Australian National University
Research School of Computer Science

COMP3600/COMP6466 in 2016 – Quiz One

Due: 5pm Friday, July 29

Submit your work electronically through Wattle. The total mark of this quiz worths 10 points, which is worth of 3 points of the final mark.

Question 1 (2 points).
Given the following sequence, order them into a sorted sequence in the order of **growth** when n approaches infinity.

$\sqrt{n}$, $n^4 \log n$, $n^{3.5} \log n + n^3$, $3^{0.01n}$, n^6, $5^{75} \log^2 n$, $n^3 + n \log \log n$, $n^2 \log^{0.5} n$.

Answer: , $5^{75} \log^2 n$ $\sqrt{n}$, $n^2 \log^{0.5} n$, $n^3+n \log \log n$, $n^{3.5} \log n+n^3$, $n^4 \log n$, n^6, $3^{0.01n}$.

Question 2 (2 points).
Let $f(n) = an^3+bn^2+cn+d$ and $g(n) = n^3$ where a, b, c and d are nonnegative constants. Show that

$$f(n) = \Theta(g(n)).$$

Answer: We first show $f(n) = O(g(n))$. Following the big-O notation, clearly for any $n \geq n_0 = 1$, there is a constant $e_1 = a + b + c + d$ such that

$$0 \leq f(n) \leq e_1 \cdot g(n). \tag{1}$$

Thus, $f(n) = O(g(n))$.

We then show $f(n) = \Omega(g(n))$. Following the $\Omega()$ definition, for any $n \geq n_0 = 1$ there is a constant $e_2 = \min\{a, b, c, d\}$ such that

$$0 \leq e_2 \cdot g(n) \leq f(n). \tag{2}$$

Thus, $f(n) = \Omega(g(n))$.

Combining inequalities 1 and 2, there are constants $e_1 = a + b + c + d > 0$ and $e_2 = \min\{a, b, c, d\} > 0$, for any $n \geq n_0 = 1$ such that

$$0 \leq e_2 \cdot g(n) \leq f(n) \leq e_1 \cdot g(n). \tag{3}$$

Thus, we have $f(n) = \Theta(g(n))$ by the $\Theta()$ definition.

Question 3 (2 points).
Provide the simplest expression for $\sum_{k=1}^{n} k^{9/4}$, using the $\Theta(\,)$ notation. Explain your reasoning clearly.

Answer:

$\sum_{k=1}^{n} k^{9/4} \leq \sum_{k=1}^{n} n^{9/4} = n \cdot n^{9/4} = O(n^{13/4})$.

On the other hand,

$$\begin{aligned}
\sum_{k=1}^{n} k^{9/4} &= \sum_{k=1}^{\lceil n/2\rceil - 1} k^{9/4} + \sum_{k=\lceil n/2\rceil}^{n} k^{9/4} & (4)\\
&\geq \sum_{k=\lceil n/2\rceil}^{n} k^{9/4} & (5)\\
&\geq (n - \lceil n/2\rceil + 1)(\lceil n/2\rceil)^{9/4} & (6)\\
&\geq (n - (n/2 + 1) + 1)(\lceil n/2\rceil)^{9/4} & (7)\\
&\geq (n/2)(\lceil n/2\rceil)^{9/4} & (8)\\
&\geq (n/2)(n/2)^{9/4} & (9)\\
&= (n/2)^{13/4} & (10)\\
&= \Omega(n^{13/4}), & (11)
\end{aligned}$$

where the lower bound in (8) follows from the fact that there are more than $n/2$ additive terms in (5) and the smallest term is $(\lceil n/2\rceil)^{9/4}$.
The asymptotic upper and lower bounds together imply $\sum_{k=1}^{n} k^{9/4} = \Theta(n^{13/4})$.

Question 4 (4 points).
Give an asymptotic upper bound on $T(n)$ for the following recurrence, using the $O(\,)$ notation. Justify your answers.

$$T(n) = 3T(2n/5) + n^2.$$

Answer: We use the substitution method (i.e., mathematical induction), starting with the guess that the answer might be $T(n) = \Theta(n^2)$. Let's first prove that $T(n) = O(n^2)$ and let's assume in the proof that $T(n)$ is defined for all positive rational numbers.

Base case: We can assume that $T(n) \leq C$ for small values of n, say, for $n \leq 10$, with some positive constant C. Then, there exists some constant $c > 0$ such that $T(n) \leq C \leq cn^2$ for every integer $1 \leq n \leq 10$. (Note that $0 < n \leq 10$ wouldn't work here. Why?)

Inductive step: Assume the hypothesis that for the constant c used in the base case, and for all $1 \leq n' < n$, we have

$$T(n') \leq cn'^2.$$

We will show that $T(n) \leq cn^2$ (note that we only need to consider cases when $n \geq 10$). Applying the recurrence, we have

$$\begin{aligned} T(n) &= 3T(2n/5) + n^2 \\ &\leq 3c(2n/5)^2 + n^2 \\ &= \frac{(12c + 25)}{5^2} n^2 \\ &\leq cn^2 \end{aligned}$$

as long as $c \geq \frac{12c+25}{5^2}$, and if the hypothesis can be used to bound $T(2n/5)$, that is, $1 \leq 2n/5 < n$. The condition on c is satisfied for any $c \geq \frac{25}{13}$, and the one on n is satisfied for any $n \geq 1$, so it is satisfied for every n that we consider in the inductive step.

We make sure that we choose a large enough c that works both in the base case and in the inductive step. Then, by induction, we have $T(n) \leq cn^2$ for every integer $n \geq 1$. If we consider these bounds only for positive integers, we obtain $T(n) = O(n^2)$.

We also have $T(n) \geq n^2$ (this is obvious by looking at the recurrence), so $T(n) = \Omega(n^2)$. The two bounds together give $T(n) = \Theta(n^2)$.

Question 5 (2 points).

(i) Show that $\ln x \leq x, \forall x \in R^+$, and $x \geq 1$, where R^+ is the set of positive real numbers.

(ii) Show that $\log n = O(n^\epsilon),\ \forall \epsilon > 0$.

Hints: (i) use the fact that $\ln 1 = 0$ and $(\ln x)' = 1/x$. (ii) use a variable substitution such as $n = k^2$, $\ln x^k = k \ln x$, and $\log x = \frac{\ln x}{\ln 2}$.

Answer: (i) Consider the function $f(x) = \ln x - x$, note that $f(1) = -1$. Since $f'(x) = \frac{1}{x} - 1$, it is easy to show that $f'(x) \leq 0,\ \forall x \geq 1$. Therefore, f is a non-increasing function, leading to $f(x) \leq -1 \leq 0,\ \forall x \geq 1$.

(ii) Show that $\log n = O(n^\epsilon),\ \forall \epsilon > 0$, which implies to show that there are constants $c > 0$ and $n_0 > 0$ for any $n \geq n_0$ such that

$$\log n \leq c \cdot n^\epsilon.$$

Let $k = n^\epsilon$, then $n = k^{1/\epsilon}$. There are constants $c > 0$ and $n_0 > 0$ such that

$$\log k^{1/\epsilon} \leq c \cdot k,\ \forall k \geq (n_0)^\epsilon,$$

which implies that there are constants $c > 0$ and $n_0 > 0$ such that

$$\frac{\log k}{\epsilon} \leq c \cdot k, \ \forall k \geq (n_0)^\epsilon.$$

Let $c = \frac{1}{\epsilon \ln 2}$ and $n_0 = 1$, then $\frac{\log k}{\epsilon} \leq c \cdot k, \ \forall k \geq (n_0)^\epsilon$, i.e,

$\frac{\ln k}{\epsilon \ln 2} \leq \frac{1}{\epsilon \ln 2} \cdot k, \ \forall k \geq 1$ as $\frac{1}{\epsilon \ln 2} > 0$. Thus,

$$\ln(k) \leq k, \ \forall k \geq 1.$$