110550085 房[illegible]

1. Initial condition: $\frac{dy}{dt} = \overset{f(y,t)}{y^2+t^2}$, $y(1)=0$, at $t=2$, $h=0.1$

$\Rightarrow y(t_0+h) = y(t_0) + h \cdot f(y,t)$

$y(1.1) = 0 + 0.1 \cdot f(0,1) = 0.1$

$y(1.2) = 0.1 + 0.1 \cdot f(0.1, 1.1) = 0.222$

$y(1.3) = 0.222 + 0.1 \cdot f(0.222, 1.2) = 0.370928$

$y(1.4) = 0.370928 + 0.1 \cdot f(0.370928, 1.3) \fallingdotseq 0.553687$

$y(1.5) = 0.553687 + 0.1 \cdot f(0.553687, 1.4) \fallingdotseq 0.780344$

$y(1.6) = 0.780343 + 0.1 \cdot f(0.780343, 1.5) \fallingdotseq 1.066237$

$y(1.7) = 1.066237 + 0.1 \cdot f(1.066237, 1.6) \fallingdotseq 1.435923$

$y(1.8) = 1.435923 + 0.1 \cdot f(1.435923, 1.7) \fallingdotseq 1.931110$

$y(1.9) = 1.931110 + 0.1 \cdot f(1.931110, 1.8) \fallingdotseq 2.628029$

$y(2) = 2.628029 + 0.1 \cdot f(2.628029, 1.9) \fallingdotseq \underline{3.679683}$.

Repeat the above method with $h=0.5$, we would get $y(2.0) \fallingdotseq \underline{4.55816}$

By Romberg integration, error $= \frac{4.55816 - 3.679683}{2^2-1} \fallingdotseq \underline{0.292826}$. #

2. Use $f(t)=1$, $f(t)=t$, $f(t)=t^2$, and $f(t)=t^3$

$$\Rightarrow \frac{h^4}{4} = C_0(-2h)^3 + C_1(-h)^3 + C_2(0) + C_3(h)^3$$

$$\frac{h^3}{3} = C_0(-2h)^2 + C_1(-h)^2 + C_2(0) + C_3(h)^2$$

$$\frac{h^2}{2} = C_0(-2h) + C_1(-h) + C_2(0) + C_3(h)$$

$$h = C_0(1) + C_1(1) + C_2(1) + C_3(1).$$

$$\Rightarrow \begin{bmatrix} -8 & -1 & 0 & 1 \\ 4 & 1 & 0 & 1 \\ -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{3} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Solve the linear system:

$6C_0 = \frac{1}{4}$, $C_0 = \frac{1}{24}$, $2C_3 = \frac{9}{12}$

$\Rightarrow -C_1 + C_3 = \frac{7}{12}$, $C_1 + C_3 = \frac{1}{6} \Rightarrow C_3 = \frac{9}{24}$, $C_1 = -\frac{5}{24}$

$\Rightarrow C_2 = 1 - \frac{1}{24} + \frac{5}{24} - \frac{9}{24} = \frac{19}{24}$. $\Rightarrow \tilde{x}_{n+1} = x_n + \frac{h}{24}[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}]$. + error.

And for the error term:

$$f_{n-2} = x'(f_n - 2h) \fallingdotseq x'(f_n) - 2hx''(f_n) + \frac{4h^2}{2!}x'''(f_n) + \frac{-8h^3}{3!}x^{(4)}(f_n) + \frac{16h^4}{4!}x^{(5)}(f_n).$$

$$f_{n-1} = x'(f_n - h) \fallingdotseq x'(f_n) - hx''(f_n) + \frac{h^2}{2!}x'''(f_n) + \frac{-h^3}{3!}x^{(4)}(f_n) + \frac{h^4}{4!}x^{(5)}(f_n).$$

$$f_n = x'(f_n)$$

$$f_{n+1} = x'(f_n + h) \fallingdotseq x'(f_n) + hx''(f_n) + \frac{h^2}{2!}x'''(f_n) + \frac{h^3}{3!}x^{(4)}(f_n) + \frac{h^4}{4!}x^{(5)}(f_n).$$

$\tilde{x}_{n+1} = x(f_n) + h x'(f_n) + \frac{h^2}{2} x''(f_n) + \frac{h^3}{6} x'''(f_n) + \frac{h^4}{24} x^{(4)}(f_n) + \frac{h^5}{120} x^{(5)}(f_n)$

$$\text{error} = x_{n+1} - \left(x_n + \frac{h}{24}[9f_{n+1} + 19 f_n - 5f_{n-1} + f_{n-2}]\right).$$

$$= \cdots\cdots + \left(\frac{h^5}{120} - \frac{h}{24}\left[9 \cdot \frac{h^4}{24} - 5 \cdot \frac{h^4}{24} + \frac{16h^4}{24}\right]\right)$$

$$= \left(\frac{h^5}{120} - \frac{h}{24}\left[\frac{20}{24} h^4\right]\right)$$

$$= \frac{h^5}{120} - \frac{5h^5}{144}$$

$$= \frac{-19}{720} h^5 x^{(5)}(f_n).$$

$\Rightarrow$ In conclusion $\tilde{x}_{n+1} = x_n + \frac{h}{24}[9f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}] - \frac{19}{720} h^5 x^{(5)}(f_n).$

3. (a). Let $y = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $f = y' = \begin{bmatrix} y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ 2x_1 - tx_2 + t \end{bmatrix}$, by $y''' + ty' - 2y = t$;

Then, solve it with the MATLAB code appended in the file (rkf.m).

We would get $y(0.2) = 0.200133$,

$y(0.4) = 0.402132$,

$y(0.6) = 0.610778$.

(b) Calculate $y(0.8)$ with the previous 4 points.

$$y(0.8) = y(0.6) + \frac{0.2}{24}(55 \cdot f_{0.6} - 59 f_{0.4} + 37 f_{0.2} - 9 f_0) = \begin{bmatrix} 0.834 \\ 1.1694 \\ 0.6296 \end{bmatrix}.$$

Compute $f_{0.8}$ by (a): $f_{0.8} = \begin{bmatrix} 1.1694 \\ 0.6296 \\ 1.5325 \end{bmatrix}$.

Then apply the corrector:

$$\tilde{y}(0.8) = y(0.6) + \frac{0.2}{24}(9 f_{0.8} + 19 f_{0.6} - 5 f_{0.4} + f_{0.2}) = \begin{bmatrix} 0.834 \\ 1.1692 \\ 0.6291 \end{bmatrix}.$$

$\Rightarrow$ Calculate $y(1.0)$ with the previous 4 points.

$$y(1.0) = y(0.8) + \frac{0.2}{24}(55 \cdot f_{0.8} - 59 f_{0.6} + 37 f_{0.4} - 9 f_{0.2}) = \begin{bmatrix} 1.0827 \\ 1.3284 \\ 0.9679 \end{bmatrix}.$$

Compute $f_{1.0}$ by (a): $f_{1.0} = \begin{bmatrix} 1.3284 \\ 0.9679 \\ 1.8370 \end{bmatrix}$.

Then apply the corrector:

$$\tilde{y}(1.0) = y(0.8) + \frac{0.2}{24}(9 f_{1.0} + 19 f_{0.8} - 5 f_{0.6} + f_{0.4}) = \begin{bmatrix} 1.0826 \\ 1.3281 \\ 0.9675 \end{bmatrix}.$$

(c) The error should be $\left|\tilde{y}(1.0) - y(1.0)\right| * \left(\frac{19}{251+19}\right) \doteqdot \underline{7.04 \times 10^{-6}}$.

4.(a). Given $h=\frac{\pi}{4}$, $y''+\frac{y}{4}=0$, $y(0)=0$, $y(\pi)=2$

$\Rightarrow \frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+\frac{y_i}{4}=0$.

Divide into 4 intervals.

$\Rightarrow$ Since $h=\frac{\pi}{4}$, $\frac{16}{\pi^2}y_{i-1}+\left(\frac{\pi^2-128}{4\pi^2}\right)y_i+\frac{16}{\pi^2}y_{i+1}=0$, $i=0,1,\cdots 4$.

Construct a system of equations, we have

$$\begin{bmatrix} \frac{16}{\pi^2} & \frac{\pi^2-128}{4\pi^2} & \frac{16}{\pi^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{16}{\pi^2} & \frac{\pi^2-128}{4\pi^2} & \frac{16}{\pi^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{16}{\pi^2} & \frac{\pi^2-128}{4\pi^2} & \frac{16}{\pi^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{16}{\pi^2} & \frac{\pi^2-128}{4\pi^2} & \frac{16}{\pi^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{16}{\pi^2} & \frac{\pi^2-128}{4\pi^2} & \frac{16}{\pi^2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{-1} \\ y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

Solve this, we'd get the tableau:

	y_{-1}	y_0	y_1	y_2	y_3	y_4	y_5
θ	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$
Estimate	-0.7702	0	0.7702	1.4215	1.8537	2	1.8379
RealValue	-0.7654	0	0.7654	1.4142	1.8478	2	1.8478
Error	0.0048	0	-0.0048	-0.0073	-0.0059	0	0.0099

(b) Current maximum error is about 0.62%. we don't need to have a huge decrease for h, try $h=\frac{\pi}{5}$, solve like the above method, we'd get:

	y_{-1}	y_0	y_1	y_2	y_3	y_4	y_5	y_6
θ	$-\frac{1}{5}\pi$	0	$\frac{1}{5}\pi$	$\frac{2}{5}\pi$	$\frac{3}{5}\pi$	$\frac{4}{5}\pi$	π	$\frac{6}{5}\pi$
Estimate	-0.6205	0	0.6205	1.1798	1.6227	1.9054	2	1.8973
RealValue	-0.6180	0	0.6180	1.1756	1.6180	1.9021	2	1.9021
Error	0.0025	0	-0.0025	-0.0042	-0.0047	-0.0032	0	0.0049

The maximum error is 0.404%, which corresponds to the need.

(c) Apply shooting method with secant method:

Let $\begin{bmatrix} y \\ y' \end{bmatrix}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}=\begin{bmatrix} x_2 \\ -\frac{1}{4}x_1 \end{bmatrix}$, $\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix}=\begin{bmatrix} 0 \\ y'(0) \end{bmatrix}$.

We have 2 IVPs:

Choose $\alpha_0=1$, $\alpha_2=1.2$

IVP_1: $y''=-\frac{y}{4}$, $y(0)=0$, $y'(0)=1$

IVP_2: $y''=-\frac{y}{4}$, $y(0)=0$, $y'(0)=1.2$

Use Euler's method for the IVPs.

$IVP_1: y' = z,\ y(0) = 0$

$z' = -\frac{y}{4},\ z(0) = 1$, set $h = \frac{\pi}{4}$

$y_{n+1} = y_n + h z_n,\ z_{n+1} = z_n + h\left(-\frac{y_n}{4}\right)$

$y_1 = y\left(\frac{\pi}{4}\right) = y_0 + h z_0 = 0 + \frac{\pi}{4}\cdot 1 = \frac{\pi}{4}$,

$z_1 = z\left(\frac{\pi}{4}\right) = z_0 + h\left(-\frac{y_0}{4}\right) = 1 + \frac{\pi}{4}(0) = 1$.

$\vdots$

Solve with the appended code (shoot.m), $y_4 \doteq 2.039597$

$IVP_2: y_4 = 2.447517$

$\phi(\alpha_0) = y(0;1) - 2 \doteq 0.039599$

$\phi(\alpha_1) = y(0;1.2) - 2 \doteq 0.447517$

$\alpha_2 = 1.2 - \frac{(1.2-1)}{0.407920} \times (0.447517) \doteq 0.980586$

$IVP_3: y_4 \doteq 2.000000627$ which is very near to 2, so we select 0.980586 to be $y'(0)$.

$\Rightarrow \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \end{bmatrix} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} + h \begin{bmatrix} x_2^{(k)} \\ -\frac{1}{4}x_1^{(k)} \end{bmatrix}$.

$\Rightarrow$ Solve with the appended code (shootingmethod.m), we found that for h at about 0.0003, it can have error less than 0.5

5. From 0 to 1 with 4 subintervals, we have $h = \frac{1}{4}$.

Given $x'' - tx' + t^2 x = t^3$

$\Rightarrow \left(\frac{x_{i+2} - 2x_{i+1} + x_i}{h^2}\right) - t_i\left(\frac{x_{i+1} - x_i}{h}\right) + t_i^2 x_i = t_i^3$, by forward difference.

Multiply by h^2, and reorder it, we get:

$x_{i+2} + (-2 - t_i h)x_{i+1} + (1 + t_i h + t_i^2 h^2)x_i = t_i^3 h^2$

Set $t_i = ih$, $i = 0,1,2,3$.

$\Rightarrow x_{i+2} + (-2 - ih^2)x_{i+1} + (1 + ih^2 + i^2h^4)x_i = i^3h^5$, $i = 0,1,2,3$.

$$\Rightarrow \begin{cases} x_2 - 2x_1 + x_0 = 0 \quad \cdots ① \quad \Rightarrow x_2 - 2x_1 = -\frac{5}{2} \\ x_3 + (-2-h^2)x_2 + (1+h^2+h^4)x_1 = h^5 \cdots ② \\ x_4 + (-2-2h^2)x_3 + (1+2h^2+4h^4)x_2 = 8h^5 \cdots ③ \\ x_5 + (-2-3h^2)x_4 + (1+3h^2+9h^4)x_3 = 27h^5. \cdots ④ \end{cases}$$

Also, by the two given relations, we can get $x(0) = \frac{5}{2}$, $x'(0) - x(1) + x'(1) = \frac{1}{2}$

$\Rightarrow \frac{x_1 - x_0}{h} - x_4 + \frac{x_5 - x_4}{h} = \frac{1}{2}$

$\Rightarrow x_1 - x_0 - hx_4 + x_5 - x_4 = \frac{1}{2}h$

$\Rightarrow x_5 = x_0 - x_1 + (h+1)x_4 + \frac{1}{2}h$.

Substitute into ④, and $x(0) = \frac{5}{2}$

$\Rightarrow \frac{5+h}{2} - x_1 + (1+3h^2+9h^4)x_3 + (-3h^2+h-1)x_4 = 27h^5$

$\Rightarrow -x_1 + (1+3h^2+9h^4)x_3 + (-3h^2+h-1)x_4 = 27h^5 - \frac{h}{2} - \frac{5}{2}$

So we can get

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1+h^2+h^4 & -2-h^2 & 1 & 0 \\ 0 & 1+2h^2+4h^4 & -2-2h^2 & 1 \\ -1 & 0 & 1+3h^2+9h^4 & -3h^2+h-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ h^5 \\ 8h^5 \\ 27h^5 - \frac{h}{2} - \frac{5}{2} \end{bmatrix}$$

$\Rightarrow (x_1, x_2, x_3, x_4) \doteq (3.1976, 3.8951, 4.6249, 5.3927)$.

#