Set theory: review

COMP2600 / COMP6260

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Why should we study set theory?

Set Theory as Foundation

All aspects of mathematics can ultimately be ‘compiled’ down to set theory.

Programming

Programs as functions that map structured sets of inputs to (structured) sets of outputs. Types are sets (of values).

Reasoning about programs

Relies on sets of data values

Set Theory Basics

Basic Constructs

- set membership: $x \in A$
- set equality: $A = B$

Membership and equality are linked: $A = B$ iff $\forall x.(x \in A \leftrightarrow x \in B)$

Derived Concept: Subsets

$A \subseteq B$ iff $\forall x.(x \in A \rightarrow x \in B)$

Notation

- explicit enumeration: $\{1, 2, 12, 17\}$
- comprehension: $A = \{x \in \mathbb{N} \mid x \text{ even } \}$

Simplest Consequence

- order doesn't matter: $\{1, 2, 12\} = \{12, 1, 2\}$
- multiplicity doesn't matter: $\{1, 1, 2\} = \{1, 2\}$

Standard sets

Numbers

$\mathbb{N}$ is the set of natural numbers; $\mathbb{Z}$ is the set of integers;
$\mathbb{Q}$ is the set of rational numbers; $\mathbb{R}$ is the set of real numbers.

Booleans and Characters

The set $\{True, False\}$ is typically called **Bool**. The set of (often ASCII) characters is typically referred to as **Char**.

The empty set $\emptyset$

- can be defined: $\emptyset = \{x \in \mathbb{N} \mid x \neq x\}$
- is a subset of every set $\forall A.\ \emptyset \subseteq A$
- should not be confused with $\{\emptyset\}$.

Caveat on Notation

The *"dot-dot-dot"* notation

- $\{1, 3, 5, 7, ...\}$ 'obviously' denotes the set of odd numbers. Or does it?
- Maybe $\{1, 3, 5, 7, ...\}$ is the set of numbers not divisible by 2 or 13?

Comprehension to the Rescue

$O = \{x \in \mathbb{N} \mid x \text{ odd } \}$

Characteristic Predicates

- $S = \{x \in D \mid P(x)\}$
- P will be called the characteristic predicate of the set S.

Russell's Paradox

Setup

Consider the set of all sets that are not members of themselves:
$S \equiv \{A \mid A \notin A\}$
Russell asked: "Is S a member of itself?"

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Solution?

Consider the 2 cases:

- Suppose $S \in S$. Then S is in the set $\{A \mid A \notin A\}$, so $S \notin S$.
- Suppose $S \notin S$. Then S does not satisfy the predicate $A \notin A$, so $S \in S$.

Hence there is no such S. The predicate does not characterise a set.

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What's going on?

Reveals a problem with an unconstrained notion of sets: even if P is a sensible predicate, it doesn't necessarily characterise a set.

New sets from Old

Subset

- $A \subseteq B \;$ iff $\; \forall x \in A.\; x \in B$
- If P is a predicate of appropriate type, and A is given, then $\{x \in A \mid P(x)\}$ is a new set (a subset of A).

Power set

- Notation: $\mathcal{P}(A)$ denotes the power set of A.
- Definition: $\mathcal{P}(A) \;\equiv\; \{s \mid s \subseteq A\}$
  i.e. $\mathcal{P}(A)$ is the set of all subsets of A.
- Example: $\mathcal{P}(\{1,2\}) \;=\; \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

Cardinality notation: $|A|$ indicates the cardinality of A.
The power set is so called because $|\mathcal{P}(A)| \;=\; 2^{|A|}$

New sets from Old

Cartesian product

Notation:

- $A \times B \ = \ \{(x, y) \mid x \in A \wedge y \in B\}$
- $A \times B \times C \ = \ \{(x, y, z) \mid x \in A \wedge y \in B \wedge z \in C\}$
- etc.

Union and intersection

Binary union and intersection (of 2 sets belonging to a universe of discourse):

- $A \cup B \ = \ \{x \in U \mid (x \in A) \vee (x \in B)\}$
- $A \cap B \ = \ \{x \in U \mid (x \in A) \wedge (x \in B)\}$

U is the universe of discourse. That is, $A \subseteq U$ and $B \subseteq U$.

Relations and Functions

Set-theoretic view: relations and functions are both sets of ordered pairs.

Definition

If A and B are sets then a subset $R \subseteq A \times B$ is a (binary) *relation* between A and B. People often write xRy to mean $(x, y) \in R$.

Flavours of Relations

A relation R between A and A is:

- *reflexive*, if $\forall x \in A.(x, x) \in R$
- *transitive*, if $\forall x, y, z \in A.(x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \in R$
- *symmetric*, if $\forall x, y \in A.(x, y) \in R \rightarrow (y, x) \in R$

Functions

Definition

A relation R between A and B is *functional* if both

- $\forall x \in A. \exists y \in B. (x, y) \in R$ (left-totality, every x maps somewhere)
- $\forall x \in A. \forall y, z \in B. (x, y) \in R \land (x, z) \in R \to y = z$ (right-uniqueness, only one function value)

Definition

If f is a functional relation between A and B, we call f a *function* and write $f : A \to B$.

Ternary relations

Definition

If A, B, C are sets then a subset $R \subseteq A \times B \times C$ is a *ternary relation* between A, B and C.

Can extend this principle to n-ary relations.