# 目录

# 一、 The Foundations: Logic and Proofs (逻辑与证明)

## 1. Proposition and Connective(命题与连接)

### 1.1 Proposition

**Definition 1** *Proposition(命题) is a statement that is either true or false, but not both.*

(1) True value: T, F.

(2) Latters are used to denote proposition:

$p, q, r, c \ldots$.

(3) Atom and Compound Proposition.

### 1.2 Logical Operators(Connective)

(1) Negation(否定, not $p$): $\neg p$.

| $p$ | $\neg p$ |
|---|---|
| T | F |
| F | T |

表 1: Negation

(2) Conjunction(合取, $p$ and $q$): $p \wedge q$

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | F | F |
| F | T | F |
| T | T | T |
| F | F | F |

表 2: Conjunction

(3) Disjunction(析取, $p$ or $q$): $p \vee q$

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | F | T |
| F | T | T |
| T | T | T |
| F | F | F |

表 3: Disjunction

(4) Implication(蕴含, If $p$ then $q$): $p \longrightarrow q$

| $p$ | $q$ | $p \longrightarrow q$ |
|---|---|---|
| T | F | F |
| F | T | T |
| T | T | T |
| F | F | T |

表 4: Implication

(5) Biconditional (当且仅当, $p$ If and only if $q$): $p \longleftrightarrow q$

| $p$ | $q$ | $p \longleftrightarrow q$ |
|---|---|---|
| T | F | F |
| F | T | F |
| T | T | T |
| F | F | T |

表 5: Biconditional

Remark:

(1) Highest Priorities: $\neg$, then $\vee, \wedge$, then $\longrightarrow, \longleftrightarrow$.

(2) $\uparrow, \downarrow$ is functionally complete.

## 2. Formula(公式)

**Definition 2** *The formal definition of a* *formula* *(also called a well formed formula, or wff) as follows:*

*(1) Each atom proposition is a formula.*

*(2) The connective of formulas is formula.*

*(3) Any other is not a formula.*

### 2.1 Classification of Proposition Formula

(1) Tautology(永真式, 重言式) e.g. $p \longrightarrow p \vee q$.

(2) Contradiction(永假式) e.g.$p \wedge \neg p$.

(3) Contingence(连接式, 有真有假) e.g. $p \longrightarrow q$.

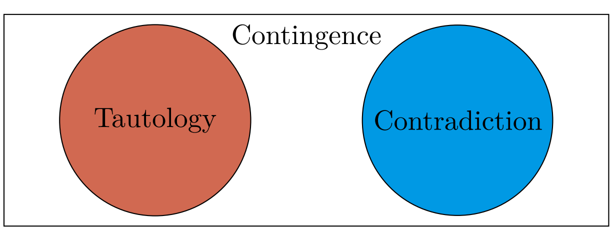


图 1: Propositional Formula

### 2.2 Calcualte

(1) truth table

(2) Calcualte

(3) Formula

## 3. Propositional Equivalences

The number of truth table involving variables $p_1, p_2, \cdots, p_n$ is only $2^{2^n}$, but the number of the formulae involving them is infinity.

**Definition 3** *Formulae $A$ and $B$ are called* *logically equivalent* *if $A \longleftrightarrow B$ is tautology, denoted by $A \Longleftrightarrow B$.*

e.g.

- $p \longrightarrow q \Longleftrightarrow \neg p \vee q$
- $p \longleftrightarrow q \Longleftrightarrow (p \longrightarrow q) \wedge (q \longrightarrow p) \Longleftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$

### 3.1 Some important equvialences

| | | |
|---|---|---|
| Identity laws(单位元) | $p \vee F \Longleftrightarrow p$ | $p \wedge T \Longleftrightarrow p$ |
| Domination laws(零) | $p \vee T \Longleftrightarrow T$ | $p \wedge F \Longleftrightarrow F$ |
| Idempotent laws(幂等) | $p \vee p \Longleftrightarrow p$ | $p \wedge p \Longleftrightarrow p$ |
| Complementation laws | $\neg(\neg p) \Longleftrightarrow p$ | |
| Commutative laws(交换) | $p \vee q \Longleftrightarrow q \vee p$ | $p \wedge q \Longleftrightarrow q \wedge p$ |
| Associative laws(结合) | $p \vee (q \vee r) \Longleftrightarrow (p \vee q) \vee r$ | |
| | $p \wedge (q \wedge r) \Longleftrightarrow (p \wedge q) \wedge r$ | |
| Distributive laws(分配) | $p \wedge (q \vee r) \Longleftrightarrow (p \wedge q) \vee (p \wedge r)$ | |
| Laws of excluded middle(排中) | $p \wedge \neg p \Longleftrightarrow F$ | $p \vee \neg p \Longleftrightarrow T$ |
| Absorption laws(吸收) | $p \wedge (p \vee q) \Longleftrightarrow p$ | |
| | $p \vee (p \wedge q) \Longleftrightarrow p$ | |
| De Morgan's laws | $\neg(p \vee q) \Longleftrightarrow \neg p \wedge \neg q$ | |
| | $\neg(p \wedge q \Longleftrightarrow \neg p \vee \neg q)$ | |

### 3.2 Disjunctive Normal Form(DNF, 析取范式)

**Definition 4** *Conjunctive Clauses and Disjunctive Normal Form*

*(1)* *Literal**: Atom proposition and its negation.*

*(2)* *Conjunctive clauses**: Conjunctions with literals.*

*(3)* *Disjunctive nromal form**: Disjunctions with conjunctive clauses.*

In general, a formula in DNF is

$$(A_{1_1} \wedge A_{1_2} \wedge \cdots A_{1_{n1}}) \vee \cdots \vee (A_{k_1} \wedge A_{k_2} \wedge \cdots A_{k_{nk}})$$

where $A_{i_j}$ are literals.

**Theorem 1** *Any formula $A$ is tautologically equivalent to some formula in disjunctive normal form.*

### 3.3 Full Disjunctive Form(主析取范式)

**Definition 5** *A minterm is a conjunction of literals in which each variable is represented exactly once.*

Properties of the Minterms:

(1) For n variables, there are only $2^n$ minterms, and each minterm is true for exactly one assignment.

(2) If $A$ and $B$ are two distinct minterms $\Longrightarrow$ $A \wedge B \Longleftrightarrow F$

**Definition 6** *If a boolean function is expressed as a disjunction of minterms, it's said to be in full disjunctive form.*

Remark:

(1) Tautology $A \Longleftrightarrow \vee_{i=0}^{2^{n-1}} m_i$ .

(2) Can obtain full disju form by using truth table.

(3) $\{\neg, \vee, \wedge\}$ is functionally complete.

Conjunctive Normal Form (CNF) and DNF are dual.

## 4. Methods of Proof

定理, 公理, 引理, 推论, 猜想 etc.

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \longrightarrow q \text{ is tauto or not.}$$
$$\Longleftrightarrow (p_1 \wedge p_2 \wedge \cdots \wedge p_n) \Longrightarrow q$$

(1) Law of detachment or modus ponens(假言推断)

$$\because p \longrightarrow q$$
$$p$$
$$\therefore q$$

(2) Modus tollens(逆否)

$$\because p \longrightarrow q$$
$$\neg q$$
$$\therefore \neg p$$

(3) Rule of Addition(附加)

$$\because p$$
$$\therefore p \vee q$$

(4) Rule of simplification(简化)

$$\because p \wedge q$$
$$\therefore p$$

(5) Rule of conjunction(合取)

$$\because p$$
$$q$$
$$\therefore p \wedge q$$

(6) Rule of hypothetical syllogism(三段论)

$$\because p \longrightarrow q$$
$$q \longrightarrow r$$
$$\therefore p \longrightarrow r$$

(7) Rule of disjunctive syllogism(析取三段论)

$$\because p \vee q$$
$$\neg p$$
$$\therefore q$$

(8) x(潘解原理)

$$\because p \vee q$$
$$\neg p \vee r$$
$$\therefore q \vee r$$

Remark:

(1)

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \longrightarrow (p \rightarrow q)$$
$$\Longleftrightarrow (p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge p) \longrightarrow q$$

(2)

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \longrightarrow q$$
$$\Longleftrightarrow \neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \vee \neg(\neg q)$$
$$\Longleftrightarrow \neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge \neg q)$$

## 5. Predicates and Quantifiers(谓词与量化)

### 5.1 Predicates

**Definition 7** *A statement of the form $P(x_1, x_2, \cdots, x_n)$ is the value of the propositional function $P$ at n-tuple $(x_1, x_2, \cdots, x_n)$, and $P$ is also called a predicate.*
*$x_1, x_2, \cdots, x_n$ is an element of a set $D$.*

### 5.2 Quantifiers

$$Predicates \xrightarrow{Quantification} Propositions$$

Domain(论域):

- Universal quantifiers: For all $x, p(x)$: $\forall x, p(x)$
- Existential quantifiers: For some $x, p(x)$: $\exists x, p(x)$

Remark:

(1) If $x_1, x_2, \cdots, x_n$, then

$$\forall x, P(x) \iff P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$$
$$\exists x, P(x) \iff P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$$

(2)

$$\forall x \forall y, p(x, y) \iff \forall y \forall x, p(x, y)$$
$$\exists x \exists y, p(x, y) \iff \exists y \exists x, p(x, y)$$
$$\text{But } \forall x \exists y, p(x, y) \not\Leftrightarrow \exists y \forall x, p(x, y)$$

### 5.3 Banding Variables(辖域)

**Definition 8** *When a quantifier is used on the variable x or when we assign a variable to this variable.*

- *this occurrence of the variable is bound.*
- *other occurrence of the variableis free.*

Remark:

(1) All the variables that occur in a propositional function must be bound to turn it into a proposition.

(2) Rename bounded variables and free variables in formula logically equivalence.

### 5.4 Classification of Predicates Formula

(1) Tautology: All true.

(2) Contradiction: All false.

(3) Contingence: neither a tautology nor a contradiction.

### 5.5 Some improtant Equivalent Predicates Formula

(1) De Morgan's laws:

a. For predicates:

$$\neg \forall x, p(x) \iff \exists x, \neg p(x)$$
$$\neg \exists x, p(x) \iff \forall x, \neg p(x)$$

b. For quantifiers:

$$\forall x, (p(x) \wedge q(x)) \iff (\forall x, p(x)) \wedge (\forall x, q(x))$$
$$\exists x, (p(x) \vee q(x)) \iff (\exists x, p(x)) \vee (\exists x, q(x))$$

But

$$\forall x, (p(x) \vee q(x)) \Longleftarrow (\forall x, p(x)) \vee (\forall x, q(x))$$
$$\exists x, (p(x) \wedge q(x)) \Longrightarrow (\exists x, p(x)) \wedge (\exists x, q(x))$$

(2) More logical equivalence: When x isn't occurring in A,

a.

$$A \wedge \forall x, q(x) \iff \forall x, (A \wedge q(x))$$
$$A \vee \forall x, q(x) \iff \forall x, (A \vee q(x))$$
$$\star \forall x, q(x) \longrightarrow A \iff \exists x, (q(x) \longrightarrow A)$$
$$A \longrightarrow \forall x, q(x) \iff \forall x, A(\longrightarrow q(x))$$

b.

$$A \wedge \exists x, q(x) \iff \exists x, (A \wedge q(x))$$
$$A \vee \exists x, q(x) \iff \exists x, (A \vee q(x))$$
$$\star \exists x, q(x) \longrightarrow A \iff \forall x, (q(x) \longrightarrow A)$$
$$A \longrightarrow \exists x, q(x) \iff \exists x, A(\longrightarrow q(x))$$

### 5.6 Prenex Normal Forms

**Definition 9** *A formula is in prenex normal form if it is of the form*

$$Q_1 x_1 Q_2 x_2 \cdots Q_n x_n B$$

*when $Q_i (i = 1, 2, \cdots, n)$ is $\forall$ or $\exists$ and $B$ is quantifier free. (Not unique)*

5.7 Methods of Proof

(1) Universal instantiation(UI)

$$\because \forall x \in D, P(x)$$
$$d \in D$$
$$\therefore P(d)$$

(2) Universal generalization(UG)

$$\because P(d) \text{ for any } d \in D$$
$$\therefore \forall x, P(x)$$

(3) Existential instantiation(EI)

$$\because \exists x \in D$$
$$P(x)$$
$$\therefore P(d) \text{ for some } d \in D$$

(4) Existential generalization(EG)

$$\because P(d) \text{ for some } d \in D$$
$$\therefore \exists x, P(x)$$

# 二、 Basic Structures: Sets, and Functions (集合与函数)

## 1. Sets

1.1 Properties of sets

- Order of elements doesn't matter.
- Repetition of elements doesn't matter.
- Certainty.

1.2 Infinite and Finite Set

Cardinality of set $S$ ($|S|$) is the number of elements in $S$.

- Infinite Countable.
- Uncountable.

1.3 Subsets

(1) Subset notation: $\subseteq$

$$S \subseteq T \Longleftrightarrow \forall x \in S \longrightarrow x \in T$$

(2) Proper Subset: $\subset$

$$S \subset T \Longleftrightarrow \forall x \in S \longrightarrow x \in T \text{ and } S \neq T$$

(3) Empty set $\emptyset$ and Universal set $U$:

For any set A:

$$A \subseteq A$$
$$\emptyset \subseteq A \subseteq U$$

## 2. Set Operations

(1) Union

$$A \cup B = \{x | x \in A \vee x \in B\}$$

(2) Intersection

$$A \cap B = \{x | x \in A \wedge x \in B\}$$

(3) Difference

$$A - B = \{x | x \in A \vee x \notin B\}$$

(4) Complement

Let $U$ be the universal set.

$$\bar{A} = U - A$$

(5) Symmetric Difference

$$A \oplus B = (A - B) \cup (B - A)$$

2.1 The Power Set

**Definition 10** *Given s set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by P(S) or $2^S$.*

$$2^S = \{T | T \subseteq S\}$$

Remark: If $|S| = n$, $|2^S| = 2^n$.

2.2 Cartesian Products

**Definition 11** *The ordered n- tuple $(a_1, a_2, \cdots, a_n)$ is the ordered collection that has $a_1$ as its first element, $\cdots$, and $a_n$ as its n-th element.*

- $(a_1, \cdots, a_n) = (b_1, \cdots, b_n) \Longleftrightarrow a_i = b_i$ for $i = 1, \cdots, n$.

- In particular, 2-tuples are called ordered pairs (序偶).

**Definition 12** *Let $A$ and $B$ be sets. The Cartesian product of $A$ and $B$ , denoted by $A \times B$, is the set of all ordered pairs $(a,b)$, where $a \in A$ and $b \in B$. Hence,*

$$A \times B = \{(a,b) | a \in A \wedge b \in B\}$$

- $A^2 = A \times A$ the Cartesian square of A.

- If $A$ and $B$ finite, $A \times B$ finite.

If $A$ has $m$, $B$ has $n$, $A \times B$ has $mn$.

- If $A$ infinite and $B$ non-empty, $A \times B, B \times A$ infinite.

Properites of Cartesian Products:

(1) $A \times \emptyset = \emptyset \times B = \emptyset$.

(2) In general, $A \times B \neq B \times A$.

(3) In general, $(A \times B) \times C \neq A \times (B \times C)$, unless we identify $((a,b),c)$ and $(a,(b,c))$.

(4)

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$
$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

**Definition 13** *The Cartesian porduct of $A_1, A_2, \cdots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of all ordered $n$-tuples $(a_1, a_2, \cdots, a_n)$, where $a_i \in A_i$ for $i = 1, 2, \cdots, n$. In other words,*

$$A_1 \times \cdots \times A_n = \{(a_1, \cdots, a_n) | a_i \in A_i \text{ for } i = 1, \cdots, n\}$$

- $A^n = A \times A \times \cdots \times A$($n$ times).

-

$$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$$
$$(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)$$

## 3. Cardinality of Finite and Infinite Sets

### 3.1 Counting Finite Sets

(1) Cardinality: $|S|$.

(2) Prnciple of Inclusion-exclutsion:

$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B| \\ \left|\bigcup_{i=1}^{n} A_i\right| &= \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i \neq j \leq n} |A_i \cap A_j| + \cdots \\ &\quad + (-1)^{n-1} \left|\bigcap_{i=1}^{n} A_i\right|\end{aligned}$$

### 3.2 Cardinality of Infinite Sets

$|A| = |B|$, $A, B$ have the same cardinality (equinumerous), iff there is an one-to-one correspondence (双射) from $A$ to $B$.

Remark:

(1) Two sets have the same cardinality is a equivalence relation (等价关系).

(2) Two sets have the same cardinality, but the one-to-one correspondence may be not unique.

(3) A set and its proper set have the same cardinality iff it is infinite set.

### 3.3 Two Types of Infinite sets

- Countable (denumberable) set ($\mathbb{N}$) is either finite or has the same cardinality as the set of natural numbers $\mathbb{N}$, $\aleph_0$ is called countable.

- And other are uncountable set.

Some special infinite sets:

(1) The set of integers is countable, $|\mathbb{N}| = |\mathbb{Z}|$.

(2) The set of rational number is countable, $|\mathbb{N}| = |\mathbb{Q}|$.

(3) The set of real nnumbers is uncountable, $|R| = \aleph > \aleph_0$.

(4) The set $\mathbb{N} \times \mathbb{N}$ is uncountable.

(5) The uncountable set always has a proper set that is countable.

**Theorem 2 (Cantor Theorem)** *The cardinality of the power set of an arbitrary set has a greater cardinality than the original arbitrary set, or*

$$\left|2^A\right| > |A|$$

.

## 4. Functions

### 4.1 Introduction

**Definition 14** *Let $A$ and $B$ be sets, a function $f$ from $A$ to $B$:*

$$f : A \longrightarrow B \Longleftrightarrow \forall a \in A\, \exists! b \in B(bunique) : f(a) = b$$

*$f$ maps $A$ to $B$.*

- *$A$ is the domain of $f$.*
- *$B$ is the codomain of $f$.*
- *$f(a) = b, a \in A, b \in B$, $b$ is the image of $a$, $a$ is a pre-image of $b$.*
- *The range of $f$ is the set:*

$$\text{Range}(f) = \{b \in B | \exists a \in A, f(a) = b\}$$

### 4.2 One-to-one and Onto Functions

**Definition 15** *Lef $f$ be a function fro $A$ to $B$:*

- *one-to-one function: (injective) (单射)*

$$\forall a, b \in A \wedge a \neq b \Longrightarrow f(a \neq f(b))$$

- *onto function: (surjective) (满射)*

$$\forall b \in B\, \exists a \in A \text{ such that } f(a) = b$$

- *bijection function: (one-to-one correspondence) (双射)*

$$one\text{-}to\text{-}one + onto$$

### 4.3 Inverse and composition of function

**Definition 16** *Let $f$ be a one-to-one correspondence from $A$ to $B$. The inverse function of $f$, $f^{-1} : B \longrightarrow A$ is*

$$\forall a \in A, b \in B(f(a) = b) \Longleftrightarrow (f^{-1}(b) = a)$$

Remark:

(1) A one-to-one correspondence is called invertible.

(2) A function is not invertible if it is not a one-to-one correspondence.

**Definition 17** *Let $g : A \longrightarrow B$ and $f : B \longrightarrow C$ are two functions. The composition of the functions $f$ and $g$, $f \circ g : A \longrightarrow C$ is*

$$\forall a \in A, (f \circ g)(a) = f(g(a))$$

### 4.4 Some Important Functions

**Definition 18** *The floor functions and the ceiling function.*

- *The floor functions $\lfloor x \rfloor$ assigns the real number $x$ the largest integer that $\leq x$.*
- *The ceiling function $\lceil x \rceil$ assigns to the real number $x$ the smallest integer that is $\geq x$.*

Remark:

(1) $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$

(2) $\lceil -x \rceil = -\lfloor x \rfloor$

(3) $\lfloor -x \rfloor = -\lceil x \rceil$

### 4.5 The Growth of Functions

**Definition 19 (Big-O, Big-Omega, Big-Theta)** *Let $f$ and $g$ be functions from the set of integers or the set of real numbers.*

- *We say that f(x) is $O(g(x))$ if there exist constants $C$ and $k$ such that*

$$|f(x) \leq C|g(x)||$$

*where x>k.*

- *We say that f(x) is $\Omega(g(x))$ if there exist constants $C$ and $k$ such that*

$$|f(x)| \geq C|g(x)|$$

*where x>k.*

- *We say that f(x) is $\Theta(g(x))$ if f(x)=O(g(x)) and $f(x) = \Omega(g(x))$. We also say that f(x) is of order $g(x)$.*

# 三、 The Fundamentals: Algorithms (算法基础)

## 1. Algorithms

### 1.1 Introduction

**Definition 20** *An algorithm: a finite set of precise instructions for performing a computation or solving problem.*

Pseudocode (伪代码)

### 1.2 Searching Algorithms

(1) Linear search.

(2) Binary search.

## 2. Complexity of Algorithms

**Definition 21** *Complexity, space Complexity and time Complexity.*

- *Complexity: the amount of time and/or space needed to execute the algorithm.*
- *Space Complexity: be tied with particular data structures of used to implement the algorithm.*
- *Time Complexity: can be expressed in terms of the number of operation used by the algorithm when the input has a particular size.*

Types of Complexity:

- Best-case time
- Worst-casr time
- Average-case time

### 2.1 Some Terminology to Describe the Time Complexity

| Complexity | Terminology |
|---|---|
| $O(1)$ | Constant complexity |
| $O(\log n)$ | Logarithmic complexity |
| $O(n)$ | Linear complexity |
| $O(n \log n)$ | $n \log n$ complexity |
| $O(n^b)$ | Polynomial complexity |
| $O(b^n), b > 1$ | Exponential complexity |
| $O(n!)$ | Factorial complexity |

表 6: Commonly used Terminology for the Complexity of Algorithms

### 2.2 NP Problem

- P class: Problems can be solved bu Polynomial time algorithm.
- NP class: Problems for which a solution can be checked in Polynomial time.
- NP-Complete Problem: If any of these probelms can be solved bu polynomial worst-case time algorithm, then all can be solvev by polynomial worst-case time algorithms.

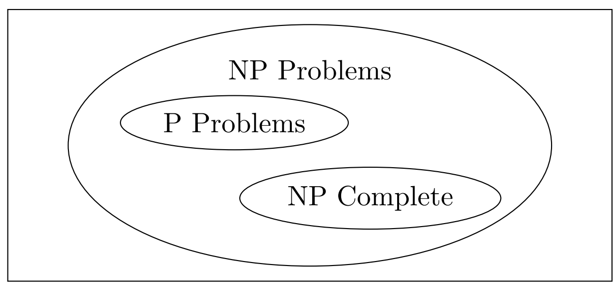


图 2: The World

# 四、 Induction and Recursion (归纳与递归)

## 1. Mathematical Induction

### 1.1 The Well-ordeing Property (良序集)

Every nonnegative integers has a least element.

### 1.2 The Proof by Mathematical Induction

A proof by mathematical induction that $P(n)$ is true for every positive integer $n$ consists of two steps:

(1) Basic step: Th proposition $P(1)$ is shown to be true.

(2) Inductive step: The implication $P(n) \longrightarrow P(n+1)$ is shown to be true for every positive integer $n$.

1.3 The Second Principle of Mathematical Induction

(1) Basic step: Th proposition $P(1)$ is shown to be true.

(2) Inductive step: The implication $[P(1) \wedge P(2) \wedge \cdots \wedge P(n)] \longrightarrow P(n+1)$ is shown to be true for every positive integer $n$.

The two forms of mathematical induction are equivalence.

### 2. Recursive Definitions

2.1 Recursively Defined Functions

To define a function with the set of nonnegative integers as its domain.

(1) Specify the value of the function at zero, $f(0)$.

(2) Give a rule for finding its value as an integer from ist values at smaller integers.

e.g. The Fibonacci numbers

$$f_0 = 1, f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2}$$

2.2 Recursively Defined Sets

e.g. Let S be defined recursively by

(1) $3 \in S$.

(2) $x + y \in S$ if $x \in S$ and $y \in S$.

$S$ is the set of positive integers divisible by 3.

The most commom uses is to define well-formed formulae.

Also can define strings.

2.3 Recursive Algorithms

An algorithm is called recursive if it solves a problem by reducing it to an instance of the same problem with smaller input.

## 五、 Counting (计数)

### 1. The Basics of Counting

1.1 The Sum Rule

**Theorem 3** *If $A_1, A_2, \cdots, A_m$ are disjoint sets, then the number of elements in the union of these sets is the sum of the number of elements in them.*

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

1.2 The Product Rule

**Theorem 4** *If $A_1, A_2, \cdots, A_m$ are disjoint sets, then the number of elements in the cartesian product of these sets is the product of the number of elements in each set.*

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|$$

### 2. The Pigeonhole Principle

**Theorem 5 (The Pigeonhole Principle)**

*If $k+1$ or more objects are placed into $k$ boxes then there is at least one box containing two or more of the objects.*

**Theorem 6 (The Generalized Pigeonhole)**

*If $N$ objects are placed into $k$ boxes then there is at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects.*

2.1 Some Elegant Applications of Pigeonhole Principle

**Definition 22** *Suppose that $a_1, \cdots, a_N$ is a sequence of real number.*

- *A subsequence of this sequence is a sequence of the form $a_{i_1}, \cdots, a_{i_m}$, where $1 \leq i_1 < \cdots < i_m \leq N$.*
- *A sequenceis called strictly increasing if each term is lager than the one that precedes it, and it is called strictly decreasing if each term is smaller than the one that precedes it.*

**Theorem 7** *Every sequence of $n^2+1$ distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.*

### 3. Permutations and Combinations

3.1 Permutations (排列)

**Definition 23** *Given a set of distinct objects $X = \{x_1, \cdots, x_n\}$*

- *a permutation of $X$ is an ordered arrangement of $x_1, \cdots, x_n$.*
- *a r-permutation, where $r \leq n$ is an ordering of a subset of r-elements of $X$.*

- *The number of r-permutations of a set of distinct elements is denoted by $P(n,r)$.*

**Theorem 8**
$$P(n,r)=\frac{n!}{(n-r)!}$$
*In particular, note taht $P(n,n)=n!$.*

3.2 Combinations

**Definition 24** *Let $X=\{x_1,x_2,\cdots,x_n\}$ be a set containing $n$ distinct elements.*

- *an r-combination of $X$ is an unordered selection of r-elements of $X$.*
- *the number of r-combinations of a set $n$ distinct elements is denoted by $C(n,r)$.*

**Theorem 9**
$$C(n,r)=\frac{n!}{(n-r)!r!}=\frac{P(n,r)}{r!}$$

3.3 Binomial Coefficients

**Theorem 10 (Binomial Theorem)** *If $a$ and $b$ are real numbers and $n$ is a positive integer, then*
$$\begin{aligned}(a+b)^n=&C(n,0)a^nb^0+C(n,1)a^{n-1}b^1+\cdots\\&+C(n,n-1)a^1b^{n-1}+C(n,n)a^0b^n\end{aligned}$$

**Theorem 11 (Pascal's Identity)** *Let $n$ and $k$ be positive integers with $n\geq k$, then*
$$C(n+1,k)=C(n,k)+C(n,k-1)$$

**Theorem 12** *Let $n$ be a positive integer, then*
$$\sum_{k=0}^{n}C(n,k)=2^n$$

**Theorem 13 (Vanderomnde's Identity)** *Let m,n and r be nonnegative integers with r not exceeding either m or n , then*
$$C(m+n,r)=\sum_{k=0}^{r}C(m,r-k)C(n,k)$$

### 4. Generalized Permutations and Combinations

4.1 Permutations with Repetition

**Theorem 14** *The number of r-permutations of a set of n objects with repetition allowed is $n^r$.*

4.2 Combinations with repetition

**Theorem 15** *There are $C(n+r-1,r)$ or $C(n+r-1,n-1)$ r-combinations from a set with n elements when repetition of elements is allowed.*

4.3 Distributing Objects into Boxes

**Theorem 16** *The number of ways to distribute n distinguishable objects into k distinguishable boxes so that $n_i$ objects are placed into box $i$, $i=1,2,\cdots,k$ equals*
$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

### 5. Generating Permutations and Combinations

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## 六、 Advanced Counting Techniques (高级算法科技)

### 1. Recurrence Relations

**Definition 25** *A recurrence relation for a sequence $\{a_n\}$ is an equation that express $a_n$ in terms of one or more of the previous terms of the sequence.*

- *A sequence is calle a solution of recurrence relation if its term satisfy the recurrence relation.*
- *Initial condition.*

e.g. Rabbits and Fibonacci Numbers, The Tower of Hanoi, DP etc.

### 2. Solving Recurrence Relations

2.1 Induction

**Definition 26** *A linear homogeneous recurrence relation of degree k with constant coefficients (k 阶线性齐次常系数递推关系) is a recurrence relation of the form*
$$a_n=c_1a_{n-1}+c_2a_{n-2}+\cdots+c_ka_{n-k}$$
*where $c_1,c_2,\cdots,c_k$ are real number, and $c_k\neq 0$.*

2.2 Solving k 阶线性齐次常系数递推关系

To look for solutions of the form $a_n=r^n$, where $r$ is a constant. And $a_n=r^n$ is a solution of $a_n=c_1a_{n-1}+c_2a_{n-2}+\cdots+c_ka_{n-k}$. Then can obtain
$$r^k-c_1r^{k-1}-c_2r^{k-2}-\cdots-c_{k-1}r-c_k=0$$

called Characteristic Equation (特征方程). The solution are called the Characteristic roots (特征根).

(1) For degree $k = 2$

**Theorem 17** *Suppose $r^2 - c_1 r - c_2 = 0$ has two distinct roots $r_1$ and $r_2$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation, iff*

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

*for $n = 0, 1, 2, \cdots$, where $\alpha_1$ and $\alpha_2$ are constants.*

**Theorem 18** *Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root $r_0$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation, iff*

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$$

*for $n = 0, 1, 2, \cdots$, where $\alpha_1$ and $\alpha_2$ are constants.*

(2) For degree $k > 2$

**Theorem 19** *Suppose that the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$ has $k$ distinct roots $r_1, \cdots, r_k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ iff*

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

*for $n = 0, 1, 2, \cdots$, where $\alpha_1, \alpha_2, \cdots, \alpha_k$ are constants.*

**Theorem 20** *Suppose that $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$ has $t$ distinct roots $r_1, \cdots, r_t$ with multiplicities $m_1, \cdots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \cdots, t$ and $m_1 + m_2 + \cdots + m_t = k$. The the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ iff*

$$\begin{aligned} a_n = &(\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1} n^{m_1 - 1}) r_1^n \\ &+ (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2} n^{m_2 - 1}) r_2^n \\ &+ \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t} n^{m_t - 1}) r_t^n \end{aligned}$$

*for $n = 0, 1, 2, \cdots$, where $\alpha_{i,j}$ are constants.*

### 2.3 Solving k 阶线性非齐次常系数递推关系

**Definition 27** *A linear nonhomogeneous recurrence relation of degree $k$ with constant coefficients ($k$ 阶线性非齐次常系数递推关系) is a recurrence relation of the form*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

*where $c_1, c_2, \cdots, c_k$ are real numbers, and $F(n)$ is a function not identically zero depending only on $n$. The recurrence relation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

*is called the associated homogeneous recurrence relation (相对齐次式子).*

**Theorem 21** *If $\left\{a_n^{(p)}\right\}$ is particular solution of the linear nonhomogeneous recurrence relation with constant coefficients*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

*then every solution is of the form $\left\{a_n^{(p)} + a_n^{(h)}\right\}$, where $\left\{a_n^{(h)}\right\}$ is a solution of the associated homogeneous recurrence relation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

**Theorem 22** *Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$ and $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n$, where $b_1, \cdots, b_t$ and $s$ are real numbers.*

*(1) When $s$ isn't a characteristic root of the associated linear homogeneous recurrence relation, there is a particular solution of the form*

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n$$

*(2) When $s$ is a characteristic root with multiplicity $m$, there is a particular solution of the form*

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n$$

## 3. Generating Function

### 3.1 Introduction

**Definition 28** *The generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series*

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

**Definition 29** *The generating functions are usually considered to be formal power series (形式化序列)*

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

### 3.2 Calculating

**Theorem 23** *Let* $f(x) = \sum_{k=0}^{\infty} a_k x^k$ *and* $g(x) = \sum_{k=0}^{\infty} b_k x^k$. *Then,*

*(1)* $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$

*(2)* $f(x) \cdot g(x) = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} a_j + b_{k-j} \right) x^k$

*(3)* $\alpha \cdot f(x) = \sum_{k=0}^{\infty} \alpha a_k x^k$

*(4)* $x \cdot f'(x) = \sum_{k=0}^{\infty} k a_k x^k$

*(5)* $f(\alpha x) = \sum_{k=0}^{\infty} \alpha^k a_k x^k$

### 3.3 Extended Binomial Coefficient

**Definition 30** *Let* $u \in \mathbb{R}$, *and* $k \in \mathbb{N}$. *Then the extended binomial coefficient (广义二项式系数) is* $\binom{u}{k}$ *defined by*

$$\binom{u}{k} = \begin{cases} \frac{u(u-1)\cdots(u-k+1)}{k!} & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}$$

Remark: When $u = -n$ is a negative integer,

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$$

**Theorem 24 (The Extended Binomial Theorem)**
*Let* $x, u \in \mathbb{R}$ *with* $|x| < 1$. *Then,*

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

### 3.4 Useful Generating Function

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### 3.5 Using Generating Functions to Solve Recurrence Relations

等式两边同乘 $x^n$, 累加求和后化求 $G(x)$, 再用 $G(x)$ 解 $a_n$.

### 3.6 Counting Problems and Generating Functions

(1) Combination:

$$G(x) = (1 + x + x^2 + x^3 + \cdots)^n = \frac{1}{(1-x)^n}$$

$a^r$ is the number of $r$-combination from a set with $n$ elements when the repetition of elements is allowed.

(2) Permutation: Using

$$\sum_{n=1}^{\infty} \frac{a_n}{n!} x^n$$

$\frac{x^r}{r!}$ is the solution.

## 4. Applications of Inclusion-Exclusion (容斥应用)

### 4.1 Introduction

**Theorem 25 (Principle of Inclusion-Exclusion)**
*Let A and B are two finite sets, then,*

$$|A \cup B| = |A| + |B| - |A \cap B|$$

*The extended principle of Inclusion-Exclusion*

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i \ne j \le n} |A_i \cap A_j| + \cdots + (-1)^{n-1} \left| \bigcap_{i=1}^{n} A_i \right|$$

# 七、 Relations (关系)

## 1. Relations and Their Properties

### 1.1 Relations

**Definition 31** *A binary relation R between A and B is a subset of Cartesian product* $A \times B$

$$R \subseteq A \times B$$

*when* $A = B$, *R is called a relation on set A.*

(1) Given a relation $R$ from $A$ to $B$.

- The domian of $R$

  $Dom(R) = \{x \in A | \exists y \in B, (x, y) \in R\}$

- The range of $R$

  $Ram(R) = \{y \in B | \exists x \in A, (x, y) \in R\}$

表 7: Useful Generating Function

| $G(x)$ | $a_k$ |
|---|---|
| $(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$ | $\binom{n}{k}$ |
| $(1+\alpha x)^n = \sum_{k=0}^{\infty} \binom{n}{k} \alpha^k x^k$ | $\binom{n}{k}\alpha^k$ |
| $(1+x^r)^n$ | $\begin{cases} \binom{n}{\frac{k}{r}} & r \mid k \\ 0 & otherwise \end{cases}$ |
| $\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k$ | $\begin{cases} 1 & k \le n \\ 0 & otherwise \end{cases}$ |
| $\frac{1}{1-x} = \sum_{k=0}^{n} x^k$ | $1$ |
| $\frac{1}{1-\alpha x} = \sum_{k=0}^{n} \alpha^k x^k$ | $a^k$ |
| $\frac{1}{1-x^r} = \sum_{k=0}^{n} x^{rk}$ | $\begin{cases} 1 & r \mid k \\ 0 & otherwise \end{cases}$ |
| $\frac{1}{(1-x)^2} = \sum_{k=0}^{n} (k+1) x^k$ | $k+1$ |
| $\frac{1}{(1-x)^n} = \sum_{k=0}^{n} \binom{n+k+1}{k} x^k$ | $\binom{n+k+1}{k}$ |
| $\frac{1}{(1+x)^n} = \sum_{k=0}^{n} \binom{n+k+1}{k} (-x)^k$ | $(-1)^k \binom{n+k+1}{k}$ |
| $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ | $\frac{1}{k!}$ |

(2) Function as relations: Recall a function $f$ from set $A$ to $B$.

(3) n-ary relations: Let $A_1, A_2, \dots, A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$. $n$ is called the degree.

### 1.2 Combining Relations

(1) Union, intersection, complement and difference of relations (并交补差).

(2) Let $R$ be a relation from a set $A$ to $B$ and $S$ a relation from $B$ to $C$. The composite (合成) of $R$ and $S$ is the relation

$$S \circ R = \left\{ (a,c) \mid a \in A, c \in C\, \exists b \in B, \begin{array}{c} (a,b) \in R \\ (b,c) \in S \end{array} \right\}$$

(3) LEt $R$ be a relation on set $A$. The power (幂) $R^n$ $n = 1, 2, \dots$ are defined inductively by

$$\begin{aligned} R^1 &= R \\ R^{n+1} &= R^n \circ R \end{aligned}$$

(4) Given a relation $R$ from $A$ to $B$, its inverse $R^{-1}$ is the relation from $B$ to $A$ defined by

$$R^{-1} = \{(y,x) \mid (x,y) \in R\}$$

### 1.3 Properties of Relations

**Definition 32** *Let $R$ be a relation on a set $A$.*

$$R \text{ is reflexive (自反)} \iff \forall x \in A, (x,x) \in R$$
$$R \text{ is irreflexive (反自反)} \iff \forall x \in A, (x,x) \notin R$$

**Definition 33** *Let $R$ be a relation on a set $A$.*

$$\begin{aligned} R \text{ is symmetric (对称)} \Longleftrightarrow & \forall x, y \in A, (x,y) \in R \\ & \Rightarrow (y,x) \in R \\ R \text{ is anti-symmetric (反对称)} \Longleftrightarrow & \forall x, y \in A, (x,y) \in R \\ & \text{and } (y,x) \in R \Rightarrow x = y \end{aligned}$$

Remark:

(1) $R$ is symmetric $\Longleftrightarrow R^{-1} = R$

(2) $R$ is anti-symmetric $\Longleftrightarrow R \cap R^{-1} \subseteq R_=$

(3) Non-symmetric $\not\Leftrightarrow$ anti-symmetric

**Definition 34** *Let $R$ be a relation on a set $A$.*

$$\begin{aligned} R \text{ is transitive (传递)} \Longleftrightarrow & \forall x, y, z \in A \\ & ((x,y) \in R \wedge (y,z) \in R) \\ & \Rightarrow (x,z) \in R \end{aligned}$$

Remark: $R$ is transitive $\Leftrightarrow R \circ R \subseteq R$.

**Theorem 26** *The relation $R$ on a set $A$ is transitive iff*

$$R^n \subseteq R$$

*for $n = 2, 3, \dots$*

## 2. Representing Relations

### 2.1 Matrices of Relations

**Definition 35** *Suppose $R$ a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$, The relation $R$ can be represented by matrix $M_R = (m_{ij})_{m \times n}$*

$$m_{ij} = \begin{cases} 1 & (a_i, b_j) \in R \\ 0 & (a_i, b_j) \notin R \end{cases}$$

Remark:

(1) Let $M_R$ be the matrix of a relation $R$ on set $A$. Let $M_R^2 = M_R \circ M_R$.

a. $R$ is reflexive $\Leftrightarrow m_{ii} = 1$.
$R$ is irreflexive $\Leftrightarrow m_{ii} = 0$.

b. $R$ is symmetric $\Leftrightarrow m_{ij} = m_{ji}$, i.e. $M_R$ is a symmetric matrix.
$R$ is anti-symmetric $\Leftrightarrow m_{ij} = 1, i \neq j \Rightarrow m_{ji} = 0$.

c. $R$ is transitive $\Leftrightarrow$ whenever $c_{ij}$ in $C = M_R^2$ is nonzero then entry $m_{ij}$ in $M_R$ is also nonzero.

$$c_{ij} = a_{i1}a_{1j} \vee a_{i2}a_{2j} \vee \cdots \vee a_{ik}a_{kj} \vee \cdots \vee a_{in}a_{nj}$$

(2) Suppose that $R_1$ and $R_2$ are relations on a set $A$ represented by matrix $M_{R_1}$ and $M_{R_2}$ respectively. Then

$$\begin{aligned} M_{R_1 \cup R_2} &= M_{R_1} \vee M_{R_2} \\ M_{R_1 \cap R_2} &= M_{R_1} \wedge M_{R_2} \\ M_{S \circ R} &= M_R \odot M_S \end{aligned}$$

where the operator are join and meet.

### 2.2 Digraphs of Relations

Each element of the set is represented by a point，and each ordered pair is represented using an arc with its direction indicated by an arrow — directed graphs or digraphs (元素为结点, 关系为有向边).

Remark: Let $R$ be a relation on set $A$.

(1) $R$ is reflexive $\Leftrightarrow$ There are loops at every vertex of digraph.

(2) $R$ is symmetric $\Leftrightarrow$ Every edge is Bi-directional edge.

## 3. Closures of Relations

### 3.1 Introduction

**Definition 36** *Let $R$ be a relation on a set $A$. If there is a relation $S$ satisfy:*

*(1) $S$ with property $P$ (reflexive, symmetricm, or transitive) and $R \subseteq S$.*

*(2) $\forall S'$ with property $P$ and $R \subseteq S'$, then $S \subseteq S'$.*

*Then $S$ is called the closure of $R$ with respect to $P$.*

### 3.2 Computing of Closures

**Theorem 27** *Let $R$ be a relation on set $A$.*

*(1) The reflexive closure of relation $R$:*

$$r(R) = R \cup \Delta$$

*where $\Delta = \{(a,a) | a \in A\}$ is diagonal relation on $A$.*

*(2) The symmetric closure of relation $R$:*

$$s(R) = R \cup R^{-1}$$

*(3) The transitive closure of $R$:*

$$t(R) = R^*$$

**Definition 37** *A path from $a$ to $b$ in the digraph $G$ is a sequence of one or more edges $(x_0, x_1)$, $(x_1, x_2)$, $\dots$, $(x_{n-1}, x_n)$ in $G$. where $x_0 = a$ and $x_n = b$. The path is denoted by $x_0, x_1, \dots, x_n$ and has length $n$.*

- *A circuit or cycle: a path that begins and ends at the same vertex.*

**Theorem 28** *Let $R$ be a relation on set $A$. There is a path of length $n$ from $a$ to $b \Leftrightarrow (a, b) \in R^n$.*

**Definition 38** *The connectivity relation*
$R^* = \{(a, b) | \text{there is a path from } a \text{ to } b\}$.

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

**Theorem 29** *The transitive closure of a relation $R$ equals the connectivity relation $R^*$, i.e.*

$$t(R) = R^*$$

Let $R$ be a relation on set $A$ with $n$ elements. Then,

$$R^* = \bigcup_{i=1}^{n} R^i$$

**Theorem 30** *Let $M_R$ be the zero-one matrix of the relation $R$ on a set with $n$ elements. Then the zero-one matrix of the transitive closure $R^*$ is*

$$M_{R^*} = M_R \vee \cdots \vee M_R^{[n]}$$

$O(n^4)$

### 3.3 Warshall's Algorithm

$O(2n^3)$

## 4. Equivalence Relations

### 4.1 Introduction of Equivalence Relations

**Definition 39** *Relation $R_{\sim} : A \leftrightarrow A$ is an equivalence relation (等价关系), if it reflexive, symmetric and transitive.*

### 4.2 Equivalence Classes and its Properties

**Definition 40** *Let $R : A \leftrightarrow A$ is an equivalence relation. $\forall a \in A$, the equivalence class (等价类) of $a$ is the set of the elements related to $a$*

$$[a]_R = \{x \in A | (x, a) \in R\}$$

*If $b \in [a]_R$, $b$ is called a representative of this equivalence class.*

The properties of equivalence classes:

(1) $\forall a \in A, [a]_R \neq \emptyset$.

(2) $(a, b) \in R \Rightarrow [a]_R = [b]_R$

(3) $(a, b) \notin R \Rightarrow [a]_R \cap [b]_R = \emptyset$

(4) $\bigcup_{a \in A} [a]_R = A$

**Definition 41** *The set of all equivalence classes of $R$ is the quotient set (商集) of $A$ with respect to $R$*

$$\frac{A}{R} = \{[a]_R | a \in A\}$$

Remark:

(1) If $A$ finite, then $\frac{A}{R}$ finite.

(2) If $A$ has $n$ elements, and if every $[a]_R$ has $m$ elements, then $m|n$, and $\frac{A}{R}$ has $\frac{n}{m}$ elements.

### 4.3 Partition

**Definition 42** *A partition (划分) $\pi$ on a set $S$ is a family $\{A_1, A_2, \dots, A_n\}$ of subsets of $S$ and*

*(1) $\bigcup_{k=1}^{n} A_k = S$.*

*(2) $A_j \cap A_k = \emptyset$ for every $j, k$ with $j \neq k$, $1 < j, k < n$.*

**Theorem 31** *Let $R$ be an equivalence relation on a set $S$. Then the equivalence classes of $R$ form a partition of $X$. Conversely, given a partition $\{A_i | i \in I\}$ of set $S$, there is an equivalence relation $R$ that has the set $A_i, i \in I$, as its equivalence classes.*

### 5. Partial Orderings

#### 5.1 Introduction

**Definition 43** *Relation $R_{\preceq} : S \leftrightarrow S$ is a partial order, if it reflexive, anti-symmetric and transitive.*

*A set $S$ together with a partial ordering $R_{\preceq}$ is called a partial order set or poset, and is denoted by $(S, R_{\preceq}), P(S)$.*

**Definition 44** *$\forall a, b \in poset(S, \preceq)$ are called comparable (可比) if either $a \preceq b$ or $b \preceq a$, otherwise they are called incomparable (不可比).*

*If $(S, \preceq)$ is a poset and every elements of $S$ are comparable, $S$ is called a totally order (全序集) or linearly order set (线性集), $\preceq$ is called a total order or linear order (全序或线性). A totally ordered set is also called a chain (链).*

#### 5.2 Lexicographic Order

#### 5.3 Hasse Diagram

Represent a partial ordering on a finite set using the following procedure:

(1) Start with the directed graph for the relation.

(2) Remove all loops.

(3) Remove all edges that must be present because of the transitivity.

(4) Finally, arrange each edges so that its initial vertex is below its terminal vertex, remove all the arrows.

The resulting diagram contains sufficient information to find the partial ordering. — Hasse Diagram.

#### 5.4 Maximal and Minimal Elements

**Definition 45** *Let $(A, \preceq)$ be a partial ordered set, $B \subseteq A$.*

*(1) maximal and minimal elements*

*a. $a$ is a maximal element (极大元) of $B$:*

$$a \in B \wedge \nexists x \in B : a \prec x$$

*b. $b$ is a minimal element (极小元) of $B$:*

$$b \in B \wedge \nexists x \in B : x \prec b$$

*(2) greatest and least elements*

*a. $a$ is the greatest element (最大元) of $B$:*

$$a \in B \wedge \forall x \in B : x \preceq a$$

*b. $b$ is the least element (最小元) of $B$:*

$$b \in B \wedge \forall x \in B : b \preceq x$$

*(3) upper and lower bound*

*a. $c$ is an upper bound (上界) of $B$:*

$$c \in A \wedge \forall x \in B : x \preceq c$$

*b. $d$ is a lower bound (下界) of $B$:*

$$d \in A \wedge \forall x \in B : d \preceq x$$

*(4) least upper and greatest lower bound*

*a. $c$ is the least upper bound (最小上界) of $B$:*

$$c \text{ is an upper bound of } B$$
$$\wedge \forall x \text{ is an upper bound of } B : c \preceq x$$

*b. $d$ is the greatest lower bound (最大下界) of $B$:*

$$d \text{ is an lower bound of } B$$
$$\wedge \forall x \text{ is an lower bound of } B : x \preceq d$$

*$(S, \preceq)$ is a well-ordered set if it is a poset such that $\preceq$ is a total ordering and such that every nonempty of $S$ has a least element.*

#### 5.5 Lattices

**Definition 46** *A partially ordered set in which pair of elements has both a least upper bound and a greatest lower bound is called lattice (格).*

## 八、 Graphs (图)

### 1. Introduction to Graphs

#### 1.1 Types of Graphs

**Definition 47** *(1) A simple graph (简单图) $G = (V, E)$ consists of $V$, a nonempty set of vertices and $E$, a set of unordered pair of distinct elements of $v$ called edges.*

*(2) A multigraph (重图) $G=(V,E)$ consists of a set $V$ of vertices and a set $E$ of edges which has multiple or parallel edges.*

*(3) A pseudograph (伪图) $G=(V,E)$ consists of a set $V$ of vertices and a set of $E$ of edges which has loops and multiple edges.*

- *The multiple or parallel edges:*

$$f: E \to \{\{u,v\} \,|u,v \in V, u \neq v\}$$

$e_1, e_2$ *are called multiple edges if* $f(e_1)=f(e_2)$.

- *The loop:*

$$f: E \to \{\{u,v\} \,|u,v \in V\}$$

$e$ *is a loop if* $f(e)=\{u,u\}=\{u\}, u \in V$.

**Definition 48** *(1) A directed graph $G=(V,E)$ consists of a set of vertices $V$, and a set of edges $E$ taht ordered pairs of elements of $V$.*

*(2) A directed multigraph $G=(V,E)$ consists of a set of vertices $V$, and a set of edges $E$ taht ordered pairs of elements of $V$ and has multiple edges.*
*The multiple edges:*

$$f: E \to \{(u,v)|u,v \in V, u \neq v\}$$

$e_1, e_2$ *are called multiple edges if* $f(e_1)=f(e_2)$.

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## 2. Graph Terminology

表 8: Graph Terminology

| Type | Edges | Multiple Edges Allowed? | Loop Allowed? |
|---|---|---|---|
| Simple Graph | Undirected | No | No |
| Multigraph | Undirected | Yes | No |
| Pseudograph | Undirected | Yes | Yes |
| Directed Graph | Directed | No | Yes |
| Directed Multigraph | Directed | Yes | Yes |