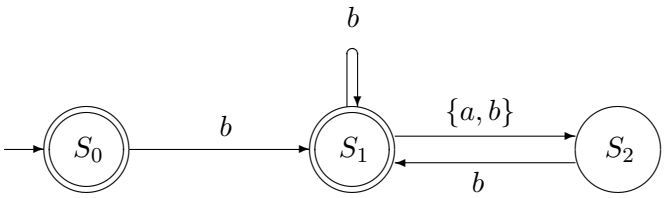
Department of Computer Science, Australian National University
COMP2600 — Formal Methods for Software Engineering
Semester 2, 2015

— Assignment 2 —
Automata, Languages, and Computability

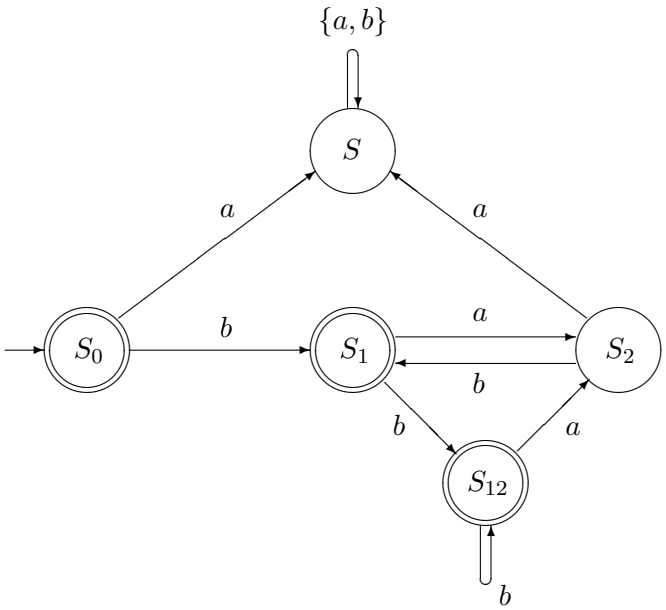
Sample Solution

1 Finite State Automata and Regular Language

The NFA A:



(i) The DFA B:



(ii) Our initial split differentiates non-final and final states:

$$[\,[\,S, S_2\,],\ [\,S_0, S_1, S_{12}\,]\,]$$

Testing the non-final group with b produces a split, as for S we stay in the non-final group, while for S_2 we go to the final group:

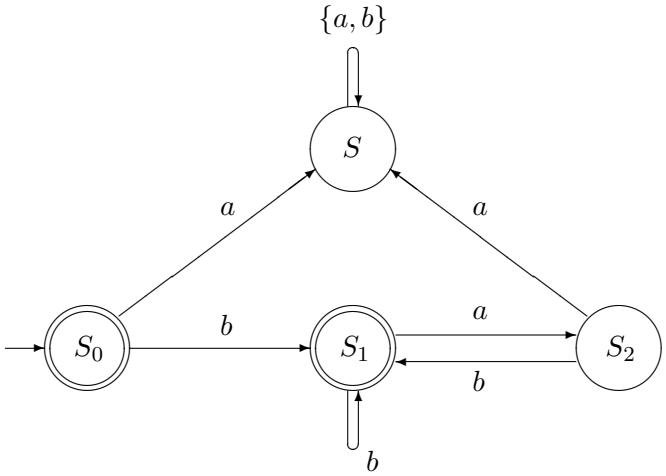
$$[\,[\,S\,],\ [\,S_2\,],\ [\,S_0, S_1, S_{12}\,]\,]$$

Testing the final group with a produces another split, as for S_0 we go to the group $[\,S\,]$, while for S_1 and S_{12} we go to the group $[\,S_2\,]$:

$$[\,[\,S\,],\ [\,S_2\,],\ [\,S_0\,],\ [\,S_1, S_{12}\,]\,]$$

The only non-singleton group left, $[\,S_1, S_{12}\,]$, cannot be split by any test: on a we go to $[\,S_2\,]$, while on b we stay in $[\,S_1, S_{12}\,]$. The algorithm therefore terminates, with S_1 and S_{12} the only equivalent states.

(iii) We convert B above into the minimal C by deleting the state S_{12} and re-assigning the arc from S_1 to S_{12} to be a loop on S_1:



(iv) Note that a string is *not* in the language L if and only if

- it has a as its first letter, or
- it has a as its last letter, or
- it contains the substring aa.

Subgoal one: If a string w is in L, then C accepts w.

Subgoal two: If C accepts a string w, then $w \in L$.

As suggested by the hint, we reformulate both these goals for proof by contrapositive:

Subgoal one (contrapositive): If C rejects a string w, then $w \notin L$.

Subgoal two (contrapositive): If $w \notin L$, then C rejects w.

Proof of subgoal one (contrapositive): If C rejects w then $N^*(S_0, w) = S_2$ or $N^*(S_0, w) = S$. We consider each case separately.

$N^*(S_0, w) = S_2$: the only arc into S_2 is labelled by a, so w must have the form αa for some string α. But any string ending in a is not in L.

$N^*(S_0, w) = S$: because S is not the start state, the DFA must have transitioned into S from other state at some stage. The only arcs into S from other states are labelled by a, so w must have the form $\alpha a \beta$ for some strings α, β. There are two subcases then to consider, depending on which incoming arc was used: $N^*(S_0, \alpha) = S_0$ or $N^*(S_0, \alpha) = S_2$. We consider each subcase separately.

$N^*(S_0, \alpha) = S_0$: there are no arcs into S_0 except the 'start state' arc, so the only way to validate this equation is if α is the empty string ϵ. But then $w = \alpha a \beta = a\beta$, and any string starting with a is not in L.

$N^*(S_0, \alpha) = S_2$: as we argued above, this can only hold if $\alpha = \gamma a$ for some string γ. But then $w = \alpha a \beta = \gamma a a \beta$, and any string w containing a substring aa is not in L.

Proof of subgoal two (contrapositive):

First, observe the lemma

$$N^*(S, w) \;=\; S \qquad \text{(Lemma)}$$

for all strings w. This holds because all arcs out of S go back to S, so there is 'no escape'.

Suppose $w \notin L$. There are, as noted above, three cases by which this could be, and we consider each separately.

w starts with a:

$$\begin{aligned} N^*(S_0, w) &= N^*(S_0, a\alpha) && \text{(for some string } \alpha) \\ &= N^*(N(S_0, a), \alpha) && \text{(Definition of } N^*) \\ &= N^*(S, \alpha) \\ &= S && \text{(Lemma)} \end{aligned}$$

and S is a non-final state, so w is rejected.

w ends with a: here $w = \alpha a$ for some α. Now $N^*(S_0, \alpha a) = N(N^*(S_0, \alpha), a)$ by the corollary to the Append Theorem. We have no idea what $N^*(S_0, \alpha)$ is, so we consider all four possible cases:

$$\begin{aligned} N^*(S_0, a) &= S & \qquad N^*(S_2, a) &= S \\ N^*(S_1, a) &= S_2 & \qquad N^*(S, a) &= S \end{aligned}$$

S and S_2 are both non-final, so w is rejected.

w contains substring aa: here $w = \alpha aa\beta$ for some α, β. Now $N^*(S_0, \alpha aa\beta) = N^*(N^*(S_0, \alpha), aa\beta) = N^*(N^*(N^*(S_0, \alpha), aa), \beta)$ by the Append Theorem. We do not know what $N^*(S_0, \alpha)$ is, so we investigate what $N^*(N^*(S_0, \alpha), aa)$ might be by all four cases again:

$$N^*(S_0, aa) = S \qquad N^*(S_2, aa) = S$$
$$N^*(S_1, aa) = S \qquad N^*(S, aa) = S$$

So S is the answer no matter what. Now

$$\begin{aligned} N^*(S_0, w) &= N^*(N^*(N^*(S_0, \alpha), aa), \beta) \\ &= N^*(S, \beta) \\ &= S \qquad \text{(Lemma)} \end{aligned}$$

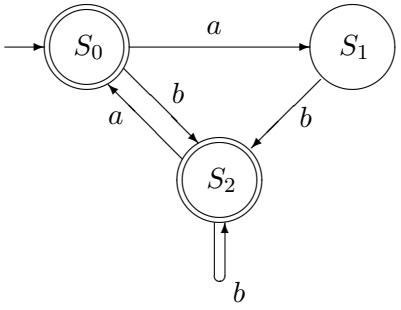
S is non-final, so w is rejected.

(v)

$$\begin{aligned} S_0 &\to bS_1 \mid \epsilon \\ S_1 &\to bS_1 \mid aS_2 \mid bS_2 \mid \epsilon \\ S_2 &\to bS_1 \end{aligned}$$

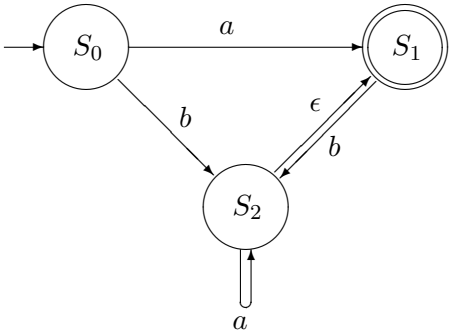
where S_0 is the start symbol.

(v) The automaton below will be at state S_2 if it has just read b; it will be at state S_1 if it has just read a *without* b to its left (in this case we will reject unless the next token is b); it will be at S_0 if the string has just started, or we have just read ba (in this case reading an a next is legal, but will put us 'in danger' of rejection by transitioning to S_1).



2 Pushdown Automata and Context-Free Language

1 Grammars



(i) Find a right-linear grammar which accepts the same language as this NFA. Your grammar may (unlike the definition in lectures) contain productions of the form $A \to B$, thus it may contain the following sorts of productions:

$$A \to a \qquad \text{or} \qquad A \to aB \qquad \text{or} \qquad A \to \epsilon \qquad \text{or} \qquad A \to B$$

(It is understood that upper case letters are non-terminals, and lower-case letters are terminals).

Solution The natural way of doing this is where a non-terminal can be expanded to the strings which would be accepted by the machine starting at the corresponding state. Thus we get

$$\begin{aligned} S &\to S_0 & S_1 &\to bS_2 \\ S_0 &\to aS_1 & S_2 &\to S_1 \\ S_0 &\to bS_2 & S_2 &\to aS_2 \\ S_1 &\to \epsilon \end{aligned}$$

(ii) Describe how to convert a grammar containing productions of the form $A \to B$ into a grammar which contains no productions of this form, but accepts the same language.

Solution Firstly, if there are any sets of productions which form a "cycle", such as

$$A \to B \quad B \to C \quad C \to A$$

(so that $A \Rightarrow^* A$), replace each of A, B, C wherever it occurs by A.

Secondly, for any production $A \to B$, where the grammar also contains productions $B \to \beta_i$ (for some set of is) replace $A \to B$ by $A \to \beta_i$ for each such i. (Do not delete $B \to \beta_i$!).

(An alternative to this second step is to say: for any production $A \to B$, where the grammar also contains productions $C_i \to \alpha_i$ where α_i contains A, let β_i be α_i with A replaced by B. Then replace $A \to B$ by $C_i \to \beta_i$ for each i. However this step doesn't quite work when A is the starting symbol S).

Question (continued) If your answer to part (a) contains productions of this form, convert your grammar to a right-linear grammar as defined in lectures (ie, not containing productions of the form $A \to B$), which accepts the same language.

Solution Considering the approach above the first step does not apply. So we get

$$\begin{aligned} S &\to aS_1 & \qquad S_1 &\to bS_2 \\ S &\to bS_2 & \qquad S_2 &\to \epsilon \\ S_0 &\to aS_1 & \qquad S_2 &\to bS_2 \\ S_0 &\to bS_2 & \qquad S_2 &\to aS_2 \\ S_1 &\to \epsilon \end{aligned}$$

Now notice that S_0 cannot appear in any sentential form (ie, you can't get a string containing S_0 from S). So you can delete productions $S_0 \to \ldots$ without changing the language. So this gives

$$\begin{aligned} S &\to aS_1 & \qquad S_1 &\to bS_2 \\ S &\to bS_2 & \qquad S_2 &\to \epsilon \\ S_1 &\to \epsilon & \qquad S_2 &\to bS_2 \\ & & \qquad S_2 &\to aS_2 \end{aligned}$$

(iii) Find a left-linear grammar which accepts the same language as this NFA. Your grammar may (unlike the definition in lectures) contain productions of the form $A \to B$, thus it may contain the following sorts of productions:

$$A \to a \quad \text{or} \quad A \to Ba \quad \text{or} \quad A \to \epsilon \quad \text{or} \quad A \to B$$

Solution The natural way of doing this is where a non-terminal can be expanded to the strings which would be accepted by the machine if the corresponding state were a final state. Thus we get

$$\begin{aligned} S &\to S_1 & \qquad S_2 &\to S_0 b \\ S_1 &\to S_0 a & \qquad S_2 &\to S_1 b \\ S_1 &\to S_2 & \qquad S_2 &\to S_2 a \\ S_0 &\to \epsilon \end{aligned}$$

2 A Content Comparison Problem

Here is a *deterministic* PDA that accepts the language over $\{a, b\}$ where each string in the language has more a's than b's. The input strings string will be terminated by a hash symbol (#). (Thus the input alphabet is $\{a, b, \#\}$.)

For this PDA, strings will be deemed to be accepted when the PDA consumes the input and empties the stack.

$$\begin{array}{ll} \delta(q_1, a, Z) \mapsto (q_1, aZ) & \delta(q_1, b, Z) \mapsto (q_1, bZ) \\ \delta(q_1, a, a) \mapsto (q_1, aa) & \delta(q_1, b, a) \mapsto (q_1, \epsilon) \\ \delta(q_1, a, b) \mapsto (q_1, \epsilon) & \delta(q_1, b, b) \mapsto (q_1, bb) \\ \delta(q_1, \#, a) \mapsto (q_2, \epsilon) & \\ \delta(q_2, \_, a) \mapsto (q_2, \epsilon) & \delta(q_2, \_, Z) \mapsto (q_2, \epsilon) \end{array}$$

The idea of the above design is that

i) The PDA will remain in state q_1 while the input is scanned only going to state q_2 when the end-of-input symbol is reached (with an a on the stack).

ii) If, in state q_1, n more of one letter have been seen that the other letter then the stack will contain n copies of that letter.

iii) When the PDA goes to state q_2 excess copies of a will be erased and the string will be accepted.

An alternative presentation of the next state relation is preferred by some people.

	a, Z	b, Z	$\#, Z$	a, a	b, a	$\#, a$	a, b	b, b	$\#, b$	ϵ, a	ϵ, Z
q_1	q_1, aZ	q_1, bZ	—	q_1, aa	q_1, ϵ	q_2, ϵ	q_1, ϵ	q_1, bb	—	—	—
q_2	—	—	—	—	—	—	—	—	—	q_2, ϵ	q_2, ϵ

3 Palindrome Acceptance

The exercise is to design an *NPDA (Nondeterministic PDA)* that recognises nonempty palindromes over the alphabet $\{a, b\}$. (Palindromes are strings that read the same forwards or backwards). The strings that it is required to accept include `ababbaba` and `ababa`.

Strings are deemed to be accepted when the input is consumed and the stack is empty.

Palindrome checking with a PDA has an obvious strategy involving one state in which the symbols up to the mid-point are pushed onto the stack and another state in which a check is made that the remaining symbols in the input match those on the stack (in reverse order). Non-determinism is inevitably required in this problem to 'guess' the midpoint.

The palindrome -recognising PDA (with stack alphabet $\{a, b, Z\}$ and initial state q_0) is defined by its next state relation.

$\delta(q_0, a, Z) \mapsto (q_0, aZ)$
$\delta(q_0, a, a) \mapsto (q_0, aa)$
$\delta(q_0, a, a) \mapsto (q_1, \epsilon)$
$\delta(q_0, a, b) \mapsto (q_0, ab)$

$\delta(q_0, \_, a) \mapsto (q_1, \epsilon)$
$\delta(q_1, a, a) \mapsto (q_1, \epsilon)$
$\delta(q_1, \_, Z) \mapsto (q_1, \epsilon)$

$\delta(q_0, b, Z) \mapsto (q_0, bZ)$
$\delta(q_0, b, a) \mapsto (q_0, ba)$

$\delta(q_0, b, b) \mapsto (q_0, bb)$
$\delta(q_0, b, b) \mapsto (q_1, \epsilon)$
$\delta(q_0, \_, b) \mapsto (q_1, \epsilon)$
$\delta(q_1, b, b) \mapsto (q_1, \epsilon)$

... and here it is in tabular form.

	a, Z	b, Z	a, a	b, a	a, b	b, b	ϵ, a	ϵ, b	ϵ, Z
q_0	q_0, aZ	q_0, bZ	q_0, aa q_1, ϵ	q_0, ba	q_0, ab	q_0, bb q_1, ϵ	q_1, ϵ	q_1, ϵ	—
q_1	—	—	q_1, ϵ	—	—	q_1, ϵ	—	—	q_1, ϵ

3 Turing Machine and Computability

1 Palindrome Acceptance

The exercise is to construct a Turing Machine that recognizes palindromes over the alphabet $\{a, b\}$.

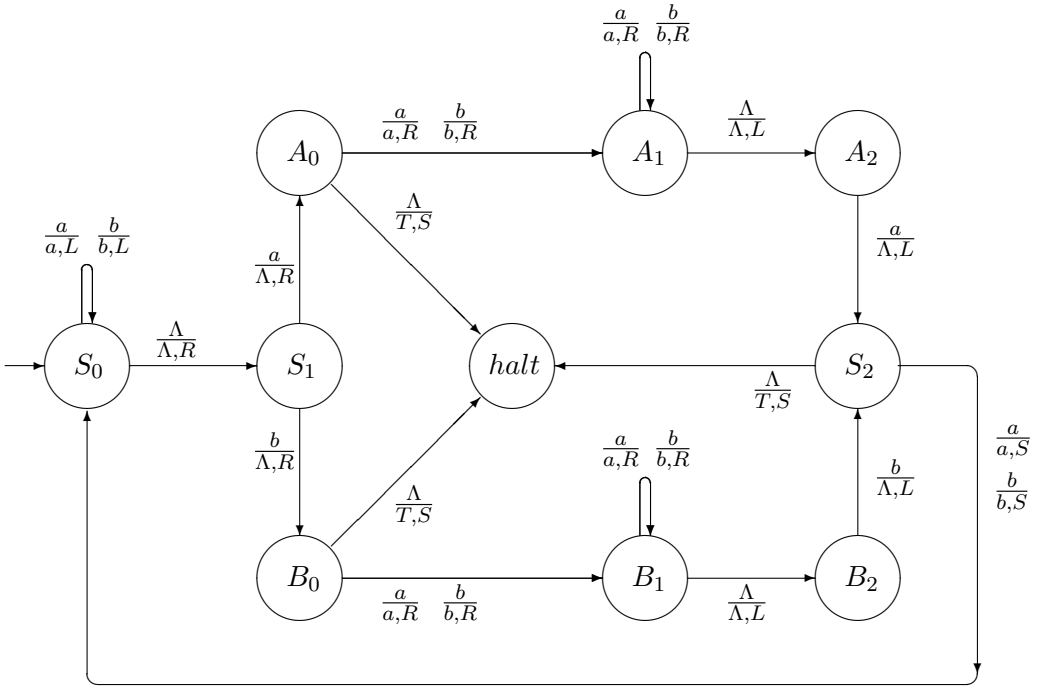
The precondition for the computation is that the tape of the TM will just have a string made up of a's and b's with the read head somewhere over the string.

The machine will halt if (and only if) the string is a palindrome. In the following solution, when the machine halts the tape will have just a single T (for $True$) on the tape and the read head above it.

The machine works by iteratively checking that the leftmost and rightmost symbols on the tape are the same. After each successful match those two symbols are erased. The looping terminates successfully when either

a) At the start of an iteration the leftmost symbol is also the rightmost one.

b) After erasure of matching symbols the tape is blank.



2 Computability

Suppose for contradiction that there exists a TM, T, that decides whether an arbitrary TM, X, ever prints a given letter when provided input, w. Let M be a TM which will construct a new TM, X', as a copy of X whose alphabet is the same as that of X with one new character, α. Let M modify all transitions in X' which cause the machine to halt to also write α to the tape. By the assumption, M could now run T with X and w as input and decide whether it ever writes α to the tape. However, this would mean that M provides a solution to the Halting Problem (since X' printing α would only occur if X had halted). Since the halting problem is undecidable, we have a contradiction, hence, this problem is undecidable.