GAME2016
Mathematical Foundation of Game Design and Animation

Lecture 2

Vectors

Dr. Paolo Mengoni
pmengoni@hkbu.edu.hk
Senior Lecturer @HKBU Department of Interactive Media

Agenda

- Mathematical properties of vectors.
- Geometric properties of vectors.
- Connecting the mathematical definition with the geometric definition.
  - Vectors vs points.
- Fundamental vector calculations

Mathematical Definition and notation

Vectors and Scalars

- An “ordinary number” is called a *scalar*.
- Algebraic definition of a vector: a list of scalars in square brackets.
  - Eg. [1, 2, 3].
- Vector *dimension* is the number of numbers in the list
  - The dimension of the vector in the example is 3.
  - Typically, we use dimension 2 for 2D work, dimension 3 for 3D work.
  - We’ll find a use for dimension 4 also, later.

Row vs. Column Vectors

- Vectors can be written in one of two different ways: horizontally or vertically.
- Row vector: [1, 2, 3]
- Column vector: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
- Mathematicians use row vectors because they're easier to write and take up less space.
  - For now, it doesn't really matter which convention you use.
  - Later it will become important.

Notation

- Bold case letters for vectors e.g., **v**.
- Scalar parts of a vector are called *components*, use subscripts for components.
- Example:
  - If **v** = [6, 19, 42],
  - its components are $\mathbf{v}_1 = 6$, $\mathbf{v}_2 = 19$, $\mathbf{v}_3 = 42$.
- Can also use *x*, *y*, *z* for subscripts.
  - 2D vectors: $[\mathbf{v}_x, \mathbf{v}_y]$.
  - 3D vectors: $[\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z]$.
  - 4D vectors $[\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z, \mathbf{v}_w]$ (We'll get to *w* later.)
- In this case, we can refer directly to the cartesian plane dimension

More Notation

- How to read some formulas.
- Scalar variables will be represented by lowercase Roman or Greek letters in italics: a, b, x, y, z, θ, α, ω, γ.
- Vector variables of any dimension will be represented by lowercase letters in boldface: **a**, **b**, **u**, **v**, **q**, **r**.
- Matrix variables will be represented using uppercase letters in boldface: **A**, **B**, **M**, **R**.

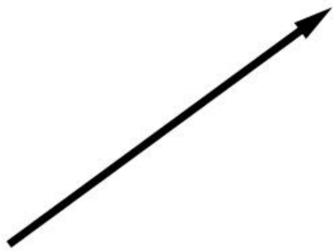
Terminology

- *Displacement* is a vector (e.g., 10 miles West)
- *Distance* is a scalar (e.g., 10 miles away)
- *Velocity* is a vector (e.g., 55mph North)
- *Speed* is a scalar (e.g., 55mph)
- Vectors are used to express relative things.
- Scalars are used to express absolute things.

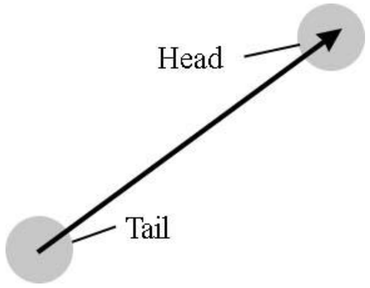
Geometric Definition

Geometric Definition of Vector

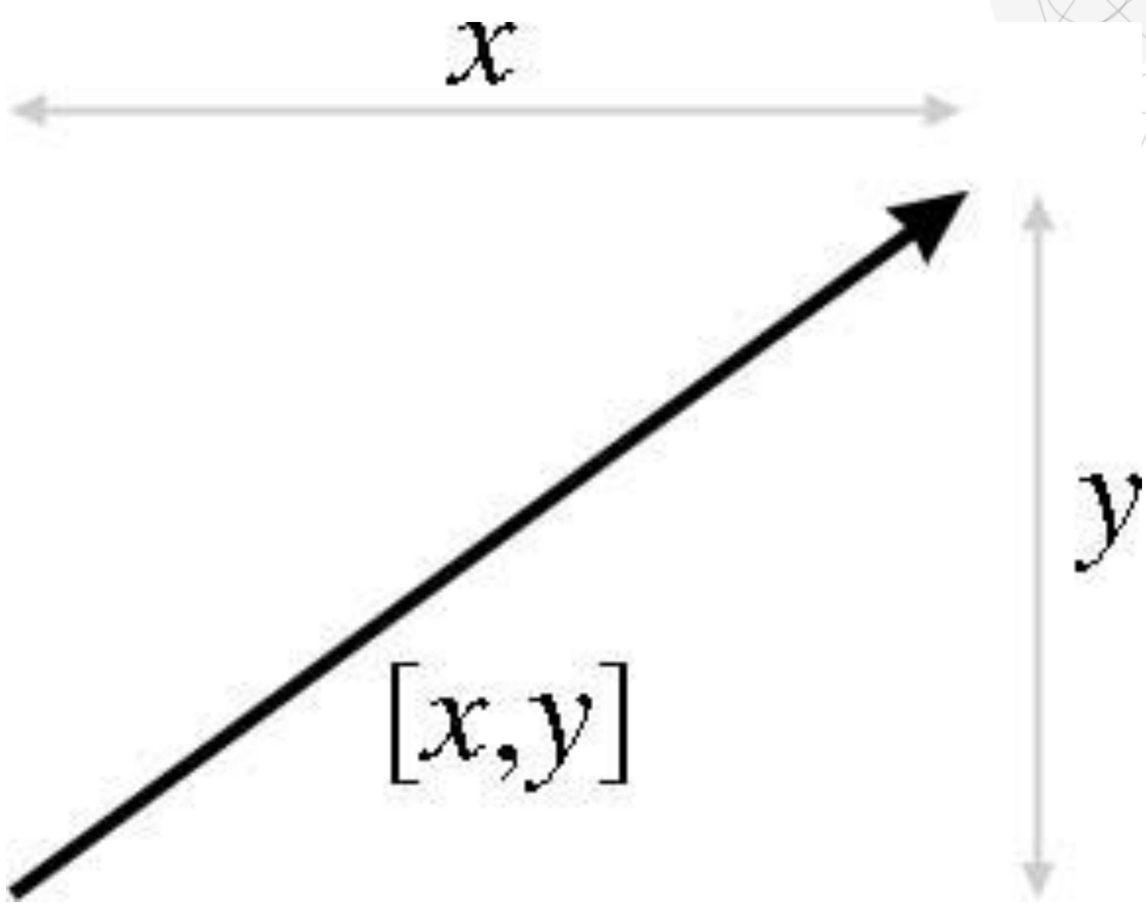
- A vector consists of a magnitude and a direction.
  - Magnitude = size.
  - Direction = orientation.
- Draw it as an arrow.

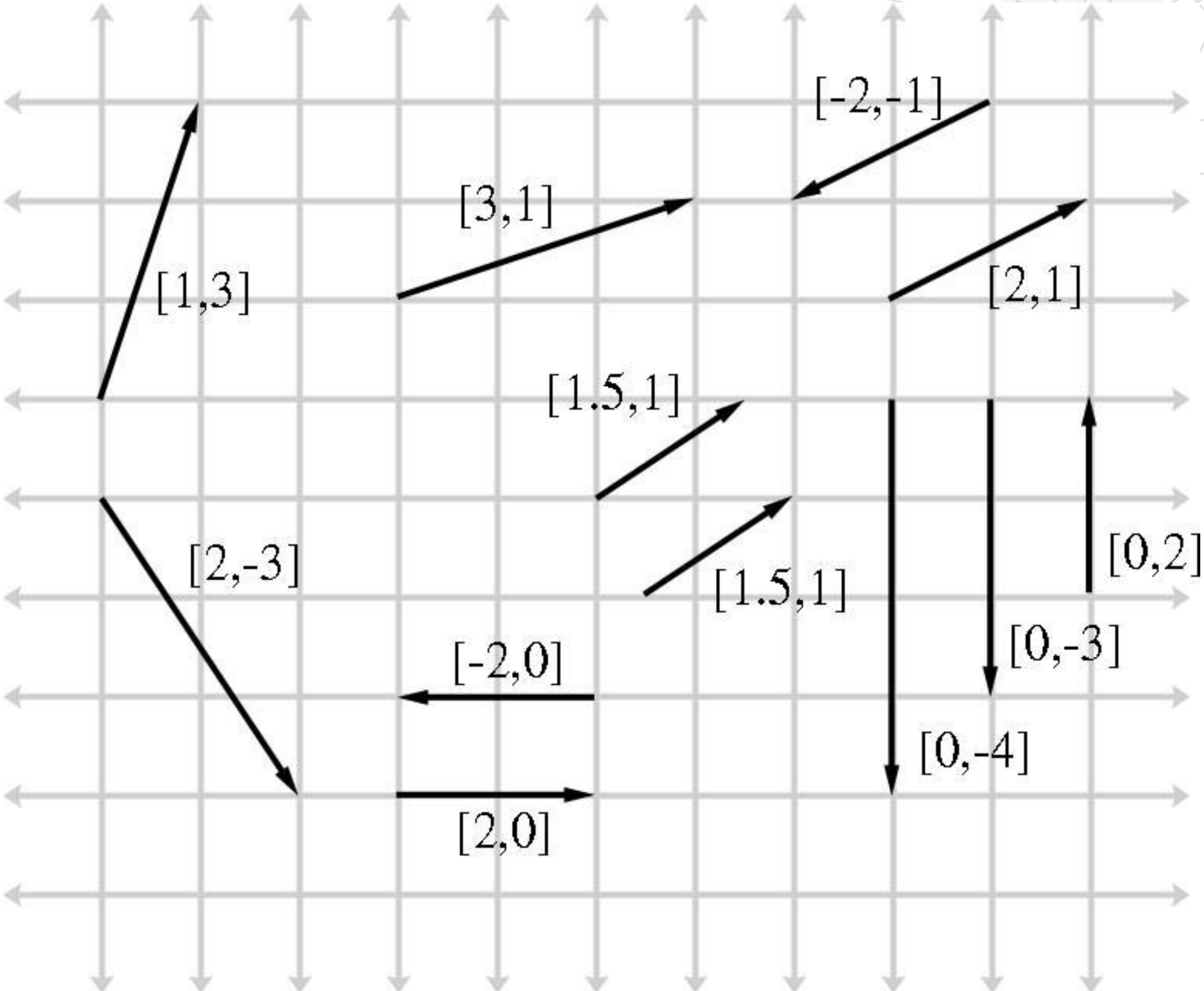


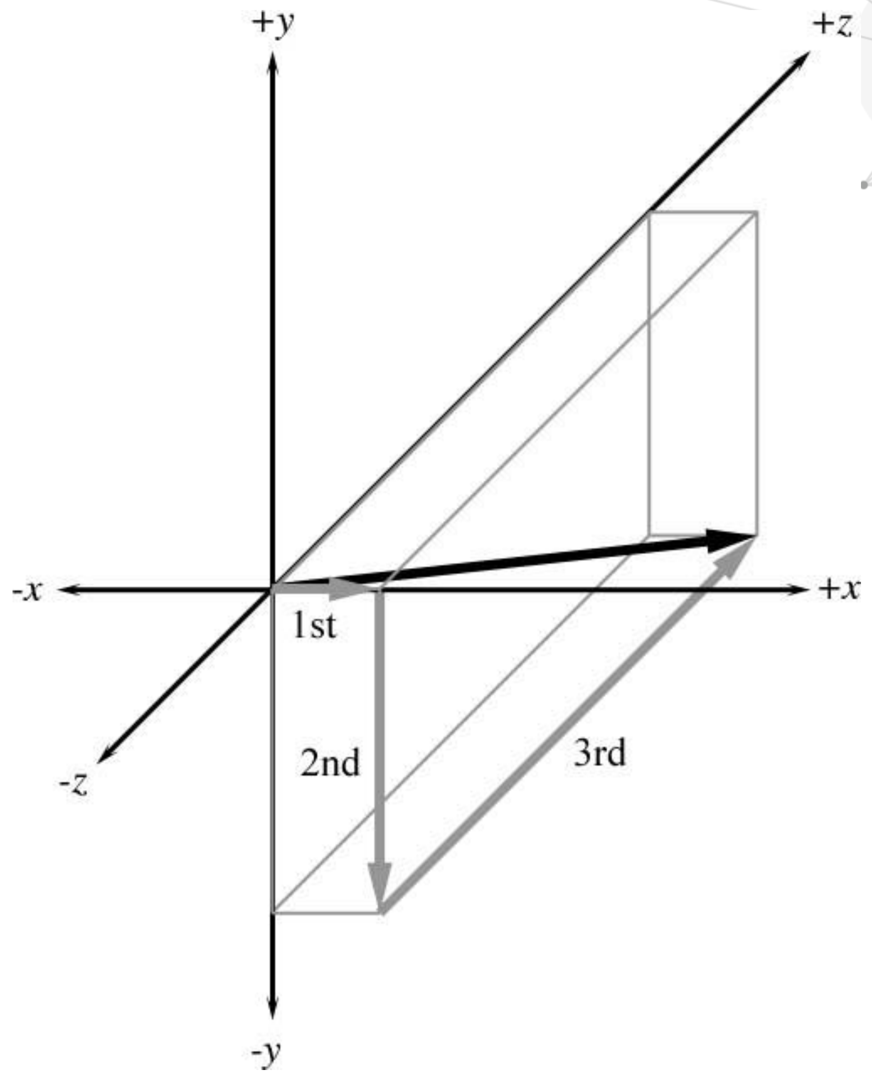
- Which End is Which?



Specifying Vectors Using Cartesian Coordinates


x
y
$[x,y]$


[-2,-1]
[3,1]
[1,3]
[2,1]
[1.5,1]
[2,-3]
[1.5,1]
[0,2]
[-2,0]
[0,-3]
[0,-4]
[2,0]


+y
+z
-x
+x
1st
2nd
3rd
-z
-y

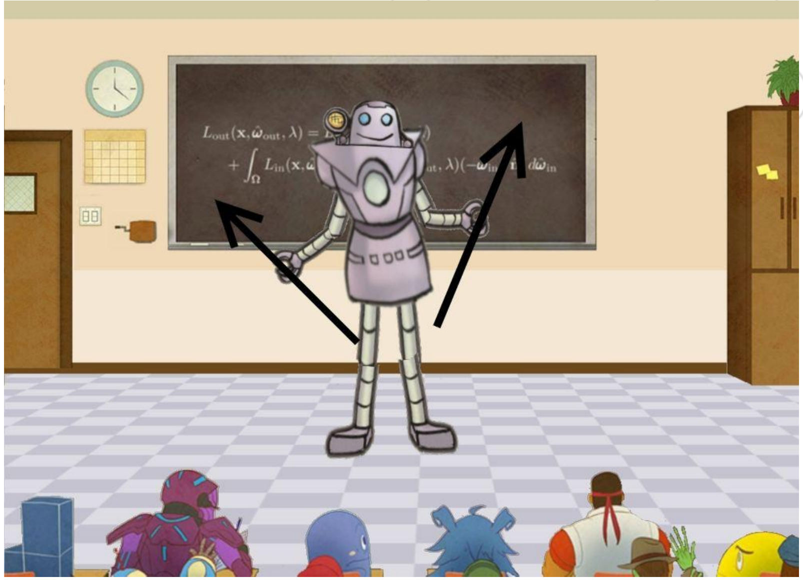
The Zero Vector

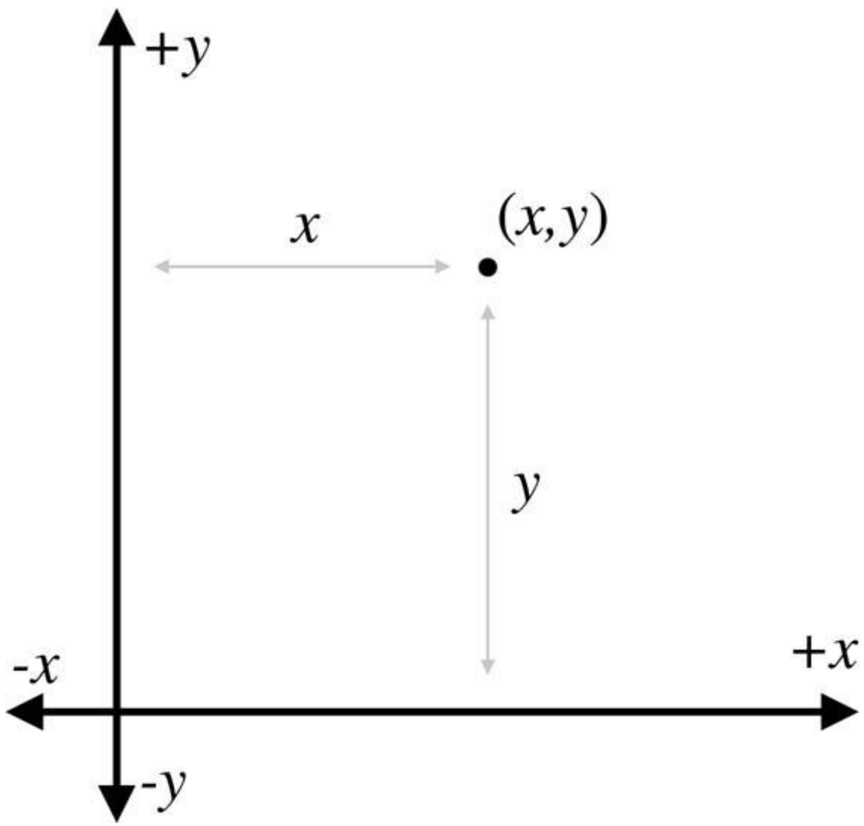
- The zero vector **0** is the additive identity, meaning that for all vectors **v**, **v** + **0** = **0** + **v** = **v**
- **0** = [0, 0,…, 0]
- The zero vector is unique: It's the only vector that doesn't have a direction

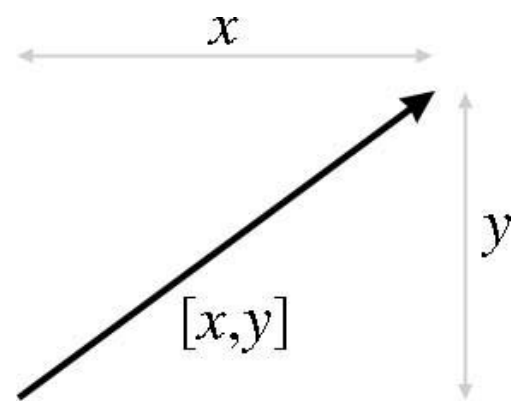
Vectors vs Points

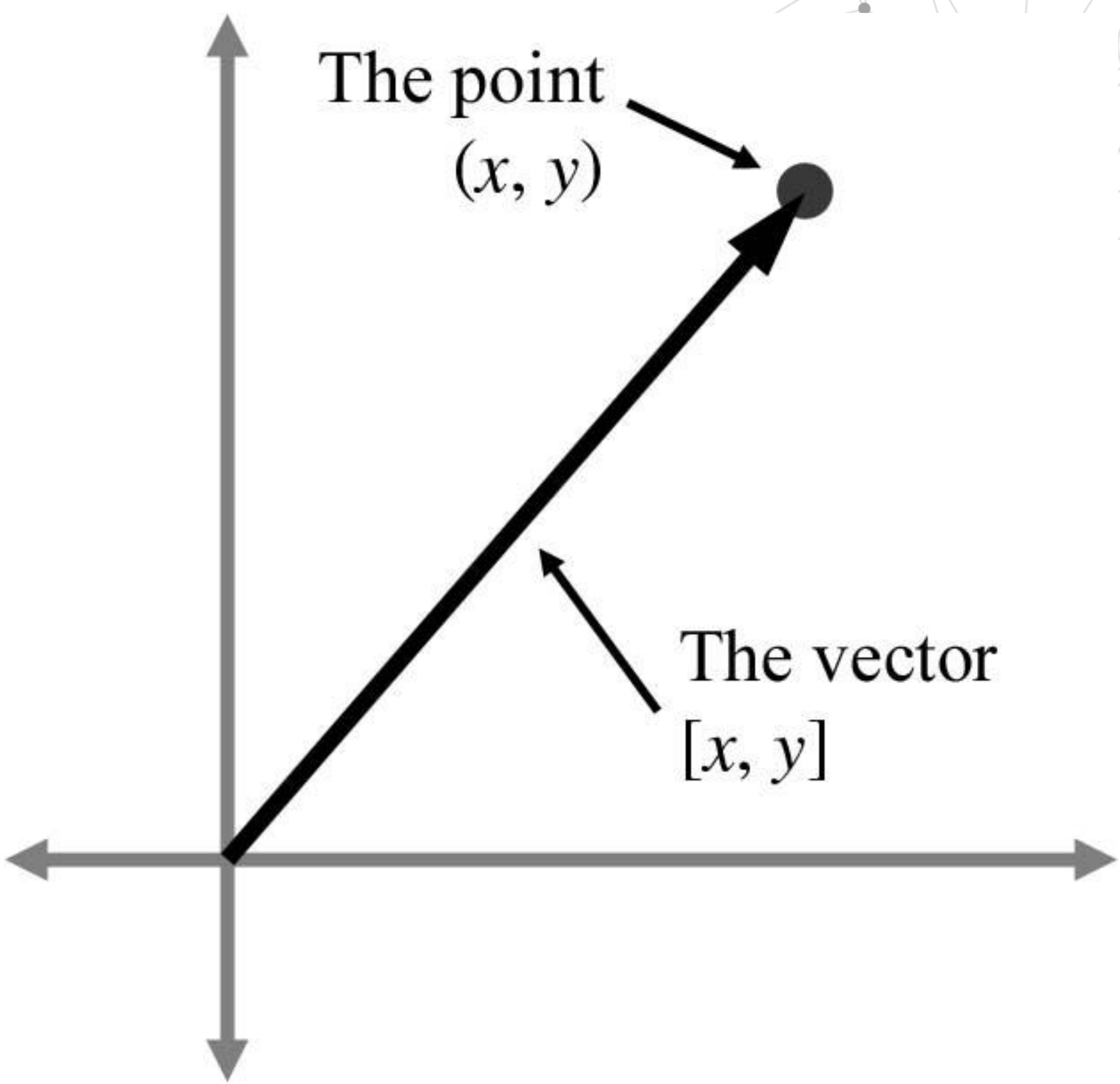
Vectors vs Points

- Points are measured relative to the origin.
- Vectors are intrinsically relative to everything.
  - So, a vector can be used to represent a point.
- The point (*x*,*y*) is the point at the head of the vector [*x*,*y*] when its tail is placed at the origin.
  - But vectors don't have a location




+y
x
(x,y)
y
-x
+x
-y


x
y
[x,y]


The point
(x, y)
The vector
[x, y]

Key Things to Remember

- Vectors don’t have a location.
- They can be dragged around the world whenever it’s convenient.
  - We will be doing that a lot.
- It’s tempting to think of them with tail at the origin.
- We can but don’t have to.
  - Be flexible.

Vector Operations

Next: Vector Operations

- Negation
- Multiplication by a scalar
- Addition and Subtraction
- Displacement
- Magnitude
- Normalization
- Dot product
- Cross product

- We will describe both the algebra and the geometry behind vector operations

Negating a Vector

Vector Negation: Algebra

- Negation is the additive inverse:
- **v** + -**v** = -**v** + **v** = **0**
- To negate a vector, negate all of its components.

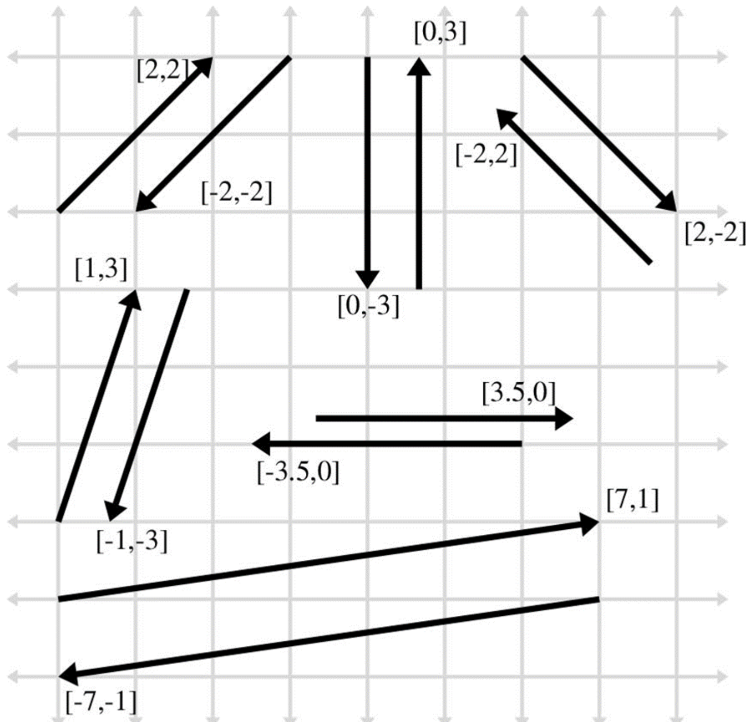
$$-\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_{n-1} \\ -a_n \end{bmatrix}$$

$$-\begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} -x & -y \end{bmatrix}$$
$$-\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} -x & -y & -z \end{bmatrix}$$
$$-\begin{bmatrix} x & y & z & w \end{bmatrix} = \begin{bmatrix} -x & -y & -z & -w \end{bmatrix}$$

$$-\begin{bmatrix} 4 & -5 \end{bmatrix} = \begin{bmatrix} -4 & 5 \end{bmatrix}$$
$$-\begin{bmatrix} -1 & 0 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sqrt{3} \end{bmatrix}$$
$$-\begin{bmatrix} 1.34 & -3/4 & -5 & \pi \end{bmatrix} = \begin{bmatrix} -1.34 & 3/4 & 5 & -\pi \end{bmatrix}$$

Vector Negation: Geometry

- To negate a vector, make it point in the opposite direction.
  - i.e., swap the head with the tail
- A vector and its negative are parallel and have the same magnitude but point in opposite directions.



Vector Multiplication by a Scalar

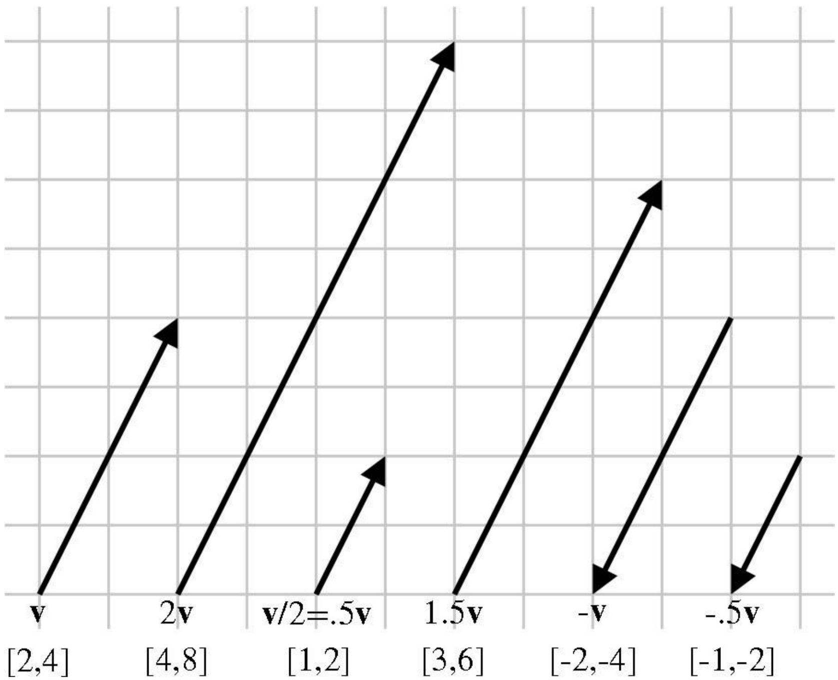
Vector Multiplication by a Scalar: Algebra

- Can multiply a vector by a scalar.
- Result is a vector of the same dimension.
- To multiply a vector by a scalar, multiply each component by the scalar.
  - For example, if $k\mathbf{a} = \mathbf{b}$, then $\mathbf{b}_1 = k\mathbf{a}_1$, etc.
  - Vector negation = multiplying by the scalar –1.
- Division by a scalar same as multiplication by the scalar multiplicative inverse.
  - i.e., multiply by 1/scalar

$$k \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} k = \begin{bmatrix} ka_1 \\ ka_2 \\ \vdots \\ ka_{n-1} \\ ka_n \end{bmatrix}$$

Vector Mult. by a Scalar: Geometry

- Multiplication of a vector **v** by a scalar *k* stretches **v** by a factor of *k*
  - In the same direction if *k* is positive.
  - In the opposite direction if *k* is negative.



Vector Addition and Subtraction

Vector Addition: Algebra

- Can add two vectors of the same dimension.
- Result is a vector of the same dimension.

- To add two vectors, add their components.
  - For example, if **a** + **b** = **c**, then $\mathbf{c}_1 = \mathbf{a}_1 + \mathbf{b}_1$, etc.
- Subtract vectors by adding the negative of the second vector, so **a** – **b** = **a** + (– **b**)

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_{n-1} + b_{n-1} \\ a_n + b_n \end{bmatrix}$$

addition

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} + \left(- \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} \right) = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ \vdots \\ a_{n-1} - b_{n-1} \\ a_n - b_n \end{bmatrix}$$

subtraction

Algebraic Identities

- Vector addition is associative.

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$$

- Vector addition is commutative.

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

- Vector subtraction is anti-commutative.

$$\mathbf{a} - \mathbf{b} = -(\mathbf{b} - \mathbf{a})$$

Vector Addition: Geometry

- To add vectors **a** and **b**: use the *triangle rule*.

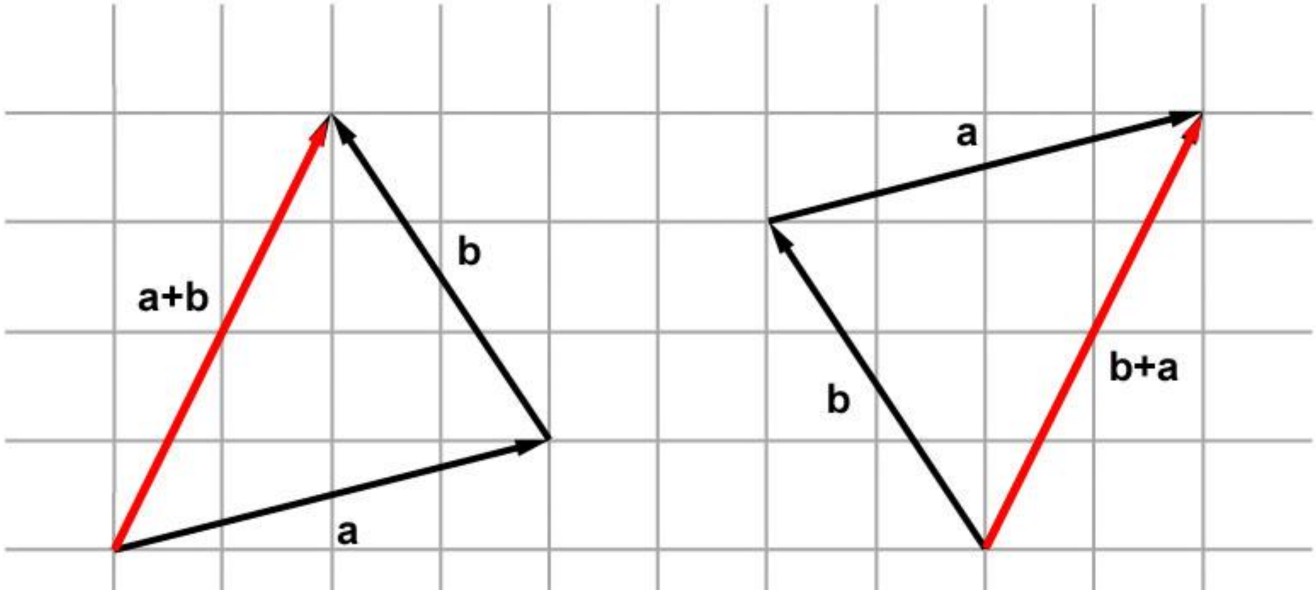
- Place the tail of **a** on the head of **b**.
- **a** + **b** is the vector from the tail of **b** to the head of **a**.

- Or the other way around: we can swap the roles of **a** and **b**.
  - because vector addition is commutative.

Triangle Rule for Addition

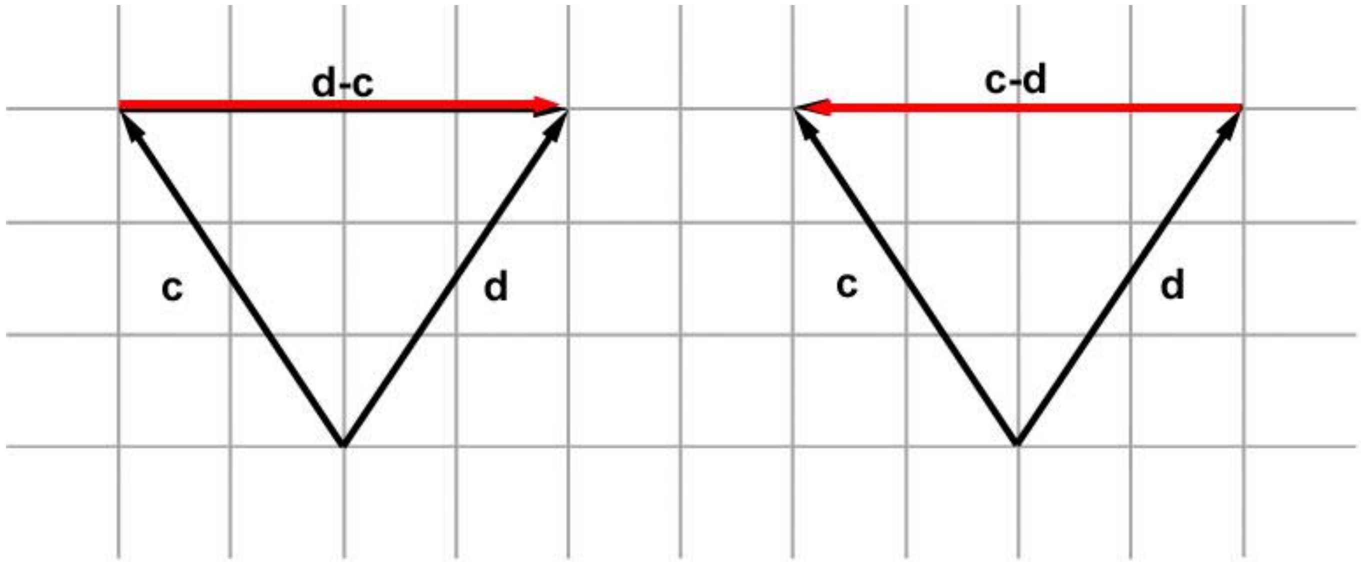
- Algebra: [4, 1] + [-2, 3] = [2, 4]

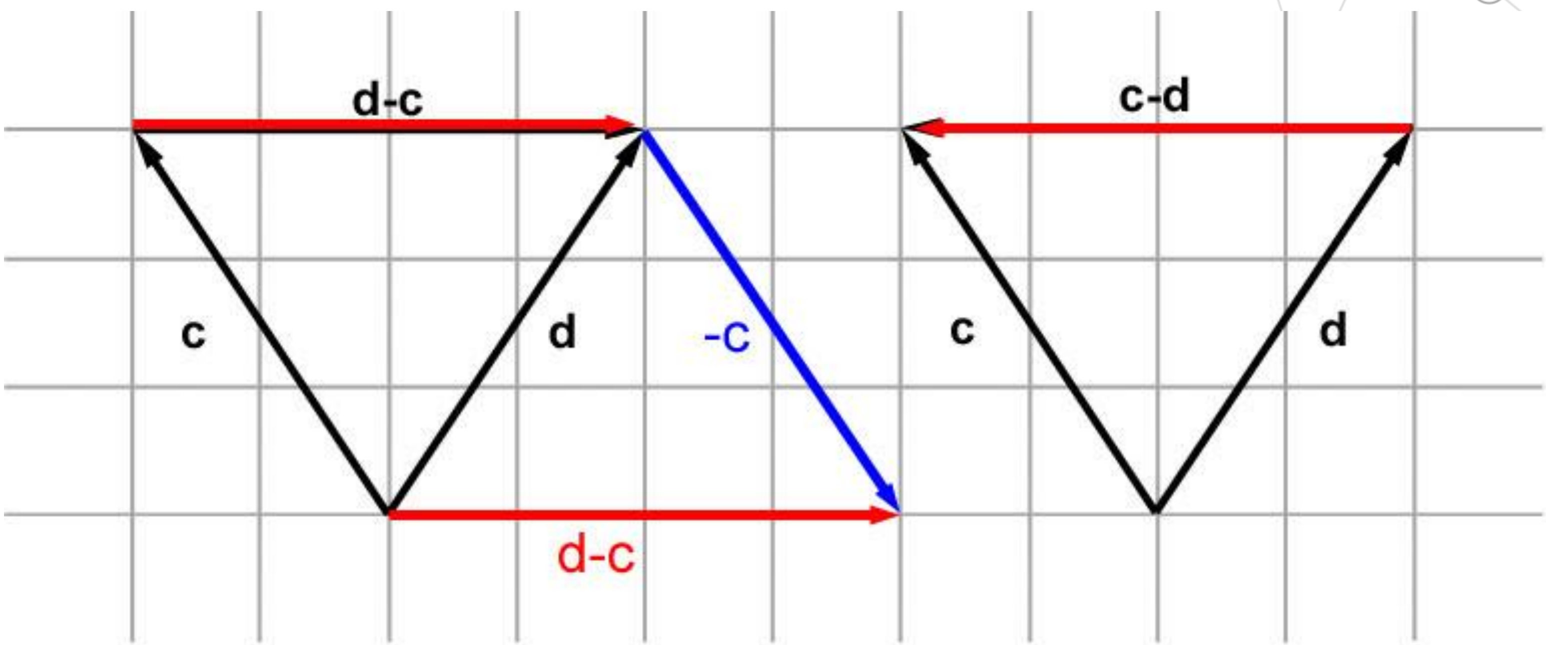
- Geometry:

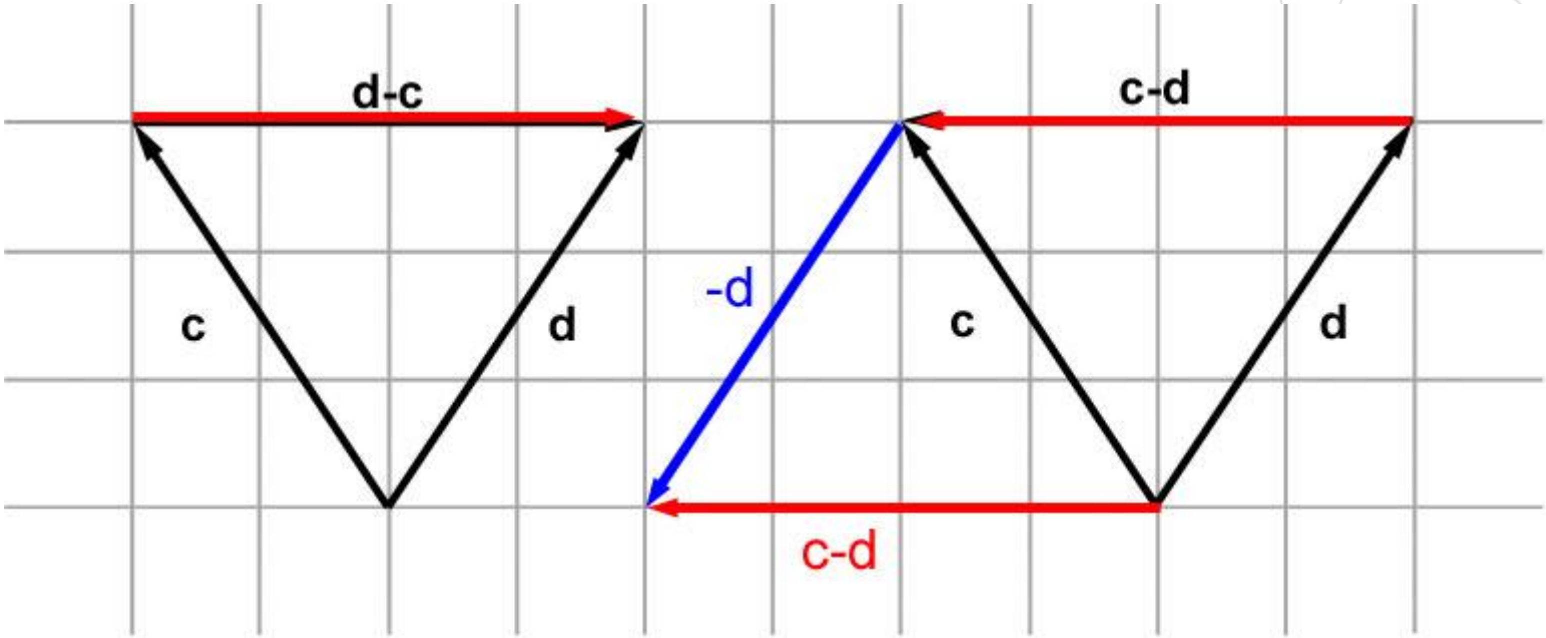


Triangle Rule for Subtraction

- Place **c** and **d** tail to tail.
- **c – d** is the vector from the head of **d** to the head of **c** (head-positive, tail-negative).

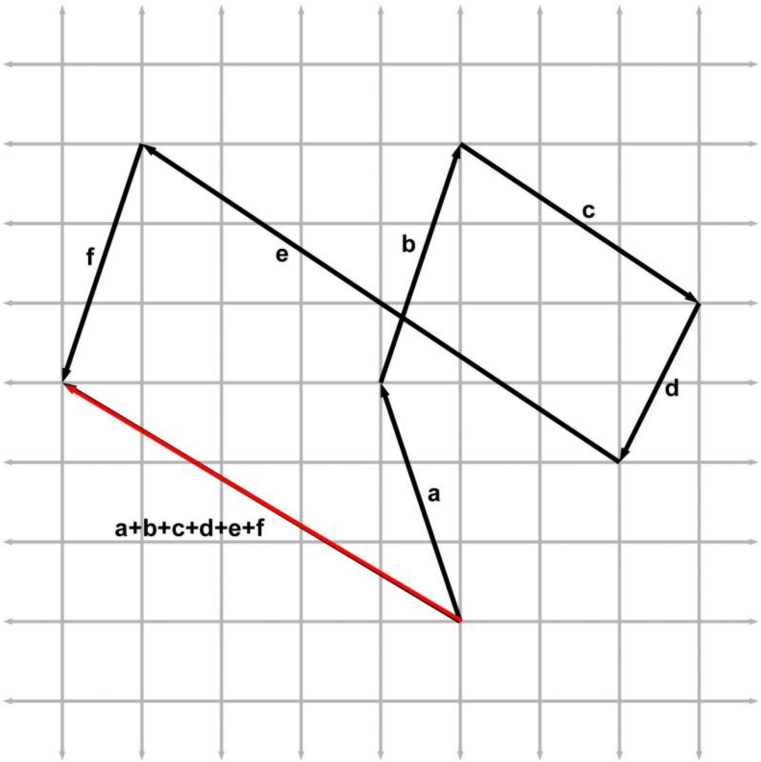



d-c
c
d
-c
d-c
c-d
c
d


d-c
c
d
c-d
-d
c
d
c-d

Adding Many Vectors

- Repeat the triangle rule as many times as necessary?
- Result: string all the vectors together.
  - Should we call this the polygon rule or the multitriangle rule?



Vector Displacement

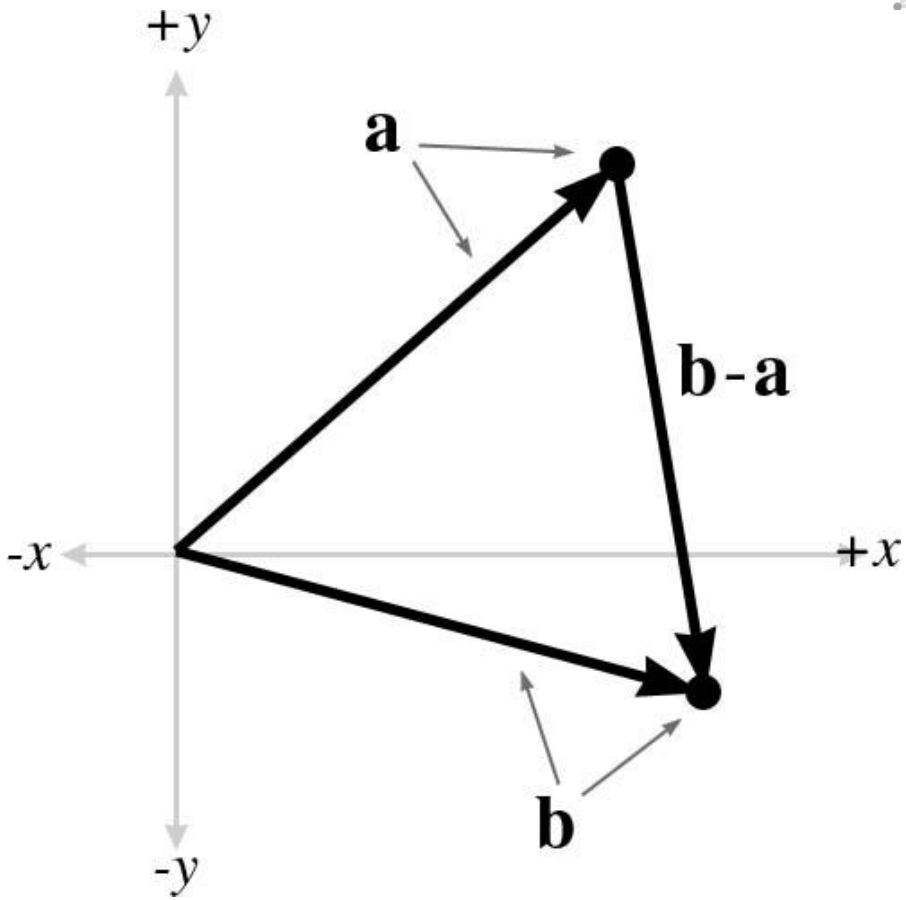
Vector Displacement: Algebra

- Here's how to get the vector displacement from point *a* to point *b*.

- Let **a** and **b** be the vectors from the origin to the respective points.

- The vector from *a* to *b* is **b** – **a** (the destination is positive)

Vector Displacement: Geometry



Vector Magnitude

Vector Magnitude: Algebra

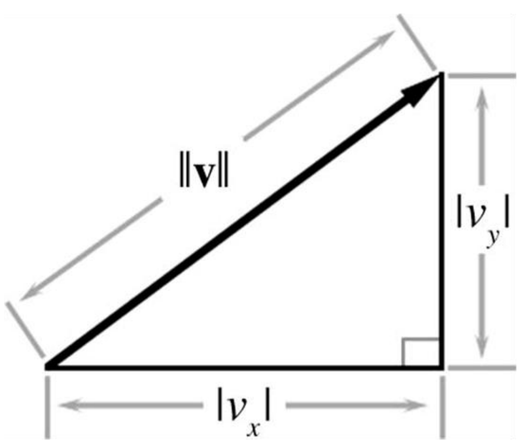
- The magnitude of a vector is a scalar.
  - Also called the “norm”.

- It is always positive

$$\|\mathbf{v}\| = \sqrt{\sum_{i=1}^{n} {v_i}^2} = \sqrt{{v_1}^2 + {v_2}^2 + \cdots + {v_{n-1}}^2 + {v_n}^2}$$

Vector Magnitude: Geometry

- Magnitude of a vector is its length.
- Use the Pythagorean theorem.
- In the next formulas, two vertical lines ||**v**|| means "magnitude of a vector **v**", one vertical line |$\mathbf{v}_x$| means "absolute value of a scalar $\mathbf{v}_x$"



$$\|\mathbf{v}\|^2 = |v_x|^2 + |v_y|^2$$

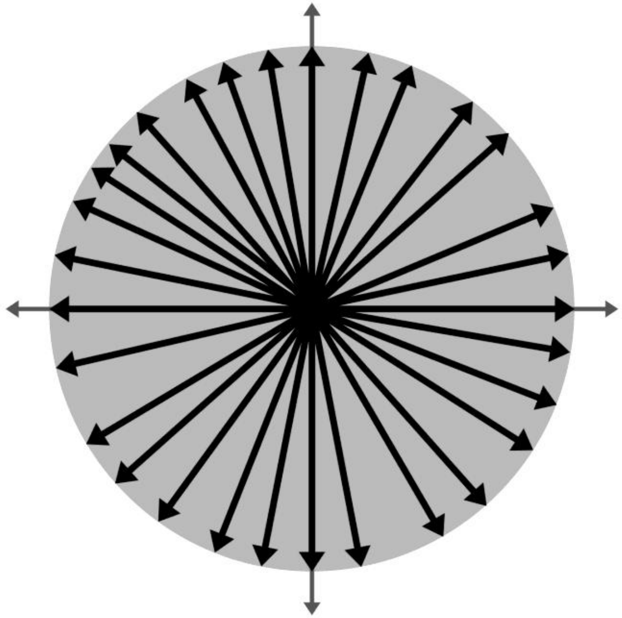
$$\|\mathbf{v}\|^2 = {v_x}^2 + {v_y}^2$$

$$\sqrt{\|\mathbf{v}\|^2} = \sqrt{{v_x}^2 + {v_y}^2}$$

$$\|\mathbf{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$$

Observations

- The zero vector has zero magnitude.
- There are an infinite number of vectors of each magnitude (except zero).



Unit Vectors

Normalization: Algebra

- A *normalized* vector always has unit length.
- To normalize a nonzero vector, divide by its magnitude.

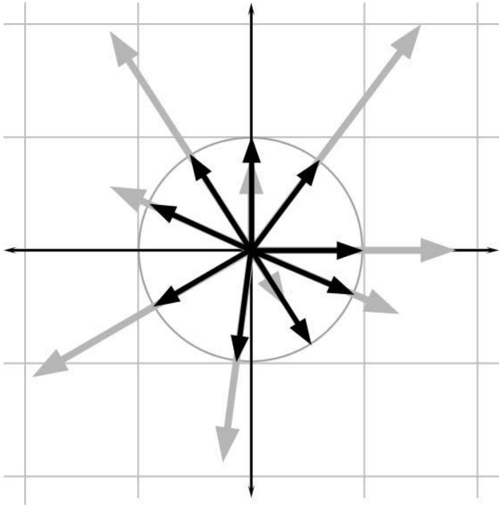
$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

- Normalize [12, -5]:

$$\frac{\begin{bmatrix}12 & -5\end{bmatrix}}{\left\|\begin{bmatrix}12 & -5\end{bmatrix}\right\|} = \frac{\begin{bmatrix}12 & -5\end{bmatrix}}{\sqrt{12^2+5^2}} = \frac{\begin{bmatrix}12 & -5\end{bmatrix}}{\sqrt{169}} = \frac{\begin{bmatrix}12 & -5\end{bmatrix}}{13} = \begin{bmatrix}\frac{12}{13} & \frac{-5}{13}\end{bmatrix}$$
$$\approx \begin{bmatrix}0.923 & -0.385\end{bmatrix}$$

Normalization: Geometry

- Normalization keeps the angle difference between vectors



- Useful operation but needs care when implemented in a computer program.
  - You need to deal with approximation errors and “aliasing”

The Distance Formula

Application: Computing Distance

- To find the geometric distance between two points a and b.
- Compute the vector **d** from **a** to **b**.
- Compute the magnitude of **d**.
- We know how to do both of those things.

Vector Dot Product

Dot Product: Algebra

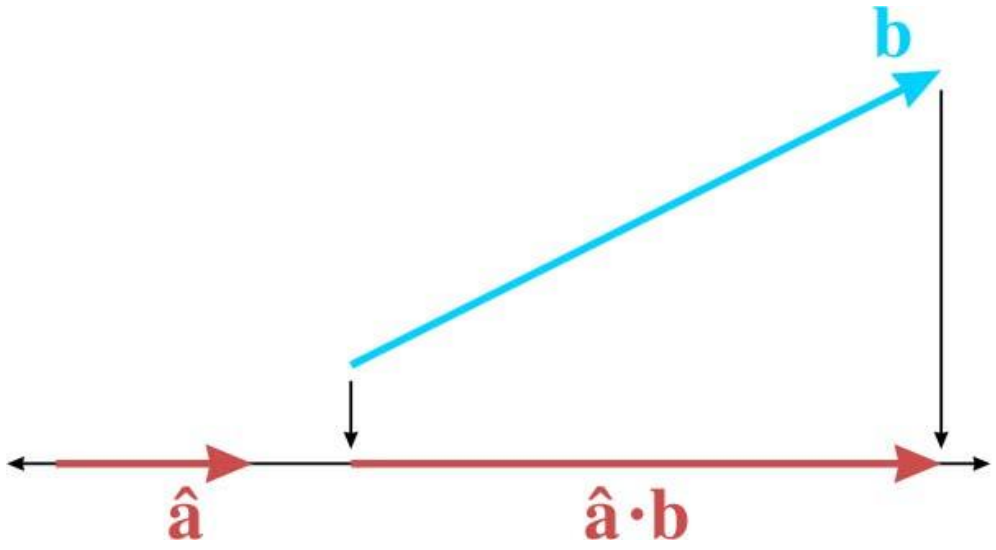
- Can take the dot product of two vectors of the same dimension.
- The result is a scalar.

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i$$

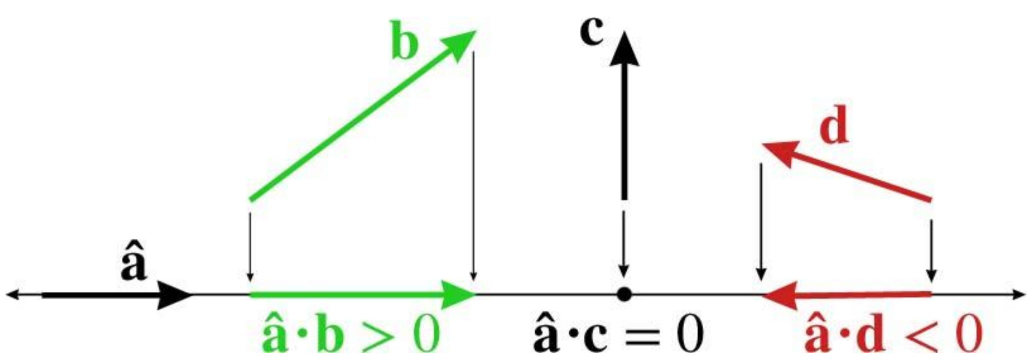
$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_{n-1} b_{n-1} + a_n b_n$$

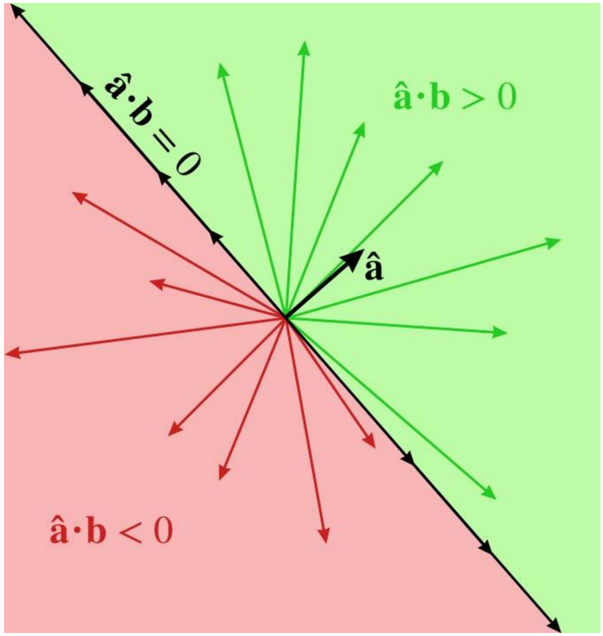
Dot Product: Geometry

- Dot product is the magnitude of the projection of one vector onto another.



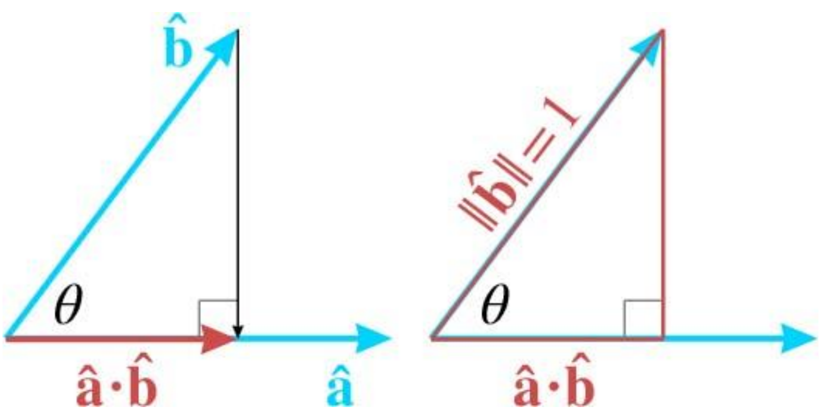
Sign of Dot Product





Dot Product: Geometry

- Dot product can be used to find the angle between two vectors a and b.
- First normalize a and b.
- The angle between them is acos $\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$
  - Arccos, or acos, or $\cos^{-1}$ is the inverse of cos



$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}}{1} = \hat{\mathbf{a}} \cdot \hat{\mathbf{b}}.$$

$$\theta = \operatorname{acos}\left(\frac{\text{adjacent}}{\text{hypothenuse}}\right) = \cdots = \operatorname{acos}(\hat{\mathrm{a}} \cdot \hat{\mathrm{b}})$$

Sign of Dot Product

$\mathbf{a} \cdot \mathbf{b}$	θ	**Angle is**	**a and b are**
> 0	$0^{\circ} \leqslant \theta < 90^{\circ}$	acute	pointing mostly in the same direction
0	$\theta = 90^{\circ}$	right	perpendicular
< 0	$90^{\circ} < \theta \leqslant 180^{\circ}$	obtuse	pointing mostly in the opposite direction

Vector Cross Product

Cross Product: Algebra

- Can take the cross product of two vectors of the same dimension.
- Result is a vector of the same dimension.

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

- Remember this formula, no easy way

Cross Pattern

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

Cross Product: Geometry

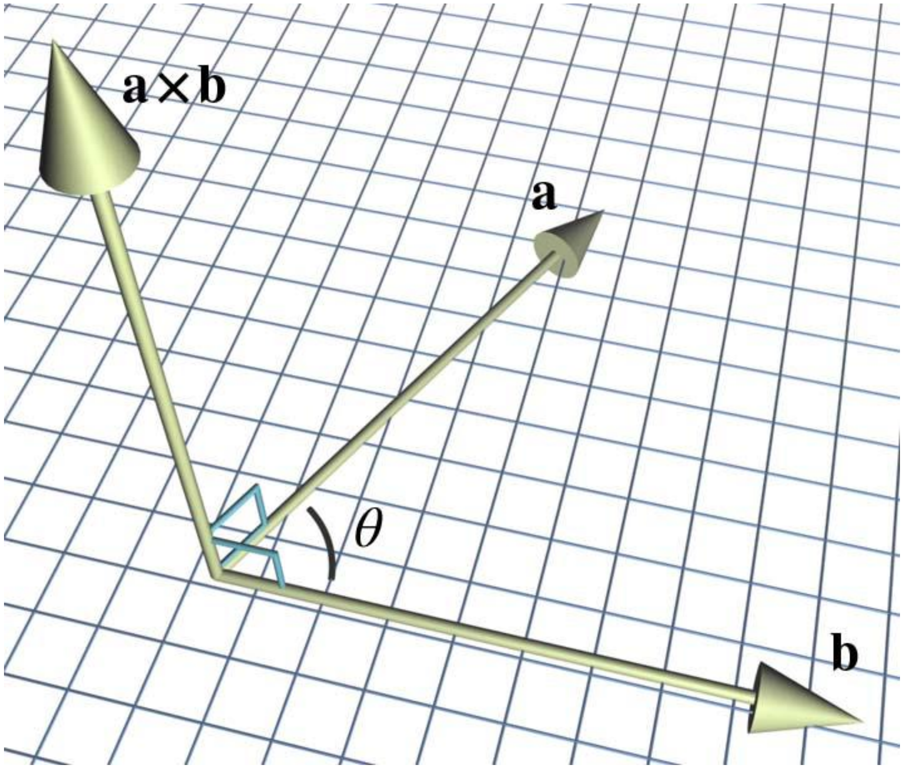
- Given 2 nonzero vectors **a**, **b**.
- They are (must be) coplanar.

- The cross product of **a** and **b** is a vector perpendicular to the plane of **a** and **b**.

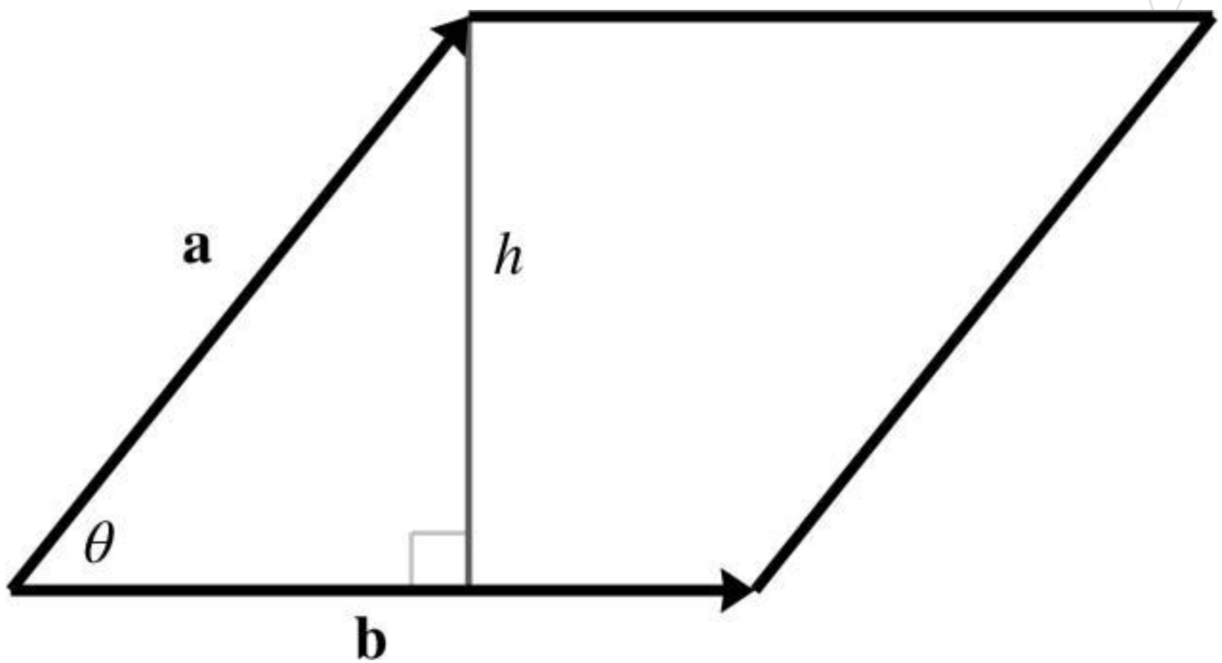
- The magnitude is related to the magnitude of **a** and **b** and the angle between **a** and **b**.

- The magnitude is equal to the area of a parallelogram with sides **a** and **b**.

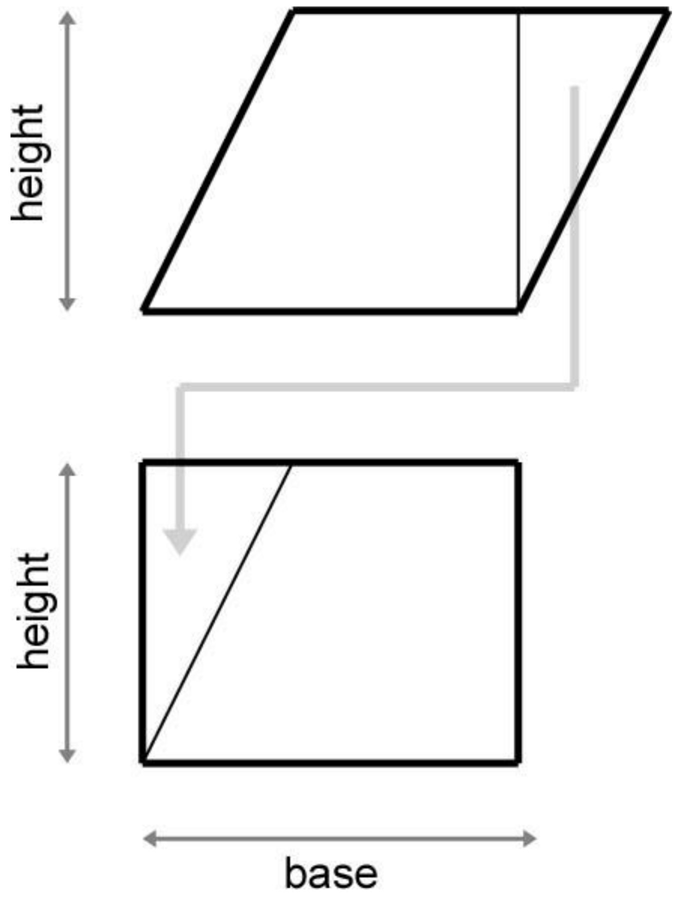
$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin\theta$$



Area of this parallellogram is ||**b**|| h



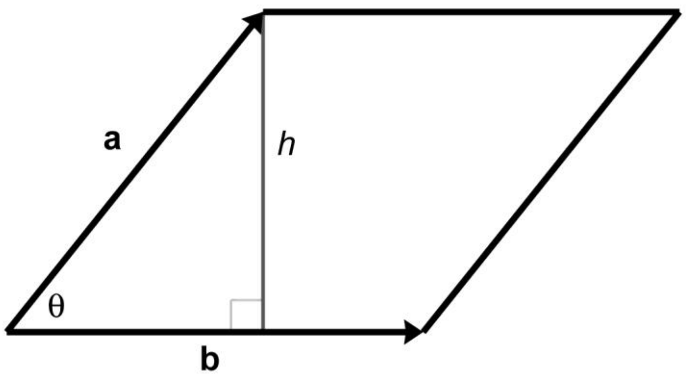
Aside: Here’s Why



Catch Your Breath

- Are you OK with the fact that the area of a parallelogram is its base times its height measured perpendicularly to the base?
- Now we'll show that the area is

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta$$



$\sin\theta = h/||\mathbf{a}||$

$h = ||\mathbf{a}|| \sin\theta$

$||\mathbf{a}\mathrm{x}\mathbf{b}||=||\mathbf{b}||\ h$

$\quad = ||\mathbf{a}||\ ||\mathbf{b}|| \sin\theta$

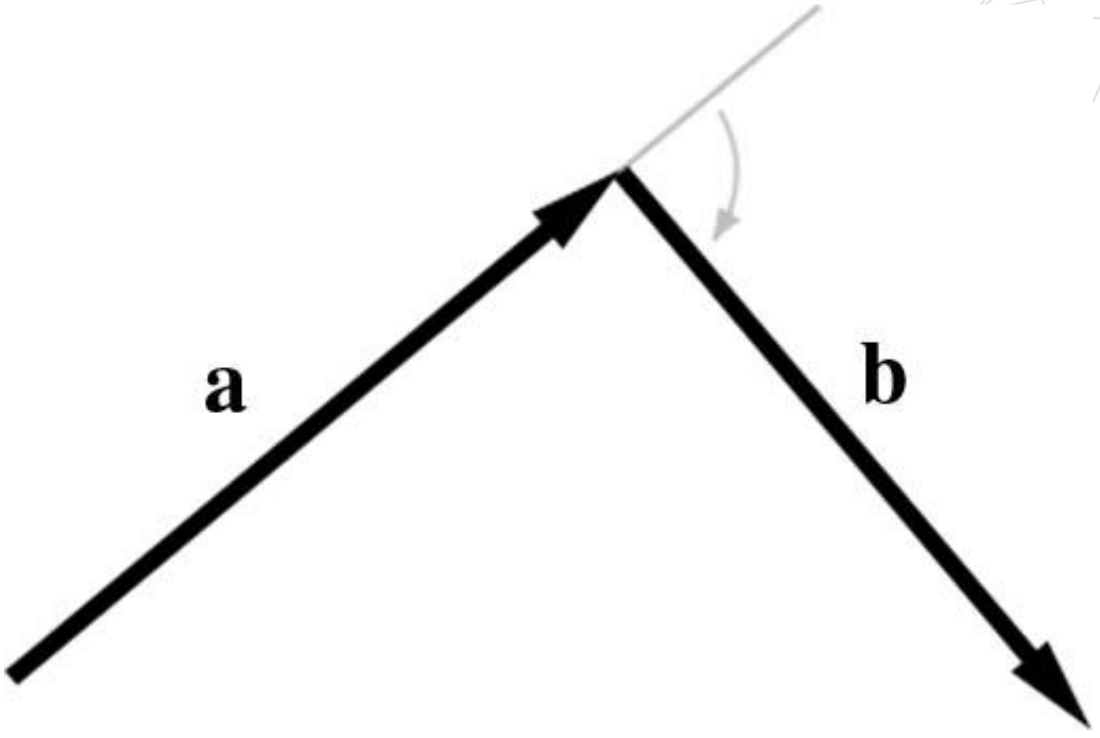
What About the Orientation?

- That’s taken care of the magnitude. Now for the direction.

- Does the vector **a** x **b** point up or down from the plane of **a** and **b**?
- Place the tail of **b** at the head of **a**.
- Look at whether the angle from **a** to **b** is clockwise or counterclockwise.

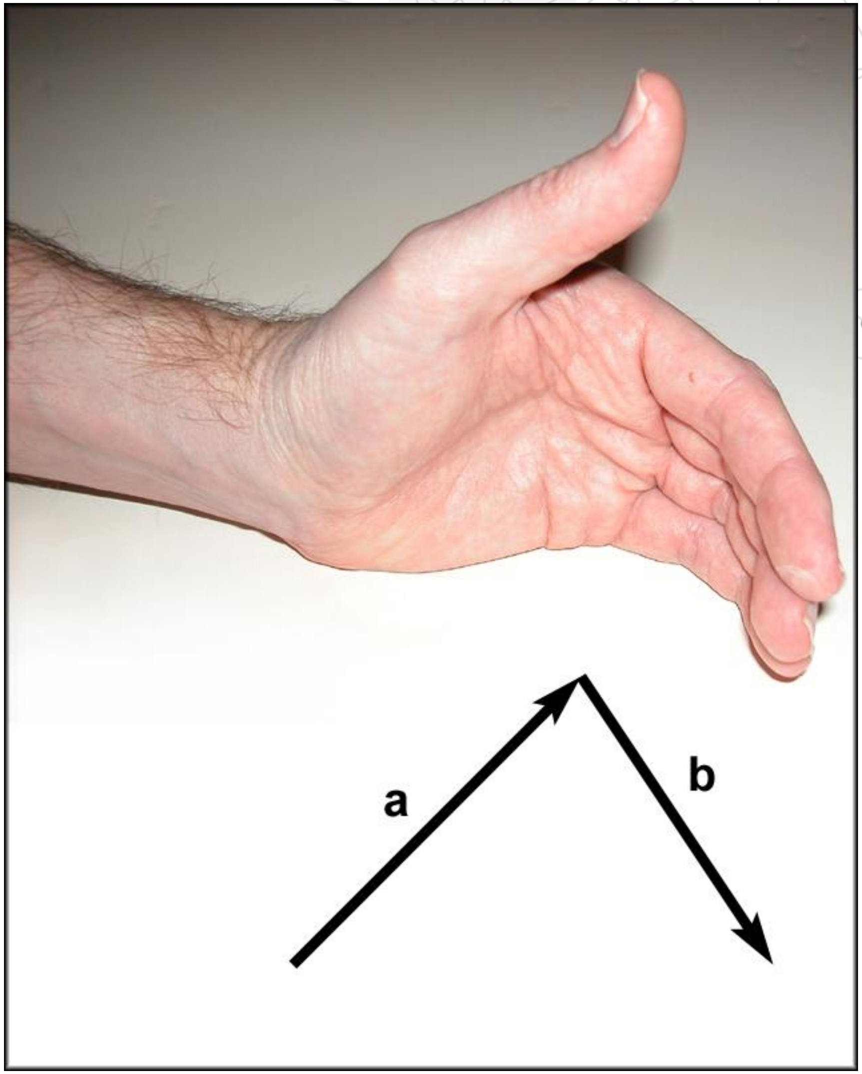
- The result depends on whether coordinate system is left- or right-handed.

Left-handed coordinate system

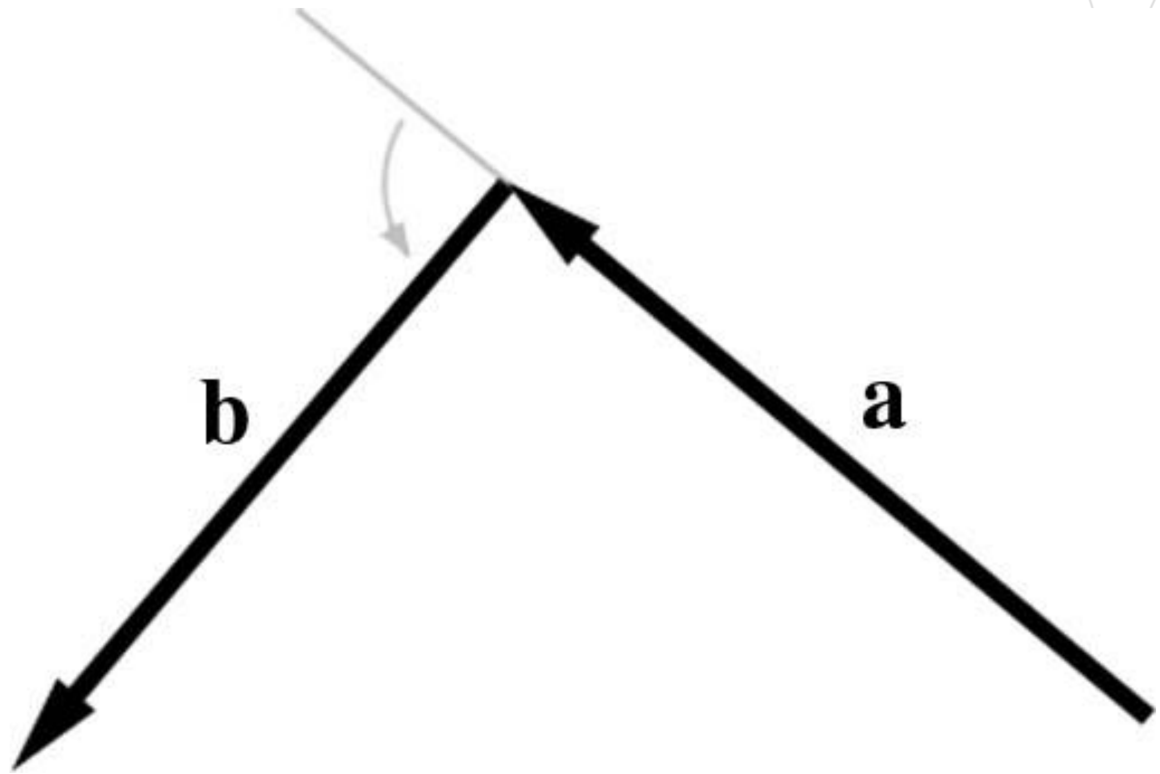


Left-handed coordinate system

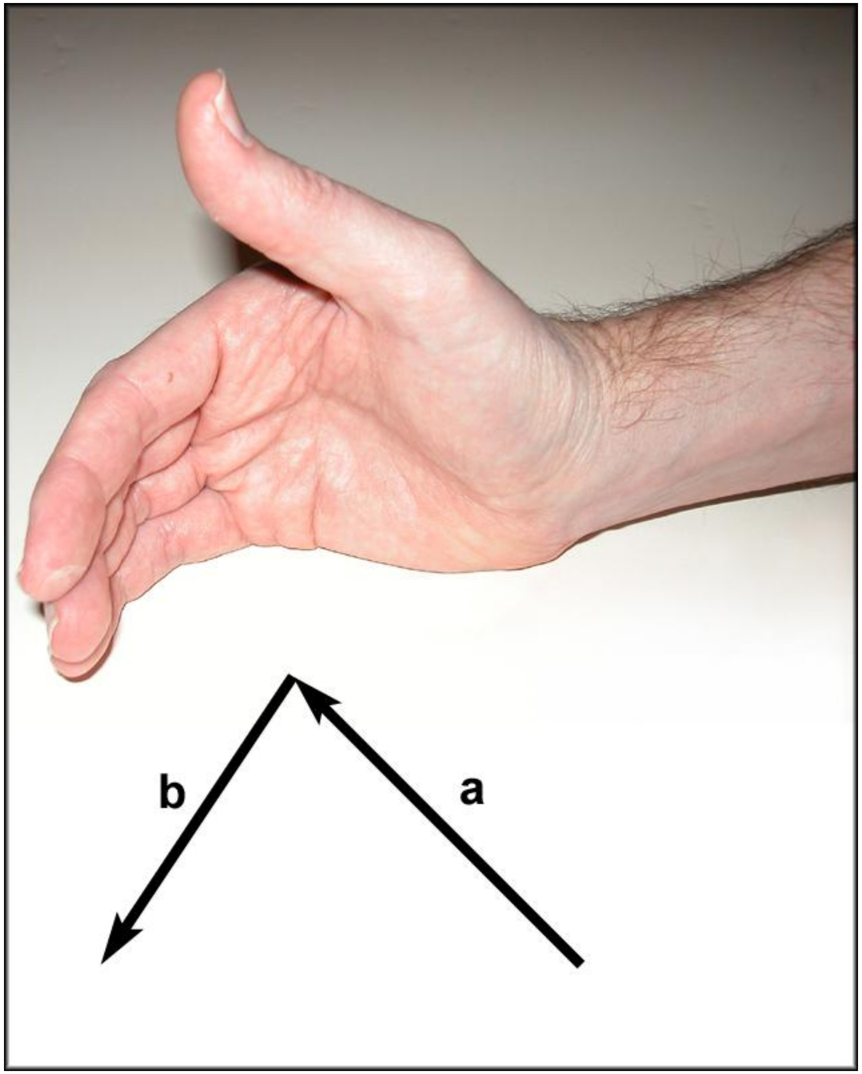
- In a left-handed coordinate system, use your left hand.
- Curl fingers in direction of vectors
- Thumb points in direction of **a x b**



Right-handed coordinate system



Right-handed coordinate system



- In a right-handed coordinate system, use your right hand
- Curl fingers in direction of vectors
- Thumb points in direction of **a x b**

Corollary

- In a left-handed coordinate system, list your triangles in clockwise order.

- Then you can compute a *surface normal* (a unit vector pointing out from the face of the triangle) by taking the cross product of two consecutive edges.
  - Very important in computer graphics to apply the correct light sources that generate shadings and other visual effects.

Computing a Surface Normal

- Given a triangle with points *a*, *b*, *c*.

- Compute the vector displacement from *a* to *b*, and the vector from *b* to *c*.

- Take their cross product.

- Normalize the resulting surface normal.

- WARNING: some modeling programs may output zero-width triangles: these have a zero cross product. Don't normalize it.

Facts About Dot and Cross Product

- If $\mathbf{a} \cdot \mathbf{b} = 0$, then **a** is perpendicular to **b**.

- If **a** x **b** = **0**, then **a** is parallel to **b**.

- Dot product interprets every vector as being perpendicular to **0**.
- Cross product interprets every vector as being parallel to **0**.