1. A die has six sides that come up with different probabilities: Pr(1) = Pr(2) = Pr(3) = Pr(4) = 1/8, Pr(5) = Pr(6) = 1/4.

   (a) You roll the die; let Z be the outcome. What is E(Z) and var(Z)?

   (b) You roll the die 10 times, independently; let X be the sum of all the rolls. What is E(X) and var(X)?

Solution: (a) $E(Z) = \frac{1}{8}(1+2+3+4) + \frac{1}{4}(5+6) = 4$

$E(Z^2) = \frac{1}{8}(1^2+2^2+3^2+4^2) + \frac{1}{4}(5^2+6^2) = 19$

$var(Z) = E(Z^2) - \left(E(Z)\right)^2 = 19 - 4^2 = 3$

(b) Let X be the sum of all ten rolls and Z_i be the outcome of the i[th] roll. Then

$X = \sum_{i=1}^{10} Z_i$

$E(X) = E\left(\sum_{i=1}^{10} Z_i\right) = 10 * 4 = 40$

$E(X^2) = E\left(\left(\sum_{i=1}^{10} Z_i\right)^2\right) = E\left(\sum_{i=1}^{10} {Z_i}^2 + \sum_{i\neq j} Z_i Z_j\right) = \sum_{i=1}^{10} E\left({Z_i}^2\right) + \sum_{i\neq j} E\left(Z_i Z_j\right)$

$E\left({Z_i}^2\right) = 10 * 19 = 190$

Since Z_i and Z_j are independent

$E\left(Z_i Z_j\right) = E(Z_i)E\left(Z_j\right) = 4 * 4 = 16$

Since there are only (10) (9) different pairs for $i \neq j$ then

$E(X^2) = \sum_{i=1}^{10} E\left({Z_i}^2\right) + \sum_{i\neq j} E\left(Z_i Z_j\right) = 190 + 90 * 16 = 1630$

$var(X) = E(ZX^2) - \left(E(X)\right)^2 = 1630 - 40^2 = 30$

2. A box contains 9 red marbles and 1 blue marbles. Nine hundred random draws are made from this box, with replacement. What is distribution of the number of red marbles seen, roughly?

Solution: The distribution formula is $N(\mu, \sigma^2) = (nP, nP(1-P))$

$P =$ probability of red marbles

$P = \frac{9}{10} = 0.9$

$E(X) = \mu = np = 900 * 0.9 = 810$

$Var(X) = \sigma^2 = nP(1-P) = 900 * 0.9 * 0.1 = 81$

Standard deviation is $= 9$

$N(\mu, \sigma^2) = (810{,}81)$

3. Suppose that in the world at large, 1% of people are left-handed. A sample of 200 people is chosen at random. Give a 99% confidence interval for the number of them that are left-handed.

Solution: $\mathrm{n} = 200$

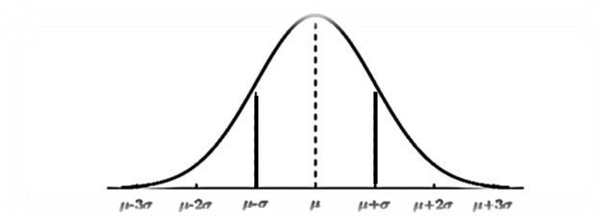
$\mathrm{P} = 0.01$

X is a binomial distribution

$\mathrm{E(X)} = \mu = \mathrm{nP} = 200 * 0.01 = 2$

$var(X) = \sigma^2 = \mathrm{nP(1-P)} = 200 * 0.01 * 0.99 \approx 1.98$

$sdev(X) = \sigma = \sqrt{\mathrm{nP(1-P)}} = \sqrt{200 * 0.01 * 0.99} \approx 1.41$



With a 99% probability, X lies within the range $\mu \mp 3\sigma$

$Pr(\mu - 3\sigma \leq X \leq \mu + 3\sigma) \approx 0.99$

$Pr(2 - 3 * 1.41 \leq X \leq 2 + 3 * 1.41) \approx 0.99$

$Pr(-2.23 \leq X \leq 6.23)$

The range as around [0,6]

4. Poisson: Calculate the maximum likelihood estimate of λ

P(X = x) = λxe−λ/ x!

Assume you have n observations: X1, X2, . . . , Xn iid observations from a Poisson random variable.

(Hint: Use log-likelihood).

Solution: A X1, X2, . . . , Xn iid observation from a Poisson random variable will have a joint frequency function that is a product of marginal frequency functions, the log likelihood is :

$L(\lambda) = \sum_{i=1}^{n}(X_i log\lambda - \lambda - logX_i!)$

$= log\lambda \sum_{i=1}^{n} X_i - n\lambda - \sum_{i=1}^{n} logX_i!$

To find the maximum likelihood, take the derivative of the function and set it to zero.

$L'(\lambda) = \frac{d}{d\lambda}[log\lambda \sum_{i=1}^{n} X_i - n\lambda - \sum_{i=1}^{n} logX_i!]$

$= \frac{1}{\lambda}\sum_{i=1}^{n} X_i - n = 0$

Which implies that the estimate should be

$\hat{\lambda} = \hat{X}$

5. A) A coin is flipped 100 times. Given that there are 55 heads, find the maximum likelihood estimate for the probability p of heads on a single toss using maximum likelihood estimation and Binomial distribution assumption.

B) Redo part A using log likelihood.

Solution: (A) The probability of getting 55 heads in the experiment is binomial probability.

$$P(55\ heads) = \binom{100}{55} P^{55}(1-P)^{45}$$

Since the probability of getting 55 heads depends on the value of P,

$$P(55\ heads|P) = \binom{100}{55} P^{55}(1-P)^{45}$$

To find the maximum likelihood estimate (MLE), take the derivative of the likelihood function and set it to zero,

$$\frac{d}{dP} P(55\ heads|P) = \binom{100}{55} [55P^{54}(1-P)^{45} - 45P^{55}(1-P)^{44}] = 0$$

$$55P^{54}(1-P)^{45} = 45P^{55}(1-P)^{44}$$

$$55(1-P) = 45P$$

$55 = 100P \rightarrow$ the MLE is $\hat{P} = 0.55$

(B) Log likelihood

$$P(55\ heads|P) = \binom{100}{55} P^{55}(1-P)^{45}$$

$$ln\big(P(55\ heads|P)\big) = ln\binom{100}{55} + 55\ln(P) + 45\ln(1-P)$$

$$\frac{d}{dP}(\log likelihood) = \frac{d}{dP}\left[ln\binom{100}{55} + 55\ln(P) + 45\ln(1-P)\right] = 0$$

$$= \frac{55}{P} - \frac{45}{1-P} = 0$$

$55(1-P) = 45P \rightarrow$ the MLE is $\hat{P} = 0.55$