Quiz 1.

1.

$5^{75}\log^2 n < \sqrt{n} < n^2\log^{0.5} n < n^3 + n\log\log n < n^{3.5}\log n + n^3$
$< n^4 \log n < n^6 < 3^{0.01 n}$

2.

To prove $f(n) = \Theta(g(n))$, firstly we need to prove big O notation.

big O: $f(n) = O(g(n))$.

$$\exists\ C_1 > 0,\ n_0 > 0 \mid an^3 + bn^2 + cn + d \le C_1 \cdot n^3 \quad \forall n \ge n_0$$

$$an^3 + bn^2 + cn + d \le C_1 \cdot n^3 \quad \forall n \ge n_0$$

$$a + \frac{b}{n} + \frac{c}{n^2} + \frac{d}{n^3} \le C_1 \quad \forall n \ge n_0.$$

So $C_1 = a+b+c+d$, $n_0 = 1$

Let $C_1 = a+b+c+d$, $n_0 = 1$

$$\Rightarrow a + \frac{b}{n_0} + \frac{c}{n_0^2} + \frac{d}{n_0^3} \le C_1 \quad \forall n \ge n_0$$

$$\Rightarrow a + \frac{b}{n} + \frac{c}{n^2} + \frac{d}{n^3} \le C_1 \quad \forall n \ge n_0.$$

$$\Rightarrow an^3 + bn^2 + cn + d \le C_1 \cdot n^3 \quad \forall n \ge n_0$$

$$\Rightarrow f(n) = O(g(n)).$$

Secondly, prove big-Omega Notation: $f(n) = \Omega(g(n))$.

$$\exists\ C_2 > 0,\ n_0 > 0 \mid an^3 + bn^2 + cn + d \ge C_2 \cdot n^3 \quad \forall n \ge n_0$$

$$an^3+bn^2+cn+d \geq c_2 \cdot n^3 \quad \forall n \geq n_0.$$

$$a+\frac{b}{n}+\frac{c}{n^2}+\frac{d}{n^3} \geq c_2 \quad \forall n \geq n_0.$$

As n gets bigger, c_2 gets smaller, so when n equal to infinite, c_2 gets the smallest value, that is a.

Let $c_2 = a$, $n_0 = +\infty \Rightarrow a+\frac{b}{n_0}+\frac{c}{n_0^2}+\frac{d}{n_0^3} \geq c_2 \quad \forall n \geq n_0$

$$\Rightarrow a+\frac{b}{n}+\frac{c}{n^2}+\frac{d}{n^3} \geq c_2 \quad \forall n \geq n_0$$

$$\Rightarrow an^3+bn^2+cn+d \geq c_2 \cdot n^3 \quad \forall n \geq n_0.$$

$$\Rightarrow f(n) = \Omega(g(n)).$$

As above, we get $c_1 = a+b+c+d$, $c_2 = a$, n_0 equal to any ~~number specific number~~ every integer between 1 and infinite. Hence, $f(n) = \Theta(g(n))$.

3.

upper bound: $\sum_{k=1}^{n} k^{\frac{9}{4}} \leq \sum_{k=1}^{n} n^{\frac{9}{4}} = n \cdot n^{\frac{9}{4}} = O(n^{\frac{13}{4}})$.

lower bound: $\sum_{k=1}^{n} k^{\frac{9}{4}} = \sum_{k=1}^{\frac{n}{2}-1} k^{\frac{9}{4}} + \sum_{k=\frac{n}{2}}^{n} k^{\frac{9}{4}}$ ①

$$\geq \sum_{k=\frac{n}{2}}^{n} k^{\frac{9}{4}} \quad ②$$

$$\geq (n-\frac{n}{2}+1) \cdot (\frac{n}{2})^{\frac{9}{4}} \quad ③$$

$$\geq [n-(\frac{n}{2}+1)+1] \cdot (\frac{n}{2})^{\frac{9}{4}} \quad ④$$

$$\geq \frac{n}{2} \cdot (\frac{n}{2})^{\frac{9}{4}} \quad ⑤$$

$$= (\frac{n}{2})^{\frac{13}{4}} \quad ⑥$$

$$= (\frac{1}{2})^{\frac{13}{4}} \cdot n^{\frac{13}{4}} \quad ⑦$$

$$= \Omega(n^{\frac{13}{4}}) \quad ⑧$$

where the lower bound in ⑤ follows from the fact that there are more than $\frac{n}{2}$ ~~ta~~ additive terms in ② and the smallest term is $(\frac{n}{2})^{\frac{13}{4}}$.

The ~~asymptotic~~ asymptotic upper and lower bounds together imply $\sum_{k=1}^{n} k^{\frac{9}{4}} = \Theta(n^{\frac{13}{4}})$.

4. We use the substitution method and guess might $T(n) = \Theta(n^2)$.
To prove $T(n) = \Theta(n^2)$, we first need to prove $T(n) = O(n^2)$. And we assume that $T(n)$ is defined for all positive rational numbers.

Base case: We assume that $T(n) \leq C$ for small values of n, for example, $n \leq 10$, with positive constant C. Then, there exists some constant $c > 0$ such that $T(n) \leq C \leq cn^2$ for every integer $1 \leq n \leq 10$.

Inductive step: We assume that for all $10 \leq n' < n$, we have $T(n') \leq cn'^2$.
Then we have:

$$\begin{aligned} T(n) &= 3T\left(\frac{2n}{5}\right) + n^2 \\ &\leq 3c\left(\frac{2n}{5}\right)^2 + n^2 \\ &= \frac{12c+25}{25} n^2 \\ &\leq cn^2 . \end{aligned}$$

Hence, as long as $c \geq \frac{12c+25}{25}$, that is $c \geq \frac{25}{13}$, and for any $n \geq 10$, it is satisfied for every n that we consider in the inductive step. Therefore, we obtain $T(n) = O(n^2)$.

We also have $T(n) = 3T(\frac{2n}{5}) + n^2 \geqslant n^2$, so $T(n) = \Omega(n^2)$.
These two together ~~app~~ imply $T(n) = \Theta(n^2)$

5.
(i) Let $f(x) = \ln x - x$.
Therefore, $f'(x) = \frac{1}{x} - 1$, and because $x \geqslant 1$, $0 < \frac{1}{x} \leqslant 1$.
Then, we get $f'(x) \leq 0$, which represents $f(x)$ is monotone decreasing, and $f(x)$ has the biggest value when $f'(x)$ equal to 0, that is when x equal to 1, $f(x)$ has the biggest value -1
Hence, $f(x) \leq -1$, that is $\ln x - x \leq -1$

$$\ln x \leq x - 1$$
$$\leq x \qquad \forall x \in R^+.$$

(ii)

$\exists\, c > 0,\ n_0 > 0 \quad | \quad \log n \leq c \cdot n^{\varepsilon} \qquad \forall n \geqslant n_0, \varepsilon > 0.$

$$\frac{\ln n}{\ln 2} \leq c \cdot n^{\varepsilon} \qquad \forall n \geqslant n_0, \varepsilon > 0$$

$$\ln n \leq c \cdot n^{\varepsilon} \cdot \ln 2. \qquad \forall n \geqslant n_0, \varepsilon > 0$$

from (i), we know $\ln n \leq n$.

here we assume $n \leq c \cdot n^{\varepsilon} \cdot \ln 2 \quad \forall n \geqslant n_0, \varepsilon > 0.$

$$\frac{1}{\ln 2} \leq c \cdot n^{\varepsilon - 1} \qquad \forall n \geqslant n_0, \varepsilon > 0$$

$$\frac{1}{\ln 2} \cdot \frac{1}{n^{\varepsilon - 1}} \leq c \qquad \forall n \geqslant n_0, \varepsilon > 0.$$

So $c = \frac{1}{\ln 2}$, $n_0 = 1$

Let $c = \frac{1}{\ln 2}$, $n_0 = 1 \Rightarrow \frac{1}{\ln 2} \cdot \frac{1}{n_0^{\varepsilon - 1}} \leq c \qquad \forall n \geqslant n_0, \varepsilon > 0$

$$\Rightarrow 1 \leq c \cdot n^{\varepsilon - 1} \cdot \ln 2 \qquad \forall n \geqslant n_0, \varepsilon > 0.$$

$$\Rightarrow \ln n \leq n \leq c \cdot n^{\varepsilon} \cdot \ln 2 \qquad \forall n \geqslant n_0, \varepsilon > 0$$

$$\Rightarrow \frac{\ln n}{\ln 2} \leq c \cdot n^{\varepsilon} \qquad \forall n \geqslant n_0, \varepsilon > 0$$

$\Rightarrow \log n \leq C \cdot n^{\varepsilon} \quad \forall n \geq n_0, \varepsilon > 0.$

$\Rightarrow \log n = O(n^{\varepsilon}) \quad \forall n \geq n_0, \varepsilon > 0.$