Australian National University
Research School of Computer Science

COMP3600/COMP6466 in 2015 – Quiz One

Due: 23:55pm Friday, July 31

Submit your work electronically through Wattle. The total mark of this quiz worths 10 points, which is worth of 3 points of the final mark.

Question 1 (2 points).
Given the following sequence, order them into a sorted sequence in the order of **growth** when n approaches infinity.

$$\sqrt{n}\log n, \quad n^4\sqrt{\log n}, \quad 2^{0.01n}, \quad n^{5.5}\log n + n^4, \quad n^5, \quad 2^{25}\log^{2.3} n, \quad n^2 + n\log\log n, \quad n\log^{0.5} n.$$

Answer: $2^{25}\log^{2.3} n$, $\sqrt{n}\log n$, $n\log^{0.5} n$, $n^2{+}n\log\log n$, $n^4\sqrt{\log n}$, n^5, $n^{5.5}\log n{+}n^4$, $2^{0.01n}$.

Question 2 (2 points).
Let $f(n)$ and $g(n)$ be positive functions. Each of the following statements is either (a) 'true' or (b) 'false', tick one of them.

(i) If $f(n) = 4n^3 + n^2\log^3 n$ and $g(n) = n^2 + n^{1.5}\log^2 n$, then $g(n) = O(f(n))$. (a) true, (b) false.
Answer: (a) true.

(ii) If $f(n) = 7n^2 + 50n - 3000$ and $g(n) = n^2/5 + n\log^3 n$, then $f(n) = \Theta(g(n))$. (a) true, (b) false.
Answer: (a) true.

(iii) If $f(n) = n^3 - 10n^2 + 5\log n$ and $g(n) = n^2\log n$, then $g(n) = \Omega(f(n))$. (a) true, (b) false.
Answer: (b) false.

(iv) If $f(n) = 45n^2$ and $g(n) = n^{7/2}/9$, then $f(n) = o(g(n))$. (a) true, (b) false.
Answer: (b) true.

Question 3 (2 points).
Provide the simplest expression for $\sum_{k=1}^{n} k^{11/5}$, using the $\Theta()$ notation. Explain your reasoning clearly.

Answer:

$\sum_{k=1}^{n} k^{11/5} \leq \sum_{k=1}^{n} n^{11/5} = n \cdot n^{11/5} = O(n^{16/5})$.

On the other hand,

$$\begin{aligned}
\sum_{k=1}^{n} k^{11/5} &= \sum_{k=1}^{\lceil n/2\rceil-1} k^{11/5} + \sum_{k=\lceil n/2\rceil}^{n} k^{11/5} && (1)\\
&\geq \sum_{k=\lceil n/2\rceil}^{n} k^{11/5} && (2)\\
&\geq (n-\lceil n/2\rceil+1)(\lceil n/2\rceil)^{11/5} && (3)\\
&\geq (n-(n/2+1)+1)(\lceil n/2\rceil)^{11/5} && (4)\\
&\geq (n/2)(\lceil n/2\rceil)^{11/5} && (5)\\
&\geq (n/2)(n/2)^{11/5} && (6)\\
&= (n/2)^{16/5} && (7)\\
&= \Omega(n^{16/5}), && (8)
\end{aligned}$$

where the lower bound in (5) follows from the fact that there are more than $n/2$ additive terms in (2) and the smallest term is $(\lceil n/2\rceil)^{11/5}$.
The asymptotic upper and lower bounds together imply $\sum_{k=1}^{n} k^{11/5} = \Theta(n^{16/5})$.

Question 4 (4 points).
Give an asymptotic upper bound on $T(n)$ for the following recurrences, using the $O()$ notation. Justify your answers.

(a) $T(n) = T(n/6) + n$

Answer: We apply the iteration method. For simplicity, we only consider values of n that are exact powers of 6 (as $T()$ is defined for integers only). Note that $k = \log_6 n$ when $\frac{n}{6^k} = 1$. Then,

$$\begin{aligned}
T(n) &= T(n/6) + n\\
&= (T(n/6^2) + n/6) + n\\
&= \cdots\\
&= n + (n/6) + (n/6^2) + \ldots + (n/6^{k-1}) + T(1).
\end{aligned}$$

Let's assume now (without proving it) that $T(n) = n + (n/6) + (n/6^2) + \ldots + (n/6^{k-1}) + T(1)$ holds for every integer n, and not only for exact powers of 6. Then, apart from $T(1)$, this is a series of geometric type so its sum is $\Theta(\text{largest term}) = \Theta(n)$. The term $T(1)$ is negligible in comparison (we assume that it's a constant), the answer thus is $T(n) = \Theta(n)$.

(b) $T(n) = 3T(2n/7) + n^2$

Answer: We use the substitution method (i.e., mathematical induction), starting with the guess that the answer might be $T(n) = \Theta(n^2)$. Let's first prove that $T(n) = O(n^2)$ and let's assume in the proof that $T(n)$ is defined for all positive rational numbers.

Base case: We can assume that $T(n) \leq C$ for small values of n, say, for $n \leq 14$, with some positive constant C. Then, there exists some constant $c > 0$ such that $T(n) \leq C \leq cn^2$ for every integer $1 \leq n \leq 14$.

Inductive step: Assume the hypothesis that for the constant c used in the base case, and for all $15 \leq n' < n$, we have

$$T(n') \leq cn'^2.$$

We will show that $T(n) \leq cn^2$ (note that we only need to consider cases when $n \geq 15$). Applying the recurrence, we have

$$\begin{aligned} T(n) &= 3T(2n/7) + n^2 \\ &\leq 3c(2n/7)^2 + n^2 \\ &= \frac{(12c + 49)}{7^2} n^2 \\ &\leq cn^2 \end{aligned}$$

as long as $c \geq \frac{12c+49}{7^2}$. The condition on c is satisfied for any $c \geq \frac{49}{37}$, and the one on n is satisfied for any $n \geq 15$, so it is satisfied for every n that we consider in the inductive step.

We make sure that we choose a large enough c that works both in the base case and in the inductive step. Then, by induction, we have $T(n) \leq cn^2$ for every integer $n \geq 15$. If we consider these bounds only for positive integers, we obtain $T(n) = O(n^2)$.

We also have $T(n) = 3T(2n/7) + n^2 \geq n^2$ (this is obvious by looking at the recurrence), so $T(n) = \Omega(n^2)$. The two bounds together give $T(n) = \Theta(n^2)$.