



GAME2016

Mathematical Foundation of Game Design and Animation

Lecture 5

Course Overview

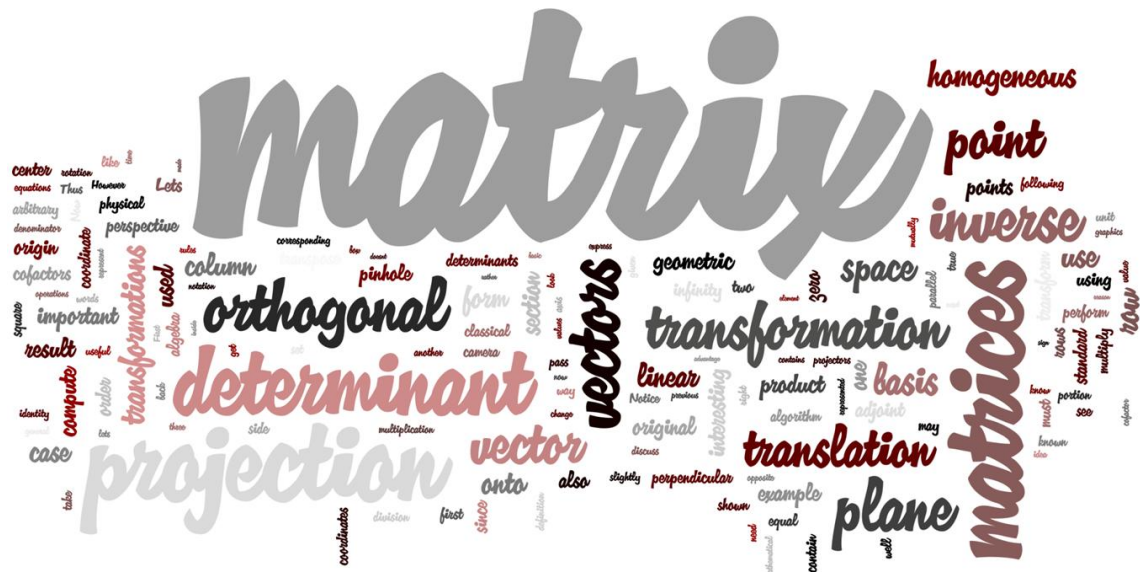
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Agenda

- Determinant of a matrix
- Inverse of a matrix
 - With adjoint matrix
- Orthogonal matrices and orthogonalization





Determinant of a Matrix

Determinant

- Determinant is defined for **square matrices**.
- Denoted $|\mathbf{M}|$ or $\det \mathbf{M}$.
- Determinant of a 2x2 matrix is

$$|\mathbf{M}| = \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = m_{11}m_{22} - m_{12}m_{21}$$

2 x 2 Example



$$\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = (2)(2) - (1)(-1) = 4 + 1 = 5$$

$$\begin{vmatrix} -3 & 4 \\ 2 & 5 \end{vmatrix} = (-3)(5) - (4)(2) = -15 - 8 = -23$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3 x 3 Determinant



$$\begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix}$$

$$= m_{11}m_{22}m_{33} + m_{12}m_{23}m_{31} + m_{13}m_{21}m_{32} \\ - m_{13}m_{22}m_{31} - m_{12}m_{21}m_{33} - m_{11}m_{23}m_{32}$$

$$= m_{11}(m_{22}m_{33} - m_{23}m_{32}) \\ + m_{12}(m_{23}m_{31} - m_{21}m_{33}) \\ + m_{13}(m_{21}m_{32} - m_{22}m_{31})$$

3 x 3 Example



$$\begin{vmatrix} -4 & -3 & 3 \\ 0 & 2 & -2 \\ 1 & 4 & -1 \end{vmatrix}$$

$$= \begin{aligned} & (-4)((-2)(-1) - (-2)(4)) \\ & + (-3)((-2)(1) - (0)(-1)) \\ & + (3)((0)(4) - (2)(1)) \end{aligned}$$

$$= \begin{aligned} & (-4)((-2) - (-8)) & (-4)(6) & (-24) \\ & + (-3)((-2) - (0)) = & + (-3)(-2) = & + (6) \\ & + (3)((0) - (2)) & + (3)(-2) & + (-6) \end{aligned}$$

$$= -24$$

Triple Product

- If we interpret the rows of a 3x3 matrix as three vectors, then the determinant of the matrix is equivalent to the so-called triple product of the three vectors:

$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \begin{aligned} & (a_y b_z - a_z b_y) c_x \\ & + (a_z b_x - a_x b_z) c_y \\ & + (a_x b_y - a_y b_x) c_z \end{aligned} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

Minors

- Let \mathbf{M} be an $r \times c$ matrix.
- Consider the matrix obtained by deleting row i and column j from \mathbf{M} .
- This matrix size will obviously be $r-1 \times c-1$.
- The **determinant of this submatrix**, denoted $M^{\{ij\}}$ is known as a **minor of \mathbf{M}** .
- For example, the minor $M^{\{12\}}$ is the determinant of the 2×2 matrix that is the result of deleting the 1st row and 2nd column from \mathbf{M} :

$$\mathbf{M} = \begin{bmatrix} -4 & -3 & 3 \\ 0 & 2 & -2 \\ 1 & 4 & -1 \end{bmatrix} \implies M^{\{12\}} = \begin{vmatrix} 0 & -2 \\ 1 & -1 \end{vmatrix} = 2$$

Cofactors

- The *cofactor* of a square matrix **M** at a given row and column is the same as the corresponding minor, only every alternating minor is negated.
- We will use the notation $C^{\{12\}}$ to denote the cofactor of **M** in row i , column j .
- Use $(-1)^{(i+j)}$ term to negate alternating minors.

$$C^{\{ij\}} = (-1)^{i+j} M^{\{ij\}}$$

Negating Alternating Minors

- The $(-1)^{i+j}$ term negates alternating minors in this pattern:

$$\begin{bmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- If the sum of $i+j$ is an even number, the minor is positive.
- If the sum of $i+j$ is an odd number, the minor is negative.

$n \times n$ Determinant

- The definition we will now consider expresses a **determinant in terms of its cofactors**.
- This definition is recursive, since **cofactors are themselves signed determinants**.
- First, we **arbitrarily select a row or column** from the matrix.
- Now, **for each element** in the row or column, we **multiply this element by the corresponding cofactor**.
- **Summing these products yields the determinant of the matrix**.

$n \times n$ Determinant

- For example, arbitrarily selecting row i , the determinant can be computed by:

$$|\mathbf{M}| = \sum_{j=1}^n m_{ij} C^{\{ij\}} = \sum_{j=1}^n m_{ij} (-1)^{i+j} M^{\{ij\}}$$

3 x 3 Determinant



$$\begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix}$$

$$= m_{11} \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix} - m_{12} \begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix} + m_{13} \begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix}$$

3 x 3 Determinant



$$\begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix}$$

$$= m_{11} \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix} - m_{12} \begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix} + m_{13} \begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix}$$

Sign of the cofactor

$$\begin{bmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \begin{bmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \begin{bmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

4 x 4 Determinant



$$\begin{vmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{vmatrix}$$

$$= m_{11} \begin{vmatrix} m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \\ m_{42} & m_{43} & m_{44} \end{vmatrix} - m_{12} \begin{vmatrix} m_{21} & m_{23} & m_{24} \\ m_{31} & m_{33} & m_{34} \\ m_{41} & m_{43} & m_{44} \end{vmatrix}$$

$$+ m_{13} \begin{vmatrix} m_{21} & m_{22} & m_{24} \\ m_{31} & m_{32} & m_{34} \\ m_{41} & m_{42} & m_{44} \end{vmatrix} - m_{14} \begin{vmatrix} m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{vmatrix}$$

Expanding Cofactors This Equals

$$\begin{vmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{vmatrix}$$

$$= m_{11} \begin{vmatrix} m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \\ m_{42} & m_{43} & m_{44} \end{vmatrix} - m_{12} \begin{vmatrix} m_{21} & m_{23} & m_{24} \\ m_{31} & m_{33} & m_{34} \\ m_{41} & m_{43} & m_{44} \end{vmatrix}$$

$$+ m_{13} \begin{vmatrix} m_{21} & m_{22} & m_{24} \\ m_{31} & m_{32} & m_{34} \\ m_{41} & m_{42} & m_{44} \end{vmatrix} - m_{14} \begin{vmatrix} m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{vmatrix}$$

$$\begin{aligned} & m_{11} [m_{22} m_{33} m_{44} - m_{34} m_{43}] + m_{23} [m_{34} m_{42} - m_{32} m_{44}] + m_{24} [m_{32} m_{43} - m_{33} m_{42}] \\ & - m_{12} [m_{21} (m_{33} m_{44} - m_{34} m_{43}) + m_{23} (m_{34} m_{41} - m_{31} m_{44}) + m_{24} (m_{31} m_{43} - m_{33} m_{41})] \\ & + m_{13} [m_{21} (m_{32} m_{44} - m_{34} m_{42}) + m_{22} (m_{34} m_{41} - m_{31} m_{44}) + m_{24} (m_{31} m_{42} - m_{32} m_{41})] \\ & - m_{14} [m_{21} (m_{32} m_{43} - m_{33} m_{42}) + m_{22} (m_{33} m_{41} - m_{31} m_{43}) + m_{23} (m_{31} m_{42} - m_{32} m_{41})] \end{aligned}$$

Important Determinant Facts

- The identity matrix has determinant 1:

$$|\mathbf{I}| = 1.$$

- The determinant of a matrix product is equal to the product of the determinants:

$$|\mathbf{AB}| = |\mathbf{A}| |\mathbf{B}|.$$

- This extends to multiple matrices:

$$|\mathbf{M}_1 \mathbf{M}_2 \dots \mathbf{M}_{n-1} \mathbf{M}_n| = |\mathbf{M}_1| |\mathbf{M}_2| \dots |\mathbf{M}_{n-1}| |\mathbf{M}_n|.$$

- The determinant of the transpose of a matrix is equal to the original.

$$|\mathbf{M}^T| = |\mathbf{M}|.$$

Zero Row or Column

- If any row or column in a matrix contains all zeros, then the determinant of that matrix is zero.

$$\begin{vmatrix} ? & ? & \dots & ? \\ ? & ? & \dots & ? \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ ? & ? & \dots & ? \end{vmatrix} = \begin{vmatrix} ? & ? & \dots & 0 & \dots & ? \\ ? & ? & \dots & 0 & \dots & ? \\ \vdots & \vdots & & \vdots & & \vdots \\ ? & ? & \dots & 0 & \dots & ? \end{vmatrix} = 0$$

Exchanging Rows or Columns

Exchanging any pair of rows or columns negates the determinant.

$$\begin{vmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & & \vdots \\ m_{i1} & m_{i2} & \cdots & m_{in} \\ \vdots & \vdots & & \vdots \\ m_{j1} & m_{j2} & \cdots & m_{jn} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{vmatrix} = - \begin{vmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & & \vdots \\ m_{j1} & m_{j2} & \cdots & m_{jn} \\ \vdots & \vdots & & \vdots \\ m_{i1} & m_{i2} & \cdots & m_{in} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{vmatrix}$$

Adding a Multiple of a Row or Column

- Adding any multiple of a row (or column) to another row (or column) does not change the value of the determinant.

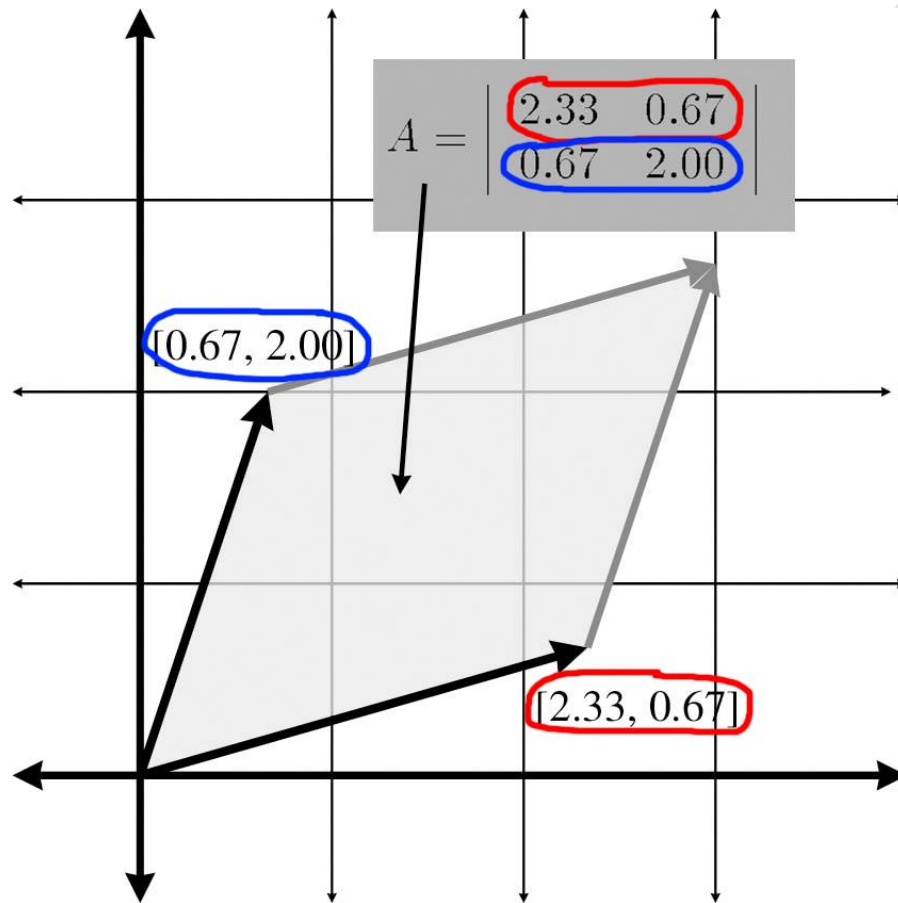
$$\begin{vmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & & \vdots \\ m_{i1} & m_{i2} & \cdots & m_{in} \\ \vdots & \vdots & & \vdots \\ m_{j1} & m_{j2} & \cdots & m_{jn} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{vmatrix} = \begin{vmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & & \vdots \\ m_{i1} + km_{j1} & m_{i2} + km_{j2} & \cdots & m_{in} + km_{jn} \\ \vdots & \vdots & & \vdots \\ m_{j1} & m_{j2} & \cdots & m_{jn} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{vmatrix}$$

- This explains why shear matrices that have a determinant of 1.

Geometric Interpretation

- In 2D, the determinant is equal to the signed area of the parallelogram or skew box that has the basis vectors as two sides.
- By signed area, we mean that the area is negative if the skew box is flipped relative to its original orientation.

2 x 2 Determinant as Area



3 x 3 Determinant as Volume

- In 3D, the determinant is the volume of the parallelepiped that has the transformed basis vectors as three edges.
- It will be negative if the object is reflected (turned inside out) as a result of the transformation.

Uses of the Determinant

- The determinant is related to the change in size that results from transforming by the matrix.
- The absolute value of the determinant is related to the change in area (in 2D) or volume (in 3D) that will occur as a result of transforming an object by the matrix.
- The determinant of the matrix can also be used to help classify the type of transformation represented by a matrix.
- If the determinant of a matrix is zero, then the matrix contains a projection.
- If the determinant of a matrix is negative, then the matrix contains a reflection.



Inverse of a Matrix

Inverse of a Matrix

- The inverse of a square matrix \mathbf{M} , denoted \mathbf{M}^{-1} is the matrix such that when we multiply by \mathbf{M}^{-1} , the result is the identity matrix.

$$\mathbf{M} \mathbf{M}^{-1} = \mathbf{M}^{-1} \mathbf{M} = \mathbf{I}.$$

- Not all matrices have an inverse.
 - E.g., a matrix with a row or column of zeros
 - no matter what you multiply this matrix by, you will end up with the zero matrix.
- If a matrix has an inverse, it is said to be *invertible* or *non-singular*.
- A matrix that does not have an inverse is said to be *non-invertible* or *singular*.

Invertibility and Linear Independence

- For any invertible matrix \mathbf{M} , the vector equality $\mathbf{vM} = \mathbf{0}$ is true only when $\mathbf{v} = \mathbf{0}$.
- Furthermore, the rows of an invertible matrix are linearly independent, as are the columns.
- The rows and columns of a singular matrix are linearly dependent.

Determinant and Invertibility

- The determinant of a non-singular matrix is non-zero.
- The determinant of a singular matrix is zero
- Checking the magnitude of the determinant is the most commonly used test for invertibility
 - it's easier and quicker.

The Classical Adjoint

- Our method for computing the inverse of a matrix is based on the *classical adjoint*.
- The classical adjoint of a matrix \mathbf{M} , denoted $\text{adj } \mathbf{M}$, is defined to be the transpose of the matrix of cofactors of \mathbf{M} .

Computing the Cofactors

- For example, let:

$$\mathbf{M} = \begin{bmatrix} -4 & -3 & 3 \\ 0 & 2 & -2 \\ 1 & 4 & -1 \end{bmatrix}$$

- Compute the cofactors of \mathbf{M} :

$$C^{\{11\}} = + \begin{vmatrix} 2 & -2 \\ 4 & -1 \end{vmatrix} = 6 \quad C^{\{12\}} = - \begin{vmatrix} 0 & -2 \\ 1 & -1 \end{vmatrix} = -2 \quad C^{\{13\}} = + \begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix} = -2$$

$$C^{\{21\}} = - \begin{vmatrix} -3 & 3 \\ 4 & -1 \end{vmatrix} = 9 \quad C^{\{22\}} = + \begin{vmatrix} -4 & 3 \\ 1 & -1 \end{vmatrix} = 1 \quad C^{\{23\}} = - \begin{vmatrix} -4 & -3 \\ 1 & 4 \end{vmatrix} = 13$$

$$C^{\{31\}} = + \begin{vmatrix} -3 & 3 \\ 2 & -2 \end{vmatrix} = 0 \quad C^{\{32\}} = - \begin{vmatrix} -4 & 3 \\ 0 & -2 \end{vmatrix} = -8 \quad C^{\{33\}} = + \begin{vmatrix} -4 & -3 \\ 0 & 2 \end{vmatrix} = -8$$

Classical Adjoint of **M**

- The classical adjoint of **M** is the transpose of the matrix of cofactors:

$$\begin{aligned}\text{adj } \mathbf{M} &= \begin{bmatrix} C^{\{11\}} & C^{\{12\}} & C^{\{13\}} \\ C^{\{21\}} & C^{\{22\}} & C^{\{23\}} \\ C^{\{31\}} & C^{\{32\}} & C^{\{33\}} \end{bmatrix}^T \\ &= \begin{bmatrix} 6 & -2 & -2 \\ 9 & 1 & 13 \\ 0 & -8 & -8 \end{bmatrix}^T = \begin{bmatrix} 6 & 9 & 0 \\ -2 & 1 & -8 \\ -2 & 13 & -8 \end{bmatrix}\end{aligned}$$

Back to the Inverse

- The inverse of a matrix is its classical adjoint divided by its determinant:

$$\mathbf{M}^{-1} = \text{adj } \mathbf{M} / |\mathbf{M}|.$$

- If the determinant is zero, the division is undefined
 - i.e., matrices with a zero determinant are non-invertible.

Example of Matrix Inverse

■ If:

$$\mathbf{M} = \begin{bmatrix} -4 & -3 & 3 \\ 0 & 2 & -2 \\ 1 & 4 & -1 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \frac{\text{adj } \mathbf{M}}{|\mathbf{M}|} = \frac{1}{-24} \begin{bmatrix} 6 & 9 & 0 \\ -2 & 1 & -8 \\ -2 & 13 & -8 \end{bmatrix} = \begin{bmatrix} -1/4 & -3/8 & 0 \\ 1/12 & -1/24 & 1/3 \\ 1/12 & -13/24 & 1/3 \end{bmatrix}$$

Gaussian Elimination

- There are other techniques that can be used to compute the inverse of a matrix, such as *Gaussian elimination*.
 - Maybe you already studied this technique
- Many linear algebra textbooks incorrectly assert that such techniques are better suited for implementation on a computer as they require fewer arithmetic operations.
 - True for large matrices, or for matrices with a structure that can be exploited.
- For arbitrary smaller order matrices like 2×2 , 3×3 , and 4×4 used most often in geometric applications, the classical adjoint method is faster.
- The reason is that the classical adjoint method provides for a branchless implementation, meaning there are no **if** statements or loops that cannot be unrolled statically.
 - Can be accelerated by today's superscalar architectures and vector processors (GPUs).

Facts About Matrix Inverse

- The inverse of the inverse of a matrix is the original matrix. If \mathbf{M} is nonsingular, $(\mathbf{M}^{-1})^{-1} = \mathbf{M}$.
- The identity matrix is its own inverse: $\mathbf{I}^{-1} = \mathbf{I}$.
- Note that there are other matrices that are their own inverse, such as any reflection matrix, or a matrix that rotates 180° about any axis.
- The inverse of the transpose of a matrix is the transpose of the inverse: $(\mathbf{M}^T)^{-1} = (\mathbf{M}^{-1})^T$

More Facts About Matrix Inverse

- The inverse of a product is equal to the product of the inverses in reverse order.

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

- This extends to more than two matrices:

$$(\mathbf{M}_1\mathbf{M}_2\ldots\mathbf{M}_{n-1}\mathbf{M}_n)^{-1} = \mathbf{M}_n^{-1}\mathbf{M}_{n-1}^{-1}\ldots\mathbf{M}_2^{-1}\mathbf{M}_1^{-1}$$

- The determinant of the inverse is the inverse of the determinant: $|\mathbf{M}^{-1}| = 1/|\mathbf{M}|$.

Geometric Interpretation of Inverse

- The inverse of a matrix is useful geometrically because it allows us to compute the reverse or opposite of a transformation – a transformation that undoes another transformation if they are performed in sequence.
- So, if we take a vector \mathbf{v} , transform it by a matrix \mathbf{M} , and then transform it by the inverse \mathbf{M}^{-1} of \mathbf{M} , then we will get \mathbf{v} back.
- We can easily verify this algebraically:
$$(\mathbf{vM})\mathbf{M}^{-1} = \mathbf{v}(\mathbf{MM}^{-1}) = \mathbf{vI} = \mathbf{v}$$



Orthogonal Matrices

Orthogonal Matrices

- A square matrix M is *orthogonal* if and only if the product of the matrix and its transpose is the identity matrix: $\mathbf{M}\mathbf{M}^T = \mathbf{I}$.
- If a matrix is orthogonal, its transpose and the inverse are equal: $\mathbf{M}^T = \mathbf{M}^{-1}$.
- If we know that our matrix is orthogonal, we can essentially *avoid computing the inverse*, which is a relatively costly computation.
- For example, *rotation and reflection matrices are orthogonal*.

Testing Orthogonality

- Let **M** be a 3 x 3 matrix. Let's see exactly what it means when **MM^T = I**.

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} m_{11} & m_{21} & m_{31} \\ m_{12} & m_{22} & m_{32} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9 Equations

- This gives us 9 equations, all of which must be true in order for **M** to be orthogonal:

$$m_{11}m_{11} + m_{12}m_{12} + m_{13}m_{13} = 1$$

$$m_{11}m_{21} + m_{12}m_{22} + m_{13}m_{23} = 0$$

$$m_{11}m_{31} + m_{12}m_{32} + m_{13}m_{33} = 0$$

$$m_{21}m_{11} + m_{22}m_{12} + m_{23}m_{13} = 0$$

$$m_{21}m_{21} + m_{22}m_{22} + m_{23}m_{23} = 1$$

$$m_{21}m_{31} + m_{22}m_{32} + m_{23}m_{33} = 0$$

$$m_{31}m_{11} + m_{32}m_{12} + m_{33}m_{13} = 0$$

$$m_{31}m_{21} + m_{32}m_{22} + m_{33}m_{23} = 0$$

$$m_{31}m_{31} + m_{32}m_{32} + m_{33}m_{33} = 1$$

Consider the Rows

- Let the vectors \mathbf{r}_1 , \mathbf{r}_2 , \mathbf{r}_3 stand for the rows of \mathbf{M} :

$$\mathbf{r}_1 = [m_{11} \quad m_{12} \quad m_{13}]$$

$$\mathbf{r}_2 = [m_{21} \quad m_{22} \quad m_{23}]$$

$$\mathbf{r}_3 = [m_{31} \quad m_{32} \quad m_{33}]$$

$$\mathbf{M} = \begin{bmatrix} -\mathbf{r}_1- \\ -\mathbf{r}_2- \\ -\mathbf{r}_3- \end{bmatrix}$$

9 Equations Using Dot Product

- Now we can re-write the 9 equations more compactly:

$$\begin{array}{lll} \mathbf{r}_1 \cdot \mathbf{r}_1 = 1 & \mathbf{r}_1 \cdot \mathbf{r}_2 = 0 & \mathbf{r}_1 \cdot \mathbf{r}_3 = 0 \\ \mathbf{r}_2 \cdot \mathbf{r}_1 = 0 & \mathbf{r}_2 \cdot \mathbf{r}_2 = 1 & \mathbf{r}_2 \cdot \mathbf{r}_3 = 0 \\ \mathbf{r}_3 \cdot \mathbf{r}_1 = 0 & \mathbf{r}_3 \cdot \mathbf{r}_2 = 0 & \mathbf{r}_3 \cdot \mathbf{r}_3 = 1 \end{array}$$

Two Observations

- First, the if and only if the vector is dot product of a vector with itself is 1 a unit vector.
- Therefore, the equations with a 1 on the right-hand side of the equals sign will only be true when \mathbf{r}_1 , \mathbf{r}_2 , and \mathbf{r}_3 are unit vectors.
- Second, the dot product of two vectors is 0 if and only if they are perpendicular.
- Therefore, the other six equations (with 0 on the right hand side of the equals sign) are true when \mathbf{r}_1 , \mathbf{r}_2 , and \mathbf{r}_3 are mutually perpendicular.

Conclusion

- So, for a matrix to be orthogonal, the following must be true:
 1. Each row of the matrix must be a unit vector.
 2. The rows of the matrix must be mutually perpendicular.
- Similar statements can be made regarding the columns of the matrix, since if \mathbf{M} is orthogonal, then \mathbf{M}^T must be orthogonal.

Orthonormal Bases Revisited

- Notice that these criteria are precisely those that are satisfied by an orthonormal set of basis vectors.
- We also noted that an orthonormal basis was particularly useful because we could perform the “opposite” coordinate transform from the one that is always available, using the dot product.
- When we say that the transpose of an orthogonal matrix is its inverse, we are just restating this fact in the formal language of linear algebra.

9 is Actually 6

- Also notice that 3 of the orthogonality equations are duplicates (since dot product is commutative), and between these 9 equations, we actually have 6 constraints, leaving 3 degrees of freedom.
- This is interesting, since 3 is the number of degrees of freedom inherent in a rotation matrix.
- Recall that rotation matrices cannot compute a reflection, so there is slightly more freedom in the set of orthogonal matrices than in the set of orientations in 3D.

Caveats

- When computing a matrix inverse, we will usually only take advantage of orthogonality if we know *a priori* that a matrix is orthogonal.
 - If we don't know in advance, it's probably a waste of time checking.
- Finally, even matrices which are orthogonal in the abstract may not be exactly orthogonal when represented in floating point
 - we must use tolerances, which need to be tuned.

A Note on Terminology

- In linear algebra, we described a set of basis vectors as *orthogonal* if they are mutually perpendicular.
 - It is not required that they have unit length.
- If they do have unit length, they are an *orthonormal* basis.
- Thus, the rows and columns of an orthogonal matrix are orthonormal basis vectors.
- However, constructing a matrix from a set of orthogonal basis vectors does not necessarily result in an orthogonal matrix (unless the basis vectors are also orthonormal).

Scary Monsters (Matrix Creep)

- Recall that rotation matrices (and products of them) are orthogonal.
- Recall that the rows of an orthogonal matrix form an orthonormal basis.
- Or at least, that's the way we'd like them to be.
- But the world is not perfect. **Floating point numbers are subject to numerical instability.**
 - Aka "matrix creep"
- We need to orthogonalize the matrix, resulting in a matrix that has mutually perpendicular unit vector axes and is (hopefully) as close to the original matrix as possible.

Gramm-Schmidt Orthogonalization

- Here's how to control matrix creep.
- Go through the rows of the matrix in order.
- For each, subtract off the component that is parallel to the other rows.
- More details: let $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ be the rows of a 3×3 matrix \mathbf{M} .
- Remember, you can also think of these as the x -, y -, and z -axes of a coordinate space.

Steps 1 and 2

- Then an orthogonal set of row vectors, $\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_3'$ can be computed as follows:
- Step 1: Normalize \mathbf{r}_1 to get a new vector \mathbf{r}_1'
 - i.e., make its magnitude 1

- Step 2: Replace \mathbf{r}_2 by

$$\mathbf{r}_2' = \mathbf{r}_2 - (\mathbf{r}_1' \cdot \mathbf{r}_2) \mathbf{r}_1'$$

- \mathbf{r}_2' is now perpendicular to \mathbf{r}_1' because

$$\begin{aligned}\mathbf{r}_1' \cdot \mathbf{r}_2' &= \mathbf{r}_1' \cdot (\mathbf{r}_2 - (\mathbf{r}_1' \cdot \mathbf{r}_2) \mathbf{r}_1') \\ &= \mathbf{r}_1' \cdot \mathbf{r}_2 - (\mathbf{r}_1' \cdot \mathbf{r}_2)(\mathbf{r}_1' \cdot \mathbf{r}_1') \\ &= \mathbf{r}_1' \cdot \mathbf{r}_2 - \mathbf{r}_1' \cdot \mathbf{r}_2 \\ &= 0\end{aligned}$$

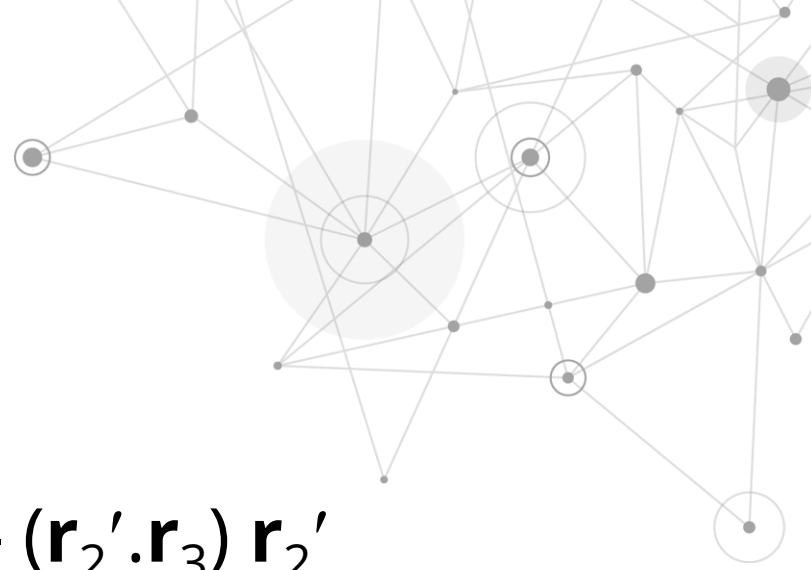
Steps 3, 4, and 5

- Step 3: Normalize \mathbf{r}_2'

- Step 4: Replace \mathbf{r}_3 by

$$\mathbf{r}_3' = \mathbf{r}_3 - (\mathbf{r}_1' \cdot \mathbf{r}_3) \mathbf{r}_1' - (\mathbf{r}_2' \cdot \mathbf{r}_3) \mathbf{r}_2'$$

- Step 5: Normalize \mathbf{r}_3'



Checking \mathbf{r}_3' and \mathbf{r}_1'

- \mathbf{r}_3' is now perpendicular to \mathbf{r}_1' because

$$\begin{aligned}\mathbf{r}_1' \cdot \mathbf{r}_3' &= \mathbf{r}_1' \cdot (\mathbf{r}_3 - (\mathbf{r}_1' \cdot \mathbf{r}_3) \mathbf{r}_1' - (\mathbf{r}_2' \cdot \mathbf{r}_3) \mathbf{r}_2') \\ &= \mathbf{r}_1' \cdot \mathbf{r}_3 - (\mathbf{r}_1' \cdot \mathbf{r}_3) (\mathbf{r}_1' \cdot \mathbf{r}_1') - (\mathbf{r}_2' \cdot \mathbf{r}_3) (\mathbf{r}_1' \cdot \mathbf{r}_2') \\ &= \mathbf{r}_1' \cdot \mathbf{r}_3 - \mathbf{r}_1' \cdot \mathbf{r}_3 - 0 \\ &= 0\end{aligned}$$

Checking \mathbf{r}_3' and \mathbf{r}_2'

- \mathbf{r}_3' is now perpendicular to \mathbf{r}_2' because

$$\begin{aligned}\mathbf{r}_2' \cdot \mathbf{r}_3' &= \mathbf{r}_2' \cdot (\mathbf{r}_3 - (\mathbf{r}_1' \cdot \mathbf{r}_3) \mathbf{r}_1' - (\mathbf{r}_2' \cdot \mathbf{r}_3) \mathbf{r}_2') \\ &= \mathbf{r}_2' \cdot \mathbf{r}_3 - (\mathbf{r}_1' \cdot \mathbf{r}_3) (\mathbf{r}_2' \cdot \mathbf{r}_1') - (\mathbf{r}_2' \cdot \mathbf{r}_3) (\mathbf{r}_2' \cdot \mathbf{r}_2') \\ &= \mathbf{r}_2' \cdot \mathbf{r}_3 - 0 - \mathbf{r}_2' \cdot \mathbf{r}_3 \\ &= 0\end{aligned}$$

Bias

- This is biased towards \mathbf{r}_1 , meaning that \mathbf{r}_1 doesn't change but the other basis vectors do change.
- Option: instead of subtracting off the whole amount, subtract off a fraction of the original axis.
 - Let k be a fraction – say $1/4$

Gramm-Schmidt in Practice

- Step 1: normalize $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$

- Step 2: repeat a dozen or so times:

$$\mathbf{r}_1' = \mathbf{r}_1 - k(\mathbf{r}_1 \cdot \mathbf{r}_2) \mathbf{r}_2 - k(\mathbf{r}_1 \cdot \mathbf{r}_3) \mathbf{r}_3$$

$$\mathbf{r}_2' = \mathbf{r}_2 - k(\mathbf{r}_1 \cdot \mathbf{r}_2) \mathbf{r}_1 - k(\mathbf{r}_2 \cdot \mathbf{r}_3) \mathbf{r}_3$$

$$\mathbf{r}_3' = \mathbf{r}_3 - k(\mathbf{r}_1 \cdot \mathbf{r}_3) \mathbf{r}_1 - k(\mathbf{r}_2 \cdot \mathbf{r}_3) \mathbf{r}_2$$

- Step 3: Do a vanilla Gramm-Schmidt to catch any remaining “abnormality”