Probability HW1.

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Problem 1

(a). Let $S=\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}S_n$, $T=\{x \mid x\in S_n$, for infinitely many $n\}$,

• $S\subseteq T$

Assume that x is an element in S, if $S\not\subseteq T$, it means that x is in only finitely many S_n, then there must be a k such that $\forall N>k$, $x\notin S_N$, so x is not in S, which yields a contradiction. So $S\subseteq T$.

• $T\subseteq S$

Let x be an element in T, assume that $T\not\subseteq S$, then there must be a k such that $\forall N>k$, $x\notin S_N$, but x must be in every S_n, which yields a contradiction, so $T\subseteq S$.

Since $S\subseteq T$ and $T\subseteq S$, $T=S$. #

(b) Let $S=\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}(A_n\cup B_n)$, $T=(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}A_n)\cup(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}B_n)$

• $S\subseteq T$

Assume that x is an element in S, then by (a), we know that x is in all $A_n\cup B_n$, where n is an integer. If $x\in S$, then $\forall k, \exists N>k$ such that $x\in\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}A_n$, or $x\in\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}B_n$. Then $\forall k, \exists N>k$ such that x is in A_n or x is in B_n. So $x\in T$, $S\subseteq T$.

• $T\subseteq S$

Assume that x is an element in T, then by (a), It must be in either all A_n or in all B_n, so for every $A_n\cup B_n$, x must be in there. i.e.: $x\in S$, so $T\subseteq S$.

Since $S\subseteq T$ and $T\subseteq S$, $S=T$. #

Problem 2.

(a) 1. Let $A_1=A$, $A_2=A^c$, $A_3=\phi$, $A_4=\phi$, ..., $A_n=\phi$, ...

Then $\bigcup_{i=1}^{\infty}A_i=\Omega$, $P(\bigcup_{i=1}^{\infty}A_i)=P(\Omega)=\sum_{i=1}^{\infty}P(A_i)=P(A)+P(A^c)$, by countable additivity.

$\Rightarrow P(\Omega)=P(A)+P(A^c)$; $1=P(A)+P(A^c)$, by normalization.

$\Rightarrow P(A)=1-P(A^c)$. #

2. Let $A_1=A-B$, $A_2=A\cap B$, $A_3=\phi$, $A_4=\phi$, ..., $A_n=\phi$, we know that $A-B$ and $A\cap B$ are mutually exclusive.

Then $\bigcup_{i=1}^{\infty}A_i=A$, $P(\bigcup_{i=1}^{\infty}A_i)=P(A)=P(\sum_{i=1}^{\infty}A_i)=P(A-B)+P(A\cap B)$ ...

$\Rightarrow P(A)=P(A-B)+P(A\cap B)$

3. By 2, we know that $P(A)=P(A-B)+P(A\cap B)$, also $P(B)=P(B-A)+P(A\cap B)$

$\Rightarrow P(A)+P(B)-P(A\cap B)=P(A-B)+P(A\cap B)+P(B-A)+P(A\cap B)-P(A\cap B)=P(A-B)+P(B-A)+P(A\cap B)$

Since $P(A-B)$, $P(B-A)$, $P(A\cap B)$ are all mutually exclusive, also $(A-B)+(B-A)+(A\cap B)=A\cup B$, we can use countable additivity to show that

$P(A\cup B)=P(A-B)+P(B-A)+P(A\cap B)$,

then $P(A\cup B)=P(A)+P(B)-P(A\cap B)$. #

(b) Since $P(\Omega)=P(\{1,2,3,4,5\})=1$, $P(\{1,2\})=P(\Omega)-P(\{3,4,5\})$, so $P(\{3,4,5\})=0.8$, $P(\{4\})=0.25$,
Also $P(\{3,4,5\})=P(\{3,4\})+P(\{5\})$, $P(\{5\})=0.35$. then $P(\{4,5\})=P(\{4\})+P(\{5\})$, $P(\{2,3\})=0.35$, $P(\{2\})=0.15$, $P(\{1,2\})=P(\{1\})+P(\{2\})$, $P(\{1\})=0.05$
$P(\{3,4,5\})=P(\{3\})+P(\{4,5\})$, $P(\{3\})=0.2$, $P(\{2,3\})=0.35$, $P(\{2\})=0.15$, $P(\{4\})=0.25$, $P(\{5\})=0.35$.

$\Rightarrow P(\{1\})=0.05$, $P(\{2\})=0.15$, $P(\{3\})=0.2$, $P(\{4\})=0.25$, $P(\{5\})=0.35$.

$\Rightarrow$ It is valid since $P(\Omega)=P(\{1,2,3,4,5\})=P(\{1\})+P(\{2\})+P(\{3\})+P(\{4\})+P(\{5\})=1$, also all the given equations are able to be solved by the distribution.

Also, we could know the distribution is unique, since there is only one solution to the linear equations, which are

$$\begin{cases} x_1+x_2+x_3+x_4+x_5=1 \\ x_1+x_2=0.2 \\ x_2+x_3=0.35 \\ x_3+x_4=0.45 \\ x_4+x_5=0.6 \end{cases}$$

, the solution can only be $x_1=0.05$, $x_2=0.15$, $x_3=0.2$, $x_4=0.25$, $x_5=0.35$. #

Problem 3.

(a) Let $B_k = \bigcup_{n=k}^{\infty} A_n$, $\{B_k\}$ is a decreasing sequence of event, then we have

$$0 \le P\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n\right) = P\left(\bigcap_{k=1}^{\infty} B_k\right) = P\left(\lim_{k\to\infty} B_k\right) \quad // \ B_k \text{ is a decreasing sequence}$$

$$= \lim_{k\to\infty} P(B_k) \quad // \text{ By continuity of probability.} \qquad \text{By union bound.}$$

$$= \lim_{k\to\infty} P\left(\bigcup_{n=k}^{\infty} A_n\right) \le \lim_{k\to\infty} \sum_{n=k}^{\infty} P(A_n) = 0. \quad // \text{ Since } \sum_{n=1}^{\infty} P(A_n) < \infty. \ *$$

For the downside one,

(b) It is sufficient to show that $1 - P(\overline{\lim_{n\to\infty}} A_n) = 0$

$$\Rightarrow 1 - P(\overline{\lim_{n\to\infty}} A_n) = 1 - P\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n\right) = P\left(\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\right)^c\right) = P\left(\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} A_n^c\right) = P\left(\underline{\lim_{n\to\infty}} A_n^c\right) = \lim_{k\to\infty} P\left(\bigcap_{n=k}^{\infty} A_n^c\right),$$

Then we want to show $P\left(\bigcap_{n=k}^{\infty} A_n^c\right) = 0$,

Since $\langle A_n \rangle_{n=1}^{\infty}$ are independent, $P\left(\bigcap_{n=k}^{\infty} A_n^c\right) = \prod_{n=k}^{\infty} P(A_n^c) = \prod_{n=k}^{\infty}(1 - P(A_n)) \le \prod_{n=k}^{\infty} e^{(-P(A_n))} = e^{-\sum_{n=k}^{\infty} P(A_n)} = 0.$ *

By (a) and this proof, Borel Zero-One Law is correct.

(c) $P(I) = P\left(\sum_{k=1}^{\infty} \frac{1}{100} k^{-N}\right)$, we want to show that when $N \le 1$, $P(I) = 1$,

i.e. we want to prove that $\sum_{k=1}^{\infty} \frac{1}{100} k^{-N} = \infty$, i.e. $\sum_{k=1}^{\infty} \frac{1}{100k^N}$,

Since we know that $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$, then $\sum_{k=1}^{\infty} \frac{1}{100k} = \frac{1}{100}\sum_{k=1}^{\infty} \frac{1}{k} = \infty$.

also for $N<1$, $\frac{1}{100k^N} > \frac{1}{100k}$, by comparison test, $\frac{1}{100}\sum_{k=1}^{\infty} k^{-N} = \infty$. So $P(I) = 1$ if $N \le 1$. #

If we want $P(I) = 0$, it means that $\sum_{k=1}^{\infty} \frac{1}{100} k^{-N} < \infty$, i.e. $\sum_{k=1}^{\infty} \frac{1}{100} k^{-N}$ converges, then we can know that it converges when $N>1$

(d). By Borel-Cantelli Lemma, if $\sum_{k=1}^{\infty} \frac{1}{100} k^{-N} < \infty$, then $P(I) = 0$,

also it is shown in (c) that if $N>1$, $\sum_{k=1}^{\infty} \frac{1}{100} k^{-N} < \infty$, it doesn't matter whether the events are independent or not.

- If the probability would be affected by the first toss, then the $P(I)$ might be between 0 and 1.

Let $P_{2\sim k}(\text{Yes}) = \frac{1}{100} k^{-2}$ if first toss is "No" and 2 to kth tosses are independent,

also, $P_{2\sim k}(\text{Yes}) = \frac{1}{100} k^{-\frac{1}{2}}$ if first toss is Yes, and 2 to kth tosses are independent,

Then $P(I)$ is determined by the first toss, it should be between 0 and 1.

Problem 4.

(a) $P(B) = \frac{1}{3} \times 0.3 + \frac{1}{3} \times 0.6 + \frac{1}{3} \times 0.3 = 0.4$.

$$P(A_1|B) = \frac{P(A_1)\cdot P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)} = \frac{\frac{1}{3}\times 0.3}{\frac{1}{3}\times 0.3 + \frac{1}{3}\times 0.6 + \frac{1}{3}\times 0.3} = \frac{1}{4}.$$

$P(A_2|B) = \frac{1}{2}$, $P(A_3|B) = \frac{1}{4}$.

(b)
$$P(A_1|C) = \frac{P(A_1)\cdot P(C|A_1)}{P(A_1)\cdot P(C|A_1) + P(A_2)\cdot P(C|A_2) + P(A_3)P(C|A_3)} = \frac{\frac{1}{3}\times(0.1)^3\times(0.3)^6\times(0.6)^1}{\frac{1}{3}\times(0.1)^3\times(0.3)^6\times(0.6)^1 + \frac{1}{3}\times(0.3)^3\times(0.6)^6\times(0.1)^1 + \frac{1}{3}\times(0.6)^3\times(0.3)^6\times(0.1)^1} = \frac{4.374\times10^{-7}}{4.374\times10^{-7} + 1259\times10^{-7} + 157\times10^{-7}}$$

$\fallingdotseq 0.307\%$

$P(A_2|C) \fallingdotseq 88.63\%$, $P(A_3|C) \fallingdotseq 11.05\%$

- $\Rightarrow$ The most probable value is $(0.6, 0.3, 0.1)$.

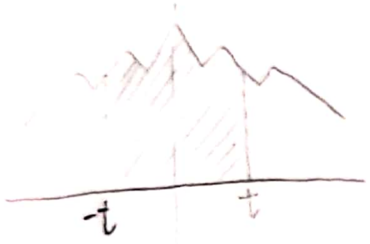
(c) It requires that $P(A_2|C) > P(A_1|C)$ && $P(A_2|C) > P(A_3|C)$, also α should be in $[0, \frac{2}{3}]$.

$$\frac{\alpha \times 3^3 \times 6^6}{\frac{1}{3}\times 3^6 + \alpha\times 3^3\times 6^6 + (\frac{2}{3}-\alpha)\times 6^3 \times 3^6} > \frac{\frac{1}{3}\times 3^6}{\frac{1}{3}\times 3^6 + \alpha\times 3^3\times 6^6 + (\frac{2}{3}-\alpha)\times 6^3\times 3^6} \Rightarrow \alpha > 3^5 \times \frac{1}{3^3\times 6^6} = \frac{9}{6^6}$$

and $\alpha \times 3^3 \times 6^6 > (\frac{2}{3} - \alpha) \times 6^3 \times 3^6 \Rightarrow 6^3 \times \alpha > 18 - 27\alpha \Rightarrow 243\alpha > 18$, $\alpha > \frac{18}{243} = \frac{2}{27}$

Conclusively, if $\alpha \in (\frac{2}{27}, \frac{2}{3})$, then $(0.3, 0.6, 0.1)$ would be the most probable value.

Problem 5



- $P(|X| \le t) = F_X(t) - P(X < -t)$

$P(X < -t) = 1 - F_X(t)$, since it is symmetric about 0.

$\Rightarrow P(|X| \le t) = F_X(t) - P(X < -t) = F_X(t) - [1 - F_X(t)] = 2F_X(t) - 1$.

- $P(X = t) = F_X(t) - F_X(t^-)$

Then, $F_X(t^-) = 1 - F_X(-t)$, since it is symmetric, can be observed by the graph.

So $P(X = t) = F_X(t) - (1 - F_X(-t)) = F_X(t) + F_X(-t) - 1$. #