

Bound states and QCD*

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A perturbative expansion for hadrons and soft QCD processes is formulated. Bound states must be expanded around initial states that are non-polynomial in the coupling. Atomic physics suggests to use states which are bound by the classical field of their constituents. Including the classical field of the *in* and *out* states allows to calculate bound state *S*-matrix elements.

Quark states are bound by an $\mathcal{O}(\alpha_s^0)$ classical gluon field A_a^0 which is characterized by a universal scale Λ . It confines color in $q\bar{q}$ mesons and qqq baryons, giving relativistic dynamics with the characteristics of dual models. Massless bound states allow to include a $J^{PC} = 0^{++}$ condensate in the perturbative vacuum, thus breaking chiral symmetry spontaneously.

Contents

I. Introduction	2
II. Phenomenological motivations	5
III. The perturbative expansion for atoms	6
A. The Potential Picture	6
B. Positronium at Born level	7
IV. The QCD confinement potential	8
A. The instantaneous color field \hat{A}_a^0	8
B. The classical potential for mesons	8
C. The classical potential for baryons	10
V. Mesons in the rest frame	11
A. Pair production effects	11
B. Bound state equation	12
C. Separation of radial and angular variables	13
D. The 0^{-+} trajectory: $\eta_P = (-1)^{j+1}$, $\eta_C = (-1)^j$	14
E. The 0^{--} trajectory: $\eta_P = (-1)^{j+1}$, $\eta_C = (-1)^{j+1}$	15
F. The 0^{++} trajectory: $\eta_P = (-1)^j$, $\eta_C = (-1)^j$	15
VI. Properties of the meson states	16
A. Features of the radial wave function	17
B. Orthogonality	17
C. Normalization	18
D. Parton picture and duality for $M \gg V(r)$	19
E. Mass spectrum of the 0^{-+} trajectory for $m = 0$	20

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VII. Bound states in motion	20
A. Active vs. Passive boosts	20
B. Poincaré covariance of QED bound states in $D = 1 + 1$ dimensions	21
1. Active boost in $A^1 = 0$ gauge	21
2. Actively boosted QED ₂ bound states	22
3. Passively boosted QED ₂ bound states	23
C. \mathbf{P} -dependence of free $q\bar{q}$ states	24
D. Active boost in $D = 3 + 1$	25
VIII. Spontaneous breaking of chiral invariance	26
A. Vanishing quark mass, $m = 0$	26
B. Limit of small quark mass, $m \rightarrow 0$	27
IX. Outlook	29
Acknowledgments	30
A. Symmetries of the $q\bar{q}$ wave function	30
1. Gauge transformations	30
2. Space translations	31
3. Rotations	31
4. Parity	31
5. Charge conjugation	32
B. Structure of Dirac states	32
C. Passive boost in $D = 3 + 1$	33
References	36

I. INTRODUCTION

Hadrons are non-perturbative, in the sense that color confinement and chiral symmetry breaking effects are not present in Feynman diagrams of any finite order. Hadrons are also QCD bound states, and Feynman diagrams do not have poles even for atoms. Perturbation theory for bound states differs from scattering amplitudes. The binding energies of QED atoms can be expanded in powers of α and $\log \alpha$, whereas their wave functions are non-polynomial even in a first approximation.

The non-perturbative aspects of hadrons need not imply that a perturbative expansion is inapplicable. This question deserves attention, since perturbation theory is the main analytic tool for the Standard Model. Hadrons moreover have perturbative features as well (section II). The aspects of hadrons as $q\bar{q}$ and qqq states are simpler than expected for strong coupling. Dokshitzer [1, 2] has emphasized the perturbative features of soft hadron dynamics.

The Taylor expansion of an ordinary function, $f(x) = f(x_0) + (x - x_0)f'(x_0) + \dots$, is determined once the point of expansion (x_0) is chosen. Analogously, a perturbative expansion in field theory requires to choose the initial (“lowest order”) state. Scattering amplitudes are successfully expanded around free, $\mathcal{O}(\alpha^0)$ *in* and *out* states at asymptotic times, $t = \pm\infty$. This is not possible for atoms, whose stationary time evolution requires interactions at all times.

A perturbative expansion around an initial atomic state of $\mathcal{O}(\alpha^\infty)$ cannot be unique. Parts of the power corrections in α can be incorporated into the initial wave function, defining a new, equivalent expansion. This was explicitly demonstrated by Caswell and Lepage [3] in the context of the QED Bethe-Salpeter equation [4]. A change in the

B-S propagator can be compensated by a corresponding change in the kernel, yielding an equally valid perturbative expansion. The choice of initial state requires a principle beyond perturbation theory.

Precision calculations of atomic properties, such as the hyperfine splitting of Positronium [5], use Non-Relativistic QED [6–8]. This effective theory relies on the low, $\mathcal{O}(\alpha m_e) e^\pm$ momentum to expand the QED action in powers of ∇/m_e . The non-relativistic Schrödinger equation with the classical $V(r) = -\alpha/r$ potential is chosen to determine the initial Positronium state.

We adopt this principle: *Initial states of the perturbative expansion are bound by their classical gauge field.* The stationary action $\mathcal{S}[A_{cl}^\mu]$ gives the dominant contribution to the functional integral in the $\hbar \rightarrow 0$ limit,

$$\int [dA^\mu] \exp(i\mathcal{S}[A^\mu]/\hbar) \quad (1.1)$$

We refer to classically bound states as Born states, of lowest order in \hbar .

In QED scattering processes the colliding e^\pm , γ are initially widely separated. Their classical field may be neglected when the momentum transfers are large, and the infrared divergences are regularized at each order in α . The Feynman diagram expansion of a scattering amplitude is defined by the standard expression (3.1) of the perturbative S -matrix in the Interaction Picture. The expansion is formally exact since the free *in* and *out* states at $t = \pm\infty$ are dressed into the physical scattering states by the interaction Hamiltonian \mathcal{H}_I .

The expansion (3.1) fails for bound states, since the infinitely separated constituents of the *in* and *out* states have vanishing overlap with finite-sized bound states (section III A). There is a simple remedy, however. Taking into account the classical field enables Born level bound states in the *in* and *out* states. With this modification the “Potential Picture” formula (3.5) defines the perturbative corrections for bound asymptotic states. The propagators of Feynman diagrams will now be evaluated in the classical field, as required for bound states (see, *e.g.*, Ch. 14 of [9]). This picture has the potential to define bound state perturbation theory as uniquely as the Interaction Picture for scattering amplitudes, and thus to remove some of the “art” from bound state calculations (section 10-3 of [10]). Here we study the expansion only at $\mathcal{O}(\mathcal{H}_I^0)$. In section III B we demonstrate that the asymptotic states of Positronium agree with those given by the Schrödinger equation.

Each loop correction (field fluctuation) brings a factor of $\alpha\hbar$ (section 6-2-1 of [10], and [11]). For scattering amplitudes one usually sets $\hbar = 1$ and counts only powers of α . However, for atomic states any initial approximation is of $\mathcal{O}(\alpha^\infty)$. This requires to pay attention to \hbar . Since \hbar is a fundamental parameter one expects Poincaré and other symmetries, as well as unitarity, to hold at each order of \hbar . The fact that the powers of α and \hbar match in loop corrections makes the \hbar expansion relevant only when perturbation theory applies.

Considering now QCD (section IV), we note a qualitative difference between the classical fields generated by the constituents of Positronium and hadrons. Both states are singlets under *global* gauge transformations only, given that the wave function has no gauge link connecting the constituents. The $|e^-(\mathbf{x}_1) e^+(\mathbf{x}_2)\rangle$ component of Positronium creates a dipole photon field, $A^0(\mathbf{x})$ (3.13), which is invariant under global $U(1)$ transformations. On the other hand, a color singlet hadron cannot give rise to a color octet field, hence $A_a^\mu(\mathbf{x}) = 0$ for all \mathbf{x} . The classical field of each color component $|q^A(\mathbf{x}_1) \bar{q}^A(\mathbf{x}_2)\rangle$ of a meson is weighted by T_a^{AA} . The vanishing trace of the color generator ensures that the gauge field cancels in the sum over quark colors A . The classical field can thus bind each color component, while the field vanishes for an external particle (quark), which interacts with the entire, color singlet hadron state.

Color confinement implies a scale Λ_{QCD} which defines the hadron radius, but this scale is not present in the QCD Lagrangian. A classical field can have such a scale only through a boundary condition. There is an essentially unique¹ homogeneous, $\mathcal{O}(\alpha_s^0)$ operator solution $\hat{A}_a^0(\mathbf{x})$ (4.2) of the QCD field equations which satisfies basic physical requirements. The classical field is given by the expectation value (4.6) for each component of the bound state. Since $\hat{A}_a^0(\mathbf{x})$ is linear in \mathbf{x} the field energy density is constant throughout space. Provided the value of the energy density is universal the total (infinite) field energy does not affect the quark dynamics. This fixes the normalization κ of the homogeneous solution, up to an overall scale Λ . Coherence over all space is maintained because the \hat{A}_a^0 field is instantaneous. The classical field (4.7) vanishes when summed over quark colors.

The existence of the homogeneous solution (4.2) with its universal scale Λ means that a perturbative expansion for hadrons needs to be considered. In this paper we explore some properties of the $\mathcal{O}(\hbar^0 \alpha_s^0)$ bound states that serve as the initial states of the perturbative expansion defined by (3.5). Formally, the asymptotic states only need to have a non-vanishing overlap with physical hadrons. In practice, the match must be close enough for the corrections to be perturbative. The Born hadrons thus should confine color and spontaneously break chiral symmetry.

¹ For states without constituent (propagating) gluons.

Convergence of the QCD perturbative expansion requires that α_s stays perturbative at low scales. Insofar as the classical field contribution dominates over loops at hadronic scales (as we assume) the coupling will in fact freeze. Experimentally, $\alpha_s(m_\tau) \simeq 0.32$ [12], and may not increase much further if the classical field starts to dominate at a scale of $\mathcal{O}(1 \text{ GeV})$. Gribov [13] estimated $\alpha_s^{crit.} \simeq 0.43$ in his confinement scheme, whereas a phenomenological analysis [14] gave an average $\alpha_s(Q) \simeq 0.51$ in the interval $0 \leq Q \leq 2 \text{ GeV}$. In the present framework Λ and $\alpha_s(0)$ are parameters in a fit to hadron properties, constrained also by the measured values of $\alpha_s(Q)$ in the perturbative regime.

The $q\bar{q}$ meson state in the rest frame (4.3) has the (globally) color singlet wave function (4.4). The classical field gives the linear potential (4.10) for each color component of the meson. Its slope is $g\Lambda^2$, with the factor g arising from the coupling of the $\mathcal{O}(\alpha_s^0)$ field to the quarks. Single gluon exchange is of $\mathcal{O}(\mathcal{H}_I^2)$ in the perturbative S -matrix (3.5) and neglected here. The gluon will interact with the classical field, so its contribution will differ from that of PQCD without a classical field.

The analogous potential for qqq baryons is $g\Lambda^2 d(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ (4.19), with the d -function given in (4.17). It reduces to the meson potential when two quarks coincide, $d(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2) = |\mathbf{x}_1 - \mathbf{x}_2|$. The expectation value of the operator solution (4.2) projects out diagonal generators T_a^{AA} (no sum on quark color A). Consequently only the $a = 3, 8$ gluon field components are non-vanishing. For baryons the A_3^0 field confines in the $\mathbf{x}_1 - \mathbf{x}_2$ direction, while A_8^0 confines along $\mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3$. There is no analogous confinement for states with four or more quarks. This may explain the prominence of $q\bar{q}$ and qqq hadrons.

The (color reduced) meson wave function should satisfy the Bound State Equation (BSE) (5.12) for the meson state to be an eigenstate with eigenvalue M of the QCD Hamiltonian (section V). We assume a single quark flavor with mass m . Since the potential $V = V(|\mathbf{x}_1 - \mathbf{x}_2|)$ is rotationally invariant the radial and angular variables may be separated. The Dirac structures can be classified according to their behavior under parity and charge conjugation, see (5.21). Despite the relativistic kinematics there are no states whose quantum numbers would be exotic in the quark model. The states may be grouped into three “trajectories”, with wave functions given in (5.26), (5.33) and (5.40).

For large quark masses ($m \gg \sqrt{V'}$) the BSE reduces to the non-relativistic Schrödinger equation. Heavy quarkonia are known to be well described using a phenomenological linear + gluon exchange potential [15], which agrees with the static potential found in lattice QCD (see Fig. 4.2 of [16]). The gluon exchange contribution remains to be determined in the present framework.

Hadrons composed of light quarks are highly relativistic ($m_{u,d} \ll \Lambda_{QCD}$). Chiral symmetry breaking concerns the dynamics of nearly massless quarks. Relativistic dynamics may be addressed in the \hbar expansion. The instantaneous classical field $\hat{A}_a^0(\mathbf{x})$ (4.2) is determined by the positions of the quarks, regardless of their momenta. Relativity brings novel issues of particle production and decay, which need to be carefully considered (section V A).

The classical field creates $q\bar{q}$ pairs, which prevents the linear rise of the potential (“string breaking”). This enables the hadron decays and loops of Fig. 1(a,b), which are required for unitarity to hold at $\mathcal{O}(\hbar^0)$. The pair creation effects can be treated through an expansion in inverse powers of the number of colors N_c . This is compatible with the “narrow resonance approximation” of dual models and with quenched lattice calculations, which find rough agreement with the hadron spectrum without quark loops [17]. Here we consider only contributions of leading order in $1/N_c$. The zero-width states determine the decay (5.6) and hadron loop corrections.

Time ordering implies in addition virtual pair contributions arising from “ Z ”-diagrams (Fig. 1(c)). They arise also in the Dirac equation describing an electron bound in a fixed external field (appendix B). The Dirac eigenstates are created by a Bogoliubov superposition (B.6) of the free b^\dagger and d operators. Hence the Dirac vacuum (B.7) has e^+e^- pairs. Analogously, the b and d operators cannot be neglected in the relativistic bound state (4.3). The virtual pairs form a sea quark distribution.

The Z -contributions give the wave function a constant, non-vanishing local norm at large values of the potential (section VI). For the Dirac wave function this was noticed already in the 1930’s [18]. Since the wave function cannot (and should not) be normalized to unit norm the Dirac energy spectrum is continuous. However, the solutions of the $q\bar{q}$ bound state equation have factors of $1/(M - V)$ (see, e.g., (5.26)) which generally make the wave function singular at $M = V$. Requiring local normalizability, as seems physically reasonable, leads to a discrete mass spectrum. For vanishing quark mass ($m = 0$) the states lie on linear Regge trajectories with parallel daughters (Fig. 2).

Equal-time bound states with momentum \mathbf{P} (section VII) are expressed as in (5.2). The classical field is determined by the instantaneous positions of the quarks and is thus the same (4.7) as in the rest frame. The wave functions satisfy the BSE (7.2), where \mathbf{P} breaks rotational invariance. The solution along the \mathbf{P} -axis can be analytically related (7.34) to the rest frame wave function. This boundary condition to the BSE defines the wave function for all \mathbf{x} . We have not demonstrated general Lorentz covariance, e.g., for scattering amplitudes. However, the gauge invariance of the electromagnetic form factor (7.33) in an arbitrary frame is an encouraging sign.

The rest frame BSE allows locally normalizable solutions with vanishing mass M , for any quark mass m (section VIII).

The corresponding states have vanishing four-momenta $P = (E, \mathbf{P}) = (0, \mathbf{0})$, and thus do not correspond to physical states. However, the $M = 0$, $J^{PC} = 0^{++}$ “sigma” state can form a vacuum condensate, spontaneously breaking chiral symmetry when $m = 0$. As expected, a chiral transformation of the condensate vacuum creates massless 0^{-+} Goldstone pions. The pion is annihilated by the axial vector current and its divergence.

The work presented here builds on earlier studies, see [19–24] and references therein.

II. PHENOMENOLOGICAL MOTIVATIONS

Experiments have revealed many intriguing features of hadrons. Some are very different from the properties of atoms, suggesting the need for a completely new approach to QCD bound states. The well-known ones include:

- a1. Color charge is confined at a scale $\Lambda_{QCD} \simeq 200$ MeV. Confinement is not described by Feynman diagrams of any finite order, and is thus a non-perturbative, $\mathcal{O}(\alpha^\infty)$ phenomenon.
- a2. Chiral symmetry breaking, indicated by the absence of parity doublets and the small mass of the pion. Chiral symmetry is preserved by all Feynman diagrams in the limit of vanishing quark mass.
- a3. Hadron binding energies are large, *e.g.*, $M_\rho \gg 2m_{u,d}$. This requires relativistic bound state dynamics. The binding energy of Positronium is perturbative, $2m_e - M \simeq \frac{1}{4}\alpha^2 m_e \ll 2m_e$, where $\alpha \simeq 1/137$.
- a4. Strong binding implies that hadron states have Fock components with many $q\bar{q}$ pairs. Sea quarks and gluons give important contributions to hard scattering processes.

Despite this novel QCD dynamics hadrons also have features which resemble those of atoms:

- b1. The charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) spectra resemble those of Positronium². Quarkonia are well described by the Schrödinger equation with a linear plus single gluon exchange potential [15].
- b2. The quark model classification [12] of hadrons as $q\bar{q}$ mesons and qqq baryons works even for light quarks. The relevance of valence quark degrees of freedom indicates weak coupling.
- b3. No glueballs (gg) or hybrid ($q\bar{q}g$) states have been clearly identified. Dynamical gluon degrees of freedom are not manifest in the spectra of hadrons.
- b4. The $\phi(1020) \rightarrow \pi\pi\pi$ decay can proceed via $s\bar{s} \rightarrow ggg \rightarrow u\bar{u}$, but is much suppressed compared to $\phi(1020) \rightarrow K\bar{K}$ (the OZI rule [27, 28]). This and similar selection rules are difficult to understand if α_s is large.

These features, and the central role that perturbative expansions have both in QED and QCD, motivate us to study a perturbative approach to hadrons. The possibilities are restricted by requiring an explicit framework that is consistent with general features of the data and with the theoretical principles of gauge theory.

Higher order perturbative corrections should not qualitatively change a lowest order approximation. Hence the non-perturbative features (a1, a2) must be present already in the first approximation of the bound state.

The large hadron binding energy (a3) is a measure of the confinement scale. In contrast to atoms, it is unrelated to quark masses and the perturbative coupling α_s .

The suppression of gluonic dof’s in hadron spectra (b3) suggests that the A_a^0 field dominates at the scale of hadron binding, as A^0 does in QED atoms. The A_a^0 gluon field acts as an instantaneous potential, not as a constituent.

There are no loops at Born level, so the Λ_{QCD} scale cannot arise through renormalization. A scale can be introduced through a boundary condition on the field equations that express A_a^0 in terms of the propagating fields.

The Klein paradox [29] shows that Dirac states have more than a single constituent. For a linear potential $V(r)$ the local norm of the Dirac wave function is constant at large r [18], inclusively describing a sea of constituents (a4).

Without gluon constituents there is no gluon sea. The large gluon distribution at high scales decreases rapidly when evolved downwards in Q^2 [30]. The gluons could be generated entirely from $\mathcal{O}(\alpha_s)$ radiative effects.

² In recent years multi-quark states have been discovered, the X, Y, Z and Pentaquark states [25, 26]. In a perturbative framework these states would be classified as molecules.

III. THE PERTURBATIVE EXPANSION FOR ATOMS

A. The Potential Picture

A perturbative expansion introduces, as implied by its name, small corrections to the lowest order, unperturbed state. In the case of atoms this means that already the unperturbed state should be bound. An atomic wave function is exponential in the QED coupling α . The perturbative series cannot then be unique, since powers of α may be shifted between the unperturbed wave function and its perturbative corrections.

In 1978 Caswell and Lepage [3] demonstrated the non-uniqueness of the approach introduced by Salpeter and Bethe in 1951 [4]. This led to the development of more efficient perturbative methods for atoms. Expanding the QED action in powers of ∇/m_e (NRQED, [6–8]) takes advantage of the non-relativistic momenta of atomic electrons, $\nabla \sim \alpha m_e$. This still does not determine the choice of unperturbed state. It is natural and practical to perturb around solutions of the Schrödinger equation with the classical $V(r) = -\alpha/r$ potential. The effectiveness of this approach is demonstrated by applications to the hyperfine splitting of Positronium [5].

The subtleties involved in applying perturbation theory to bound states are apparent in the expression for the S -matrix in the Interaction Picture,

$$S_{fi} = {}_{out}\langle f, t \rightarrow \infty | \left\{ \text{T exp} \left[-i \int_{-\infty}^{\infty} dt \mathcal{H}_I(t) \right] \right\} | i, t \rightarrow -\infty \rangle_{in} \quad (3.1)$$

In the IP the Hamiltonian is divided into the free and interacting part,

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I \quad (\text{Interaction Picture}) \quad (3.2)$$

The asymptotic *in* and *out* states at $t = \pm\infty$ are eigenstates of the free Hamiltonian \mathcal{H}_0 ,

$$\mathcal{H}_0 |i\rangle_{in} = E_i |i\rangle_{in} \quad \mathcal{H}_0 |f\rangle_{out} = E_f |f\rangle_{out} \quad (3.3)$$

The asymptotic states are not eigenstates of the full \mathcal{H} , but the interactions generated by \mathcal{H}_I cause them to relax (in the infinite time available) to the physical states of the S -matrix. This requires only that the asymptotic and physical states have a non-vanishing overlap. The possibility to freely choose the asymptotic states is exploited also in the numerical bound state calculations of lattice QCD. There the relaxation is exponential in the euclidean time, and the initial states are optimized for maximal overlap with the physical bound states.

Atoms have no overlap with the free *in* and *out* states of (3.3), since bound state wave functions vanish at large separations of the constituents. Hence the expression (3.1) for the S -matrix is not applicable when the initial *i* or final *f* state contains a bound state. The textbook derivation of the Schrödinger equation through an infinite sum of “ladder diagrams” (e.g., §125. of [31]) is thus not quite correct.

Perturbing around free states is known to be problematic even for QED scattering amplitudes. In the presence of charges a stationary action requires a non-vanishing, classical gauge field. Using free fields gives rise to infrared singularities [32], which need to be regularized by adding soft photons. In hard QED processes, where momentum transfers are large compared to the Bohr momentum αm_e , the IR regularization can be handled order by order. In soft processes such as bound states, on the other hand, expanding around free states is inappropriate.

The division (3.2) of the Hamiltonian into a free and interacting part is arbitrary but convenient. The eigenstates of \mathcal{H}_0 are known and the Feynman diagrams have free propagators. In perturbative corrections to atoms one needs, however, to consider contributions where the electrons propagate in the classical field, see Ch. 14 of [9]. It is then appropriate to include the classical field with the free Hamiltonian. We shall call this the “Potential Picture”,

$$\mathcal{H} = \mathcal{H}_V + \mathcal{H}_I \quad \mathcal{H}_V = \mathcal{H}_0 + \mathcal{H}_I(A_{cl}^0) \quad (\text{Potential Picture}) \quad (3.4)$$

In the Potential Picture the S -matrix is expressed in terms of asymptotic states $| \rangle_V$,

$$S_{fi} = {}_V\langle f, t \rightarrow \infty | \left\{ \text{T exp} \left[-i \int_{-\infty}^{\infty} dt \mathcal{H}_I(t) \right] \right\} | i, t \rightarrow -\infty \rangle_V \quad (3.5)$$

which are eigenstates of \mathcal{H}_V ,

$$\mathcal{H}_V |i\rangle_V = E_{iv} |i\rangle_V \quad \mathcal{H}_V |f\rangle_V = E_{fv} |f\rangle_V \quad (3.6)$$

The classical potential A_{cl}^0 in \mathcal{H}_V binds atoms, as we shall next illustrate for Positronium. This allows an overlap between the $| \rangle_V$ -states and physical atoms. We shall not here prove that (3.5) is a formally valid expression for

the S -matrix, nor discuss the higher order corrections due to \mathcal{H}_I . Care obviously needs to be taken to avoid double counting. This is well-known in bound state perturbation theory, where the Coulomb interaction that binds the Schrödinger atom needs to be subtracted from higher order corrections.

The Potential Picture expresses our principle mentioned in the Introduction: *Initial states of the perturbative expansion are bound by their classical gauge field.*

B. Positronium at Born level

Let us illustrate the approach with Positronium. The QED action is

$$\mathcal{S}_{QED} = \int d^4y \mathcal{L}(y) = \int d^4y \left[\bar{\psi}(i\hat{\partial} - e\hat{A} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right] \quad (3.7)$$

We use a hat on the electromagnetic field operator \hat{A}^μ to distinguish it from the c -number, classical field A^μ . In Coulomb gauge ($\nabla \cdot \hat{\mathbf{A}} = 0$) the equation of motion for the \hat{A}^0 field gives Gauss' law,

$$\frac{\delta \mathcal{S}_{QED}}{\delta \hat{A}^0(t, \mathbf{x})} = 0 \quad \implies \quad -\nabla^2 \hat{A}^0(t, \mathbf{x}) = e\psi^\dagger(t, \mathbf{x})\psi(t, \mathbf{x}) \quad (3.8)$$

This allows to express \hat{A}^0 in terms of the electron field at each instant of time,

$$\hat{A}^0(t, \mathbf{x}) = \int d^3\mathbf{y} \frac{e}{4\pi|\mathbf{x} - \mathbf{y}|} \psi^\dagger(t, \mathbf{y})\psi(t, \mathbf{y}) \quad (3.9)$$

The \hat{A}^0 dependence of \mathcal{S}_{QED} through the field tensor term is contained in

$$\int d^4y \left(-\frac{1}{2}F_{i0}F^{i0} \right) = \frac{1}{2} \int d^4y \left[\bar{\psi} e\gamma^0 \hat{A}^0 \psi - (\partial_0 \hat{A}^i)^2 \right] \quad (3.10)$$

where we used (3.8). The first term cancels half of the \hat{A}^0 contribution to the fermion part of \mathcal{S}_{QED} . The \hat{A}^i field gives a contribution of higher order in α for non-relativistic bound states at rest.

The Potential Hamiltonian \mathcal{H}_V in (3.4) includes the \hat{A}^0 part of \mathcal{H}_{QED} . With $\boldsymbol{\alpha} \equiv \gamma^0 \boldsymbol{\gamma}$,

$$\mathcal{H}_V(\hat{A}^0) = \int d\mathbf{x} \psi^\dagger(t, \mathbf{x}) H_V \psi(t, \mathbf{x}) \quad H_V = -i\nabla \cdot \boldsymbol{\alpha} + m\gamma^0 + \frac{1}{2}e\hat{A}^0 \quad (3.11)$$

The classical A^0 field is given by the expectation value of \hat{A}^0 in a state where the electron and positron are at specific positions \mathbf{x}_1 and \mathbf{x}_2 ,

$$|\mathbf{x}_1, \mathbf{x}_2\rangle \equiv \bar{\psi}_\alpha(t, \mathbf{x}_1) \psi_\beta(t, \mathbf{x}_2) |0\rangle \quad (3.12)$$

Using (3.9) and the canonical anticommutation relations of the electron field,

$$\frac{\langle \mathbf{x}_1, \mathbf{x}_2 | e\hat{A}^0(\mathbf{x}) | \mathbf{x}_1, \mathbf{x}_2 \rangle}{\langle \mathbf{x}_1, \mathbf{x}_2 | \mathbf{x}_1, \mathbf{x}_2 \rangle} = \frac{\alpha}{|\mathbf{x} - \mathbf{x}_1|} - \frac{\alpha}{|\mathbf{x} - \mathbf{x}_2|} \equiv eA^0(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2) \quad (3.13)$$

It is important to note that the classical field depends on the state $|\mathbf{x}_1, \mathbf{x}_2\rangle$, *i.e.*, on the positions $\mathbf{x}_1, \mathbf{x}_2$ of the charges.

A Positronium state may be expressed as a superposition of the states (3.12), weighted by a wave function Φ . For a rest frame state of mass M at $t = 0$,

$$|M\rangle = \sum_{\alpha, \beta} \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}_\alpha(t=0, \mathbf{x}_1) \Phi_{\alpha\beta}(\mathbf{x}_1 - \mathbf{x}_2) \psi_\beta(t=0, \mathbf{x}_2) |0\rangle \quad (3.14)$$

In order to qualify as an asymptotic state $|\rangle_V$ of the S -matrix (3.5), $|M\rangle$ should be an eigenstate of \mathcal{H}_V evaluated with the classical potential (3.13),³

$$\mathcal{H}_V(A^0) |M\rangle = M |M\rangle \quad (3.15)$$

³ This requires adjusting A^0 according to the state component on which \mathcal{H}_V is acting. Equivalently, we may use the operator valued gauge field in the Hamiltonian given by (3.11), $\mathcal{H}_V^{int}(\hat{A}^0) = \frac{1}{2} \int d\mathbf{x} d\mathbf{y} \psi^\dagger(t, \mathbf{x}) \alpha / |\mathbf{x} - \mathbf{y}| \psi^\dagger(t, \mathbf{y})$.

The canonical anticommutation relations and $\mathcal{H}_V |0\rangle = 0$ (no e^+e^- pair production in the non-relativistic limit) gives the bound state equation for Φ . The potential energy is determined by the value of A^0 at $\mathbf{x} = \mathbf{x}_1, \mathbf{x}_2$. Subtracting $\mathbf{x}_1, \mathbf{x}_2$ -independent (infinite) constants,

$$V(|\mathbf{x}_1 - \mathbf{x}_2|) = \frac{1}{2} [eA^0(\mathbf{x}_1; \mathbf{x}_1, \mathbf{x}_2) - eA^0(\mathbf{x}_2; \mathbf{x}_1, \mathbf{x}_2)] = -\frac{\alpha}{|\mathbf{x}_1 - \mathbf{x}_2|} \quad (3.16)$$

Adding the kinetic \mathcal{H}_0 contribution and denoting $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ the bound state equation becomes

$$[i\boldsymbol{\alpha} \cdot \vec{\nabla} + m\gamma^0]\Phi(\mathbf{x}) + \Phi(\mathbf{x})[i\boldsymbol{\alpha} \cdot \overleftarrow{\nabla} - m\gamma^0] = [M - V(|\mathbf{x}|)]\Phi(\mathbf{x}) \quad (3.17)$$

For non-relativistic dynamics this reduces to the Schrödinger equation (see [24] for details). Adding the perturbative interactions due to \mathcal{H}_I in (3.5) gives contributions of higher order in α to the Positronium mass M . Here we limit ourselves to binding induced by the classical A^0 field. We refer to this as “Born level” in view of the \hbar expansion.

IV. THE QCD CONFINEMENT POTENTIAL

A. The instantaneous color field \hat{A}_a^0

The operator equation of motion for \hat{A}_a^0 in QCD is,

$$\begin{aligned} \frac{\delta}{\delta \hat{A}_a^0} S_{QCD} = 0 \quad \implies \quad \partial_i F_a^{i0} &= -gf_{abc}A_b^i F_c^{i0} + g\psi_A^\dagger T_a^{AB}\psi_B \\ F_a^{i0} &= \partial^i A_a^0 - \partial^0 A_a^i - gf_{abc}A_b^i A_c^0 \end{aligned} \quad (4.1)$$

A confining potential involves a scale $\sim \Lambda_{QCD}$, which is not present in (4.1). In the absence of loop contributions a scale can only arise from a boundary condition. The field must be instantaneous to avoid retardation effects, thus the propagating vector field $\hat{\mathbf{A}}$ is excluded.

Consider the $\mathcal{O}(g^0)$ homogeneous and instantaneous solution

$$\hat{A}_a^0(t, \mathbf{x}) = \kappa \sum_{B,C} \int d\mathbf{y} (\mathbf{x} \cdot \mathbf{y}) \psi_B^\dagger(t, \mathbf{y}) T_a^{BC} \psi_C(t, \mathbf{y}) \quad \hat{A}_a^i(t, \mathbf{x}) = 0 \quad (4.2)$$

with a normalization $\kappa \neq \kappa(\mathbf{x})$ of dimension GeV^3 . The field equations (4.1) are satisfied at $\mathcal{O}(g^0)$ since $\nabla^2 \hat{A}_a^0(\mathbf{x})$ and the 3-vector components $\hat{A}_a^i(\mathbf{x})$ are of $\mathcal{O}(g)$. Here and in the following we only consider fields of $\mathcal{O}(g^0)$. Higher order contributions in g (including gluon exchange) arise from \mathcal{H}_I in (3.5). They generate dynamical (transverse) gluon constituents in hadrons which will require to include also gluon fields in the expression (4.2) for \hat{A}_a^0 .

The expression (4.2) for \hat{A}_a^0 is mandated by symmetry considerations. The dot product $\mathbf{x} \cdot \mathbf{y}$ is required by rotational invariance, and global gauge covariance is ensured by the color generators T_a^{BC} . A higher power of $\mathbf{x} \cdot \mathbf{y}$ would not satisfy $\nabla^2 \hat{A}_a^0(\mathbf{x}) = 0$. As we shall see, the linear dependence on \mathbf{x} gives a translation invariant potential for color singlet meson and baryon states.

The color electric field $\nabla \hat{A}_a^0(\mathbf{x})$ is independent of \mathbf{x} . Consequently the integral over the field energy density will diverge with the volume of space. This contribution is irrelevant only if all of meson and baryon state components have the same energy density. This determines (for each state) the coefficient κ up to a universal scale Λ , which characterizes the field energy density.

B. The classical potential for mesons

We express the Born level meson state at $t = 0$ analogously to Positronium (3.14),

$$|M\rangle = \sum_{A,B} \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}^A(t=0, \mathbf{x}_1) \Phi^{AB}(\mathbf{x}_1 - \mathbf{x}_2) \psi^B(t=0, \mathbf{x}_2) |0\rangle \quad (4.3)$$

There are no gluon constituents since the meson is at $\mathcal{O}(\hbar^0)$ bound by the instantaneous A_a^0 field. The wave function is taken to have the standard color singlet structure⁴

$$\Phi^{AB}(\mathbf{x}_1 - \mathbf{x}_2) = \frac{1}{\sqrt{N_C}} \delta^{AB} \Phi(\mathbf{x}_1 - \mathbf{x}_2) \quad (N_C = 3) \quad (4.4)$$

Analogously to QED (3.13) the classical potential is given by the expectation value of \hat{A}_a^0 (4.2) for each space $(\mathbf{x}_1, \mathbf{x}_2)$ and color (A) component of the meson state⁵,

$$|\mathbf{x}_1^A, \mathbf{x}_2^A\rangle \equiv \bar{\psi}_\alpha^A(t, \mathbf{x}_1) \psi_\beta^A(t, \mathbf{x}_2) |0\rangle \quad (4.5)$$

(no sum on A)

$$\frac{\langle \mathbf{x}_1^A, \mathbf{x}_2^A | \hat{A}_a^0(t, \mathbf{x}) | \mathbf{x}_1^A, \mathbf{x}_2^A \rangle}{\langle \mathbf{x}_1^A, \mathbf{x}_2^A | \mathbf{x}_1^A, \mathbf{x}_2^A \rangle} = \kappa(\mathbf{x}_1, \mathbf{x}_2) \mathbf{x} \cdot (\mathbf{x}_1 - \mathbf{x}_2) T_a^{AA} \quad (4.6)$$

As indicated, the normalization κ may depend on the state component (4.5). In order to have a universal field energy density we need to choose $\kappa \propto 1/|\mathbf{x}_1 - \mathbf{x}_2|$. Thus we get for the classical field,

$$\bar{\psi}_\alpha^A(\mathbf{x}_1) \psi_\beta^A(\mathbf{x}_2) |0\rangle : \quad A_a^0(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2, A) = \left[\mathbf{x} - \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2) \right] \cdot \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|} T_a^{AA} 6\Lambda^2 \quad (\text{no sum on } A) \quad (4.7)$$

where Λ is a universal constant of dimension GeV. The \mathbf{x} -independent term $\propto \mathbf{x}_1 + \mathbf{x}_2$ was added for esthetical reasons. It does not affect the dynamics but ensures translation invariance for A_a^0 itself ($\mathbf{x} \rightarrow \mathbf{x} + \boldsymbol{\ell}$ with $\mathbf{x}_{1,2} \rightarrow \mathbf{x}_{1,2} + \boldsymbol{\ell}$).

Only A_3^0 and A_8^0 are $\neq 0$, in the standard convention where $T_3 = \text{diag}(1, -1, 0)/2$ and $T_8 = \text{diag}(1, 1, -2)/2\sqrt{3}$ are the only generators with diagonal elements. Conversely, this ensures that interactions preserve the quark colors and thus the color singlet structure of the state. Using the generator identity

$$\sum_a T_a^{AB} T_a^{CD} = \frac{1}{2} \delta^{AD} \delta^{BC} - \frac{1}{6} \delta^{AB} \delta^{CD} \quad (4.8)$$

we see that the field energy density is independent of \mathbf{x} , \mathbf{x}_1 , \mathbf{x}_2 and the quark color A ,

$$\sum_a [\nabla_x A_a^0(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2, A)]^2 = 12\Lambda^4 \quad (4.9)$$

The integral of this energy density over all space (the total field energy) is \propto the volume of space but irrelevant because it is the same (infinite) constant for all components of the meson.

The $\mathcal{O}(g^0)$ constant $\Lambda \sim \Lambda_{QCD}$ defines the confinement scale. The potential energy for the state (4.7) may be anticipated (including a conventional factor $\frac{1}{2}$ motivated by the QED Hamiltonian (3.11)),

$$V(\mathbf{x}_1 - \mathbf{x}_2) = \frac{1}{2}g \sum_a T_a^{AA} [A_a^0(\mathbf{x}_1; \mathbf{x}_1, \mathbf{x}_2, A) - A_a^0(\mathbf{x}_2; \mathbf{x}_1, \mathbf{x}_2, A)] = g\Lambda^2 |\mathbf{x}_1 - \mathbf{x}_2| \equiv V' |\mathbf{x}_1 - \mathbf{x}_2| \quad (\text{no sum on } A) \quad (4.10)$$

Since the potential (and kinetic) energy is independent of the quark color A the color dependence will factorize in the bound state equation.

A homogenous solution for \hat{A}_a^0 thus leads to a linear classical potential for mesons rather uniquely, considering Poincaré and color symmetry, as well as the universal value of the field energy density. Since $\text{Tr } T_a = 0$ we also have

$$\sum_A A_a^0(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2, A) = 0 \quad (4.11)$$

The total classical field of a color singlet meson (summed over quark colors) vanishes at all points \mathbf{x} . An external probe (another hadron) therefore experiences no classical field. This differs from QED, where there is no sum over charges and the classical (dipole) field (3.13) is non-zero.

⁴ There is no gauge link between the quarks so Φ^{AB} is a singlet only under global gauge transformations.

⁵ Alternatively, we may determine the potential using the operator valued gauge field $\hat{A}_a^0(\mathbf{x})$ in the Hamiltonian, see section VI B of [22].

C. The classical potential for baryons

A baryon qqq state at $t = 0$ is expressed as

$$|M\rangle = \sum_{A,B,C} \int d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 \psi_A^\dagger(t=0, \mathbf{x}_1) \psi_B^\dagger(t=0, \mathbf{x}_2) \psi_C^\dagger(t=0, \mathbf{x}_3) \Phi^{ABC}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) |0\rangle \quad (4.12)$$

The wave function is assumed to have the standard color singlet structure

$$\Phi^{ABC}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \epsilon^{ABC} \Phi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \quad (4.13)$$

The fully antisymmetric color tensor is defined with $\epsilon^{123} = \epsilon_{123} = 1$. A given component of the baryon state

$$|\mathbf{x}_1^A, \mathbf{x}_2^B, \mathbf{x}_3^C\rangle \equiv \psi_A^\dagger(t, \mathbf{x}_1) \psi_B^\dagger(t, \mathbf{x}_2) \psi_C^\dagger(t, \mathbf{x}_3) |0\rangle \quad (A \neq B \neq C) \quad (4.14)$$

gives the expectation value for \hat{A}_a^0 defined in (4.2),

$$\frac{\langle \mathbf{x}_1^A, \mathbf{x}_2^B, \mathbf{x}_3^C | \hat{A}_a^0(t, \mathbf{x}) | \mathbf{x}_1^A, \mathbf{x}_2^B, \mathbf{x}_3^C \rangle}{\langle \mathbf{x}_1^A, \mathbf{x}_2^B, \mathbf{x}_3^C | \mathbf{x}_1^A, \mathbf{x}_2^B, \mathbf{x}_3^C \rangle} = \kappa(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \mathbf{x} \cdot (\mathbf{x}_1 T_a^{AA} + \mathbf{x}_2 T_a^{BB} + \mathbf{x}_3 T_a^{CC}) \quad (4.15)$$

The coefficient κ may then be chosen so that, for ABC a permutation of 123,

$$\psi_A^\dagger(\mathbf{x}_1) \psi_B^\dagger(\mathbf{x}_2) \psi_C^\dagger(\mathbf{x}_3) |0\rangle : \quad (4.16)$$

$$A_a^0(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ABC) = [\mathbf{x} - \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)] \cdot (T_a^{AA} \mathbf{x}_1 + T_a^{BB} \mathbf{x}_2 + T_a^{CC} \mathbf{x}_3) \frac{6\Lambda^2}{d(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)} \quad (\text{no sum on colors})$$

where
$$d(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \frac{1}{\sqrt{2}} \sqrt{(\mathbf{x}_1 - \mathbf{x}_2)^2 + (\mathbf{x}_2 - \mathbf{x}_3)^2 + (\mathbf{x}_3 - \mathbf{x}_1)^2} \quad (4.17)$$

The \mathbf{x} -independent term $\propto \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3$ does not affect the dynamics and was added to make the classical field invariant under translations. The normalization κ in (4.15) was chosen so that the field energy density for each baryon component agrees with that of mesons in (4.9). For $A \neq B \neq C$ we thus have, using (4.8),

$$\sum_a |\nabla_x A_a^0(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ABC)|^2 = 12\Lambda^4 \quad (4.18)$$

The classical A_a^0 field is non-vanishing only for $a = 3, 8$ and invariant under translations since $T_a^{AA} + T_a^{BB} + T_a^{CC} = \text{Tr } T_a = 0$. It confines the quarks in the two relative directions of the three quarks. For $ABC = 123$ we have $A_3^0 \propto \mathbf{x}_1 - \mathbf{x}_2$ and $A_8^0 \propto \mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3$. It appears that no analogous confining solution exists for four or more quarks, as the number of relative directions then exceeds the number of diagonal color generators.

The potential energy arising from the interactions of the classical field (4.16) with the quarks is, including the conventional factor $\frac{1}{2}$ as in (4.10),

$$\begin{aligned} V_B(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= \frac{1}{2}g \sum_a [T_a^{AA} A_a^0(\mathbf{x} = \mathbf{x}_1) + T_a^{BB} A_a^0(\mathbf{x} = \mathbf{x}_2) + T_a^{CC} A_a^0(\mathbf{x} = \mathbf{x}_3)] \\ &= \frac{3g\Lambda^2}{d(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)} \sum_a (T_a^{AA} \mathbf{x}_1 + T_a^{BB} \mathbf{x}_2 + T_a^{CC} \mathbf{x}_3)^2 = g\Lambda^2 d(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \end{aligned} \quad (4.19)$$

The fact that V_B is independent of the quark color permutation ABC implies, as for mesons, that the interactions maintain the color structure of the wave function (4.13). The baryon potential reduces to the meson potential (4.10) when two quarks are in the same position, $d(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2) = |\mathbf{x}_1 - \mathbf{x}_2|$.

As expected due to color symmetry, the color singlet baryon state described by the wave function (4.13) does not generate a color field. Summing over the quark color permutations ABC gives the total color a gluon field,

$$\begin{aligned} \sum_{A,B,C} \epsilon^{ABC} A_a^0(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ABC) \\ = \frac{6\Lambda^2}{d(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)} [\mathbf{x} - \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)] \cdot \sum_{A,B,C} \epsilon^{ABC} (T_a^{AA} \mathbf{x}_1 + T_a^{BB} \mathbf{x}_2 + T_a^{CC} \mathbf{x}_3) = 0 \end{aligned} \quad (4.20)$$

which vanishes since the sum of even (as well as odd) permutations of ABC gives for each \mathbf{x}_i ($i = 1, 2, 3$) a coefficient $\propto \text{Tr } T_a = 0$. Thus the classical color field is invisible to an external observer (another hadron).

We do not consider the bound state dynamics of baryons further in this work.

V. MESONS IN THE REST FRAME

We now consider $q\bar{q}$ bound states at rest, in the Born approximation outlined in section III A where they appear as asymptotic states of the S -matrix (3.5). They are eigenstates of the potential Hamiltonian \mathcal{H}_V which includes interactions due to the classical gluon field only, similarly as in (3.15) for QED. The quark Hamiltonian at $t = 0$ is⁶

$$\mathcal{H}_V(A^0) = \int d\mathbf{x} \psi^\dagger(\mathbf{x}) H_V \psi(\mathbf{x}) \quad H_V(A^0) = -i\boldsymbol{\alpha} \cdot \vec{\nabla} + m\gamma^0 + \frac{1}{2}g \sum_a A_a^0(\mathbf{x}) T_a \quad (5.1)$$

where the classical gluon field A_a^0 is given in (4.7) for the $q^A(\mathbf{x}_1)\bar{q}^A(\mathbf{x}_2)$ meson component. The field energy density (4.9) is independent of \mathbf{x}_1 , \mathbf{x}_2 and the quark color A and thus irrelevant for the bound state dynamics. Eigenstates of the momentum operator (A.3) with any eigenvalue \mathbf{P} can (at $t = 0$) be expressed as

$$|M, P\rangle \equiv \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}_A(t=0, \mathbf{x}_1) e^{i\mathbf{P} \cdot (\mathbf{x}_1 + \mathbf{x}_2)/2} \Phi_{AB}^{(\xi)}(\mathbf{x}_1 - \mathbf{x}_2) \psi_B(t=0, \mathbf{x}_2) |0\rangle \quad (5.2)$$

where ξ denotes the boost parameter ($E = M \cosh|\xi|$) and the wave function is a (global) color singlet,

$$\Phi_{AB}^{(\xi)}(\mathbf{x}_1 - \mathbf{x}_2) = \frac{1}{\sqrt{N_C}} \delta_{AB} \Phi^{(\xi)}(\mathbf{x}_1 - \mathbf{x}_2) \quad (5.3)$$

In appendix A we note how the rest frame wave function $\Phi^{(0)} \equiv \Phi$ transforms under gauge transformations, rotations, parity and charge conjugation of the state. In subsection V A we discuss the effects of quark pair creation (string breaking and the $q\bar{q}$ sea). We then derive the bound state equation for $\Phi(\mathbf{x}_1 - \mathbf{x}_2)$ in subsection V B and show how the radial and angular variables can be separated in subsection V C. The rest frame wave functions are determined for states of specific J^{PC} in subsections V D, V E and V F.

A. Pair production effects

The linear growth of the potential (4.10) is quenched by “string breaking”, *i.e.*, quark pair production by the $\mathcal{O}(g^0)$ classical gluon field (4.7). As shown in Fig. 1(a) a quark pair can be produced at any separation $\delta_1 + \delta_2$ between the quarks of hadron a , and may fuse again to give the hadron loop contribution of Fig. 1(b). These processes are required for unitarity at the $\mathcal{O}(\hbar^0)$ Born level.

Here we neglect string breaking and hadron loop effects, working at leading order in $1/N_c$ (N_c is the number of colors). The bound states will be stable as in dual amplitudes, and the Hamiltonian annihilates the vacuum,

$$\mathcal{H}_V |0\rangle = 0 \quad (5.4)$$

Let us note that the hadron states defined at leading order in $1/N_c$ determine contributions like Fig. 1(a,b). The decay matrix element $\langle b c | a \rangle$ at $t = 0$ can be expressed in terms of the wave functions defined in (5.2). Summing over colors using (5.3) and indicating closed Dirac structures by square brackets,

$$\begin{aligned} \langle b c | a \rangle_{t=0} &= \frac{1}{\sqrt{N_c}} \int \left[\prod_{k=a,b,c} d\mathbf{x}_1^k d\mathbf{x}_2^k \right] \exp \left\{ \frac{1}{2} i [(\mathbf{x}_1^a + \mathbf{x}_2^a) \cdot \mathbf{P}_a - (\mathbf{x}_1^b + \mathbf{x}_2^b) \cdot \mathbf{P}_b - (\mathbf{x}_1^c + \mathbf{x}_2^c) \cdot \mathbf{P}_c] \right\} \\ &\quad \times \langle 0 | [\psi^\dagger(\mathbf{x}_2^b) \Phi_b^{(\xi_b)\dagger} \gamma^0 \psi(\mathbf{x}_1^b)] [\psi^\dagger(\mathbf{x}_2^c) \Phi_c^{(\xi_c)\dagger} \gamma^0 \psi(\mathbf{x}_1^c)] [\psi^\dagger(\mathbf{x}_1^a) \gamma^0 \Phi_a^{(\xi_a)} \psi(\mathbf{x}_2^a)] | 0 \rangle \\ &= -\frac{(2\pi)^3}{\sqrt{N_c}} \delta^3(\mathbf{P}_a - \mathbf{P}_b - \mathbf{P}_c) \int d\delta_1 d\delta_2 \exp \left\{ \frac{1}{2} i [\delta_1 \cdot \mathbf{P}_c - \delta_2 \cdot \mathbf{P}_b] \right\} \text{Tr} \left[\gamma^0 \Phi_b^{(\xi_b)\dagger}(\delta_1) \Phi_a^{(\xi_a)}(\delta_1 + \delta_2) \Phi_c^{(\xi_c)\dagger}(\delta_2) \right] \end{aligned} \quad (5.5)$$

$$(5.6)$$

The field contractions in (5.5) are according to Fig. 1(a) and we denoted $\delta_1 = \mathbf{x}_1^a - \mathbf{x}_2^b$, $\delta_2 = \mathbf{x}_2^b - \mathbf{x}_2^c$. When integrated over time the factor $\exp[it(E_b + E_c - E_a)]$ gives energy conservation for real decays $a \rightarrow b + c$.

⁶ We include the factor $\frac{1}{2}$ in the interaction term for analogy with the QED case (3.11).

Higher orders in $1/N_c$, such as the hadron loop contribution Fig. 1(b), can be evaluated similarly. Thus it should be possible to establish whether unitarity is satisfied at $\mathcal{O}(\hbar^0)$, for each order of $1/N_c$.

Even when string breaking is neglected there are virtual $q\bar{q}$ pair contributions due to time ordering, as illustrated by the “Z”-diagram of Fig. 1(c). The “Z”-contributions grow with the strength of the potential. As discussed in section VI A, the local norm $|\mathbf{x}|^2 \text{Tr} [\Phi^\dagger(\mathbf{x})\Phi(\mathbf{x})]$ of the wave function tends to a constant at large $|\mathbf{x}|$. Thus the number of virtual pairs is proportional to the linear potential. They give rise to a sea quark distribution in deep inelastic scattering, as demonstrated in $D = 1 + 1$ dimensions [21]. When hadron loop corrections like Fig. 1(b) are taken into account the sea quarks will be constituents of the hadrons in the loop.

At $\mathcal{O}(g^0)$ only the instantaneous gluons of the classical field (4.7) contribute. Constituent gluon contributions (glueballs, hybrids) to the observed hadron spectrum are suppressed. The gluon distribution measured in DIS will here arise from radiative effects at higher scales, due to contributions of \mathcal{H}_I in (3.5). DIS phenomenology indicates that a sea of $q\bar{q}$ pairs is present already at low scales, whereas gluons may arise through DGLAP evolution [30].

“Z”-diagram contributions like Fig. 1(c) can be studied also in the simpler context of Dirac bound states (see appendix B). The fermion interacts only with a fixed external field, without string breaking. The local norm of the Dirac wave function tends to a constant at large values of a linear potential. The strong field causes a Bogoliubov transform (B.6) of the creation and annihilation operators. Hence the Dirac vacuum (B.7) has Fock components with any number of pairs in the free operator basis. The b and d annihilation operators analogously contribute to the state (5.2).

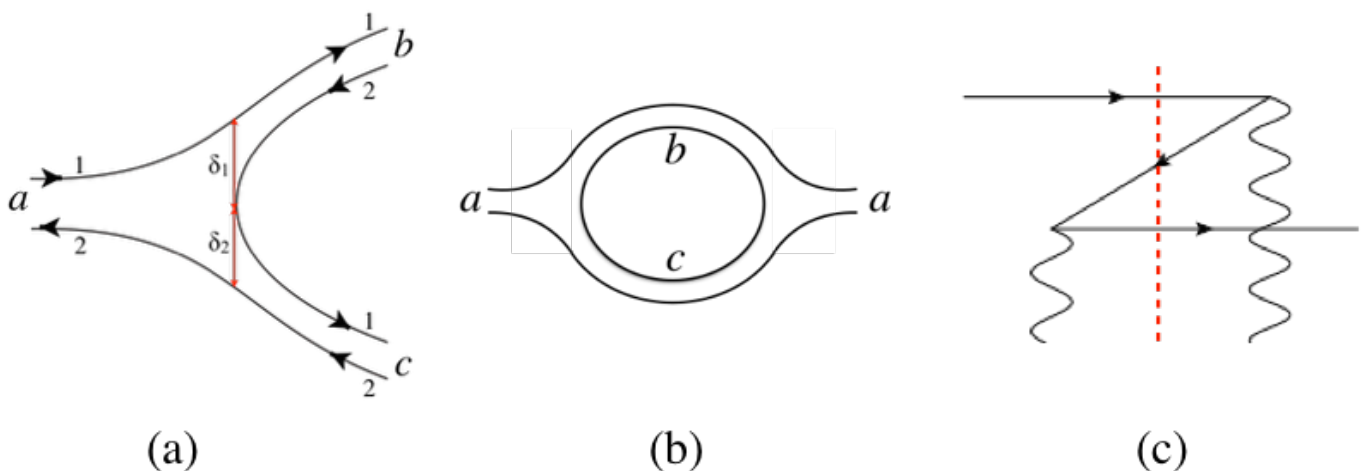


FIG. 1: (a) $a \rightarrow b + c$ through quark pair creation in the confining field. (b) Combination of splitting and fusion leading to a hadron loop correction to bound state a . (c) Time-ordered Z-diagram contribution with an intermediate $q\bar{q}$ pair.

B. Bound state equation

The state $|M\rangle \equiv |M, \mathbf{P} = 0\rangle$ (5.2) with the color singlet wave function (5.3) should be an eigenstate of the Hamiltonian (5.1) with the classical gluon field (4.7),

$$\mathcal{H}_V |M\rangle = M |M\rangle \quad (5.7)$$

According to the discussion in the previous subsection we use an iterative approach (expansion in $1/N_c$) where particle production (string breaking) is neglected at lowest order, as in (5.4). The free part of the Hamiltonian

$$\mathcal{H}_0 = \sum_C \int d\mathbf{x} \psi^{\dagger C}(\mathbf{x}) (-i\boldsymbol{\alpha} \cdot \vec{\nabla} + m\gamma^0) \psi^C(\mathbf{x}) \quad (5.8)$$

acts on the fields similarly as in QED,

$$[\mathcal{H}_0, \bar{\psi}(\mathbf{x})] = \bar{\psi}(\mathbf{x}) (-i\boldsymbol{\alpha} \cdot \overleftarrow{\nabla} + m\gamma^0) \quad [\mathcal{H}_0, \psi(\mathbf{x})] = -(-i\boldsymbol{\alpha} \cdot \vec{\nabla} + m\gamma^0) \psi(\mathbf{x}) \quad (5.9)$$

The potential term operating on the component $\bar{\psi}^A(\mathbf{x}_1)\psi^A(\mathbf{x}_2)|0\rangle$ (no sum on A) gives, since $A_a^0 = 0$ for $a \neq 3, 8$

$$\begin{aligned} & \frac{1}{2} \int d\mathbf{x} \sum_{a,C,D} [\psi^{C\dagger}(\mathbf{x})gA_a^0(\mathbf{x})T_a^{CD}\psi^D(\mathbf{x}), \bar{\psi}^A(\mathbf{x}_1)\psi^A(\mathbf{x}_2)]|0\rangle \\ &= \frac{1}{2} \left[\sum_{a,C} \bar{\psi}^C(\mathbf{x}_1)gA_a^0(\mathbf{x}_1)T_a^{CA}\psi^A(\mathbf{x}_2) - \sum_{a,D} \bar{\psi}^A(\mathbf{x}_1)gA_a^0(\mathbf{x}_2)T_a^{AD}\psi^D(\mathbf{x}_2) \right] |0\rangle \\ &= \sum_{a=3,8} \frac{1}{2} g T_a^{AA} [A_a^0(\mathbf{x}_1) - A_a^0(\mathbf{x}_2)] \bar{\psi}^A(\mathbf{x}_1)\psi^A(\mathbf{x}_2)|0\rangle = V(\mathbf{x}_1 - \mathbf{x}_2) \bar{\psi}^A(\mathbf{x}_1)\psi^A(\mathbf{x}_2)|0\rangle \end{aligned} \quad (5.10)$$

with the linear potential energy $V(\mathbf{x}_1 - \mathbf{x}_2) = g\Lambda^2|\mathbf{x}_1 - \mathbf{x}_2|$ anticipated in (4.10). The color dependence factorizes from the bound state equation since V is independent of the quark color A .

The bound state condition (5.7) thus requires, for the color reduced wave function Φ (5.3) (with no sum on A),

$$\begin{aligned} & \int d\mathbf{x}_1 d\mathbf{x}_2 \left[\bar{\psi}^A(\mathbf{x}_1) (-i\boldsymbol{\alpha} \cdot \overleftarrow{\nabla}_1 + m\gamma^0) \Phi(\mathbf{x}_1 - \mathbf{x}_2) \psi^A(\mathbf{x}_2) - \bar{\psi}^A(\mathbf{x}_1) \Phi(\mathbf{x}_1 - \mathbf{x}_2) (-i\boldsymbol{\alpha} \cdot \overrightarrow{\nabla}_2 + m\gamma^0) \psi^A(\mathbf{x}_2) \right] |0\rangle \\ &= \int d\mathbf{x}_1 d\mathbf{x}_2 [M - V(\mathbf{x}_1 - \mathbf{x}_2)] \bar{\psi}^A(\mathbf{x}_1) \Phi(\mathbf{x}_1 - \mathbf{x}_2) \psi^A(\mathbf{x}_2) |0\rangle \end{aligned} \quad (5.11)$$

where $\nabla_j \equiv \partial/\partial\mathbf{x}_j$. After partial integrations the bound state equation for $\Phi(\mathbf{x}_1 - \mathbf{x}_2)$ becomes, with $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$,

$$i\nabla \cdot \{\boldsymbol{\alpha}, \Phi(\mathbf{x})\} + m[\gamma^0, \Phi(\mathbf{x})] = [M - V(\mathbf{x})]\Phi(\mathbf{x}) \quad (5.12)$$

The (anti-)commutators refer only to the Dirac algebra. The derivatives are assumed to act on functions to their right, unless indicated to act to the left ($\overleftarrow{\nabla}$, $\overleftarrow{\partial}$). The bound state equation (5.12) may equivalently be expressed as

$$\left[\frac{2}{M - V} (i\boldsymbol{\alpha} \cdot \overrightarrow{\nabla} + m\gamma^0) - 1 \right] \Phi(\mathbf{x}) + \Phi(\mathbf{x}) \left[(i\boldsymbol{\alpha} \cdot \overleftarrow{\nabla} - m\gamma^0) \frac{2}{M - V} - 1 \right] = 0 \quad (5.13)$$

$$V(\mathbf{x}) = g\Lambda^2|\mathbf{x}| \quad (5.14)$$

It is convenient to define

$$\overrightarrow{\Lambda}_{\pm} \equiv \frac{2}{M - V} (i\boldsymbol{\alpha} \cdot \overrightarrow{\nabla} + m\gamma^0) \pm 1 \quad \overleftarrow{\Lambda}_{\pm} \equiv (i\boldsymbol{\alpha} \cdot \overleftarrow{\nabla} - m\gamma^0) \frac{2}{M - V} \pm 1 \quad (5.15)$$

which satisfy

$$\begin{aligned} \overrightarrow{\Lambda}_- \overrightarrow{\Lambda}_+ &= \frac{4}{(M - V)^2} (-\overrightarrow{\nabla}^2 + m^2) - 1 + \frac{4iV'}{r(M - V)^3} \boldsymbol{\alpha} \cdot \mathbf{x} (i\boldsymbol{\alpha} \cdot \overrightarrow{\nabla} + m\gamma^0) \\ \overleftarrow{\Lambda}_+ \overleftarrow{\Lambda}_- &= (-\overleftarrow{\nabla}^2 + m^2) \frac{4}{(M - V)^2} - 1 + (i\boldsymbol{\alpha} \cdot \overleftarrow{\nabla} - m\gamma^0) \boldsymbol{\alpha} \cdot \mathbf{x} \frac{4iV'}{r(M - V)^3} \end{aligned} \quad (5.16)$$

Using this notation the bound state equation (5.13) is

$$\overrightarrow{\Lambda}_- \Phi(\mathbf{x}) + \Phi(\mathbf{x}) \overleftarrow{\Lambda}_- = 0 \quad (5.17)$$

C. Separation of radial and angular variables

The (color reduced) wave function of the meson state (4.3) may be expressed as a sum of terms with distinct Dirac structures $\Gamma_i(\mathbf{x})$ and radial functions $F_i(r)$, having a common spherical harmonic $Y_{j\lambda}(\hat{\mathbf{x}})$:

$$\Phi(\mathbf{x}) = \sum_i \Gamma_i(\mathbf{x}) F_i(r) Y_{j\lambda}(\hat{\mathbf{x}}) \quad (5.18)$$

where $r = |\mathbf{x}|$ and $\hat{\mathbf{x}} = \mathbf{x}/r$. Provided the Dirac structures are rotationally invariant, $[\mathbf{J}, \Gamma_i(\mathbf{x})] = 0$ with $\mathbf{J} = \mathbf{L} + \mathbf{S}$ (A.6), the meson state will be an eigenstate (A.7) of the angular momentum operators of \mathcal{J}^2 and \mathcal{J}^z with eigenvalues $j(j+1)$ and λ , respectively.

The $\Gamma_i(\mathbf{x})$ need contain at most one power of the Dirac vector $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$ since higher powers may be reduced using $\alpha^i \alpha^j = \delta^{ij} + i\epsilon_{ijk} \alpha^k \gamma_5$. Rotational invariance requires that $\boldsymbol{\alpha}$ be dotted into a vector. We choose as basis the three orthogonal vectors \mathbf{x} , $\mathbf{L} = \mathbf{x} \times (-i\nabla)$ and $\mathbf{x} \times \mathbf{L}$. Each of the four Dirac structures 1, $\boldsymbol{\alpha} \cdot \mathbf{x}$, $\boldsymbol{\alpha} \cdot \mathbf{L}$ and $\boldsymbol{\alpha} \cdot \mathbf{x} \times \mathbf{L}$ can be multiplied by the rotationally invariant Dirac matrices γ^0 and/or γ_5 . This gives altogether $4 \times 2 \times 2 = 16$ possible $\Gamma_i(\mathbf{x})$. Other invariants may be expressed in terms of these, *e.g.*,

$$i\boldsymbol{\alpha} \cdot \nabla = (\boldsymbol{\alpha} \cdot \mathbf{x}) \frac{1}{r} i\partial_r + \frac{1}{r^2} \boldsymbol{\alpha} \cdot \mathbf{x} \times \mathbf{L} \quad (5.19)$$

$$(\boldsymbol{\alpha} \cdot \nabla)(\boldsymbol{\alpha} \cdot \mathbf{x}) = 3 + r\partial_r + \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{L} \quad (5.20)$$

The $\Gamma_i(\mathbf{x})$ may be grouped according to the parity η_P (A.12) and charge conjugation η_C (A.16) quantum numbers that they imply for the wave function. Since $Y_{j\lambda}(-\hat{\mathbf{x}}) = (-1)^j Y_{j\lambda}(\hat{\mathbf{x}})$ states of spin j can belong to one of four “trajectories”, here denoted by the parity and charge conjugation quantum numbers of their $j = 0$ member⁷:

$$\begin{aligned} 0^{-+} \text{ trajectory } [s=0, \ell=j] : & \quad -\eta_P = \eta_C = (-1)^j \gamma_5, \gamma^0 \gamma_5, \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{x}, \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{x} \times \mathbf{L} \\ 0^{--} \text{ trajectory } [s=1, \ell=j] : & \quad \eta_P = \eta_C = -(-1)^j \gamma^0 \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{x}, \gamma^0 \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{x} \times \mathbf{L}, \boldsymbol{\alpha} \cdot \mathbf{L}, \gamma^0 \boldsymbol{\alpha} \cdot \mathbf{L} \\ 0^{++} \text{ trajectory } [s=1, \ell=j \pm 1] : & \quad \eta_P = \eta_C = +(-1)^j 1, \boldsymbol{\alpha} \cdot \mathbf{x}, \gamma^0 \boldsymbol{\alpha} \cdot \mathbf{x}, \boldsymbol{\alpha} \cdot \mathbf{x} \times \mathbf{L}, \gamma^0 \boldsymbol{\alpha} \cdot \mathbf{x} \times \mathbf{L}, \gamma^0 \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{L} \\ 0^{+-} \text{ trajectory } [\text{exotic}] : & \quad \eta_P = -\eta_C = (-1)^j \gamma^0, \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{L} \end{aligned} \quad (5.21)$$

The non-relativistic spin s and orbital angular momentum ℓ are indicated in brackets. Relativistic effects mix the $\ell = j \pm 1$ states on the 0^{++} trajectory, resulting in a pair of coupled radial equations. The $j = 0$ state on the 0^{--} trajectory and the entire 0^{+-} trajectory are incompatible with the s, ℓ assignments and thus exotic in the quark model. They turn out to be missing also in the relativistic case. The bound state equation (5.12) has no solutions for states on the 0^{+-} trajectory ($\Gamma_i = \gamma^0$ or $\gamma_5 \boldsymbol{\alpha} \cdot \mathbf{L}$) since

$$i\nabla \cdot \{\boldsymbol{\alpha}, \gamma^0\} = i\nabla \cdot \{\boldsymbol{\alpha}, \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{L}\} = m [\gamma^0, \gamma^0] = m [\gamma^0, \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{L}] = 0 \quad (5.22)$$

D. The 0^{-+} trajectory: $\eta_P = (-1)^{j+1}$, $\eta_C = (-1)^j$

According to the classification (5.21) we expand the wave function $\Phi_{-+}(\mathbf{x})$ of the 0^{-+} trajectory states as

$$\Phi_{-+}(\mathbf{x}) = \left[F_1(r) + i\boldsymbol{\alpha} \cdot \mathbf{x} F_2(r) + \boldsymbol{\alpha} \cdot \mathbf{x} \times \mathbf{L} F_3(r) + \gamma^0 F_4(r) \right] \gamma_5 Y_{j\lambda}(\hat{\mathbf{x}}) \quad (5.23)$$

Using this in the bound state equation (5.12), noting that $i\nabla \cdot \mathbf{x} \times \mathbf{L} = \mathbf{L}^2$ and comparing terms with the same Dirac structure we get the conditions:

$$\begin{aligned} \gamma_5 : & \quad -(3 + r\partial_r)F_2 + j(j+1)F_3 + mF_4 = \frac{1}{2}(M - V)F_1 \\ \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{x} : & \quad \frac{1}{r}\partial_r F_1 = \frac{1}{2}(M - V)F_2 \\ \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{x} \times \mathbf{L} : & \quad \frac{1}{r^2}F_1 = \frac{1}{2}(M - V)F_3 \\ \gamma^0 \gamma_5 : & \quad mF_1 = \frac{1}{2}(M - V)F_4 \end{aligned} \quad (5.24)$$

Expressing F_2 , F_3 and F_4 in terms of F_1 we find the radial equation (denoting $F'_1 \equiv \partial_r F_1$)

$$F_1'' + \left(\frac{2}{r} + \frac{V'}{M - V} \right) F_1' + \left[\frac{1}{4}(M - V)^2 - m^2 - \frac{j(j+1)}{r^2} \right] F_1 = 0 \quad (5.25)$$

in agreement with the corresponding result in Eq. (2.24) of [33]. The wave function (5.23) may be expressed as

$$\Phi_{-+}(\mathbf{x}) = \left[\frac{2}{M - V} (i\boldsymbol{\alpha} \cdot \vec{\nabla} + m\gamma^0) + 1 \right] \gamma_5 F_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) = F_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) \gamma_5 \left[(i\boldsymbol{\alpha} \cdot \overleftarrow{\nabla} - m\gamma^0) \frac{2}{M - V} + 1 \right] \quad (5.26)$$

$$= \overrightarrow{\Lambda}_+ \gamma_5 F_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) = F_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) \gamma_5 \overleftarrow{\Lambda}_+ \quad (5.27)$$

⁷ The first three trajectories were named π , A_1 and ρ in [33].

The radial equation (5.25) may be verified using the identities (5.16) in the bound state equation (5.17). Both the quark and antiquark contributions have a spin-dependent interaction which cancels in their sum. The contribution from the quark term is, taking into account the radial equation,

$$\vec{\Lambda}_- \Phi_{-+}(\mathbf{x}) = \frac{8V'}{r(M-V)^3} \mathbf{S} \cdot (\vec{\mathbf{L}} \gamma_5 - im \mathbf{x} \gamma^0) F_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) \quad (5.28)$$

where the spin $\mathbf{S} = \frac{1}{2} \gamma_5 \boldsymbol{\alpha}$.

E. The 0^{--} trajectory: $\eta_P = (-1)^{j+1}$, $\eta_C = (-1)^{j+1}$

According to the classification (5.21) we expand the wave function $\Phi_{--}(\mathbf{x})$ of the 0^{--} trajectory states as

$$\Phi_{--}(\mathbf{x}) = \left[\gamma^0 \boldsymbol{\alpha} \cdot \mathbf{L} G_1(r) + i \gamma^0 \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{x} G_2(r) + \gamma^0 \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{x} \times \mathbf{L} G_3(r) + m \boldsymbol{\alpha} \cdot \mathbf{L} G_4(r) \right] Y_{j\lambda}(\hat{\mathbf{x}}) \quad (5.29)$$

Collecting terms with distinct Dirac structures in the bound state equation (5.12),

$$\begin{aligned} \gamma^0 \boldsymbol{\alpha} \cdot \mathbf{L} : \quad & G_2 - (2 + r \partial_r) G_3 + m^2 G_4 = \frac{1}{2} (M - V) G_1 \\ \gamma^0 \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{x} : \quad & \frac{j(j+1)}{r^2} G_1 = \frac{1}{2} (M - V) G_2 \\ \gamma^0 \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{x} \times \mathbf{L} : \quad & \frac{1}{r^2} (1 + r \partial_r) G_1 = \frac{1}{2} (M - V) G_3 \\ m \boldsymbol{\alpha} \cdot \mathbf{L} : \quad & G_1 = \frac{1}{2} (M - V) G_4 \end{aligned} \quad (5.30)$$

Expressing G_2 , G_3 and G_4 in terms of G_1 we find the radial equation for the 0^{--} trajectory,

$$G_1'' + \left(\frac{2}{r} + \frac{V'}{M-V} \right) G_1' + \left[\frac{1}{4} (M - V)^2 - m^2 - \frac{j(j+1)}{r^2} + \frac{V'}{r(M-V)} \right] G_1 = 0 \quad (5.31)$$

in agreement with the corresponding result in Eq. (2.38) of [33]. The 0^{--} radial equation differs from the 0^{-+} one (5.25) only by the term $\propto V'/r(M-V)$. Using

$$i \boldsymbol{\alpha} \cdot \nabla \boldsymbol{\gamma} \cdot \mathbf{L} = \gamma^0 \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{x} \frac{i \mathbf{L}^2}{r^2} + \gamma^0 \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{x} \times \mathbf{L} \frac{1}{r^2} (1 + r \partial_r) \quad (5.32)$$

allows the wave function to be expressed in terms of G_1 . With the projectors (5.15),

$$\Phi_{--}(\mathbf{x}) = \vec{\Lambda}_+ \boldsymbol{\gamma} \cdot \vec{\mathbf{L}} G_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) = G_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) \boldsymbol{\gamma} \cdot \overleftarrow{\mathbf{L}} \overleftarrow{\Lambda}_+ \quad (5.33)$$

where $\overleftarrow{L}^i = -i \overleftarrow{\partial}_k x^j \varepsilon_{ijk}$. The $j = 0$ state on the 0^{--} trajectory is missing since $\mathbf{L} Y_{00}(\hat{\mathbf{x}}) = 0$. The quark contribution to the bound state equation (5.17) is, with $\mathbf{S} = \frac{1}{2} \gamma_5 \boldsymbol{\alpha}$,

$$\vec{\Lambda}_- \Phi_{--}(\mathbf{x}) = \frac{4V'}{r(M-V)^3} [\vec{\mathbf{L}}^2 \gamma^0 \gamma_5 - 2m \mathbf{S} \cdot \mathbf{x} \times \vec{\mathbf{L}}] G_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) \quad (5.34)$$

F. The 0^{++} trajectory: $\eta_P = (-1)^j$, $\eta_C = (-1)^j$

According to the classification (5.21) we expand the wave function $\Phi_{++}(\mathbf{x})$ of the 0^{++} trajectory states in terms of six Dirac structures⁸,

$$\Phi_{++}(\mathbf{x}) = \left\{ \left[F_1(r) + i \boldsymbol{\alpha} \cdot \mathbf{x} F_2(r) + \boldsymbol{\alpha} \cdot \mathbf{x} \times \mathbf{L} F_3(r) \right] + \gamma^0 \left[\gamma_5 \boldsymbol{\alpha} \cdot \mathbf{L} G_1(r) + i \boldsymbol{\alpha} \cdot \mathbf{x} G_2(r) + \boldsymbol{\alpha} \cdot \mathbf{x} \times \mathbf{L} G_3(r) \right] \right\} Y_{j\lambda}(\hat{\mathbf{x}}) \quad (5.35)$$

⁸ The radial functions F_i and G_i are unrelated to those in sections VD and VE.

Collecting terms with distinct Dirac structures in the bound state equation (5.12),

$$\begin{aligned}
1 : & \quad -(3 + r\partial_r)F_2 + j(j+1)F_3 = \frac{1}{2}(M-V)F_1 \\
\boldsymbol{\alpha} \cdot \mathbf{x} : & \quad \frac{1}{r}\partial_r F_1 + mG_2 = \frac{1}{2}(M-V)F_2 \\
\boldsymbol{\alpha} \cdot \mathbf{x} \times \mathbf{L} : & \quad \frac{1}{r^2}F_1 + mG_3 = \frac{1}{2}(M-V)F_3 \\
\gamma^0 \gamma_5 \boldsymbol{\alpha} \cdot \mathbf{L} : & \quad G_2 - (2 + r\partial_r)G_3 = \frac{1}{2}(M-V)G_1 \\
\gamma^0 \boldsymbol{\alpha} \cdot \mathbf{x} : & \quad \frac{1}{r^2}j(j+1)G_1 + mF_2 = \frac{1}{2}(M-V)G_2 \\
\gamma^0 \boldsymbol{\alpha} \cdot \mathbf{x} \times \mathbf{L} : & \quad \frac{1}{r^2}(1 + r\partial_r)G_1 + mF_3 = \frac{1}{2}(M-V)G_3
\end{aligned} \tag{5.36}$$

It turns out to be convenient to express the above radial functions in terms of two new ones, $H_1(r)$ and $H_2(r)$:

$$\begin{aligned}
F_1 &= -\frac{2}{(M-V)^2} \left[\frac{1}{4}(M-V)^2 - m^2 \right] H_1 - \frac{4m}{M-V} \partial_r(rH_2) \\
F_2 &= -\frac{1}{r(M-V)} \partial_r H_1 + 2mH_2 \\
F_3 &= -\frac{1}{r^2(M-V)} H_1 \\
G_1 &= 2H_2 \\
G_2 &= \frac{2}{r} \partial_r \left[-\frac{m}{(M-V)^2} H_1 + \frac{2}{M-V} \partial_r(rH_2) \right] + (M-V)H_2 \\
G_3 &= \frac{2}{r^2} \left[-\frac{m}{(M-V)^2} H_1 + \frac{2}{M-V} \partial_r(rH_2) \right]
\end{aligned} \tag{5.37}$$

The bound state conditions (5.36) are satisfied provided $H_{1,2}$ satisfy the coupled radial equations,

$$H_1'' + \left(\frac{2}{r} + \frac{V'}{M-V} \right) H_1' + \left[\frac{1}{4}(M-V)^2 - m^2 - \frac{j(j+1)}{r^2} \right] H_1 = 4m(M-V)H_2 \tag{5.38}$$

$$H_2'' + \left(\frac{2}{r} + \frac{V'}{M-V} \right) H_2' + \left[\frac{1}{4}(M-V)^2 - m^2 - \frac{j(j+1)}{r^2} + \frac{V'}{r(M-V)} \right] H_2 = \frac{mV'}{r(M-V)^2} H_1 \tag{5.39}$$

These agree with Eqs. (2.48) and (2.49) for F_2^{GS} and G_1^{GS} of [33], when $H_1 = (M-V)F_2^{GS}$ and $H_2 = -iG_1^{GS}/(M-V)$.

The wave function $\Phi_{++}(\mathbf{x})$ (5.35) can be expressed in terms of the $H_{1,2}(r)$ radial functions and the Λ_+ operators (5.15) as

$$\begin{aligned}
\Phi_{++}(\mathbf{x}) &= \vec{\Lambda}_+ \left[-\frac{1}{2}H_1 + 2\boldsymbol{\gamma} \cdot \vec{\mathbf{L}} \gamma_5 H_2 + 2im\boldsymbol{\alpha} \cdot \mathbf{x} H_2 \right] Y_{j\lambda}(\hat{\mathbf{x}}) + \frac{m}{M-V} \left[\vec{\Lambda}_+ \gamma^0 H_1 + 8H_2 \right] Y_{j\lambda}(\hat{\mathbf{x}}) \\
&= Y_{j\lambda}(\hat{\mathbf{x}}) \left[-\frac{1}{2}H_1 - 2H_2 \gamma_5 \boldsymbol{\gamma} \cdot \vec{\mathbf{L}} + 2imH_2 \boldsymbol{\alpha} \cdot \mathbf{x} \right] \vec{\Lambda}_+ - Y_{j\lambda}(\hat{\mathbf{x}}) \left[H_1 \gamma^0 \vec{\Lambda}_+ - 8H_2 \right] \frac{m}{M-V}
\end{aligned} \tag{5.40}$$

The quark contribution to the bound state equation (5.17) is, with $\mathbf{S} = \frac{1}{2}\gamma_5\boldsymbol{\alpha}$,

$$\vec{\Lambda}_- \Phi_{++}(\mathbf{x}) = -\frac{4V'}{r(M-V)^3} \left[\mathbf{S} \cdot \vec{\mathbf{L}} + \frac{m}{M-V} \gamma^0 r \partial_r \right] H_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) + \frac{8V'}{r(M-V)^3} \left[\vec{\mathbf{L}}^2 + m^2 r^2 \right] \gamma^0 H_2(r) Y_{j\lambda} \tag{5.41}$$

When $m = 0$ chiral symmetry implies that $\Phi(\mathbf{x})$ and $\gamma_5\Phi(\mathbf{x})$ define bound states with the same mass M , as is apparent from the bound state equation (5.12). The radial equations (5.38) and (5.39) in fact decouple and coincide with the radial equations of the 0^{-+} (5.25) and 0^{--} (5.31) trajectories, respectively. The Φ_{++} wave functions correspondingly reduce to $\gamma_5\Phi_{-+}$ and $\gamma_5\Phi_{--}$. We discuss the case of spontaneously broken chiral symmetry in section VIII.

VI. PROPERTIES OF THE MESON STATES

In this section we illustrate some properties of the meson wave functions using the 0^{-+} trajectory. The qualitative features are the same for all trajectories.

A. Features of the radial wave function

The radial wave function $F_1(r)$ that solves (5.25) for the linear potential (5.14) ($V' = g\Lambda^2$) behaves as follows in the limits $r \rightarrow 0$, $M - V(r) \rightarrow 0$ and $r \rightarrow \infty$ (for $M \neq 0$):

(i) $r \rightarrow 0$. The standard analysis gives $F_1(r) \sim r^\beta$, with $\beta = j$ or $\beta = -j - 1$. $\Phi_{-+}(\mathbf{x})$ in (5.26) involves $\partial_r F_1(r)$, so only the solution with $F_1(r) \sim r^j$ is normalizable near $r = 0$.

(ii) $M - V(r) \rightarrow 0$. For $F_1(r) \sim (M - V)^\gamma$ the radial equation allows $\gamma = 0$ and $\gamma = 2$. The normalizing integral

$$\int d\mathbf{x} \text{Tr} [\Phi_{-+}^\dagger(\mathbf{x}) \Phi_{-+}(\mathbf{x})] = 8 \int_0^\infty dr r^2 F_1^*(r) \left[1 - \frac{2V'}{(M - V)^3} \partial_r \right] F_1(r) \quad (6.1)$$

is regular at $M - V = 0$ only if $\gamma = 2$. Having already required $F_1(r) \sim r^j$ for $r \rightarrow 0$ the condition $\gamma = 2$ can be fulfilled only for discrete bound state masses M .

In the non-relativistic limit, $M = 2m + E_b$ with ∂_r^2 , ∂_r/r , E_b and V all of the same order and $\ll M$ the radial equation (5.25) turns into the Schrödinger equation,

$$\left[\partial_r^2 + \frac{2}{r} \partial_r + m(E_b - V) - \frac{j(j+1)}{r^2} \right] F_1^{NR}(r) = 0 \quad (6.2)$$

The physical solutions $F_1^{NR}(r)$ are exponentially damped for $r \rightarrow \infty$, and vanish at $V(r) = 2m + E_b$ in the limit of large m . Relativistic wave functions which satisfy $F_1(r = M/V') = 0$ correspond to physical solutions in the Schrödinger limit. This was explicitly verified for the analytic (Airy function) solutions in $D = 1 + 1$ dimensions, see section III A of [21].

The solutions of the Dirac equation with a linear $V(r)$ potential are locally normalizable for all r . Hence the Dirac energy spectrum M is continuous [18]. If the Dirac states for a single fermion are viewed as the limit of an $f\bar{f}$ system where one of the fermion masses $M_T \rightarrow \infty$, the point where $V(r) = M + M_T$ moves to $r = \infty$.

(iii) $r \rightarrow \infty$. Assuming $F_1(r) \sim r^\gamma \exp(ar^2 + br + c)$ and comparing the terms of $r^n F_1(r)$ for $n = 2, 1, 0$ we find

$$F_1(r \rightarrow \infty) \sim \frac{1}{r} r^{-im^2/V'} \exp[i(M - V)^2/4V'] \quad \text{and} \quad c.c. \quad (6.3)$$

Consequently the integrand of the normalizing integral (6.1) tends to a constant at large r . This feature is common to states of all quantum numbers. The probability density similarly tends to a constant also in lower spatial dimensions ($D = 1 + 1$ and $D = 2 + 1$). As discussed below in subsection VIC this may be interpreted as describing the virtual $q\bar{q}$ pairs due to Z -diagrams, Fig. 1(c).

B. Orthogonality

The overlap of two hadron states $|M_1\rangle$ and $|M_2\rangle$ (4.3) is given by the annihilation of both quark fields,

$$\langle M_2 | M_1 \rangle = \int d\mathbf{x}_1 d\mathbf{x}_2 \text{Tr} [\Phi_2^\dagger(\mathbf{x}_1 - \mathbf{x}_2) \Phi_1(\mathbf{x}_1 - \mathbf{x}_2)] = [2\pi\delta(0)]^3 \int d\mathbf{x} \text{Tr} [\Phi_2^\dagger(\mathbf{x}) \Phi_1(\mathbf{x})] \quad (6.4)$$

where the trace is over the Dirac indices – the color trace is unity for color singlet states (4.4). The factors $\delta^3(0)$ appear because both states are at rest⁹.

Orthogonality follows in the standard way from the bound state equations (5.12) satisfied by Φ_1 and Φ_2^\dagger [21],

$$\begin{aligned} i\nabla \cdot \{\boldsymbol{\alpha}, \Phi_1(\mathbf{x})\} + m[\gamma^0, \Phi_1(\mathbf{x})] &= [M_1 - V(\mathbf{x})] \Phi_1(\mathbf{x}) \\ -i\nabla \cdot \{\boldsymbol{\alpha}, \Phi_2^\dagger(\mathbf{x})\} - m[\gamma^0, \Phi_2^\dagger(\mathbf{x})] &= [M_2 - V(\mathbf{x})] \Phi_2^\dagger(\mathbf{x}) \end{aligned} \quad (6.5)$$

⁹ For states of general momenta $\mathbf{P}_A, \mathbf{P}_B$ as in (5.2) this factor would be $(2\pi)^3 \delta(\mathbf{P}_A - \mathbf{P}_B)$.

Multiplying the first equation by $\Phi_2^\dagger(\mathbf{x})$ from the left and the second by $-\Phi_1(\mathbf{x})$ from the right and taking the trace of their sum gives

$$2i \nabla \cdot \text{Tr} \left(\alpha \left\{ \Phi_2^\dagger, \Phi_1 \right\} \right) = (M_1 - M_2) \text{Tr} \left(\Phi_2^\dagger \Phi_1 \right) \quad (6.6)$$

Integrating both sides over \mathbf{x} we get (assuming the integrations over space components to commute)

$$2i \sum_{j \neq k \neq \ell} \int dx^k dx^\ell \Big|_{x^j=-\infty}^{x^j=\infty} \text{Tr} \left(\alpha^j \left\{ \Phi_2^\dagger, \Phi_1 \right\} \right) = (M_1 - M_2) \int d\mathbf{x} \text{Tr} \left(\Phi_2^\dagger \Phi_1 \right) \quad (6.7)$$

The Dirac structure of Φ_{-+} (5.26) gives

$$\text{Tr} \left(\alpha^j \left\{ \Phi_2^\dagger, \Phi_1 \right\} \right) = 16i \left(\frac{1}{M_1 - V} \nabla_1^j - \frac{1}{M_2 - V} \nabla_2^j \right) F_1^{(1)}(r) F_1^{(2)*}(r) Y_{j_1 \lambda_1}(\hat{\mathbf{x}}) Y_{j_2 \lambda_2}^*(\hat{\mathbf{x}}) \quad (6.8)$$

where ∇_1 and ∇_2 differentiate $F_1^{(1)} Y_{j_1 \lambda_1}$ and $F_1^{(2)} Y_{j_2 \lambda_2}$, respectively. The leading contribution for $|x^j| \rightarrow \infty$ arises when ∇^j differentiates the factor $\exp(iV'r^2/4)$ in F_1 (6.3), making $\nabla^j \propto x^j$. The factors $1/r$ in $F_1^{(1)}$ and $F_1^{(2)*}$ as well as the factor $1/(M - V)$ makes the integrand (6.8) $\propto x^j/|x^j|^3$, causing the lhs. of (6.7) to vanish. Thus $\langle M_2 | M_1 \rangle = 0$ in (6.4) when $M_1 \neq M_2$.

C. Normalization

For $|M_1\rangle = |M_2\rangle = |M\rangle$ (6.4) gives (omitting the momentum conserving δ -functions),

$$\langle M | M \rangle = \int d\mathbf{x} \text{Tr} \left[\Phi^\dagger(\mathbf{x}) \Phi(\mathbf{x}) \right] \quad (6.9)$$

The expression (6.1) for the integral (when $\Phi = \Phi_{-+}$) together with the asymptotic behavior (6.3) of the radial function implies that the integrand (probability density) tends to a constant at large r . This behavior of the wave function in the region where $V(r) \gg M$ is due to our omission of string breaking, *i.e.*, $\mathcal{H}_V |0\rangle = 0$ (5.4). The constant density may be interpreted as describing a constant rate of virtual pairs that emerge with increasing $V(r)$. In a physical process components of the wave function with high internal momentum $p \propto r$ cannot be resolved, regularizing the divergence. The situation is analogous to the UV renormalization of wave functions.

The $q\bar{q}$ bound state $|M\rangle$ in (4.3) generally has $b^\dagger d^\dagger$, $b^\dagger b$, $d d^\dagger$ and $d b$ operator contributions. The annihilation operators contribute due to a Bogoliubov transformation as in the Dirac case (B.6), reflecting Z -diagrams such as Fig. 1(c). The operator structure of $|M\rangle$ simplifies in the $r \rightarrow \infty$ limit. The derivative ∇ in the expression (5.26) for the wave function $\Phi_{-+}(\mathbf{x}_1 - \mathbf{x}_2)$ is equivalent to $\partial/\partial \mathbf{x}_1$. After partial integration in (4.3) the derivative acts on $\bar{\psi}(\mathbf{x}_1)$ (the contribution from $\nabla_1(M - V)^{-1}$ can be neglected for large r). If $\bar{\psi}_{b^\dagger}$ ($\bar{\psi}_d$) denotes the b^\dagger (d) term in $\bar{\psi}$ we have

$$\left. \begin{array}{l} \bar{\psi}_{b^\dagger}(\mathbf{x}_1) \\ \bar{\psi}_d(\mathbf{x}_1) \end{array} \right\} (i\alpha \cdot \vec{\nabla}_1 + m\gamma^0) = \left\{ \begin{array}{l} \bar{\psi}_{b^\dagger}(\mathbf{x}_1) \sqrt{-\nabla_1^2 + m^2} \\ -\bar{\psi}_d(\mathbf{x}_1) \sqrt{-\nabla_1^2 + m^2} \end{array} \right. \quad (6.10)$$

The asymptotic behavior (6.3) implies at leading order for $r \rightarrow \infty$,

$$\sqrt{-\nabla_1^2 + m^2} F_1(r) \simeq \frac{1}{2} V F_1(r) \quad (6.11)$$

Consequently the bracket in the first form of the wave function Φ_{-+} in (5.26) becomes

$$\left[\frac{2}{M - V} (i\alpha \cdot \vec{\nabla} + m\gamma^0) + 1 \right] \simeq \left\{ \begin{array}{ll} -1 + 1 = 0 & \text{for } \bar{\psi}_{b^\dagger}(\mathbf{x}_1) \\ +1 + 1 = 2 & \bar{\psi}_d(\mathbf{x}_1) \end{array} \right. \quad (6.12)$$

Thus only the d operator in $\bar{\psi}(\mathbf{x}_1)$ contributes in the $r \rightarrow \infty$ limit. Similarly it can be seen that only the b operator in $\psi(\mathbf{x}_2)$ contributes to the state (4.3). The dominant bd contribution reflects the virtual $q\bar{q}$ sea due to Z -diagrams (Fig. 1(c)) in the linearly increasing potential $V(r)$. The “valence” (b^\dagger , d^\dagger) contributions are suppressed at large r .

D. Parton picture and duality for $M \gg V(r)$

We expect the parton model to be applicable when the kinetic energy of a quark is large compared to its binding energy. For example, $e^+e^- \rightarrow \text{hadrons}$ starts (at lowest order) with the production of nearly free quarks, $e^+e^- \rightarrow q\bar{q}$. The subsequent hadronization process is unitary, allowing the total hadronic cross section to be calculated in terms of the initial quark production.

According to duality $\sigma(e^+e^- \rightarrow \text{hadrons})$ is saturated by resonances in the direct channel. This requires that the wave function of a bound state with high $M \simeq E_{CM}(e^+e^-)$ agrees with that of a free $q\bar{q}$ pair, at separations for which $V(r) \ll M$. It is instructive to verify this in the present approach.

Neglecting $V(r)$ the radial equation (5.25) implies

$$\sqrt{-\nabla^2 + m^2} F_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) = \frac{1}{2} M F_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) \quad (6.13)$$

This allows the factor in brackets in the expression (5.26) to be squared,

$$\left[\sqrt{-\nabla^2 + m^2} + i\boldsymbol{\alpha} \cdot \vec{\nabla} + m\gamma^0 \right] = \frac{1}{M} \left(\sqrt{-\nabla^2 + m^2} + i\boldsymbol{\alpha} \cdot \vec{\nabla} + m\gamma^0 \right)^2 \quad (6.14)$$

The derivatives operate on the $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ dependence of $F_1(r) Y_{j\lambda}(\hat{\mathbf{x}})$. Replacing $\nabla \rightarrow \nabla_1$ in one of the factors on the rhs. of (6.14) and $\nabla \rightarrow -\nabla_2$ in the other, the state can after partial integrations be expressed as

$$\begin{aligned} |M\rangle_{V=0} &= \frac{2}{M^2} \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}(\mathbf{x}_1) \left[\gamma^0 \sqrt{-\nabla_1^2 + m^2} + i\boldsymbol{\gamma} \cdot \vec{\nabla}_1 + m \right] \gamma^0 \gamma_5 F_1 Y_{j\lambda} \gamma^0 \left[\gamma^0 \sqrt{-\nabla_2^2 + m^2} + i\boldsymbol{\gamma} \cdot \vec{\nabla}_2 - m \right] \psi(\mathbf{x}_2) |0\rangle \\ &= \frac{2}{M^2} \int d\mathbf{x}_1 d\mathbf{x}_2 \int \frac{d\mathbf{k}_1 d\mathbf{k}_2}{(2\pi)^6} \sum_{\lambda_1, \lambda_2} e^{-i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \mathbf{k}_2 \cdot \mathbf{x}_2)} F_1 Y_{j\lambda} [\bar{u}(\mathbf{k}_1, \lambda_1) \gamma_5 v(\mathbf{k}_2, \lambda_2)] b_{\mathbf{k}_1, \lambda_1}^\dagger d_{\mathbf{k}_2, \lambda_2}^\dagger |0\rangle \end{aligned} \quad (6.15)$$

The factors in brackets on the first line project out $b^\dagger d^\dagger$ from the field operators, giving a state with just the free valence quark and antiquark. The expression can be further simplified using

$$\mathbf{k}_1 \cdot \mathbf{x}_1 + \mathbf{k}_2 \cdot \mathbf{x}_2 = \frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2)(\mathbf{x}_1 + \mathbf{x}_2) + \frac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2)(\mathbf{x}_1 - \mathbf{x}_2) \quad (6.16)$$

The integration over $\mathbf{x}_1 + \mathbf{x}_2$ gives momentum conservation,

$$|M\rangle_{V=0} = \frac{2}{M^2} \int d\mathbf{x} \int \frac{d\mathbf{k}}{(2\pi)^3} \sum_{\lambda_1, \lambda_2} e^{-i\mathbf{k} \cdot \mathbf{x}} [\bar{u}(\mathbf{k}, \lambda_1) \gamma_5 v(-\mathbf{k}, \lambda_2)] F_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) b_{\mathbf{k}, \lambda_1}^\dagger d_{-\mathbf{k}, \lambda_2}^\dagger |0\rangle \quad (6.17)$$

We may use the relation $v(-\mathbf{k}, \lambda) = i\gamma^2 u^*(-\mathbf{k}, \lambda)$ which is implied by charge conjugation (A.14) to evaluate the helicity dependence of the states on the 0^{-+} trajectory when $V \ll M$,

$$|M\rangle_{V=0} = \int d\mathbf{x} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{4E_k}{M^2} e^{-i\mathbf{k} \cdot \mathbf{x}} F_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) \sum_{\lambda_1} (-1)^{\lambda_1+1/2} b_{\mathbf{k}, \lambda_1}^\dagger d_{-\mathbf{k}, -\lambda_1}^\dagger |0\rangle \quad (6.18)$$

Expressing $E_k e^{-i\mathbf{k} \cdot \mathbf{x}} = \sqrt{-\nabla^2 + m^2} e^{-i\mathbf{k} \cdot \mathbf{x}}$ and partially integrating over \mathbf{x} using (6.13) gives

$$|M\rangle_{V=0} = \frac{2}{M} \int d\mathbf{x} \int \frac{d\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{x}} F_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) \sum_{\lambda_1} (-1)^{\lambda_1+1/2} b_{\mathbf{k}, \lambda_1}^\dagger d_{-\mathbf{k}, -\lambda_1}^\dagger |0\rangle \quad (6.19)$$

To illuminate the structure of the state we now consider the special case of $j = m = 0$. The radial wave function is then, for $V(r) \ll M$ and arbitrarily normalized,

$$F_1(r) = \frac{1}{r} \sin\left(\frac{1}{2}Mr\right) \quad (j = m = 0) \quad (6.20)$$

The integral over \mathbf{x} becomes

$$\int d\mathbf{x} e^{-i\mathbf{k} \cdot \mathbf{x}} F_1(r) = \int_0^R dr r^2 \frac{1}{r} \sin\left(\frac{1}{2}Mr\right) \frac{4\pi}{kr} \sin(kr) = \frac{2\pi}{k} \int_0^R dr \left\{ \cos\left[\left(\frac{1}{2}M - k\right)r\right] - \cos\left[\left(\frac{1}{2}M + k\right)r\right] \right\} \quad (6.21)$$

where the range R of the r -integration is limited by $V'R \ll M$. For $M \rightarrow \infty$ also $R \rightarrow \infty$ and the term $\cos[(\frac{1}{2}M+k)r]$ in the integrand is suppressed. Thus

$$\int d\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} F_1(r) \simeq \frac{2\pi}{k} \int_0^R dr \cos[(\frac{1}{2}M-k)r] \simeq \frac{2\pi^2}{k} \delta(k - \frac{1}{2}M) \quad (6.22)$$

where the δ -function is understood to limit $|k - \frac{1}{2}M| \lesssim 1/R$. Using this in the expression (6.19) gives

$$|M\rangle_{V \ll M} \simeq \frac{1}{(4\pi)^{3/2}} \int d\Omega_{\mathbf{k}} \sum_{\lambda} (-1)^{\lambda+1/2} b_{\mathbf{k},\lambda}^{\dagger} d_{-\mathbf{k},-\lambda}^{\dagger} |0\rangle \quad \text{where } k = \frac{1}{2}M \quad (6.23)$$

Thus the bound state wave function reduces to that of a free $q\bar{q}$ pair, isotropically distributed since we considered a $J^{PC} = 0^{-+}$ state. Similarly in $e^+e^- \rightarrow \text{hadrons}$ the coupling of the virtual photon to a bound state in the direct channel will be the same as the coupling to a free $q\bar{q}$ pair, as required by duality.

E. Mass spectrum of the 0^{-+} trajectory for $m = 0$

The radial equation (5.25) can readily be solved numerically, subject to the boundary conditions $F_1(r \rightarrow 0) \sim r^j$ and $F_1(r \rightarrow M/V') \sim (M - V')^2$. As seen in Fig. 2, for the linear potential (5.14) and quark mass $m = 0$ the states lie on nearly linear Regge trajectories and their parallel daughter trajectories. The mass spectra of the 0^{--} and 0^{++} trajectories are similar [23].

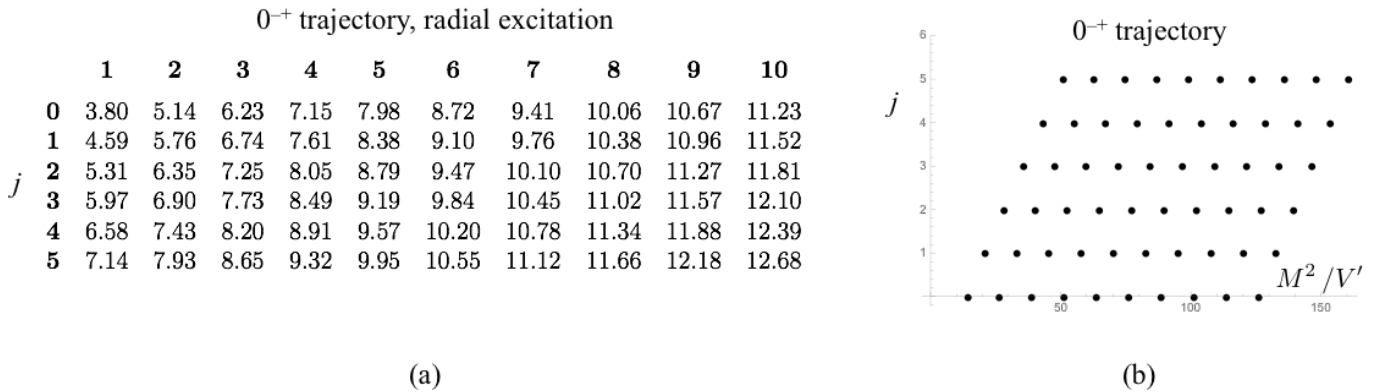


FIG. 2: (a) Masses M of the mesons on the 0^{-+} trajectory for $m = 0$, in units of $\sqrt{V'}$. (b) Plot of the spin j vs. M^2/V' for the states listed in (a). Figure taken from [23].

VII. BOUND STATES IN MOTION

A. Active vs. Passive boosts

In the previous section we considered $q\bar{q}$ states at rest, bound by the classical field (4.7). Here we study corresponding states with non-vanishing CM momenta. There is limited experience of how equal-time wave functions transform under boosts [34, 35]. An explicit demonstration that such covariance is non-trivial may be found in [36].

We take the bound state momentum \mathbf{P} to be along the z -axis¹⁰ and define the boost parameter ξ according to

$$\mathbf{P} = (0, 0, P) \quad P = M \sinh \xi \quad (7.1)$$

Eigenstates of the momentum operator \mathcal{P} (A.3) with eigenvalue \mathbf{P} have (at time $t = 0$) the general form given in (5.2), with the (globally) color singlet wave function (5.3).

¹⁰ In $D = 1 + 1$ dimensions (section VII B below) the momentum is along the $x^1 = x$ -axis.

Coordinate transformations may be active or passive. In active transformations the object itself is transformed, whereas passive transformations change the frame, or coordinate system. An active boost gives a non-vanishing momentum \mathbf{P} to a particle at rest. A passive boost sets the observer in motion, without changing the state. For non-interacting particles the two views are equivalent.

Bound states involve interacting particles, which requires to distinguish between active and passive transformations. Consider the Dirac description of an electron bound in an external field. The equation is covariant under boosts provided the external field is transformed together with the wave function. This is a passive transformation, in the sense that a moving observer sees a transformed field. An active transformation would change only the momentum of the particle, keeping the external field unchanged. This is actually ill-defined since the external field breaks translation invariance: the electron has no conserved momentum \mathbf{P} .

Here we consider the translationally covariant states (5.2), which are eigenstates of the momentum operator and bound by the classical field of their constituents. The issue is then how the classical field should behave under boosts.

The primary motivation for considering bound states with non-vanishing momenta \mathbf{P} is to enable calculations of scattering amplitudes. All states in a scattering amplitude must refer to the same frame. Thus we need bound states with non-vanishing momentum in the same frame as the $\mathbf{P} = 0$ states considered in section V. We are dealing with active Lorentz transformations which do not boost the classical field. This is consistent with our determination of the classical field in section IV A, where the gluon field (4.2) depends on the instantaneous positions of the quarks, not on their momenta.

Given that the classical gluon field (4.7) is independent of \mathbf{P} the state (5.2) should be an eigenstate of the Hamiltonian (5.1). The bound state condition is

$$i\nabla \cdot \{\boldsymbol{\alpha}, \Phi^{(\xi)}(\mathbf{x})\} - \frac{1}{2}\mathbf{P} \cdot [\boldsymbol{\alpha}, \Phi^{(\xi)}(\mathbf{x})] + m[\gamma^0, \Phi^{(\xi)}(\mathbf{x})] = [E - V(\mathbf{x})]\Phi^{(\xi)}(\mathbf{x}) \quad (7.2)$$

where $V(\mathbf{x}) = g\Lambda^2|\mathbf{x}| = V'|\mathbf{x}|$ as in (4.10) and E is the eigenvalue. The radial and angular variables cannot be separated as in section V C, since \mathbf{P} breaks rotational invariance. It is also not obvious that the energy eigenvalue will depend correctly on the momentum, $E = \sqrt{M^2 + \mathbf{P}^2}$.

It is instructive to first consider QED in $D = 1 + 1$ dimensions (section VII B). The solution of the BSE (7.2) for QED₂ agrees with the wave function obtained by boosting the rest frame state in $A^1 = 0$ gauge [20]. Active and passive boosts are gauge equivalent, in the sense that an A^1 potential may be set to zero by a gauge transformation. The BSE with a boosted potential ($A^1 \neq 0$) gives the same eigenvalues (E, P) but a different wave function.

In $D = 3 + 1$ dimensions we begin with the simple exercise of free quarks, $V = 0$ (section VII C). This allows to determine the boost dependence of the wave function $\Phi^{(\xi)}(\mathbf{x})$ at the origin, since $V(\mathbf{x} = 0) = 0$. When $V \neq 0$ (section VII D) the locally normalizable wave function on the z -axis, $\Phi^{(\xi)}(x = y = 0, z)$, is determined (7.34) by the $\mathbf{P} = 0$ wave function since the potential is linear. This defines a boundary condition for the BSE (7.2) which (numerically) determines the wave function for all \mathbf{x} .

It is also of interest to consider the wave function for a passive boost in $D = 3 + 1$, which transforms the rest frame A^0 field. We consider this case in appendix C. The moving observer sees a magnetic field $\mathbf{B} \perp \mathbf{P}$, which affects the quark spins.

B. Poincaré covariance of QED bound states in $D = 1 + 1$ dimensions

1. Active boost in $A^1 = 0$ gauge

In a Hamiltonian formulation Poincaré invariance is implemented by generators of transformations that obey the Lie algebra. For QED in $D = 1 + 1$ dimensions the generator of space translations (momentum) \mathcal{P} , Hamiltonian \mathcal{H} and boost generator \mathcal{K} are in $A^1 = 0$ gauge [20],

$$\mathcal{P} = \int dx \psi^\dagger(t, x)(-i\partial_x)\psi(t, x) \quad (7.3)$$

$$\mathcal{H} = \int dx \psi^\dagger(t, x)(-i\gamma^0\gamma^1\partial_x + m\gamma^0)\psi(t, x) - \frac{e^2}{4} \int dx dy \psi^\dagger\psi(t, x)|x - y|\psi^\dagger\psi(t, y) \quad (7.4)$$

$$\mathcal{K} = t\mathcal{P} + \int dx \psi^\dagger(t, x)\left[x(i\gamma^0\gamma^1\partial_x - m\gamma^0) + \frac{i}{2}\gamma^0\gamma^1\right]\psi(t, x) + \frac{e^2}{8} \int dx dy \psi^\dagger\psi(t, x)(x + y)|x - y|\psi^\dagger\psi(t, y) \quad (7.5)$$

where $\psi^\dagger\psi(t, x) \equiv \psi^\dagger(t, x)\psi(t, x)$. In $D = 1 + 1$ dimensions the Dirac matrices may be represented using the 2×2 Pauli matrices, with $\gamma^0\gamma^1 = \sigma_1$, $\gamma^0\gamma^2 = \sigma_2$ and $\gamma^0 = \sigma_3$.

The boost generator (7.5) includes a gauge transformation which ensures that $A^1 = 0$ in all frames. Since A^0 depends on the separation of the fermions the $U(1)$ gauge parameter θ is operator valued,

$$\theta(t, x) = -\frac{e^2}{4} \int dy (x - y) |x - y| \psi^\dagger\psi(t, y) \quad (7.6)$$

The canonical commutation relations of the fermion fields ensure¹¹ the Poincaré Lie algebra of QED₂,

$$[\mathcal{H}, \mathcal{P}] = 0 \quad [\mathcal{H}, \mathcal{K}] = i\mathcal{P} \quad [\mathcal{P}, \mathcal{K}] = i\mathcal{H} \quad (7.7)$$

A state with momentum $P = M \sinh \xi$ is obtained by boosting the rest frame state,

$$|M, P\rangle = \exp(-i\xi\mathcal{K}) |M, 0\rangle \quad (7.8)$$

The Lie algebra (7.7) ensures that $\mathcal{P} |M, P\rangle = P |M, P\rangle$ and $\mathcal{H} |M, P\rangle = \sqrt{M^2 + P^2} |M, P\rangle$. The state $|M, P\rangle$ was confirmed [20, 22] to have the general form (5.2) with wave function $\Phi^{(\xi)}$ related to the rest frame one as

$$\Phi^{(\xi)}(\tau_P) = \exp\left(-\frac{1}{2}\zeta\sigma_1\right) \Phi^{(0)}(\tau_0) \exp\left(\frac{1}{2}\zeta\sigma_1\right) \quad \text{for } \tau_P = \tau_0 \quad (7.9)$$

The variable τ_P is a P -dependent function of x , defined in terms of the kinetic momentum Π ,

$$\Pi(x) \equiv (E - V(x), P) \quad \tau_P(x) \equiv \Pi^2/V' = [(E - V)^2 - P^2]/V' \quad (7.10)$$

where $E = \sqrt{M^2 + P^2}$ and $V' = g\Lambda^2$. The variable $\zeta(x)$ is defined by

$$\cosh \zeta = \frac{E - V(x)}{\sqrt{\Pi^2}} \quad \sinh \zeta = \frac{P}{\sqrt{\Pi^2}} \quad (7.11)$$

In (7.9) τ_P appears as a boost-invariant variable, in terms of which the wave function is frame-independent up to the factors $\exp(\pm \frac{1}{2}\zeta\sigma_1)$. The x -dependence of the boosted wave function $\Phi^{(\xi)}$ is found by expressing τ_P (7.10), $\tau_0 = (M - V)^2/V'$ and ζ (7.11) in terms of x .

In the non-relativistic limit ($V \ll M$) we have $\tau_P(x) \simeq M^2/V' - 2E|x|$. A wave function that is a P -independent function of τ_P , *i.e.*, of $E|x|$, Lorentz contracts as a function of $x \propto 1/E$. In the same limit ζ reduces to the boost parameter, $\zeta \simeq \xi$. For relativistic binding the x -dependence of $\Phi^{(\xi)}$ differs from Lorentz contraction. This is a consequence of the gauge transform (7.6) which was included in the boost to keep $A^1 = 0$. Without the gauge transform ($\theta = 0$) the QED₂ wave functions do Lorentz contract (section VIIB3 below).

2. Actively boosted QED₂ bound states

Instead of applying the boost (7.8) we can determine the wave function of the QED₂ state $|M, P\rangle$ by solving the bound state equation (7.2) in $D = 1 + 1$ dimensions,

$$i\partial_x \{\sigma_1, \Phi^{(\xi)}\} - \frac{1}{2}P[\sigma_1, \Phi^{(\xi)}] + m[\sigma_3, \Phi^{(\xi)}] = (E - V)\Phi^{(\xi)} \quad (7.12)$$

It is not obvious that the wave function which satisfies the BSE agrees with the boosted one (7.9). The generators are exactly determined by the QED₂ action, whereas the bound state equations are Born approximations. A comparison of the boost and BSE methods tests Poincaré invariance at lowest order in \hbar , as here defined.

The QED₂ BSE (7.12) was studied in [21], see also [23]. Here we only recall the relevant aspects. The 2×2 wave function can be expanded in the unit and Pauli matrices,

$$\Phi^{(\xi)}(x) = \phi_0(x) + \phi_1(x)\sigma_1 + \phi_2(x)i\sigma_2 + \phi_3(x)\sigma_3 \quad (7.13)$$

¹¹ The corresponding generators for QCD₂ obey the Poincaré algebra only up to a color octet term [20, 37, 38]. The Lie algebra is still satisfied for transformations of (globally) color singlet states.

The BSE implies a second order differential equation for ϕ_1 , which for $x > 0$ is

$$\partial_x^2 \phi_1 + \frac{V'}{E - V} \partial_x \phi_1 + \frac{1}{4}(E - V)^2 \left[1 - \frac{4m^2}{(E - V)^2 - P^2} \right] \phi_1 = 0 \quad (7.14)$$

Since $V(x) = V(-x)$ we may choose $\phi_1(x)$ to be either symmetric or antisymmetric in x . Correspondingly, either $\partial_x \phi_1(0) = 0$ or $\phi_1(0) = 0$. A second constraint is due to local normalizability, as previously discussed in section VII A. Solutions of (7.14) typically give wave functions $\Phi^{(\xi)}$ that are singular at $(E - V)^2 - P^2 = 0$, i.e., at $\tau = 0$ (7.10). Only for discrete energies E are there (anti)symmetric $\phi_1(x)$ with a locally normalizable $\Phi^{(\xi)}$.

In general we should not expect any simple dependence of the discrete energies E on the momentum P , which appears as a parameter of the differential equation (7.14). However, for a linear potential ($V'' = 0$), and only in this case, it so happens that expressing (7.14) in terms of τ rather than x eliminates all explicit P -dependence,

$$\partial_\tau^2 \phi_1 + \frac{1}{16} \left(1 - \frac{4m^2}{V'\tau} \right) \phi_1 = 0 \quad (7.15)$$

The constraint on $\phi_1(\tau = 0)$ is thus P -independent, and the wave functions $\Phi^{(\xi)}$ of different P are related as in (7.9). The symmetry constraint on $\phi_1(\tau)$ at $x = 0$ must similarly correspond to a fixed value of τ . Since $\tau(x = 0) = E^2 - P^2$ when $V(0) = 0$ this imposes $E^2 - P^2 = M^2$ to be P -independent, ensuring the expected $E(P)$.

Thus the BSE solution reduces to that of the boost for the QED₂ linear potential. The non-trivial agreement between the boost and BSE supports the expectation that Poincaré invariance holds at each order of \hbar , and that we have correctly identified the lowest order contribution in \hbar .

3. Passively boosted QED₂ bound states

To a moving observer the rest frame state appears to be bound by a boosted potential [23],

$$eA^0 = V_R \cosh \xi \equiv V'|x_1 - x_2|_R \cosh \xi \quad eA^1 = V_R \sinh \xi \quad (7.16)$$

where ξ is the boost parameter ($P = M \sinh \xi$) and $(x_1 - x_2)_R$ is the rest frame separation corresponding to the Lorenz contracted separation $x_1 - x_2$ perceived by the observer,

$$x_1 - x_2 = \frac{(x_1 - x_2)_R}{\cosh \xi} \quad (7.17)$$

We indicate the state as perceived by the moving observer by an overline,

$$\overline{|M, P\rangle} \equiv \int dx_1 dx_2 \bar{\psi}(t = 0, x_1) e^{iP(x_1 + x_2)/2} \bar{\Phi}^{(\xi)}(x_1 - x_2) \psi(t = 0, x_2) |0\rangle \quad (7.18)$$

Given the potentials (7.16) the wave function satisfies the BSE,

$$i\partial_x \{ \sigma_1, \bar{\Phi}^{(\xi)}(x) \} - \frac{1}{2}(P - eA^1) [\sigma_1, \bar{\Phi}^{(\xi)}(x)] + m [\sigma_3, \bar{\Phi}^{(\xi)}(x)] = (E - eA^0) \bar{\Phi}^{(\xi)}(x) \quad (7.19)$$

where $x = x_1 - x_2$. Let us assume that $E = M \cosh \xi$, and verify later that this is the consistent choice. Then

$$(E - eA^0) = (M - V_R) \cosh \xi \quad (P - eA^1) = (M - V_R) \sinh \xi \quad (7.20)$$

Expanding the wave function in terms of Pauli matrices as in (7.13),

$$\bar{\Phi}^{(\xi)}(x) = \bar{\phi}_0(x) + \bar{\phi}_1(x) \sigma_1 + \bar{\phi}_2(x) i\sigma_2 + \bar{\phi}_3(x) \sigma_3 \quad (7.21)$$

the BSE (7.19) implies

$$\partial_x^2 \bar{\phi}_1 + \frac{V'}{M - V_R} \cosh \xi \partial_x \bar{\phi}_1 + \frac{1}{4}(M - V_R)^2 \left[1 - \frac{4m^2}{(M - V_R)^2 - P^2} \right] \cosh^2 \xi \bar{\phi}_1 = 0 \quad (7.22)$$

The invariant variable $\bar{\tau}(x)$ is defined as in (7.10), but in terms of the present kinetic momentum $\bar{\Pi}(x)$,

$$\bar{\Pi}(x) \equiv (E - eA^0, P - eA^1) \quad \bar{\tau}(x) \equiv \bar{\Pi}^2 / V' = (M - V_R)^2 / V' \quad (7.23)$$

The variable $\bar{\tau}$ is P -independent when expressed in terms of the rest frame coordinate x_R , since $V_R = V'|x_R|$. In terms of $\bar{\tau}$ the differential equation (7.22) assumes the same form as (7.15) for ϕ_1 in $A^1 = 0$ gauge,

$$\partial_{\bar{\tau}}^2 \bar{\phi}_1 + \frac{1}{16} \left(1 - \frac{4m^2}{V'\bar{\tau}} \right) \bar{\phi}_1 = 0 \quad (7.24)$$

Thus $\bar{\phi}_1(\bar{\tau}) = \phi_1(\tau)$, but the two functions differ as functions of x . The symmetry constraint at $x = 0$ occurs at the fixed value $\bar{\tau}(x = 0) = \tau(x = 0) = M^2/V'$, verifying our choice of E . The P -dependence of the full wave function

$$\bar{\Phi}^{(\xi)}(\bar{\tau}) = \exp\left(-\frac{1}{2}\xi\sigma_1\right)\bar{\Phi}^{(0)}(\bar{\tau}_0)\exp\left(\frac{1}{2}\xi\sigma_1\right) \quad \text{for } \bar{\tau}_P = \bar{\tau}_0 \quad (7.25)$$

may be compared with (7.9) in $A^1 = 0$ gauge: here the exponential factors are independent of x . Because $\bar{\tau}$ is a P -independent function of $x_R = x \cosh \xi$ the wave function $\bar{\Phi}^{(\xi)}$ Lorentz contracts as a function of x . This agrees with the general notion of how objects appear to a moving observer.

States in a scattering amplitude must all be evaluated in a common frame. This requires active Lorentz transformations, which transform the states rather than the observer. For QED₂ this corresponds to $A^1 = 0$ gauge. In higher dimensions the active and passive transformations are not gauge equivalent, since it is not possible to set all 3-vector components of the gauge field to $\mathbf{A} = 0$.

C. P -dependence of free $q\bar{q}$ states

As a warm-up in $D = 3 + 1$ dimensions we first consider the case of a free $q\bar{q}$ pair, *i.e.*, $V = 0$. This solution is relevant for the interacting case at small separations, since $V(r \rightarrow 0) = 0$ for a linear potential. Active and passive transformations are equivalent when there are no interactions.

Let the momenta of a free quark and antiquark be $\mathbf{p}_{10} \equiv \mathbf{p}_0$ and $\mathbf{p}_{20} = -\mathbf{p}_0$, respectively, in the rest frame of the pair. The total energy is then $M = 2\sqrt{\mathbf{p}_0^2 + m^2}$. In the frame (7.1) where the total energy is $E = M \cosh \xi$ and momentum $\mathbf{P} = M \sinh \xi (0, 0, 1)$ the quark momenta are $\mathbf{p}_1 \equiv \mathbf{p}$ and $\mathbf{p}_2 = \mathbf{P} - \mathbf{p}$, with

$$\begin{aligned} p^3 &= \frac{1}{2}M \sinh \xi + p_0^3 \cosh \xi \\ \mathbf{p}^\perp &= \mathbf{p}_0^\perp \end{aligned} \quad (7.26)$$

The state of a single quark at $t = 0$ is expressed in terms of the field $\bar{\psi}(\mathbf{x})$ as

$$b_{\mathbf{p}, \lambda_1}^\dagger |0\rangle = \int d\mathbf{x}_1 \frac{d\mathbf{k}_1}{(2\pi)^3 2E_1} \sum_{\mu_1} \bar{u}(\mathbf{k}_1, \mu_1) \gamma^0 u(\mathbf{p}, \lambda_1) e^{-i(\mathbf{k}_1 - \mathbf{p}) \cdot \mathbf{x}_1} b_{\mathbf{k}_1, \mu_1}^\dagger |0\rangle = \int d\mathbf{x}_1 \bar{\psi}(\mathbf{x}_1) e^{i\mathbf{p} \cdot \mathbf{x}_1} \gamma^0 u(\mathbf{p}, \lambda_1) |0\rangle \quad (7.27)$$

With an analogous expression for the antiquark state we have

$$|M, P\rangle \equiv b_{\mathbf{p}, \lambda_1}^\dagger d_{\mathbf{P} - \mathbf{p}, \lambda_2}^\dagger |0\rangle = \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}(\mathbf{x}_1) e^{i\mathbf{p} \cdot \mathbf{x}_1 + i(\mathbf{P} - \mathbf{p}) \cdot \mathbf{x}_2} \gamma^0 u(\mathbf{p}, \lambda_1) \bar{v}(\mathbf{P} - \mathbf{p}, \lambda_2) \gamma^0 \psi(\mathbf{x}_2) |0\rangle \quad (7.28)$$

The exponent can be written

$$\mathbf{p} \cdot \mathbf{x}_1 + (\mathbf{P} - \mathbf{p}) \cdot \mathbf{x}_2 = \frac{1}{2}\mathbf{P} \cdot (\mathbf{x}_1 + \mathbf{x}_2) + (\mathbf{p} - \frac{1}{2}\mathbf{P}) \cdot (\mathbf{x}_1 - \mathbf{x}_2) = \frac{1}{2}\mathbf{P} \cdot (\mathbf{x}_1 + \mathbf{x}_2) + \mathbf{p}_0 \cdot \mathbf{x}_R \quad (7.29)$$

where the rest frame separation \mathbf{x}_R corresponding to the separation $\mathbf{x} \equiv \mathbf{x}_1 - \mathbf{x}_2$ in the moving frame was denoted

$$\mathbf{x}_R \equiv (\mathbf{x}^\perp, x^3 \cosh \xi) \quad (7.30)$$

The free Dirac spinors in (7.28) are related to their rest frame expressions by a boost,

$$\begin{aligned} \gamma^0 u(\mathbf{p}, \lambda_1) &= \gamma^0 \exp(\frac{1}{2}\xi\alpha^3) u(\mathbf{p}_0, \lambda_1) = \exp(-\frac{1}{2}\xi\alpha^3) \gamma^0 u(\mathbf{p}_0, \lambda_1) \\ v^\dagger(\mathbf{P} - \mathbf{p}, \lambda_2) &= v^\dagger(-\mathbf{p}_0, \lambda_2) \exp(\frac{1}{2}\xi\alpha^3) \end{aligned} \quad (7.31)$$

Using this in (7.28) the state takes the general form (5.2), with

$$\Phi^{(\xi)}(\mathbf{x}) = \exp(-\frac{1}{2}\xi\alpha^3) \Phi^{(0)}(\mathbf{x}_R) \exp(\frac{1}{2}\xi\alpha^3) \quad \text{when } V = 0 \quad (7.32)$$

Thus the boosted wave function, after extracting the factors $\exp[i\mathbf{P} \cdot (\mathbf{x}_1 + \mathbf{x}_2)/2]$ and $\exp(\pm \frac{1}{2}\xi\alpha^3)$, is given by the wave function of the rest frame at the corresponding (Lorentz dilated) quark separation (7.30).

D. Active boost in $D = 3 + 1$

The boost generators which satisfy the Lie algebra with the momentum \mathcal{P} (A.3) and Hamiltonian \mathcal{H} with the gluon field (4.2) have not been constructed. Hence we consider only the solutions of the bound state equation (7.2).

The rest frame separation of radial and angular variables (section V C) is inapplicable because the momentum \mathbf{P} in (7.2) leaves rotational symmetry only around the z -axis. This necessitates solving a (two-dimensional) partial differential equation. It turns out that the solution of (7.2) at $\mathbf{x}^\perp = 0$ can be expressed in terms of the rest frame wave function when the potential is linear [19]. This provides a boundary condition for the partial differential equation that is locally normalizable and has the correct P -dependence of the energy eigenvalue. The solution of the BSE (7.2) is thus uniquely determined.

Solutions of the BSE do not ensure that scattering amplitudes are Lorentz covariant. This remains a topic for future studies. An encouraging indication is provided by the gauge invariance of the transition electromagnetic form factor $\gamma^* a \rightarrow b$, for bound states a and b of any momenta. The matrix element of the electromagnetic current $j^\mu(z) = \bar{\psi}(z)\gamma^\mu\psi(z)$ is,

$$F_{ab}^\mu(z) = \langle M_b, \mathbf{P}_b | j^\mu(z) | M_a, \mathbf{P}_a \rangle \quad (7.33)$$

$$= e^{iz \cdot (P_b - P_a)} \int d\mathbf{x} \left\{ e^{-i\mathbf{x} \cdot (P_a - P_b)/2} \text{Tr} [\Phi_b^{(\xi_b)\dagger}(\mathbf{x}) \gamma^\mu \gamma^0 \Phi_a^{(\xi_a)}(\mathbf{x})] - e^{i\mathbf{x} \cdot (P_a - P_b)/2} \text{Tr} [\Phi_b^{(\xi_b)\dagger}(\mathbf{x}) \Phi_a^{(\xi_a)}(\mathbf{x}) \gamma^0 \gamma^\mu] \right\}$$

The condition $\partial F_{ab}^\mu(z)/\partial z^\mu = 0$ was shown to hold (see section V B of [21]) when the wave functions $\Phi_a^{(\xi_a)}(\mathbf{x})$ and $\Phi_b^{(\xi_b)}(\mathbf{x})$ satisfy the BSE (7.2).

The $\mathbf{x}^\perp = 0$ solution is analogous to the one in $D = 1 + 1$ (7.9),

$$\Phi^{(\xi)}(\mathbf{x}^\perp = 0, \tau_P) = \exp(-\tfrac{1}{2}\zeta\alpha^3) \Phi^{(0)}(\mathbf{x}^\perp = 0, \tau_0) \exp(\tfrac{1}{2}\zeta\alpha^3) \quad \text{for } \tau_P = \tau_0 \quad (7.34)$$

where τ_P and ζ are defined as in (7.10) and (7.11) with $V = V'|x^3|$ at $\mathbf{x}^\perp = 0$.

The BSE (7.2) may be expressed as

$$\left[i\vec{\nabla} \cdot \boldsymbol{\alpha} - \tfrac{1}{2}(E - V + P\alpha^3) + m\gamma^0 \right] \Phi^{(\xi)} + \Phi^{(\xi)} \left[i\overleftarrow{\nabla} \cdot \boldsymbol{\alpha} - \tfrac{1}{2}(E - V - P\alpha^3) - m\gamma^0 \right] = 0 \quad (7.35)$$

Multiplying by $\exp(\tfrac{1}{2}\zeta\alpha^3)$ from the left and by $\exp(-\tfrac{1}{2}\zeta\alpha^3)$ from the right the relation (7.34) requires, at $\mathbf{x}^\perp = 0$,

$$e^{\zeta\alpha^3/2} \left[i\vec{\nabla} \cdot \boldsymbol{\alpha} - \tfrac{1}{2}(E - V + P\alpha^3) + m\gamma^0 \right] e^{-\zeta\alpha^3/2} \Phi^{(0)} + \Phi^{(0)} e^{\zeta\alpha^3/2} \left[i\overleftarrow{\nabla} \cdot \boldsymbol{\alpha} - \tfrac{1}{2}(E - V - P\alpha^3) - m\gamma^0 \right] e^{-\zeta\alpha^3/2} = 0 \quad (7.36)$$

According to (7.10) and (7.11),¹²

$$-\tfrac{1}{2}(E - V \pm P\alpha^3) = -\tfrac{1}{2}\Pi \exp(\pm\zeta\alpha^3) \quad \Pi = \sqrt{\Pi^2} \quad (7.37)$$

Since $\nabla^\perp V(|\mathbf{x}|) = 0$ at $\mathbf{x}^\perp = 0$ we may in the first term of (7.36) use

$$i\vec{\nabla}^\perp \cdot \boldsymbol{\alpha}^\perp e^{-\zeta\alpha^3/2} = e^{\zeta\alpha^3/2} i\vec{\nabla}^\perp \cdot \boldsymbol{\alpha}^\perp \quad \text{as well as} \quad m\gamma^0 e^{-\zeta\alpha^3/2} = e^{\zeta\alpha^3/2} m\gamma^0 \quad (7.38)$$

and analogously bring the factor $\exp(\tfrac{1}{2}\zeta\alpha^3)$ to the right in the second term of (7.36).

The contribution $-i(\partial_3 \zeta/2)\Phi^{(0)}$ in the first term of (7.36) cancels with the corresponding contribution from the second term. In the ansatz (7.34) $\Phi^{(\xi)}(\mathbf{x}_\perp = 0, x^3)$ depends on x^3 only via $\tau_P = \Pi^2/V'$, and $\nabla^\perp \tau_P = 0$. Hence we may change to the boost-invariant variable Π ,

$$\partial_3 = -2V'\Pi \cosh \zeta \partial_{\Pi^2} = -V' \cosh \zeta \partial_\Pi \quad (7.39)$$

¹² For simplicity we consider here only values of \mathbf{x} for which $x^3 > 0$ and $\Pi^2 \geq 0$.

The ∂_3 contributions in the first and second term of (7.36) are then, respectively,

$$\begin{aligned} i\alpha^3 \vec{\partial}_3 \Phi^{(0)} &= -iV' \alpha^3 \cosh \zeta \vec{\partial}_\Pi \Phi^{(0)} = -iV' e^{\zeta \alpha^3} \alpha^3 \vec{\partial}_\Pi \Phi^{(0)} + iV' \sinh \zeta \vec{\partial}_\Pi \Phi^{(0)} \\ i\Phi^{(0)} \overleftarrow{\partial}_3 \alpha^3 &= -i\Phi^{(0)} \overleftarrow{\partial}_\Pi \alpha^3 \cosh \zeta V' = -i\Phi^{(0)} \overleftarrow{\partial}_\Pi \alpha^3 e^{-\zeta \alpha^3} V' - iV' \sinh \zeta \vec{\partial}_\Pi \Phi^{(0)} \end{aligned} \quad (7.40)$$

The terms $\propto \sinh \zeta$ cancel. For $\mathbf{x}^\perp = 0$ the condition (7.36) is equivalent to

$$e^{\zeta \alpha^3} \left(-iV' \alpha^3 \vec{\partial}_\Pi + i\vec{\nabla}^\perp \cdot \boldsymbol{\alpha}^\perp - \frac{1}{2}\Pi + m\gamma^0 \right) \Phi^{(0)} + \Phi^{(0)} \left(-i\overleftarrow{\partial}_\Pi \alpha^3 V' + i\vec{\nabla}^\perp \cdot \boldsymbol{\alpha}^\perp - \frac{1}{2}\Pi - m\gamma^0 \right) e^{-\zeta \alpha^3} = 0 \quad (7.41)$$

Provided V' is independent of x^3 , *i.e.*, for a linear potential, only the factors $e^{\pm \zeta \alpha^3}$ depend explicitly on P . The coefficient of $\cosh \zeta$ is the BSE of the rest frame, which $\Phi^{(0)}$ solves by definition. For (7.36) to be satisfied at all P also the coefficient of $\sinh \zeta$ must vanish. Given that the $\cosh \zeta$ coefficient vanishes the $\sinh \zeta$ condition becomes an anticommutator with α^3 . Expressed in terms of x^3 , which in the rest frame is related to $\Pi = M - V$,

$$\left\{ \alpha^3, [i\vec{\nabla} \cdot \boldsymbol{\alpha} + m\gamma^0 - \frac{1}{2}(M - V)] \Phi^{(0)}(\mathbf{x}^\perp = 0, x^3) \right\} = \frac{1}{2}(M - V) \left\{ \alpha^3, \vec{\Lambda}_- \Phi^{(0)}(\mathbf{x}^\perp = 0, x^3) \right\} = 0 \quad (7.42)$$

where $\vec{\Lambda}_-$ is defined in (5.15). From the expressions for $\vec{\Lambda}_- \Phi(\mathbf{x})$ in (5.28), (5.34) and (5.41) it is clear that (7.42) holds for all wavefunctions. Thus (7.34) solves the BSE (7.2) for all P at $\mathbf{x}^\perp = 0$ when the potential is linear. The wave function $\Phi^{(\xi)}(\mathbf{x}^\perp = 0, \tau)$ is frame independent, apart from the factors $\exp(\pm \frac{1}{2}\zeta \alpha^3)$.

Unless an analytic relation between $\Phi^{(\xi)}$ and $\Phi^{(0)}$ is found for $\mathbf{x}^\perp \neq 0$ the wave function must be extended to all \mathbf{x} by solving the BSE (7.2) numerically, using (7.34) as a boundary condition. We assume that the local normalizability of the wave function at $\mathbf{x}^\perp = 0$ ensures local normalizability for all \mathbf{x} . We also have not addressed the analytic continuation required for values of x^3 that make τ_P negative. In the rest frame $\tau_0 = (M - V)^2/V' \geq 0$ for all real \mathbf{x} .

VIII. SPONTANEOUS BREAKING OF CHIRAL INVARIANCE

We have required that the solutions of the $\mathbf{P} = 0$ bound state equation (5.12) be locally normalizable. One of the two independent solutions is square integrable at $|\mathbf{x}| \equiv r = 0$, similarly as for the Schrödinger equation. The Schrödinger wave function $\Phi(\mathbf{x})$ determines the probability distribution of a single particle and so should have unit norm, $\int |\Phi(\mathbf{x})|^2 d\mathbf{x} = 1$. This implies a discrete mass spectrum. The relativistic equation (5.12) includes Z -contributions like in Fig. 1(c), which increase the number of constituents. According to (6.3) the normalizing integral grows linearly with an asymptotic cutoff in r . A unit norm is thus neither motivated nor possible. This is equally the case for the Dirac equation with a linear potential [18].

Instead there arises another constraint, discussed in section VI A. The wave functions (*e.g.*, (5.26)) have factors $1/(M - V)$ which make them locally normalizable only if the radial wave function vanishes at $V(r) = V'r = M$. Together with the constraint at $r = 0$ this allows only discrete bound state masses M . The local normalizability criterion coincides with that of a finite norm in the non-relativistic (Schrödinger) limit [21].

There is a special case that we did not discuss so far, namely $M = 0$. Then the singular points at $r = 0$ and $r = M/V'$ coincide. Locally normalizable, massless solutions in fact exist for any quark mass m [21]. An $M = 0$ rest frame state has vanishing four-momentum in all frames and thus does not correspond to a physical particle. However, the massless $J^{PC} = 0^{++}$ state may mix with the perturbative vacuum while preserving Poincaré invariance. When $m = 0$ this causes a spontaneous breaking of chiral invariance. In this section we make an exploratory study of chiral symmetry breaking with a single quark flavor (the chiral anomaly arises only at loop level). We set the scale such that $V(r) = r$,

$$V' = 1 \quad (8.1)$$

A. Vanishing quark mass, $m = 0$

The states discussed in section V have exact chiral symmetry for vanishing quark mass. The coupled radial equations of the 0^{++} trajectory (5.38) and (5.39) decouple when $m = 0$, with the equation for $H_1(r)$ reducing to (5.25) for $F_1(r)$ of the 0^{-+} trajectory. The radial equation of the 0^{++} and 0^{-+} states with $M = 0$,

$$H_1''(r) + \frac{1}{r} H_1'(r) + \frac{1}{4} r^2 H_1(r) = 0 \quad (8.2)$$

can be solved analytically. The wave functions of the 0^{++} “sigma” (5.35) and 0^{-+} “pion” (5.23) states are

$$\frac{1}{N_\sigma} \Phi_\sigma(\mathbf{x}) = J_0(\tfrac{1}{4}r^2) + i \boldsymbol{\alpha} \cdot \mathbf{x} \frac{1}{r} J_1(\tfrac{1}{4}r^2) \quad (m = M = 0) \quad (8.3)$$

$$\frac{1}{N_\pi} \Phi_\pi(\mathbf{x}) = \frac{1}{N_\sigma} \gamma_5 \Phi_\sigma(\mathbf{x}) \quad (8.4)$$

where $J_{0,1}$ are Bessel functions and N_σ, N_π normalization constants. The sigma state is thus (at $t = 0$ and with color and Dirac indices suppressed),

$$|\sigma\rangle = \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}(\mathbf{x}_1) \Phi_\sigma(\mathbf{x}_1 - \mathbf{x}_2) \psi(\mathbf{x}_2) |0\rangle \equiv \hat{\sigma} |0\rangle \quad (8.5)$$

Similarly Φ_π determines the pion state $|\pi\rangle = \hat{\pi} |0\rangle$. These states have vanishing four-momentum in all frames,

$$\hat{P}^\mu |\sigma\rangle = 0 \quad (8.6)$$

The 0^{++} sigma state has vacuum quantum numbers and thus can mix with the perturbative vacuum. Let us consider the “chiral condensate”

$$|\chi\rangle = \exp(\hat{\sigma}) |0\rangle \equiv \hat{\chi} |0\rangle \quad (8.7)$$

where $\hat{\sigma}$ is defined in (8.5). Since $\hat{\sigma}$ commutes with itself we may neglect contractions of the quark fields in $\hat{\chi}$, *i.e.*, $\{\psi(t, \mathbf{x}), \bar{\psi}(t, \mathbf{y})\} = 0$. Formally discretizing the integrals over $\mathbf{x}_1, \mathbf{x}_2$ (and still suppressing color indices) we can express $\hat{\chi}$ as

$$\hat{\chi} = \exp \left[\sum_{\alpha, \beta} \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}_\alpha(\mathbf{x}_1) \Phi_\sigma^{\alpha\beta}(\mathbf{x}_1 - \mathbf{x}_2) \psi_\beta(\mathbf{x}_2) \right] = \prod_{\mathbf{x}_1, \mathbf{x}_2, \alpha, \beta} [1 + d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}_\alpha(\mathbf{x}_1) \Phi_\sigma^{\alpha\beta}(\mathbf{x}_1 - \mathbf{x}_2) \psi_\beta(\mathbf{x}_2)] \quad (8.8)$$

since the square of $\bar{\psi}_\alpha(\mathbf{x}_1) \psi_\beta(\mathbf{x}_2)$ vanishes. In the absence of contractions only the “1” part of each factor contributes in $\langle \chi | \chi \rangle = 1$. On the other hand, when other fields such as $\bar{\psi}_\alpha(\mathbf{x}) \psi_\alpha(\mathbf{x})$ operate on $|\chi\rangle$ they may contract with a corresponding pair in the product, giving the vacuum expectation value

$$\langle \chi | \bar{\psi} \psi | \chi \rangle = \text{Tr} [\gamma^0 \Phi_\sigma(\mathbf{0}) \gamma^0] = 4N_\sigma \quad (8.9)$$

This implies that chiral symmetry is spontaneously broken.

An infinitesimal chiral transformation $U_\chi(\beta)$ ($\beta \ll 1$) transforms the quark fields as

$$U_\chi(\beta) \bar{\psi}(\mathbf{x}) U_\chi^\dagger(\beta) = \bar{\psi}(\mathbf{x}) (1 - i\beta \gamma_5) \quad U_\chi(\beta) \psi(\mathbf{x}) U_\chi^\dagger(\beta) = (1 - i\beta \gamma_5) \psi(\mathbf{x}) \quad (8.10)$$

With $\Phi_\pi = \gamma_5 \Phi_\sigma = \frac{1}{2} \{\gamma_5, \Phi_\sigma\}$ (absorbing a relative normalization in β) we get

$$U_\chi(\beta) |\chi\rangle = \exp \left[\int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}(\mathbf{x}_1) [\Phi_\sigma - i\beta \{\gamma_5, \Phi_\sigma\}] \psi(\mathbf{x}_2) \right] |\chi\rangle = (1 - 2i\beta \hat{\pi}) |\chi\rangle \quad (8.11)$$

where the second equality is valid in the absence of field contractions. Thus a chiral transformation of $|\chi\rangle$ creates massless pions.

In bound states built on the chiral condensate,

$$|M\rangle_\chi = \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}(\mathbf{x}_1) \Phi(\mathbf{x}_1 - \mathbf{x}_2) \psi(\mathbf{x}_2) |\chi\rangle \quad (8.12)$$

contractions of $\bar{\psi}(\mathbf{x}_1) \psi(\mathbf{x}_2)$ with fields in $|\chi\rangle$ as in (8.9) will break chiral invariance, removing the parity doubling. This may be described by considering emissions of $\hat{\sigma}$ from chirally symmetric bound states as in Fig. 1(a), with the $\hat{\sigma}$ then being absorbed in the chiral condensate. We shall not pursue this further here.

B. Limit of small quark mass, $m \rightarrow 0$

In QCD the small u, d quark masses break chiral invariance explicitly and give the pion its physical mass. Let us consider the case $0 < m \ll 1$ (in units of $\sqrt{V'}$ defined in (8.1)). The exact massless ($M = 0$) solution for the 0^{++}

sigma wave function for any m is¹³

$$\begin{aligned}\Phi_\sigma(\mathbf{x}) &= f_1(r) + i\boldsymbol{\alpha} \cdot \mathbf{x} f_2(r) + i\boldsymbol{\gamma} \cdot \mathbf{x} g_2(r) \quad (M=0) \\ \frac{1}{N_\sigma} f_1(r) &= -\frac{1}{r^2} e^{-ir^2/4} \left[(2m^2 - r^2) L_{(im^2-1)/2}(\tfrac{1}{2}ir^2) - 2m^2 L_{(im^2+1)/2}(\tfrac{1}{2}ir^2) \right] = J_0(\tfrac{1}{4}r^2) + \mathcal{O}(m^2) \\ \frac{1}{N_\sigma} f_2(r) &= -\frac{2}{r^3} e^{-ir^2/4} \left[(\tfrac{1}{2}ir^2 - 1) L_{(im^2-1)/2}(\tfrac{1}{2}ir^2) + L_{(im^2+1)/2}(\tfrac{1}{2}ir^2) \right] = \frac{1}{r} J_1(\tfrac{1}{4}r^2) + \mathcal{O}(m^2) \\ g_2(r) &= -\frac{2m}{r} f_2(r)\end{aligned}\tag{8.13}$$

where the $L_\nu(x)$ are Laguerre functions. Since the sigma state remains massless when $m \neq 0$ it may form a chiral condensate as above, without breaking Poincaré invariance.

For the record, let us note that the massless ($M=0$) 0^{-+} state for finite quark mass m has the wave function

$$\begin{aligned}\Phi_\pi(\mathbf{x}) &= [F_1(r) + i\boldsymbol{\alpha} \cdot \mathbf{x} F_2(r) + \gamma^0 F_4(r)] \gamma_5 \\ \frac{1}{N_\pi} F_1(r) &= e^{-ir^2/4} L_{(im^2-1)/2}(\tfrac{1}{2}ir^2) = J_0(\tfrac{1}{4}r^2) + \mathcal{O}(m^2) \quad (M=0) \\ \frac{1}{N_\pi} F_2(r) &= \frac{2}{r^3} e^{-ir^2/4} \left[(im^2 + 1 - \tfrac{1}{2}ir^2) L_{(im^2-1)/2}(\tfrac{1}{2}ir^2) - (im^2 + 1) L_{(im^2+1)/2}(\tfrac{1}{2}ir^2) \right] = \frac{1}{r} J_1(\tfrac{1}{4}r^2) + \mathcal{O}(m^2) \\ F_4(r) &= -\frac{2m}{r} F_1(r)\end{aligned}\tag{8.14}$$

However, we are interested in a 0^{-+} state with a non-zero mass $M \neq 0$ that may describe the physical pion. For $m \ll 1$ it should be close to the $m=0$ solution (8.4). According to (5.23) also the $M \neq 0$ pion wave function has the form (8.14) where F_1 satisfies the radial equation (5.25)

$$F_1'' + \left(\frac{2}{r} + \frac{1}{M-r} \right) F_1' + \left[\frac{1}{4}(M-r)^2 - m^2 \right] F_1 = 0\tag{8.15}$$

For the physical solution $F_1(0)$ is a non-vanishing constant. An important difference from the $M=0$ solution (8.14) is seen from the constraint (5.24), $mF_1(r) = \frac{1}{2}(M-r)F_4(r)$. It implies that the F_4 radial function is finite at $r=0$,

$$F_4(0) = \frac{2m}{M} F_1(0)\tag{8.16}$$

The pion should, as a Goldstone boson, be annihilated by the axial vector current $j_5^\mu(x) = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$ and by its divergence $\partial_\mu j_5^\mu(x) = 2im\bar{\psi}(x)\psi(x)$,

$$\langle \chi | \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) \hat{\pi} | \chi \rangle = iP^\mu f_\pi e^{-iP \cdot x}\tag{8.17}$$

$$\langle \chi | \bar{\psi}(x)\gamma_5\psi(x) \hat{\pi} | \chi \rangle = -i \frac{M^2}{2m} f_\pi e^{-iP \cdot x}\tag{8.18}$$

Since the pion is an eigenstate of the Hamiltonian with eigenvalue $E = P^0$ we have at time t , using (5.2) for a general momentum \mathbf{P} (and neglecting any chiral corrections due to $|0\rangle \rightarrow |\chi\rangle$),

$$\hat{\pi}(t) | \chi \rangle = e^{-iP^0 t} \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}(t, \mathbf{x}_1) e^{i\mathbf{P} \cdot (\mathbf{x}_1 + \mathbf{x}_2)/2} \Phi_\pi^\dagger(\boldsymbol{\xi})(\mathbf{x}_1 - \mathbf{x}_2) \psi(t, \mathbf{x}_2) | \chi \rangle\tag{8.19}$$

Contracting the quark fields the lhs. of (8.17) becomes

$$\langle \chi | \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) \hat{\pi} | \chi \rangle = \text{Tr} [\gamma^\mu\gamma_5\gamma^0\Phi_\pi^\dagger(\boldsymbol{\xi})(0)\gamma^0] e^{-iP \cdot x}\tag{8.20}$$

¹³ We use lowercase letters for the radial functions of the 0^{++} trajectory defined in (5.35), to distinguish them from the radial functions of the 0^{-+} trajectory (5.23).

According to (7.32) the momentum dependence of the wave function at the origin is

$$\Phi_{\pi}^{(\xi)}(\mathbf{x} = 0) = \exp(-\frac{1}{2}\xi \cdot \alpha) \Phi_{\pi}^{(\xi=0)}(\mathbf{x} = 0) \exp(\frac{1}{2}\xi \cdot \alpha) \quad (8.21)$$

and from (8.14), since $F_2(0)$ is finite for the physical solution,

$$\Phi_{\pi}^{(0)}(0) = [F_1(0) + \gamma^0 F_4(0)] \gamma_5 \quad (8.22)$$

giving

$$\gamma^0 \Phi_{\pi}^{(\xi)}(0) \gamma^0 = -[F_1(0) + \not{P} F_4(0)/M] \gamma_5 \quad (8.23)$$

Substituting this into (8.20) gives

$$\langle \chi | \bar{\psi}(x) \gamma^{\mu} \gamma_5 \psi(x) \hat{\pi} | \chi \rangle = \text{Tr} \{ \gamma^{\mu} [-F_1(0) + \not{P} F_4(0)/M] \} e^{-iP \cdot x} = 4P^{\mu} F_4(0)/M e^{-iP \cdot x} \quad (8.24)$$

Similarly

$$\langle \chi | \bar{\psi}(x) \gamma_5 \psi(x) \hat{\pi} | \chi \rangle = \text{Tr} \{ [-F_1(0) + \not{P} F_4(0)/M] \} e^{-iP \cdot x} = -4F_1(0) e^{-iP \cdot x} \quad (8.25)$$

Comparing with the rhs. of (8.17) and (8.18) we have the two conditions

$$F_1(0) = i \frac{M^2}{8m} f_{\pi} \quad F_4(0) = \frac{2m}{M} F_1(0) \quad (8.26)$$

The first one determines the normalization of F_1 and the latter agrees with the relation (8.16) which followed from the bound state equation.

A smooth $m \rightarrow 0$ chiral limit requires $M^2 \propto m$ in (8.18). In section VIA we saw that $F_1(r \rightarrow 0) \propto r^0$ for the physical solution, and $F_1(r \rightarrow M) \propto (M - r)^{\gamma}$, with $\gamma = 0, 2$. At small m and M continuity requires the same behavior at the two points, hence $F_1(r \rightarrow M) \propto (M - r)^0$ for the pion. Previously we excluded this solution because it makes Φ_{π} locally unnormalizable at $M - r = 0$. At small M the normalization integral is suppressed: $\int dr r^2 \simeq M^2 \int dr$. The normalizability constraint thus allows a smooth $m \rightarrow 0$ limit, with $M^2 \propto m$ at lowest order in m .

IX. OUTLOOK

Bound state physics is often neglected in textbooks and courses on field theory. The Hydrogen atom is taught in introductory level courses on quantum mechanics by postulating the Schrödinger equation. Contrary to custom in physics, in more advanced courses we rarely return to derive the Schrödinger equation from the underlying theory of QED. Students are inadequately prepared for the concepts and methods of bound states.

Here we focussed on some principles of perturbative bound states, which complement the application of perturbation theory to scattering amplitudes. Bound state wave functions cannot be approximated by polynomials in the perturbative coupling α . In this sense even atoms are non-perturbative, yet their binding energies are well described by a power series in α and $\log \alpha$ [5]. A Feynman diagram expansion of bound states is in principle incorrect since the initial, free *in* and *out* states are orthogonal to bound states. The failure of the sum of QCD ladder diagrams to describe confinement may have contributed to the belief that perturbation theory is irrelevant for hadrons.

A perturbative expansion can formally be developed around any initial state that has a finite overlap with the physical bound state. State-of-the-art NRQED calculations use initial states defined by the Schrödinger equation with the classical potential. We adopted the principle that initial states of the perturbative expansion should be bound by their classical gauge field. Including the classical field in the *in* and *out* states of the perturbative S -matrix (3.5) (“Potential Picture”) defines a unique bound state expansion, analogous to that using free states for scattering amplitudes. This may remove some of the “art” from perturbative bound state calculations.

In applying the Potential Picture to hadrons the QCD scale can arise through a boundary condition on the classical field equations. Fortunately there is an essentially unique, $\mathcal{O}(\alpha_s^0)$ homogeneous solution (4.2). The issue is then whether the bound states generated by this field are close enough to physical hadrons to make a perturbative expansion relevant. The results presented here and in previous work [19–24] seem encouraging enough to merit further studies.

Including single gluon exchange at $\mathcal{O}(H_f^2)$ in (3.5) will allow comparisons with data on heavy quarkonia. The gluon propagates in, and hopefully will be confined by, the classical field. This will require to include a gluon source on the

rhs. of (4.2). It could also be interesting to study non-conventional hadrons such as glueballs, replacing the quarks in the state (4.3) with two gluons (see appendix A of [22] for the case of scalar QED).

The classical field creates light quark pairs, as required for hadron decays and unitarity. The zero-width bound states, of leading order in the limit of a large number of colors, define “string breaking” corrections such as Fig. 1(a,b). The decay amplitudes also determine hadron scattering with $q\bar{q}$ (Regge?) exchange. The hadron loops allow to study hadronic threshold effects and, possibly, hadron “molecules”. The contributions remind of dual diagrams, and the massless quark spectrum (Fig. 2) is similar to that of dual models.

The virtual $q\bar{q}$ pairs due to Z -diagrams (Fig. 1(c)) define a sea quark distribution at leading order in $1/N_c$. The number of sea quarks that can be resolved depends on the physical process. String breaking limits the hadron size and rearranges the $q\bar{q}$ pairs into hadron constituents through loops such as Fig. 1(b). This need not alter the sea quark distribution significantly. A smooth transition between partons and hadrons is suggested by the remarkable duality features observed in data (see Figs. 3-5 in [2]). The gluon sea is assumed to arise through perturbative contributions.

The equal-time wave functions of bound states with non-vanishing momenta \mathbf{P} are related dynamically to the rest frame ones. The analytic relation given in (7.34) on the \mathbf{P} -axis serves as a boundary condition for the bound state equation (7.2). This defines the wave function for all \mathbf{x} . As in the rest of this work we assumed a single quark flavor. Bound states of quarks with unequal mass were studied in $D = 1 + 1$ dimensions [21]. We did not yet study the Lorentz covariance of scattering amplitudes. Rest frame states as seen by a moving observer (“passive” boosts) are discussed in appendix C.

Our results for the hadron spectrum with massless quarks (*e.g.*, Fig. 2) were obtained for a chirally symmetric vacuum, and thus include parity doublets. The existence of massless bound states allowed to consider (section VIII) the consequences of a vacuum with a 0^{++} condensate. The general properties agree with expectations for the spontaneous breaking of chiral symmetry, but much remains to be explored.

We did not consider baryon states, beyond noting their confining potential (4.19). It appears that the classical A_a^0 field (4.2) confines only $q\bar{q}$ and qqq states. The fact that the classical field vanishes when summed over quark colors (4.20) means that baryons (like mesons) do not interact with each other via their confining fields. This may help explain why the shell model of nuclei works so well.

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Appendix A: Symmetries of the $q\bar{q}$ wave function

In this appendix we note the transformation of a rest frame meson wave function $\Phi_{\alpha\beta}^{AB}(\mathbf{x}_1 - \mathbf{x}_2)$ under gauge transformations, space translations, rotations, parity and charge conjugation. The wave function describes the state

$$|M\rangle = \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}(\mathbf{x}_1) \Phi(\mathbf{x}_1 - \mathbf{x}_2) \psi(\mathbf{x}_2) |0\rangle \quad (\text{A.1})$$

1. Gauge transformations

The absence of a $\partial_0 \hat{A}_a^0$ term in the action allows to express $\hat{A}_a^0(t, \mathbf{x})$ in terms of the propagating (quark) fields for all \mathbf{x} at each instant of time. Our states have no propagating (constituent) gluons at $\mathcal{O}(\alpha_s^0)$ so we could omit a gluon source in the homogeneous solution (4.2). Global gauge transformations maintain $\hat{\mathbf{A}}_a = 0$ and do not affect the meson wave function (4.4).

The meson state is invariant in local, time independent gauge transformations $U(\mathbf{x})$ provided its wave function is

transformed accordingly,

$$\psi^A(\mathbf{x}) \rightarrow \sum_B U^{AB}(\mathbf{x}) \psi^B(\mathbf{x}) \quad \text{implies} \quad \Phi(\mathbf{x}_1 - \mathbf{x}_2) \rightarrow U(\mathbf{x}_1) \Phi(\mathbf{x}_1 - \mathbf{x}_2) U^\dagger(\mathbf{x}_2) \quad (\text{A.2})$$

The loss of explicit translation invariance as well as $\hat{\mathbf{A}}_a \neq 0$ makes the bound state dynamics appear quite different in a general gauge. The same is true for Positronium in QED.

The gauge transformation may depend on the component of the state to which it is applied, $U(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2)$. An example of this is provided by boosts in $D = 1 + 1$ dimensions, which must be combined with the state dependent (operator valued) gauge transformation (7.6) to maintain $A^1(x; x_1, x_2) = 0$. This causes the different $x_1 - x_2$ -dependence of the $P \neq 0$ wave function (7.9), compared to the standard Lorentz contraction (7.25) for boosts without a gauge transformation.

2. Space translations

Under space translations $\mathbf{x} \rightarrow \mathbf{x} + \boldsymbol{\ell}$ the quark fields are transformed by the operator

$$U(\mathbf{a}) = \exp[-i\boldsymbol{\ell} \cdot \mathbf{P}] \quad \text{where} \quad \mathbf{P} = \int d\mathbf{x} \psi^\dagger(\mathbf{x}) (-i\boldsymbol{\nabla}) \psi(\mathbf{x}) \quad (\text{A.3})$$

The momentum operator satisfies

$$[\mathbf{P}, \psi(\mathbf{x})] = i\vec{\nabla} \psi(\mathbf{x}) \quad [\mathbf{P}, \bar{\psi}(\mathbf{x})] = \bar{\psi}(\mathbf{x}) i\overleftarrow{\nabla} \quad (\text{A.4})$$

With $\mathbf{P}|0\rangle = 0$ we may verify that the state (A.1) is at rest, $\mathbf{P}|M\rangle = 0$.

3. Rotations

Rotations are generated by the angular momentum operators

$$\mathcal{J} = \int d\mathbf{x} \psi^\dagger(\mathbf{x}) \mathbf{J} \psi(\mathbf{x}) \quad \mathcal{J}^2 = \int d\mathbf{x} \psi^\dagger(\mathbf{x}) \mathbf{J}^2 \psi(\mathbf{x}) \quad (\text{A.5})$$

$$\mathbf{J} = \mathbf{L} + \mathbf{S} = \mathbf{x} \times (-i\boldsymbol{\nabla}) + \frac{1}{2} \gamma_5 \gamma^0 \boldsymbol{\gamma} \quad (\text{A.6})$$

The angular momentum quantum numbers j, λ of the bound state are defined as usual,

$$\mathcal{J}^2 |M, j, \lambda\rangle = j(j+1) |M, j, \lambda\rangle \quad \mathcal{J}^z |M, j, \lambda\rangle = \lambda |M, j, \lambda\rangle \quad (\text{A.7})$$

This requires the wave function to satisfy

$$[\mathbf{J}^2, \Phi(\mathbf{x})] = j(j+1) \Phi(\mathbf{x}) \quad [J^z, \Phi(\mathbf{x})] = \lambda \Phi(\mathbf{x}) \quad (\text{A.8})$$

4. Parity

The operator \mathbb{P} reverses 3-momenta \mathbf{p} but leaves the spin components λ invariant:

$$\mathbb{P} b(\mathbf{p}, \lambda) \mathbb{P}^\dagger = b(-\mathbf{p}, \lambda) \quad \mathbb{P} d(\mathbf{p}, \lambda) \mathbb{P}^\dagger = d(-\mathbf{p}, \lambda) \quad (\text{A.9})$$

The intrinsic parity of the quarks is irrelevant for $q\bar{q}$ states. The *relative* intrinsic parity -1 of quarks and antiquarks in (A.9) ensures that the field transforms as

$$\mathbb{P} \psi(t, \mathbf{x}) \mathbb{P}^\dagger = \gamma^0 \psi(t, -\mathbf{x}) \quad \mathbb{P} \bar{\psi}(t, \mathbf{x}) \mathbb{P}^\dagger = \bar{\psi}(t, -\mathbf{x}) \gamma^0 \quad (\text{A.10})$$

A $q\bar{q}$ state is an eigenstate of parity,

$$\mathbb{P} |M\rangle = \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}(\mathbf{x}_1) \gamma^0 \Phi(-\mathbf{x}_1 + \mathbf{x}_2) \gamma^0 \psi(\mathbf{x}_2) |0\rangle = \eta_P |M\rangle \quad (\text{A.11})$$

if its wave function satisfies

$$\gamma^0 \Phi(-\mathbf{x}) \gamma^0 = \eta_P \Phi(\mathbf{x}) \quad (\eta_P = \pm 1) \quad (\text{A.12})$$

5. Charge conjugation

The charge conjugation operator \mathbb{C} transforms particles into antiparticles,

$$\mathbb{C}b(\mathbf{p}, \lambda)\mathbb{C}^\dagger = d(\mathbf{p}, \lambda) \quad \mathbb{C}d(\mathbf{p}, \lambda)\mathbb{C}^\dagger = b(\mathbf{p}, \lambda) \quad (\text{A.13})$$

In the standard Dirac matrix representation this implies (here T indicates transpose and $\alpha^2 \equiv \gamma^0 \gamma^2$)

$$\mathbb{C}\psi(t, \mathbf{x})\mathbb{C}^\dagger = -i\alpha^2 \bar{\psi}^T(t, \mathbf{x}) \quad \mathbb{C}\bar{\psi}(t, \mathbf{x})\mathbb{C}^\dagger = -i\psi^T(t, \mathbf{x})\alpha^2 \quad (\text{A.14})$$

For a meson state to be an eigenstate of charge conjugation,

$$\mathbb{C}|M\rangle = \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}(\mathbf{x}_1)\alpha^2 \Phi^T(\mathbf{x}_2 - \mathbf{x}_1)\alpha^2 \psi(\mathbf{x}_2)|0\rangle = \eta_C |M\rangle \quad (\text{A.15})$$

its wave function should satisfy

$$\alpha^2 \Phi^T(-\mathbf{x})\alpha^2 = \eta_C \Phi(\mathbf{x}) \quad (\eta_C = \pm 1) \quad (\text{A.16})$$

Appendix B: Structure of Dirac states

The Dirac equation

$$[i\nabla \cdot \boldsymbol{\alpha} + m\gamma^0 + eA^0(\mathbf{x})]\chi(\mathbf{x}) = M\chi(\mathbf{x}) \quad (\text{B.1})$$

where $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$ is the condition for an electron to form a bound state in a given potential $A^0(\mathbf{x})$. The Dirac four-spinor wave function $\chi(\mathbf{x})$ has both positive and negative energy eigenvalues M .

The Dirac dynamics may be regarded as a limit of QED where the electron interacts only with the potential $A^0(\mathbf{x})$ of an infinitely massive particle T at rest. The time ordered Feynman diagrams of Fig. 3(a,b) then describe second order contributions to the interactions of the electron (upper line) with the massive particle T (lower line). Electron loop contributions such as Fig. 3(c) are neglected in the Dirac approximation.

Dirac dynamics allows to study relativistic bound state effects, since the potential is assumed to be commensurate with the electron mass, $A^0 \sim m$. Z-diagram contributions like Fig. 3(b) imply that a Dirac state has Fock components with any number of e^+e^- pairs. The Klein paradox [29] is a well-known consequence of this.

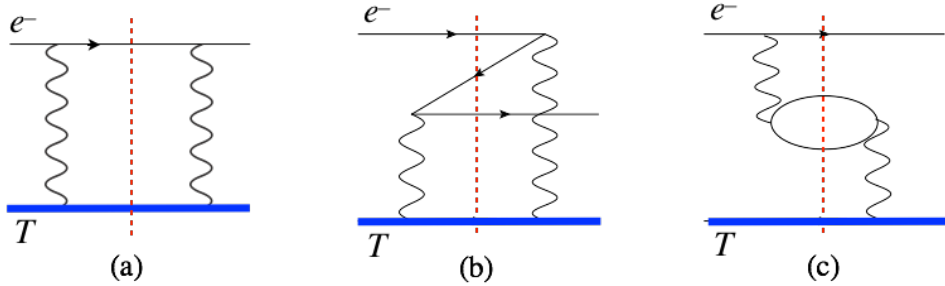


FIG. 3: Time-ordered diagrams in e^-T scattering. Time increases from left to right, with wavy lines representing instantaneous photon exchange. (a) The intermediate state (vertical dashed line) has a single e^- . (b) The electron moves backward in time between scatterings. The intermediate state has an additional e^+e^- pair. (c) An electron loop contribution also gives an e^+e^- pair, but involves a photon that is not attached to T .

A Dirac wave function $\chi(\mathbf{x})$ with positive eigenvalue $M > 0$ defines (at $t = 0$) a state,

$$|M\rangle = \int d\mathbf{x} \bar{\psi}(\mathbf{x}) \chi(\mathbf{x}) |0\rangle \quad (\text{B.2})$$

The ground state $|0\rangle$ is by definition an eigenstate of the Dirac Hamiltonian, $\mathcal{H}|0\rangle = 0$ where

$$\mathcal{H} = \int d\mathbf{x} \psi^\dagger(\mathbf{x}) [-i\boldsymbol{\alpha} \cdot \vec{\nabla} + m\gamma^0 + eA^0(\mathbf{x})] \psi(\mathbf{x}) \quad (\text{B.3})$$

The bound state condition

$$\mathcal{H}|M\rangle = \int d\mathbf{x} \bar{\psi}(\mathbf{x}) [i\boldsymbol{\alpha} \cdot \vec{\nabla} + m\gamma^0 + eA^0(\mathbf{x})] \chi(\mathbf{x}) |0\rangle = M|M\rangle \quad (\text{B.4})$$

is satisfied since $\chi(\mathbf{x})$ satisfies the Dirac equation (B.1).

The bound state (B.2) appears to be a single electron state, created by the field operator $\bar{\psi}(\mathbf{x})$, and has the corresponding quantum numbers. However, the ground state $|0\rangle$ has Fock components with any number of e^+e^- pairs, arising from Z -diagrams such as Fig. 3(b). Let $\{\chi_n\}$ be a complete set of positive energy solutions to the Dirac equation, defining the states

$$|n\rangle = \int d\mathbf{x} \bar{\psi}(\mathbf{x}) \chi_n(\mathbf{x}) |0\rangle \equiv c_n^\dagger |0\rangle \quad (\text{B.5})$$

In terms of the matrices B_{np}, D_{np} in

$$c_n = \int \frac{d\mathbf{p}}{(2\pi)^3 2E_p} \sum_{\lambda} \bar{\chi}_n(\mathbf{p}) [u(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} + v(-\mathbf{p}, \lambda) d_{-\mathbf{p}, \lambda}^\dagger] \equiv B_{np} b_p + D_{np} d_p^\dagger \quad (\text{B.6})$$

the ground state can be expressed as [23]

$$|0\rangle = N_0 \exp \left[-b_p^\dagger (B^{-1})_{pn} D_{nq} d_q^\dagger \right] |0\rangle_{free} \quad (\text{B.7})$$

Sums over the 3-momenta \mathbf{p}, \mathbf{q} and helicities are implied, as well as over the states n . The perturbative vacuum satisfies $b_{\mathbf{p}, \lambda} |0\rangle_{free} = d_{\mathbf{p}, \lambda} |0\rangle_{free} = 0$. The strong potential thus creates e^+e^- pairs in the perturbative vacuum. The operator c_n annihilates the ground state, $c_n |0\rangle = 0$, and is related to the free operators b, d^\dagger by the Bogoliubov transform (B.6).

In the case of a linear potential $eA^0(\mathbf{x}) = V|\mathbf{x}|$ the Dirac equation has a continuous mass spectrum M_n , with wave functions χ_n that are not square integrable [18]. Illustrations in $D = 1 + 1$ dimensions are given in [23]. At large $|\mathbf{x}|$ (and hence large A^0) the coefficient D_{np} of the d^\dagger operator in c_n (B.6) dominates over B_{np} . Hence there are only positrons in the region where $eA^0 \gg m$. The reason is that a potential which confines electrons repels positrons. The large negative potential energy $-eA^0$ of positrons is cancelled by a correspondingly large kinetic energy, allowing a fixed total energy M .

Solutions $\chi(\mathbf{x})$ of the Dirac equation (B.1) with negative eigenvalues $M < 0$ define states $\int d\mathbf{x} \bar{\chi}(\mathbf{x}) \psi(\mathbf{x}) |0\rangle$ with a valence positron. They are also eigenstates of \mathcal{H} with *positive* eigenvalues $-M$.

Appendix C: Passive boost in $D = 3 + 1$

In this appendix we consider the dynamics of a bound state at rest from the point of view of an observer in motion. The state appears to have momentum $\mathbf{P} = (0, 0, M \sinh \xi)$ to the observer, and to be bound by a classical field $A_\xi^\mu(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2)$ which is related to the rest frame field A_R^0 (4.7) by a boost. The case of $D = 1 + 1$ dimensions is discussed in section VII B 3.

With a short-hand notation for the field at the positions of the quarks,

$$A_\xi^\mu(\mathbf{x}_1 - \mathbf{x}_2) \equiv A_\xi^\mu(\mathbf{x}_1; \mathbf{x}_1, \mathbf{x}_2) = -A_\xi^\mu(\mathbf{x}_2; \mathbf{x}_1, \mathbf{x}_2) \quad (\text{C.1})$$

we have

$$A_\xi^0(\mathbf{x}_1 - \mathbf{x}_2) = \cosh \xi A_R^0(\mathbf{x}_R) \quad A_\xi^3(\mathbf{x}_1 - \mathbf{x}_2) = \sinh \xi A_R^0(\mathbf{x}_R) \quad A_\xi^\mu(\mathbf{x}_1 - \mathbf{x}_2) = 0 \quad (\mu = 1, 2) \quad (\text{C.2})$$

The rest frame separation $\mathbf{x}_R = \mathbf{x}_{1R} - \mathbf{x}_{2R}$ corresponds to the Lorentz contracted separation $\mathbf{x}_1 - \mathbf{x}_2$,

$$\mathbf{x}_{iR} = (x_i, y_i, z_i \cosh \xi) \quad (i = 1, 2) \quad (\text{C.3})$$

The boost (C.2) of the rest frame A^0 field gives rise to a magnetic field, $\mathbf{B}_\xi = \nabla_{\mathbf{x}} \times \mathbf{A}_\xi(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2)$. With $\mathbf{A}_\xi = \sum_a \mathbf{A}_\xi^a T_a^{AA}$ and the expression (4.7) for A_{Ra}^0 the magnetic field has a unit color matrix,

$$B_\xi^1 = 2\Lambda^2 \frac{x_R^2}{|\mathbf{x}_R|} \sinh \xi \quad B_\xi^2 = -2\Lambda^2 \frac{x_R^1}{|\mathbf{x}_R|} \sinh \xi \quad B_\xi^3 = 0 \quad (\text{C.4})$$

The magnetic field affects the quark spins. To the moving observer the spins will appear to precess, in a manner which depends on the quantum numbers and is proportional to the magnetic field. There is no spin and hence no precession in $D = 1 + 1$ and $D = 2 + 1$ dimensions (where the Dirac matrices are 2×2).

The gauge field (C.2) in the Hamiltonian

$$\mathcal{H} = \int d\mathbf{x} \psi^\dagger(0, \mathbf{x}) \left[-i\boldsymbol{\alpha} \cdot \vec{\nabla} + m\gamma^0 + \frac{1}{2}\gamma^0 g\mathcal{A}_\xi \right] \psi(0, \mathbf{x}) \quad (\text{C.5})$$

acting on the state $\bar{\psi}_\alpha^A(\mathbf{x}_1)\psi_\beta^A(\mathbf{x}_2)|0\rangle$ is

$$\frac{1}{2}\gamma^0 g\mathcal{A}_\xi(\mathbf{x}_1 - \mathbf{x}_2) = \frac{1}{2}V_R(\mathbf{x}_R) \exp(-\xi\alpha^3) \quad (\text{C.6})$$

where the rest frame potential is $V_R(\mathbf{x}_R) = g\Lambda^2|\mathbf{x}_R|$.

We define the state as perceived by the moving observer in terms of a (color singlet) wave function $\Phi^{(\xi)}$ with the boost dependent factors $e^{\pm\xi\alpha^3/2}$ extracted,¹⁴

$$|\overline{M}, P\rangle \equiv \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}(t=0, \mathbf{x}_1) e^{i\mathbf{P} \cdot (\mathbf{x}_1 + \mathbf{x}_2)/2} e^{-\xi\alpha^3/2} \Phi^{(\xi)}(\mathbf{x}_1 - \mathbf{x}_2) e^{\xi\alpha^3/2} \psi(t=0, \mathbf{x}_2) |0\rangle \quad (\text{C.7})$$

Operating with \mathcal{H} on this state and subtracting $E = M \cosh \xi$ we get,

$$(\mathcal{H} - E)|\overline{M}, P\rangle \equiv \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}(\mathbf{x}_1) e^{i\mathbf{P} \cdot (\mathbf{x}_1 + \mathbf{x}_2)/2} e^{-\xi\alpha^3/2} \left[\delta\Phi^{(\xi)}(\mathbf{x}_1 - \mathbf{x}_2) \right] e^{\xi\alpha^3/2} \psi(\mathbf{x}_2) |0\rangle \quad (\text{C.8})$$

With $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ and $\nabla = \partial/\partial\mathbf{x}$,

$$\begin{aligned} e^{-\xi\alpha^3/2} \delta\Phi^{(\xi)}(\mathbf{x}) e^{\xi\alpha^3/2} = & \left(i\boldsymbol{\alpha} \cdot \vec{\nabla} + m\gamma^0 - \frac{1}{2}M \cosh \xi - \frac{1}{2}M\alpha^3 \sinh \xi + \frac{1}{2}V_R e^{\xi\alpha^3} \right) e^{-\xi\alpha^3/2} \Phi^{(\xi)}(\mathbf{x}) e^{\xi\alpha^3/2} \\ & + e^{-\xi\alpha^3/2} \Phi^{(\xi)}(\mathbf{x}) e^{\xi\alpha^3/2} \left(i\boldsymbol{\alpha} \cdot \overleftarrow{\nabla} - m\gamma^0 - \frac{1}{2}M \cosh \xi + \frac{1}{2}M\alpha^3 \sinh \xi + \frac{1}{2}V_R e^{-\xi\alpha^3} \right) \end{aligned} \quad (\text{C.9})$$

Combining

$$\begin{aligned} -\frac{1}{2}M(\cosh \xi + \alpha^3 \sinh \xi) + \frac{1}{2}V_R e^{\xi\alpha^3} &= -\frac{1}{2}(M - V_R) e^{\xi\alpha^3} \\ -\frac{1}{2}M(\cosh \xi - \alpha^3 \sinh \xi) + \frac{1}{2}V_R e^{-\xi\alpha^3} &= -\frac{1}{2}(M - V_R) e^{-\xi\alpha^3} \end{aligned} \quad (\text{C.10})$$

we have

$$\begin{aligned} e^{-\xi\alpha^3/2} \delta\Phi^{(\xi)}(\mathbf{x}) e^{\xi\alpha^3/2} = & \left[i\boldsymbol{\alpha} \cdot \vec{\nabla} + m\gamma^0 - \frac{1}{2}(M - V_R) e^{\xi\alpha^3} \right] e^{-\xi\alpha^3/2} \Phi^{(\xi)}(\mathbf{x}) e^{\xi\alpha^3/2} \\ & + e^{-\xi\alpha^3/2} \Phi^{(\xi)}(\mathbf{x}) e^{\xi\alpha^3/2} \left[i\boldsymbol{\alpha} \cdot \overleftarrow{\nabla} - m\gamma^0 - \frac{1}{2}(M - V_R) e^{-\xi\alpha^3} \right] \end{aligned} \quad (\text{C.11})$$

We may regard $\Phi^{(\xi)}(\mathbf{x})$ as a function of the Lorentz dilated rest frame coordinate \mathbf{x}_R (7.30),

$$\Phi^{(\xi)}(\mathbf{x}) \equiv \Phi_R^{(\xi)}(\mathbf{x}_R) \quad (\text{C.12})$$

and express ∇ in terms of $\nabla_R = \partial/\partial\mathbf{x}_R$,

$$\begin{aligned} i\alpha^3 \frac{\vec{\partial}}{\partial z} &= i\alpha^3 \cosh \xi \vec{\partial}_{zR} = i\alpha^3 e^{\xi\alpha^3} \vec{\partial}_{zR} - i \sinh \xi \vec{\partial}_{zR} \\ i\alpha^3 \frac{\overleftarrow{\partial}}{\partial z} &= i\alpha^3 \cosh \xi \overleftarrow{\partial}_{zR} = i\alpha^3 e^{-\xi\alpha^3} \overleftarrow{\partial}_{zR} + i \sinh \xi \overleftarrow{\partial}_{zR} \end{aligned} \quad (\text{C.13})$$

¹⁴ For conciseness we omit the bar over $\Phi^{(\xi)}$. However, it is distinct from the wave function of the state (5.2) where $A^3 = 0$.

The $\pm i \sinh \xi \partial_z$ contributions cancel in (C.11). For the x, y components $\nabla^\perp = \nabla_R^\perp$ and

$$i\alpha^\perp \cdot \nabla^\perp e^{-\xi\alpha^3/2} = e^{\xi\alpha^3/2} i\alpha^\perp \cdot \nabla^\perp \quad (\text{C.14})$$

Similarly $m\gamma^0 e^{-\xi\alpha^3/2} = e^{\xi\alpha^3/2} m\gamma^0$. Thus

$$\frac{2}{M - V_R} \delta\Phi^{(\xi)}(\mathbf{x}) = e^{\xi\alpha^3} \left[\frac{2}{M - V_R} (i\vec{\nabla}_R \cdot \boldsymbol{\alpha} + m\gamma^0) - 1 \right] \Phi_R^{(\xi)}(\mathbf{x}_R) + \Phi_R^{(\xi)}(\mathbf{x}_R) \left[(i\vec{\nabla}_R \cdot \boldsymbol{\alpha} - m\gamma^0) \frac{2}{M - V_R} - 1 \right] e^{-\xi\alpha^3} \quad (\text{C.15})$$

The factors in square brackets are the same as in the rest frame (5.13).

In $D = 1 + 1$ dimensions the P -dependence of the wave function (7.25) is simply given by Lorentz contraction. We make the corresponding ansatz in $D = 3 + 1$,

$$\Phi^{(\xi)}(\mathbf{x}) \equiv \Phi_R^{(\xi)}(\mathbf{x}_R) = \Phi^{(0)}(\mathbf{x}_R) \quad \mathbf{x}_R = (x, y, z \cosh \xi) \quad (\text{C.16})$$

The expression (C.15) for $\delta\Phi^{(\xi)}$ then equals the rest frame BSE (5.13), but with factors $\exp(\pm\xi\alpha^3)$ multiplying the quark and antiquark contributions. In $D = 1 + 1$ and $D = 2 + 1$ dimensions one may verify that those factors do not affect the BSE, *i.e.*, $\delta\Phi^{(\xi)} = 0$.

The non-relativistic Schrödinger wave functions of Positronium states in $D = 3 + 1$ are independent of the electron spin, and the quark and antiquark contributions to the rest frame BSE vanish separately [24],

$$\left[\frac{2}{M - V_R} (i\vec{\nabla}_R \cdot \boldsymbol{\alpha} + m\gamma^0) - 1 \right] \Phi^{(0)}(\mathbf{x}_R) = \Phi^{(0)}(\mathbf{x}_R) \left[(i\vec{\nabla}_R \cdot \boldsymbol{\alpha} - m\gamma^0) \frac{2}{M - V_R} - 1 \right] = 0 \quad (e^+ e^-) \quad (\text{C.17})$$

where $V_R(\mathbf{x}_R) = -\alpha/|\mathbf{x}_R|$. Hence the factors $e^{\pm\xi\alpha^3}$ in (C.15) are irrelevant and the Lorentz contracted Positronium bound states are stationary for any \mathbf{P} at $\mathcal{O}(\alpha^2)$ in the binding energy. The fact that atomic wave functions contract similarly as in classical relativity was first demonstrated in 2004 by Järvinen [35], using a Bethe-Salpeter approach.

The dynamics of strongly bound states in $D = 3 + 1$ is spin dependent, as seen from (5.28). The magnetic field (C.4) may then give rise to spin precession, *i.e.*, to time dependence in the wave function.

The coefficient of $\cosh \xi$ in (C.15) is the rest frame BSE (5.13), which $\Phi^{(0)}(\mathbf{x}_R)$ satisfies by definition. The two terms of this coefficient being therefore equal and opposite the coefficient of $\sinh \xi$ can be expressed as an anticommutator with α^3 ,

$$\frac{2}{M - V} \delta\Phi^{(\xi)}(\mathbf{x}) = \sinh \xi \left\{ \alpha^3, \left[\frac{2}{M - V} (i\boldsymbol{\alpha} \cdot \vec{\nabla} + m\gamma^0) - 1 \right] \Phi^{(0)}(\mathbf{x}) \right\} = \sinh \xi \left\{ \alpha^3, \vec{\Lambda}_- \Phi^{(0)}(\mathbf{x}) \right\} \quad (\text{C.18})$$

where $\vec{\Lambda}_-$ is defined in (5.15) and the subscript R is suppressed since all quantities in the anti-commutator refer to the rest frame. From the expressions for $\vec{\Lambda}_- \Phi(\mathbf{x})$ in (5.28), (5.34) and (5.41) it is readily seen that only the terms proportional to the quark spin \mathbf{S} contribute to the anticommutator. The result for states on each of the trajectories defined in (5.21) are:

$$\begin{aligned} \delta\Phi_{-+}^{(\xi)}(\mathbf{x}) &= \sinh \xi \frac{4V'}{r(M - V)^2} [L^3 - m(x^1\gamma^2 - x^2\gamma^1)] F_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) = \frac{2g}{(M - V)^2} \mathbf{B}_\xi \cdot (i\vec{\nabla} + m\boldsymbol{\gamma}) F_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) \\ \delta\Phi_{--}^{(\xi)}(\mathbf{x}) &= -\sinh \xi \frac{4mV'}{r(M - V)^2} \gamma_5 (x^1 L^2 - x^2 L^1) G_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) = \frac{2mg}{(M - V)^2} \mathbf{B}_\xi \cdot \mathbf{L} \gamma_5 G_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) \\ \delta\Phi_{++}^{(\xi)}(\mathbf{x}) &= -\sinh \xi \frac{2V'}{r(M - V)^2} \gamma_5 L^3 H_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) = -\frac{g}{(M - V)^2} \mathbf{B}_\xi \cdot i\vec{\nabla} \gamma_5 H_1(r) Y_{j\lambda}(\hat{\mathbf{x}}) \end{aligned} \quad (\text{C.19})$$

We interpret the time dependence as a spin precession. It is in all cases proportional to the magnetic field \mathbf{B}_ξ of (C.4), and vanishes when $m = 0$ and/or $\lambda = 0$, depending on the trajectory. Furthermore, $\delta\Phi^{(\xi)}(\mathbf{x}^\perp = 0, x^3) = 0$ for all states.

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- [1] Y. L. Dokshitzer, In *Vancouver 1998, High energy physics, vol. 1* 305-324 [[hep-ph/9812252](#)];
Y. L. Dokshitzer, *QCD phenomenology*, [hep-ph/0306287](#).
 - [2] Yuri Dokshitzer, *Hadrons from quarks, gluons and first principles*, PoS CERP **2010** (2010) 001.
 - [3] W. E. Caswell and G. P. Lepage, Phys. Rev. A **18** (1978) 810. doi:10.1103/PhysRevA.18.810;
G. P. Lepage, “Two-body Bound States In Quantum Electrodynamics” (PhD thesis), SLAC-R-0212 (1978).
 - [4] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84** (1951) 1232.
 - [5] G. S. Adkins, Hyperfine Interact. **233** (2015) no.1-3, 59. doi:10.1007/s10751-015-1137-9
 - [6] W. E. Caswell and G. P. Lepage, Phys. Lett. **167B** (1986) 437. doi:10.1016/0370-2693(86)91297-9
 - [7] T. Kinoshita and M. Nio, Phys. Rev. D **53** (1996) 4909 doi:10.1103/PhysRevD.53.4909 [[hep-ph/9512327](#)].
 - [8] P. Labelle, Phys. Rev. D **58** (1998) 093013 doi:10.1103/PhysRevD.58.093013 [[hep-ph/9608491](#)].
 - [9] S. Weinberg, “The Quantum theory of fields. Vol. 1: Foundations,” Cambridge University Press, 1995.
 - [10] C. Itzykson and J. B. Zuber, “Quantum Field Theory”, New York, USA: McGraw-hill (1980)
 - [11] J. Iliopoulos, C. Itzykson and A. Martin, Rev. Mod. Phys. **47** 165 (1975).
 - [12] C. Patrignani *et al.* [Particle Data Group], Chin. Phys. C **40** (2016) no.10, 100001. doi:10.1088/1674-1137/40/10/100001
 - [13] V. N. Gribov, Eur. Phys. J. C **10** (1999) 91 doi:10.1007/s100529900052 [[hep-ph/9902279](#)].
 - [14] T. Gehrmann, M. Jaquier, G. Luisoni, Eur. Phys. J. **C67** (2010) 57-72. [[arXiv:0911.2422](#) [hep-ph]].
 - [15] E. Eichten, S. Godfrey, H. Mahlke and J. L. Rosner, Rev. Mod. Phys. **80** (2008) 1161 doi:10.1103/RevModPhys.80.1161 [[hep-ph/0701208](#)].
 - [16] G. S. Bali, Phys. Rept. **343** (2001) 1 doi:10.1016/S0370-1573(00)00079-X [[hep-ph/0001312](#)].
 - [17] S. Aoki *et al.* [CP-PACS Collaboration], Phys. Rev. Lett. **84** (2000) 238 doi:10.1103/PhysRevLett.84.238 [[hep-lat/9904012](#)].
 - [18] M. S. Plesset, Phys. Rev. **41** (1932) 278.
 - [19] P. Hoyer, Phys. Lett. B **172** (1986) 101. doi:10.1016/0370-2693(86)90224-8
 - [20] D. D. Dietrich, P. Hoyer and M. Järvinen, Phys. Rev. D **85** (2012) 105016
doi:10.1103/PhysRevD.85.105016 [[arXiv:1202.0826](#) [hep-ph]].
 - [21] D. D. Dietrich, P. Hoyer and M. Järvinen, Phys. Rev. D **87** (2013) 065021
doi:10.1103/PhysRevD.87.065021 [[arXiv:1212.4747](#) [hep-ph]].
 - [22] P. Hoyer, “Bound states – from QED to QCD,” [arXiv:1402.5005](#) [hep-ph].
 - [23] P. Hoyer, “Lectures on Bound states”, [arXiv:1605.01532](#) [hep-ph].
 - [24] P. Hoyer, PoS (Hadron217) 151 [[arXiv:1711.10851](#) [hep-ph]].
 - [25] H. X. Chen, W. Chen, X. Liu and S. L. Zhu, Phys. Rept. **639** (2016) 1 doi:10.1016/j.physrep.2016.05.004 [[arXiv:1601.02092](#) [hep-ph]].
 - [26] R. F. Lebed, R. E. Mitchell and E. S. Swanson, Prog. Part. Nucl. Phys. **93** (2017) 143 doi:10.1016/j.ppnp.2016.11.003 [[arXiv:1610.04528](#) [hep-ph]].
 - [27] S. Okubo, Phys. Lett. **5** (1963) 165;
G. Zweig, CERN-TH-412 (1964);
J. Iizuka, Prog. Theor. Phys. Suppl. **37** (1966) 21.
 - [28] V. P. Nomokonov and M. G. Sapozhnikov, Phys. Part. Nucl. **34** (2003) 94 [Fiz. Elem. Chast. Atom. Yadra **34** (2003) 189] [[hep-ph/0204259](#)].
 - [29] O. Klein, Z. Phys. **53** (1929) 157. doi:10.1007/BF01339716
 - [30] A. M. Cooper-Sarkar, doi:10.3204/DESY-PROC-2009-01/110-2 [arXiv:0901.4001](#) [hep-ph];
A. M. Cooper-Sarkar and K. Wichmann, [arXiv:1803.00968](#) [hep-ex].
 - [31] V. B. Berestetskii, E. M. Lifshitz and L. P. Pitaevskii, “Quantum Electrodynamics”, (1982). ISBN: 978-0-7506-3371-0.
 - [32] P. P. Kulish and L. D. Faddeev, Theor. Math. Phys. **4** (1970) 745 [Teor. Mat. Fiz. **4** (1970) 153]. doi:10.1007/BF01066485
 - [33] D. A. Geffen and H. Suura, Phys. Rev. D **16** (1977) 3305.
 - [34] S. J. Brodsky and J. R. Primack, Annals Phys. **52** (1969) 315. doi:10.1016/0003-4916(69)90264-4
 - [35] M. Järvinen, Phys. Rev. D **71** (2005) 085006 [[arXiv:hep-ph/0411208](#)].
 - [36] X. Artru, Phys. Rev. D **29** (1984) 1279. doi:10.1103/PhysRevD.29.1279
 - [37] I. Bars and M. B. Green, Phys. Rev. D **17** (1978) 537. doi:10.1103/PhysRevD.17.537
 - [38] Y. Jia, S. Liang, L. Li and X. Xiong, JHEP **1711** (2017) 151 doi:10.1007/JHEP11(2017)151 [[arXiv:1708.09379](#) [hep-ph]].