TP4_14_04_2021_14h30

$f: D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$(x_1, \dots, x_n) \longmapsto (f_1, f_2, \dots, f_m)$

$$D f(x_1, \cdots, x_n) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

$n=2 \qquad m=2$

(c) $f(x,y) = (\underbrace{2x+3y}_{f_1}, \underbrace{4x+5y}_{f_2})$;

$$\longrightarrow D f(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}_{2\times 2}$$

ou $J f(x,y)$

ou $J_{(x,y)} f$

matriz Jacobiana ou matriz das derivadas parciais

(e) $f(x,y,z) = (\underbrace{x+e^z+y}_{f_1}, \underbrace{yx^2}_{f_2})$;

$$D f(x,y,z) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix}_{2\times 3} = \begin{bmatrix} 1 & 1 & e^z \\ 2xy & x^2 & 0 \end{bmatrix}$$

(f) $g(x,y) = (xye^{xy}, x\sin y, 5xy^2)$

$Dg(x,y)$ ou $Jg(x,y)$ ou $J_{(x,y)}\, g$

$$Dg(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix}_{3\times 2} = \begin{bmatrix} ye^{xy} + xy^2 \cdot e^{xy} & xe^{xy} + x^2 y e^{xy} \\ \sin y & x\cos y \\ 5y^2 & 10xy \end{bmatrix}$$

$(Kf)' = K \cdot f'$

$\frac{\partial}{\partial x}(x \sin y) = \sin y \cdot \underbrace{\frac{\partial}{\partial x}(x)}_{=1}$

$xy \cdot e^{xy}$

$\frac{\partial}{\partial x} e^{xy} = y\, e^{xy}$

25. A10 Determine as funções derivadas parciais de primeira ordem de:

(a) $f(x,y) = \begin{cases} 0 & \text{se} \quad (x,y) = (0,0) \\ \frac{x \text{ seny}}{x^2+y^2} & \text{se} \quad (x,y) \neq (0,0) \end{cases}$;

$f(h,0) = \frac{h \text{ sen } 0}{h^2+0^2} = \frac{0}{h^2} = 0$

$f(0,h) = \frac{0 \cdot \text{sen } h}{0^2+h^2} = 0$

AM I

$f(x) = \begin{cases} 0 & \text{se } x=0 \\ \bigcirc & \text{se } x \neq 0 \end{cases}$

$f'(x) = \begin{cases} \lim_{h\to 0} \frac{f(x)-f(0)}{x-0} & \text{se } x=0 \\ \text{Regras} & \text{se } x \neq 0 \end{cases}$

• Se $(x,y) \neq (0,0)$, $f(x,y) = \frac{x \text{ sen } y}{x^2+y^2}$

(1) $\longrightarrow$ $$\boxed{\frac{\partial f}{\partial x} = \frac{(\text{sen } y)\cdot(x^2+y^2) - x \cdot \text{sen } y \cdot (2x+0)}{(x^2+y^2)^2} = \frac{\text{sen } y \, (x^2+y^2-2x^2)}{(x^2+y^2)^2} = \frac{\text{sen } y \cdot (y^2-x^2)}{(x^2+y^2)^2}}$$

(2) $\longrightarrow$ $$\frac{\partial f}{\partial y} = \frac{(x \cdot \cos y)\cdot(x^2+y^2) - x \cdot \text{sen } y \cdot (2y)}{(x^2+y^2)^2} = \frac{x \cos y \, (x^2+y^2) - 2xy \text{ sen } y}{(x^2+y^2)^2}$$

• Se $(x,y) = (0,0)$,

$$\xrightarrow{(3)} \quad \frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{\overbrace{f(0+h,0)}^{f(h,0)=0} - \overbrace{f(0,0)}^{0}}{h} = \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} 0 = 0$$

$$\xrightarrow{(4)} \quad \frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{\overbrace{f(0,0+h)}^{f(0,h)=0} - \overbrace{f(0,0)}^{0}}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

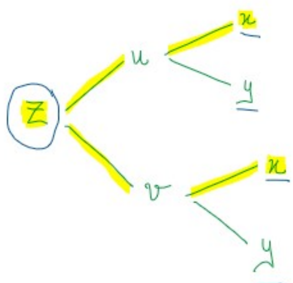
Concluimos que:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} 0 & \text{se } (x,y) = (0,0) \\ \dfrac{\operatorname{sen} y \cdot (y^2 - x^2)}{(x^2+y^2)^2} & \text{se } (x,y) \neq (0,0) \end{cases} \qquad \text{(resulta de (1) e (3))}$$

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} 0 & \text{se } (x,y) = (0,0) \\ \dfrac{x \cos y\,(x^2+y^2) - 2xy \operatorname{sen} y}{(x^2+y^2)^2} & \text{se } (x,y) \neq (0,0) \end{cases}$$

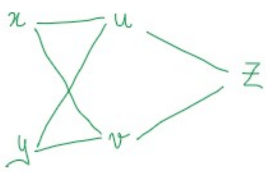
29. Usando a regra da cadeia, calcule $\dfrac{\partial z}{\partial x}$ e $\dfrac{\partial z}{\partial y}$ para:

(a) $z = e^{3u+v}$, onde $u = x + y^3$ e $v = \ln(x+y)$;



$$\frac{\partial z}{\partial x} = \underbrace{\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}}_{} + \underbrace{\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}}_{}$$

$$= 3e^{3u+v} \cdot \underbrace{1} + e^{3u+v} \underbrace{\frac{1}{x+y}}$$



$$= 3 \cdot e^{3u+v} + e^{3u+v} \cdot \frac{1}{x+y}$$

$$= e^{3u+v}\left(3 + \frac{1}{x+y}\right) = \underbrace{e^{3(x+y^3) + \ln(x+y)}}_{e^{3(x+y^3)} \cdot e^{\ln(x+y)}} \cdot \left(3 + \frac{1}{x+y}\right) = e^{3(x+y^3)}\big(3(x+y)+1\big)$$

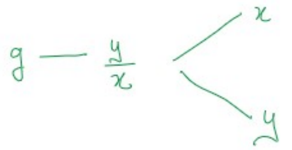
$$e^{3(x+y^3)} \cdot (x+y)$$

31. Considere $\boxed{w = g\left(\frac{y}{x}\right),\ x \neq 0}$, para alguma função g derivável em $\mathbb{R}$. Mostre que $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0$.

$w = g\left(\frac{y}{x}\right)$ g é derivável em $\mathbb{R}$

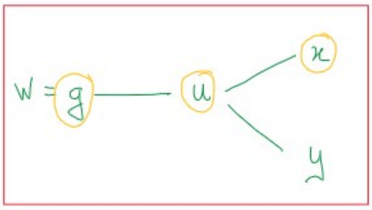
u

$g(\cancel{x,y})$



$\boxed{x}$ $\boxed{y}$ $\boxed{\ }\left(\frac{y}{x}\right)$

$W = g(u)$ $\boxed{u = \frac{y}{x}} = \frac{1}{x}\cdot y$



$g(u)$ $g'(u) = \frac{dg}{du}$

$$\frac{\partial w}{\partial x} = \frac{\partial g}{\partial x} = \frac{dg}{du}\frac{\partial u}{\partial x} = g'(u)\underbrace{\frac{\partial u}{\partial x}}_{-\frac{1}{x^2}\cdot y} = g'(u)\cdot\left(-\frac{1}{x^2}\cdot y\right)$$

$$\frac{\partial w}{\partial y} = \frac{\partial g}{\partial y} = \frac{dg}{du}\cdot\frac{\partial u}{\partial y} = g'(u)\cdot\frac{1}{x}$$

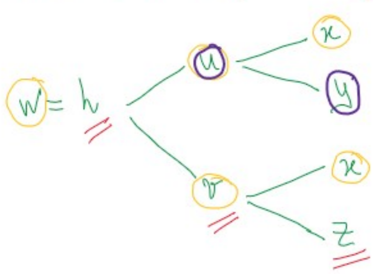
$$\underbrace{x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y}}= 0$$

$$x\cdot g'(u)\cdot\left(-\frac{1}{x^2}\cdot y\right) + y\cdot g'(u)\cdot\frac{1}{x} = -\frac{1}{x}y\cdot g'(u) + \frac{1}{x}y\,g'(u) = \left(-\cancel{\frac{y}{x}} + \cancel{\frac{y}{x}}\right)g'(u) = 0$$

32. Considere $\boxed{w = h(y^2 - x^2, x^2 - z^2)}$, sendo h diferenciável em $\mathbb{R}^2$. Prove a igualdade

$$yz\frac{\partial w}{\partial x} + xz\frac{\partial w}{\partial y} + xy\frac{\partial w}{\partial z} = 0\,.$$

$W = h(u, v)$ $\boxed{u = y^2 - x^2}$ $\boxed{v = x^2 - z^2}$



$$\frac{\partial w}{\partial x} = \frac{\partial h}{\partial x} = \underbrace{\frac{\partial h}{\partial u}}\underbrace{\frac{\partial u}{\partial x}}_{-2x} + \underbrace{\frac{\partial h}{\partial v}}\underbrace{\frac{\partial v}{\partial x}}_{2x} = 2x\left(-\frac{\partial h}{\partial u} + \frac{\partial h}{\partial v}\right)$$

$$\frac{\partial w}{\partial y} = \frac{\partial h}{\partial y} = \frac{\partial h}{\partial u}\underbrace{\frac{\partial u}{\partial y}}_{2y} = 2y\,\frac{\partial h}{\partial u}$$

$$\frac{\partial w}{\partial z} = \frac{\partial h}{\partial v}\underbrace{\frac{\partial v}{\partial z}}_{-2z} = -2z\cdot\frac{\partial h}{\partial v}$$

$$yz\frac{\partial w}{\partial x}+xz\frac{\partial w}{\partial y}+xy\frac{\partial w}{\partial z} = yz.\,2x\left(-\frac{\partial h}{\partial u}+\frac{\partial h}{\partial v}\right) + xz.\,2y\,\frac{\partial h}{\partial u} + xy.(-2z)\frac{\partial h}{\partial v}$$

$$= \cancel{-2xyz\frac{\partial h}{\partial u}} + \cancel{2xyz\frac{\partial h}{\partial v}} + \cancel{2xyz\frac{\partial h}{\partial u}} - \cancel{2xyz\frac{\partial h}{\partial v}} = 0 \;//$$