GAME2016
Mathematical Foundation of Game Design and Animation

Lecture 6

Polar coordinate systems
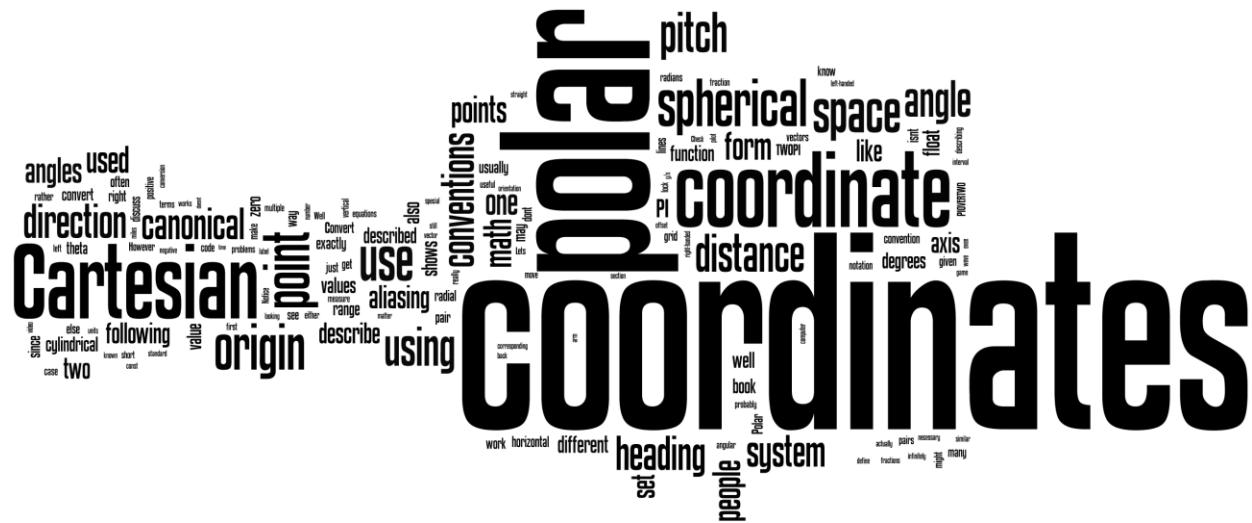
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Agenda

- Why Use Polar Coordinates?
  - Some examples where polar coordinates are preferable to Cartesian coordinates.

- 2D polar coordinates.
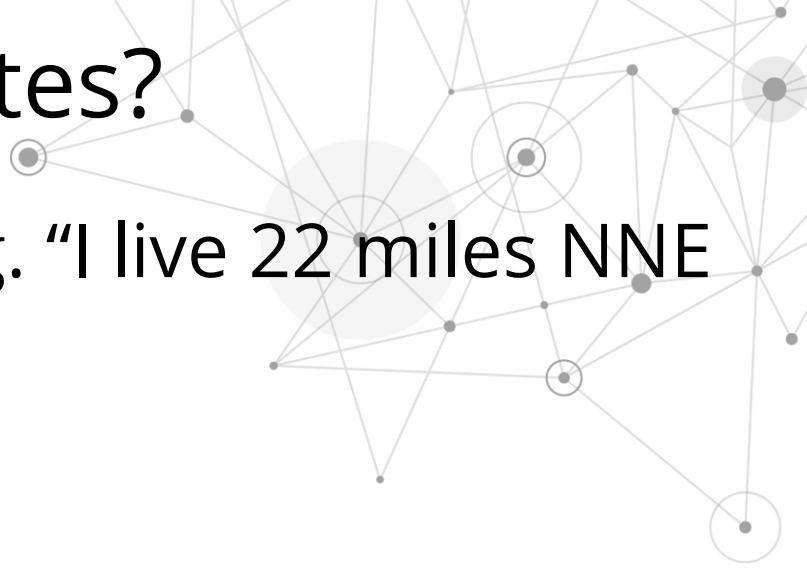
Why Use Polar Coordinates?

Why Use Polar Coordinates?

- They're better for humans (eg. "I live 22 miles NNE of Dallas, TX")

- They're useful in video games
  - Cameras
  - Turrets
  - Position the assassin's arms

- Sometimes we even use 3D spherical coordinates for locating things on the globe
  - Latitude and longitude.
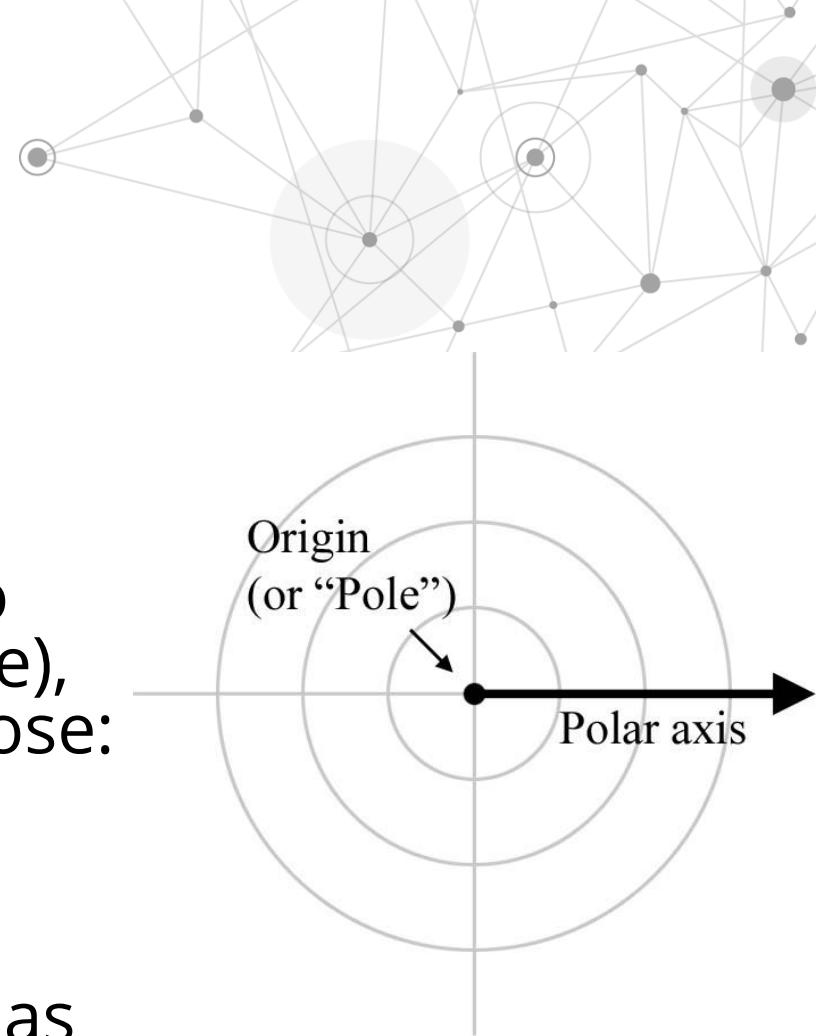
2D Polar Coordinates
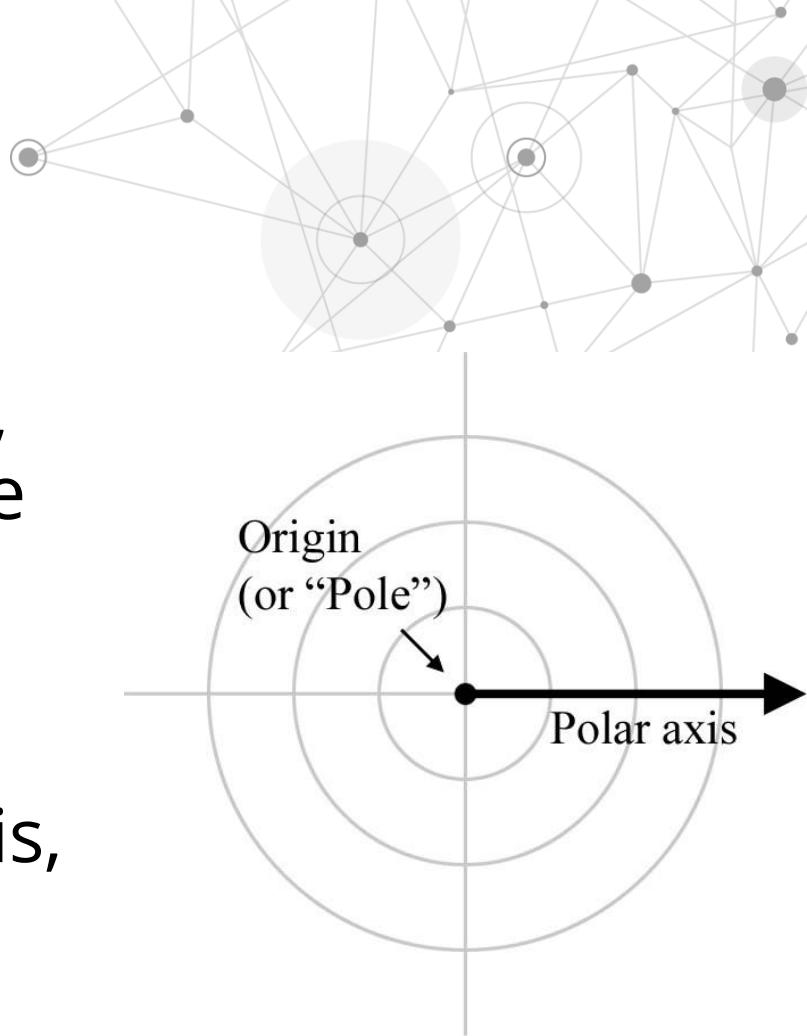
Polar Coordinate Space

- Recall that 2D Cartesian coordinate space has an origin and two axes that pass through the origin.

- A 2D *polar coordinate space* also has an origin (known as the pole), which has the same basic purpose: it defines the center of the coordinate space.

- A polar coordinate space only has one axis, sometimes called the *polar axis*, which is usually depicted as a ray from the origin.

Origin
(or "Pole")
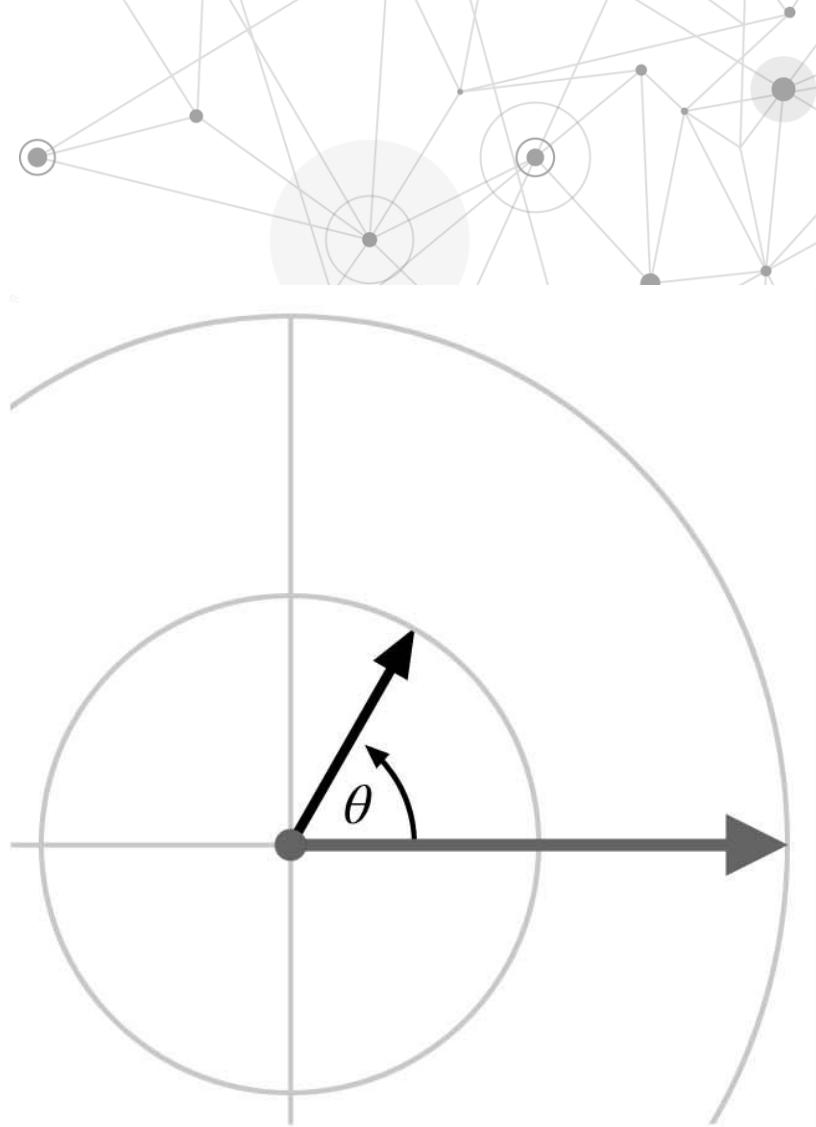
Polar axis

Polar Coordinate Space

- It is customary in math literature for the polar axis to point to the right in diagrams, and thus it corresponds to the +*x* axis in a Cartesian system.

- It's often convenient to use different conventions than this, as we'll discuss later.

Origin
(or "Pole")

Polar axis
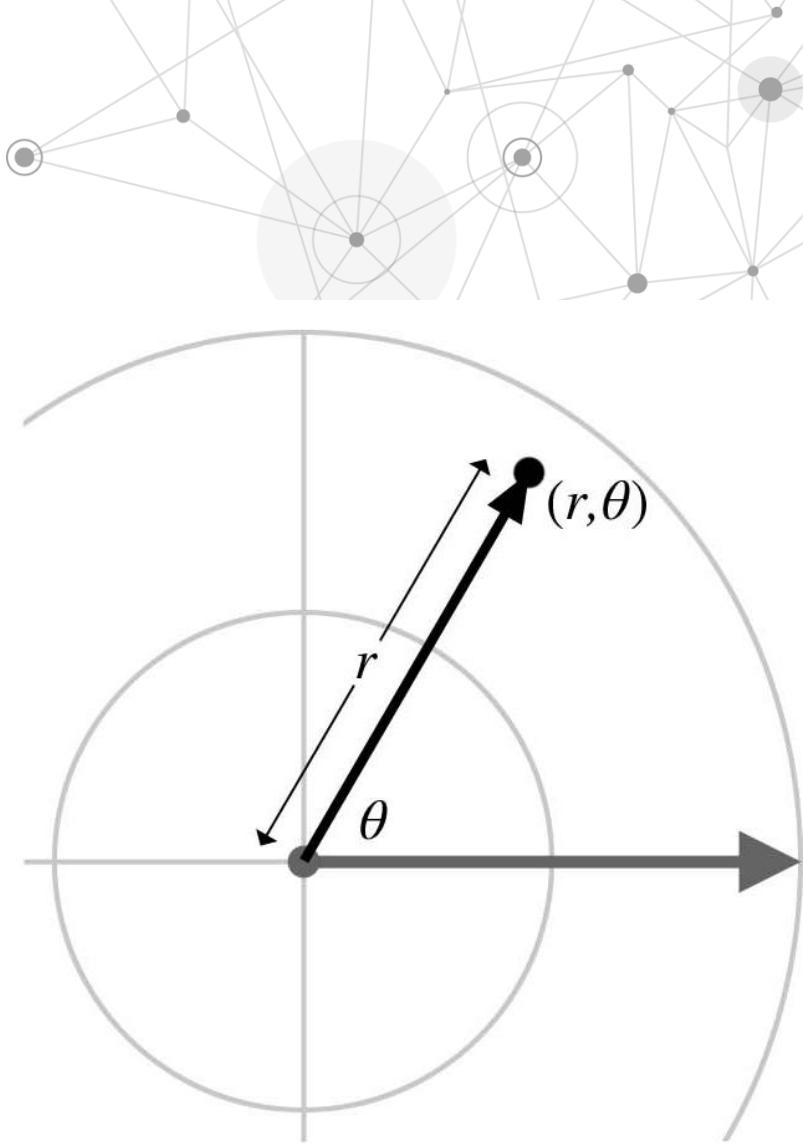
Polar Coordinates

- In Cartesian coordinates we described a 2D point using the using two signed distances, *x* and *y*.

- Polar coordinates use a distance and an angle.

Polar Coordinates
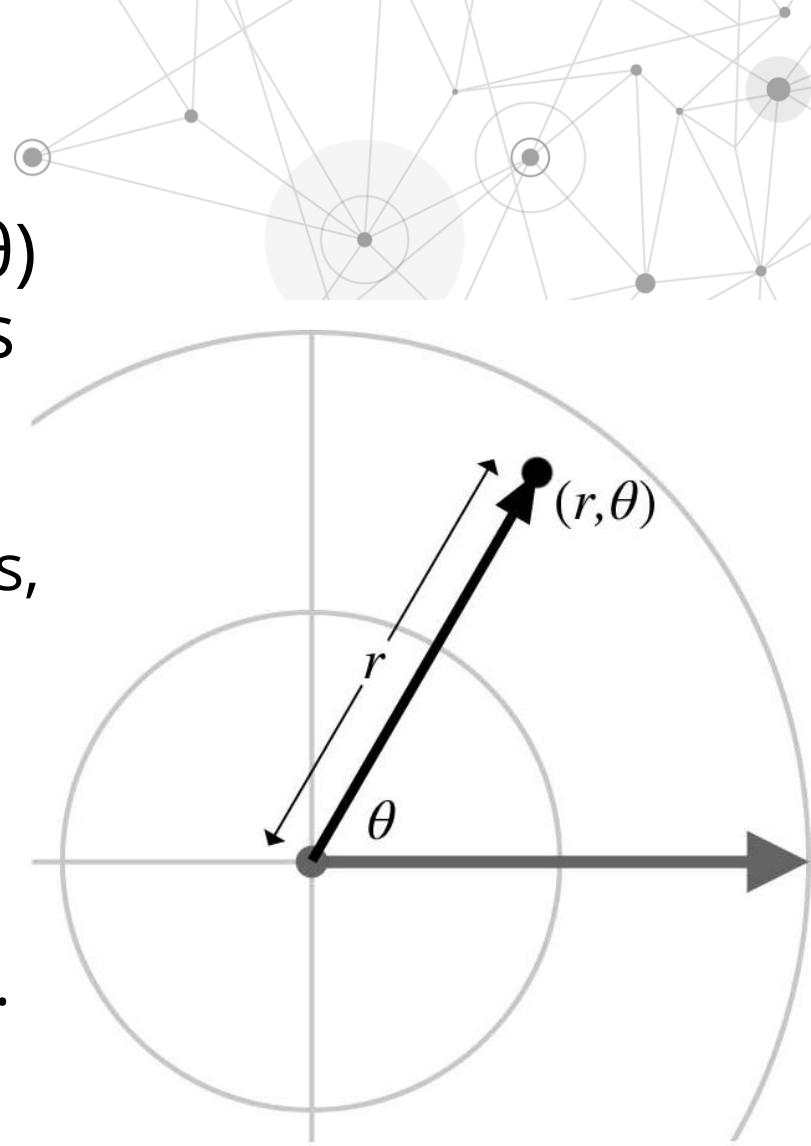
By convention, the distance is usually called *r* (which is short for *radius*) and the angle is usually called θ.

(r,θ)

r

θ
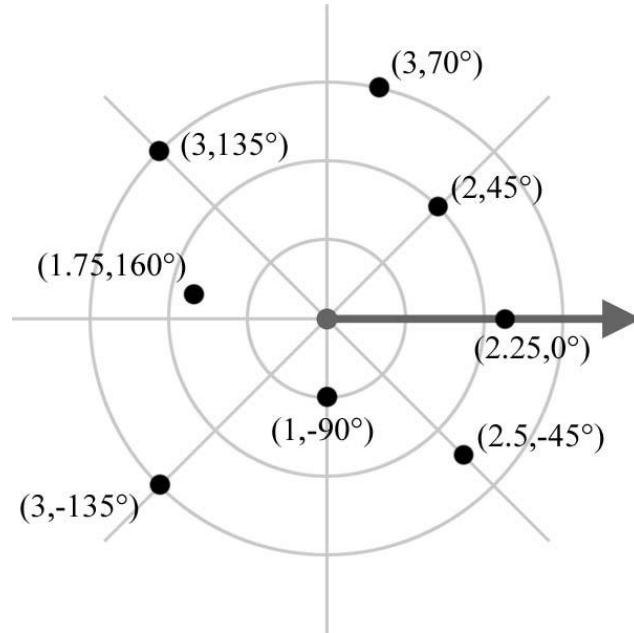
Polar Coordinates

- The polar coordinate pair (r, θ) species a point in 2D space as follows:

    1. Start at the origin, facing in the direction of the polar axis, and rotate by angle θ. Positive values of θ are usually interpreted to mean counterclockwise rotation, with negative values indicating clockwise rotation.
    2. Now move forward from the origin a distance of r units.
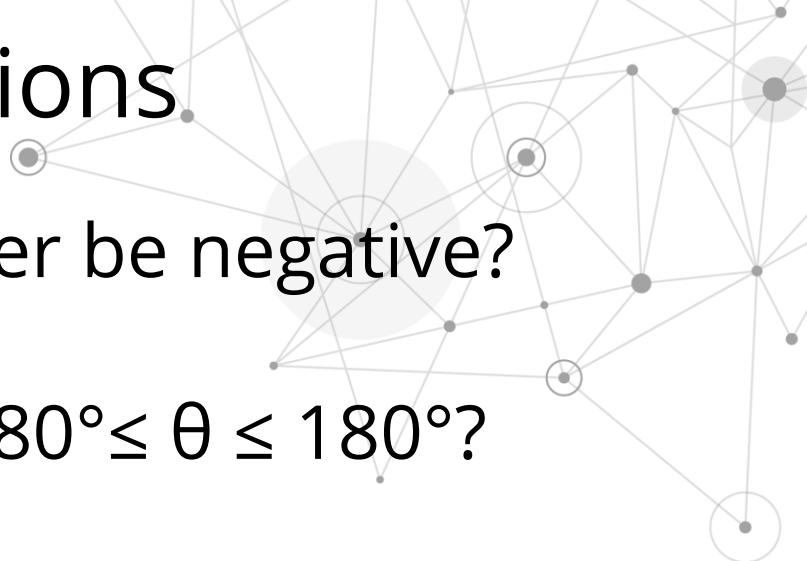
Polar Diagrams
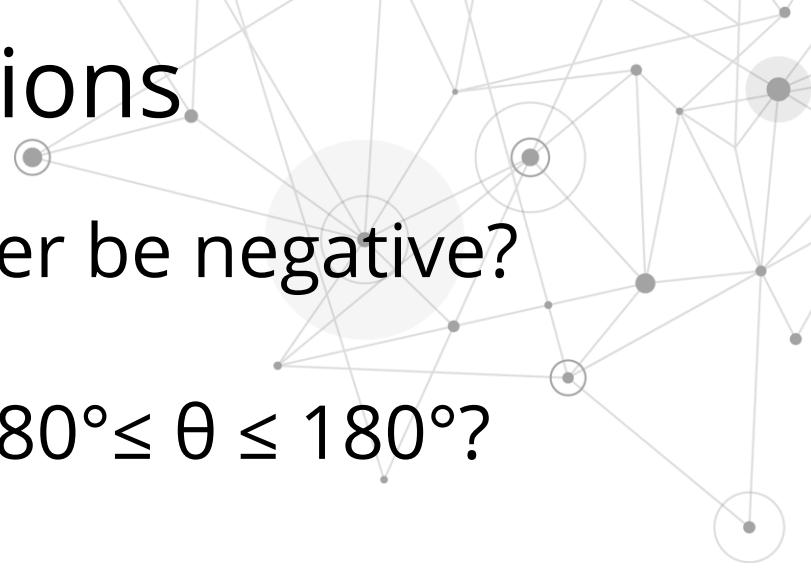
- The grid circles show lines of constant *r*.

- The straight grid lines that pass through the origin show lines of constant θ, consisting of points that are the same direction from the origin.
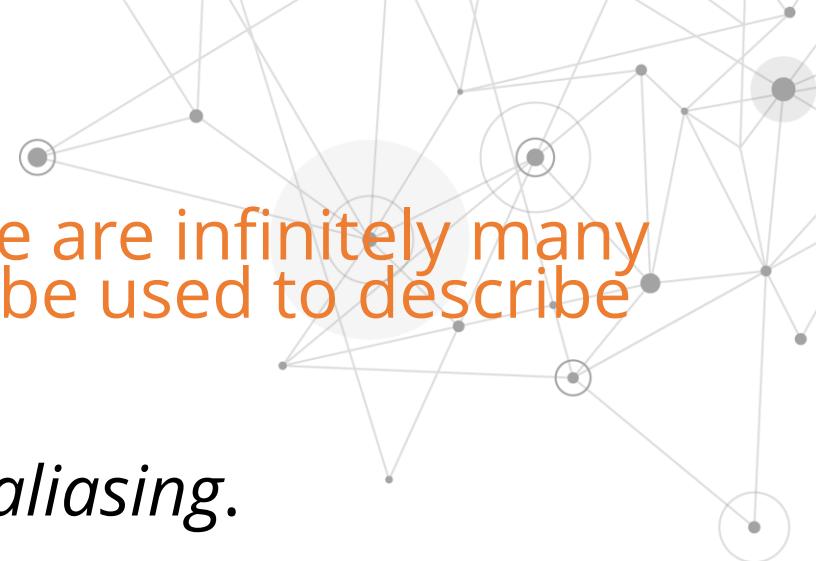
Some Ponderable Questions

1. Can the radial distance *r* ever be negative?

2. Can θ ever go outside of –180° ≤ θ ≤ 180°?

3. The value of the angle directly west of the origin (i.e. for points where *x* < 0 and *y* = 0 using Cartesian coordinates) is ambiguous. Is θ equal to +180° or –180° for these points?

4. The polar coordinates for the origin itself are also ambiguous. Clearly *r* = 0, but what value of θ should we use? Wouldn't any value work?

Some Ponderable Questions
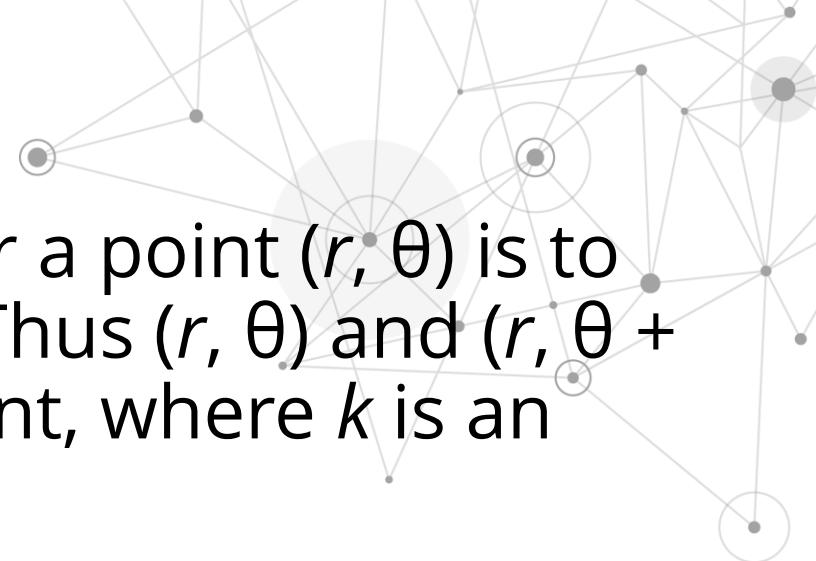
1. Can the radial distance *r* ever be negative?
   - Yes.
2. Can θ ever go outside of –180°≤ θ ≤ 180°?
   - Yes.
3. The value of the angle directly west of the origin (i.e. for points where *x* < 0 and *y* = 0 using Cartesian coordinates) is ambiguous. Is θ equal to +180° or –180° for these points?
   - Yes.
4. The polar coordinates for the origin itself are also ambiguous. Clearly *r* = 0, but what value of θ should we use? Wouldn't any value work?
   - Yes.

Aliasing

- In fact, for any given point, there are infinitely many polar coordinate pairs that can be used to describe that point.

- This phenomenon is known as *aliasing*.

- Two coordinate pairs are said to be *aliases* of each other if they have different numeric values but refer to the same point in space.

- Aliasing doesn't happen in Cartesian space. Each point in space is assigned exactly one (*x*, *y*) coordinate pair.

- A given point in polar space corresponds to many coordinate pairs, but a coordinate pair unambiguously designates exactly one point.

Creating Aliases

- One way to create an alias for a point (r, θ) is to add a multiple of 360° to θ. Thus (r, θ) and (r, θ + k360°) describe the same point, where k is an integer.

- We can also generate an alias by adding 180° to θ and negating r
  - We face the other direction, but we displace by the opposite amount.

- In general, for any point (r, θ) other than the origin, all of the polar coordinates that are aliases for (r, θ) be expressed as:

$$\left((-1)^k\, r, \theta + k180^{\mathrm{o}} \right)$$

Canonical Polar Coordinates

- A polar coordinate pair (r, θ) is in canonical form if all of the following are true:
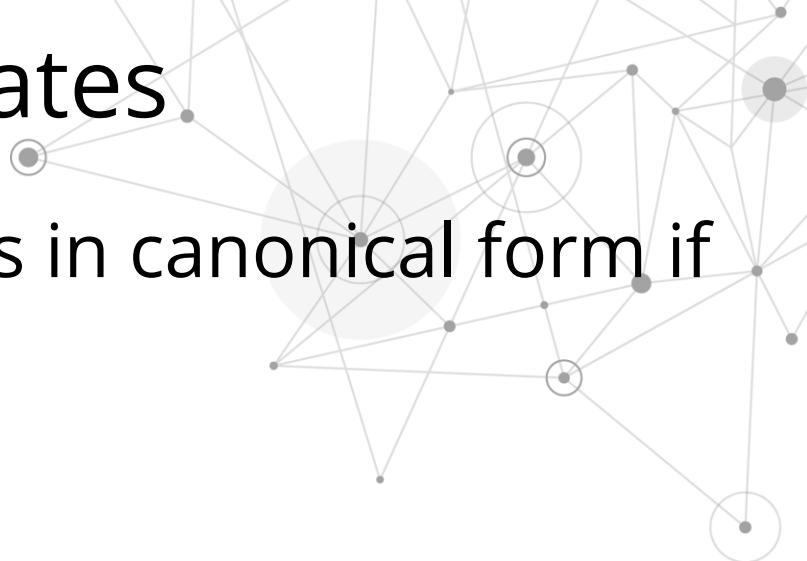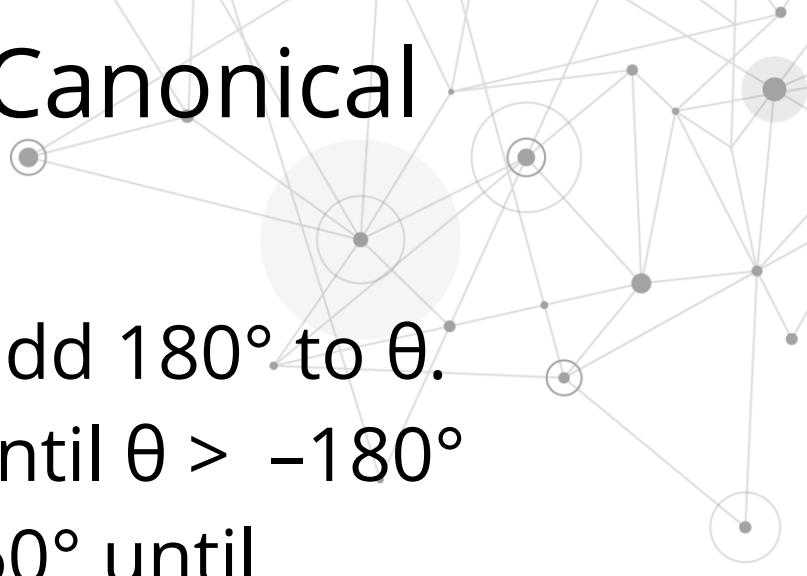
$r \geqslant 0$ — We don't measure distances "backwards."

$-180° < \theta \leqslant 180°$ — The angle is limited to 1/2 revolution, and use $+180°$ for "West."

$r = 0 \implies \theta = 0$ — At the origin, set the angle to zero.

Algorithm to Make (*r*, θ) Canonical
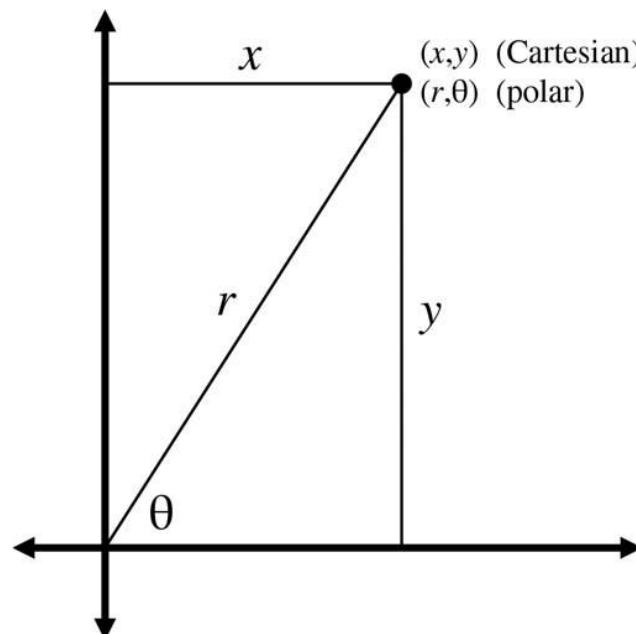
1. If *r* = 0, then assign θ = 0.
2. If *r* < 0, then negate *r*, and add 180° to θ.
3. If θ ≤ 180°, then add 360° until θ > −180°
4. If θ > 180°, then subtract 360° until θ ≤ 180°.

Converting from Polar to Cartesian Coordinates in 2D
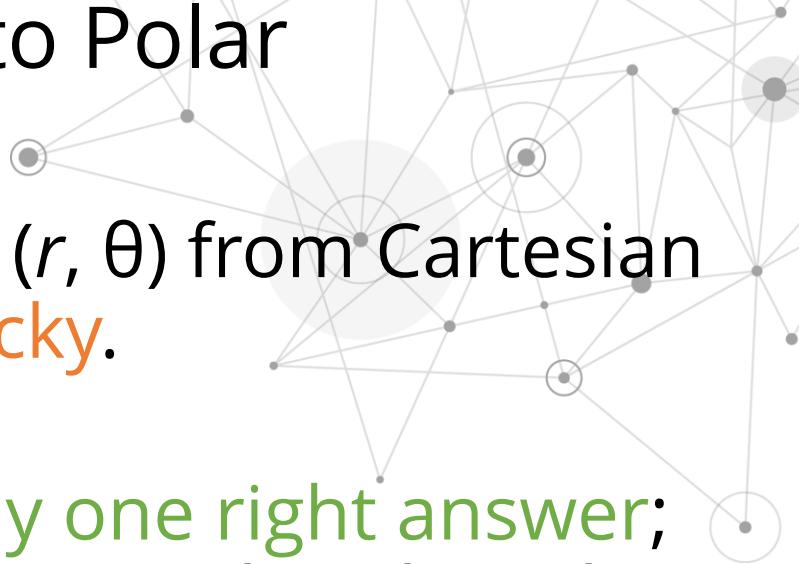
- **Straightforward**

- Converting polar coordinates (r, θ) to the corresponding Cartesian coordinates (x, y) follows from the definition of sin and cos.

$$x = r \cos θ$$
$$y = r \sin θ$$

Converting from Cartesian to Polar Coordinates in 2D

- Computing polar coordinates (*r*, θ) from Cartesian coordinates (*x*, *y*) is slightly tricky.

- Due to aliasing, there isn't only one right answer; there are infinitely many (*r*, θ) pairs that describe the point (*x*, *y*).
  - Usually, we want canonical coordinates.

- We can easily compute *r* using Pythagoras's theorem

$$r = \sqrt{x^2 + y^2}.$$

Converting from Cartesian to Polar Coordinates in 2D

- Computing θ for the point (*x*, *y*) seems not difficult.

- Solve the following equation:

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$$

$$\frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

$$y/x = \tan\theta$$

$$\theta = \arctan(y/x)$$
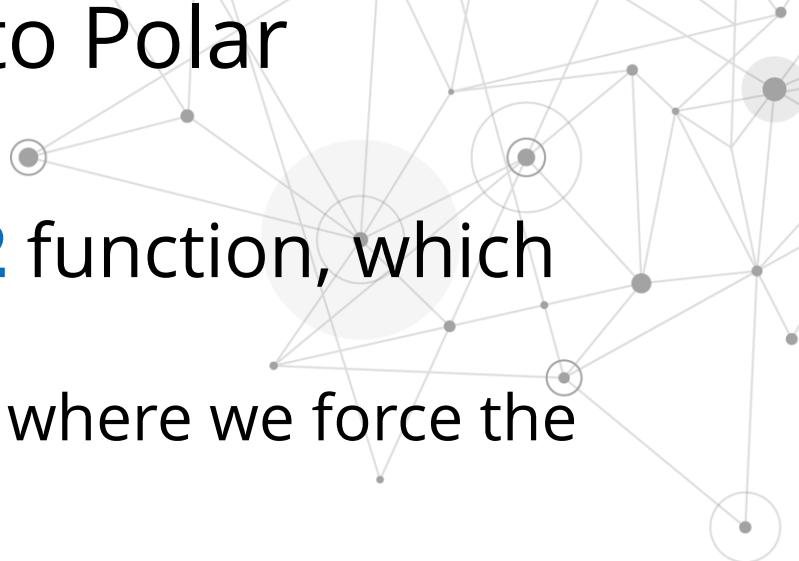
Converting from Cartesian to Polar Coordinates in 2D

$$\theta = \arctan(y/x)$$

- There are two problems with this approach.

  1. The first is that if $x = 0$, then the division is undefined.
  2. The second is that arctan has a range from –90° to +90°.

- Additionally, the division y/x effectively discards the sign of x and y
  - Both x and y can either be positive or negative, resulting in four different possibilities, corresponding to the four different quadrants that may contain the point. But the division y/x results in a single value.
  - If we negate both x and y, we move to a different quadrant in the plane, but the ratio x/y doesn't change.

- Because of these problems, the complete equation for conversion from Cartesian to polar coordinates requires some **if** statements to handle each quadrant

Converting from Cartesian to Polar Coordinates in 2D

- Programmers have the **atan2** function, which properly computes the angle
  - Except for the case at the origin where we force the result to be 0.

$$
\text{atan2}(y, x) = \begin{cases}
0 & x = 0, y = 0 \\
+90° & x = 0, y > 0 \\
-90° & x = 0, y < 0 \\
\arctan(y/x) & x > 0 \\
\arctan(y/x) + 180° & x < 0, y \geqslant 0 \\
\arctan(y/x) - 180° & x < 0, y < 0
\end{cases}
$$

Converting from Cartesian to Polar Coordinates in 2D

- Finally, to recap the discussion we can compute the 2D polar coordinates using the following equations:

$$r = \sqrt{x^2 + y^2} \qquad \theta = \text{atan2}(y, x)$$