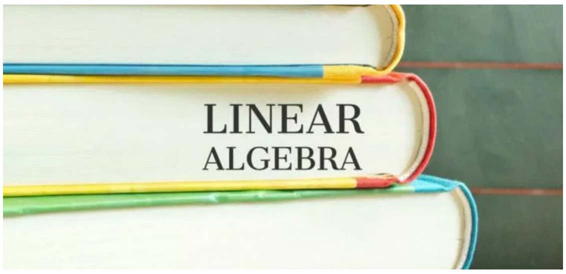
LINEAR ALGEBRA

Eigenvalues of a Stochastic Matrix is

Always Less than or Equal to 1

BY YU · PUBLISHED 11/17/2016 · UPDATED 07/06/2017



Problem 185

Let $A = (a_{ij})$ be an $n \times n$ matrix. We say that $A = (a_{ij})$ is a **right stochastic matrix** if each entry a_{ij} is nonnegative and the sum of the entries of each row is 1. That is, we have

$$a_{ij} \geq 0 \quad \text{and} \quad a_{i1} + a_{i2} + \cdots +$$

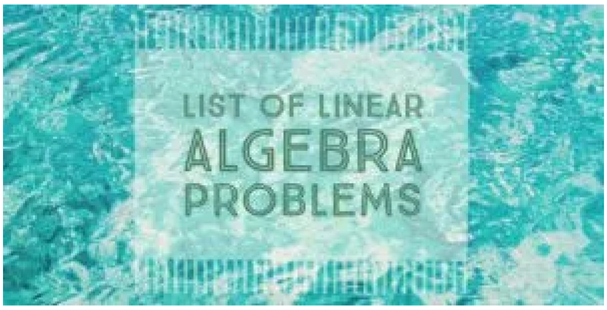
for $1 \leq i, j \leq n$.

Let $A = (a_{ij})$ be an $n \times n$ right stochastic matrix. Then show the following statements.

(a) The stochastic matrix A has an eigenvalue 1.

(b) The absolute value of any eigenvalue of the stochastic matrix A is less than or equal to 1.

The list of linear algebra problems is available here.



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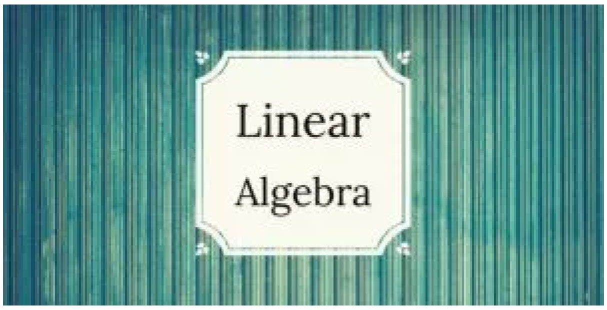

♡ Add to solve later

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Problem 185
Proof.
(a) The stochastic matrix A has an eigenvalue 1.
(b) The absolute value of any eigenvalue of the stochastic matrix A is less than or equal to 1.
Remark.

Proof.

(a) The stochastic matrix A has an eigenvalue 1.

We compute that

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} =$$

Here the second equality follows from the definition of a right stochastic matrix.
(Each row sums up to 1.)
This computation shows that 1 is

LINEAR ALGEBRA

- Introduction to Matrices
- Elementary Row Operations
- Gaussian-Jordan Elimination
- Solutions of Systems of Linear Equations
- Linear Combination and Linear Independence
- Nonsingular Matrices
- Inverse Matrices
- Subspaces in $\mathbb{R}^n$
- Bases and Dimension of Subspaces in $\mathbb{R}^n$

an eigenvector of A and $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 1.

> (b) The absolute value of any eigenvalue of the stochastic matrix A is less than or equal to 1.

Let λ be an eigenvalue of the stochastic matrix A and let $\mathbf{v}$ be a corresponding eigenvector.
That is, we have

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Comparing the i-th row of the both sides, we obtain

$$a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{in}v_n =$$

for $i = 1, \dots, n$.
Let

$$|v_k| = \max\{|v_1|, |v_2|, \dots, |v_n|\},$$

namely v_k is the entry of $\mathbf{v}$ that has the maximal absolute value.

Note that $|v_k| > 0$ since otherwise we have $\mathbf{v} = \mathbf{0}$ and this contradicts that an eigenvector is a nonzero vector.
Then from (*) with $i = k$, we have

- General Vector Spaces
- Subspaces in General Vector Spaces
- Linearly Independency of General Vectors
- Bases and Coordinate Vectors
- Dimensions of General Vector Spaces
- Linear Transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$
- Linear Transformation Between Vector Spaces
- Orthogonal Bases
- Determinants of Matrices
- Computations of Determinants
- Introduction to Eigenvalues and Eigenvectors
- Eigenvectors and Eigenspaces
- Diagonalization of Matrices
- The Cayley-Hamilton Theorem
- Dot Products and Length of Vectors
- Eigenvalues and Eigenvectors of Linear Transformations
- Jordan Canonical Form

CATEGORIES

- Elementary Number Theory (1)
- Field Theory (26)

$$
\begin{aligned}
|\lambda| \cdot |v_k| &= |a_{k1} v_1 + a_{k2} v_2 + \cdots + \\
&\leq a_{k1} |v_1| + a_2 |v_2| + \cdots \\
&\leq a_{k1} |v_k| + a_2 |v_k| + \cdots \\
&= (a_{k1} + a_{k2} + \cdots + a_{kn}
\end{aligned}
$$

Since $|v_k| > 0$, it follows that

$$\lambda \leq 1$$

as required.

Remark.

A stochastic matrix is also called probability matrix, transition matrix, substitution matrix, or Markov matrix.

More from my site

Stochastic Matrix (Markov Matrix) and its Eigenvalues and Eigenvectors

(a) Let

$$A = \begin{bmatrix} a_{11} & a_1 \\ a_{21} & a_2 \end{bmatrix}$$

be a matrix such that $a_{11} + a_{12} = 1$ and $a_{21} + a_{22} = 1$. Namely, the sum of the entries in each

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Find All Values of x such that the Matrix is Invertible

Find All Eigenvalues and Corresponding Eigenvectors for the 3×3 matrix

row is 1. (Such a matrix is called (right) stochast matrix (also termed [...]

Find the Limit of a Matrix Let

$$A = \begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{3}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{3}{7} \end{bmatrix}$$

be 3×3 matrix. Find

$$\lim_{n \to \infty} A^n.$$

(Nagoya University Linear [...]

Basis with Respect to Which the Matrix for Linear Transformation is Diagonal Let P_1 be the vector space of all real polynomials of degree 1 or less. Consider the linear transformation $T : P_1 \to P_1$ defined by

$$T(ax + b) = ($$

for any $ax + b \in P_1$. (a) With respect to the basis $B = \{1, x\}$, find the matrix of the linear transformation [...]

- Find All Values of a which Will Guarantee that A Has Eigenvalues 0, 3, and -3.
- Compute the Determinant of a Magic Square

LINEAR ALGEBRA

If A is an Idempotent Matrix, then When $I - kA$ is an Idempotent Matrix?

LINEAR ALGEBRA

Express a Hermitian Matrix as a Sum of Real Symmetric Matrix and a Real Skew-Symmetric Matrix

LINEAR ALGEBRA

Quiz 3. Condition that Vectors are Linearly Dependent/ Orthogonal Vectors are Linearly Independent

LINEAR ALGEBRA

Trace of the Inverse Matrix of a Finite Order Matrix

LINEAR ALGEBRA

Find Matrix Representation of Linear Transformation From $\mathbb{R}^2$ to $\mathbb{R}^2$

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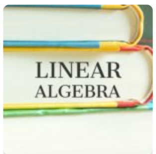
3. The Intersection of Two Subspaces is also a Subspace
4. The Matrix for the Linear Transformation of the Reflection Across a Line in the Plane
5. Eigenvalues of Real Skew-Symmetric Matrix are Zero or Purely Imaginary and



If Every Vector is Eigenvector, then Matrix is a Multiple of Identity Matrix Let A be an $n \times n$ matrix. Assume that every vector $\mathbf{x}$ in $\mathbb{R}^n$ is an eigenvector for some eigenvalue of A. Prove that there exists $\lambda \in \mathbb{R}$ such that $A = \lambda I$, where I is the $n \times n$ identity matrix. Proof. Let us write [...]

Given All Eigenvalues and Eigenspaces, Compute a Matrix Product Let C be a 4×4 matrix with all eigenvalues $\lambda = 2, -1$ and eigensapces \[E_2=\Span\left \{\quad \begin{bmatrix 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad\right \} \text{ and } E_{-1}=\Span\le \{ \quad\begin{br 1 \\ 2 \\ 1 \\ 1 [...]

5 the Rank is Even

6 Express a Vector as a Linear Combination of Other Vectors

7 Prove a Group is Abelian if $(ab)^2 = a^2b^2$

8 Eigenvalues of a Hermitian Matrix are Real Numbers

9 Rotation Matrix in the Plane and its Eigenvalues and Eigenvectors

10 Vector Form for the General Solution of a System of Linear Equations

SITE MAP & INDEX

Site Map

Index

Eigenvalues of a Hermitian Matrix are Real Numbers

Show that eigenvalues of a Hermitian matrix A are real numbers. (The Ohio State University Linear Algebra Exam Problem)   We give two proofs. These two proofs are essentially the same. The second proof is a bit simpler and concise compared to the first one. [...]

Determine Eigenvalues, Eigenvectors, Diagonalizable From a Partial Information of a Matrix

Suppose the following information is known about a 3×3 matrix A.\ [A\begin{bmatr
1 \\ 2 \\ 1
\end{bmatrix}=
1 \\ 2 \\ 1
\end{bmatrix},
\quad
A\begin{bmatri
1 \\ -1 \\ 1 [...]

Given Eigenvectors and Eigenvalues, Compute a Matrix Product (Stanford University Exam)

Suppose that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of a matrix A corresponding to the eigenvalue 3 and that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue -2. Compute $A^2\begin{bm 4 [...]

Tags: eigenvalue eigenvector linear algebra Markov matrix matrix probability matrix stochastic matrix substitution matrix transition matrix triangle inequality vector

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NEXT STORY
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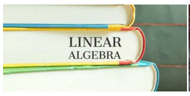

Finite Order Matrix and its Trace

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