

Parallel and Distributed Computing in Statistics

eigenspaces and representation decomposition for \mathfrak{sl}_3

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Example 1: $\mathfrak{sl}_3(\mathbb{R})$

\mathfrak{sl}_3 is a vector space of 3×3 matrices with trace 0, in other words:

$$\mathfrak{sl}_3 = \left\{ \begin{bmatrix} a & c & d \\ e & -a+b & f \\ g & h & -b \end{bmatrix} \mid a, b, c, d, e, f, g, h \in \mathbb{R} \right\}$$

equipped with $[A, B] = AB - BA$

Andersen Ang (个人主页)

Example 2: $\mathfrak{sl}_3(\mathbb{R})$

\mathfrak{sl}_3 is a set of 3×3 matrices with trace 0, in other words:

$$\mathfrak{sl}_3(\mathbb{R}) = \left\{ \begin{bmatrix} a & c & d \\ e & -a+b & f \\ g & h & -b \end{bmatrix} \mid a, b, c, d, e, f, g, h \in \mathbb{R} \right\}$$

the basis is:

$$\mathcal{B}_{\mathfrak{sl}_3} = \left\{ H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\} \cup \{E_{ij} \mid i \neq j\}.$$

E_{ij} is a matrix where $E_{i,jij} = 1$ and other places are 0.

Definition 1: subspace $\mathfrak{h} \subseteq \mathfrak{sl}_3(\mathbb{R})$

Let \mathfrak{h} be the two-dimensional subspace of all diagonal matrices, namely:

$$\mathfrak{h} = \text{span} \left\{ H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

What does it mean to be an eigenspace for \mathfrak{h} ?

For all $H \in \mathfrak{h}$, an eigenspace \mathfrak{g}_λ has all $M \in \mathfrak{sl}_3(\mathbb{R})$ s.t.

$$\text{ad}_H(M) = [H, M] = HM - MH = \lambda M?$$

Let's pick an arbitrary $H \in \mathfrak{h}$ and $M \in \mathfrak{sl}_3(\mathbb{R})$, namely:

$$H = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}, M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

Let's see the result under adjoint action, i.e.

$$ad_H(M) = [H, M] = HM - MH = ?$$

$$\begin{aligned} [H, M] &= \begin{bmatrix} (a_1 - a_1)m_{11} & (a_1 - a_2)m_{12} & (a_1 - a_3)m_{13} \\ (a_2 - a_1)m_{21} & (a_2 - a_2)m_{22} & (a_2 - a_3)m_{23} \\ (a_3 - a_1)m_{31} & (a_3 - a_2)m_{32} & (a_3 - a_3)m_{33} \end{bmatrix} \\ &= \begin{bmatrix} 0 \cdot m_{11} & (a_1 - a_2)m_{12} & (a_1 - a_3)m_{13} \\ (a_2 - a_1)m_{21} & 0 \cdot m_{22} & (a_2 - a_3)m_{23} \\ (a_3 - a_1)m_{31} & (a_3 - a_2)m_{32} & 0 \cdot m_{33} \end{bmatrix} \end{aligned}$$

Since we are asking for eigenspaces of \mathfrak{h} , we need a way to get $a_i - a_j$ from any $H \in \mathfrak{h}$ for us, thus we could define a linear functional:

Definition 2: L_i

L_i is a linear functional s.t.

$$L_i \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} = a_i$$

it satisfies the requirement of a linear functional:

$$L_i(A + B) = L_i(A) + L_i(B), L_i(aA) = aL_i(A).$$

Notice that for the adjoint representation of $\mathfrak{sl}_3(\mathbb{R})$, we have the following decomposition:

$$\mathfrak{sl}_3(\mathbb{R}) = \mathfrak{h} \oplus (\oplus \mathfrak{g}_\alpha)$$

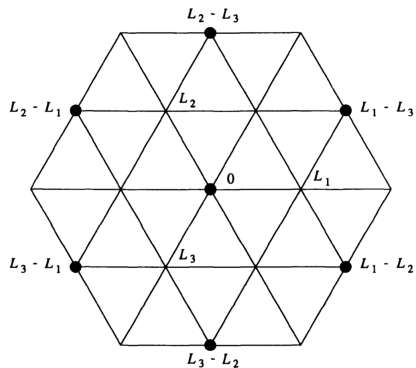
where α ranges over \mathfrak{h}^* and \mathfrak{g}_α is the eigenspace w.r.t α

this is a special case of a general statement:

Theorem 1

For any semisimple Lie algebra \mathfrak{g} (it is a direct sum of simple Lie algebras (non-abelian Lie algebras without any non-zero proper ideals), we could find an abelian subalgebra (closed under bracket) $\mathfrak{h} \subset \mathfrak{g}$ s.t. the action of \mathfrak{h} on any representation V could be written as a direct sum decomposition of V into eigenspaces V_α for \mathfrak{h} .

graph of eigenspaces



Given any $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$, where would ad_X sends Y to? i.e.
 $ad_X(Y) = [X, Y] = ?$

\mathfrak{h} could help us, namely:

$$[H, [X, Y]] = ((\alpha + \beta)(H)) \cdot [X, Y]$$

in other words, $ad_X(Y)$ is an eigenvector for \mathfrak{h} with eigenvalue $\alpha + \beta$.

$$ad_{\mathfrak{g}_\alpha} : \mathfrak{g}_\beta \rightarrow \mathfrak{g}_{\alpha+\beta}$$

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