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ECE 407 Mid-term Exam 2 - Solution

Q1 a: $\omega^T x + b > 0$

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$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} [x_1 \ x_2] + b > 0$$

$$2x_1 - x_2 + b > 0$$

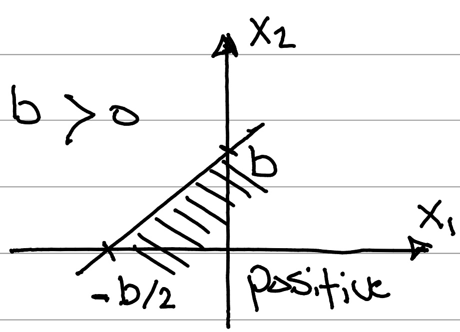
$$x_2 < b + 2x_1$$

To find the intersection points

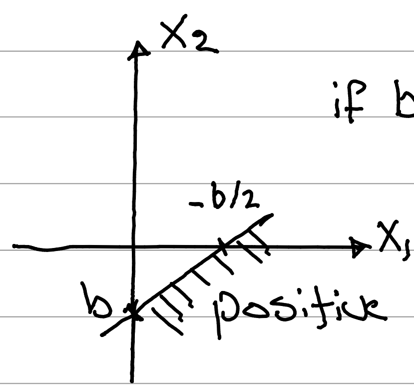
let $x_1 = 0 \Rightarrow x_2 = b$ $\quad (0, b)$

let $x_2 = 0 \Rightarrow x_1 = \frac{-b}{2}$ $\quad (\frac{-b}{2}, 0)$

if $b > 0$



if $b < 0$



b) Multivariate Gaussian distribution
" diagonal Covariance Matrix Case "

$\frac{2}{5}$

$$p(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma_1} \exp\left(\frac{1}{-2\sigma_1^2}(x_1 - \mu_1)^2\right)$$

$$\cdot \frac{1}{\sqrt{2\pi}\,\sigma_2} \exp\left(\frac{1}{-2\sigma_2^2}(x_2 - \mu_2)^2\right)$$

$$\sigma_1 = 1 \;;\; \sigma_2 = 1 \qquad \mu_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \;;\; \mu_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$p(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-1}{2}(x_1 - \mu_1)^2 - \frac{1}{2}(x_2 - \mu_2)^2\right)$$

$$p_1(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_1 - 1)^2 - \frac{1}{2}(x_2 + 1)^2\right)$$

$$p_2(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_1 - 1)^2 - \frac{1}{2}(x_2 - 1)^2\right)$$

Since $\pi_1 = \pi_2 = 0.5$

$$p_1(x) = p_2(x)$$

$$\cancel{\frac{1}{\sqrt{2\pi}}} \exp\left(-\frac{1}{2}(x_1 - 1)^2 - \frac{1}{2}(x_2 + 1)^2\right) = \cancel{\frac{1}{\sqrt{2\pi}}} \exp\left(-\frac{1}{2}(x_1 - 1)^2 - \frac{1}{2}(x_2 - 1)^2\right)$$

by taking ln for both sides

$$\cancel{-(x_1 - 1)^2} - (x_2 + 1)^2 = \cancel{-(x_1 - 1)^2} - (x_2 - 1)^2$$

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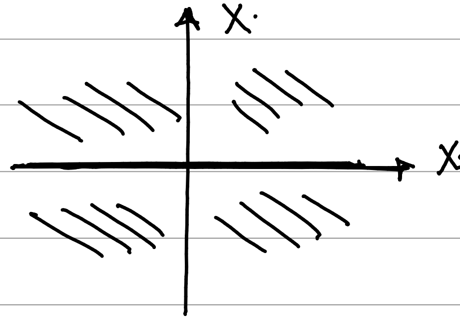
$$(X_2+1)^2 = (X_2-1)^2$$

$$X_2^2 + 2X_2 + 1 = X_2^2 - 2X_2 + 1$$

$$\boxed{X_2 = 0}$$



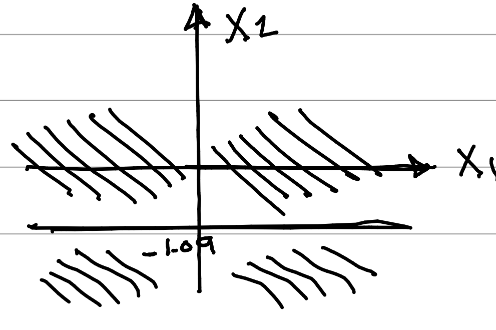
c) $\pi_1 = 0.75$; $\pi_2 = 0.25$

$$\log \pi_1 - \frac{1}{2}|X-\mu_1|^2 = \log \pi_2 - \frac{1}{2}|X-\mu_2|^2$$

$$\frac{|X-\mu_1|^2}{|X-\mu_2|^2} = \log \frac{\pi_2}{\pi_1}$$

decision boundary

$$x_2 = \log \frac{\pi_2}{\pi_1} = -1.09$$



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Q2: Choose an initial values for
-5 your centriod

$C_1 = 1$ $C_2 = -1$

Find the Euclidean distance to assign the label cluster for each observation

$\text{cluster}_1 = \{1, 2, 1, 0.5, 1.5\}$
$\text{cluster } 2 = \{-1.5, -1, -1, -1, -2\}$

Find the new centriods

$$C_1 = \frac{1+2+1+0.5+1.5}{5} = \boxed{1.2}$$

$$C_2 = \frac{-1.5-1-1-1-2}{5} = \boxed{-1.3}$$

Find the Euclidian distance to assign the label clusters again

b) Estimated decision rule: $\frac{C_1+C_2}{2}$

$\boxed{5}$

Q3 $\frac{30}{30}$ $$X = \begin{bmatrix} 2 & 1 & 1 & 1 \\ -0.1 & -0.1 & 0.1 & -0.1 \end{bmatrix}$$

$$\mu = \begin{bmatrix} 5/4 \\ -0.2/4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ -0.05 \end{bmatrix}$$

$$X - \mu = \begin{bmatrix} 0.75 & -0.25 & -0.25 & -0.25 \\ -0.05 & -0.05 & 0.15 & 0.05 \end{bmatrix}$$

The Covariance Matrix $\Sigma = \frac{1}{N-1} (X-\mu)(X-\mu)^T$

$$\Sigma = \begin{vmatrix} Cov(X,X) & Cov(X,Y) \\ Cov(Y,X) & Cov(Y,Y) \end{vmatrix}$$

$$\Sigma = \begin{vmatrix} 0.25 & -0.0167 \\ -0.0167 & 0.01 \end{vmatrix}$$

b) To find the eigen values

$$|\Sigma - \lambda I| = 0$$

$$\begin{vmatrix} 0.25-\lambda & -0.0167 \\ -0.0167 & 0.01-\lambda \end{vmatrix} = 0$$

$\boxed{6}$

$$\lambda = \begin{bmatrix} 0.0088 \\ 0.2512 \end{bmatrix}$$

For $\lambda_1 = 0.0088$ the eign vector is equal to

$$(\Sigma - \lambda_1 I) u = 0$$

$$\Sigma u = \lambda_1 u \Rightarrow \vec{u}_1 = [\ ]$$

$$\Sigma u = \lambda_2 u \Rightarrow \vec{u}_2 = [\ ]$$

c) $x = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

The projection

$$\vec{u}\,\vec{u}^T x = [\ ]$$

The largest vector is the vector that has the largest λ

$\boxed{7}$
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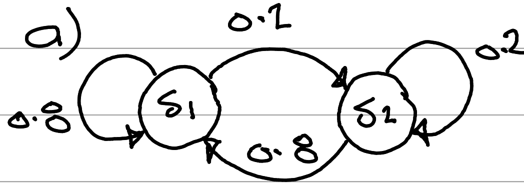
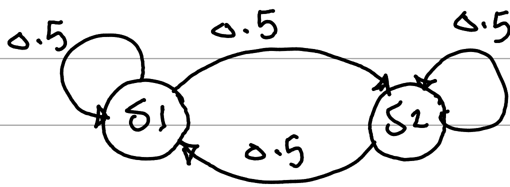
Q4: a)



d_1



d_2

b) $P(O|d_1) = 0.5 * 0.8 * 0.2$

$P(O|d_2) = 0.5 * 0.5 * 0.5$