GAME2016
Mathematical Foundation of Game Design and Animation

Lecture 4

Classes of transformations at a glance

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Classes of Transformations

- Linear transformations
- Affine transformations
- Invertible transformations
- Angle preserving transformations
- Orthogonal transformations
- Rigid body transformations

Disclaimer

- When we discuss transformations in general, we make use of the synonymous terms *mapping* or *function*.
- A mapping is simply a rule that takes an input and produces an output.
  - We denote that the mapping F maps a to b by writing $F(a) = b$. (Read "F of a equals b.")
- We are mostly interested in the transformations that can be expressed as matrix multiplication, but others are possible.
- In this section we introduce the determinant of a matrix. We will give a full explanation of determinants next week.
  - For now, just know that that the determinant of a matrix is a scalar quantity that is very useful for making certain high-level, shall we say, *determinations* about the matrix.

Linear Transformations

Linear Transformations

- A mapping $F(\mathbf{a})$ is linear if

$$F(\mathbf{a} + \mathbf{b}) = F(\mathbf{a}) + F(\mathbf{b})$$

and

$$F(k\mathbf{a}) = kF(\mathbf{a})$$

- The mapping $F(\mathbf{a}) = \mathbf{aM}$, where $\mathbf{M}$ is any square matrix, is a linear transformation as matrix multiplication satisfies the previous equations:

$$F(\mathbf{a} + \mathbf{b}) = (\mathbf{a} + \mathbf{b})\mathbf{M} = \mathbf{aM} + \mathbf{bM} = F(\mathbf{a}) + F(\mathbf{b})$$

and

$$F(k\mathbf{a}) = (k\mathbf{a})\mathbf{M} = k(\mathbf{aM}) = kF(\mathbf{a})$$

The Zero Vector

- Any linear transformation will transform the zero vector into the zero vector.
- If $F(\mathbf{0}) = \mathbf{a}$ and $\mathbf{a} \neq 0$, then F cannot be a linear transformation, since $F(k\mathbf{0}) = \mathbf{a}$ and therefore $F(k\mathbf{0}) \neq kF(\mathbf{0})$.
- Therefore:
  - Any transformation that can be accomplished with matrix multiplication is a linear transformation.
  - Linear transformations do not include translation.

Caveats

- A linear transformation is defined as one in which parallel lines remain parallel after transformation.

- This is almost completely accurate, with two exceptions:
  1. Parallel lines remain parallel after translation, but translation is not a linear transformation.
  2. What about projection? When a line is projected and becomes a single point, can we consider that point parallel to anything?

- Excluding these technicalities, we can say that a linear transformation may stretch things, but straight lines are not warped and parallel lines remain parallel.

Affine Transformations

Affine Transformations

- An *affine* transformation is a linear transformation followed by translation.

- Thus, the set of affine transformations is a superset of the set of linear transformations: any linear transformation is an affine translation, but not all affine transformations are linear transformations.

- Since all of the transformations we discussed so far are linear transformations, they are all also affine transformations.
  - Though none of them have a translation portion.

- Any transformation of the form $\mathbf{v}' = \mathbf{v}\mathbf{M} + \mathbf{b}$ is an affine transformation.

Invertible transformations

Invertible Transformations

- A transformation is *invertible* if there exists an opposite transformation, known as the *inverse* of F (i.e., F^{-1}), that undoes the original transformation.

- In other words, a mapping $F(\mathbf{a})$ is invertible if there exists an inverse mapping F^{-1} such that for all $\mathbf{a}$, $F^{-1}(F(\mathbf{a})) = \mathrm{F}(F^{-1}(\mathbf{a})) = \mathbf{a}$.

- This implies that F^{-1} is also invertible.

- There are non-affine invertible transformations, but we will not consider them for the moment.

Are All Affine Transforms Invertible?

- An affine transformation is a linear transformation followed by a translation.
- Obviously, we can always undo the translation portion by simply translating by the opposite amount.
- So, the question becomes whether or not the linear transformation is invertible.

Are All Linear Transforms Invertible?

- Intuitively, we know that all of the transformations other than projection can be undone
  - if we rotate, scale, reflect, or skew, we can always unrotate, unscale, unreflect, or unskew.

- But when an object is projected, we effectively discard one or more dimensions‘ worth of information
- This information cannot be recovered.

- So, all the primitive transformations other than projection are invertible.

Are All Matrices Invertible? No.

- Since any linear transformation can be expressed as multiplication by a matrix, finding the inverse of a linear transformation is equivalent to finding the inverse of a matrix.
  - If the matrix has no inverse, we say that it is *singular*, and the transformation is *non-invertible*.

- We can use the *determinant* (a matrix special value) to determine whether a matrix is invertible.
  - The determinant of an invertible matrix is nonzero.
  - The determinant of a non-invertible matrix is zero.

Angle preserving transformations

Angle Preserving Transformations

- A transformation is *angle-preserving* if the angle between two vectors is not altered in either magnitude or direction after transformation.
- Only translation, rotation, and uniform scale are angle-preserving transformations.
- An angle-preserving matrix preserves proportions.
- We do not consider reflection an angle-preserving transformation because even though the magnitude of angle between two vectors is the same after transformation, the direction of angle may be inverted.
- All angle-preserving transformations are affine and invertible.

Orthogonal transformations

Orthogonal Transformations

- *Orthogonal* is a term that describes a matrix whose rows form an orthonormal basis (i.e., the axes are perpendicular to each other and have unit length).
- Orthogonal transformations are interesting because it is easy to compute their inverse, and they arise frequently in practice.
- Translation, rotation, and reflection are the only orthogonal transformations.
- Orthogonal matrices preserve the *magnitudes* of angles, areas, and volumes, but possibly not the signs.
- The determinant of an orthogonal matrix is 1.
- All orthogonal transformations are affine and invertible.

Rigid body transformations

Rigid Body Transformations

- A *rigid body transformation* is one that changes the location and orientation of an object, but not its shape.
- All angles, lengths, areas, and volumes are preserved.
- Translation and rotation are the only rigid body transformations.
- Reflection is not considered a rigid body transformation.

Rigid Body Transformations

- Rigid body transformations are also known as *proper transformations*.
- All rigid body transformations are orthogonal, angle-preserving, invertible, and affine.
- Rigid body transforms are the most restrictive class of transforms, but they are also extremely common in practice.
- The determinant of a rigid body transformation matrix is 1.