GAME2016
Mathematical Foundation of Game Design and Animation

Lecture 6

Polar coordinate systems

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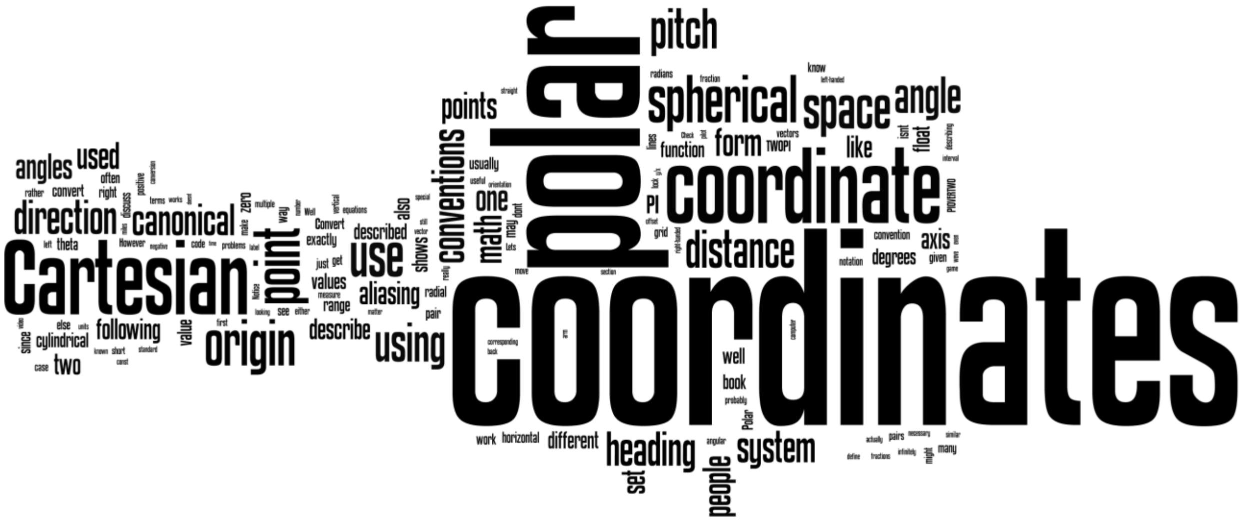
Agenda

- 3D polar space.

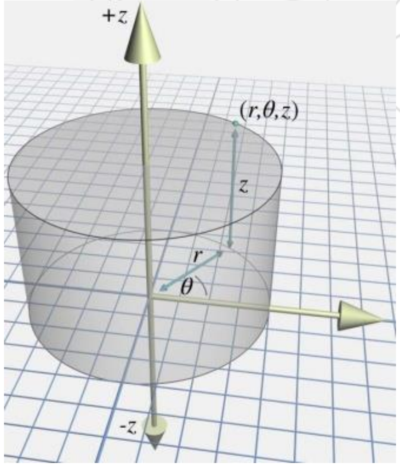


3D Polar Space

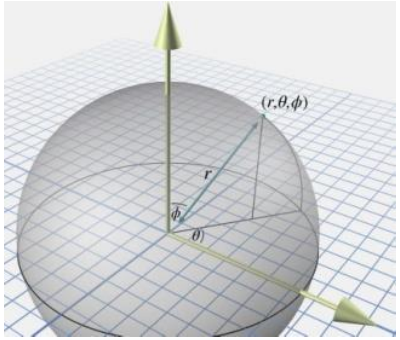
3D Polar Space

There are two kinds in common use:

1. Cylindrical coordinates
   - 1 angle and 2 distances

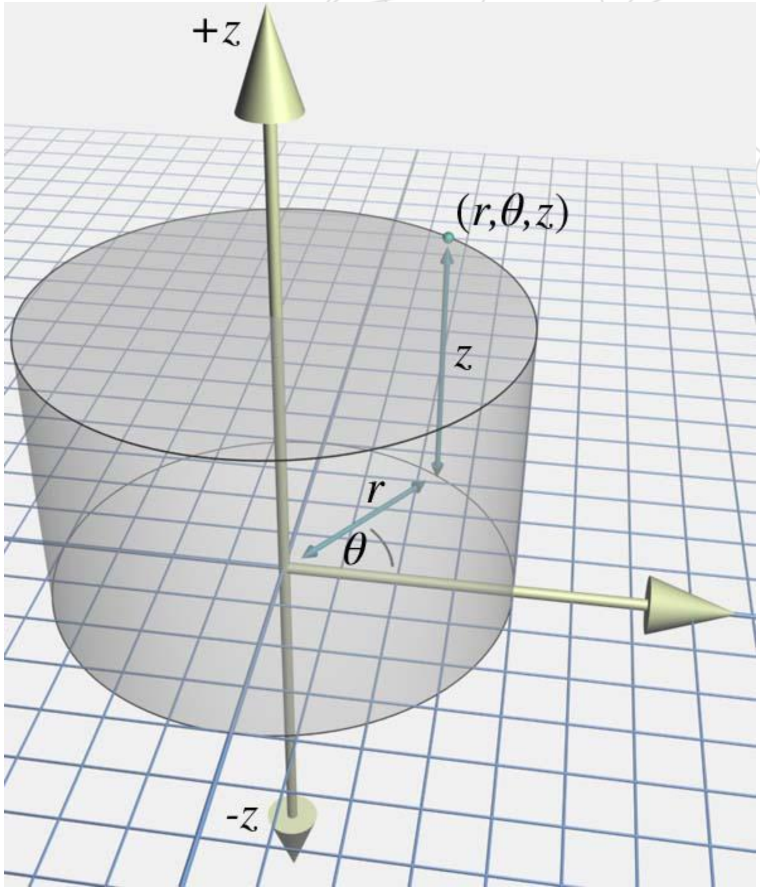


2. Spherical coordinates
   - 2 angles and 1 distance



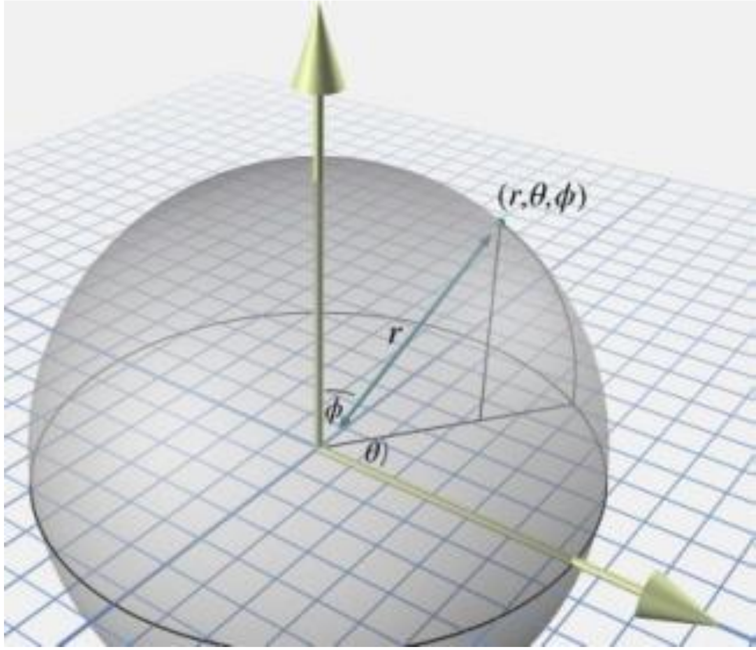
3D Cylindrical Space

- To locate the point described by cylindrical coordinates (r, θ, z), start by processing r and θ just like we would for 2D polar coordinates, and then move up or down the z axis by z.



3D Spherical Coordinates

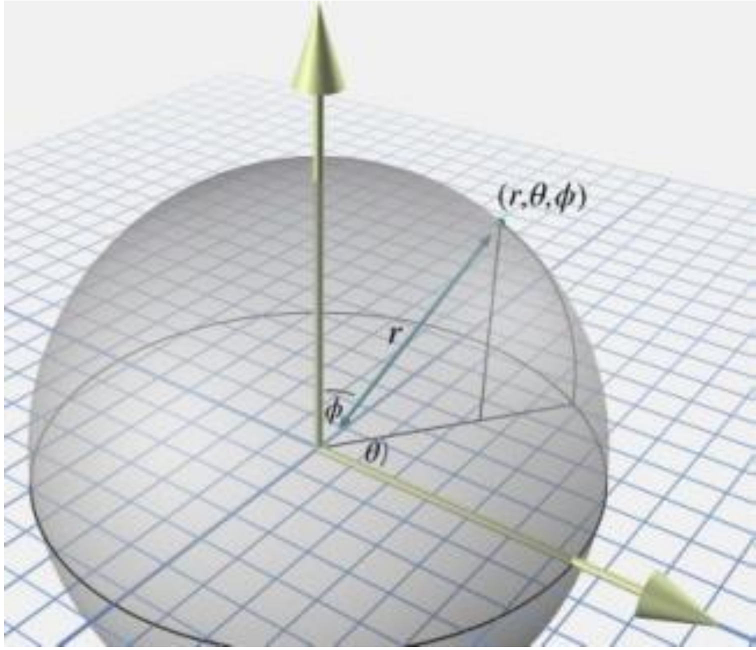
- As with 2D polar coordinates, 3D spherical coordinates also work by defining a direction and distance.
- The only difference is that in 3D it takes *two angles* to define a direction.
- Two polar axes in 3D spherical space.
  1. The first axis is horizontal and corresponds to the polar axis in 2D polar coordinates or +*x* in our 3D Cartesian conventions.
  2. The other axis is vertical, corresponding to +*y* in our 3D Cartesian conventions.



Notational Confusion

- Math people also are in general agreement about how these two angles are to be interpreted to define a direction.

- You can imagine it like this:

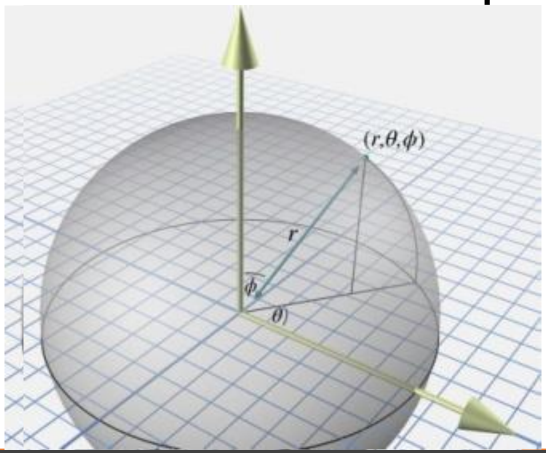


Finding the Point (r, θ, φ)

- The two angles are named θ and φ.

- Finding θ:

1. Stand at the origin, face the direction of the horizontal polar axis. The vertical axis points from your feet to your head.
2. Rotate counterclockwise by the angle θ
   - The same way that we did for 2D polar coordinates.

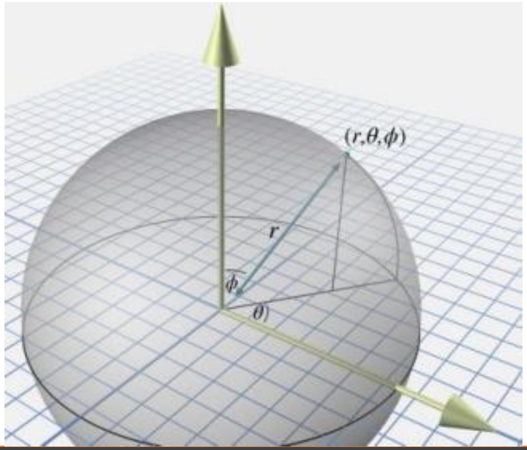


Finding the Point (r, θ, φ)

- Finding φ :

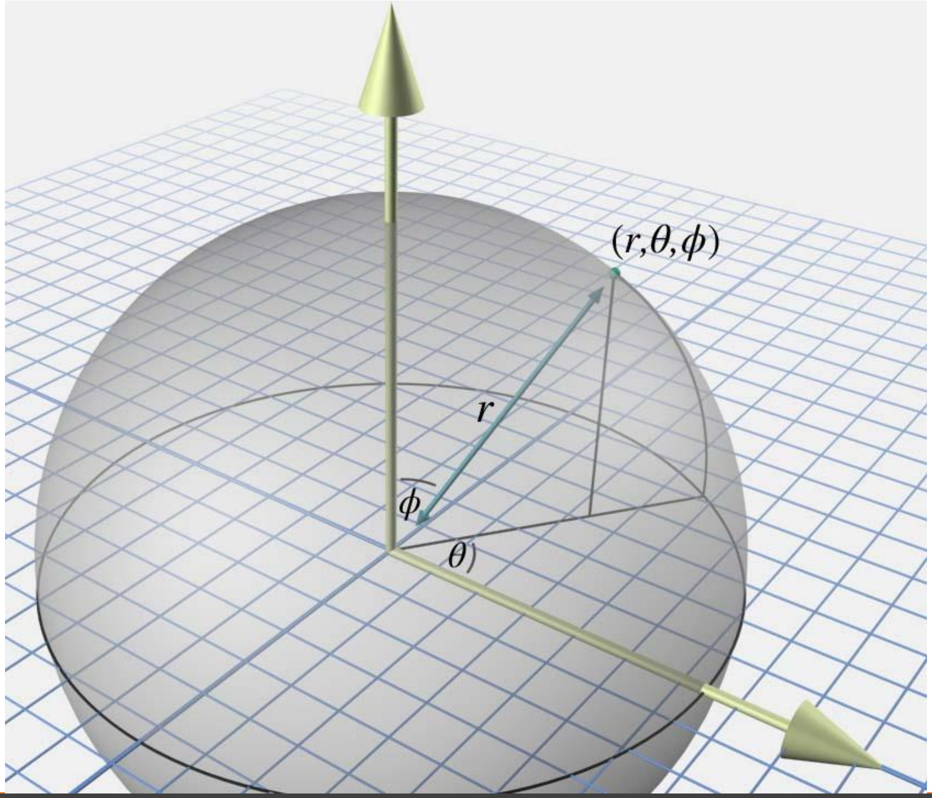
1. Point your arm straight up, in the direction of the vertical polar axis.
2. Rotate your arm downward by the angle φ.

- Your arm now points in the direction specified by the polar angles θ, φ.



Finding the Point (r, θ, φ)

- To find the point (r, θ, φ):
- Displace from the origin along this direction by the distance r, and we've arrived at the point described by the spherical coordinates (r, θ, φ).



Azimuth, Zenith, Lat, and Long

- The horizontal angle θ is known as the *azimuth*, and φ is the *zenith*.
- Other terms that you've probably heard are *longitude* and *latitude*.
- *Longitude* is basically θ, and *latitude* is the angle of inclination, 90° – φ.

- Physical coordinates
  - The latitude/longitude system for describing locations on planet Earth is actually a type of spherical coordinate system.
  - We're often only interested in describing points on the planet's surface, and so the radial distance *r*, which would measure the distance to the center of the Earth, isn't necessary
    - We may assume that it’s fixed (somewhat).

Visualizing Polar Coordinates

- The spherical coordinate system described in the previous section is the traditional right-handed system used by “math people.”

- We'll soon see that the formulas for converting between Cartesian and spherical coordinates are rather elegant under these assumptions.

- However, if you are like most people in the video game industry, you probably spend more time visualizing geometry than manipulating equations, and for our purposes these conventions carry a few irritating disadvantages:

Irritating Disadvantage 1

1. The default horizontal direction at $\theta = 0$ points in the direction of $+x$.

- This is unfortunate, since for us, $+x$ points "to the right" or "east," neither of which are the default directions in most people's mind.

- Similar to the way that numbers on a clock start at the top, it would be nicer for us if the horizontal polar axis pointed towards $+z$, which is "forward" or "north."

Irritating Disadvantage 2

2. The conventions for the angle φ are unfortunate in several respects.

- It would be nicer if the 2D polar coordinates (r, θ) were extended into 3D simply by adding a third coordinate of zero (r, θ, 0)
  - Like we extend the Cartesian system from 2D to 3D.
  - But (r, θ, 0) don't correspond to (r, θ) as we'd like.
- In fact, assigning φ = 0 puts us in the awkward situation of *Gimbal lock*
  - a singularity we'll describe soon.
- Instead, the points in the 2D plane are represented as (r, θ, 90°).
  - i.e., the 2D plane defined by the coordinates
- It might have been more intuitive to measure latitude, rather than zenith. Most people think of the default as “horizontal” and “up” as the extreme case.

Irritating Disadvantage 3

3. No offense to the Greeks, but θ and φ take a little while to get used to.

- The symbol r isn't so bad because at least it stands for something meaningful: “radial distance” or “radius.”

- Wouldn't it be great if the symbols we used to denote the angles were similarly short for English words, rather than completely arbitrary Greek symbols?

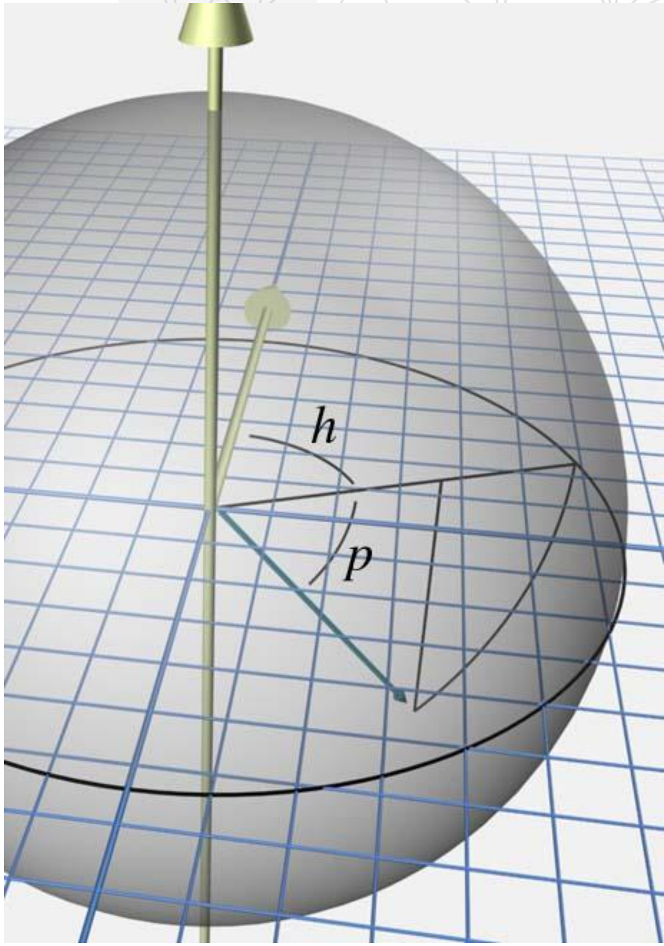
Irritating Disadvantages 4 and 5

4. For a better comparison it would be nice if the two angles for spherical coordinates were the same as the first two angles we will use for *Euler angles*
   - Which are used to describe orientation in 3D.
   - We're going to discuss Euler angles soon in the next lectures.

5. The polar coordinates use a right-handed system, and we use a left-handed system.

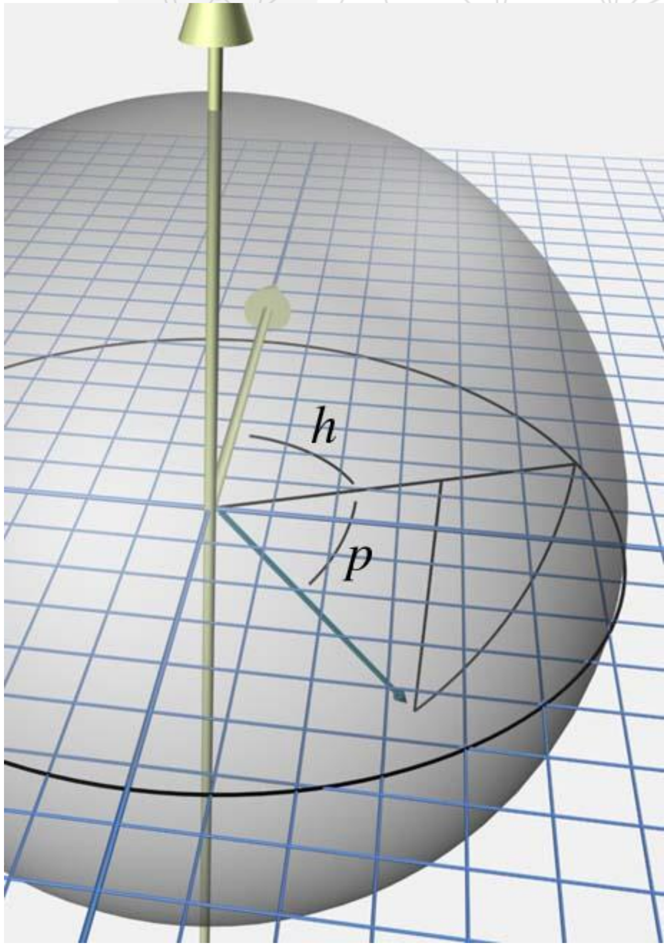
Spherical Coordinates for Gamers

- The horizontal angle θ we will rename to *h*, which is short for *heading*, similar to a compass heading.
- A heading of zero indicates a direction of “forward” or “to the north”, depending on the context.
  - This matches the standard aviation conventions.
  - In our convention the heading of zero (and thus our primary polar axis) corresponds to +*z*.
- Also, since we prefer a left-handed coordinate system, positive rotation will rotate clockwise when viewed from above.



Spherical Coordinates for Gamers

- The vertical angle φ is renamed to *p*, which is short for *pitch* and measures how much we are looking up or down.
- The default pitch value of zero indicates a horizontal direction, which is what most of us intuitively expect.
- Perhaps not-so-intuitively, positive pitch rotates downward, which means that pitch actually measures the *angle of declination*.
  - It is consistent with the left-hand rule.
  - Soon it will bear fruit.



Aliasing in 3D Spherical Coordinates

- The first way to generate an alias is to add a multiple of 360° to either angle.
  - Trivial, it's caused by the cyclic nature of angular measurements.

- The other two forms of aliasing are caused by the interdependence of the coordinates.
- i.e., the meaning of one coordinate *r* depends on the values of the other coordinate(s) – the angles
- This dependency creates:
  - A form of aliasing and
  - A singularity

The Aliasing and the Singularity

- The aliasing in 2D polar space could be triggered by negating the radial distance r and adjusting the angle so that the opposite direction is indicated.

- We can do the same with spherical coordinates.
- Using our heading and pitch conventions, all we need to do is:
  - Flip the heading by adding an odd multiple of 180°, and
  - Negate the pitch afterwards.

- The singularity in 2D occurred at the origin
  - The angular coordinate is irrelevant when $r = 0$.
- With spherical coordinates, *both* angles are irrelevant at the origin.

That’s Not All, Folks

- Spherical coordinates also suffer additional forms of aliasing because the pitch angle rotates about an axis that varies depending on the heading angle.

- This creates an additional form of aliasing and an additional singularity, analogous to those caused by the dependence of *r* on the direction.

More Aliasing

- Different heading and pitch values can result in the same direction, even excluding trivial aliasing of each individual angle.
- An alias of (h, p) can be generated by $(h \pm 180°, 180° - p)$.
  - For example, instead of turning right 90°(facing east) and pitching down 45°, we could turn left 90°(facing west) and then pitch down 135°.
  - Although we would be upside down, we would still be looking in the same direction.

More Singularity (Gimbal lock)

- A singularity occurs when the pitch angle is set to 90° (or any alias of these values).

- In this situation, known as Gimbal lock, the direction indicated is purely vertical (straight up or straight down), and the heading angle is irrelevant.
  - We'll have a great deal more to say about Gimbal lock when we discuss Euler angles.

Canonical Spherical Coordinates

- Just as we did in 2D, we can define a set of *canonical spherical coordinates* such that any given point in 3D space maps unambiguously to exactly one coordinate triple within the canonical set.

- We place similar restrictions on *r* and *h* as we did for polar coordinates.
- Two additional constraints are added related to the pitch angle.
  1. Pitch is restricted to be in the range –90° to 90°.
  2. Since the heading value is irrelevant when pitch reaches the extreme values of Gimbal lock, we force *h* = 0 in that case.

Conditions for Canonical Spherical Coordinates

$r \geqslant 0$	We don't measure distances "backwards."
$-180^{\circ} < h \leqslant 180^{\circ}$	Heading is limited to 1/2 revolution, and use $+180^{\circ}$ for "South."
$-90^{\circ} \leqslant p \leqslant 90^{\circ}$	Pitch limits are straight up and down. Can't "pitch over backwards."
$r = 0 \quad \Rightarrow \quad h = p = 0$	At the origin, set angles to zero.
$\lvert p \rvert = 90^{\circ} \quad \Rightarrow \quad h = 0$	When looking directly up or down, set heading to zero.

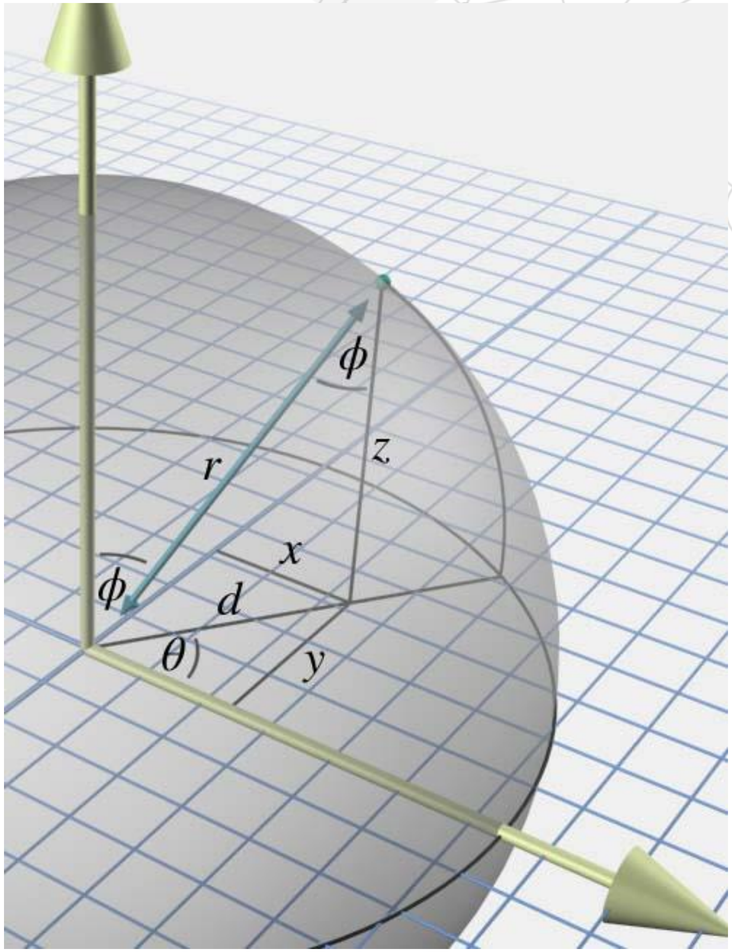
Algorithm to Make (r, p, h) Canonical

1. If $r = 0$, then assign $h = p = 0$
2. If $r < 0$, then negate r, add 180° to h, and negate p
3. If $p < -90°$, then add 360° to p until $p \geq -90°$
4. If $p > 270°$, then subtract 360° from p until $p \leq 270°$
5. If $p > 90°$, add 180° to h and set $p = 180° - p$
6. If $h \leq -180°$, then add 360° to h until $h > -180°$
7. If $h > 180°$, then subtract 360° from h until h $\leq 180°$

Converting Spherical Coordinates to 3D Cartesian Coordinates.

- Let's see how to convert spherical coordinates to 3D Cartesian coordinates.

- The figure shows both spherical and Cartesian coordinates.



Converting Spherical Coordinates to 3D Cartesian Coordinates.

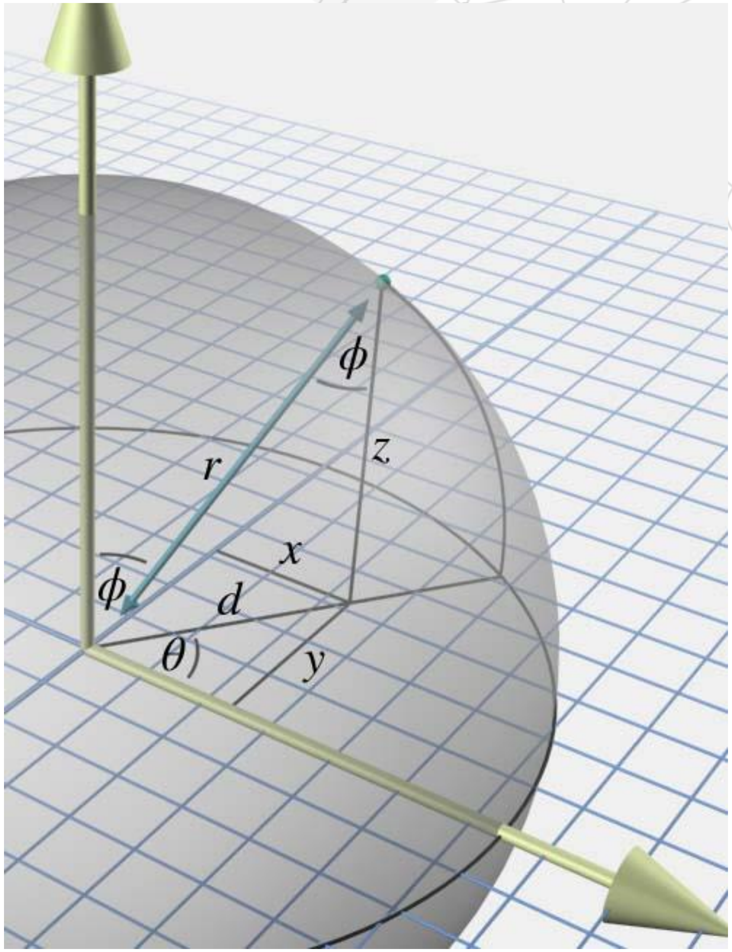
… after long math discussion about angles and right triangles…

- Convert spherical coordinates to 3D cartesian coordinates using the formulas:

$$x = r \sin \varphi \cos \theta$$
$$y = r \sin \varphi \sin \theta$$
$$z = r \cos \theta$$

- Using our gamer conventions:

$$x = r \cos p \sin h$$
$$y = -r \sin p$$
$$z = r \cos p \cos h$$



Converting 3D Cartesian Coordinates to Spherical Coordinates

- Compute (r, p, h) from (x, y, z)

- *r* is easy: $$r = \sqrt{x^2 + y^2 + z^2}$$
- As before, the singularity at the origin when $r = 0$ is handled as a special case.

- The heading angle *h* is surprisingly simple to compute using our atan2 function,

$$h = \text{atan2}(x, z).$$

Converting 3D Cartesian Coordinates to Spherical Coordinates

- Finally, once we know *r*, we can solve for *p* from *y*.

$$y = -r \sin p$$

$$-y/r = \sin p$$

$$p = \arcsin(-y/r)$$

- The arcsin function has a range of –90° to 90°, which fortunately coincides with the range for *p* within the canonical set.