac{r-b}{r-a}\cdot\frac{p-1}{p}\right)} + (1-p)\cdot e^{\frac{\beta-r}{\alpha-p}\ln\left(\frac{r-b}{r-a}\cdot\frac{p-1}{p}\right)}\right]}_{\text{define as } N}$$

by def

$$= -b_t\cdot e^{-c_t W_t}$$

$$\Rightarrow \begin{cases} b_{t+1}\cdot N = b_t \\ c_{t+1}\cdot(1+r) = c_t \end{cases}$$

For $T-1$: $V_{T-1}^*(W_{T-1}) = \max_{x_{T-1}} Q_{T-1}(x_{T-1}, W_{T-1}) = \max_{x_{T-1}}\left\{E_{Y\sim\text{Bernoulli}(\alpha,\beta,p)} \frac{-e^{-a[x_{T-1}\cdot(Y_{T-1}-r)+W_{T-1}\cdot(1+r)]}}{a}\right\}$

$$= \max\left\{\frac{-e^{-aW_T(1+r)}}{a}\cdot E_{Y\sim\text{Bernoulli}(\alpha,\beta,p)}\left[e^{-a x_{T-1}(Y_{T-1}-r)}\right]\right\}$$

from previous, we just need to change $c_{t+1}\to a$ in this case

$\Rightarrow x_{T-1}^* = \frac{1}{-a(\alpha-\beta)}\ln\left(\frac{r-\beta}{r-\alpha}\cdot\frac{p-1}{p}\right)$ plug in x_{T-1}^*

$\Rightarrow V_{T-1}^*(W_{T-1}) = -b_{T-1}\cdot e^{-c_{T-1}W_{T-1}} = \cancel{-b_T\cdot e^{-c_T W_{T-1}(1+r)}\cdot N} - \frac{e^{-aW_T(1+r)}}{a}\cdot N$

$\Rightarrow \begin{cases} b_{T-1} = \frac{N}{a} & \text{~~where~~ } \cancel{b_T = a} \\ c_{T-1} = a(1+r) \end{cases}$

$\Rightarrow$ $b_t = \frac{1}{a}\cdot N^{T-t}$

$c_t = a\cdot(1+r)^{T-t}$

$$\Rightarrow V_t^*(W_t) = -\frac{1}{a}\cdot N^{T-t}\cdot e^{-a(1+r)^{T-t}\cdot W_t}$$

$$\Rightarrow \pi_t^*(W_t) = x_t^* = \frac{1}{-a\cdot(1+r)^{T-t-1}(\alpha-\beta)}\ln\left(\frac{r-b}{r-a}\cdot\frac{p-1}{p}\right)$$

$$Q_t(W_t, x_t) = -\frac{1}{a}\cdot N^{T-t-1} e^{-a(1+r)^{T-t}W_t}\cdot\left[p\cdot e^{-a(1+r)^{T-t-1}(\alpha-r)} + (1-p)\cdot e^{-a(1+r)^{T-t-1}(\beta-r)}\right]$$