ECE 407 HW 7-Solution

Question 1)

In this question, we need to find all the possible hidden states so as to find the probability of the set of observation V = {**'abbbaba'**} given the HMM Θ. P(V|Θ)

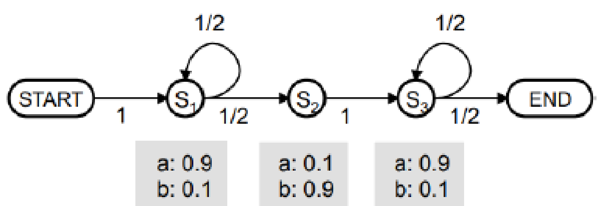

Figure 1: A hidden Markov model.

Looking at the Markov chain and the transition states, we can see that there are a total of only 5 possible state-sequences $\boldsymbol{W}$'s , which are:

$\boldsymbol{W^1}$	$s1 - s1 - s1 - s1 - s1 - s2 - s3 - end$
$\boldsymbol{W^2}$	$s1 - s1 - s1 - s1 - s2 - s3 - s3 - end$
$\boldsymbol{W^3}$	$s1 - s1 - s1 - s2 - s3 - s3 - s3 - end$
$\boldsymbol{W^4}$	$s1 - s1 - s2 - s3 - s3 - s3 - s3 - end$
$\boldsymbol{W^5}$	$s1 - s2 - s2 - s3 - s3 - s3 - s3 - end$

Therefore, the observation probability is given as:

$$P(V|\Theta) = \sum_{i=1}^{5} P(V, W^i|\Theta)$$

Where $P(V, W^i|\Theta)$ is calculated as follows:

$$P\left(V, W^i\middle|\Theta\right) = \pi_1 b_{1a} \prod_{j=2}^{7} b_j\left(O_j\right) a_{(j-1)j} * p_{end|s3}$$

This is because we always start with $s = s1$, and $p_{end|s3} = \frac{1}{2}$. Therefore, for each $\boldsymbol{W}$**,** the calculations using Matlab are given in the following table:

$\boldsymbol{W^1}$	$1.025e - 05$
$\boldsymbol{W^2}$	$1.265e - 07$
$\boldsymbol{W^3}$	$1.025e - 05$
$\boldsymbol{W^4}$	$1.025e - 05$
$\boldsymbol{W^5}$	$1.025e - 05$

Therefore, $P(V|\Theta)$, which is the summation of the above answers, is $4.1133e - 05$

Question 2

In this question, we are given the set of observations $V = \{TTHH\}$. Since the states are connected with non-zeros transition state, the number of possible state sequences is $2^4 = 16$, starting with {F,F,F,F} all the way to {L, L, L, L}.

The transition matrix for the hidden states is

	F	L
F	0.9	0.1
L	0.5	0.5

The probabilities of the observations given the states are

observation	H	T
State F	0.5	0.5
State L	0.2	0.8

The Probability $P\ (V|\Theta)$ is $\mathbf{0.0794.}$

I used the following code to find the probabilities:

```
close all;
clear all;
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```
clc;

seq= {'T', 'T', 'H', 'H'};

%observable probabilities
P_O = [0.5 0.5;
       0.2 0.8];

states = [1, 2];

%transition matrix
T = [0.9 0.1; %<=F
     0.5 0.5]; %<=L

N = 16;
seq_len = 4;

%here I generate the 16 possible sequence by producing a matrix of 1's (Fair)
and
%2's (loaded)
stats_M = de2bi([0:N-1])+1;

%a vector for storig the probability for each state
prob_state = ones(16,1);
for i=1:N
    for j=1:seq_len
        if seq{j} =='H'
            P_obs = P_O(stats_M(i, j), 1);
        else
            P_obs = P_O(stats_M(i, j), 2);
        end
        if j==1
            %starting probability is equal
            P_start = 0.5;
            prob_state(i) = prob_state(i).*P_obs.*P_start;
        else

            P_T = T(stats_M(i, j-1), stats_M(i, j));
            if seq{j} =='T'
                P_obs = P_O(stats_M(i, j), 1);
            else
                P_obs = P_O(stats_M(i, j), 2);
            end
            prob_state(i) = prob_state(i).*P_obs.*P_T;
        end
    end
end

%finally sum up the probabilites for all the states
sum(prob_state)
```

Question 3

In this question, the transition matrix is:

$$a_{ij} = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$$

Part i

The sequence given is $W^8 = \{\omega 3, \omega 2, \omega 3, \omega 1, \omega 1, \omega 3, \omega 2, \omega 3\}$. Assuming prior probabilities are equal, i.e. 1/3, the log probability $P(W^8|\Theta)$ is:

$$\log_{10} P(W^8|\Theta) = \log \pi_{w3} + \log a_{32} + \log a_{23} + \log a_{31} + \log a_{11} + \log a_{13} + \log a_{32} + \log a_{23}$$

$$\log_{10} P(W^8|\Theta) = \log 1/3 + \log 0.1 + \log 0.2 + \log 0.1 + \log 0.4 + \log 0.3 + \log 0.1 + \log 0.2$$

$$\log_{10} P(W^8|\Theta) = -0.478 - 1 - 0.699 - 1 - 0.398 - 0.522 - 1 - 0.699 = \mathbf{-5.7959}$$

Therefore, $P(W^8|\Theta) = pow(10, -5.7959) = 1.60e - 06$

Part ii

$P(w\ same\ for\ d\ days|w(1) = w) = P(w_1, w_1, \dots., w_1)\ \ d - 1$ times

$$P(rainy\ for\ d\ days|\ rainy\ at\ day = 1) = a_{11}^{d-1} = 0.4^{d-1}$$

$$P(cloudy\ for\ d\ days|cloudy\ at\ day = 1) = a_{11}^{d-1} = 0.6^{d-1}$$

$$P(sunny\ for\ d\ days|sunny\ at\ day = 1) = a_{11}^{d-1} = 0.4^{d-1}$$

Question 4

The transition matrix $B = \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$

Part i

Using Matlab to iteratively find P, the equation $BP^{k+1} = P^k$ converged at k=53, the solution obtained was $P = \left[0.3\dot{3}\ 0.2\dot{2}\ 0.2\dot{2}\ 0.2\dot{2}\right]^T$, the vector is a valid probability vector since it adds up to unity.

Part ii

In this part, we need to explicitly solve $BP = P$, in other words we need to solve

$$BP - P = 0 \rightarrow (B - I)P = 0 \rightarrow B'P = 0$$

Where $B' \equiv I - B$ is:

$$B' = \begin{bmatrix} -1 & 1/2 & 1 & 0 \\ 1/3 & -1 & 0 & 1/2 \\ 1/3 & 0 & -1 & 1/2 \\ 1/3 & 1/2 & 0 & -1 \end{bmatrix}$$

As it can be seen, B' is not full rank since the summation of the last three rows is equal to the first row, therefore the system $B'P = 0$ has a non-trivial solution,

Using Matlab, the rank of B' is 3, therefore a homogeneous solution of dimension 1 exists.

The solution is the null space of the matrix B', using Matlab, I obtained:

$$P = \ [\,0.6547 \quad 0.4364 \quad 0.4364 \quad 0.4364]^T$$

Which when normalized gives

$$P = \left[0.33\dot{3}\ 0.22\dot{2}\ 0.22\dot{2}\ 0.22\dot{2}\right]^T$$

Which is equal to the solution found in Part i.