HW 4. 110550085 房天越

1. The divided difference table:

I	x_i	f_i	$f[x_i, x_{i+1}]$	$f[x_i \cdots x_{i+2}]$	$f[x_i \cdots x_{i+3}]$.
0	0.15	0.1761	2.4355	-5.7505	15.3476
1	0.21	0.3222	1.9754	-3.9088	8.7492
2	0.23	0.3617	1.7409	-2.9464	5.9642.
3	0.27	0.4314	1.4757	-2.2307	
4	0.32	0.5051	1.2973		
5	0.35	0.5441			

⇒ The best three points are the points that are the nearest to $x=0.268$, which are: $x=0.23, 0.27, 0.32$.

The quardratic through these points is:

$p(x) = -2.9833x^2 + 3.2342x - 0.2243.$

$p'(x) = -5.9666x + 3.2342.$

$p'(0.268) \doteqdot 1.6352.$ #

2. (a) Choose $i=1$.

$$S = \frac{x - x_i}{h} = \frac{0.72 - 0.5}{0.2} = 1.1$$

$$p_3'(s) = \frac{1}{h}\left[\Delta f_i + \sum_{j=2}^{n}\left\{\sum_{k=0}^{j-1}\prod_{\substack{l=0\\ l\neq k}}^{j-1}(s-l)\right\}\frac{\Delta^j f_i}{j!}\right].$$

$$= \frac{1}{0.2}\left[0.2549 + \frac{\overset{0.11}{[(s-1)+s]} \times \overset{-0.0043}{(-0.0086)}}{2!} + \frac{[\overset{-0.09}{(s-1)(s-2)} + \overset{-0.99}{s(s-2)} + \overset{0.11}{s(s-1)}] \times (-0.0018)}{3!}\right]$$

$$= \frac{1}{0.2}\left[0.2549 - 0.00516 + 0.000291\right]$$

$$\doteqdot 1.250155.$$

(b) Choose $i=4$,

$$S = \frac{1.33 - 1.1}{0.2} = 1.15,$$

$$p_2'(s) = \frac{1}{0.2}\left[0.2241 + \frac{[(s-1)+s] \times (-0.0128)}{2!}\right]$$

$$= \frac{1}{0.2} \times 0.21578$$

$$= 1.0789.$$

* The function difference table is:

i	x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
0	0.3	0.3985	0.2613	-0.0064	-0.0022	0.0003
1	0.5	0.6598	0.2549	-0.0086	-0.0018	0.0004
2	0.7	0.9147	0.2464	-0.0104	-0.0014	0.0005
3	0.9	1.1611	0.2360	-0.0118	-0.0010	
4	1.1	1.3971	0.2241	-0.0128		
5	1.3	1.6212	0.2113			
6	1.5	1.8325				

(c) Since x is one of the x_i's, we can apply the simpler formula:

$$f'(x_i)=\frac{1}{h}\left[\Delta f_i-\frac{1}{2}\Delta^2 f_i+\frac{1}{3}\Delta^3 f_i-\frac{1}{4}\Delta^4 f_i\right]$$

Chose $i=1$.

$$P_4'(x_i)=\frac{1}{0.2}\left[0.2549+0.0043-0.0006-0.0001\right]$$

$$=\frac{1}{0.2}\left[0.2585\right]$$

$$=\underline{1.2925}.\ \#$$

3. Consider the five special cases for $f''(x)$, which are $P(u)=1, P(u)=u, P(u)=u^2, P(u)=u^3, P(u)=u^4$, we would get

$$\begin{bmatrix}1&1&1&1&1\\-2h&-h&0&h&2h\\4h^2&h^2&0&h^2&4h^2\\-8h^3&-h^3&0&h^3&8h^3\\16h^4&h^4&0&h^4&16h^4\end{bmatrix}\begin{bmatrix}C_{-2}\\C_{-1}\\C_0\\C_1\\C_2\end{bmatrix}=\begin{bmatrix}0\\0\\2\\0\\0\end{bmatrix}$$

$\Rightarrow$ Solve this, we would get $(C_{-2},C_{-1},C_0,C_1,C_2)=(-\frac{1}{12},\frac{4}{3},-\frac{5}{2},\frac{4}{3},-\frac{1}{12})$.

$\Rightarrow$ The formula is $\underline{f''(x_0)=\dfrac{-f_2+16f_1-30f_0+16f_{-1}-f_{-2}}{12h^2}}$.

Also, by Taylor series of $f(x-2h)=f(x)-f'(x)\cdot 2h+\frac{f''(x)}{2!}\cdot 4h^2-\frac{f'''(x)}{3!}8h^3+\frac{f^{(4)}(x)}{4!}\cdot 16h^4-\frac{f^{(5)}(x)}{5!}\cdot 32h^5+\frac{f^{(6)}(x)}{6!}64h^6$.

$$f(x-h)=f(x)-f'(x)\cdot h+\frac{f''(x)}{2!}\cdot h^2-\frac{f'''(x)}{3!}h^3+\frac{f^{(4)}(x)}{4!}\cdot h^4-\frac{f^{(5)}(x)}{5!}\cdot h^5+\frac{f^{(6)}(x)}{6!}\cdot h^6$$

$$f(x)=f(x)$$

$$f(x+h)=f(x)+f'(x)\cdot h+\frac{f''(x)}{2!}\cdot h^2+\frac{f'''(x)}{3!}h^3+\frac{f^{(4)}(x)}{4!}h^4+\frac{f^{(5)}(x)}{5!}\cdot h^5+\frac{f^{(6)}(x)}{6!}\cdot h^6$$

$$f(x+2h)=f(x)+f'(x)2h+\frac{f''(x)}{2!}\cdot 4h^2+\frac{f'''(x)}{3!}\cdot 8h^3+\frac{f^{(4)}(x)}{4!}16h^4+\frac{f^{(5)}(x)}{5!}\cdot 32h^5+\frac{f^{(6)}(x)}{6!}\cdot 64h^6.$$

$\Rightarrow f''(x_0)=f''(x)-\underline{\frac{f^{(6)}(x)}{90}h^4}$. $\Rightarrow$ $\underline{\text{The error term is } O(h^4)}$.

Then, for $f'''(x)$, we would get

$$\begin{bmatrix}1&1&1&1&1\\-2h&-h&0&h&2h\\4h^2&h^2&0&h^2&4h^2\\-8h^3&-h^3&0&h^3&8h^3\\16h^4&h^4&0&h^4&16h^4\end{bmatrix}\begin{bmatrix}C_{-2}\\C_{-1}\\C_0\\C_1\\C_2\end{bmatrix}=\begin{bmatrix}0\\0\\0\\6\\0\end{bmatrix}.$$

$\Rightarrow$ Solve this, we would get $(C_{-2},C_{-1},C_0,C_1,C_2)=(-\frac{1}{2},1,0,-1,\frac{1}{2})$.

The formula is $f'''(x_0)=\dfrac{f_2-2f_1+2f_{-1}-f_{-2}}{2h^3}$

$\Rightarrow$ The error term can be analyzed like above. it's $\underline{O(h^2)}$. #

4. It is better to use the Simpson's 3/8 rule at where the function is most nearly linear, and apply the 1/3 rule at where it is not.
We can observe that the second difference becomes larger as x grows up. so we had better use the 3/8 rule from $x=1.0$ to $x=1.6$, and use the 1/3 rule from $x=1.6$ to $x=1.8$. thus we can get the most accurate answer. *

5. The trapezoidal rule is:

$$\int_a^b f(x)dx = \frac{h}{2}(f_0+2f_1+2f_2+\cdots+2f_{n-1}+f_n).$$

For the first iteration, $h_1=0.4$, $S_n \doteqdot 6.311111$

For the second iteration, $h_2=\frac{0.4}{2}=0.2$, $S_n \doteqdot 4.718055$, $|S_{n+1}-S_n|=1.593055$. so continue.

For the third iteration, $h_3=\frac{0.2}{2}=0.1$, $S_n \doteqdot 4.197677$, $|S_{n+1}-S_n|=0.520378$, so continue.

For the fourth iteration, $h_4=\frac{0.1}{2}=0.05$, $S_n \doteqdot 4.051042$, $|S_{n+1}-S_n|=0.146634$, so continue.

For the fifth iteration, $h_5=\frac{0.05}{2}=0.025$, $S_n \doteqdot 4.012876$, $|S_{n+1}-S_n|=0.038166$, so continue.

For the sixth iteration, $h_6=\frac{0.025}{2}=0.0125$, $S_n \doteqdot 4.003226$ $|S_{n+1}-S_n|=0.009649$, which is less than 0.02, so we stop here.

$\Rightarrow$ So, at $h=0.125$, the computation terminates. *

6. $$\int_{-0.2}^{1.4}\int_{0.4}^{2.6} e^x \sin(2y)\,dy\,dx$$

$$\Rightarrow y=\frac{2.6+0.4}{2}+\frac{2.6-0.4}{2}\eta = 1.5+1.1\eta, \quad dy=1.1d\eta$$

$$x=\frac{1.4-0.2}{2}+\frac{1.4+0.2}{2}\xi = 0.6+0.8\xi, \quad dx=0.8d\xi$$

$$\Rightarrow \int_{-1}^{1}\int_{-1}^{1} e^{(0.6+0.8\xi)}\sin(3+2.2\eta)\cdot 1.1d\eta\cdot 0.8d\xi = 0.88\int_{-1}^{1}\int_{-1}^{1} e^{(0.6+0.8\xi)}\sin(3+2.2\eta)\,d\eta\,d\xi.$$

$$\Rightarrow 0.88\sum_{j=1}^{3}\sum_{i=1}^{3} w_i w_j\left[e^{(0.6+0.8\xi_i)}\sin(3+2.2\eta_j)\right]$$

$$=0.88\sum_{j=1}^{3}\left\{w_1w_j\left[e^{(0.6+0.8\xi_1)}\sin(3+2.2\eta_j)\right]+w_2w_j\left[e^{(0.6+0.8\xi_2)}\sin(3+2.2\eta_j)\right]+w_3w_j\left[e^{(0.6+0.8\xi_3)}\sin(3+2.2\eta_j)\right]\right.$$

$$=0.88\left\{w_1w_1\left[e^{(0.6+0.8\xi_1)}\sin(3+2.2\eta_1)\right]+w_1w_2\left[e^{(0.6+0.8\xi_1)}\sin(3+2.2\eta_2)\right]+w_1w_3\left[e^{(0.6+0.8\xi_1)}\sin(3+2.2\eta_3)\right]\right.$$

$$+w_2w_1\left[e^{(0.6+0.8\xi_2)}\sin(3+2.2\eta_1)\right]+w_2w_2\left[e^{(0.6+0.8\xi_2)}\sin(3+2.2\eta_2)\right]+w_2w_3\left[e^{(0.6+0.8\xi_2)}\sin(3+2.2\eta_3)\right].$$

$$+w_3w_1\left[e^{(0.6+0.8\xi_3)}\sin(3+2.2\eta_1)\right]+w_3w_2\left[e^{(0.6+0.8\xi_3)}\sin(3+2.2\eta_2)\right]+w_3w_3\left[e^{(0.6+0.8\xi_3)}\sin(3+2.2\eta_3)\right].$$

$$= 0.88\{0.30864\times[0.98052\times 0.96245] + 0.49382[0.98052\times 0.14112] + 0.30864[0.98052\times(-0.99997)]$$

$$+ 0.49382\times[1.82212\times 0.96245] + 0.79011[1.82212\times 0.14112] + 0.49392[1.82212\times(-0.99997)]$$

$$+ 0.30864\times[3.38608\times 0.96245] + 0.49382[3.38608\times 0.14112] + 0.30864[3.38608\times(-0.99997)]\}.$$

$$= 0.88\{0.29126 + 0.06833 - 0.30262 + 0.86601 + 0.20317 - 0.89977 + 1.00584 + 0.23597 - 1.04505\}$$

$$= 0.88\times 0.42304$$

$$= \underline{0.3722752}. \#$$