1. (a) The polynomial gives:

$$P_n(x_s) \approx f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0, \text{ here } s = \frac{0.231-0.12}{0.12} = 0.925$$

$$P_n(0.231) \approx 0.79168 + 0.925 \cdot (-0.01834) + \frac{0.925(-0.075)}{2} \cdot (-0.01129)$$

$$= 0.79168 - 0.0169645 + 0.00039162187$$

$$\doteq \underline{0.7751072187}.$$

(b)
$$0.7751072187 + \frac{0.925(-0.075)(-1.075)}{3!} \cdot (0.00134)$$

$$\doteq 0.7751072187 + 0.00001665578$$

$$= \underline{0.7751238765}.$$

(c) Error ≈ Next Term

For (a), it equals to $\underline{1.665578 \times 10^{-5}}$

For (b), it is $\frac{0.925(-0.075)(-1.075)(-2.075)}{4!} \cdot (0.000381) \approx \underline{2.6 \times 10^{-10}}$.

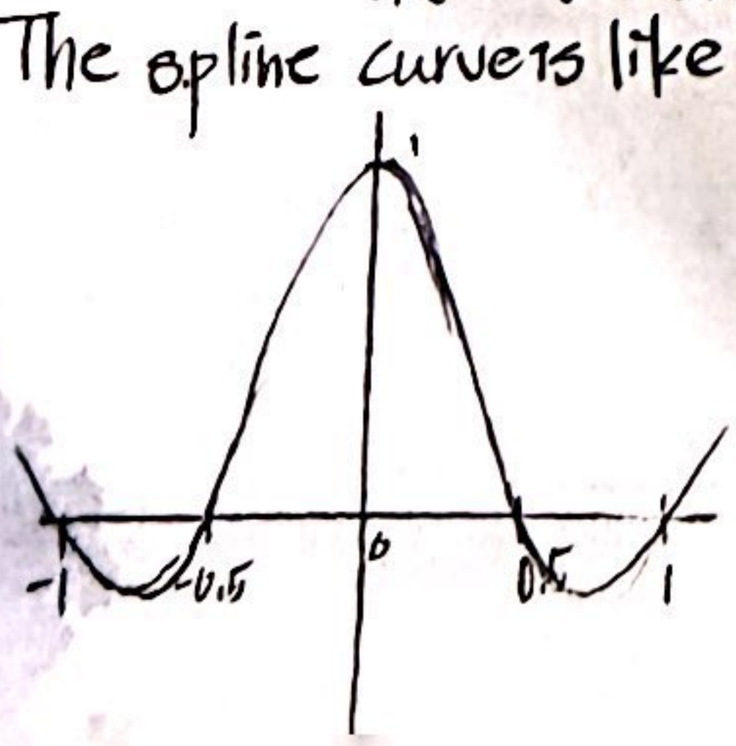
(d) Using $x_0 = 0.36$ is better, because the interpolating points with $x_0 = 0.36$ are 0.24, 0.36, 0.48, and 0.42 lies in the smallest interval containing 0.24, 0.36, and 0.48, which means using the row with $x_0 = 0.36$ is better centered.

A: (a) 0.7751072187
(b) 0.7751238765
(c) For (a): 1.665578×10^{-5}
For (b): 2.6×10^{-10}
(d) $x_0 = 0.36$.

2. Solve this problem with the MATLAB program in the file (Problem3_2.m), we get the functions:

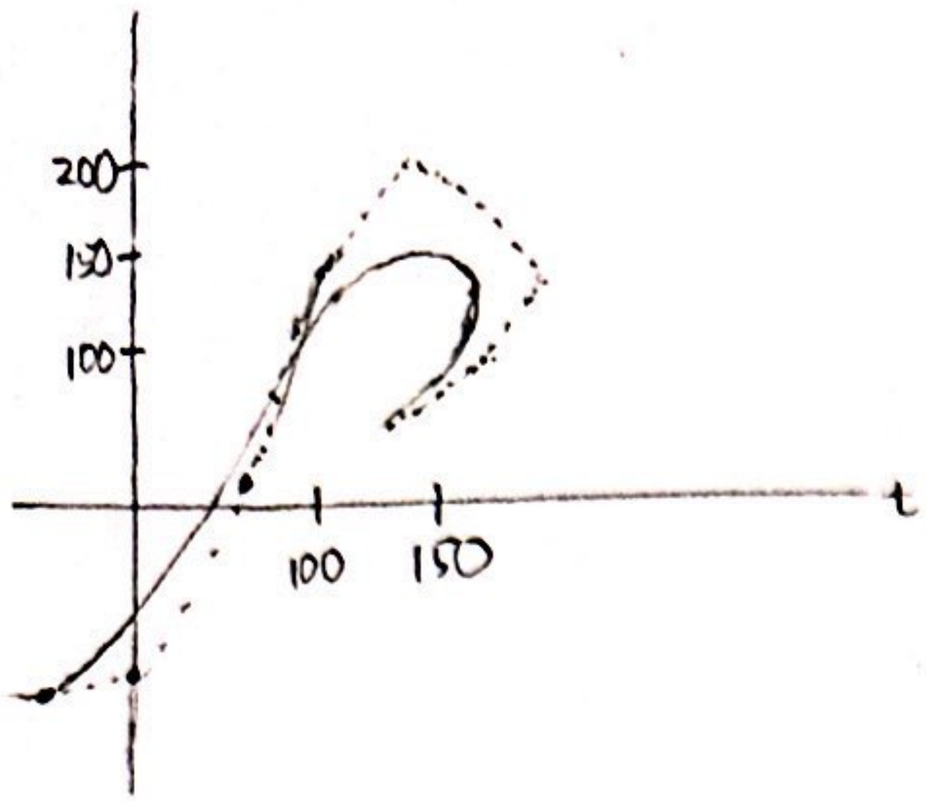
$$\begin{cases} 3.4286(x+1)^3 - 0.8571(x+1), & \text{for } -1 \le x \le -0.5 \\ -9.1429(x+0.5)^3 + 5.1429(x+0.5)^2 + 1.7143(x+0.5), & \text{for } -0.5 \le x \le 0 \\ 9.1429x^3 - 8.5714x^2 + 1, & \text{for } 0 \le x \le 0.5 \\ -3.4286(x-0.5)^3 + 5.1429(x-0.5)^2 - 1.7143(x-0.5), & \text{for } 0.5 \le x \le 1. \end{cases}$$

The spline curves like:



3. (a)

4. The graph is like:

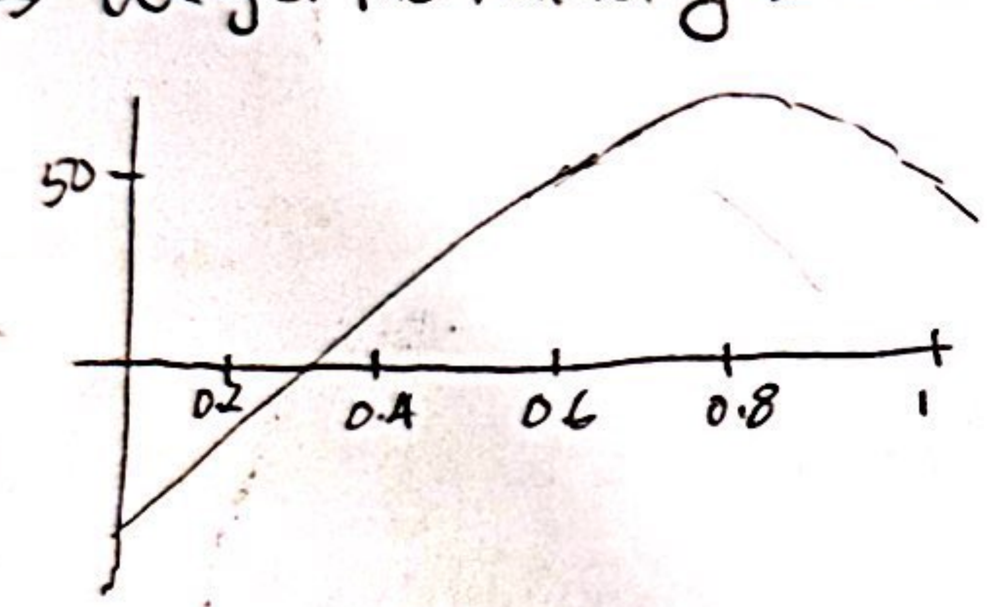


(b) The two points lie on straight lines determined by two other points, so the derivative at these points are constant, which means that these points are "Smoothly connected" because differentiability implies continuity

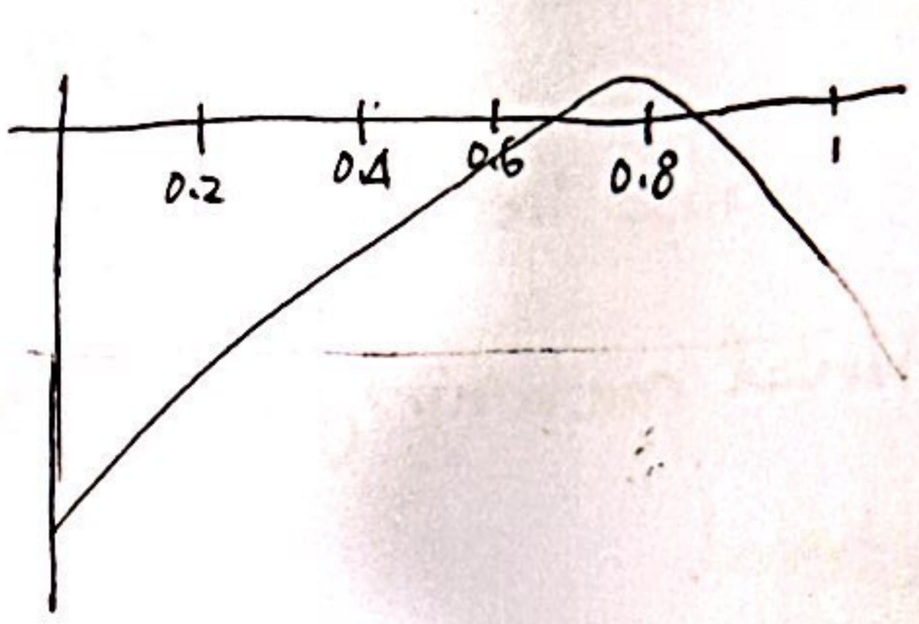
(c) For the graph above, the Bezier curve near control point 3 passes through $x \doteq 90$, and the control point 6 passes through $x \doteq 160$, make plot of:

$$\begin{cases} f(u) = \sum_{i=0}^{9} C_i^9 (1-u)^{9-i} u^i p_i = 90 \\ f(u) = \sum_{i=0}^{9} C_i^9 (1-u)^{9-i} u^i p_i = 160. \end{cases}$$

⇒ We get the following 2 curves:



and



Find the root for both, with starting values of 0.3 and 0.6, respectively.

We get the roots are $u \to 0.319966$ and $u \to 0.659739$, respectively.

⇒ Then mapping: $[0,1] \to [0, 0.319966]$,

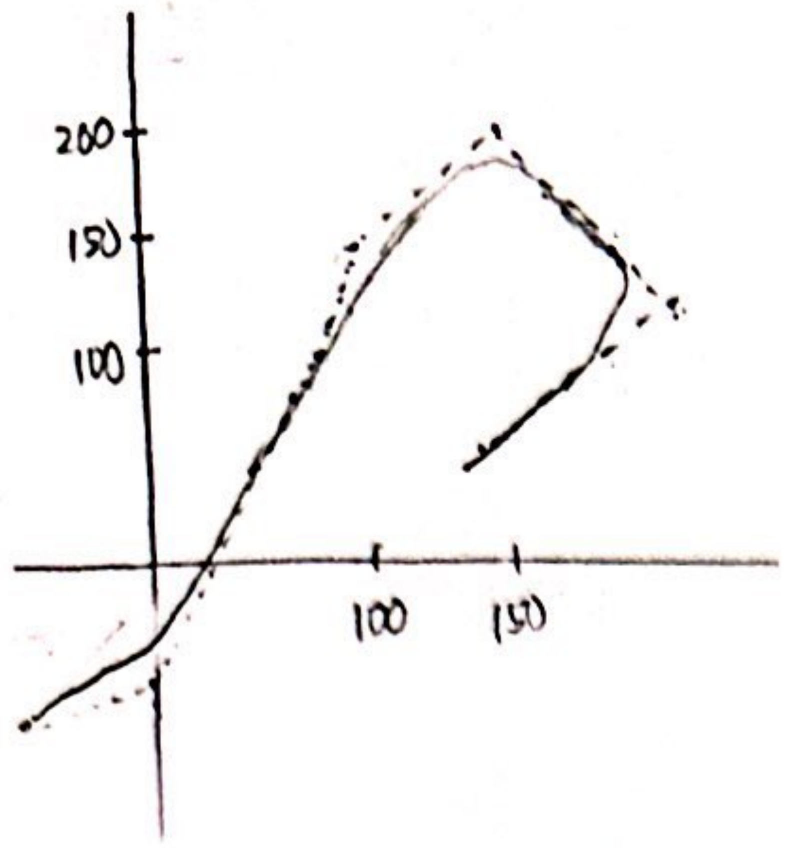
$[1,2] \to [0.319966, 0.659739]$,

$[2,3] \to [0.659739, 1]$.

⇒ The piecewise substitutions of u into the Bezier function is:

$$u = \begin{cases} 0.319966\,\tilde{u}, & 0 \le \tilde{u} \le 1. \\ 0.339773\,\tilde{u} - 0.19807, & 1 \le \tilde{u} \le 2. \\ 0.340261\,\tilde{u} - 0.20783, & 2 \le \tilde{u} \le 3. \end{cases}$$

4. (a) The graph is like



(b) The two points lie on straight lines determined by two other points, so the derivative at these points are constant, and differentiabrlity implies continuity.

(c) We cannot do this on B-splines, because they are piecewise continuous constructions where each individual piece is dependent on the parameter $u \in [0,1]$,

5. (a) Let $A = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 1 & 1.2 & 2.1 \\ 1 & 3.4 & 4 \\ 1 & 4.1 & 4.9 \\ 1 & 5.7 & 6.3 \\ 1 & 7.2 & 8.1 \\ 1 & 9.3 & 8.9 \end{bmatrix}$, solve this with $A^TAa = A^Tz$, then $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0.4 & 1.2 & 3.4 & 4.1 & 5.7 & 7.2 & 9.3 \\ 0.7 & 2.1 & 4 & 4.9 & 6.3 & 8.1 & 8.9 \end{bmatrix} \begin{bmatrix} 1 & 0.4 & 0.7 \\ 1 & 1.2 & 2.1 \\ 1 & 3.4 & 4 \\ 1 & 4.1 & 4.9 \\ 1 & 5.7 & 6.3 \\ 1 & 7.2 & 8.1 \\ 1 & 9.3 & 8.9 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 1 & 1.2 & 2.1 \\ 1 & 3.4 & 4 \\ 1 & 4.1 & 4.9 \\ 1 & 5.7 & 6.3 \\ 1 & 7.2 & 8.1 \\ 1 & 9.3 & 8.9 \end{bmatrix}^T \begin{bmatrix} 0.031 \\ 0.933 \\ 3.058 \\ 3.349 \\ 4.87 \\ 5.757 \\ 8.921 \end{bmatrix}$

(b), Solve this, we canget

$\alpha_1 \doteqdot 0.2206660284$

$\alpha_2 \doteqdot 1.5960921815$

$\alpha_3 \doteqdot -0.7023813566.$

$\Rightarrow \underline{z = 0.2206660284 + 1.5960921815x - 0.7023813566y}.$

(c) The sum of squares of the deviations of the points from the plane is

$$\frac{\sum \|z - A\alpha\|^2}{7-3} = \frac{0.3193951298}{4} \doteqdot \underline{0.07984878245}. \#$$

6.

$$\cos^2(x) \approx 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \frac{1}{315}x^8$$

$$\frac{(1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \frac{1}{315}x^8)(1 + b_1x + b_2x^2 + b_3x^3) - (a_0 + a_1x + a_2x^2 + a_3x^3)}{(1 + b_1x + b_2x^2 + b_3x^3)}$$

$$= \frac{1 + b_1x + (b_2 - 1)x^2 + (b_3 - b_1)x^3 + (\frac{1}{3} - b_2)x^4 + (\frac{1}{3}b_1 - b_3)x^5 + (\frac{1}{3}b_2 - \frac{2}{45})x^6 + \cdots - a_0 - a_1x - a_2x^2 - a_3x^3}{1 + b_1x + b_2x^2 + b_3x^3}$$

$\Rightarrow a_0 = 1,\ b_1 - a_1 = 0,\ b_2 - a_2 - 1 = 0,\ b_3 - b_1 - a_3 = 0,\ \frac{1}{3} - b_2 = 0,\ \frac{1}{3}b_1 - b_3 = 0,$

$\Rightarrow a_0 = 1,\ b_1 = b_3 = 0,\ a_1 = 0,\ b_2 = \frac{1}{3},\ a_2 = -\frac{2}{3},\ a_3 = 0$

$$\Rightarrow \cos^2(x) \approx \frac{1 - \frac{2}{3}x^2}{1 + \frac{2}{3}x^2}.$$

$$\sin(x^4-x) \approx -x+\frac{1}{6}x^3+x^4-\frac{1}{120}x^5-\frac{1}{2}x^6.$$

$$\frac{\left(-x+\frac{1}{6}x^3+x^4-\frac{1}{120}x^5-\frac{1}{2}x^6\right)(1+b_1x+b_2x^2+b_3x^3)-(a_0+a_1x+a_2x^2+a_3x^3)}{1+b_1x+b_2x^2+b_3x^3}$$

$\Rightarrow a_0=0,$ $\quad -b_3+\frac{1}{6}b_1+1=0$

$a_1=-1,$ $\quad \frac{1}{6}b_2+b_1-\frac{1}{120}=0$

$a_2=-b_1$ $\quad \frac{b_3}{6}+b_2-\frac{1}{120}b_1-\frac{1}{2}=0$

$a_3=-b_2+\frac{1}{6},$

$$\Rightarrow \begin{cases} \frac{1}{6}b_1-b_3+1=0 \\ b_1+\frac{1}{6}b_2-\frac{1}{120}=0 \\ -\frac{1}{120}b_1+b_2+\frac{1}{6}b_3-\frac{1}{2}=0 \end{cases}$$

$\Rightarrow -\frac{1}{20}b_1+6b_2+b_3=3$

$\Rightarrow \frac{14}{120}b_1+6b_2=2$

$36b_1+6b_2=\frac{36}{120}$

$36 \times 12 = 72,\ 36,\ 432$

$\frac{4306}{120}b_1=-\frac{84}{120}$, $b_1=-\frac{204}{4306}=-\frac{102}{2153}.$

$b_2=\frac{14393}{43060}$, $b_3=\frac{2136}{2153}.$

$\Rightarrow a_2=\frac{102}{2153}$, $a_3=-\frac{21649}{129180}$

$$\Rightarrow \sin(x^4-x) \approx \frac{-x+\frac{102}{2153}x^2-\frac{21649}{129180}x^3}{1-\frac{102}{2153}x+\frac{14393}{43060}x^2+\frac{2136}{2153}x^3}.$$

$$xe^x \approx x+x^2+\frac{x^3}{2}+\frac{x^4}{6}+\frac{x^5}{24}+\frac{x^6}{120}$$

$$\frac{\left(x+x^2+\frac{x^3}{2}+\frac{x^4}{6}+\frac{x^5}{24}+\frac{x^6}{120}\right)(1+b_1x+b_2x^2+b_3x^3)-(a_0+a_1x+a_2x^2+a_3x^3)}{1+b_1x+b_2x^2+b_3x^3}.$$

$-\frac{1}{2}b_2+\frac{9}{120}=0$

$b_2=\frac{9}{60}$

$-\frac{1}{2}b_2+\frac{1}{5}-\frac{3}{24}=0$

$b_3+\frac{9}{60}-\frac{3}{10}+\frac{1}{6}=0$

$\Rightarrow a_0=0,$ $\quad b_3+b_2+\frac{1}{2}b_1+\frac{1}{6}=0$

$a_1=1,$ $\quad b_3+\frac{1}{2}b_2+\frac{1}{6}b_1+\frac{1}{24}=0$

$a_2=b_1+1$ $\quad \frac{b_3}{2}+\frac{1}{6}b_2+\frac{1}{24}b_1+\frac{1}{120}=0$

$a_3=b_1+b_2+\frac{1}{2},$

$$\Rightarrow \begin{cases} b_3+b_2+\frac{1}{2}b_1+\frac{1}{6}=0 \\ b_3+\frac{1}{2}b_2+\frac{1}{6}b_1+\frac{1}{24}=0 \\ b_3+\frac{1}{3}b_2+\frac{1}{12}b_1+\frac{1}{60}=0 \end{cases} \Rightarrow \begin{cases} b_3+b_2+\frac{1}{2}b_1+\frac{1}{6}=0 \\ 0-\frac{1}{2}b_2-\frac{1}{3}b_1-\frac{3}{24}=0 \\ 0-\frac{2}{3}b_2-\frac{5}{12}b_1-\frac{9}{60}=0 \end{cases} \times -\frac{4}{3}$$

$\frac{4}{9}-\frac{5}{12}$, $\frac{48-45}{108}$

$\frac{3}{108}b_1+\frac{1}{60}=0$, $b_1=-\frac{3}{5}$

$-\frac{9}{60}+\frac{1}{6}$

$-\frac{9}{20}+\frac{1}{2}$

$\Rightarrow b_1=-\frac{3}{5},$ $\quad \Rightarrow a_2=\frac{2}{5}$

$b_2=\frac{3}{20},$ $\quad a_3=\frac{1}{20}$

$b_3=-\frac{1}{60}$

$$\Rightarrow xe^x \approx \frac{x+\frac{2}{5}x^2+\frac{1}{20}x^3}{1-\frac{3}{5}x+\frac{3}{20}x^2-\frac{1}{60}x^3}$$

7. (a)

x	$f(x)$ (xe^{-x})	$f'(x)$ ($e^{-x}-xe^{-x}$)
1	$\frac{1}{e}$	0
2	$\frac{2}{e^2}$	$-\frac{1}{e^2}$
3	$\frac{3}{e^3}$	$\frac{-2}{e^3}$

$\approx$

x	$f(x)$	$f'(x)$
1	0.36788	0
2	0.27067	-0.13534
3	0.14936	0.09957

$$\Rightarrow L_0(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{1}{2}x^2 - \frac{5}{2}x + 3, \quad L_0'(x) = x - \frac{5}{2}$$

$$L_1(x) = \frac{(x-1)(x-3)}{(2-1)(2-3)} = -x^2 + 4x - 3, \quad L_1'(x) = -2x + 4$$

$$L_2(x) = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{1}{2}x^2 - \frac{3}{2}x + 1, \quad L_2'(x) = x - \frac{3}{2}.$$

$H(x) = \sum u_i(x)\cdot y_i + \sum v_i(x)\cdot y_i'$, where

$$u_i(x) = [1 - 2(x - x_i)L_i'(x_i)][L_i(x)]^2, \quad v_i(x) = (x - x_i)[L_i(x)]^2.$$

$$\Rightarrow u_0(x) = (3x-2)\left(\frac{1}{2}x^2 - \frac{5}{2}x + 3\right)^2, \quad v_0(x) = (x-1)\left(\frac{1}{2}x^2 - \frac{5}{2}x + 3\right)^2.$$

$$u_1(x) = (-x^2 + 4x - 3)^2, \quad v_1(x) = (x-2)(-x^2 + 4x - 3)^2.$$

$$u_2(x) = (-3x+10)\left(\frac{1}{2}x^2 - \frac{3}{2}x + 1\right)^2, \quad v_2(x) = (x-3)\left(\frac{1}{2}x^2 - \frac{3}{2}x + 1\right)^2$$

$$H(x) = (3x-2)\left(\frac{1}{2}x^2 - \frac{5}{2}x + 3\right)^2 \times 0.36788 + (-x^2 + 4x - 3)^2 \times 0.27067 + (x-2)(x^2 + 4x - 3)^2 \times (-0.13534)$$
$$+ (-3x+10)\left(\frac{1}{2}x^2 - \frac{3}{2}x + 1\right)^2 + (x-3)\left(\frac{1}{2}x^2 - \frac{3}{2}x + 1\right)^2 \times 0.09957$$

(b)

$\Rightarrow H(1.5) \approx 0.33015.$ #