Introducão á Análise Complexa - Teoria

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Abstract

This text is a very complete coverage in Portuguese of the theory of Complex Analysis. One should go from not knowing nothing at all to be at research level.

1 Pre-requisites

To read this text the only thing the readers need is a good knowledge on real analysis in one variable, we recommend for a graduate read in the subbject we recommend [2] and for an undergraduate approach we recommend [1].

2 Historical Context

To be made after. I have first to know the specifics of the subject and only after write the history of the subject.

3 The Complex Number System

3.1 What are Complex Numbers?

Definition 3.1 (Complex Numbers)**.** We define the complex numbers as the algebraic field $(\mathbb{R}^2, +, *)$ where

$$\mathbb{R}^2 := \{(a, b) : a, b \in \mathbb{R}\}$$

that is equiped with the following operations

$$(a, b) + (c, d) = (a + b, c + d)$$
$$(a, b) * (c, d) = (ac - bd, bc + ad)$$

Remark 1. If we define a set

$$\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}$$

where $i = \sqrt{-1}$ is called a "imaginary" number, and where the $+, *$ are defined as

$$(a + bi) + (c + di) := a + c + i(b + d)$$
$$(a + bi) * (c + di) := ac - bd + i(ad + bc)$$

we can actually make an isomorphism $\mathbb{R}^2 \cong \mathbb{C}$ via $(a, b) \mapsto a + bi$ which is what is commonly called a complex number, there are good reasons to adopt this notation therefore that is what we will do from now onwards in the text.

We said in the definition that $\mathbb{C}$ was an algebraic field, therefore there must exist multiplicative inverses, to find such objects we notice that for a complex number z the following indentity holds

$$z^{-1} := \frac{a}{a^2 + b^2} - i(\frac{b}{a^2 + b^2})$$

Just notice that $zz^{-1} = z^{-1}z = 1$ which is to say that indeed we have a multiplicative inverse for any $z \neq 0$.

Remark 2. We define the real part Re and the imaginary part Im of a complex number $z := a + bi$ as the following $\operatorname{Re} z = a$ and $\operatorname{Im} z = b$.

Definition 3.2 (Absolute Value and Conjugate)**.** Let z be a complex number we define the absolute value $|z|$ of z as the real number $|z| := a^2 + b^2$ and the conjugate $\bar{z}$ of z as the value $\bar{z} := a - ib$.

Remark 3. The algebraic identity $z\bar{z} = |z|^2$ is good to know from times to times.

4 The Geometry of Complex Numbers

Bibliography

References

[1] Stephen Abbott. *Understanding Analysis*. 2nd. Undergraduate Texts in Mathematics. New York: Springer, 2015. ISBN: 978-1493927111.

[2] Walter Rudin. *Principles of Mathematical Analysis*. 3rd. New York: McGraw-Hill, 1976. ISBN: 978-0070542358.