GAME2016
Mathematical Foundation of Game Design and Animation

Lecture 1

Cartesian Coordinate Systems

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Agenda

- 1D Mathematics
- 2D Cartesian mathematics
- 3D Cartesian mathematics
- Angles and Trigonometric functions

1D Mathematics

Introduction

- 3D math is all about measuring
  - Locations
  - Distances
  - Angles
- precisely and mathematically in 3D space.

- The most frequently used framework to perform such calculations using a computer is called the Cartesian coordinate system.

René Descartes, 1596 - 1650

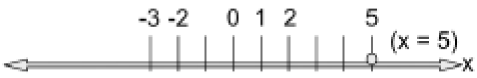
- Cartesian mathematics was invented by René Descartes (1596–1650)
  - French philosopher, physicist, physiologist, and mathematician.

- Cartesian mathematics is a unification of algebra and geometry.

Source: Frans Hals, Portrait of René Descartes, Wikimedia Commons

1D Mathematics

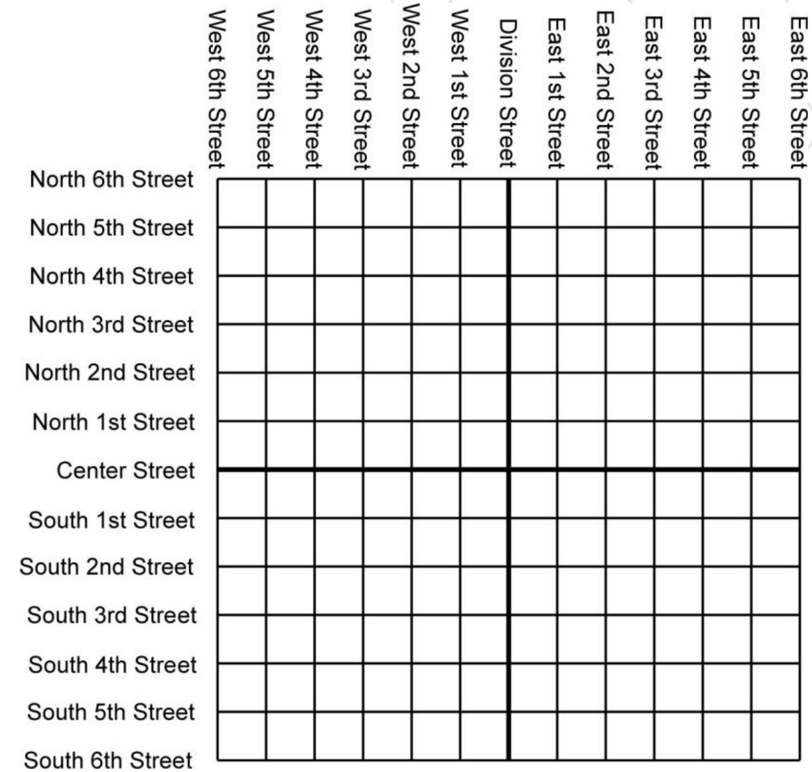
- Assumptions
- What are natural numbers, integers, rational numbers, and real numbers.
  - Corresponding to short, int, float, and double on a computer (with a limited precision).
- Basic understanding about how numbers are represented on a computer.
  - Remember the First Law of Computer Graphics: If it looks right, it is right.
- See below a Cartesian plots of x = 5 in one dimension



2D Cartesian Space

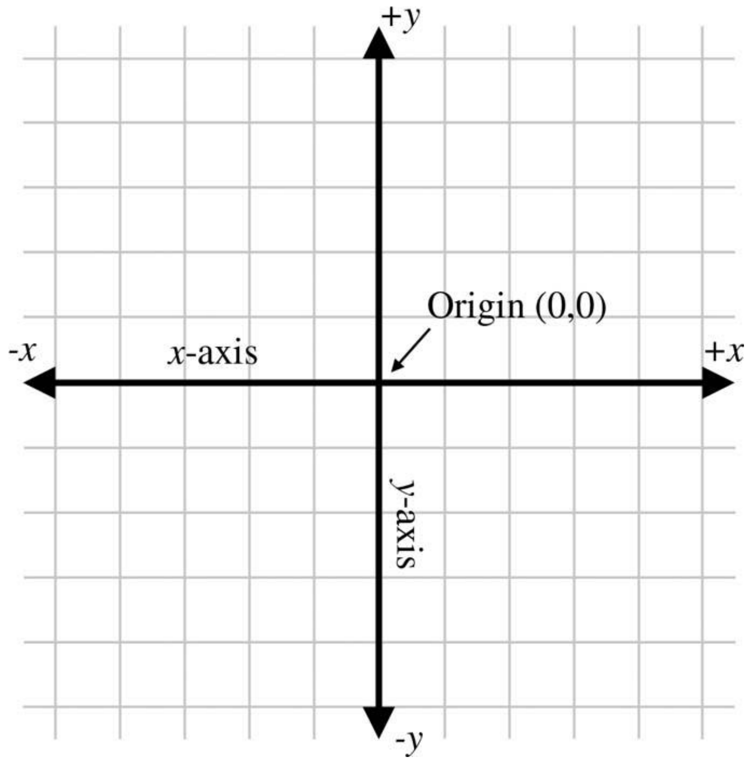
The City of Cartesia

- Center Street runs east-west through the middle of town
  - All other east-west streets are named based on whether they are north or south of Center Street, and how far away they are from Center Street.
  - Examples of streets which run east-west are North 3rd Street and South 15th Street.
- Division Street runs north-south through the middle of town.
  - All other north-south streets are named based on whether they are east or west of Division street, and how far away they are from Division Street.
  - So, we have streets such as East 5th Street and West 22nd Street.



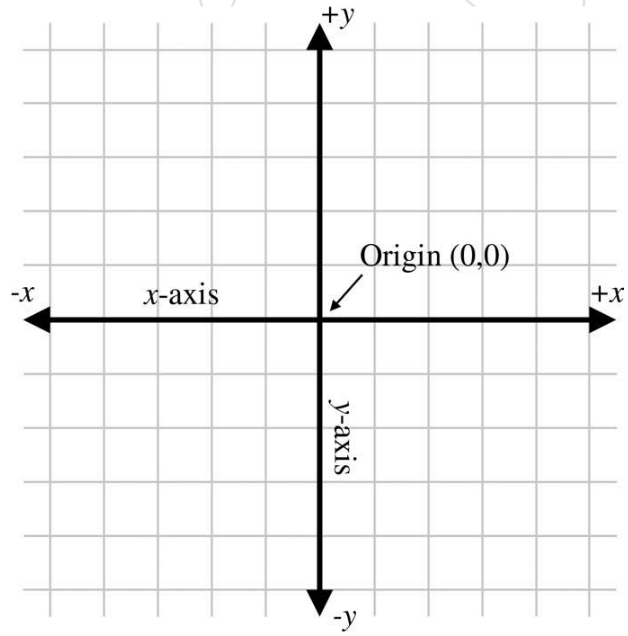
2D Coordinate Spaces

- All that really matters are the numbers.
- The abstract version of this is called a 2D Cartesian coordinate space.



Origin and Axes

- Every 2D Cartesian coordinate space has a special location, called the *origin*, which is the center of the coordinate system.
  - The origin is analogous to the center of the city in Cartesia.
- Every 2D Cartesian coordinate space has two straight lines that pass through the origin. Each line is known as an *axis* and extends infinitely in both directions.
  - The two axes are perpendicular to each other.
    - They don't have to be, but most of the coordinate systems we will look at will have perpendicular axes.

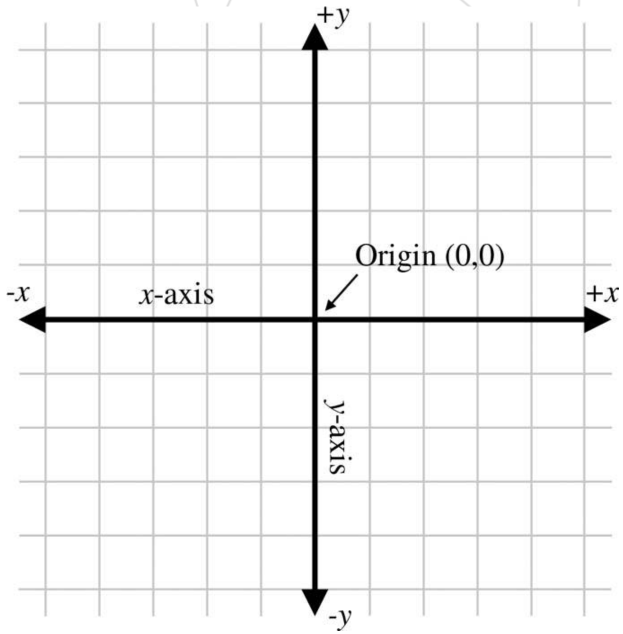


Axes

- The horizontal axis is called the *x*-axis, with positive *x* pointing to the right, and the vertical axis is the *y-axis*, with positive *y* pointing up.

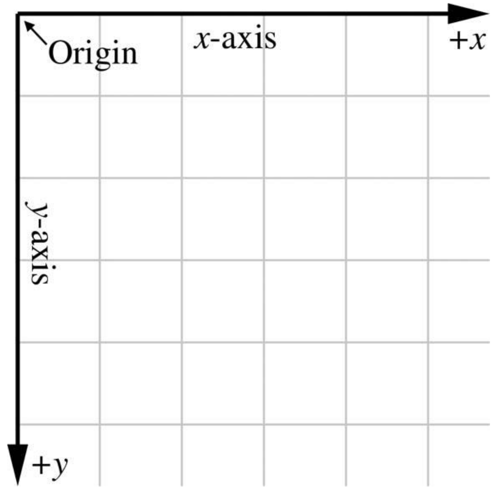
- This is the customary orientation for the axes in a diagram.

- But it doesn't have to be this way. It's only a convention.



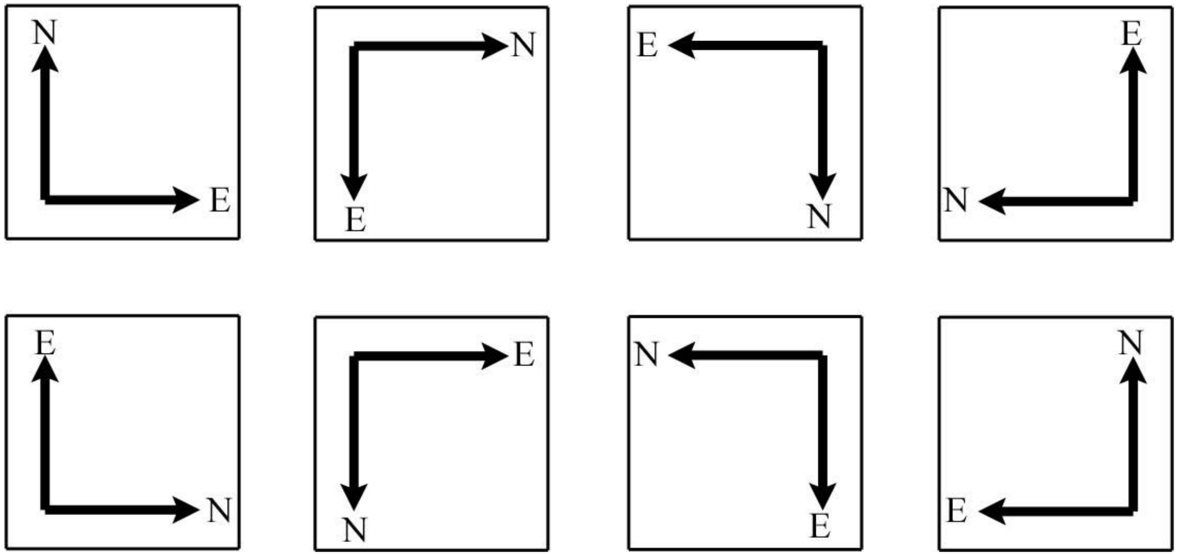
Screen Space

- In screen space, for example, $+y$ points down.
- Screen space is how you measure on a computer screen, with the origin at the top left corner.



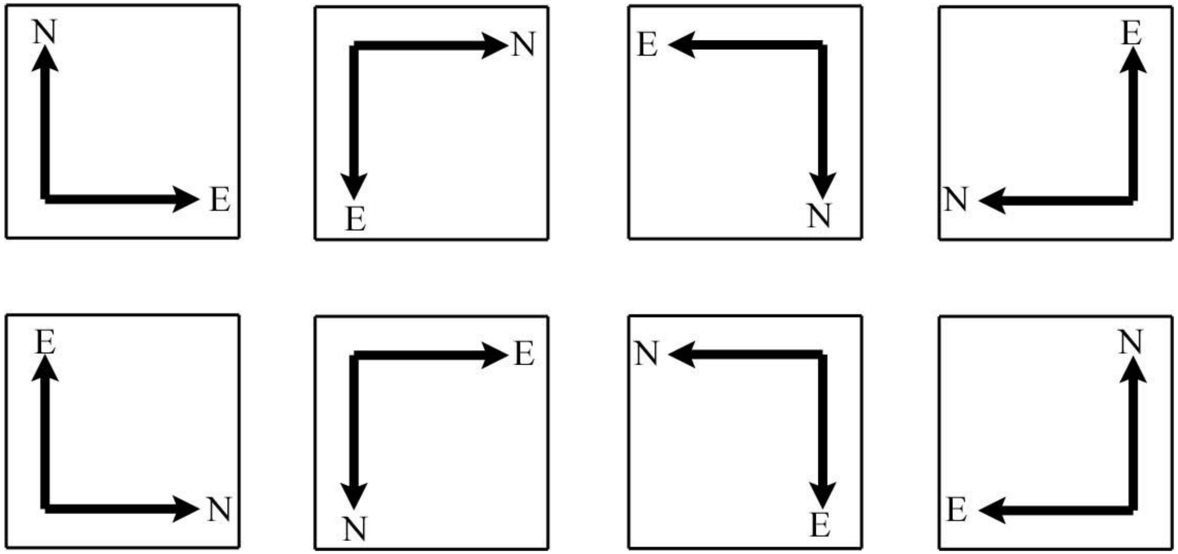
Axis Orientation

- There are 8 possible ways of orienting the Cartesian axes.
  - You need to know this to adjust your math



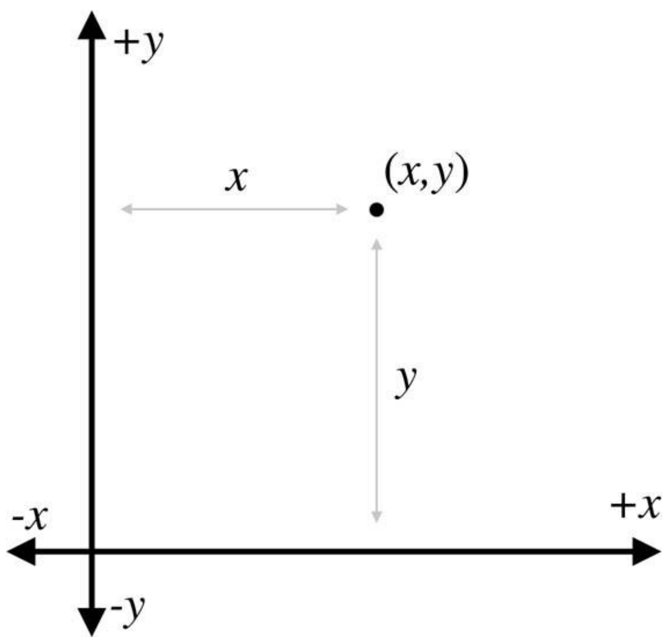
Axis Equivalence in 2D

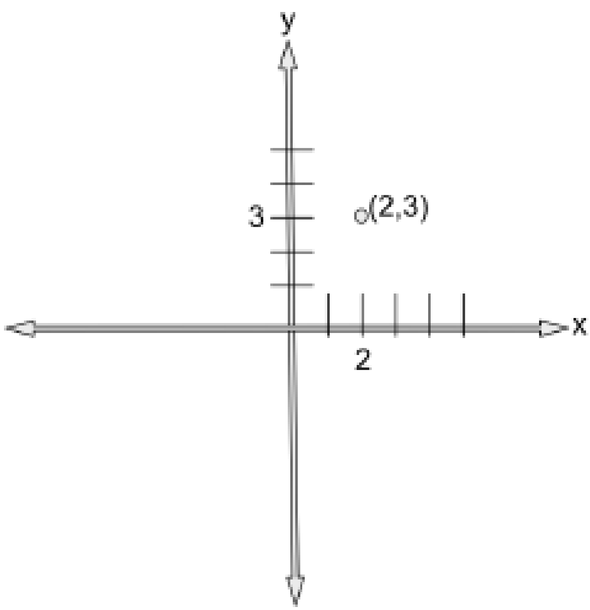
- These 8 alternatives can be obtained by rotating the map around 2 axes (any 2 will do).
  - This is not true of 3D coordinate space



Locating Points in 2D

- Point (*x*,*y*) is located *x* units across and *y* units up from the origin.

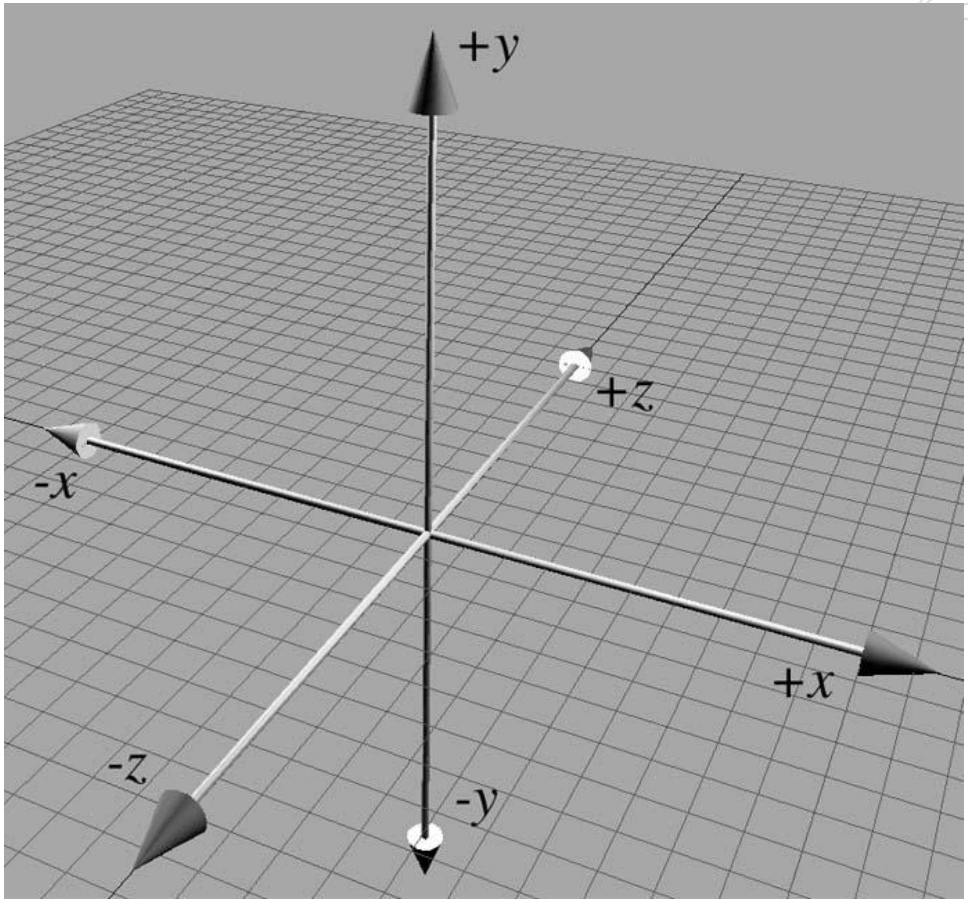




Cartesian plot of (2,3) in two dimensions

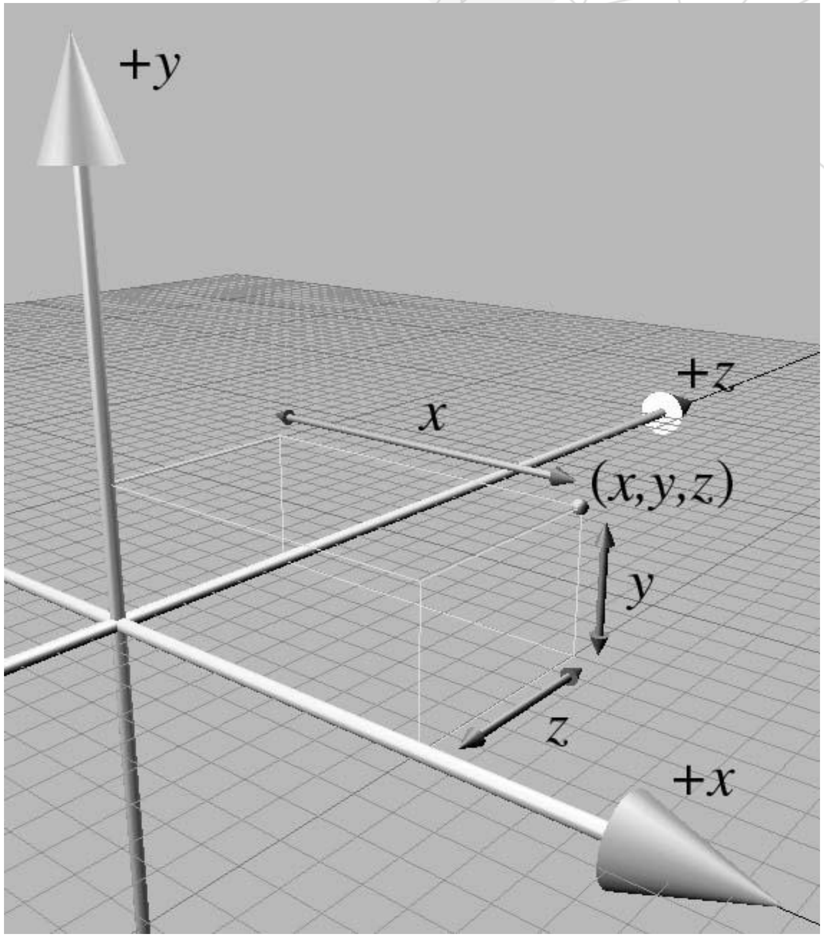
3D Cartesian Space

3D Cartesian Space



Locating Points in 3D

- Point (x,y,z) is located
  - x units along the x-axis,
  - y units along the y-axis,
  - z units along the z-axis
- from the origin.

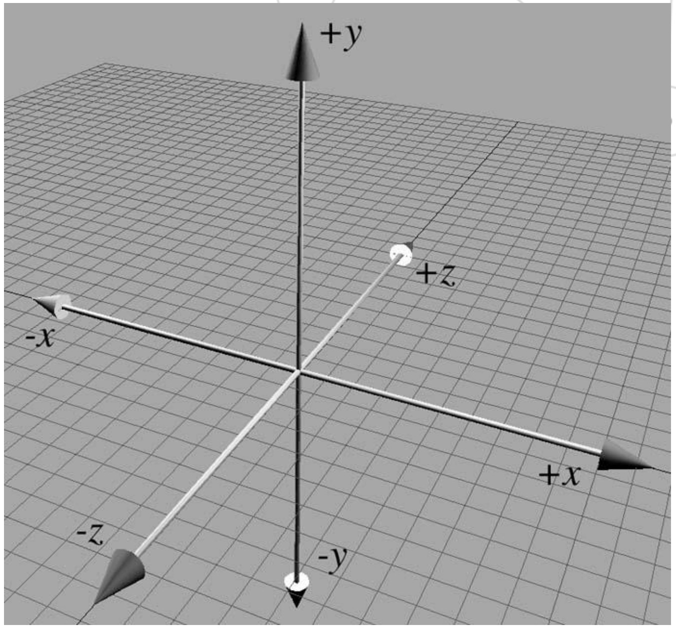


Visualizing 3D Space

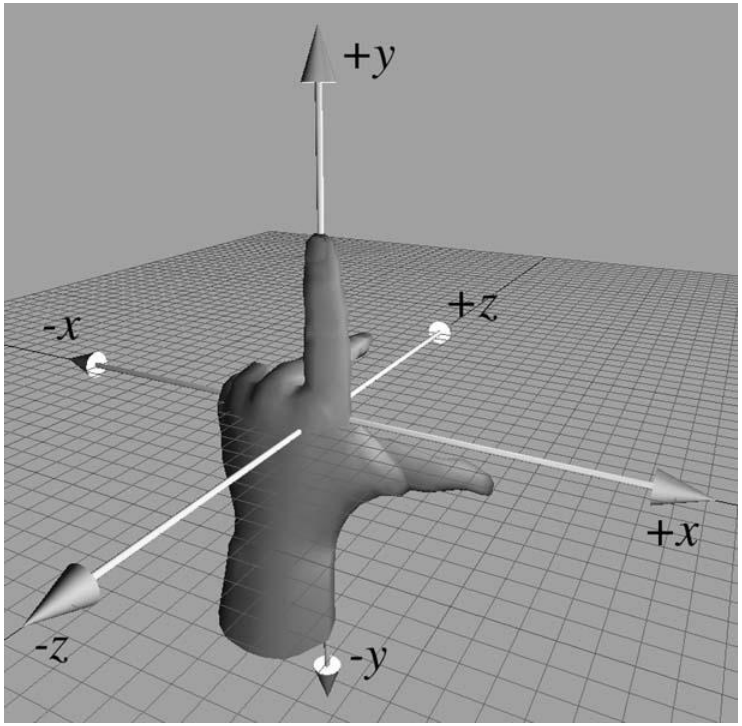
- The usual convention is that the *x*-axis is horizontal and positive is right, and that the *y*-axis is vertical and positive is up

- The *z*-axis is depth, but should the positive direction go
  - forwards “into” the screen
  - or backwards “out from” the screen?

- No correct answer!
  - No standard
  - Just use your hands

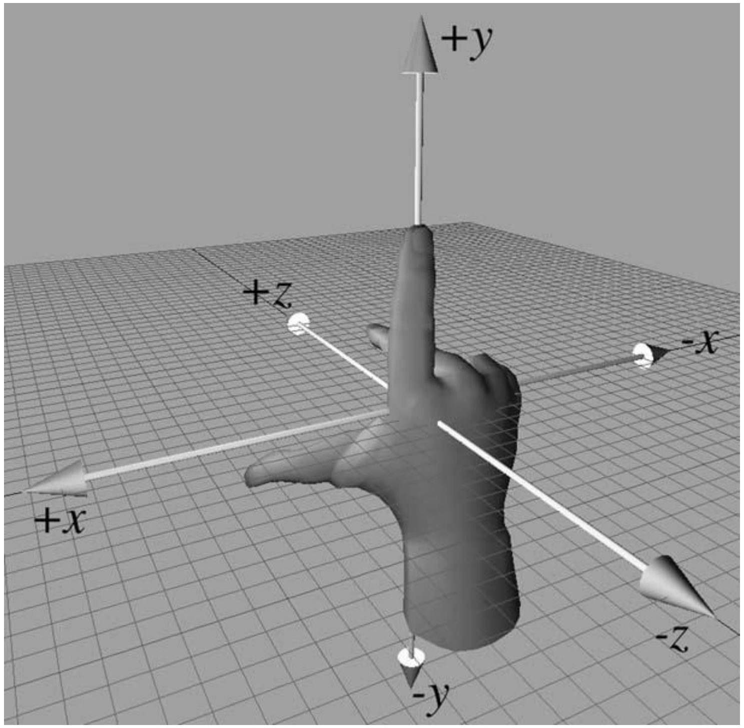


Left-handed Coordinates



- +*z* goes "into" screen
- Use your left hand
- Thumb is +*x*
- Index finger is +*y*
- Second finger is +*z*

Right-handed Coordinates



- +*z* goes "out from" screen
- Use your right hand
- Thumb is +*x*
- Index finger is +*y*
- Second finger is +*z*
- (Same fingers, different hand)

Changing Conventions

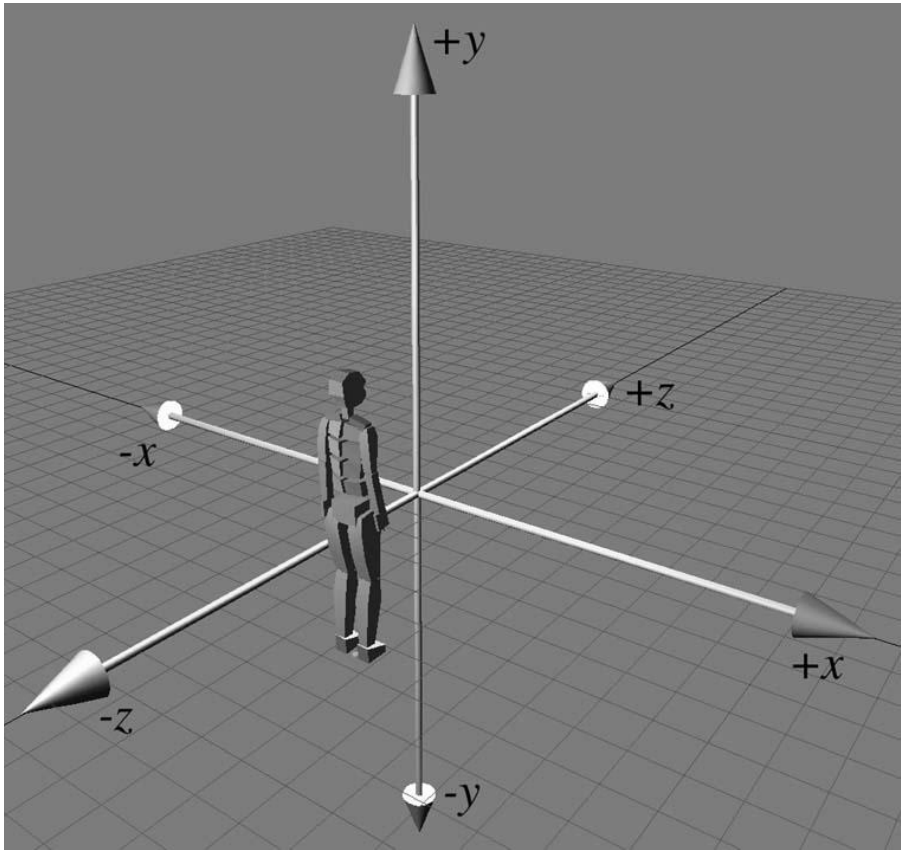
- To swap between left and right-handed coordinate systems, negate the *z*.

- Left-handed systems
  - Graphics books usually use left-handed
  - Unity3D graphics engine
  - OpenGL graphics library (in window space)

- Right-handed systems
  - Linear algebra books usually use right-handed
  - DirectX graphics library
  - Blender 3D modeling software
  - OpenGL graphics library (in world and object space)

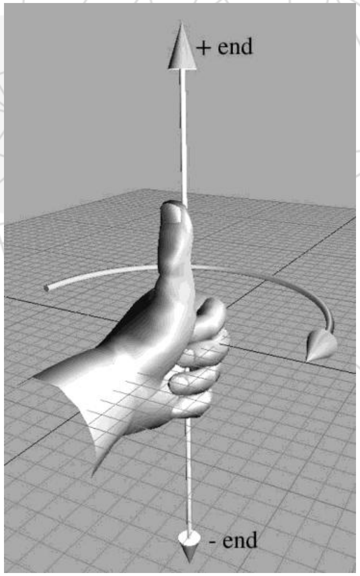
Our Convention

- We'll use left-handed.

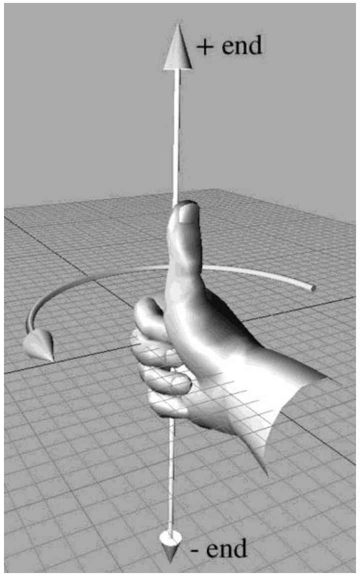


Positive Rotation

- How to visualize a rotation
  - Use your left hand for a left-handed coordinate space
  - Use your right hand for a right-handed coordinate space.



- Point your thumb in the positive direction of the axis of rotation
  - Note: it may not be one of the principal axes.



- Your fingers curl in the direction of positive rotation.

Angles and Trigonometric functions

Odds and Ends of Math Used

- Summation and product notation:

$$\sum_{i=1}^{6} a_i = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$$

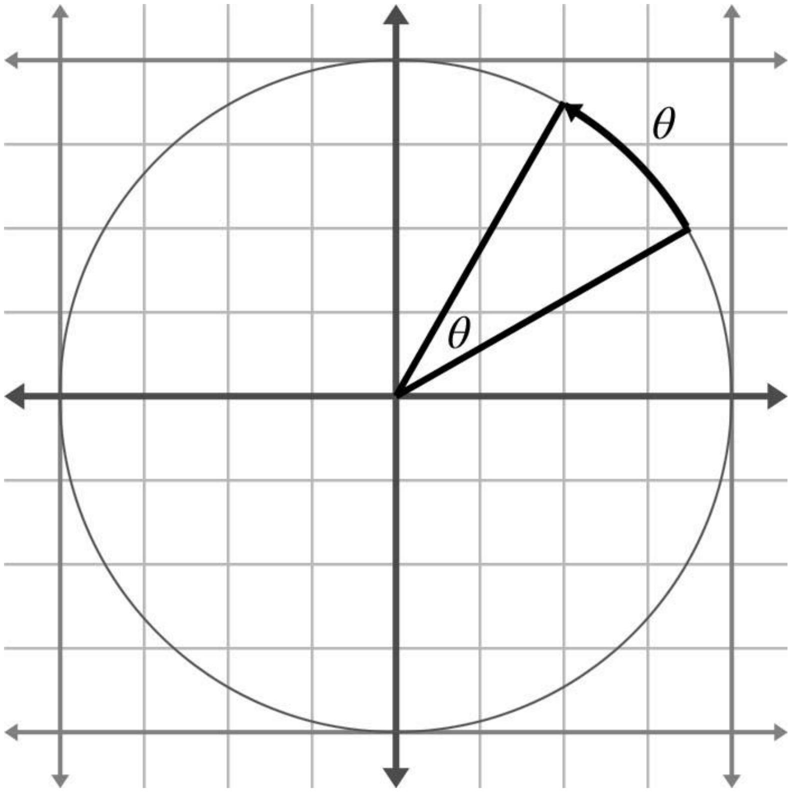
$$\prod_{i=1}^{n} a_i = a_1 \times a_2 \times \cdots \times a_{n-1} \times a_n$$

Angles

- An angle measures an amount of rotation in the plane.
  - Variables for angles are often given the Greek letter θ (theta).
- Measured in degrees (°) and radians (rad)
- Humans usually measure angles using degrees.
  - One degree measures 1/360th of a revolution.
  - 360° is a complete revolution.
- Mathematicians, prefer to measure angles in radians.
  - Based on the properties of a circle.

θ Radians

- When we specify the angle between two rays in radians, we are measuring the length of the intercepted arc of a unit circle
  - The angle θ in figure



Radians and Degrees

- The circumference of a unit circle is 2π radians
  - Circumference = 2πr = 2π1 = 2π
- 2π radians represents a complete revolution
- 360° = 2π rad
- 180° = π rad

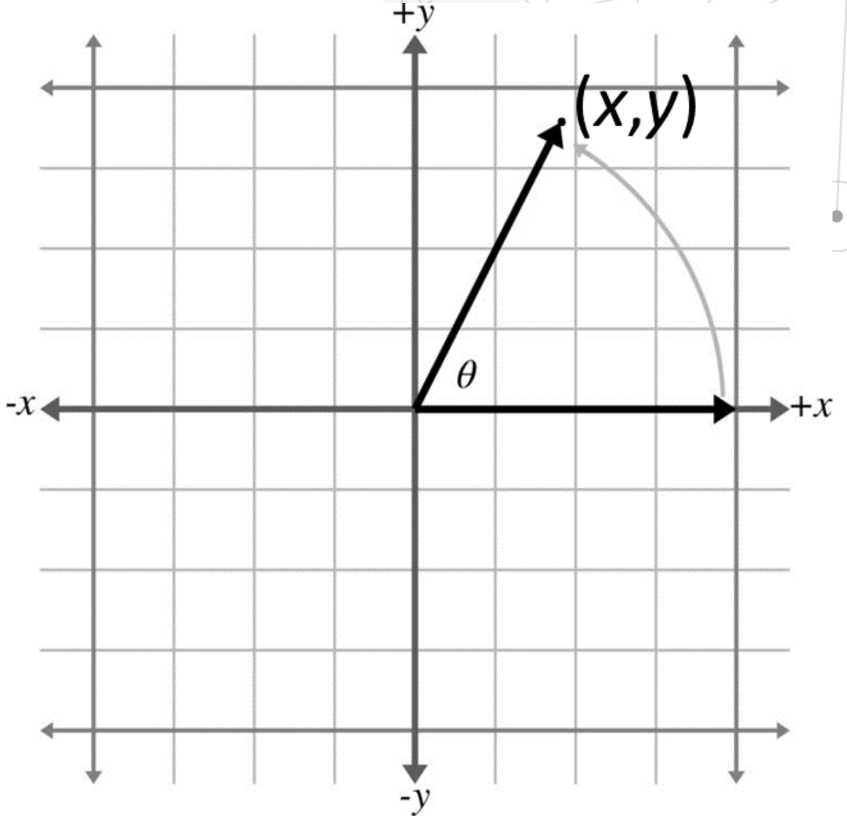
- π approximately equal to 3.14159265359

- To convert an angle θ from radians to degrees, we multiply by 180/π (i.e., ≈ 57.29578)
  - E.g., with θ = 1.5 rad, θ * 180/π = 1.5 * 180/π ≈ 85.94367°

- To convert an angle θ from degrees to radians, we multiply by π/180 (i.e., ≈ 0.01745329)
  - E.g., with θ = 90, θ * π/180 = 90 * π/180 ≈ 1.5707961 rad

Trig Functions

- Consider the angle θ between the +*x* axis and a ray to the point (*x*,*y*) in this diagram.



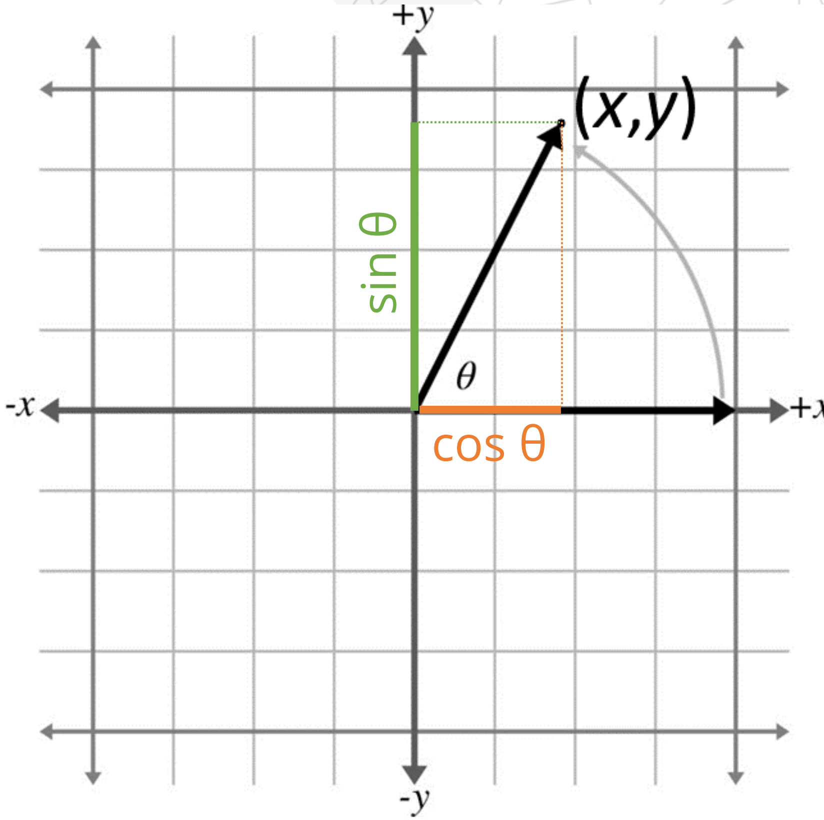
Cosine & Sine

- The coordinates of the endpoint of the ray *x* and *y* have special properties

- They are mathematically significant and have been assigned special functions: the cosine and sine of the angle.

$$\cos \theta = x$$
$$\sin \theta = y$$

- Which is which?
- They are in alphabetical order: *x* comes before *y*, and cos comes before sin.



More Trig Functions

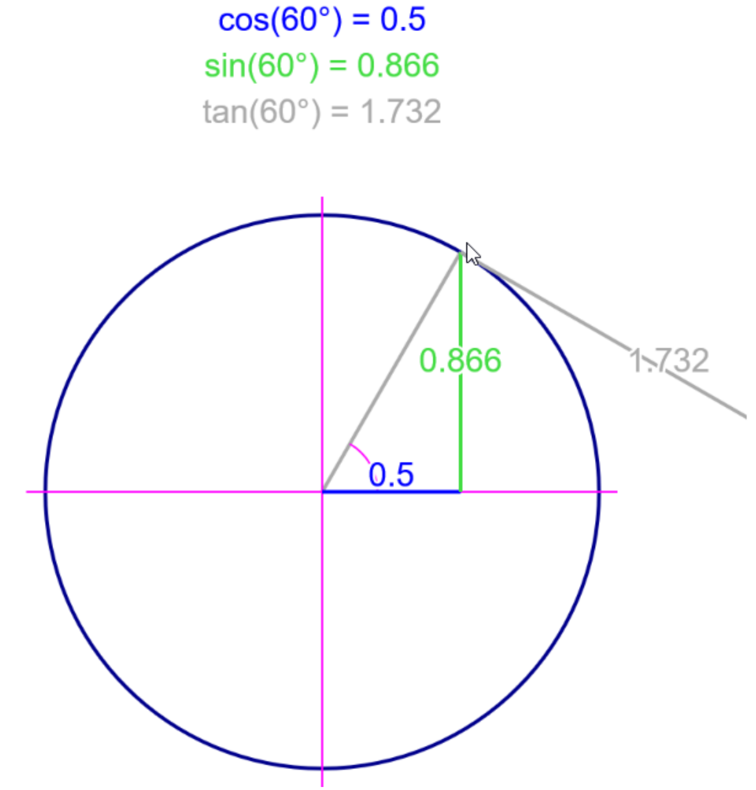
- The secant, cosecant, tangent, and cotangent.

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

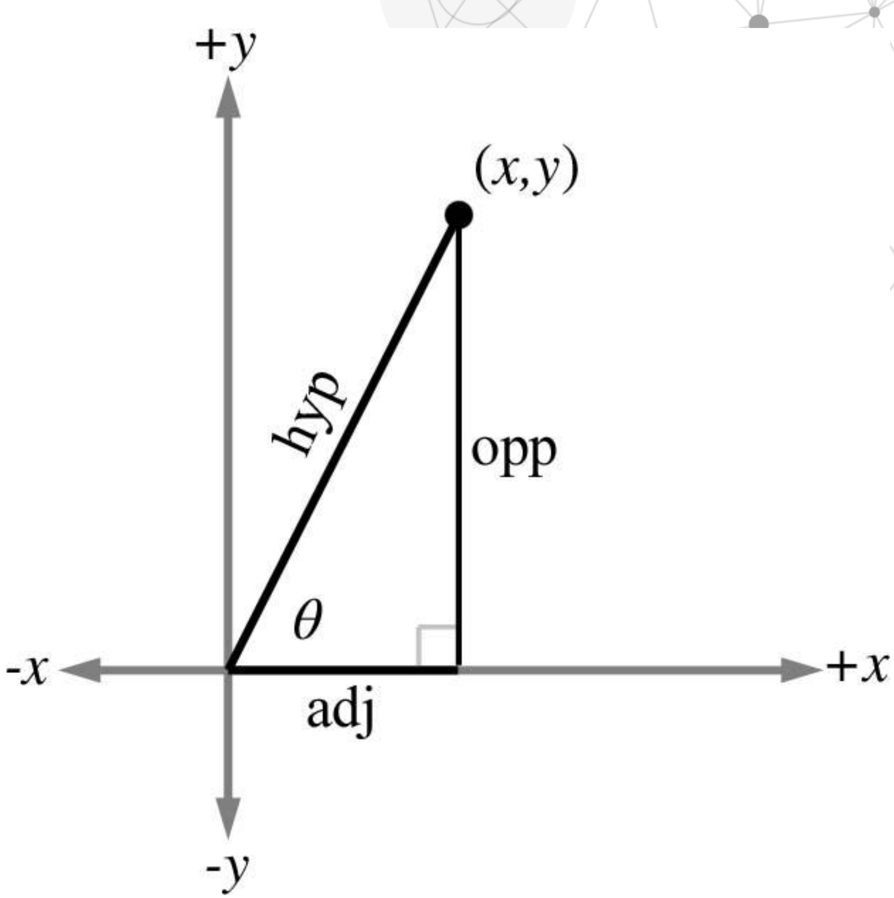
$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$



More on Sine and Cosine

- If we form a right triangle using the rotated ray as the hypotenuse, we see that *x* and *y* give the lengths of the adjacent and opposite legs of the triangle, respectively.

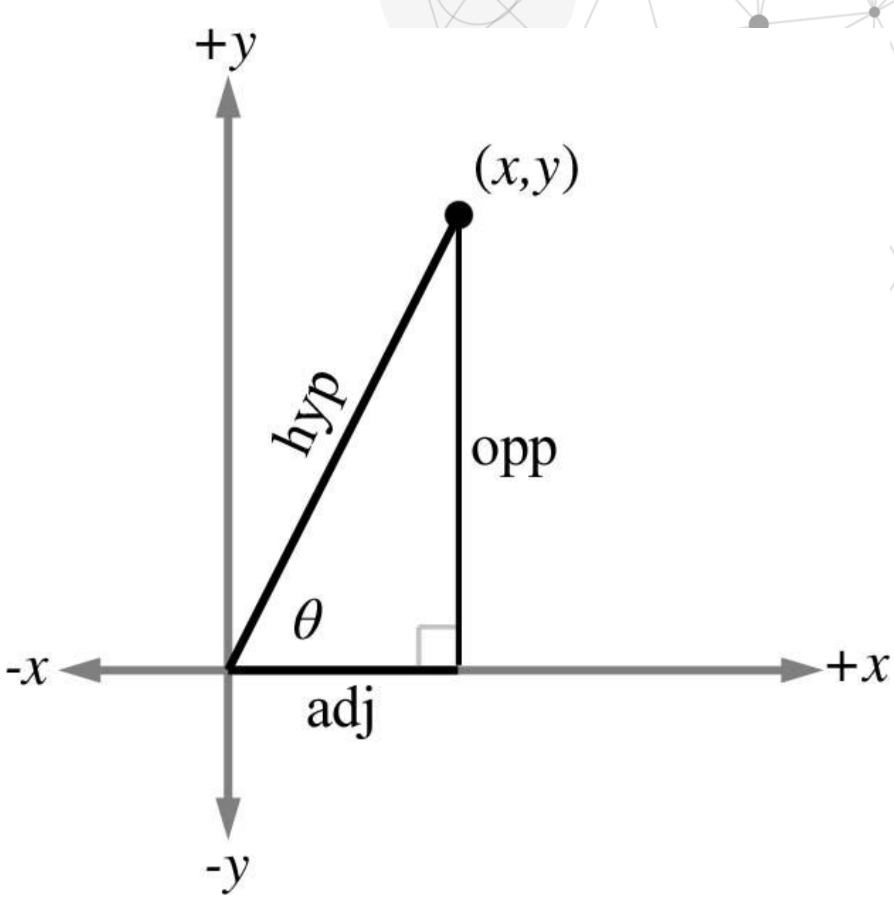
- The terms *adjacent* and *opposite* are relative to the angle θ.



More on Sine and Cosine

- Alphabetical order is again a useful memory aid: *adjacent* and *opposite* are in the same order as the corresponding cosine and sine.

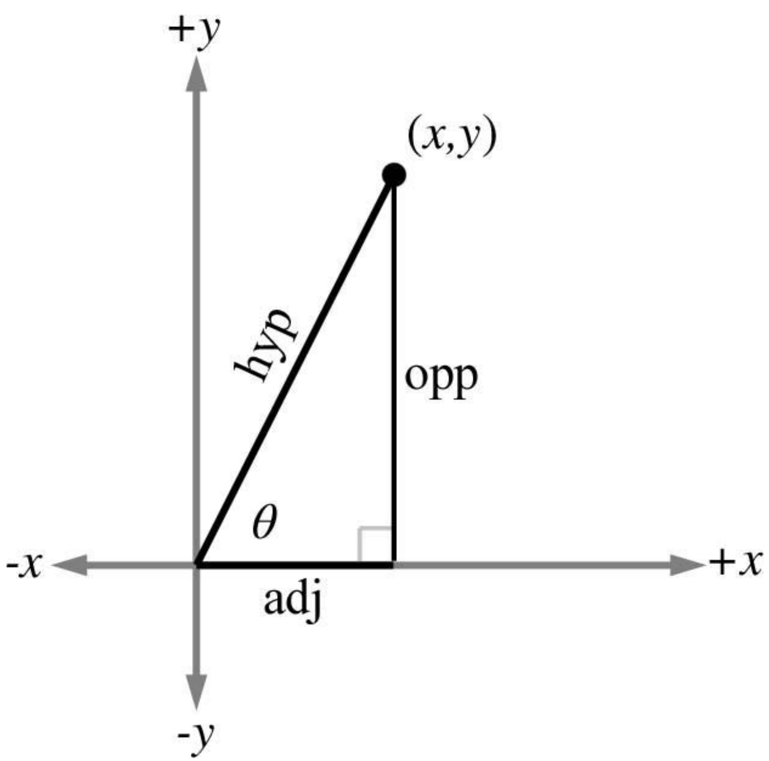
- Let the variables hypotenuse, adjacent, and opposite stand for the lengths of the hypotenuse, adjacent leg, and opposite leg, respectively.



Primary Trig Functions

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

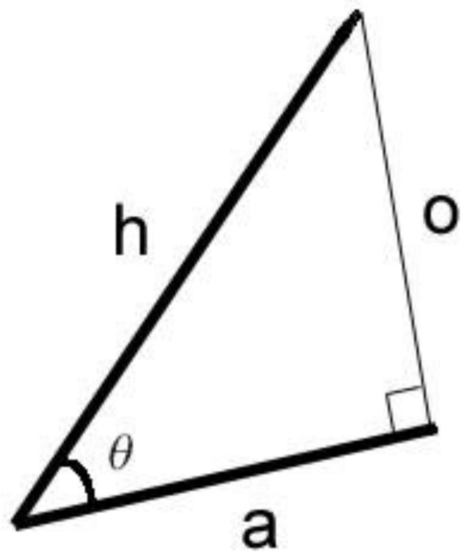
$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$



Mnemonics for Trig Functions

"SohCahToa"

$\sin\theta$ = o/h
$\cos\theta$ = a/h
$\tan\theta$ = o/a



h is for "hypotenuse"
o is for "opposite"
a is for "adjacent"

Alternative Forms

Some **o**ld **h**orse
Caught **a**nother **h**orse
Taking **o**ats **a**way

Some **o**ld **h**ippy
Caught **a**nother **h**ippy
Tripping **o**n **a**cid