HW2. 110550085 房天越

Problem 1. Special Random Variables

(a) $P(X=n)=(1-p)^{n-1}\cdot p$; X can be $p, (1-p)p, (1-p)^2p, \dots$

all possibilities.

$P(X<p)=\left(\sum_{k=2}^{\infty}(1-p)^{k-1}\cdot p\right)^n \Rightarrow P(X=p)=1-\left(\sum_{k=2}^{\infty}(1-p)^{k-1}\cdot p\right)^n=\left(\sum_{k=1}^{\infty}(1-p)^{k-1}\cdot p\right)^n-\left(\sum_{k=2}^{\infty}(1-p)^{k-1}\cdot p\right)^n$

$P(X=(1-p)\cdot p)=1-\left[1-\left(\sum_{k=2}^{\infty}(1-p)^{k-1}\cdot p\right)^n\right]-\left(\sum_{k=3}^{\infty}(1-p)^{k-1}\cdot p\right)^n=\left(\sum_{k=2}^{\infty}(1-p)^{k-1}\cdot p\right)^n-\left(\sum_{k=3}^{\infty}(1-p)^{k-1}\cdot p\right)^n$

$P(X=(1-p)^2\cdot p)=1-\left[1-\left(\sum_{k=2}^{\infty}(1-p)^{k-1}\cdot p\right)^n\right]-\left[\left(\sum_{k=2}^{\infty}(1-p)^{k-1}p\right)^n-\left(\sum_{k=3}^{\infty}(1-p)^{k-1}\cdot p\right)^n\right]-\left(\sum_{k=4}^{\infty}(1-p)^{k-1}\cdot p\right)^n=\left(\sum_{k=3}^{\infty}(1-p)^{k-1}\cdot p\right)^n-\left(\sum_{k=4}^{\infty}(1-p)^{k-1}\cdot p\right)^n$

$\Rightarrow$ Thus we know that PMF of X is $\left(\sum_{k=x}^{\infty}(1-p)^{k-1}\cdot p\right)^n-\left(\sum_{k=x+1}^{\infty}(1-p)^{k-1}\cdot p\right)^n$, where x is a positive integer, otherwise it equals to 0.

For Y, it can also be $p, (1-p)p, (1-p)^2p, \dots$

$P(Y=p)=p^n$, $P(Y=(1-p)p)=(p+(1-p)p)^n-p^n=\left(\sum_{k=1}^{2}(1-p)^{k-1}\cdot p\right)^n-\left(\sum_{k=1}^{1}(1-p)^{k-1}\cdot p\right)^n$

$P(Y=(1-p)^2p)=(p+(1-p)p+(1-p)^2p)^n-(p+(1-p)\cdot p)^n=\left(\sum_{k=1}^{3}(1-p)^{k-1}p\right)^n-\left(\sum_{k=1}^{2}(1-p)^{k-1}\cdot p\right)^n$

$\Rightarrow$ Thus we know that PMF of Y is $\begin{cases}\left(\sum_{k=1}^{x}(1-p)^{k-1}\cdot p\right)^n-\left(\sum_{k=1}^{x-1}(1-p)^{k-1}\cdot p\right)^n, & x\in\{2,3,4,\dots\} \\ p^n, & x=1 \\ 0, & \text{otherwise.}\end{cases}$

By 等比級數, it's $\frac{p\cdot[1-(1-p)^x]}{(1-1+p)}-\frac{p(1-(1-p)^{x-1})}{(1-1+p)}=1-(1-p)^x-(1-(1-p)^{x-1})=(1-p)^{x-1}-(1-p)^x=(1-p)^{x-1}(1-1+p)=(1-p)^{x-1}\cdot p$

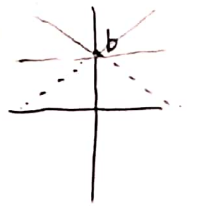
$\Rightarrow$ So it's Geometric Random Variable.

(b) For $n=1$, $p(1)=p(2)=\frac{1}{2}=\frac{1}{(1+1)-1}$, it agrees.

Assume that for $n=k$, it agrees, i.e. $p(1)=p(2)=\dots=p(k+1)=\frac{1}{(k+1)-1}=\frac{1}{k}$, and there are $k+1$ balls

Then for $n=k+1$, $p(1)=\frac{1}{k}\times\frac{k}{k+1}=\frac{1}{k+1}$, $p(2)=\frac{1}{k}\times\frac{1}{k+1}+\frac{1}{k}\times\frac{k-1}{k+1}=\frac{k}{k(k+1)}=\frac{1}{k+1}$, $p(n)=\frac{1}{k}\times\frac{(n-1)}{k+1}+\frac{1}{k}\times\frac{k+1-n}{k+1}=\frac{k}{k(k+1)}=\frac{1}{k+1}$

$\Rightarrow$ By the induction steps, it agrees. #



(c) X has PDF like $\frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$, Y is $\begin{cases}ax+b, & \text{if } x\geq 0 \\ -ax+b, & \text{if } x<0.\end{cases}$

and X's CDF be like $\Phi\left(\frac{t-\mu}{\sigma}\right)$.

's CDF like:

Case 1: $a>0$

$P(Y\leq t)$
$=P(Y\leq t, X\geq 0)+P(Y\leq t, X<0)$
$=P(ax+b\leq t)+P(-ax+b\leq t)$
$=P\left(X\leq\frac{t-b}{a}\right)+P\left(X\leq\frac{t-b}{-a}\right)$
$=\Phi\left(\frac{t-b}{a}\right)+\Phi\left(\frac{t-b}{-a}\right)$

Case 2: $a=0$.

$\Rightarrow P(Y\leq t)$
$=\begin{cases}0 & \text{for } Y<b \\ 1 & \text{for } Y\geq b\end{cases}$

Case 3: $a<0$.

$\Rightarrow P(Y\leq t)$
$=P(Y\leq t, X\geq 0)+P(Y\leq t, X<0)$
$=P(ax+b\leq t)+P(-ax+b\leq t)$
$=P\left(X\leq\frac{t-b}{a}\right)+P\left(X\leq\frac{t-b}{-a}\right)$
$=\Phi\left(\frac{t-b}{a}\right)+\Phi\left(\frac{t-b}{-a}\right)$

$\Rightarrow$ And Y's PDF is like:

If $a\neq 0$, $\Phi'\left(\frac{t-b}{a}\right)\cdot\frac{1}{a}-\Phi'\left(\frac{t-b}{-a}\right)\cdot\frac{1}{a}$

If $a=0$, $\begin{cases}1, & \text{for } Y=b \\ 0, & \text{for elsewhere.}\end{cases}$

$\Rightarrow$ Y cannot be a normal random variable, since normal random variables has PDF symmetric about μ, while taking absolute breaks the characteristic.

Problem 2 (PMF and Entropy)

(a)

$$H(X) = -\sum_{i=1}^{n} P_i \ln P_i = \sum_{i=1}^{n} P_i \cdot (-\ln P_i)$$

By the weighted inequality of arithmetic and geometric means, $\frac{P_1(-\ln P_1)+P_2(-\ln P_2)+\cdots+P_n(-\ln P_n)}{(P_1+\cdots+P_n)} \geq \left((-\ln P_1)^{P_1} \cdot (-\ln P_2)^{P_2} \cdots (-\ln P_n)^{P_n}\right)^{\frac{1}{P_1+\cdots+P_n}}$

Since $P_1+\cdots+P_n = 1$, $-P_1\ln P_1 - P_2\ln P_2 - \cdots - P_n \ln P_n \geq (-\ln P_1)^{P_1}(-\ln P_2)^{P_2}\cdots(-\ln P_n)^{P_n}$

The "=" holds when $P_1 \ln P_1 = P_2 \ln P_2 = \cdots = P_n \ln P_n$, then $P_1 = P_2 = \cdots = P_n = \frac{1}{n}$

$\Rightarrow \sum_{i=1}^{n} \frac{1}{n}\left(-\ln\left(\frac{1}{n}\right)\right) = \sum_{i=1}^{n} \frac{1}{n} \cdot \ln\left(\frac{1}{n}\right)^{-1} = \ln n$, $\ln n$ is the maximum value of entropy.

The PMF is $\begin{cases} \frac{1}{n}, & \text{if } i = 1,2,\ldots,n \\ 0, & \text{otherwise.} \end{cases}$

(b)

The minimum value happens when only for one i, $P_i = 1$, otherwise $P_i = 0$, then the value is $-1\ln 1 = 0$, since $P_i(-\ln P_i)$ always larger than or equals to 0, 0 is the minimum value for $H(X)$.

$\Rightarrow$ The PMFs are $\begin{cases} 1, & \text{if } i = k, \\ 0, & \text{otherwise} \end{cases}$; $k \in \{1, 2, \ldots, n\}$.

Problem 3 (Expectation and Moments)

(a) (i) $E[X] = 1\cdot p + 2\cdot(1-p)\cdot p + \cdots + n(1-p)^{n-1}p + \cdots = \sum_{n=1}^{\infty} n(1-p)^{n-1}\cdot p = \frac{p}{1-p}\sum_{n=1}^{\infty} n\cdot(1-p)^n.$

Let $A = \sum_{n=1}^{\infty} n(1-p)^n = (1-p) + 2(1-p)^2 + 3(1-p)^3 + \cdots$

$$\begin{array}{rl} & A = (1-p) + 2(1-p)^2 + 3(1-p)^3 + \cdots \\ -) & (1-p)A = \qquad (1-p)^2 + 2(1-p)^3 + \cdots \\ \hline & pA = (1-p) + (1-p)^2 + (1-p)^3 + \cdots \end{array}$$

$\Rightarrow pA = \frac{(1-p)}{1-1+p} \Rightarrow A = \frac{1-p}{p^2} \Rightarrow E[X] = \frac{p}{1-p} \times \frac{1-p}{p^2} = \frac{1}{p}$

(ii)

$$E[e^{tX}] = e^t\cdot p + e^{2t}\cdot(1-p)\cdot p + e^{3t}\cdot(1-p)^2\cdot p + \cdots = \sum_{n=1}^{\infty} e^{nt}\cdot(1-p)^{n-1}\cdot p = \frac{p}{1-p}\sum_{n=1}^{\infty} e^{nt}\cdot(1-p)^n$$

Let $B = \sum_{n=1}^{\infty} e^{nt}(1-p)^n = e^t\cdot(1-p) + e^{2t}\cdot(1-p)^2 + e^{3t}\cdot(1-p)^3 + \cdots$

$$\begin{array}{rl} -) & e^t(1-p)B = \qquad e^{2t}(1-p)^2 + e^{3t}\cdot(1-p)^3 + \cdots \\ \hline & (1-e^t+e^tp)B = e^t(1-p) \end{array}$$

$\Rightarrow B = \frac{e^t(1-p)}{(1-e^t+e^tp)}$

$\Rightarrow E[e^{tX}] = \frac{p}{1-p} \cdot \frac{e^t(1-p)}{1-(1-p)e^t} = \frac{pe^t}{1-(1-p)e^t}$ #

(b) Prove:

$E[X^m] = E[Y^m]$, $\forall m \in \{1, 2, \ldots, n-1\} \Leftrightarrow P(X=t) = P(Y=t)$, $\forall t \in \{a_1, \ldots, a_n\}$.

"$\Leftarrow$": It is trivial since $X = Y$.

"$\Rightarrow$": $E[X^m] = 1^m P_1 + 2^m P_2 + 3^m P_3 + \cdots n^m P_n$

We could write all possibilities into matrices multiplexing:

$$[P_1\, P_2\, P_3 \cdots P_n]\begin{bmatrix} 1 & 1 & 1^2 & 1^3 & \cdots & 1^{n-1} \\ 1 & 2 & 2^2 & 2^3 & \cdots & 2^{n-1} \\ 1 & 3 & 3^2 & 3^3 & \cdots & 3^{n-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & n & n^2 & n^3 & \cdots & n^{n-1} \end{bmatrix} = [1\ E[X^1]\ E[X^2] \cdots E[X^{n}]]$$

(notice that the right matrix is the Vandermonde matrix, it has property that $\det[V] \neq 0$ when $\alpha_1, \alpha_2, \cdots \alpha_n$ ($1, 2, 3, \ldots n$ in this question), it implies that the result of $E[X^1], E[X^2], \cdots E[X^{n-1}]$ is unique, i.e. $E[X^m] = E[Y^m]$ only when X and Y are identically distributed, so it has been proved.

By "$\Rightarrow$" and "$\Leftarrow$", $E[X^m] = E[Y^m]$, $\forall m \in \{1, 2, \ldots, n-1\} \Leftrightarrow P(X=t) = P(Y=t)$, $\forall t \in \{a_1, \ldots, a_n\}$. #

(c) $Var[Z] = E[Z^2] - E[Z]^2$

$E[Z^2] = \sum_{n=1}^{\infty} \left((-1)^n \sqrt{n}\right)^2 \cdot \frac{6}{(\pi n)^2} = \sum_{n=1}^{\infty} n \cdot \frac{6}{\pi^2 n^2} = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty$. $\Rightarrow$ $Var[Z]$ doesn't exist.

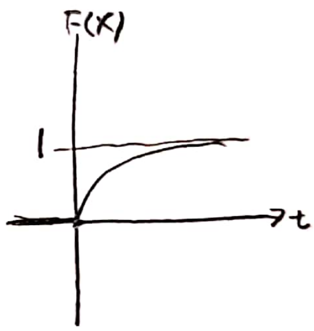
$\sum_{n=1}^{\infty} z_n^3 \cdot p_Z(z_n) = \sum_{n=1}^{\infty} (-1)^{3n} \cdot n^{\frac{3}{2}} \cdot \frac{6}{\pi^2 n^2} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{6}{\pi^2} \cdot \frac{1}{n^{\frac{1}{2}}} = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} \cdot (-1)^n \fallingdotseq \frac{6}{\pi^2} \times -0.605 \fallingdotseq -0.3678.$

$E[Z^3]$ exists, and above is its appoximated value

Problem 4 (Inverse Transform Sampling)

(a) Its CDF $F(x)$ is $\int_0^t \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^t = -e^{-\lambda t} + 1$, for $t \geq 0$

0, otherwise.



$y = 1 - e^{-\lambda t}$

$e^{-\lambda t} = 1 - y$

$-\lambda t = \ln(1-y)$

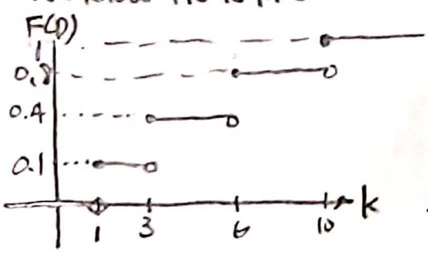
$t = -\frac{1}{\lambda} \ln(1-y)$

$F^{-1}(x) = \inf\{z : F(z) \geq z\}$

$$= \begin{cases} -\infty, & x \leq 0 \\ -\frac{1}{\lambda}\ln(1-y), & 0 < x < 1 \\ \infty & x \geq 1. \end{cases}$$

\#

(b) We know its is like



, then we could write $F^{-1}(p) = \begin{cases} -\infty, & \text{for } p \leq 0 \\ 1, & \text{for } 0 < p \leq 0.1 \\ 3, & \text{for } 0.1 < p \leq 0.4 \\ 6, & \text{for } 0.4 < p \leq 0.8 \\ 10, & \text{for } 0.8 < p \leq 1 \\ \infty, & \text{for } p > 1. \end{cases}$ \#

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