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Problem 1

(a) Marginal CDF of X: $F_X(t) = F_{XY}(t,\infty) = \underline{1-e^{-t}}$,

Marginal CDF of Y: $F_Y(u) = F_{XY}(\infty,u) = \underline{1-e^{-u}}$.

For X, Y are independent, $F_{XY}(t,u) = F_X(t)\cdot F_Y(u)$

$\Rightarrow 1-e^{-t}-e^{-u}+e^{-(t+u+\theta tu)} = (1-e^{-t})(1-e^{-u})$

$\Rightarrow 1-e^{-t}-e^{-u}+e^{-(t+u+\theta tu)} = 1-e^{-t}-e^{-u}+e^{-(t+u)}$

$\Rightarrow e^{-(t+u+\theta tu)} = e^{-(t+u)}$

$\Rightarrow e^{-\theta tu} = 1 \Rightarrow \underline{\theta = 0}$.

(b) Take partial derivatives.

$$\Rightarrow \frac{\partial^2}{\partial t\partial u}F_{XY}(t,u) = \frac{\partial}{\partial t}\frac{\partial}{\partial u}\left(1-e^{-t}-e^{-u}+e^{(-(t+u+\theta tu))}\right)$$

$$= \frac{\partial}{\partial t}\left(e^{-u}+e^{-(t+u+\theta tu)}\cdot(-1-\theta t)\right)$$

$$= \frac{\partial}{\partial t}\left(e^{-u}-e^{-t-u-\theta tu}-\theta t\cdot e^{-t-u-\theta tu}\right)$$

$$= -e^{-t-u-\theta tu}\cdot(-1-\theta u)-\theta\left(e^{-t-u-\theta tu}+t\cdot e^{-t-u-\theta tu}\cdot(-1-\theta u)\right)$$

$$= e^{-t-u-\theta tu}+\theta u\cdot e^{-t-u-\theta tu}-\theta e^{-t-u-\theta tu}+\theta t\cdot e^{-t-u-\theta tu}+ut\theta^2 e^{-t-u-\theta tu}.$$

$\Rightarrow \underline{(1+\theta u+\theta t-\theta+ut\theta^2)e^{-t-u-\theta tu}}$ can be one joint PDF.

(c)

Marginal PDF of X: $f_X(t) = \frac{d}{dt}F_X(t) = e^{-t}$

Marginal PDF of Y: $f_Y(u) = \frac{d}{du}F_Y(u) = e^{-u}$

(d) First by $F_{XY}(\infty,\infty)=1$, we know that $\theta \geq 0$, if not, F_{XY} would explode.

Also, the joint PDF must be greater than 0.

$\Rightarrow (1+\theta u+\theta t-\theta+ut\theta^2) > 0$.

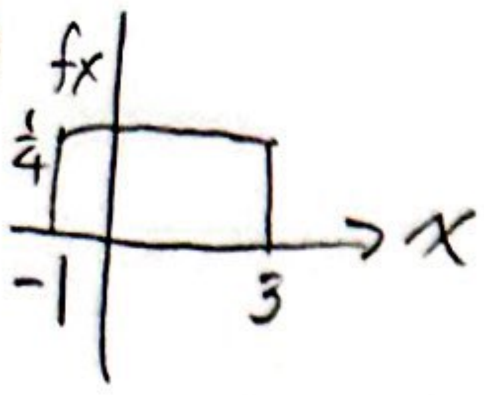
If $\theta > 1$, $1-\theta < 0$, for every $\theta > 1$, we can find $\theta u+\theta t+\theta^2 ut < \theta - 1$, since we only ask u and t to be greater th

$\Rightarrow$ So, θ also cannot be greater than 1,

$\Rightarrow$ Thus θ can only $\in [0,1]$. *

Problem 2

(a)



$\Rightarrow M_X(t)=\int_{-1}^{3}\frac{1}{4}\cdot e^{tx}dx$

$=\frac{1}{4t}e^{tx}\Big|_{-1}^{3}=\frac{e^{3t}-e^{-t}}{4t}.$

$9-1=?$

$E[X]=\frac{d}{dt}M_X(t)\Big|_{t=0}=\frac{(3e^{3t}+e^{-t})4t-(e^{3t}-e^{-t})\cdot 4}{16t^2}\Big|_{t=0}=\frac{3te^{3t}+te^{-t}-e^{3t}+e^{-t}}{4t^2}\Big|_{t=0}\overset{L.H.}{=}\frac{3e^{3t}+9te^{3t}+e^{-t}-te^{-t}-3e^{3t}-e^{-t}}{8t}\Big|_{t=0}\overset{L.H.}{=}\frac{9e^{3t}+27te^{3t}-e^{-t}+te^{-t}}{8}\Big|_{t=0}=1.$

$E[X^2]=\frac{(9te^{3t}-te^{-t})(4t^2)-(3te^{3t}+te^{-t}-e^{3t}-e^{-t})(8t)}{16t^4}\Big|_{t=0}=\frac{36t^2e^{3t}-4t^2e^{-t}-24t^2e^{3t}-8t^2e^{-t}+8te^{3t}+8te^{-t}}{16t^3}\Big|_{t=0}$

$=\frac{9te^{3t}-t^2e^{-t}-6te^{3t}-2te^{-t}+2e^{3t}+2e^{-t}}{4t^3}\Big|_{t=0}\overset{L.H.}{=}\frac{9(2te^{3t}+t^2\cdot 3e^{3t})-(2te^{-t}-t^2e^{-t})-6(e^{3t}+3te^{3t})-2(e^{-t}-te^{-t})+6e^{3t}-2e^{-t}}{12t^2}\Big|_{t=0}$

$=\frac{\cancel{18te^{3t}}+27t^2e^{3t}-\cancel{2te^{-t}}+t^2e^{-t}-\cancel{6e^{3t}}-\cancel{18te^{3t}}-2e^{-t}+\cancel{2te^{-t}}+\cancel{6e^{3t}}-2e^{-t}}{12t^2}\Big|_{t=0}=\frac{27t^2e^{3t}+t^2e^{-t}-4e^{-t}}{12t^2}\Big|_{t=0}\overset{L.H.}{=}\frac{27(2te^{3t}+t^2\cdot 3e^{3t})+2te^{-t}-t^2e^{-t}+4e^{-t}}{24t}$

$=\frac{54te^{3t}+81t^2e^{3t}+2te^{-t}-t^2e^{-t}+4e^{-t}}{24t}\Big|_{t=0}=\frac{54(e^{3t}+3te^{3t})+81(2te^{3t}+t^2\cdot 3e^{3t})+2(e^{-t}-te^{-t})-(2te^{-t}-t^2e^{-t})-4e^{-t}}{24}\Big|_{t=0}$

$=\frac{54+2-4}{24}=\frac{52}{24}=\frac{13}{6}.$

$Var[X]=\left(\frac{13}{6}\right)^2-1^2=\frac{133}{36}.$

$\Rightarrow E[X]=1,\ Var[X]=\frac{133}{36}.$

(b) $M_Y(t)$ is $\sum_{k=0}^{\infty}\frac{6}{\pi^2k^2}\cdot e^{tk}=\frac{6}{\pi^2}\sum_{k=0}^{\infty}\frac{e^{tk}}{k^2}=\frac{6}{\pi^2}\cdot e^t\cdot\sum_{k=0}^{\infty}\frac{e^k}{k^2}$

Since $e^k\gg k^2$ when $k\to\infty$, $\sum_{k=0}^{\infty}\frac{e^k}{k^2}$ would definitely be ∞, so no matter what value t is, $M_Y(t)$ would be ∞.

According to the definition, $M_Y(t)$ doesn't exist. #

Problem 3.

(a) $M_{X_1}(t)=e^{7(e^t-1)}=e^{7e^t}\cdot e^{-7}=e^{-7}\times\sum_{x_1=0}^{\infty}\frac{(7e^t)^{x_1}}{x_1!}=\sum_{x_1=0}^{\infty}e^{tx_1}\cdot\frac{e^{-7}\cdot 7^{x_1}}{x_1!}$

$\Rightarrow$ Then we know the PMF of X_1 is $\begin{cases}\frac{e^{-7}\cdot 7^{x_1}}{x_1!}, & \text{where } x_1\in\mathbb{N}\\ 0, & \text{where } x_1\notin\mathbb{N}\end{cases}$, and that X_1 is a Poisson random variable.

$M_{X_2}(t)$ is a uniform random variable by observing (a) in Problem 2, and it has upper bound 2 and lower bound 1.

$\Rightarrow$ PDF of X_2 is $\begin{cases}1, & 1<X_2<2\\ 0, & \text{otherwise}\end{cases}$

(b) // i.i.d means identically independent distributed.

$M_{X_1}(t)=e^{(e^t-1)},\ M_{X_2}(t)=\sum_{x_2=0}^{\infty}e^{t\cdot 2\cdot x_2}\cdot\frac{e^{-1}\cdot 1}{x_2!}=e^{-1}\sum_{x_2=0}^{\infty}\frac{e^{2tx_2}}{x_2!}=e^{-1}\sum_{x_2=0}^{\infty}\frac{(e^{2t})^{x_2}}{x_2!}=e^{-1}\cdot e^{e^{2t}}=e^{(e^{2t}-1)}$

$M_{X+2Y}(t)=E[e^{tX}]\times E[e^{t\cdot 2Y}]=M_X(t)\cdot M_{2Y}(t)=e^{(e^{2t}+e^t-2)}$, which is not the form of MGF of Poisson r.v.s.

$\Rightarrow X+2Y$ is not a poisson random variable. #

Problem 4

(a) $\begin{cases} x_1 = \sigma_1 Z + \mu_1 \\ X_2 = \sigma_2(\rho Z + \sqrt{1-\rho^2}\,W) + \mu_2 \end{cases}$ $\qquad y_1 = \sigma Z + \mu_{0}$, $\quad y_2 = d$

$(x_1 + y_1) = \sigma$

$$f_{ZW}(Z,W) = f_Z(Z)\cdot f_W(W)$$
$$= \frac{1}{\sqrt{2\pi\sigma_1^2}}\exp\left(-\frac{1}{2}\frac{(x_1-\mu_1)^2}{\sigma_1^2}\right)\cdot\frac{1}{\sqrt{2\pi\sigma_2^2}}\exp\left(-\frac{1}{2}\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)$$
$$= \frac{1}{2\pi\sigma_1\sigma_2}\exp\left(-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\right)$$

By Linear Transformation of 2 Random Variables:

$$f_{V_1V_2}(v_1,v_2) = \frac{1}{|\det(A)|}f_{U_1U_2}(A^{-1}[v_1\ v_2]^T)$$

$$\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ \sigma_2\rho & \sigma_2\sqrt{1-\rho^2} \end{bmatrix}}_{A}\begin{bmatrix} z \\ w \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\begin{bmatrix} \sigma_2\sqrt{1-\rho^2} & 0 \\ -\sigma_2\rho & \sigma_1 \end{bmatrix} \Rightarrow A^{-1}\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix} = \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\begin{bmatrix} \sigma_2\sqrt{1-\rho^2}(x_1-\mu_1) \\ -\sigma_2\rho(x_1-\mu_1)+\sigma_1(x_2-\mu_2) \end{bmatrix} = \begin{bmatrix} \frac{x_1-\mu_1}{\sigma_1} \\ \frac{-\sigma_2\rho(x_1-\mu_1)+\sigma_1(x_2-\mu_2)}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \end{bmatrix}$$

$$f_{X_1X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2}\cdot\frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\cdot\exp\left(-\frac{1}{2}\left(\frac{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}{\sigma_1^2}+\frac{\left(\frac{-\sigma_2\rho(x_1-\mu_1)+\sigma_1(x_2-\mu_2)}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right)^2}{\sigma_2^2}\right)\right)$$
$$= \frac{1}{2\pi\sigma_1^2\sigma_2^2\sqrt{1-\rho^2}}\cdot\exp\left(-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^4}+\frac{\sigma_1^2(x_2-\mu_2)^2-2\sigma_1\sigma_2\rho(x_1-\mu_1)(x_2-\mu_2)+\sigma_2^2\rho^2(x_1-\mu_1)^2}{\sigma_1^2\sigma_2^4(1-\rho^2)}\right)\right)$$
$$= \frac{1}{2\pi\sigma_1^2\sigma_2^2\sqrt{1-\rho^2}}\cdot\exp\left(-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^4}+\frac{(x_2-\mu_2)^2}{\sigma_2^4(1-\rho)}-\frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2^3(1-\rho^2)}+\frac{\rho^2(x_1-\mu_1)^2}{\sigma_1^2(1-\rho^2)\sigma_2^2}\right)\right)$$

Since Z and W are standard normal, $\sigma_1^2=\sigma_2^2=1$, we can replace them with 1 if needed.

$$\Rightarrow f_{X_1X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\cdot\exp\left(-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{\rho^2(x_1-\mu_1)^2}{\sigma_1^2(1-\rho^2)}-\frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2(1-\rho^2)}+\frac{(x_2-\mu_2)^2}{\sigma_2^2(1-\rho^2)}\right)\right)$$
$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\cdot\exp\left(-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2(1-\rho^2)}-\frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2(1-\rho^2)}+\frac{(x_2-\mu_2)^2}{\sigma_2^2(1-\rho^2)}\right)\right)$$
$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{\frac{(x_1-\mu_1)^2}{\sigma_1^2}-2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}}{2(1-\rho^2)}\right]. \#$$

Problem 4

(b) I would like to do (ii) first.

(ii) $\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $A^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

$$f_{Z_1Z_2}(z_1,z_2) = \frac{1}{2} f_{XY}\left(\frac{1}{2}z_1+\frac{1}{2}z_2, \frac{1}{2}z_1-\frac{1}{2}z_2\right)$$

$$= \frac{1}{2}\cdot\frac{1}{2\pi\cdot1\cdot1\sqrt{1-\rho^2}}\cdot\exp\left[-\frac{(\frac{1}{2}z_1+\frac{1}{2}z_2)^2-2\rho\cdot(\frac{1}{2}z_1+\frac{1}{2}z_2)(\frac{1}{2}z_1-\frac{1}{2}z_2)+(\frac{1}{2}z_1-\frac{1}{2}z_2)^2}{2(1-\rho^2)}\right]$$

$$= \frac{1}{4\pi\sqrt{1-\rho^2}}\cdot\exp\left[-\frac{\frac{1}{4}z_1^2+\frac{1}{8}z_1z_2+\frac{1}{4}z_2^2-2\rho(\frac{1}{4}z_1^2-\frac{1}{4}z_2^2)+\frac{1}{4}z_1^2-\frac{1}{8}z_1z_2+\frac{1}{4}z_2^2}{2(1-\rho^2)}\right]$$

$$= \frac{1}{4\pi\sqrt{1-\rho^2}}\cdot\exp\left[-\frac{\frac{1}{2}z_1^2-2\rho(\frac{1}{4}z_1^2-\frac{1}{4}z_2^2)+\frac{1}{2}z_2^2}{2(1-\rho^2)}\right]$$

$$= \frac{1}{2\pi\sqrt{2}\cdot\sqrt{2}\cdot\sqrt{1-\rho^2}}\cdot\exp\left[-\frac{\frac{(\frac{1}{\sqrt{2}}z_1+\frac{1}{\sqrt{2}}z_2)^2}{(\sqrt{2})^2}-2\rho(\frac{1}{\sqrt{2}}z_1+\frac{1}{\sqrt{2}}z_2)(\frac{1}{\sqrt{2}}z_1-\frac{1}{\sqrt{2}}z_2)+\frac{(\frac{1}{\sqrt{2}}z_1-\frac{1}{\sqrt{2}}z_2)^2}{(\sqrt{2})^2}}{2(1-\rho^2)}\right].$$

(i) Seeing the result of (ii), we know that $f_{Z_1Z_2}$ is bivariate normal, because it has form of bivariate normal, and it is constructed by $\frac{1}{\sqrt{2}}z_1+\frac{1}{\sqrt{2}}z_2$ and $\frac{1}{\sqrt{2}}z_1-\frac{1}{\sqrt{2}}z_2$.

So two linear combinations of Z_1 and Z_2 are bivariate normal.

That means that Z_1 and Z_2 are also bivariate normals. #

Problem 5

(a) No, because he uses 3.5 to count the expected value, while we can't do this because we have certain stopping conditions. so the rolling times would be affected by the number we get in one roll. Which makes it not 3.5 for both dices, So the argument must be incorrect.

(b) No, because it is not the same for not only the last roll but also the previous rolls. The values for previous rolls for the green die can only be 2~6, while the orange die can only be 1~5. So it's not the same, we cannot say $E[T_1]<E[T_6]$ by simply taking the last roll's result.

(c) $E[T_1] = \frac{5}{6}\times4+\frac{1}{6}\times1+\frac{5}{6}\times\frac{5}{6}\times4+\frac{5}{6}\times\frac{1}{6}\times1+\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times4+\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}\times1+\cdots$

$$= \frac{(\frac{5}{6}\times4)\times(1-(\frac{5}{6})^{\infty})}{1-\frac{5}{6}} + \frac{(\frac{1}{6}\times1)\times(1-(\frac{5}{6})^{\infty})}{1-\frac{5}{6}}$$

$= \frac{20}{6}\times6+\frac{1}{6}\times6 = 21.$ (The value 4 is because $\frac{2+3+4+5+6}{5}=4$, which is the condition for T_1 can continue to be rolle

$E[T_6] = \frac{5}{6}\times3+\frac{1}{6}\times6+\frac{5}{6}\times\frac{5}{6}\times3+\frac{5}{6}\times\frac{1}{6}\times6+\cdots$

$$= \frac{(\frac{5}{6}\times3)\times(1-(\frac{5}{6})^{\infty})}{1-\frac{5}{6}} + \frac{(\frac{1}{6}\times6)\times(1-(\frac{5}{6})^{\infty})}{1-\frac{5}{6}}$$

$= 15+6=21.$

Thus, we have $E[T_1]=E[T_6]$. #