$F(x,y,z)=0$

41. Mostre que a equação $\boxed{xz + y + z^2 = 7}$ define $z$ como função de $(x, y)$ numa vizinhança de $(1, 1, 2)$ e determine $\frac{\partial z}{\partial x}(1,1)$, $\frac{\partial z}{\partial y}(1,1)$ e $\frac{\partial^2 z}{\partial x \partial y}(1,1)$.

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \qquad z = f(x,y)$$

$$\frac{\partial z}{\partial x}(1,1) \qquad \frac{\partial z}{\partial y}(1,1)$$

$$xz + y + z^2 = 7 \iff z = f(x,y)$$

$$\underbrace{xz + y + z^2 - 7}_{F(x,y,z)} = 0$$

(i) $F(1,1,2) = 0$ $\quad 1.2 + 1 + 2^2 - 7 = 0 \Leftrightarrow 2+1+4-7=0 \Leftrightarrow 0=0$ ✓

(ii) $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ e $\frac{\partial F}{\partial z}$ são contínuas numa vizinhança de $(1,1,2)$ ✓

$$\frac{\partial F}{\partial x} = z, \quad \frac{\partial F}{\partial y} = 1, \quad \boxed{\frac{\partial F}{\partial z} = x + 2z} \longrightarrow$$

são contínuas em $\mathbb{R}^3$, visto serem funções polinomiais, donde são contínuas numa qualquer vizinhança de $(1,1,2)$

(iii) $\frac{\partial F}{\partial z}(1,1,2) \neq 0$

$$\frac{\partial F}{\partial z}(1,1,2) = (x+2z)\Big|_{(1,1,2)} = 1 + 2.2 = 5 \neq 0$$

Pelo T. da função implícita concluímos que a eq. $F(x,y,z)=0$ define $z$ como função de $(x,y)$.

$$\frac{\partial z}{\partial x}(1,1) = -\frac{\frac{\partial F}{\partial x}(1,1,2)}{\frac{\partial F}{\partial z}(1,1,2)} = -\frac{z\big|_{(1,1,2)}}{(x+2z)\big|_{(1,1,2)}} = -\frac{2}{5}$$

$$\frac{\partial z}{\partial y}(1,1) = -\frac{\frac{\partial F}{\partial y}(1,1,2)}{\frac{\partial F}{\partial z}(1,1,2)}$$

$$\frac{\partial z}{\partial y}(1,1) = -\frac{1}{(x+2z)\big|_{(1,1,2)}} = -\frac{1}{5}$$

$$\frac{\partial z}{\partial x} = -\frac{z}{x+2z} \qquad \frac{\partial z}{\partial y} = -\frac{1}{x+2z}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{1}{x+2z}\right) = -\frac{0 - 1.\frac{\partial}{\partial x}(x+2z)}{(x+2z)^2} = \frac{\frac{\partial}{\partial x}(x+2z)}{(x+2z)^2}$$

$$= \frac{1 + 2\frac{\partial z}{\partial x}}{(x+2z)^2}$$

$z = f(x,y)$

$z - 1 \quad x$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}$$

$$z = f \begin{cases} x \\ y \end{cases} \qquad \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}$$

$$(x+2z)^2$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1 + 2 \cdot \frac{\partial z}{\partial x}}{(x+2z)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y}(1,1) = \frac{1 + 2 \overbrace{\frac{\partial z}{\partial x}(1,1)}^{=-\frac{2}{5}}}{(x+2z)^2\big|_{(1,1)}} = \frac{1 + 2 \cdot \left(-\frac{2}{5}\right)}{(1+2\cdot 2)^2} = \frac{1 - \frac{4}{5}}{5^2} = \frac{\frac{1}{5}}{5^2} = \frac{1}{125}$$

$$\left.\begin{array}{l} x = 1 \\ y = 1 \end{array}\right\} \Rightarrow z = 2$$

42. Seja $g : \mathbb{R} \longrightarrow \mathbb{R}$ uma função com derivada contínua tal que $g(\pi) = -1$.

$F(x,y,z) = 0$

(a) Prove que a equação $x^2 + 2\cos(yz) - g(\underbrace{\tfrac{y}{z}}_{g(u)}) = 0$ define $x$ como função implícita de $(y, z)$ numa vizinhança de $(-1, \pi, 1)$.

$\boxed{\frac{y}{z} = u}$ $\quad g \longrightarrow u \begin{cases} y \\ z \end{cases}$

(b) Mostre que $\left(y\,\dfrac{\partial x}{\partial y} + z\,\dfrac{\partial x}{\partial z}\right)\Big|_{(\pi,1)} = 0.$

$$\frac{\partial g}{\partial y} = \underbrace{\frac{dg}{du}}_{g'(u)} \frac{\partial u}{\partial y}$$

$$F(x,y,z) = x^2 + 2\cos(yz) - g\left(\frac{y}{z}\right) = x^2 + 2\cos(yz) - g(u)$$

a) (i) $F(-1,\pi,1) = (-1)^2 + 2\cos(\pi \cdot 1) - g\left(\frac{\pi}{1}\right) = 1 - 2 - g(\pi) = 1 - (-1) = -1 + 1 = 0$ ✓

(ii) $\dfrac{\partial F}{\partial x} = 2x \quad$ contínua

$\boxed{u = \frac{y}{z} = \frac{1}{z} \cdot y}$ $\quad \dfrac{\partial u}{\partial z} = y \cdot \left(-\dfrac{1}{z^2}\right)$

$$\frac{\partial F}{\partial y} = 0 - 2z\sin(yz) - \overbrace{\frac{dg}{du}}^{g'(u)} \cdot \overbrace{\frac{\partial u}{\partial y}}^{\frac{1}{z}} = \underbrace{-2z \cdot \sin(yz)}_{\text{contínua}} - \underbrace{g'(u)}_{\text{contínua pelo enunciado}} \cdot \underbrace{\frac{1}{z}}_{\text{contínua se } z \neq 0}$$

$$\frac{\partial F}{\partial z} = 0 - 2y\sin(yz) - \overbrace{\frac{dg}{du}}^{g'(u)} \cdot \overbrace{\frac{\partial u}{\partial z}}^{-\frac{y}{z^2}}$$

$$= \underbrace{-2y\sin(yz)}_{\text{contínua}} + \underbrace{g'(u)}_{\text{contínua}} \cdot \underbrace{\frac{y}{z^2}}_{\text{contínua se } z \neq 0}$$

$\dfrac{\partial F}{\partial x}$, $\dfrac{\partial F}{\partial y}$ e $\dfrac{\partial F}{\partial z}$ são contínuas no conjunto $\{(x,y,z) \in \mathbb{R}^3 : z \neq 0\}$

$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ e $\frac{\partial F}{\partial z}$ são contínuas no conjunto $\{(x,y,z) \in \mathbb{R}^3 : z \neq 0\}$

donde são contínua numa vizinhança do ponto $(-1, \pi, 1)$

(iii) $\frac{\partial F}{\partial x}(-1,\pi,1) = 2x\Big|_{(-1,\pi,1)} = -2 \neq 0$ ✓ $\qquad P_0(-1,\pi,1)$

$F(x,y,z)=0 \overset{?}{\Leftrightarrow} x = f(y,z)$

Pelo T. da função implícita concluímos que a eq. $F(x,y,z)=0$ define $x$ como função de $(y,z)$.

b) $\left(y\frac{\partial x}{\partial y} + z\frac{\partial x}{\partial z}\right)\Big|_{(\pi,1)} = 0$ ???

$\pi \cdot \left(-\frac{g'(\pi)}{2}\right) + 1 \cdot \frac{g'(\pi)}{2}\pi = 0$ ✓

$$\frac{\partial x}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}} = -\frac{-2z\sin(yz) - g'(u)\cdot\frac{1}{z}}{2x}$$

$$\frac{\partial x}{\partial z} = -\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial x}} = -\frac{-2y\sin(yz) + g'(u)\cdot\frac{y}{z^2}}{2x}$$

$$\frac{\partial x}{\partial y}\Big|_{(\pi,1)} = -\frac{-2\cdot 1\cdot \overset{0}{\overbrace{\sin\pi}} - g'(\pi)\cdot\frac{1}{1}}{2\cdot(-1)} = -\frac{g'(\pi)}{2}$$

$$\frac{\partial x}{\partial z}\Big|_{(\pi,1)} = -\frac{-2\cdot\pi\,\overset{0}{\overbrace{\sin\pi}} + g'(\pi)\cdot\frac{\pi}{1}}{2\cdot(-1)} = \frac{g'(\pi)\cdot\pi}{2}$$

$x(y,z)$

$(-1, \pi, 1)$: $x$, $y$, $z$

$u = \frac{y}{z} = \frac{\pi}{1} = \pi$

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28. Sejam $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ e $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ caracterizadas pelas expressões designatórias:

$$f(x,y,z) = (\underbrace{e^{x-z}}_{f_1}, \underbrace{\cos(x+y) + \sin(x+y+z)}_{f_2}) \quad \text{e} \quad g(x,y) = (e^x, \cos(y-x), e^{-y}) .$$

$g(0,0) = (e^0, \cos 0, e^{-0}) = (1,1,1)$

Calcule $J_{(0,0)}(f \circ g)$.

$J_{(0,0)}(f \circ g)$
$\parallel$
$D(f\circ g)(0,0) = Df(g(0,0)) \cdot Dg(0,0) = Df(1,1,1) \cdot Dg(0,0)$

$\mathbb{R}^2 \xrightarrow{g} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}^2$

$$
\begin{tikzcd}
(x,y) \arrow[r, mapsto] \arrow[rr, bend right] & g(x,y) \arrow[r, mapsto] & (f\circ g)(x,y)
\end{tikzcd}
$$

$$Df(1,1,1) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix} = \begin{bmatrix} e^{x-z} & 0 & -e^{x-z} \\ -\sin(x+y)+\cos(x+y+z) & -\sin(x+y)+\cos(x+y+z) & \cos(x+y+z) \end{bmatrix}\Bigg|_{(1,1,1)}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ -\sin 2+\cos 3 & -\sin 2 + \cos 3 & \cos 3 \end{bmatrix}$$

$$g(x,y) = (\underbrace{e^{x}}_{g_1}, \underbrace{\cos(y-x)}_{g_2}, \underbrace{e^{-y}}_{g_3})$$

$$Dg(0,0) = \begin{bmatrix} e^{x} & 0 \\ \sin(y-x) & -\sin(y-x) \\ 0 & -e^{-y} \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$J_{(0,0)}(f\circ g) = \underline{\underline{Df(1,1,1)\cdot Dg(0,0)}} =$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ -\sin 2+\cos 3 & -\sin 2+\cos 3 & \cos 3 \end{bmatrix}_{2\times 3} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 1 & 1 \\ -\sin 2+\cos 3 & -\cos 3 \end{bmatrix}$$