Australian National University
Research School of Computer Science

# COMP3600/COMP6466 in 2015 – Quiz Three

**Due:** 23:55pm Friday, September 25

Submit your work electronically through Wattle. The total mark of this quiz worths 20 points, which is worth of 4.5 points of the final mark.

**Question 1** (4 points).
Given $n$ items with item $i$ having weight $w_i > 0$ and a profit $p_i > 0$ for all $i$, $1 \leq i \leq n$, assume that each item can be cut into an arbitrary fraction if needed. The fractional knapsack problem is to pack as many items as possible to a knapsack with capacity $W$ such that the total profit of items (or a fraction of an item) in the knapsack is maximized. Show how to solve this fractional knapsack problem in $O(n)$ time. Notice that you are asked to devise a linear running time algorithm for the problem, while an $O(n \log n)$ time algorithm can easily be devised. (*Hint: adopt the greedy strategy and the linear selection algorithm*).

**Answer:** Let the knapsack capacity be $W$ and item $i$ have weight $w_i$ with the profit $p_i$ for all $i$ with $1 \leq i \leq n$. We order the items by the *decreasing order* of $p_i/w_i$. Let $i_1, i_2, \ldots, i_n$ be the sorted order of items. The greedy strategy adopted is to pack items $i_1, i_2, \ldots, i_{j-1}$ and item $i_j$ with $(W - \sum_{l=1}^{j-1} w_{i_l})$ if $w_{i_j} \geq W - \sum_{l=1}^{j-1} w_{i_l}$. Then, the maximum profit of this solution is $S = \sum_{l=1}^{j-1} p_{i_l} + \frac{p_{i_j}}{w_{i_j}} \cdot (W - \sum_{l=1}^{j-1} w_{i_l})$. We claim that this solution $S$ is an optimal solution, by contradiction.

Assume that part of an item $i_{j'}$ with $j' > j$, $w'_{i_{j'}}$, is added to the solution, then the same amount of an item (e.g, $i_k$ with $k \leq j$) will be removed from the optimal solution, then the profit delivered by the updated solution is

$$\begin{aligned} S' &= \sum_{l=1}^{j-1} p_{i_l} + \frac{p_{i_j}}{w_{i_j}} \cdot (W - \sum_{l=1}^{j-1} w_{i_l}) + (\frac{p_{i_{j'}}}{w_{i_{j'}}} \cdot w'_{i_k} - \frac{p_{i_k}}{w_{i_k}} \cdot w'_{i_k}) \\ &= S + w'_{i_k}(\frac{p_{i_{j'}}}{w_{i_{j'}}} - \frac{p_{i_k}}{w_{i_k}}) \\ &\leq S, \quad \text{as } \frac{p_{i_k}}{w_{i_k}} \geq \frac{p_{i_{j'}}}{w_{i_{j'}}} \ . \end{aligned} \tag{1}$$

Thus, the solution $S$ is optimal. Clearly if the items are sorted, it takes $O(n)$ time. Otherwise, the sorting takes $O(n \log n)$ time.

The above solution with $O(n)$ time is assumed that the ratios of profits to weights of items are sorted. We now show that it still takes $O(n)$ time without this assumption. The detailed steps are as follows. Let $R$ be the set of $n$ items.

- Step 1. Calculate the ratio $r_i = p_i/w_i$ of each item $i$ for all $i$ with $1 \leq i \leq n$;
- Step 2. Find **the median** of the ratio sequence, let $\gamma$ be the median;
- Step 3. Partition the items ( their weights) into 3 disjoint subsets: $R_1 = \{w_i \mid p_i/w_i < \gamma\}$, $R_2 = \{w_i \mid p_i/w_i = \gamma\}$, $R_3 = \{w_i \mid p_i/w_i > \gamma\}$;
- Step 4. if the weighted sum of all items in $R_3$ is larger than $W$, then apply the algorithm on $R_3$ recursively. Otherwise, if the weighted sum of items in $R_2$ and $R_3$ is no less than $W$, make use of all items in $R_3$ and and add fractional items in $R_2$ to make up $W$. Otherwise, apply the algorithm on $R_1$ with the knapsack capacity of $W' = W - \sum_{item_i \in R_2 \cup R_3} w_i$ recursively. Clearly, it is an application of the linear selection algorithm, which takes $O(n)$ time.

**Question 2** (4 points).
Show that if we order the characters in an alphabet so that their frequencies are monotonically decreasing, then, there exists an optimal code whose codeword lengths are monotonically increasing. (*Hint: adopt the greedy strategy*)

**Answer:** We show this by contradiction. Assume that there are two characters $x$ and $y$ with frequencies $f_x$ and $f_y$, we further assume that $f_x < f_y$.Let $c_x$ and $c_y$ be the length of code words of $x$ and $y$. Let $\mathcal{A}$ be the set of character in a document. Following the proposed approach, we assume there is a method for the character coding $c(\cdot)$ for the characters in $\mathcal{A}$ such that the length of the document after coding is $\sum_{z \in \mathcal{A}} f_z \cdot c_z$ is minimized. This coding has a property, that is, if $f_x < f_y$, then

$$c_x > c_y \tag{2}$$

Now, we assume that there is another coding $c'$ for the document, in which all other character code words are identical to their corresponding ones in the original method, the only difference lies in the two characters $x$ and $y$ with $\{x, y\} \subseteq \mathcal{A}$, i.e., we swap the coding between $x$ and $y$, in other words, for any $z \in \mathcal{A} \setminus \{x, y\}$ we have $c'_z = c_z$, while $c'_x = c_y$ and $c'_y = c_x$. Now, we have $c'_x < c'_y$ as $c_x > c_y$, the length of the document under this new coding thus is

$$\begin{aligned}
\sum_{z \in \mathcal{A}} f_z \cdot c'_z &= \sum_{z \in \mathcal{A}} f_z \cdot c_z - (f_x \cdot c_x + f_y \cdot c_y) + (f_x \cdot c_y + f_y \cdot c_x) \\
&= \sum_{z \in \mathcal{A}} f_z \cdot c_z + f_y(c_x - c_y) - f_x(c_x - c_y) \\
&> \sum_{z \in \mathcal{A}} f_z \cdot c_z, \text{since } f_x < f_y \text{ and } c_x > c_y,
\end{aligned} \tag{3}$$

which means that the new coding delivers a solution that has a longer length of the document, this leads to a contradiction that the length is the smallest one. Thus, for any coding $c''$ if we aim to minimize the encoding length of the document, we must have $c''_x > c''_y$ if $f_x < f_y$.

**Question 3** (3 points).
Generalize Huffman's algorithm to ternary codewords (i.e., codewords using the symbols 0, 1, 2), and prove that it yields optimal ternary codes.

**Answer:** Following the construction of Huffman's tree, each internal node has at most 3 children, the links to these three children are labeled as 0, 1, and 2. Assume that there are three characters $a$, $b$ and $c$ are the three least frequent characters with frequencies $f_a$, $f_b$ and $f_c$. Assume that $f_a \leq f_b \leq c$. Let $T$ be an optimal coding tree, assume that characters $x$, $y$ and $z$ are siblings at the greatest depth, i.e., $d_T(x) = d_T(y) = d_T(z) > \max\{d_T(a), d_T(b), d)T(c)\}$. We now construct another tree $T'$ based on $T$, that is, we swap the positions of characters $a$ and $x$, the positions of characters $b$ and $y$, and the positions of characters $c$ and $z$. Then, the difference of the sum of the lengths of codewords in a document $\mathcal{A}$ between $T'$ and $T$ is

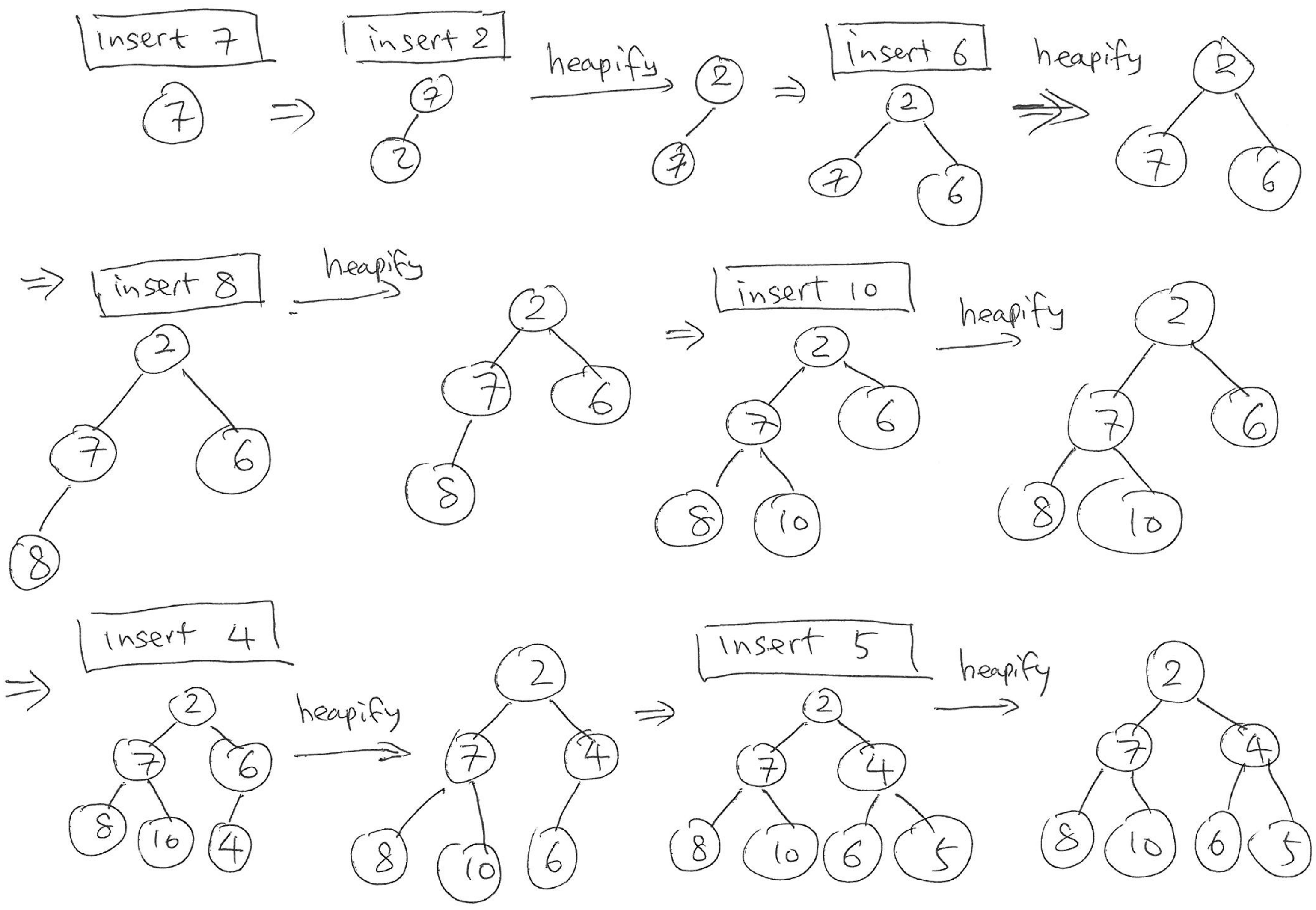
$$\begin{aligned} B(T,\mathcal{A}) - B(T',\mathcal{A}) &= f(a)d_T(a) + f(b)d_T(b) + f(c)d_T(c) + f(x)d_T(x) + f(y)d_T(y) + f(z)d_T(z) \\ &- [f(a)d_T(x) + f(b)d_T(y) + f(c)d_T(z) + f(x)d_T(a) + f(y)d_T(b) + f(z)d_T(c)] \\ &= f(a)(d_T(a) - d_T(x)) + f(b)(d_T(b) - d_T(y)) + f(c)(d_T(c) - d_T(z)) \\ &= -f(x)(d_T(a) - d_T(x)) - f(y)(d_T(b) - d_T(y)) - f(z)(d_T(c) - d_T(z)) \\ &= (f(a) - f(x))(d_T(a) - d_T(x) + (f(b) - f(y))(d_T(b) - d_T(y)) \\ &= +(f(c) - f(c))(d_T(c) - d_T(z)) \\ &> 0, \quad \text{as } d_T(x) = d_T(y) = d_T(z) > \max\{d_T(a), d_T(b), d_T(c)\} \\ &\quad \text{and } f(a) - f(x) < 0,\ f(b) - f(y) < 0, \text{ and } f(c) - f(z) < 0, \end{aligned}$$

which implies that the $T'$ results in a smaller size document coding. We can perform the swapping operations iteratively until no swapping can be performed. The resulting tree is the Huffman's tree. Thus, the solution delivered by the Huffman's coding is the optimal one.

**Question 4** (6 points).
(a) Assuming an initial min-heap is empty, insert the keys $7, 2, 6, 8, 10, 4, 5, 9, 4, 10$ into the min-heap one by one (once a time) until all elements are inserted, then remove the key in the root repeatedly until the heap is empty again. (3 points)

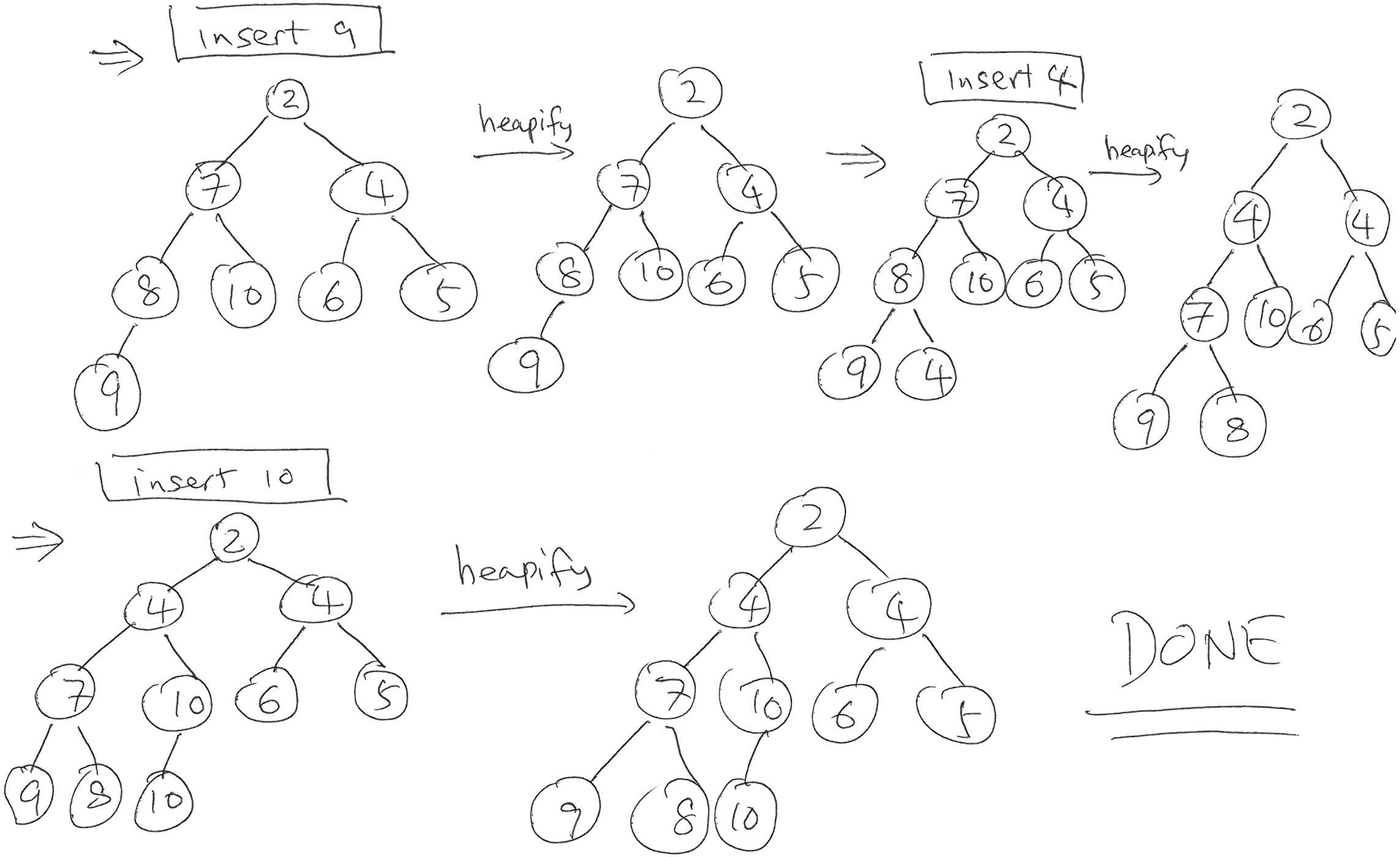
1. Use diagrams to illustrate each step of the insertion and deletion procedure.

Q3(a) Page 1



insert 7
insert 2
heapify
insert 6
heapify
insert 8
heapify
insert 10
heapify
insert 4
heapify
insert 5
heapify

Q3(a) Page 2.

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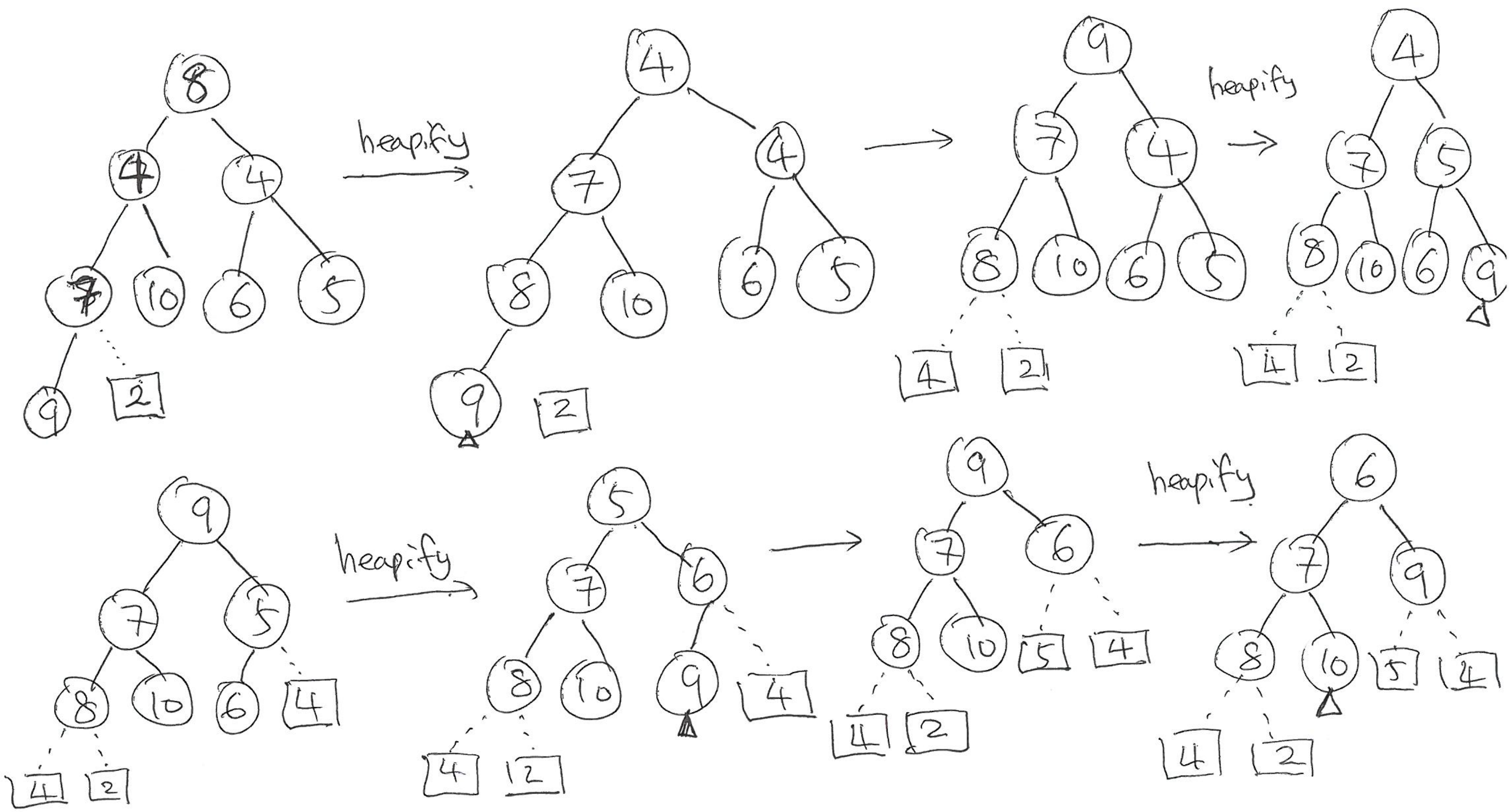
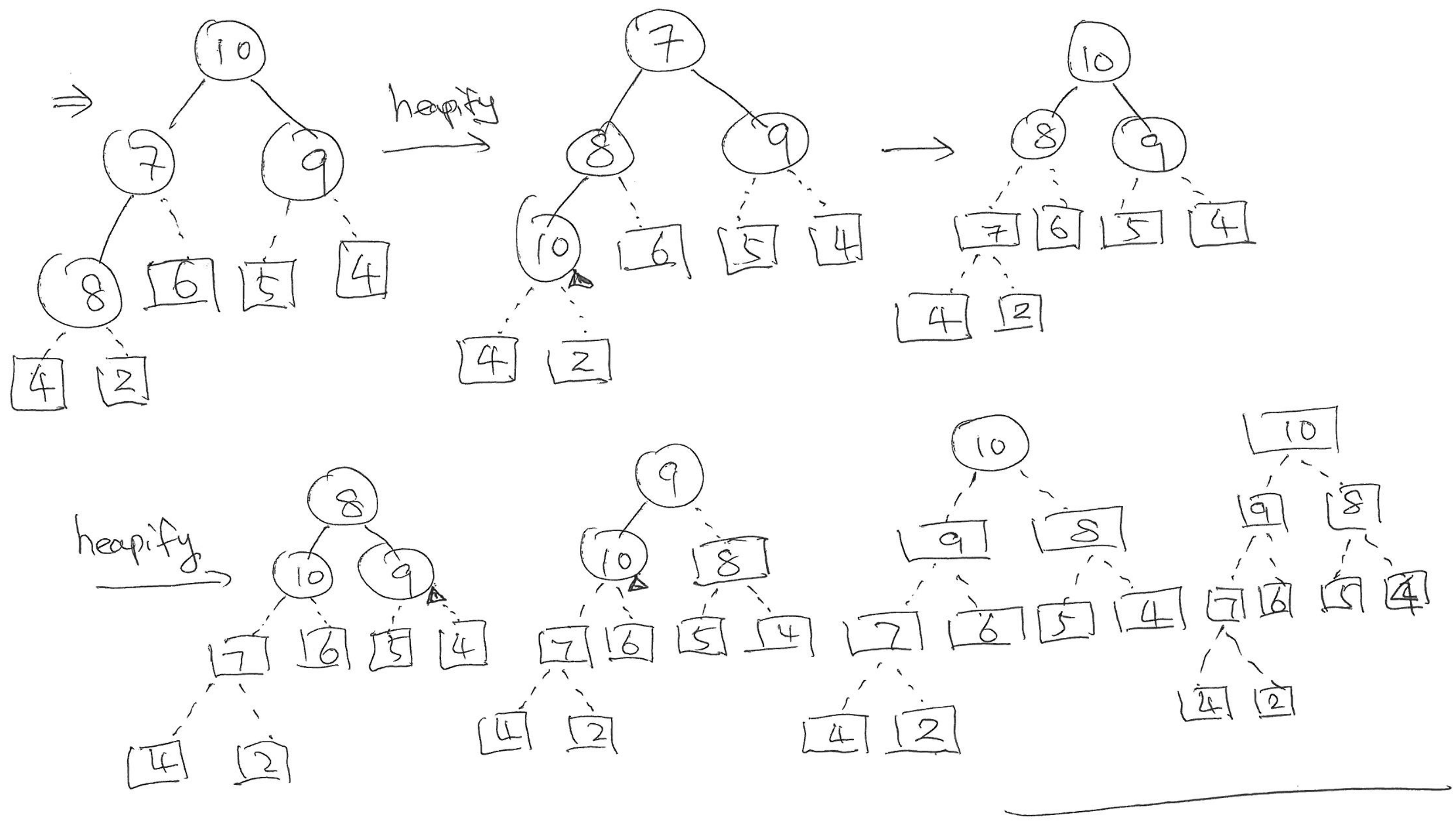
Q 3(a) Page 3.

Delete/Remove the root, put the last element (in the array) into the root position, then perform the heapify operation.



Q3(a) Page 4



2. What is the time complexity of sorting in this fashion if there are $n$ keys to be inserted to and then removed from the min-heap?

   **Answer:** The time complexity is $O(n \log n)$ if there are $n$ elements. Since the initialization of a min-heap takes $O(1)$ time, performing heapifying operation takes $O(\log n)$ time when inserting an element into the heap. It takes $O(1)$ time to remove an element from the root and $O(\log n)$ time to add another element to the root as heapifying the min-heap is needed after that.

(b) In the open addressing schema, three probing techniques: `linear probing`, `quadratic probing`, and `double hashing` have been introduced. (1) How many different probing sequences can be generated for each of the schemes? justify your answer. (2) What are the advantages and disadvantages among these techniques? (3 points)

**Answer:** Assume that there are $m$ slots in the hash table. Then,

- there are $O(m)$ probing sequences for `linear probing`,

  $$h(k,i) = (h'(k) + i) \mod m,$$

  $0 \leq h'(k) \leq m-1$ and $0 \leq i \leq m$, as a different $h'(k)$ leads to a different probing sequence.

- There are $O(m)$ probing sequences for `quadratic probing`,

  $$h(k,i) = (h'(k) + c_1 i + c_2 i^2) \mod m,$$

  $0 \leq h'(k) \leq m-1$, $0 \leq i \leq m$, $c_1$ and $c_2$ are constants, as a different $h'(k)$ leads to a different probing sequence.

- There are $O(m^2)$ probing sequences for `double hashing`.

  $$h(k,i) = (h_1(k) + ih_2(k)) \mod m,$$

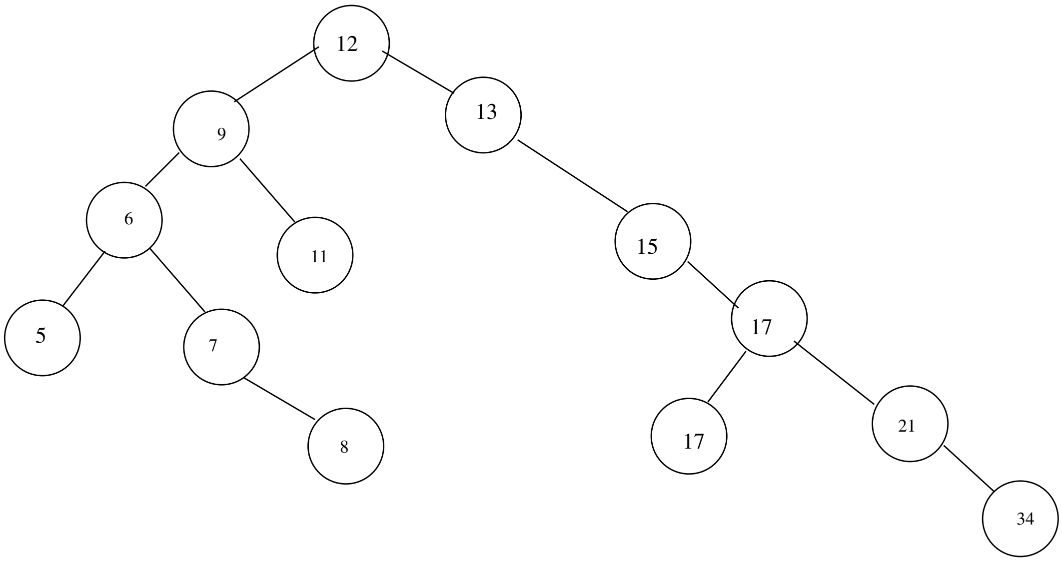
  $0 \leq h_1(k), h_2(k) \leq m-1$ and $0 \leq i \leq m$, as a different pair of $(h_1(k), h_2(k))$ leads to a different probing sequence and there are $O(m^2)$ such pairs.

`Linear probing` is very simple, and takes less time, but it suffices the primary clustering problem. `Quadratic probing` avoids the primary clustering problem, its computation is easy, but it suffices the secondary clustering problem. `Double hashing` is an ideal hash approach. Although it prevents both primary and secondary clustering problems, it takes a much longer running time.

**Question 5** (3 points).
Given an element sequence 12, 13, 15, 9, 6, 17, 21, 11, 34, 7, 8, 5, 17.

- Illustrate the final binary search tree by inserting the elements into the sequence one by one.

  **Answer:**




- Assume that a new element 14 will be inserted to the tree, show the resulting tree after the insertion.

  **Answer:**