Cpts 515. 12/2/2020

TSP: impossibility of approximation

Given: a graph $G = (V, V \times V)$ (a complete graph)

where each edge (u, v) has a nonnegative weight $w(u, v) \geq 0$

Goal: Find a HC^* with minimal total weight.

Let $\rho \geq 1$ be a number. Can we design a poly-time alg such that we can find a HC with

$$\text{Weight}(HC) \leq \rho \cdot \text{Weight}(HC^*) ?$$

This is called ρ-approximation.

Thm. TSP doesn't have ρ-approximation unless P = NP.

Proof. Recall the original HC problem which is NP-complete. That is, given a graph $\hat{G}$, Is there a HC for $\hat{G}$?

We construct a complete graph G with the following weight assignments:

$w(u,v) = 1$ when (u,v) is an edge in $\hat{G}$;

$w(u,v) = \underbrace{\rho|V| + 1}$ o.w.

↑ a super large weight

Now, if $\hat{G}$ has a HC, then the HC^* for G with minimal total weight sat:

$$\text{Weight}(HC^*) = |V|.$$

if $\hat{G}$ doesn't have a HC, then the HC^* for G with minimal total weight must sat:

$$\begin{aligned}\text{Weight}(HC^*) &\geq \rho \cdot |V| + 1 \\ &> \rho \cdot |V|\end{aligned}$$

Hence, the result of ρ-approximation of G can tell whether $\hat{G}$ has a HC.

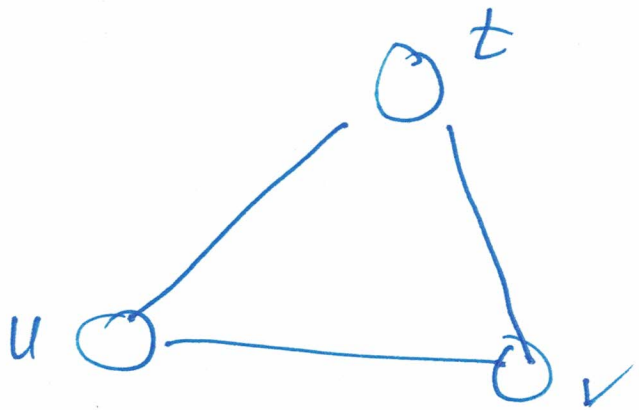
"$> \rho \cdot |V|$?"

□.

A special case of TSP:

metric-TSP where the weight on edges sat. triangle inequality:

$$w(u,v) \leq w(u,t) + w(t,v)$$



① metric-TSP is NP-complete.

② metric-TSP has 2-approx.

Recall: an instance for metric-TSP is still a complete graph as before, where each edge has a weight ≥ 0.

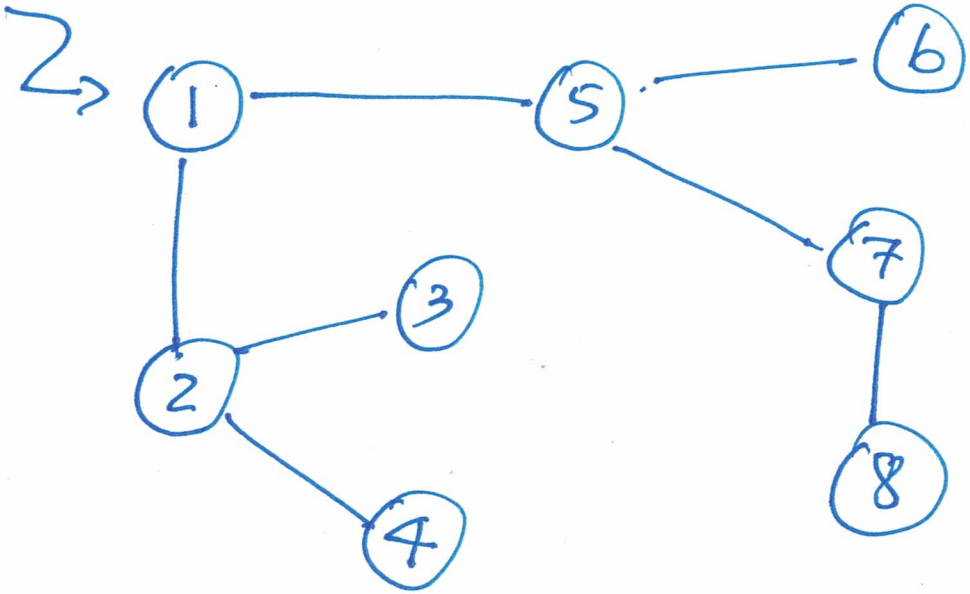
ALG: (for 2-approx)

Run minimal spanning tree alg (Prim's Alg) on the complete graph G;

"Traverse" the minimal spaning tree to obtain the HC:

↳ next page.

Suppose that the following is the minimal spanning tree obtained.



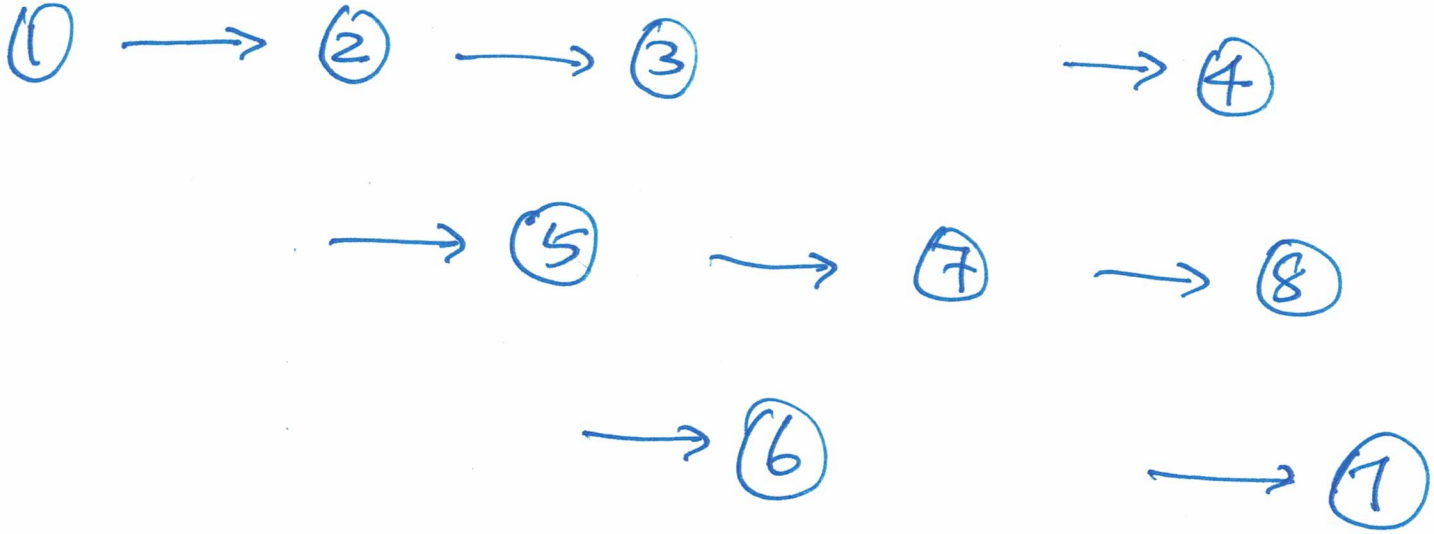
traverse the tree using DFS:

1 → 2 → 3 → 2 → 4 → 2

→ 1 → 5 → 7 → 8 → 7

→ 5 → 6 → 5 → 1

delete all the repeated nodes:



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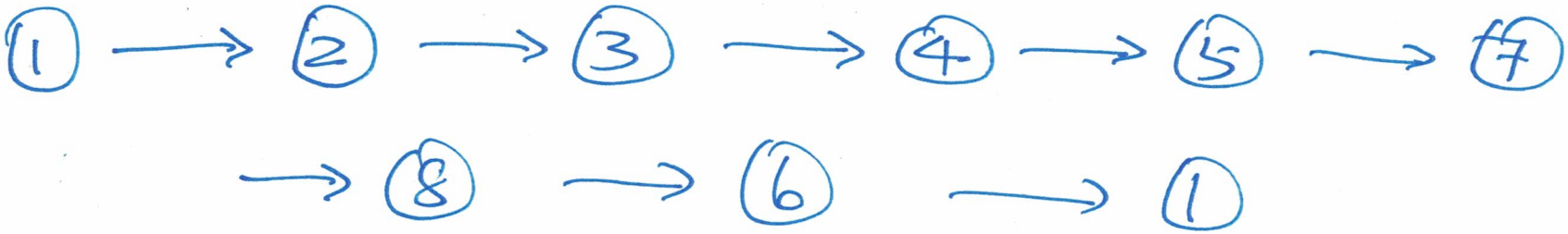
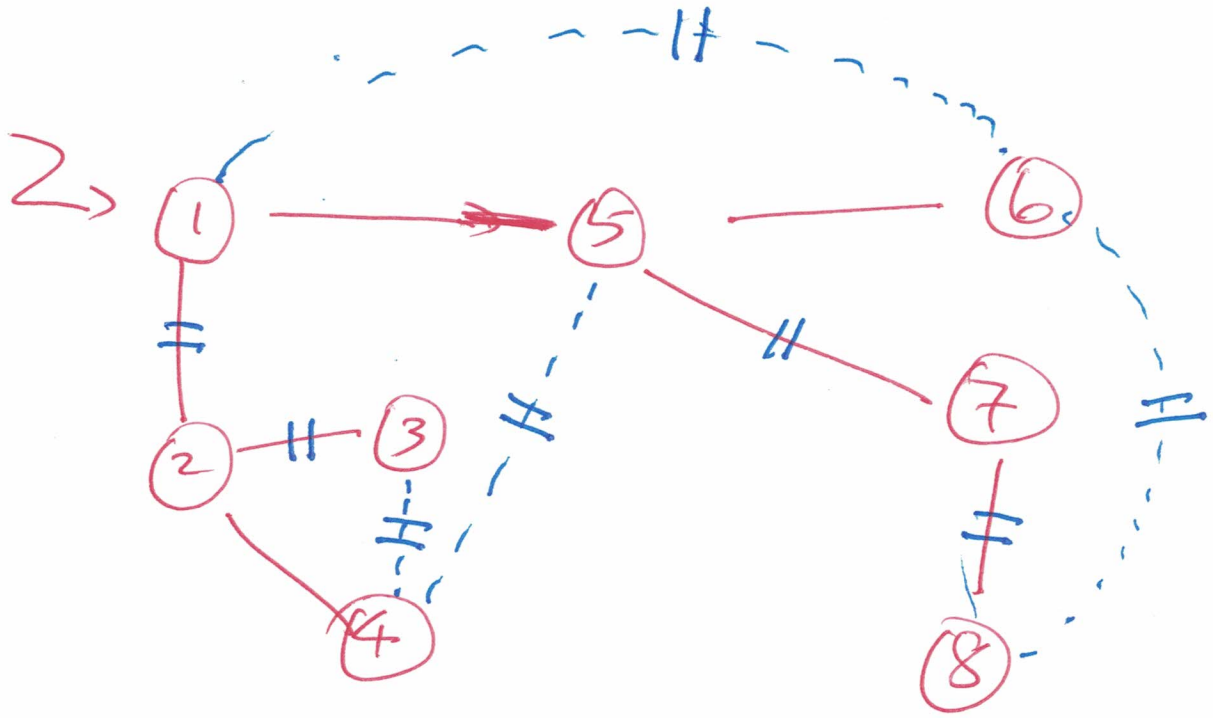
recount it:





This is part of the complete graph G, and the newly obtained sequence is an HC of G.

The total weight of this HC $\leq$ 2 · total weight of the minimal spaning tree.

(for instance, Weight (③, ④) $\leq$ weight (②, ③) + weight (②, ④))

and total weight of the minimal spanning tree

$\leq$ total weight of HC^*

(why? Since the HC^*, after dropping an edge, is a ~~minimal~~ spanning tree).

"Rounding" ⟵ Full poly-time approximation

Which is to find $(1 \pm \varepsilon)$-approx

in poly-time (in $\frac{1}{\varepsilon}$ and the size of the problem), for any $\varepsilon > 0$.

Knapsack problem: (weight, value)

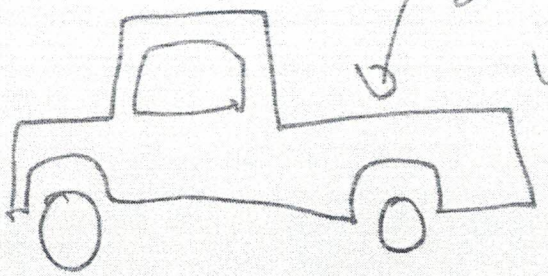
Given: $w_i, v_i, 1 \le i \le n, W > 0$

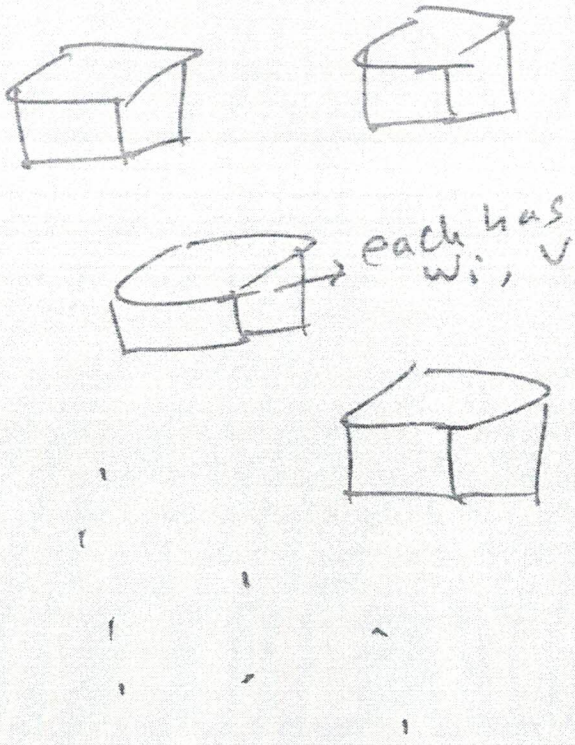
(w_i, v_i: positive integers)

Goal: Find a subset I such that

$$\sum_{i \in I} v_i \text{ is maximal with}$$

$$\sum_{i \in I} w_i \le W$$





Knapsack is NP-complete.

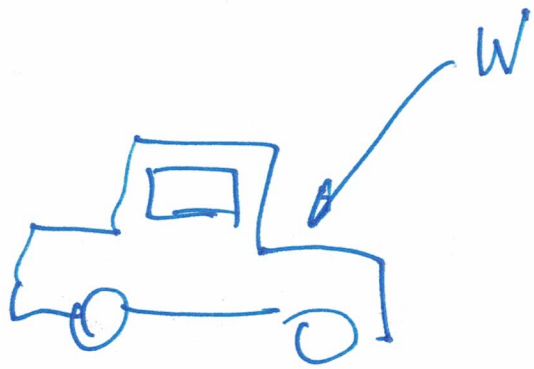
↕ the decision version.

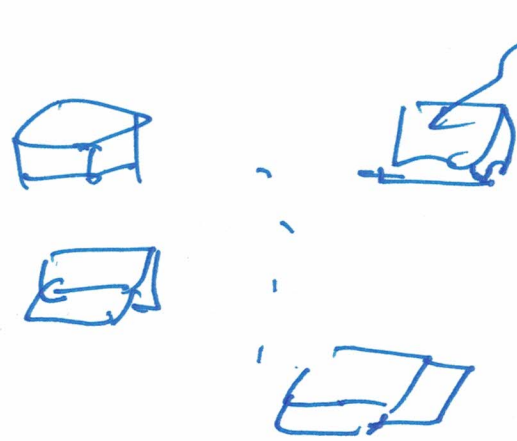
↕ V is achievable: $\sum_{i \in I} v_i \not\leq V$ &

↑ given

$\sum_{i \in I} w_i \leq W$ for some I.

(3SAT → subset-sum → Knapsack)





Knapsack is NP-complete. (Google: 3SAT → subset sum → knapsack).

↑ the decision version

given V, whether V is achievable:

$$\sum_{i \in I} v_i \geq V \quad \text{and}$$

$$\sum_{i \in I} w_i \leq W \quad \text{for some } I.$$

Pay attention: input size = log of all the numbers in the problem.

For instance, for the knapsack problem, input size

$$= \sum_i \log w_i + \sum_i \log v_i + \log W.$$

Set-up: We are given n boxes, each of which has a weight w_i and a value v_i.

Dynamic programing:

observation: if I choose a subset S_1 of boxes and a subset S_2 of boxes

with total weight W_1, value V_1 and ath total weight W_2, value V_2, respectively,

and when $W_1 \leq W_2$ but $V_1 \geq V_2$,
(the first set of boxes are lighter and more valuable), then I would certainly put S_1 on my car! In this case, I write

$$S_1 \geq S_2$$

$\uparrow$ dormmates.

Observation: Given S_i, and a box $(w, v) = Box$ that is not in S_i, we write

$S_i + (w, v)$ to denote the new subset of boxes: $S_i \cup \{Box\}$.

Observation: original set-up of the problem

$Box_1, Box_2, \ldots, Box_n$

$\|$

$(w_1, v_1) \quad \ldots \quad (w_n, v_n)$

we define $S_\emptyset$ is the empty set of Boxes with 0 total weight and 0 total value.