Q1: We have 8 red balls and 8 blue balls. We store them in two urns. In urn1, we have 2 red ... and 6 blue balls. In urn2, we have 2 blue balls and 6 red balls. We pick urn1 with probability 0.75. We pick a ball from one of the two urns, and its red. Determine the probabilities:

a) $P(R|Urn1)$, $P(R|Urn2)$.

b) $P(Urn1|R)$, $P(Urn2|R)$.

c) $P(R)$:Total probability of picking a red ball. (25) Marks

---

a) $P(R|Urn1) = \frac{2}{2+6} = \boxed{\frac{1}{4}}$

$P(R|Urn2) = \frac{6}{2+6} = \boxed{\frac{3}{4}}$


$P[urn1] = 0.75 = 3/4$

$P[Urn2] = 0.25 = 1/4$

b) $P(Urn1|R)$ : using bayes' rule

$P(Urn1|R) = \frac{P(R|Urn1) \cdot P(Urn1)}{P(R)} = \frac{1/4 \cdot 3/4}{3/8} = \frac{3/16}{3/8} = \boxed{\frac{1}{2}}$

$P(R)$ ← found in C

again using bayes' rule;

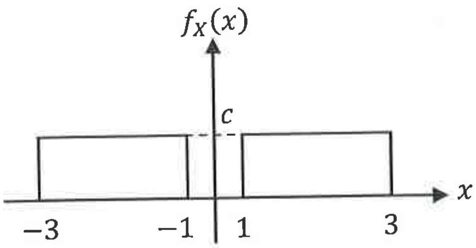
$P(Urn2|R) = \frac{P(R|Urn2) . P(Urn2)}{P(R)} = \frac{3/4 \cdot 1/4}{3/8} = \frac{3/16}{3/8} = \boxed{\frac{1}{2}}$

$P(R)$ ← found in C

c) $P(R) = P(R, Urn1) + P(R, Urn2) = P(R|Urn1)P(Urn1) + P(R|Urn2)P(Urn2)$

$= \frac{1}{4} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{1}{4} = \frac{6}{16} = \frac{3}{8}$

Q2: Given the pdf




a) Determine c.

b) Calculate $E[x]$

(25) Marks

a) In order to find c, we need the CDF = $F_X$ to be 1 as $X \to \infty$

$$P(X<\infty) = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-3}^{-1} c\,dx + \int_{1}^{3} c\,dx = c\,x\Big|_{-3}^{-1} + c\,x\Big|_{1}^{3}$$

$$= c \cdot 2 + c \cdot 2 = c \cdot 4$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 4c = 1 \Rightarrow \boxed{c = 1/4}$$

b) $$E[x] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \int_{-3}^{-1} c\,x\,dx + \int_{1}^{3} c\,x\,dx$$

$$= c\,\frac{x^2}{2}\Big|_{-3}^{-1} + c\,\frac{x^2}{2}\Big|_{1}^{3} = \frac{c}{2}(1-9) + \frac{c}{2}(9-1)$$

$$= 0$$

$EX = 0$

Q3: We observe the following data corresponding to different classes:

| | | | |
|---|---|---|---|
| **Class 1:** | 10 | 11 | 12 |
| **Class 2:** | 5 | 7 | 6 |

Prier probabilities of the above two classes are equal.

a) Construct a Gaussian based generative method for the above two classes.

b) Given the observation x=9. Estimate its class. (25) Marks

$P_1(X) \equiv P(X \mid y=1) \Leftarrow$ for class 1

$P_2(X) \equiv P(X \mid Y=2) \Leftarrow$ for class 2

let $Y$ ba a r.v. s.t $Y=1 \Rightarrow$ class 1, $Y=2 \Rightarrow$ class 2

for class 1

$$\hat{\mu}_1 = \frac{10+11+12}{2} = 11, \quad \hat{\sigma}_1^2 = \frac{1}{3-1}\left[(10-11)^2 + (11-11)^2 + (12-11)^2\right]$$

$$= \frac{1}{2}[1+0+1] = 1$$

for class 2

$$\hat{\mu}_2 = \frac{5+7+6}{3} = 6, \quad \hat{\sigma}_2^2 = \frac{1}{3-1}\left[(5-6)^2 + (7-6)^2 + (6-6)^2\right] = 1$$

$$P_1(X) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x-11)^2\right), \quad P_2(X) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-1}{2}(x-6)^2\right)$$

$$P(Y=1 \mid X) = \pi_1 \cdot \frac{P_1(X)}{P(X)} \propto 0.5\, P_1(X), \quad P(Y=2 \mid X) \propto \pi_2 P_2(X) = 0.5\, P_2(X)$$

not important $\rightarrow P(X)$

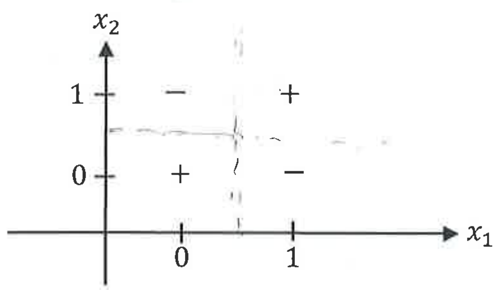
$$\boxed{y_{est} = \arg\max_i (\pi_1 P_1(x), \pi_2 P_2(x))}$$

b) $x=9$, $P(Y=1 \mid X=9) = 0.5\, P_1(X=9) = 0.5 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(9-11)^2\right) = 0.072$

$P(Y=2 \mid X=9) = 0.5\, P_2(X=9) = 0.5 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(9-6)^2\right) = 0.044$

$P(Y=1 \mid X=9) > P(Y=2 \mid X=9) \Rightarrow$ class estimate is Class 1

Q4: Consider a classification problem with two variables. In Figure 1 we show two positive "+" and two negative " – "examples.



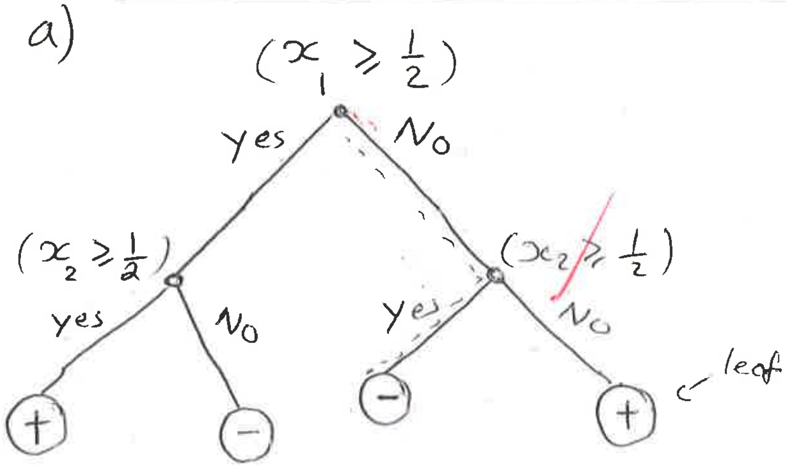

| | $x_1$ | $x_2$ | *Result* |
|---|---|---|---|
| Pt1 | 0 | 0 | + |
| Pt2 | 1 | 0 | − |
| Pt3 | 0 | 1 | − |
| Pt4 | 1 | 1 | + |

Figure 1. Two positive and two negative examples.

a) Draw a decision tree that can perfectly classify the four examples in Figure 1.
b) We observe the vector $[0,2]^T$. What is the decision using the decision tree?
c) We observe the vector $[0,2]^T$. What is the decision according to the nearest-neighbor classifier?

(25) Marks

a)




b) $v = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $x_1 = 0$, $x_2 = 2$

following the tree:

if $x_1 \geq \frac{1}{2}$ : false $\Rightarrow$ $x_2 \geq \frac{1}{2}$ : True

$\Rightarrow$ branch (3) $\Rightarrow$ $\boxed{\text{Class Negative}}$

c) we need to find the distances between $v = [0,2]^T$ and each point (points are denoted above)

$d^2_{v,Pt_1} = \| [0,2]^T - [0,0]^T \|_2^2 = \| [0,2] \|_2^2 = \boxed{4}$

$d^2_{v,Pt_2} = \| [0,2]^T - [1,0]^T \|_2^2 = \| [-1,2] \|_2^2 = \boxed{5}$

$d^2_{v,Pt_3} = \| [0,2]^T - [0,1]^T \|_2^2 = \| [0,1] \|_2^2 = \boxed{1}$

$d^3_{v,Pt_4} = \| [0,2]^T - [1,1] \|_2^2 = \| [-1,1] \|_2^2 = \boxed{2}$

The nearest neighbor to $v$ is point $(0,1)$ whose class is class Negative

$\Rightarrow$ class of $[0,2]^T$ is class Negative $(-)$