problem 1

$course_1$	$course_2$	GPA (G_2)
2	2	2
2	3	2.5
3	2	2.5
3	3	3

Problem 2

(a) To find c, we recall that the PMF must sum to 1. That is,

$$\sum_{n=1}^{3} P_N(n) = c\left(1 + \frac{1}{2} + \frac{1}{3}\right) = 1. \tag{1}$$

This implies $c = 6/11$. Now that we have found c, the remaining parts are straightforward.

(b) $\mathrm{P}[N = 1] = P_N(1) = c = 6/11$.

(c)
$$\begin{aligned}\mathrm{P}[N \geq 2] &= P_N(2) + P_N(3)\\ &= c/2 + c/3 = 5/11.\end{aligned}$$

(d) $\mathrm{P}[N > 3] = \sum_{n=4}^{\infty} P_N(n) = 0$.

Problem 3

Decoding each transmitted bit is an independent trial where we call a bit error a "success." Each bit is in error, that is, the trial is a success, with probability p. Now we can interpret each experiment in the generic context of independent trials.

(a) The random variable X is the number of trials up to and including the first success. Similar to Example 3.9, X has the geometric PMF

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

(b) If $p = 0.1$, then the probability exactly 10 bits are sent is

$$P_X(10) = (0.1)(0.9)^9 = 0.0387. \tag{1}$$

The probability that at least 10 bits are sent is

$$\mathrm{P}[X \geq 10] = \sum_{x=10}^{\infty} P_X(x). \tag{2}$$

This sum is not too hard to calculate. However, its even easier to observe that $X \geq 10$ if the first 10 bits are transmitted correctly. That is,

$$\mathrm{P}[X \geq 10] = \mathrm{P}[\text{first 10 bits correct}] = (1-p)^{10}. \tag{3}$$

For $p = 0.1$,

$$\mathrm{P}[X \geq 10] = 0.9^{10} = 0.3487. \tag{4}$$

(c) The random variable Y is the number of successes in 100 independent trials. Just as in Example 3.11, Y has the binomial PMF

$$P_Y(y) = \binom{100}{y} p^y (1-p)^{100-y}. \tag{5}$$

If $p = 0.01$, the probability of exactly 2 errors is

$$P_Y(2) = \binom{100}{2}(0.01)^2(0.99)^{98} = 0.1849. \tag{6}$$

(d) The probability of no more than 2 errors is

$$\begin{aligned} \mathrm{P}[Y \le 2] &= P_Y(0) + P_Y(1) + P_Y(2) \\ &= (0.99)^{100} + 100(0.01)(0.99)^{99} + \binom{100}{2}(0.01)^2(0.99)^{98} \\ &= 0.9207. \end{aligned} \tag{7}$$

(e) Random variable Z is the number of trials up to and including the third success. Thus Z has the Pascal PMF (see Example 3.13)

$$P_Z(z) = \binom{z-1}{2} p^3(1-p)^{z-3}. \tag{8}$$

Note that $P_Z(z) > 0$ for $z = 3, 4, 5, \ldots$.

(f) If $p = 0.25$, the probability that the third error occurs on bit 12 is

$$P_Z(12) = \binom{11}{2}(0.25)^3(0.75)^9 = 0.0645. \tag{9}$$

Problem 4

Each of these probabilities can be read from the graph of the CDF $F_Y(y)$. However, we must keep in mind that when $F_Y(y)$ has a discontinuity at y_0, $F_Y(y)$ takes the upper value $F_Y(y_0^+)$.

(a) $\mathsf{P}[Y < 1] = F_Y(1^-) = 0$.

(b) $\mathsf{P}[Y \leq 1] = F_Y(1) = 0.6$.

(c) $\mathsf{P}[Y > 2] = 1 - \mathsf{P}[Y \leq 2] = 1 - F_Y(2) = 1 - 0.8 = 0.2$.

(d) $\mathsf{P}[Y \geq 2] = 1 - \mathsf{P}[Y < 2] = 1 - F_Y(2^-) = 1 - 0.6 = 0.4$.

(e) $\mathsf{P}[Y = 1] = \mathsf{P}[Y \leq 1] - \mathsf{P}[Y < 1] = F_Y(1^+) - F_Y(1^-) = 0.6$.

(f) $\mathsf{P}[Y = 3] = \mathsf{P}[Y \leq 3] - \mathsf{P}[Y < 3] = F_Y(3^+) - F_Y(3^-) = 0.8 - 0.8 = 0$.

Problem 5

(a) With probability $1/3$, the subscriber sends a text and the cost is $C = 10$ cents. Otherwise, with probability $2/3$, the subscriber receives a text and the cost is $C = 5$ cents. This corresponds to the PMF

$$P_C(c) = \begin{cases} 2/3 & c = 5, \\ 1/3 & c = 10, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) The expected value of C is

$$\mathrm{E}[C] = (2/3)(5) + (1/3)(10) = 6.67 \text{ cents.} \tag{2}$$

(c) For the next two parts we think of each text as a Bernoulli trial such that the trial is a "success" if the subscriber sends a text. The success probability is $p = 1/3$. Let R denote the number of texts received before sending a text. In terms of Bernoulli trials, R is the number of failures before the first success. R is similar to a geometric random variable except $R = 0$ is possible if the first text is sent rather than received. In general $R = r$ if the first r trials are failures (i.e. the first r texts are received) and trial $r + 1$ is a success. Thus R has PMF

$$P_R(r) = \begin{cases} (1-p)^r p & r = 0, 1, 2 \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

[Continued]

The probability of receiving four texts before sending a text is

$$P_R(4) = (1-p)^4 p. \tag{4}$$

(d) The expected number of texts received before sending a text is

$$\mathrm{E}[R] = \sum_{r=0}^{\infty} r P_R(r) = \sum_{r=0}^{\infty} r(1-p)^r p. \tag{5}$$

Letting $q = 1-p$ and observing that the $r=0$ term in the sum is zero,

$$\mathrm{E}[R] = p \sum_{r=1}^{\infty} r q^r. \tag{6}$$

Using Math Fact B.7, we have

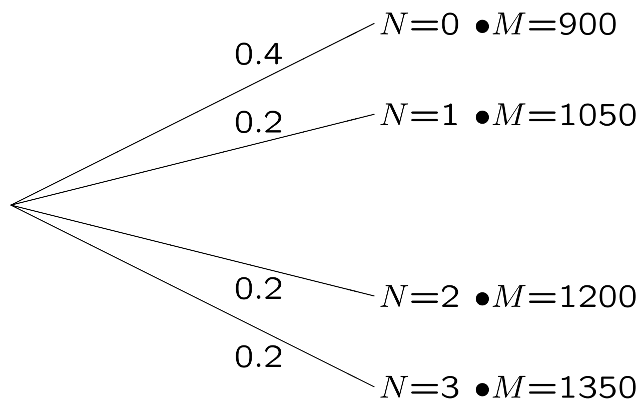
$$\mathrm{E}[R] = p \frac{q}{(1-q)^2} = \frac{1-p}{p} = 2. \tag{7}$$

Problem 6

(a) As a function of N, the money spent by the tree customers is

$$M = 450N + 300(3 - N) = 900 + 150N.$$

(b) To find the PMF of M, we can draw the following tree and map the outcomes to values of M:



From this tree,

$$P_M(m) = \begin{cases} 0.4 & m = 900, \\ 0.2 & m = 1050, 1200, 1350 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

From the PMF $P_M(m)$, the expected value of M is

$$\begin{aligned} \mathrm{E}[M] = 900P_M(900) + 1050P_M(1050) & \\ + 1200P_M(1200) + 1350P_M(1350) & \qquad (2) \\ = (900)(0.4) + (1050 + 1200 + 1350)(0.2) = 1080. & \qquad (3) \end{aligned}$$