On the maximum number of edges in k-critical graphs

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Abstract

A graph is called k-critical if its chromatic number is k but every proper subgraph has chromatic number less than k. An old and important problem in graph theory asks to determine the maximum number of edges in an n-vertex k-critical graph. This is widely open for every integer $k \geq 4$. Using a structural characterization of Greenwell and Lovász and an extremal result of Simonovits, Stiebitz proved in 1987 that for $k \geq 4$ and sufficiently large n, this maximum number is less than the number of edges in the n-vertex balanced complete $(k-2)$-partite graph. In this paper we obtain the first improvement on the above result in the past 35 years. Our proofs combine arguments from extremal graph theory as well as some structural analysis. A key lemma we use indicates a partial structure in dense k-critical graphs, which may be of independent interest.

1 Introduction

All graphs we consider are finite and simple. A graph G is *k-colorable* if we can assign k colors to its vertices such that no adjacent vertices receive the same color. We say a graph G is *k-chromatic* if it is k-colorable but not $(k-1)$-colorable. A graph G is called *k-critical* if G is k-chromatic but each of its proper subgraphs is $(k-1)$-colorable. For $k \in \{1, 2\}$ the only k-critical graph is K_k, and the family of 3-critical graphs is precisely the family of odd cycles. In this paper, we consider k-critical graphs for $k \geq 4$.

A central problem in graph theory asks to determine the maximum number of edges $f_k(n)$ in an n-vertex k-critical graph (see [6]). Before we discuss the literature on $f_k(n)$, we point out a relevant yet easy fact that the *Turán graph* $T_k(n)$ (that is, the n-vertex balanced complete

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k-partite graph) has the maximum number of edges among all n-vertex k-chromatic graphs. Dirac [2] gave $f_6(n) \geq \frac{1}{4}n^2+n$ by considering the graphs obtained by joining two vertex-disjoint odd cycles with the same number of vertices. Toft [12] proved that for every $k \geq 4$, there exists a positive constant c_k such that $f_k(n) \geq c_k n^2$ holds for all integers $n \geq k$ (except $n = k+1$). In the most basic and interesting cases $k = 4, 5$, the constants are given by

$$c_4 \geq \frac{1}{16} = 0.0625 \quad \text{and} \quad c_5 \geq \frac{4}{31} \geq 0.129.$$

In the general case when $k \geq 6$, explicit constructions in [12] show that there exist infinitely many values of n such that

$$f_k(n) \geq \left(\frac{1}{2} - \frac{3}{2k - \delta_k}\right) n^2,$$

where $\delta_k = 0$ if $k \equiv 0 \pmod 3$, $\delta_k = 8/7$ if $k \equiv 1 \pmod 3$, and $\delta_k = 44/23$ if $k \equiv 2 \pmod 3$. To our best knowledge, no construction for giving better constants $f_k(n)/n^2$ have been found since. It is also an open question if $\lim_{n\to\infty} \frac{f_k(n)}{n^2}$ exists for each $k \geq 4$. In 2013, Pegden [8] considered dense triangle-free k-critical graphs. He constructed infinitely many n-vertex triangle-free 4-critical graphs with at least $\left(\frac{1}{16} - o(1)\right) n^2$ edges, triangle-free 5-critical graphs with at least $\left(\frac{4}{31} - o(1)\right) n^2$ edges, and triangle-free k-critical graphs with at least $\left(\frac{1}{4} - o(1)\right) n^2$ edges for every $k \geq 6$. The last bound is asymptotically best possible by Turán's theorem. He also showed the existence of dense k-critical graphs without any odd cycle of length at most ℓ for any ℓ, which is again asymptotically tight for $k \geq 6$.

Turning to the upper bound of $f_k(n)$, since any n-vertex k-critical graph with $n > k$ does not contain K_k as a subgraph, by Turán's theorem one can easily obtain that $f_k(n) < e(T_{k-1}(n))$ for any $n > k \geq 4$. Using a characterization of Greenwell and Lovász [5] for subgraphs of k-critical graphs and a classical theorem of Simonovits [10], Stiebitz [11] improved this trivial bound in 1987 by showing that

$$f_k(n) < e(T_{k-2}(n)) \text{ for sufficiently large integer } n. \tag{1}$$

It has been 35 years since then and as far as we are aware, this remains the best upper bound.

There is a natural relation between $f_k(n)$ and the problem of determining the maximum number of copies of K_{k-1} in k-critical graphs. Abbott and Zhou [1] generalized an earlier result of Stiebitz [11] on 4-critical graphs and showed that for each $k \geq 4$ every k-critical graph on n vertices contains at most n copies of K_{k-1}. The bound was further improved in [7]. Recently, Gao and Ma [4] proved a sharp result that for each $n > k \geq 4$, every k-critical graph on n vertices contains at most $n - k + 3$ copies of K_{k-1}. If we delete one edge for every K_{k-1} in a k-critical graph on n vertices, then this can result in a graph without containing K_{k-1}. Using Turán's theorem and the above result of [4], we can derive that

$$f_k(n) \leq e(T_{k-2}(n)) + n - k + 3 \text{ for any } n > k \geq 4.$$

In this paper, we focus on the upper bound of $f_k(n)$. Our first result improves the long-standing upper bound (1) of Stiebitz [11].

Theorem 1.1. *For every integer $k \geq 4$ there exist constants n_k and $c_k \geq \frac{1}{36(k-1)^2}$ such that if $n \geq n_k$ then $f_k(n) \leq e(T_{k-2}(n)) - c_k n^2$.*

Our second result considers 4-critical graphs. A better upper bound for $f_4(n)$ than Theorem 1.1 is obtained in the following.

Theorem 1.2. *There exists a constant n_4 such that if $n \geq n_4$ then $f_4(n) < 0.164n^2$.*

The proofs of both theorems rely on arguments from extremal graph theory (such as the stability lemma of Füredi [3]) and a structural lemma (Lemma 2.1) given in the coming section. Lemma 2.1 indicates a partial structure in dense critical graphs (under certain constraints), which can be witnessed in many classical constructions of dense critical graphs (see the discussion at the beginning of Section 2). For that, we would like to give a full construction for the well-known *Toft graph* (see Figure 1 or [12]). The vertex set of the Toft graph is formed by 4 disjoint sets A, B, C, D with the same odd size, where A and D are odd cycles, B and C are independent sets, the edges between B and C form a complete bipartite graph, and both of the edges in (A, B) and in (C, D) form perfect matchings. It is easy to check that the n-vertex Toft graph is 4-critical and has $\frac{1}{16}n^2 + n$ edges. We remark that the Toft graph remains the best construction for dense 4-critical graphs.

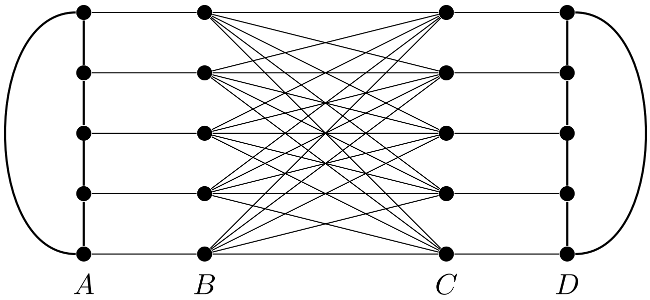

Figure 1: Toft graph: $|A| = |B| = |C| = |D| = 5$

We use standard notation in graph theory. Let $\overline{G}$ denote the complement of the graph G. For a vertex v in a graph G, let $N_G(v)$ denote the neighborhood of v in G, and let $d_G(v) := |N_G(v)|$ denote the degree of v in G. When G is clear from the context, we often drop the subscript. Let $d(G)$ denote the average degree of the graph G. Also, for any $S \subseteq V(G)$, let $G[S]$ denote the induced subgraph of G on the vertex set S. For any disjoint sets $A, B \subseteq V(G)$, let $G[A, B]$ denote the induced bipartite subgraph of G with bipartition (A, B).

The rest of the paper is organized as follows. In Section 2, we prove a lemma which is key for the coming proofs. Then we prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4.

2 Key lemma

In this section we prove our key lemma, which roughly says that if a k-critical graph G contains certain t copies of K_{k-2} sharing $k-3$ common vertices, then there exists an "induced" matching of size t in G which are connected to these cliques (see Figure 2). This indicates a substructure similar to the Toft graph (and many other examples of k-critical graphs). In particular, it reveals that the structure of k-critical graphs cannot be close to the Turán graph $T_{k-2}(n)$ and thus the inequality (1) should not be tight.

Lemma 2.1. *Let $k \geq 4$ and let G be a k-critical graph. Suppose that $G[\{x_1, x_2, \ldots, x_{k-3}\}]$ forms a copy of K_{k-3} and there exists a set $W \subseteq N(x_1) \cap \cdots \cap N(x_{k-3}) \cap N(u)$ for some vertex $u \notin \{x_1, x_2, \ldots, x_{k-3}\}$. Then there exist a set W' and a bijection $\varphi : W \to W'$ such that $N(\varphi(w)) \cap W = \{w\}$ and $N(w) \cap W' = \{\varphi(w)\}$ hold for each $w \in W$. Moreover, if $|W| \geq 3$, then W is an independent set in G, and $W' \cap W = \emptyset$.*

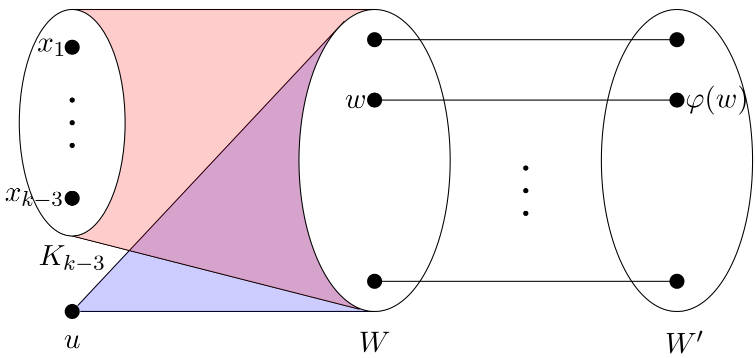

Figure 2: The red shading indicates $W \subseteq N(x_1)\cap\cdots\cap N(x_{k-3})$, and the blue shading indicates $W \subseteq N(u)$. Note that W' and W may intersect when $|W| \leq 2$.

Proof. For each vertex $w \in W$, by deleting the edge uw from the k-critical graph G, we can get a $(k-1)$-chromatic graph G'. We denote the color classes of G' by $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{k-1}$. It is easy to see the vertices u and w are in the same color class. Since $G[\{x_1, x_2, \ldots, x_{k-3}, w\}]$ is a $(k-2)$-clique, we can assume $x_1 \in \mathcal{C}_1$, $x_2 \in \mathcal{C}_2$,..., $x_{k-3} \in \mathcal{C}_{k-3}$, and $u, w \in \mathcal{C}_{k-2}$. The fact $W \subseteq N(x_1)\cap\cdots\cap N(x_{k-3})\cap N(u)$ tells us that the set $W\backslash\{w\}$ (if not empty) must be contained in $\mathcal{C}_{k-1}$, and thus $W\backslash\{w\}$ is an independent set in G. We claim $N(w) \cap \mathcal{C}_{k-1}$ must contain a vertex, say $\varphi(w)$. Since otherwise $\mathcal{C}_1, \ldots, \mathcal{C}_{k-3}, \mathcal{C}_{k-2} - \{w\}, \mathcal{C}_{k-1} \cup \{w\}$ can be a $(k-1)$-coloring of G, which contradicts the fact that G is k-critical. Besides, $\{\varphi(w)\} \cup (W\backslash\{w\}) \subseteq \mathcal{C}_{k-1}$ tells us that $N(\varphi(w)) \cap W = \{w\}$. Now we define $W' := \{\varphi(w) : w \in W\}$. As we have shown that $N(\varphi(w)) \cap W = \{w\}$ holds for each $w \in W$, it is easy to see $|W'| = |W|$, $\varphi : W \to W'$ is a bijection, and $N(w) \cap W' = \{\varphi(w)\}$ holds for each $w \in W$.

Moreover, if $|W| \geq 3$, then W is an independent set in G (since $W\backslash\{v\}$ is an independent set in G for each vertex $v \in W$). By the fact that the edges between W' and W precisely form a matching, we can see $W' \cap W = \emptyset$ in this case. $\square$

It would be very interesting to see if this lemma (or its proof) can be extended further.

3 The general case: k-critical

Providing a simple and new proof of the stability for the Turán number $\mathrm{ex}(n, K_{r+1})$, Füredi [3] showed that if an n-vertex graph G is K_{r+1}-free and has at least $e(T_r(n)) - t$ edges where $0 \leq t < e(T_r(n)) < n^2$, then there exists a partition $V_1, \ldots, V_r$ of $V(G)$ such that $\sum_{i=1}^{r} e(G[V_i]) \leq t$. The statement following Corollary 3 in [3] also suggests that the partition $V_1, \ldots, V_r$ is approximately balanced. We summarize this observation in the following lemma.

Lemma 3.1 (Füredi [3]). *Suppose that G is an n-vertex K_{r+1}-free graph with $e(G) \geq e(T_r(n)) - t$ where $0 \leq t < e(T_r(n)) < n^2$. Then there exists a complete r-chromatic graph $K := K(V_1, \ldots, V_r)$ with $V(K) = V(G)$ such that*

$$|E(K)\backslash E(G)| \leq 2t,$$

and

$$\sum_{i=1}^{r} \left(|V_i| - \frac{n}{r}\right)^2 < 4t + o(n^2).$$

We are ready to use Lemmas 2.1 and 3.1 to prove Theorem 1.1.

Proof of Theorem 1.1. Fix $k \geq 4$ and let $C = \frac{1}{36(k-1)^2}$. Let G be a k-critical graph on n vertices with $e(G) > e(T_{k-2}(n)) - Cn^2$. In the rest of the proof, we will always assume that n is large enough, and we denote $V(G)$ by V for convenience. The result in [1] tells us the number of copies of K_{k-1} in G is at most n. So by deleting at most n edges in G, we obtain a spanning subgraph G' which is K_{k-1}-free. Obviously we have $e(G') \geq e(G) - n > e(T_{k-2}(n)) - (Cn^2 + n)$.

With the application of Lemma 3.1, we get a partition $V_0, V_1, \ldots, V_{k-3}$ of V and a complete $(k-2)$-chromatic graph $K := K(V_0, \ldots, V_{k-3})$ such that $|E(K)\backslash E(G')| \leq 2(Cn^2 + n)$ and

$$\left||V_i| - \frac{n}{k-2}\right| < \sqrt{4Cn^2 + o(n^2)} < \frac{n}{3(k-1)} + o(n) \text{ for each } 0 \leq i \leq k-3.$$

Without loss of generality, we assume $|V_0| \geq \cdots \geq |V_{k-3}|$. Thus $|V_0| \geq n/(k-2)$, and $|V_i| \geq \frac{n}{k-2} - \frac{n}{3(k-1)} - o(n)$ for each $0 \leq i \leq k-3$. We call the edges in $E(K)\backslash E(G')$ *missing edges*. And the number of missing edges incident to the vertex v in K is called the *missing degree* of v. For each $0 \leq i \leq k-3$, we define B_i to be the set of $\left\lceil \frac{n}{3(k-1)} \right\rceil$ vertices in V_i satisfying that there exists some m_i such that the missing degree of any vertex in B_i is at least m_i, and the missing degree of any vertex in $U_i := V_i - B_i$ is at most m_i. Note that m_i is unique, while B_i may be not. Since there are at most $2(Cn^2+n)$ missing edges in total, we have $\sum_{i=0}^{k-3} m_i|B_i| < 4(Cn^2+n)$, and thus we can get

$$\sum_{i=0}^{k-3} m_i < 4(Cn^2+n) \Big/ \left\lceil \frac{n}{3(k-1)} \right\rceil \leq \frac{n}{3(k-1)} + 12(k-1).$$

And we can check that for each $0 \leq i \leq k-3$, we have

$$|U_i| = |V_i| - |B_i| > \frac{n}{k-2} - \frac{n}{3(k-1)} - \frac{n}{3(k-1)} - o(n) > \frac{n}{3(k-2)} > \sum_{i=0}^{k-3} m_i + 2. \quad (2)$$

Fix an arbitrary vertex $x_0 \in U_0$ and let $Y := N_{G'}(x_0) \backslash V_0$. It is clear that

$$|Y| \geq n - |V_0| - m_0.$$

Next, we want to find a copy of K_{k-3} in G' on vertices $x_1, x_2, \ldots, x_{k-3}$ with $x_i \in U_i \cap Y = U_i \cap N_{G'}(x_0)$ by greedily choosing the vertex $x_i \in U_i \cap N_{G'}(x_0) \cap \cdots \cap N_{G'}(x_{i-1})$ for $1 \leq i \leq k-3$ one by one. By the definition of U_j, we know each vertex $x \in U_j$ has at most m_j missing degree, which means $|S \backslash N_{G'}(x)| \leq m_i$ for any $S \subseteq V \backslash V_j$. Then in the i'th iteration, we have $|U_i \cap N_{G'}(x_0) \cap \cdots \cap N_{G'}(x_{i-1})| \geq |U_i| - \sum_{j=0}^{i-1} |U_i \backslash N_{G'}(x_j)| \geq |U_i| - \sum_{j=0}^{i-1} m_j > 2$ choices of x_i, where the last inequality comes from (2). Thus this algorithm will give us a copy of K_{k-3} as desired.

Then, since $|U_i| - m_{k-2} \geq |U_i| - \sum_{i=0}^{k-3} m_j > 2$ holds for each $1 \leq i \leq k-3$ by (2), we can find a vertex $u \in U_{i_0} \cap Y$ distinct from $x_1, x_2, \ldots, x_{k-3}$, where we choose i_0 such that $m_{i_0} = \min\{m_1, \ldots, m_{k-3}\}$. Let $W := N_{G'}(x_1) \cap \cdots \cap N_{G'}(x_{k-3}) \cap N_{G'}(u) \cap V_{k-2}$. We can see $W \ni x_0$, $W \cap Y = \emptyset$, and

$$|W| \geq |V_{k-2}| - \sum_{i=1}^{k-3} m_j - m_{i_0} \geq |V_{k-2}| - \left(1 + \frac{1}{k-3}\right) \sum_{i=1}^{k-3} m_j.$$

Then by using Lemma 2.1, we get a set W' with $|W'| = |W|$ such that $|N_G(w) \cap W'| = 1$ for each $w \in W'$, and $|W' \cap W| \leq 2$. Note that all vertices in Y are adjacent to the vertex $x_0 \in W$ in $G' \subseteq G$, so we can see $|W' \cap Y| \leq 1$.

As $W \cap Y = \emptyset$, $|W' \cap W| \leq 2$, $|W' \cap Y| \leq 1$ and $|W'| = |W|$, we get $n \geq |W \cup Y \cup W'| \geq 2|W| + |Y| - 3$. Thus

$$2|W| + |Y| \leq n + 3.$$

But on the other hand, we can check that

$$\begin{aligned} 2|W| + |Y| &\geq 2\left(|V_0| - \left(1 + \frac{1}{k-3}\right)\sum_{j=1}^{k-3} m_j\right) + (n - |V_0| - m_0) \\ &\geq n + |V_0| - 2\left(1 + \frac{1}{k-3}\right)\sum_{j=0}^{k-3} m_j \\ &\geq n + \frac{n}{k-2} - 2\left(1 + \frac{1}{k-3}\right)\left(\frac{n}{3(k-1)} + 12(k-1)\right) > n + 3. \end{aligned}$$

This derives a contradiction. So we have $f_k(n) \leq e(T_{k-2}(n)) - Cn^2$ for n sufficiently large. $\square$

We would like to remark that the above proof relies on the existence of K_{k-2}. (Recall that in Lemma 2.1, $G[\{w, x_1, x_2, \ldots, x_{k-3}\}]$ forms a copy of K_{k-2} for each vertex $w \in W$.) So using this approach, we will not be able to improve the upper bound to the following

$$e(G) \leq \mathrm{ex}(n, K_{k-2}) = e(T_{k-3}(n)) \leq e(T_{k-2}(n)) - \frac{n^2}{2(k-2)(k-3)};$$

that says, we are not able to obtain a constant c_k better than the order of magnitude k^{-2}.

4 The 4-critical case

In this section we consider 4-critical graphs and prove Theorem 1.2.

Before presenting the proof of Theorem 1.2, we give a short proof of a slightly weaker bound (see Theorem 4.1) than Theorem 1.2 to illustrate the proof ideas. In doing this, we study certain local structure based on 2-paths (i.e., a path of length two) in the proof of Theorem 4.1, while we consider 4-cycles (i.e., a cycle of length four) in place of 2-paths in the proof of Theorem 1.2.

4.1 A weaker upper bound

We first show the following result.

Theorem 4.1. *For any integer $n \geq 4$, it holds that $f_4(n) < \frac{1}{6}n^2 + 10n \leq 0.167n^2 + 10n$.*

We also need two lemmas as follows. For a graph G, we let $t(G)$ be the number of triangles in G. Note that Stiebitz [11] found out that

$$t(G) \leq n \text{ holds for every 4-critical graph } G \text{ on } n \text{ vertices.} \tag{3}$$

For a vertex v, let $t_G(v)$ be the number of triangles containing the vertex v in G. When G is clear, we often drop the subscript.

Lemma 4.2. *Suppose G has at most n triangles and minimum degree at least 3. Then G contains a 2-path xyz such that*

$$d(x) + d(y) + d(z) - 3t(x) - 3t(z) \geq \frac{6e(G)}{n} - \frac{9n^2}{e(G)}.$$

Proof. For some vertex $v \in V(G)$, write $N(v) = \{v_1, v_2, \ldots, v_s\}$ for some $s \geq 3$. Let

$$\mathcal{P}_v := \{v_1vv_2, \ldots, v_{s-1}vv_s, v_svv_1\}$$

be a family of 2-paths with center v. We have $|\mathcal{P}_v| = d(v)$, and

$$\sum_{xyz \in \mathcal{P}_v} (d(x) + d(y) + d(z)) = d(v)^2 + 2\sum_{u \in N(v)} d(u),$$

$$\sum_{xyz\in\mathcal{P}_v}(t(x)+t(z)) = 2\sum_{u\in N(v)} t(u).$$

Then let $\mathcal{P} := \bigcup_{v\in V(G)}\mathcal{P}_v$. We have

$$|\mathcal{P}| = \sum_{v\in V(G)} d(v) = 2e(G).$$

Using Jensen's inequality, we get

$$\begin{aligned}\sum_{xyz\in\mathcal{P}}(d(x)+d(y)+d(z)) &= \sum_{v\in V(G)} d(v)^2 + 2\sum_{v\in V(G),u\in N(v)} d(u) = \sum_{v\in V(G)} d(v)^2 + 2\sum_{u\in V(G),v\in N(u)} d(u)\\ &= \sum_{v\in V(G)} d(v)^2 + 2\sum_{u\in V(G)} d(u)^2 = 3\sum_{v\in V(G)} d(v)^2 \geq 12e(G)^2/n.\end{aligned}$$

Since every vertex in G has degree at most $n-1$ and $\sum_{u\in V(G)} t(u) = 3t(G) \leq 3n$, we get

$$\sum_{xyz\in\mathcal{P}}(t(x)+t(z)) = 2\sum_{v\in V(G)}\sum_{u\in N(v)} t(u) = 2\sum_{u\in V(G)} d(u)t(u) \leq 2n\sum_{u\in V(G)} t(u) \leq 6n^2.$$

So by picking a 2-path xyz in $\mathcal{P}$ uniformly and randomly, we see

$$\mathbb{E}[d(x)+d(y)+d(z)-3t(x)-3t(z)] \geq \frac{12e(G)^2/n - 18n^2}{|\mathcal{P}|} = \frac{6e(G)}{n} - \frac{9n^2}{e(G)}.$$

Thus we can find a 2-path xyz as desired. $\square$

Lemma 4.3. *For any 2-path xyz in a 4-critical graph G, we have*

$$d(x)+d(y)+d(z)-3t(x)-3t(z) \leq n+1.$$

Proof. Let $X := N(x)$, $Y := N(y)$, $Z := N(z)$, and $W := X\cap Z$. If $u \in X\cap Y$, uxy is a triangle. So $|X\cap Y| \leq t(x)$. Similarly, $|Z\cap Y| \leq t(z)$. Then we have

$$|X\cup Y\cup Z| \geq |X|+|Y|+|Z|-|X\cap Y|-|Z\cap Y|-|X\cap Z| \geq d(x)+d(y)+d(z)-t(x)-t(z)-|W|.$$

By Lemma 2.1, we can find a set $W' \subseteq V(G)$ and a bijection $\varphi : W \to W'$ such that $W' = \{\varphi(w) : w \in W'\}$, and for each $w \in W$, we have both $N(\varphi(w))\cap W = \{w\}$ and $N(w)\cap W' = \{\varphi(w)\}$.

We consider the size of $W'\cap(X\cup Y\cup Z)$. Since both $N(\varphi(w))\cap W = \{w\}$ and $N(w)\cap W' = \{\varphi(w)\}$ hold for each $w \in W$, and we know $y \in W$, we can see $|W'\cap Y| \leq |W'\cap N(y)| \leq 1$. Suppose $v' \in W'\cap X$. There is a vertex $v \in W$ such that vv' is an edge. Then we see xvv' is a triangle. So $|W'\cap X| \leq 2t(x)$. Similarly, $|W'\cap Z| \leq 2t(z)$. Totally, we have

$$|W'\cap(X\cup Y\cup Z)| \leq |W'\cap X| + |W'\cap Y| + |W'\cap Z| \leq 2t(x)+2t(z)+1.$$

Finally, we get

$$\begin{aligned}n &\geq |X \cup Y \cup Z \cup W'| = |X \cup Y \cup Z| + |W'| - |W' \cap (X \cup Y \cup Z)| \\ &\geq (d(x) + d(y) + d(z) - t(x) - t(z) - |W|) + |W| - (2t(x) + 2t(z) + 1) \\ &= d(x) + d(y) + d(z) - 3t(x) - 3t(z) - 1,\end{aligned}$$

completing the proof of this lemma. □

Now we can finish the proof of this subsection.

Proof of Theorem 4.1. Let G be an n-vertex 4-critical graph. It is easy to see that the minimum degree of G is at least 3. By (3), G contains at most n copies of triangles, so we can apply Lemma 4.2 to G and get a 2-path xyz with

$$d(x) + d(y) + d(z) - 3t(x) - 3t(z) \geq \frac{6e(G)}{n} - \frac{9n^2}{e(G)}.$$

Together with Lemma 4.3, we have

$$\frac{6e(G)}{n} - \frac{9n^2}{e(G)} \leq n + 1.$$

This implies that $e(G) < n^2/6 + 10n$. □

4.2 The proof of Theorem 1.2

To show Theorem 1.2, we need some new lemmas. The coming lemma can be easily obtained by averaging, which says that every graph contains an edge such that the sum of the degrees of its two endpoints is at least twice the average degree of the graph.

Lemma 4.4. *Any graph G contains an edge xy such that*

$$d(x) + d(y) \geq 2d(G).$$

Proof. By Jensen's inequality, we can get

$$\sum_{xy \in E} (d(x) + d(y)) = \sum_{v \in V} d(v)^2 \geq nd(G)^2.$$

Note that $|E| = (nd(G))/2$. Thus there exists an edge $xy \in E$ such that

$$d(x) + d(y) \geq \frac{nd(G)^2}{(nd(G))/2} = 2d(G),$$

proving the lemma. □

We now give the following lemma about 4-cycles, which can be viewed as a generalization of the previous lemma. Recall the well-known result of Reiman [9] that any n-vertex graph without containing 4-cycles has at most $\frac{n}{4}(1 + \sqrt{4n-3}) < n^{\frac{3}{2}}$ edges.

Lemma 4.5. *Any n-vertex graph G with $e(G) > \frac{n}{4}(1+\sqrt{4n-3})$ contains a 4-cycle $v_1v_2v_3v_4$ such that*

$$d(v_1)+d(v_2)+d(v_3)+d(v_4) \geq 4d(G) - O(n^{\frac{3}{4}}).$$

Proof. Fix $\epsilon := 9n^{-\frac{1}{4}}$. Note that G must contain 4-cycles by the result of Reiman [9]. Suppose to the contrary that any 4-cycle $v_1v_2v_3v_4$ in G satisfies $d(v_1)+d(v_2)+d(v_3)+d(v_4) < 4d(G)-4\epsilon n$. Let $A := \{v \in V : d(v) < d(G)\}$ and $B := \{v \in V : d(v) \geq d(G)\}$. Then $A \cup B$ forms a partition of $V(G)$ such that $G[B]$ does not contain any 4-cycle.

For each $1 \leq i \leq d(G)/\epsilon n$, let $A_i := \{v \in V : d(G) - i\epsilon n \leq d(v) < d(G) - (i-1)\epsilon n\}$. Then these A_i's form a partition of A. For each $1 \leq i \leq (n-d(G))/\epsilon n$, let $B_i := \{v \in V : d(G) + (i-1)\epsilon n \leq d(v) < d(G) + i\epsilon n\}$. Then these B_i's form a partition of B. It is not hard to check that $G[A_1]$ does not contain any 4-cycle, and for each $1 \leq i \leq (n-d(G))/\epsilon n$, $G\left[\bigsqcup_{j=1}^{i+1} A_j, B_i\right]$ does not contain any 4-cycle.

We delete all edges in $G[B]$, $G[A_1]$ and $G\left[\bigsqcup_{j=1}^{i+1} A_j, B_i\right]$ for each $1 \leq i \leq (n-d(G))/\epsilon n$ to get a spanning subgraph G' of G. By the result of Reiman [9], we can obtain

$$e(G') \geq e(G) - (2 + (n-d(G))/\epsilon n)\, n^{\frac{3}{2}} \geq e(G) - 2n^{\frac{3}{2}} - \frac{1}{9}n^{\frac{7}{4}} \geq e(G) - \frac{19}{9}n^{\frac{7}{4}}.$$

Thus we have

$$d(G') \geq d(G) - \frac{38}{9}n^{\frac{3}{4}}.$$

Note that any edge of G' is either contained in A, or between A_j and B_i for some $j \geq i+2$; moreover, $e(G'[A_1]) = 0$. Let xy be an edge in G'. If $x, y \in A$, as $e(G'[A_1]) = 0$, we can assume that $y \in A \backslash A_1$, which tells us $d_{G'}(y) \leq d(G) - \epsilon n$, so we have $d_{G'}(x) + d_{G'}(y) \leq d(G) + d(G) - \epsilon n = 2d(G) - \epsilon n$. If $x \in A_j, y \in B_i$ for some $j \geq i+2$, then we have $d_{G'}(x) + d_{G'}(y) \leq d(G) - (j-1)\epsilon n + d(G) + i\epsilon n \leq 2d(G) - \epsilon n$. Thus, as n is large enough, for any edge xy in G',

$$d_{G'}(x) + d_{G'}(y) < 2d(G) - \epsilon n = 2d(G) - 9n^{\frac{3}{4}} < 2d(G').$$

This contradicts Lemma 4.4, thus proving Lemma 4.5. □

The following lemma is derived from Lemma 2.1, which provides an essential structure to the proof of Theorem 1.2. To enhance comprehension of the lemma, referencing Figure 3 can be particularly helpful in gaining a better understanding of the concepts involved.

Lemma 4.6. *Let G be a 4-critical graph. Suppose $v_1v_2v_3v_4$ is a 4-cycle in G, and V_1, V_2, V_3, V_4 are four sets such that $\{v_2, v_4\} \subseteq V_1 \subseteq N(v_1)$, $\{v_1, v_3\} \subseteq V_2 \subseteq N(v_2)$, $\{v_2, v_4\} \subseteq V_3 \subseteq N(v_3)$, and $\{v_1, v_3\} \subseteq V_4 \subseteq N(v_4)$. Let $X = V_1 \cap V_3$ and $Y = V_2 \cap V_4$. Then there exist sets X'' and Y'' such that*

- $X'' \cap (V_1 \cup V_2 \cup V_3 \cup V_4) = \emptyset = Y'' \cap (V_1 \cup V_2 \cup V_3 \cup V_4)$,

- $e(G[X'', X]) \le |X|$ *and* $e(G[Y'', Y]) \le |Y|$, *and*
- $|X''| \ge |X| - 2t_G(v_1) - 2t_G(v_3) - 2$ *and* $|Y''| \ge |Y| - 2t_G(v_2) - 2t_G(v_4) - 2$.

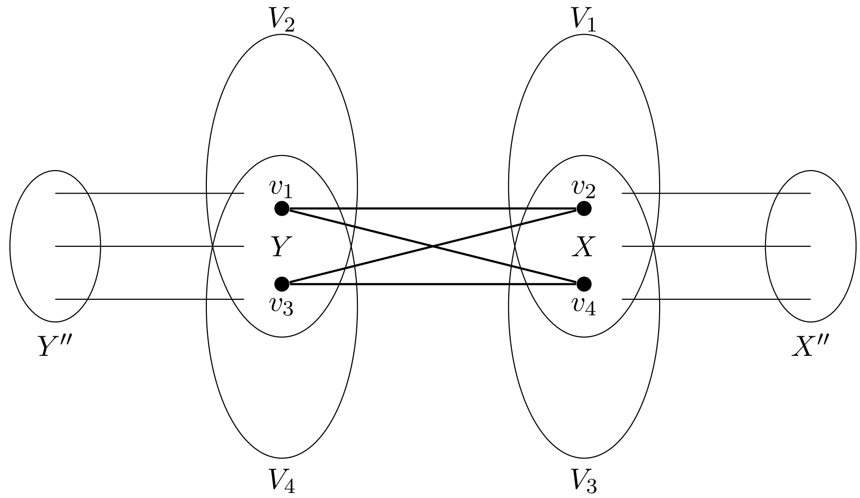

Figure 3: Note that $V_1 \cup V_3$ and $V_2 \cup V_4$ may intersect in Lemma 4.6. However, in the proof of Theorem 1.2, we only consider the case where they are disjoint. Additionally, it is important to note that X'' and Y'' may also intersect.

Proof. As $X \subseteq N(v_1) \cap N(v_3)$, by Lemma 2.1 for $k = 4$, there exists a set $X' \subseteq V(G)$ and a bijection $\varphi : X \to X'$ such that $X' = \{\varphi(x) : x \in X\}$, and for each $x \in X$, we have both $N(\varphi(x)) \cap X = \{x\}$ and $N(x) \cap X' = \{\varphi(x)\}$. We define $X'' := X' \setminus (V_1 \cup V_2 \cup V_3 \cup V_4)$, then obviously $X'' \cap (V_1 \cup V_2 \cup V_3 \cup V_4) = \emptyset$ and $e(G[X'', X]) \le |X|$.

As $Y \subseteq N(v_2) \cap N(v_4)$, by Lemma 2.1 for $k = 4$, there exists a set $Y' \subseteq V(G)$ and a bijection $\phi : Y \to Y'$ such that $Y' = \{\phi(y) : y \in Y\}$, and for each $y \in Y$, we have both $N(\phi(y)) \cap Y = \{y\}$ and $N(y) \cap Y' = \{\phi(y)\}$. We define $Y'' := Y' \setminus (V_1 \cup V_2 \cup V_3 \cup V_4)$, then obviously $Y'' \cap (V_1 \cup V_2 \cup V_3 \cup V_4) = \emptyset$ and $e(G[Y'', Y]) \le |Y|$.

Then we want to show the last property.

All vertices in V_2 are adjacent to the vertex $v_2 \in X$. Then we have $|X' \cap V_2| \le 1$ since $|N(x) \cap X'| = 1$ for each $x \in X$. Similarly, we have $|X' \cap V_4| \le 1$, $|Y' \cap V_1| \le 1$, and $|Y' \cap V_3| \le 1$.

All vertices in V_1 are adjacent to the vertex v_1. Since each vertex in X' has a neighbor in $X \subseteq N(v_1)$, we can check that $|X' \cap V_1| \le 2t(v_1)$. Similarly, we have $|X' \cap V_3| \le 2t(v_3)$, $|Y' \cap V_2| \le 2t(v_2)$, $|Y' \cap V_4| \le 2t(v_4)$. Therefore,

$$|X''| = |X'| - |X' \cap (V_1 \cup V_2 \cup V_3 \cup V_4)| \ge |X| - 2t(v_1) - 2t(v_3) - 2,$$

and

$$|Y''| = |Y'| - |Y' \cap (V_1 \cup V_2 \cup V_3 \cup V_4)| \ge |Y| - 2t(v_2) - 2t(v_4) - 2,$$

completing the proof. $\square$

Now we are ready to prove Theorem 1.2. It said that $f_4(n) < 0.164n^2$ for $n \geq n_4$, where n_4 is a constant.

Proof of Theorem 1.2. Throughout this proof, we assume that n is sufficiently large, and the subscripts of the notation such as v_i's and V_i's are under module 4. Suppose for a contradiction that there exists an n-vertex 4-critical graph G with $e(G) \geq 0.164n^2$. By (3), $t(G) \leq n$. Let $V_0 := \{v \in V(G) : t_G(v) \geq \sqrt{n}\}$. Then clearly we have $|V_0| < 3\sqrt{n}$. Let $G' := G[V(G) - V_0]$. It is not hard to see $e(G') \geq e(G) - n|V_0| > e(G) - 3n^{\frac{3}{2}} \geq 0.164n^2 - o(n^2)$. Note that $t(G') \leq t(G) \leq n$. Therefore, by deleting at most n edges from G', we can get a subgraph $G'' \subseteq G'$ such that $t(G'') = 0$, $e(G'') \geq e(G') - n \geq 0.164n^2 - o(n^2)$, and $t_G(v) < \sqrt{n}$ for each $v \in V(G'') = V(G) - V_0$. By applying Lemma 4.5 to G'', we can get a 4-cycle $v_1v_2v_3v_4$ in G'' such that

$$|V_1| + |V_2| + |V_3| + |V_4| \geq 8e(G'')/n - o(n) \geq 1.312n - o(n), \tag{4}$$

where $V_i := N_{G''}(v_i)$ for each $1 \leq i \leq 4$. Note that for each $1 \leq i \leq 4$, every vertex in $V_i \cap V_{i+1}$ must form a triangle with the vertices v_i, v_{i+1} in G'', which contradicts the fact $t(G'') = 0$. So it is clear that

$$V_i \cap V_{i+1} = \emptyset \text{ for each } 1 \leq i \leq 4.$$

Also it is easy to check that $\{v_{i-1}, v_{i+1}\} \subseteq V_i \subseteq N_G(v_i)$ for each $1 \leq i \leq 4$. Define $X = V_1 \cap V_3$ and $Y = V_2 \cap V_4$. Applying Lemma 4.6, we can get two sets X'', Y'' satisfying the three properties of Lemma 4.6. Note that X'' and Y'' are disjoint from $V_1 \cup V_2 \cup V_3 \cup V_4$, $V_1 \cap V_3 = X$, $V_2 \cap V_4 = Y$, and $V_i \cap V_{i+1} = \emptyset$ for each $1 \leq i \leq 4$. So we can see that

$$|V_1| + |V_2| + |V_3| + |V_4| - |X| - |Y| + |X'' \cup Y''| \leq n. \tag{5}$$

Besides, by using the last property in Lemma 4.6, we have

$$|X'' \cup Y''| \geq \max\{|X''|, |Y''|\} \geq \frac{|X''| + |Y''|}{2} \geq \frac{|X| + |Y|}{2} - O(\sqrt{n}). \tag{6}$$

By substituting inequalities (4) and (6) into inequality (5), we get

$$\frac{|X| + |Y|}{2} \geq |V_1| + |V_2| + |V_3| + |V_4| - n - O(\sqrt{n}) \geq 0.312n - o(n). \tag{7}$$

Then we consider the non-edges of the graph G, i.e., the edges of the graph $\overline{G}$. First, since $V_i = N_{G''}(v_i) \subseteq N_G(v_i)$ and $v_i \in V(G'')$, we can see $e(G[V_i]) \leq t_G(v_i) \leq \sqrt{n}$ for each $1 \leq i \leq 4$. So

$$e(\overline{G}[V_i]) \geq \binom{|V_i|}{2} - o(n^2) = \frac{1}{2}|V_i|^2 - o(n^2) \text{ for each } 1 \leq i \leq 4.$$

Thus by noting $V_1 \cap V_3 = X$, $V_2 \cap V_4 = Y$, and $V_i \cap V_{i+1} = \emptyset$ for each $1 \leq i \leq 4$, we can get

$$\left|\bigcup_{i=1}^{4} E(\overline{G}[V_i])\right| \geq \sum_{i=1}^{4} e(\overline{G}[V_i]) - \binom{|X|}{2} - \binom{|Y|}{2} \geq \frac{1}{2}\left(\sum_{i=1}^{4} |V_i|^2 - |X|^2 - |Y|^2\right) - o(n^2). \tag{8}$$

Next, since any m-vertex triangle-free graph has at most $\frac{1}{4}m^2$ edges, we can see $e(G[X''\cup Y'']) \leq \frac{1}{4}|X''\cup Y''|^2 + n$ by (3), and thus

$$e(\overline{G}[X''\cup Y'']) \geq \frac{1}{4}|X''\cup Y''|^2 - o(n^2). \tag{9}$$

By properties of X'', Y'' ensured by Lemma 4.6, we can obtain

$$e(\overline{G}[X'', X]) \geq |X''||X| - |X| \geq |X|^2 - o(n^2), \tag{10}$$

$$e(\overline{G}[Y'', Y]) \geq |Y''||Y| - |Y| \geq |Y|^2 - o(n^2). \tag{11}$$

So we can deduce that

$$\begin{aligned}
e(G) = \binom{n}{2} - e(\overline{G}) &\leq \binom{n}{2} - \frac{1}{2}\left(\sum_{i=1}^{4}|V_i|^2 - |X|^2 - |Y|^2\right) - \frac{1}{4}|X''\cup Y''|^2 - |X|^2 - |Y|^2 + o(n^2) \\
&\leq \frac{1}{2}n^2 - \frac{1}{8}\left(|V_1| + |V_2| + |V_3| + |V_4|\right)^2 - \frac{1}{4}\left(\frac{|X|+|Y|}{2}\right)^2 - \left(\frac{|X|+|Y|}{2}\right)^2 + o(n^2) \\
&\leq \frac{1}{2}n^2 - \frac{1}{8}(1.312n)^2 - \frac{5}{4}(0.312n)^2 + o(n^2) < 0.1632n^2 + o(n^2).
\end{aligned}$$

The inequalities are derived as follows: the first inequality obtained from inequalities (8)(9)(10) and (11); the second inequality is derived from inequality (6) and the convexity of the square; and the third inequality is based on inequalities (4) and (7). This contradicts the assumption that $e(G) \geq 0.164n^2$, completing the proof of Theorem 1.2. $\square$

Our understanding of the functions $f_k(n)$ is generally poor, and it is not even known if

$$f_4(n) < f_5(n) \text{ holds for sufficiently large integers } n. \tag{12}$$

So it seems to be a natural next step to pursue the question of whether $f_4(n) \leq cn^2$ holds for some constant $c < \frac{4}{31}$ and sufficiently large n. Note that if this is true, then it would imply (12).

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