Robot kinematics

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March 2024

1 Kinematic chain and homogeneous transformations

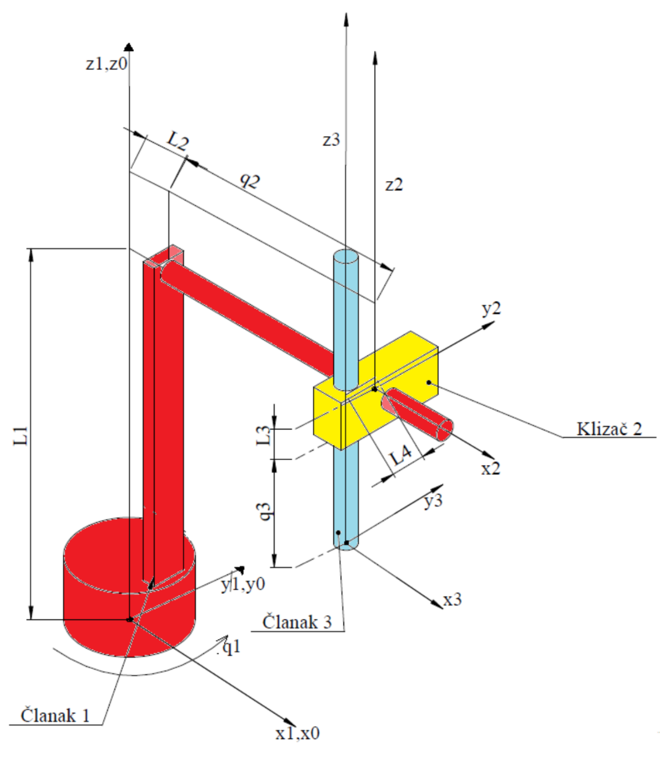

Figure 1: Schematic representation of robot[1]

[1] Tugomir Šurina, Mladen Crneković, (1990). *Industrijski roboti.* Zagreb : Školska knjiga

The homogeneous transformation matrix from coordinate system 1 to the base is given as:

$$^{0}A_1 = \mathrm{Rot}(z, q_1) = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

The transition matrix from the second to the first coordinate system is:

$$A_2 = \mathrm{Tran}(L_2 + q_2, 0, L_1) = \begin{bmatrix} 1 & 0 & 0 & L_2 + q_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

The transition matrix from the third to the second coordinate system is:

$$A_3 = \mathrm{Tran}(0, -L_4, -(L_3 + q_3)) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -L_4 \\ 0 & 0 & 1 & -(L_3 + q_3) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

The notation for writing trigonometric functions is introduced:

$$s_1 = \sin(q_1), \quad c_1 = \cos(q_1) \tag{4}$$

Relative transformations are obtained by multiplying the basic homogeneous transformations; hence, the transformation from the first to the base coordinate system is equal to:

$$^{0}T_1 = A_1 \tag{5}$$

The transformation from the second to the base coordinate system is obtained by multiplying the first two homogeneous transformations:

$$^{0}T_2 = A_1 A_2 = \begin{bmatrix} c_1 & -s_1 & 0 & c_1(L_2 + q_2) \\ s_1 & c_1 & 0 & s_1(L_2 + q_2) \\ 0 & 0 & 1 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

The transformation from the third to the base coordinate system is obtained by multiplying the first three homogeneous transformations:

$$^{0}T_3 = A_1 A_2 A_3 = \begin{bmatrix} c_1 & -s_1 & 0 & s_1 L_4 + c_1(L_2 + q_2) \\ s_1 & c_1 & 0 & -c_1 L_4 + s_1(L_2 + q_2) \\ 0 & 0 & 1 & -L_3 - q_3 + L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

The meaning of the elements in the matrix ${}^0T_3$ is as follows:

$$
{}^0T_3 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}
$$

By comparing expressions (1.8) and (1.7) we see that the vector of external coordinates for the given robot is:

$$
\mathbf{r} = \begin{bmatrix} p_x & p_y & p_z \end{bmatrix}^T \tag{9}
$$

where:

$$
\begin{aligned}
p_x &= s_1 L_4 + c_1(L_2 + q_2) \\
p_y &= -c_1 L_4 + s_1(L_2 + q_2) \\
p_z &= -L_3 - q_3 + L_1
\end{aligned} \tag{10}
$$

With this, the direct kinematics problem is solved.

```
syms q1 L1 L2 L4 q2 q3 real
syms c1 s1

c1 = cos(q1);
s1 = sin(q1);

A01 =[c1 -s1 0 0;
      s1  c1 0 0;
      0 0 1 0;
      0 0 0 1];
A12 =[1 0 0 L2 + q2;
      0 1 0 0;
      0 0 1 L1;
      0 0 0 1];
A23 =[1 0 0 L4;
      0 1 0 0;
      0 0 1 q3;
      0 0 0 1];

T02 = A01 * A12;
T03 = A01 * A12 * A23;

T03_simplifed = simplify(T03);
disp('Transformation matrices');
pretty(T03_simplifed);


% end-effector position vector as a function of joint variables
p = T03(1:3,4);

P_simplified = simplify(p);
disp('End-effector position vector');
pretty(P_simplified);
```

Listing 1: MATLAB symbolic calculation of matrix multiplication

2 Jacobian Matrix

The Jacobian matrix J relates the velocity of the control vector in joint coordinates to the velocity in the external coordinates. The expression for calculating the Jacobian matrix is as follows:

$$J = \frac{\partial \mathbf{r}}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial p_x}{\partial q_1} & \frac{\partial p_x}{\partial q_2} & \frac{\partial p_x}{\partial q_3} \\ \frac{\partial p_y}{\partial q_1} & \frac{\partial p_y}{\partial q_2} & \frac{\partial p_y}{\partial q_3} \\ \frac{\partial p_z}{\partial q_1} & \frac{\partial p_z}{\partial q_2} & \frac{\partial p_z}{\partial q_3} \end{bmatrix} \tag{11}$$

The elements of the Jacobian matrix are thus:

$$\begin{aligned}
J_{11} &= \frac{\partial p_x}{\partial q_1} = c_1 L_4 - s_1(L_2 + q_2) \\
J_{12} &= \frac{\partial p_x}{\partial q_2} = c_1 \\
J_{13} &= \frac{\partial p_x}{\partial q_3} = 0 \\
J_{21} &= \frac{\partial p_y}{\partial q_1} = s_1 L_4 + c_1(L_2 + q_2) \\
J_{22} &= \frac{\partial p_y}{\partial q_2} = s_1 \\
J_{23} &= \frac{\partial p_y}{\partial q_3} = 0 \\
J_{31} &= \frac{\partial p_z}{\partial q_1} = 0 \\
J_{32} &= \frac{\partial p_z}{\partial q_2} = 0 \\
J_{33} &= \frac{\partial p_z}{\partial q_3} = -1
\end{aligned}$$

By substituting expressions (1.12) into (1.11) we obtain the Jacobian matrix:

$$J = \begin{bmatrix} c_1 L_4 - s_1(L_2 + q_2) & c_1 & 0 \\ s_1 L_4 + c_1(L_2 + q_2) & s_1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{12}$$

3 Robot dynamics

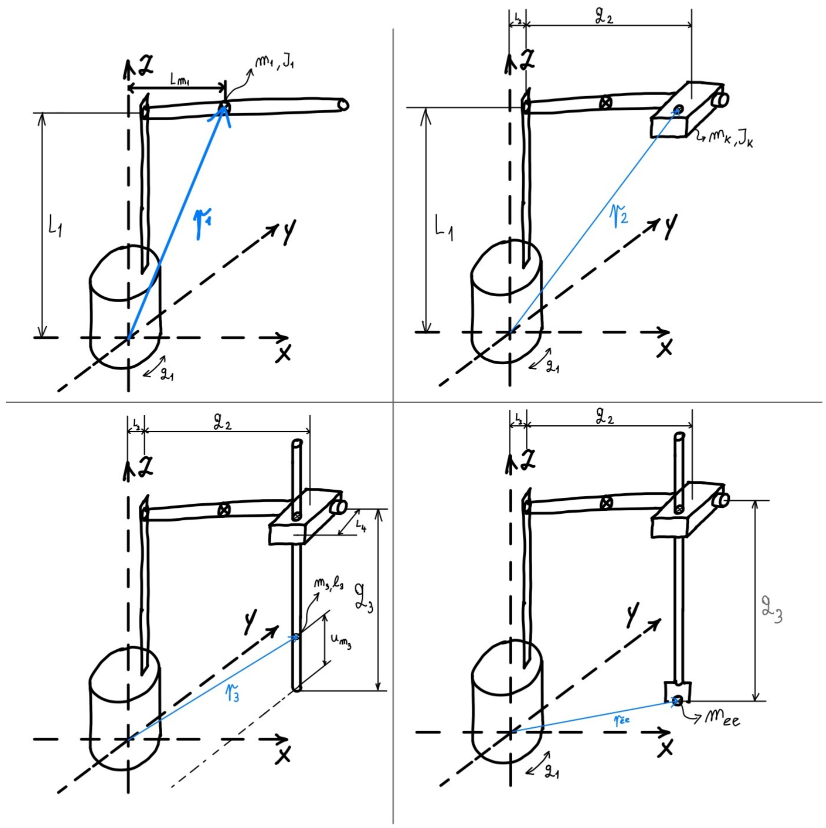

Figure 2: Position vectors of four bodies defined for dynamic analysis

3.1 First body

The position vector of the mass center for the first body consisting of cylinder and linear guide:

$$p_1 = {}^0T_1 R_1 = \begin{bmatrix} c_1(L_{m1}) \\ s_1(L_{m1}) \\ 0 \\ 1 \end{bmatrix} \tag{13}$$

The velocity of the first body mass is:

$$v = (-s_1 \dot{q}_1 (L_{m1}))\mathbf{i} + (c_1 \dot{q}_1 (L_{m1}))\mathbf{j} + 0\mathbf{k} \tag{14}$$

The square of the velocity is then equal to:

$$v^2(u) = \dot{q_1}^2 L_{m1} \tag{15}$$

Kinetic energy consist of two parts:

- Rotational of the cilinder

$$K_c = \frac{1}{2} J_c \dot{q_1}^2 \tag{16}$$

- Mass velocity

$$K_v = \frac{1}{2} m_v v^2(q_1) = \frac{1}{2} m_v \dot{q_1}^2 L_{m1}^2 \tag{17}$$

Kinetic energy of the first is then:

$$K_1 = \frac{1}{2} \dot{q_1}^2 (J_c + L_{m1}^2 m_v) = \frac{1}{2} \dot{q_1}^2 J_1 \tag{18}$$

Potential energy is:

$$P_1 = m_1 g L_1 \tag{19}$$

As potential energy is not dependent on the first revolute joint only torque dynamics comes from kinetic energy of the rotational mass:

$$T_{11} = J_1 \ddot{q}_1 \tag{20}$$

3.2 Second body

The position vector of slider (CoM):

$$p_2 = {}^0T_2 R_2 = \begin{bmatrix} c_1(L_2 + q_2) \\ s_1(L_2 + q_2) \\ L_1 \\ 1 \end{bmatrix} \tag{21}$$

The velocity of the mass is:

$$v_2 = (\dot{q}_1 - s_{q_1}(L_2 + q_2))i + (\dot{q}_2 + s_{q_1}(L_2 + q_2))j + 0k \tag{22}$$

The square of the velocity is then equal to:

$$v_2^2 = \dot{q}_1^2 + \dot{q}_2^2(L_2 + q_2)^2 \tag{23}$$

Kinetic energy is:

$$K_2 = \frac{1}{2} m_k(\dot{q}_1^2 + \dot{q}_2^2(L_2 + q_2)^2) \tag{24}$$

Potential energy is:

$$P_2 = m_k g L_1 \tag{25}$$

From two equations defining lagrangian mechanics:

$$L(q, \dot{q}) = K - P \tag{26}$$

$$\tau_i = \frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q} \tag{27}$$

The torque in the first controlled coordinate for slider displacement is:

$$\tau_{12} = \frac{d}{dt}\left(\frac{\partial K_2}{\partial \dot{q}_1}\right) - \frac{\partial K_2}{\partial q_1} - \frac{\partial P_2}{\partial q_1} \tag{28}$$

$$= \frac{d}{dt}\left(m_k(L_2 + q_2)\dot{q}_1\right) \tag{29}$$

$$= m_k(L_2 + q_2)\ddot{q}_1 + 2m_k(L_2 + q_2)\dot{q}_1\dot{q}_2 \tag{30}$$

The torque in the second controlled coordinate for slider displacement is:

$$\tau_{22} = \frac{d}{dt}\left(\frac{\partial K_2}{\partial \dot{q}_2}\right) - \frac{\partial K_2}{\partial q_2} - \frac{\partial P_2}{\partial q_2} \tag{31}$$

$$= \frac{d}{dt}\left(-m_k\dot{q}_2\right) - m_k\dot{q}_1^2(L_2 + q_2) \tag{32}$$

$$= m_k\ddot{q}_2 - m_k\dot{q}_1^2(L_2 + q_2) \tag{33}$$

3.3 Third body

The third position vector is defined as:

$$\mathbf{p}_3^0 = {}^0\mathbf{T}_3 R_3 = \begin{bmatrix} s_1 L_4 + c_1(L_2+q_2) \\ -c_1 L_4 + s_1(L_2+q_2) \\ u_{m3} + L_1 - q_3 \\ 1 \end{bmatrix} \tag{34}$$

The velocity is time derivative of position vector:

$$v_3 = (c_1\dot{q}_4 L_4 + c_1\dot{q}_2 - s_1\dot{q}_1(L_2+q_2))\mathbf{i} + (s_1\dot{q}_1 L_4 + s_1\dot{q}_2 + c_1\dot{q}_1(L_2+q_2))\mathbf{j} - \dot{q}_3\mathbf{k} \tag{35}$$

Then the square of the velocity is:

$$v_3^2 = \dot{q_3}^2 + \dot{q_2}^2 + \dot{q_1}^2(L_2+q_2)^2 + \dot{q_4}^2 L_4^2 + 2\dot{q}_1\dot{q}_2 L_4 \tag{36}$$

Kinetic energy of the third body is then defined as:

$$K_3 = \frac{1}{2} m_3 \left(\dot{q}_3^2 + \dot{q}_2^2 + \dot{q}_1^2(L_4+q_2)^2 + q_1^2 L_4^2 + 2\dot{q}_1\dot{q}_2 L_4\right) \tag{37}$$

Potential energy:

$$P_3 = m_3 g(\frac{1}{2} l_3 + L_1 - q_3) \tag{38}$$

Torques in controlled coodinates are then:

$$\tau_{13} = \frac{d}{dt}\left(\frac{\partial K_3}{\partial \dot{q}_1}\right) - \frac{\partial K_3}{\partial q_1} - \frac{\partial P_3}{\partial q_1} \tag{39}$$

$$= \frac{d}{dt}\left(m_3(L_4+q_2)^2\dot{q}_1 + m_3\dot{q}_2 L_4\right) \tag{40}$$

$$= m_3((L_4+q_2)^2 + L_4^2)\ddot{q}_1 + 2m_3(L_4+q_2)\dot{q}_2\dot{q}_2 + m_3\ddot{q}_2 L_4 \tag{41}$$

$$\tau_{23} = \frac{d}{dt}\left(\frac{\partial K_3}{\partial \dot{q}_2}\right) - \frac{\partial K_3}{\partial q_2} - \frac{\partial P_3}{\partial q_2} \tag{42}$$

$$= m_3\ddot{q}_2 - m_3\dot{q_1}^2(L_4+q_2) + m_3\ddot{q}_1 L_4 \tag{43}$$

$$\tau_{33} = \frac{d}{dt}\left(\frac{\partial K_3}{\partial \dot{q}_3}\right) - \frac{\partial K_3}{\partial q_3} - \frac{\partial P_3}{\partial q_3} \tag{44}$$

$$= \frac{d}{dt}(m_3 q_3) + m_3 g = m_3\ddot{q}_3 + m_3 g \tag{45}$$

3.4 End-effector

The end-effector position vector is defined as:

$$\mathbf{p}_{ee}^{0} = {}^{0}\mathbf{T}_{3} R_{ee} = \begin{bmatrix} s_1 L_4 + c_1(L_2 + q_2) \\ -c_1 L_4 + s_1(L_2 + q_2) \\ L_1 - q_3 \\ 1 \end{bmatrix} \tag{46}$$

The velocity is the same as for third body:

$$v_3 = (c_1\dot{q}_4 L_4 + c_1\dot{q}_2 - s_1\dot{q}_1(L_2+q_2))\mathbf{i} + (s_1\dot{q}_1 L_4 + s_1\dot{q}_2 + c_1\dot{q}_1(L_2+q_2))\mathbf{j} - \dot{q}_3\mathbf{k} \tag{47}$$

Then the square of the velocity is:

$$v_3^2 = \dot{q_3}^2 + \dot{q_2}^2 + \dot{q_1}^2(L_2 + q_2)^2 + \dot{q_4}^2 L_4^2 + 2\dot{q}_1\dot{q}_2 L_4 \tag{48}$$

Kinetic energy is defined similar as third body but the mass it is used is from end-effector:

$$K_{ee} = \frac{1}{2} m_{ee} \left(\dot{q}_3^2 + \dot{q}_2^2 + \dot{q}_1^2(L_4 + q_2)^2 + q_1^2 L_4^2 + 2\dot{q}_1\dot{q}_2 L_4\right) \tag{49}$$

Potential energy:

$$P_3 = m_3 g(L_1 - q_3) \tag{50}$$

$$\tau_{1ee} = \frac{d}{dt}\left(\frac{\partial K_e e}{\partial \dot{q}_1}\right) - \frac{\partial K_e e}{\partial q_1} - \frac{\partial P_e e}{\partial q_1} \tag{51}$$

$$= m_{ee}\left((L_2 + q_2)^2 + L_2^2\right)\ddot{q}_1 + 2m_{ee}(L_2 + q_2)\dot{q}_2\dot{q}_2 + m_{ee}\ddot{q}_2 L_2 \tag{52}$$

$$\tau_{2ee} = \frac{d}{dt}\left(\frac{\partial K_e e}{\partial \dot{q}_2}\right) - \frac{\partial K_{ee}}{\partial q_2} - \frac{\partial P_{ee}}{\partial q_2} \tag{53}$$

$$= m_{ee}\ddot{q}_2 - m_{ee}\dot{q_1}^2(L_2 + q_2) + m_{ee}\ddot{q}_1 L_4 \tag{54}$$

$$\tau_{3ee} = \frac{d}{dt}\left(\frac{\partial K_{ee}}{\partial \dot{q}_3}\right) - \frac{\partial K_{ee}}{\partial q_3} - \frac{\partial P_{ee}}{\partial q_3} \tag{55}$$

$$= \frac{d}{dt}(m_{ee} q_3) + m_{ee} g = m_{ee}\ddot{q}_3 + m_{ee} g \tag{56}$$

4 Dynamics

Combining derived equations for torques on every joint depending on body which is in motion we can represent robot dynamics in general form:

$$\mathbf{M(q)}\ddot{q} + \mathbf{C}(\dot{q}, \ddot{q}) + \mathbf{h}(q) = \tau \tag{57}$$

Added torques together for every joint separately are:

$$\tau_1 = (J_1 + (m_k + m_3 + m_{ee})((L_2 + q_2)^2 + L_2^2))\ddot{q}_1 + 2(m_k + m_3 + m_{ee})L_2(L_2 + q_2)\dot{q}_1\dot{q}_2 + (m_{ee} + m_3)\ddot{q}_2 L_2 \tag{58}$$

$$\tau_2 = (m_k + m_3 + m_p)\ddot{q}_2 = (m_k + m_3 + m_p)\dot{q}_1^2(L_2 + q_2) + (m_p + m_3)q_1\ddot{L}_4 \tag{59}$$

$$\tau_3 = (m_3 + m_{ee})\ddot{q}_3 + (m_3 + m_{ee})g \tag{60}$$

Expanding the equation (57) for general robot dynamics we can defines parts of the matrices from the torques parts of the every joint (58)

$$\begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\end{bmatrix} = \begin{bmatrix}H_1\\ H_2\\ H_3\end{bmatrix} + \begin{bmatrix}C_1\\ C_2\\ C_3\end{bmatrix} + \begin{bmatrix}M_{11} & M_{12} & M_{13}\\ M_{21} & M_{22} & M_{23}\\ M_{31} & M_{32} & M_{33}\end{bmatrix}\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\\ \ddot{q}_3\end{bmatrix} \tag{61}$$

Now we can specify each member of the matrix:

$$\begin{aligned} &H_1 = 0 && C_1 = 2(m_k + m_3 + m_{ee})(L_2 + q_2)\dot{q}_1\dot{q}_1\\ &H_2 = 0 && C_2 = -(m_k + m_3 + m_{ee})\dot{q}_1{}^2(L_2 + q_2)\\ &H_3 = (m_1 + m_{ee})g && C_3 = 0 \end{aligned}$$

Mass matrix is then:

$$\begin{aligned} &M_{11} = J_1 + (m_k + m_3 + m_{ee})(L_2 + q_2)^2 && M_{12} = (m_3 + m_{ee})L_4 && M_{13} = 0\\ &M_{21} = (m_3 + m_{ee})L_4 && M_{22} = m_k + m_3 + m_{ee} && M_{23} = 0\\ &M_{31} = 0 && M_{23} = 0 && M_{33} = m_3 + m_{ee} \end{aligned}$$

$$\begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ (m_1 + m_{ee})g\end{bmatrix} + \begin{bmatrix}2(m_k + m_3 + m_{ee})(L_2 + q_2)\dot{q}_1\dot{q}_1\\ -(m_k + m_3 + m_{ee})\dot{q}_1{}^2(L_2 + q_2)\\ 0\end{bmatrix} + \begin{bmatrix}J_1 + (m_k + m_3 + m_{ee})(L_2 + q_2)^2 & (m_3 + m_{ee})L_4 & 0\\ (m_3 + m_{ee})L_4 & m_k + m_3 + m_{ee} & 0\\ 0 & 0 & m_3 + m_{ee}\end{bmatrix}\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\\ \ddot{q}_3\end{bmatrix}$$