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四、 误差 3

一、 Basic Knowledge 1

舍入误差, Taylor 展开, 均值定理. 以及一些迭代公式与 stable 的问题.

1. Mean value theorem

2. Taylor expansion

3. Roundoff error

二、 Solution of Equations

分析收敛与稳定 (Convergence + Stability)

1. Solution of equations with one variable 2

不动点法, 二分法, 牛顿法

1.1 二分法

Theorem 二、.1 *$f \in C[a,b]$ and $f(a)f(b) < 0$. A sequence $\{p_n\}(n = 0, 1, 2, \cdots)$ approximating a zero p of f with $|p_n - p| \le \frac{b-a}{2^n}$, where $n \ge 1$.*

不能有重根, 多根.

1.2 不动点

$$f(x) = 0 \Longleftrightarrow x = g(x)$$

Theorem 二、.2 *$g \in [a,b]$, $\exists g'(x)$ and $g'(x) = k$, $k \in (0,1)$, $p_n = g(p_{n-1})$, 数列收敛于唯一点*

有

$$|p_n - p| \le \frac{1}{1-k}|p_{n+1} - p_n|$$
$$|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0|$$

1.3 牛顿迭代

Theorem 二、.3 *$f \in C^2[a,b]$, $p \in [a,b]$, f(p)=0, $f'(p) \ne 0$, $\exists \delta > 0$, $\{p_n\}$, $p_0 \in [p-\delta, p+\delta]$*

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, n \ge 1$$

1.4 误差

Theorem 二、.4 *$\{p_n\}$ 收于 p, $\exists \alpha, \lambda > 0$,*

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda$$

$\{p_n\}$converges to p of order(阶) α, with asymptotic error constant(渐进误差常数) λ.

解决牛顿重根降阶, 将根变位:

$$g(x) = x - \frac{f(x)f'(x)}{f'^2(x) - f(x)ff''(x)}$$

2. Direct Matrix Solver 6

高斯消元 (会不可用的情况), LU 分解, 其他分解

$$A\vec{x} = b$$

2.1 高斯消元

可解性见线代.

2.2 LU 分解

若 $|L| = 1$, 则分解唯一. 用 $L\vec{y} = \vec{b}$, 再 $U\vec{x} = \vec{y}$ 求解.

3. Iterative Matrix Solver 7

雅可比, 高斯-赛德尔方法, 松弛迭代

3.1 范数

向量范数

(1) $\|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i|$

(2) $\|\vec{x}\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$

(3) $\|\vec{x}\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$

(4) $\|\vec{x}\|_\infty = \max_{1\leq i\leq n} |x_i|$

矩阵范数

(1) Frobenius Norm: $\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2}$

(2) Natural Norm(operator norm):

$$\|A\|_p = \max_{\vec{x}\neq 0} \frac{\|A\vec{x}\|_p}{\|\vec{x}\|_p} = \max_{\|\vec{x}\|_p=1} \|A\vec{x}\|_p$$

a. $\|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$
b. $\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$
c. $\|A\|_2 = \sqrt{\lambda_{max}(A^TA)}$(spectral norm)(谱范数)

3.2 Jacobi 迭代

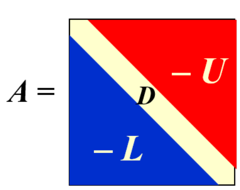

图 1: A=(D-L-U)

$\vec{x}^{(k)} = D^{-1}(L+U)\vec{x}^{(k-1)} + D^{-1}\vec{b}$

3.3 Gauss-Seidel 迭代

$\vec{x}^{(k)} = (D-L)^{-1}U\vec{x}^{(k-1)} + (D-L)^{-1}\vec{b}$

3.4 松弛迭代

$\vec{x}^{(k)} = (D-\omega L)^{-1}[(1-\omega)D+\omega U]\vec{x}^{(k-1)} + (D-\omega L)^{-1}\omega\vec{b}$

(1) 若 $a_{ii} \neq 0$, 且 $\rho(T_\omega) \geq |\omega-1|$, 迭代要求 $0 < \omega < 2$

(2) 若 A 正定, 且 $0 < \omega < 2$, 则迭代收

(3) 若 A 正定且三对角, $\rho(T_g) = [\rho(T_j)]^2 < 1$, 则 $\omega = \frac{2}{1+\sqrt{1-[\rho(T_j)]^2}}$ 最优, 且有 $\rho(T_\omega) = \omega - 1$

3.5 迭代收敛

$\vec{x}^{(k)} = T\vec{x}^{(k-1)} + \vec{C}$
$\vec{e}^{(k)} = T^k\vec{e}^{(0)}$
当且仅当 $\rho(T) < 1$ 时收. $\rho(T) = max|\lambda|$(最大特征值)

误差

$$\|\vec{x} - \vec{x}^{(k)}\| \leq \frac{\|T\|^k}{1-\|T\|}\|\vec{x}^{(1)} - \vec{x}^{(0)}\|$$

$A(\vec{x} + \delta\vec{x}) = \vec{b} + \delta\vec{b}$,

$$\frac{\|\delta\vec{x}\|}{\|\vec{x}\|} \leq \|A\|\|A^{-1}\|\frac{\|\delta\vec{b}\|}{\|\vec{b}\|}$$

$K(A) = \|A\|\|A^{-1}\|$ 被叫做条件数/放大因子.

(1) 若 A 对称, $K(A)_2 = \frac{\max|\lambda|}{\min|\lambda|}$

(2) $K(A)_p \geq 1$

(3) 若 A 正交矩 $(A^{-1} = A^T)$, $K(A)_2 = 1$

(4) $K(\alpha A) = K(A)$

(5) $K(RA)_2 = K(AR)_2 = K(A)_2$

4. Initial value problem 5

(微分方程)Euler, Runge-kutta, multi-step, 隐式与显示

三、 Interpolation and approximation

插值与逼近, 分析 Error

1. interpolation 3

Lagrange polynomial, Piecewise polynomial, 其他多项式

1.1 Lagrange 多项式

$$P(x)=\sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$
$$L_{n,k}(x)=\prod_{i=0,i\neq k}^{n}\frac{x-x_i}{x_k-x_i} = \begin{cases}0, & x=x_i, \quad i\neq k\\ 1, & x=x_k\end{cases}$$

$R_n(x)=\frac{f^{(n+1)(\xi_x)}}{(n+1)!}\prod_{i=0}^{n}(x-x_i)$
Lagrange 多项式是唯一的.

1.2 Neville’s Method

定义 $P_{m_1,\cdots,m_k}(x)$ 为 $x_{m_1},\cdots,x_{m_k}$ 插值

$$P(x)=\frac{(x-x_j)P_{0,\cdots,j-1,j+1,\cdots,k}(x)-(x-x_i)P_{0,\cdots,i-1,i+1,\cdots,k}(x)}{x_i-x_j}$$

1.3 Newton’s Divided Differences

$$f[x_0,\cdots,x_{k+1}]=\frac{f[x_0,\cdots,x_k]-f[x_0,\cdots,x_{k-1},x_{k+1}]}{x_k-x_{k+1}}$$

$f(x)=N_n(x)+R_n(x)$,

$$\begin{aligned}N_n(x)=&f\left(x_0\right)+f\left[x_0,x_1\right]\left(x-x_0\right)+f\left[x_0,x_1,x_2\right]\left(x-x_0\right)\left(x-x_1\right)\\&+\ldots+f\left[x_0,\ldots,x_n\right]\left(x-x_0\right)\ldots\left(x-x_{n-1}\right)\\R_n(x)=&f\left[x,x_0,\ldots,x_n\right]\left(x-x_0\right)\ldots\left(x-x_{n-1}\right)\left(x-x_n\right)\end{aligned}$$

1.4 Hermite 插值

$$H_{2n+1}(x)=\sum_{i=0}^{n} f(x_i)h_i(x)+\sum_{i=0}^{n} f'(x_i)\hat{h}_i(x)$$
$$h_i(x)=\delta_{ij}, h_i'(x)=0$$
$$\hat{h}_i(x)=0, \hat{h}_i'(x)=\delta_{ij}$$
$$R_n(x)=\frac{f^{(2n+2)}(\xi_x)}{(2n+2!)\prod_{i=0}^{n}(x-x_i)^2}$$

1.5 Cubic Spline

S(x)

(1) $S_i(x)$ 为 $[x_i,x_{i+1}]$ 区间

(2) $S(x_i)=f(x_i)$, (n+1) 个

(3) $S_{i+1}(x_{i+1})=S_i(x_{i+1})$, (n-1) 个

(4) $S_{i+1}'(x_{i+1})=S_i'(x_{i+1})$, (n-1) 个

(5) $S_{i+1}''(x_{i+1})=S_i''(x_{i+1})$, (n-1) 个

(具体见笔记)

更多条件:

(1) $S'(a)=f'(x_0), S'(b)=f'(x_n)$ 导数边界

(2) $S''(a)=f''(x_0), S''(b)=f''(x_n)$ 为 0 时叫 Natural Spline

(3) $S'(a^+)=S'(b^-)$

2. Numerical differentiation and Numerical integration 4

数值微分与数值积分, 很多法则

3. approximation 8

Least squares, Orthogonal polynomials, Chebyshev polynomial

四、 误差

(1) 舍入误差 (计算机带来的)

(2) 真值误差 (解方程)

(3) 截断误差 (插值与逼近)

Lagrange polynomial

$$R_n(x)=\frac{f^{(n+1)}(\xi_x)}{(n+1)!}\prod_{i=0}^{n}(x-x_i)$$

Taylor expansion

$$R_n(x)=\frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-x_0)^{n+1}$$