Experiment 81 – ELEC207 coursework

Design of a Stable Martian Segway

Report template

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1. Mathematical Modelling

A) Please define the values for *l, m* and t_s that you will use for your coursework. [1E]

According to the lab script:

l (meter) is the day of the month when I was born

m (kg) is the month of the year when I was born

t_S (seconds) is the year I was born divided by 250

Because my birthday is 5 March 2001, which means :

l (meter) = 3m

m (kg) = 5kg

$t_S = \frac{\text{year}}{250} = \frac{2001}{250} = 8.004\text{s}$

B) Now derive the transfer function, $H(s)=\theta(s)/T(s)$, of the Segway in terms of *l*, *m* and *g.* [1E]

It is assumed that the pendulum can be approximated as a point mass. And $\theta(t)$ is the angle between the line, which connects the center of mass of the person and the center of mass of the Segway, and the vertical direction. According to the lab script, the angular acceleration of pendulum can be approximated as being defined by:

$$\frac{g}{l}\theta(t) + T(t) = m\frac{d^2\theta(t)}{dt^2} \tag{1}$$

Applying the Laplace transformation to Equation (1):

$$\frac{g}{l}\theta(s) + T(s) = m * s^2 * \theta(s) \tag{2}$$

Simplifying Equation (2) gives:

$$(\frac{g}{l} - m * s^2)\theta(s) = -T(s) \tag{3}$$

According to the transfer function $\mathrm{H(s)} = \frac{\theta(s)}{T(s)}$,

$$\mathrm{H(s)} = \frac{\theta(s)}{T(s)} = \frac{1}{m * s^2 - \frac{g}{l}} \tag{4}$$

C) Using your values for *l* and *m* along with *g*=3.711 ms^{-2}, write the transfer function with the denominator and numerator of your transfer function in polynomial form. [1E]

Because the transfer function from B) is $\mathrm{H(s)} = \frac{\theta\ (s)}{\mathrm{T(s)}} = \frac{1}{\mathrm{m} * \mathrm{s}^2 - \frac{\mathrm{g}}{\mathrm{l}}}$ and the value of g is $3.711\ \mathrm{ms}^{-2}$, the final transfer function in polynomial form is :

$$\mathrm{H(s)} = \frac{\theta(s)}{T(s)} = \frac{1}{3s^2 - 0.7422} \tag{5}$$

From the Equation (5), the denominator is $3\mathrm{s}^2 - 0.7422$ and numerator is 1 in polynomial form.

D) Calculate the position of the poles for your Segway and plot the poles on the complex plane. [1E]

Characteristic Function is the denominator of the Transfer Function and the poles of the system can be calculated through the solutions of the Characteristic Equation.

Because the denominator of my transfer function is $3\mathrm{s}^2 - 0.7422$, the position of poles can be calculated:

$$3s^2 - 0.7422 = 0 \tag{6}$$

Simplifying Equation (6) gives:

$$s^2 = 0.2474 \tag{7}$$

Therefore, the poles are:

$$s_1 \approx -0.4974, s_2 \approx 0.4974 \tag{8}$$

In different way, the factorize form $H(s) = \frac{\theta\ (s)}{T(s)} = \frac{1}{(\sqrt{3}*s-0.8615)(\sqrt{3}*s+0.8615)} = \frac{0.5774}{(s-0.4974)(s+0.4974)}$ can also be used to get the positions of poles according to the Characteristic Equation. The poles calculated are the same as Equation (8). Therefore, according to the above calculation, the coordinates of the two poles are (-0.4974,0),(0.4974,0).

Then figure for plotting the poles on the complex plane is :

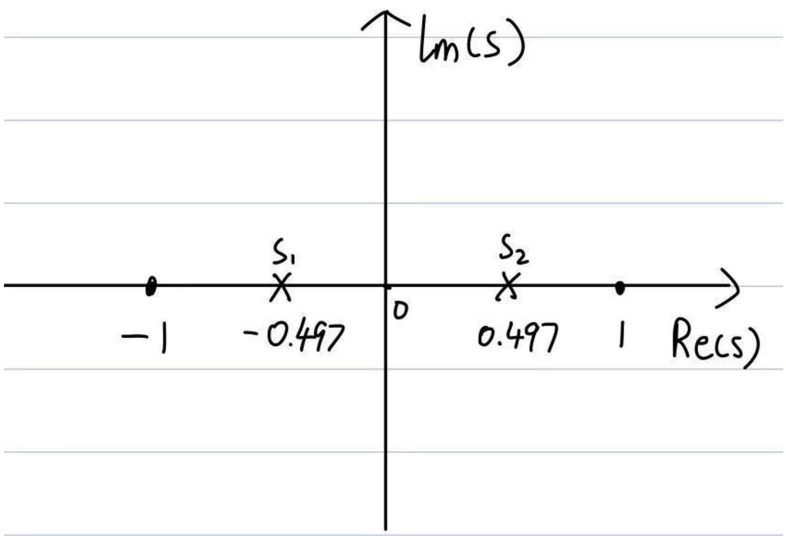

Figure 1: The poles on the complex plane

2. Validating that the Open-loop System is Unstable

E) Insert a picture of the time-response of your Segway to the unit-step. [2E]

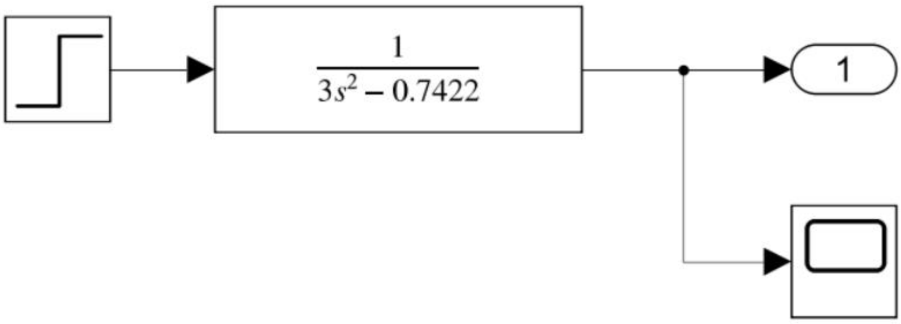

Figure 2: The simulink blocks for the system

As can be seen from the figure, the transfer fun is $\mathrm{H(s)} = \frac{\theta(s)}{T(s)} = \frac{1}{3s^2-0.7422}$. Since this system is an open loop system, the output of the transfer fun will be used directly as the final output.

In this question, I used the "step" by setting the Step Time to 1s, the start value to 0 and the final value to 1. After correctly entering my transfer function, the results obtained is as follows.

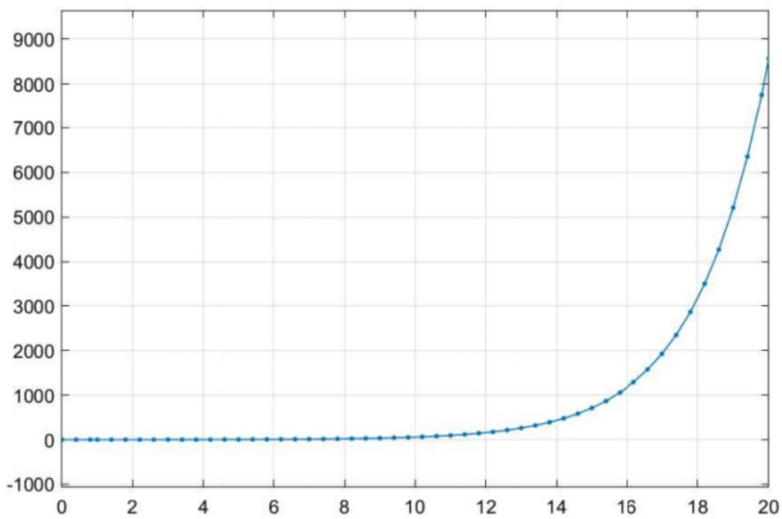

Figure 3: The time-response of my Segway to the unit-step

E) Comment on whether this time-response indicates that the open-loop system is stable. [1M]

If the output time-response is bounded for all bounded inputs for a linear system, the system is said to be stable. As can be seen from Figure 2, the value of transfer time-response increases as the time increases, which indicates that the system is unstable. A stable system, whose time-response is bound to remain **smooth** as time increases, does not tend towards infinity as this system does with increasing time-response. Therefore, the system is unstable.

From another perspective, if there is one pole in the right complex plane, which also can be said that the system is not stable. From the above question, there is a pole $s_2 \approx 0.4974$, which is in the right complex plane. That also indicates the system is unstable. As to why the presence of the pole on the right side of the complex plane causes instability in the system, a short explanation is written below.

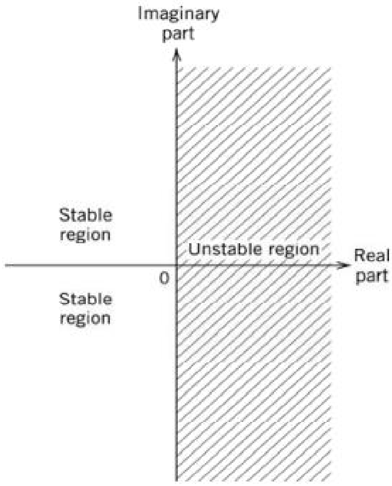

Figure 4: The stable region and the unstable region [1]

As the position of the root also provides an indication of how fast the transient response is. A real root at s = p1 corresponds to a closed-loop time constant of τ1 = 1/p1. Thus, a real root close to the imaginary axis leads to a slow response. Similarly, a complex root close to the imaginary axis corresponds to a slow response pattern. The further the complex roots are from the real axis, the more oscillatory the transient response will be. Once a root appears in the right half of the complex plane, the time response will become increasingly oscillatory as time increases. If the oscillatory amplitude of the time-response grows with time, the output time-response will not be bounded for all bounded inputs.

Alternatively, we can prove whether the system is stable or not by using mathematical equation verification. First, the output of the system will be :

$$\mathrm{Y(s)} = \frac{1}{\mathrm{s}}\frac{1}{(\sqrt{3}\mathrm{s}-0.8615)(\sqrt{3}*\mathrm{s}+0.8615)} = \frac{1}{\mathrm{s}}\frac{0.5774}{(\mathrm{s}-0.4974)(\mathrm{s}+0.4974)} \tag{9}$$

Where the input is a step function: $\frac{1}{\mathrm{s}}$,then the partial fraction need to be found:

$$\mathrm{Y(s)} = \frac{\mathrm{A}}{\mathrm{s}} + \frac{\mathrm{B}}{\mathrm{s}-0.4974} + \frac{\mathrm{C}}{\mathrm{s}+0.4974} \tag{10}$$

Therefore,

$$As(s+0.4974)(s-0.4974) + Bs(s+0.4974) + Cs(s-0.4974) = 0.5774 \tag{11}$$

Simplifying the above Equation gives:

$$(A + B + C)s^2 + (0.4974B - 0.4974C)s - 0.2474A = 0.5774 \tag{12}$$

There is no s^2 and s term for the equation, therefore,

$$\mathrm{A + B + C = 0} \tag{13}$$

$$\mathrm{B = C} \tag{14}$$

$$-0.2474\mathrm{A} = 0.5774 \tag{15}$$

Therefore, the values of A,B and C will be:

$$\mathrm{A} = \text{-2.338} \tag{16}$$

$$\mathrm{B = C} = 1.1669 \tag{17}$$

After solving the values of A,B,C, the partial fraction will be:

$$\mathrm{Y(s)} = \frac{-2.3338}{\mathrm{s}} + \frac{1.1669}{\mathrm{s}-0.4974} + \frac{1.1669}{\mathrm{s}+0.4974} \tag{18}$$

Using inverse Laplace Transform, we can get:

$$\mathrm{y(t)} =- 2.338\mathrm{u(t)} + 1.1669\mathrm{e}^{0.4974\mathrm{t}}\mathrm{u(t)} + 1.1669\mathrm{e}^{-0.4974\mathrm{t}}\mathrm{u(t)} \tag{19}$$

The time response of the system consists two responses, which are transient response and steady state response. From the above Equation, $-2.338\mathrm{u(t)}$ is the steady state response and $1.1669\mathrm{e}^{0.4974\mathrm{t}}\mathrm{u(t)} + 1.1669\mathrm{e}^{-0.4974\mathrm{t}}\mathrm{u(t)}$ is the transient response.

To control the system is stable, the following Equation need to be satisfied:

$$\lim_{n\to\infty} \text{transient(t)} = 0 \tag{20}$$

However, for our system, transient response : $1.1669e^{0.4974t}u(t) + 1.1669e^{-0.4974t}u(t)$:

$$\lim_{n\to\infty} 1.1669e^{0.4974t}u(t) + 1.1669e^{-0.4974t}u(t) \to \infty \tag{21}$$

Therefore, the system is unstable through using this method. To control the system stable, the controller must be added.

3. Ensuring that the Closed-loop System is Stable Using PID Control

G) Write the closed-loop transfer function for your Segway in terms of K_p, K_I and K_D as a ratio of polynomials in s. Ensure that the highest order term in s in the denominator has a coefficient of unity. [3M]

In this question, a PID controller is used with transfer function C_s:

$$C_s = K_p + \frac{K_i}{s} + K_d * s \tag{22}$$

where K_p, K_i and K_d are respectively the proportional, integral and derivative control constants.

For a negative feedback closed-loop transfer function:

$$\frac{\theta(s)}{X(s)} = \frac{C_s * H(s)}{1 + C_s * H(s)} \tag{23}$$

Where $C_s * H(s)$ is the open-loop transfer function with input $X(s)$, output $\theta(s)$

Substituting C_s and $H(s)$ into the above Equation (23):

$$\frac{\theta(s)}{X(s)} = \frac{(K_p + \frac{K_i}{s} + K_d * s) * \frac{1}{3s^2 - 0.7422}}{1 + (K_p + \frac{K_i}{s} + K_d * s) * \frac{1}{3s^2 - 0.7422}} \tag{24}$$

Simplifying Equation (24) gives:

$$\frac{\theta(s)}{X(s)} = \frac{\frac{1}{3}(\mathrm{K_d}*\mathrm{s}^2+\mathrm{K_p}*\mathrm{s}+\mathrm{K_i})}{s^3-0.2474*s+\frac{1}{3}(\mathrm{K_p}*\mathrm{s}+\mathrm{K_i}+\mathrm{K_d}*\mathrm{s}^2)} \tag{25}$$

Or:

$$\frac{\theta(s)}{X(s)} = \frac{\frac{1}{3}*\mathrm{K_d}*\mathrm{s}^2+\frac{1}{3}*\mathrm{K_p}*\mathrm{s}+\frac{1}{3}*\mathrm{K_i}}{s^3+\frac{1}{3}*\mathrm{K_d}*\mathrm{s}^2+(-0.2474+\frac{1}{3}\mathrm{K_p})s+\frac{1}{3}*\mathrm{K_i}} \tag{26}$$

H) What is the characteristic polynomial that would result in these pole positions? [1M]

According to the lab script, to ensure the system is stable, the placed poles will be at $s_1 =-1$, $s_2 = -2$, $s_3 =-3$. Therefore, the characteristic equation will be:

$$(s+1)*(s+2)*(s+3)=0 \tag{27}$$

Simplifying Equation (27) gives the characteristic polynomial:

$$s^3+6*s^2+11*s+6=0 \tag{28}$$

I) By equating the coefficients in the closed-loop transfer function's denominator and this characteristic function, deduce values for K_p, K_I and K_D which will ensure that the closed-loop system is stable. [3M]

The closed-loop transfer function's denominator is:

$$s^3+\frac{1}{3}*\mathrm{K_d}*\mathrm{s}^2+(-0.2474+\frac{1}{3}\mathrm{K_p})*s+\frac{1}{3}*\mathrm{K_i} \tag{29}$$

Equating the coefficients in transfer function's denominator (29) and characteristic polynomial (28) can get:

$$s^3 = s^3 \tag{30}$$

$$\frac{1}{3}*\mathrm{K_d}*\mathrm{s}^2 = 6*s^2 \tag{31}$$

$$(-0.2474 + \frac{1}{3} * \mathrm{K_p}) * s = 11 * s \tag{32}$$

$$\frac{1}{3} * \mathrm{K_i} = 6 \tag{33}$$

Equation (31) indicates:

$$\mathrm{K_d} = 18 \tag{34}$$

Equation (32) indicates:

$$\mathrm{K_p} = 33.7422 \tag{35}$$

Equation (33) indicates:

$$\mathrm{K_i} = 18 \tag{36}$$

Since we have chosen the coordinates of the poles as (-1, 0), (-2, 0), (-3, 0), which means that these poles all appear on the left half of the coordinate axis. According to the previous conclusion, the system is stable when all the poles of the system appear on the left side. Therefore, our calculation of $\mathrm{K_p}$,$\mathrm{K_d}$, $\mathrm{K_i}$ using stable values of poles will also make the system stable.

4. Validating That the Closed-loop System is Stable

J) Insert a picture of the time-response of your closed-loop system to the unit-step. [2M]

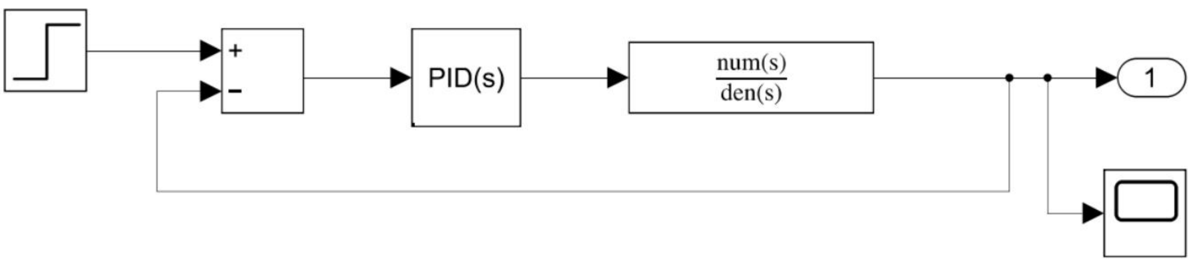

Figure 5: The simulink blocks for the closed-loop system

I put each parameter of $\mathrm{H(s)} = \frac{\theta(s)}{T(s)} = \frac{1}{3s^2-0.7422}$ in the "Transfer Fun" block correctly. Additionally, the output of the transfer fun will be back into the system as a negative feedback.

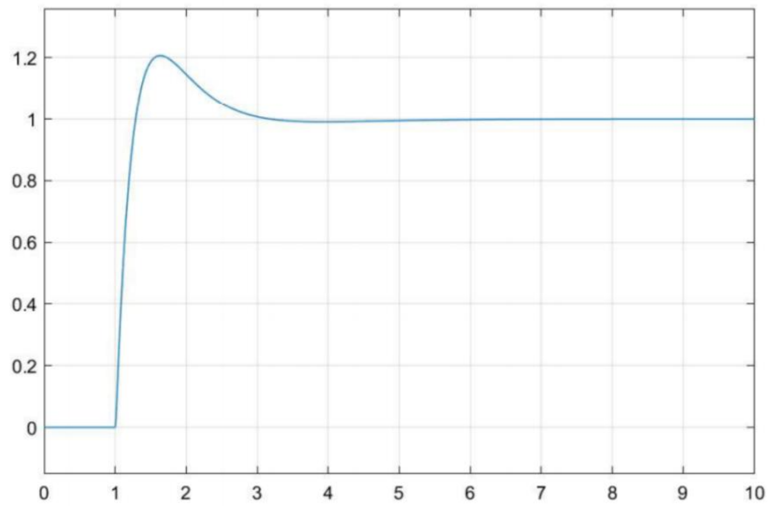

Figure 6: The time-response of my closed-loop system

The figure above shows that the time-response of my closed-loop system is now stable, because as time increases, the time-response of the system gradually converges to a constant value of 1. This means that the output time-response is bounded for all bounded inputs. In addition, since the start time of the system is 1, the figure also shows that the setting time of the system is about 2s (3-1=2s). The value of overshoot is about 21.34%, and the maximum value of time-response is about 1.2 at a time of about 1.6s. The following is the specific explanation to calculate the value of overshoot on Matlab [2]:

For a positive-going (positive-polarity) pulse, overshoot expressed as the percentage will be :

$$\text{Percentage } = 100 * \frac{\mathrm{O}-\mathrm{s}_2}{\mathrm{s}_2-\mathrm{s}_1} \tag{37}$$

Where O is the maximum deviation greater than the high level, S2 is the high level and S1 is the low level. See below for a detailed explanation.

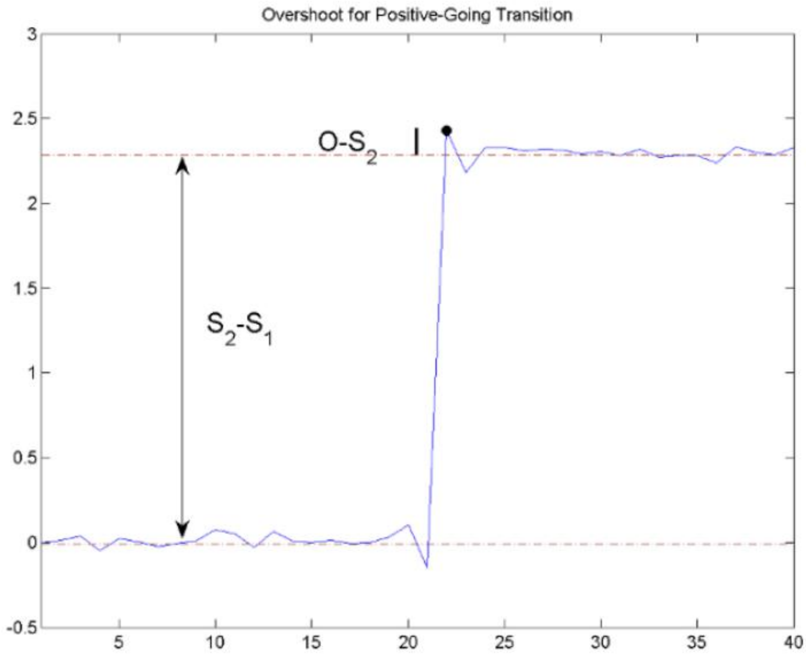

Figure 7: The calculation of overshoot [2]

The red dotted line shows the estimated stated level. The black arrows on both sides indicate the difference between the high and low levels. The solid black line shows the difference between the overshoot value and the high level.

5. Optimising the Time-Response Using Root Locus

K) Calculate the positions of the open-loop zeros (ie the zeros of $C(s)H(s)$) for the values of *l*, *m*, *K$_p$*, *K$_I$* and *K$_D$* that you have used. [1M]

From the above equations, $\mathrm{C_s} = \mathrm{K_p} + \frac{\mathrm{K_i}}{\mathrm{s}} + \mathrm{K_d} * \mathrm{s}$ and $\mathrm{H(s)} = \frac{\theta(s)}{T(s)} = \frac{1}{3s^2-0.7422}$, therefore:

$$\mathrm{C_sH(s)} = \frac{\mathrm{K_p}+\frac{\mathrm{K_i}}{\mathrm{s}}+\mathrm{K_d}*\mathrm{s}}{3s^2-0.7422} \tag{38}$$

Simplifying Equation (38) gives:

$$\mathrm{C_sH(s)} = \frac{\mathrm{K_p}*\mathrm{s}+\mathrm{K_i}+\mathrm{K_d}*\mathrm{s}^2}{3s^3-0.7422s} \tag{39}$$

Substituting the values of $\mathrm{K_p}$, $\mathrm{K_i}$ and $\mathrm{K_d}$ into Equation (39) gives:

$$\mathrm{C_sH(s)} = \frac{18*s^2+33.7422*s+18}{3s^3-0.7422s} \tag{40}$$

To get the open-loop zeros, the following Equation need to be calculated:

$$18 * s^2 + 33.7422 * s + 18 = 0 \tag{41}$$

After solve the Equation (41), the zeros are:

$$s_1 =- 0.94 - 0.35i, s_2 =- 0.94 + 0.35i \tag{42}$$

The position of the zeros in the complex plane is as follows：

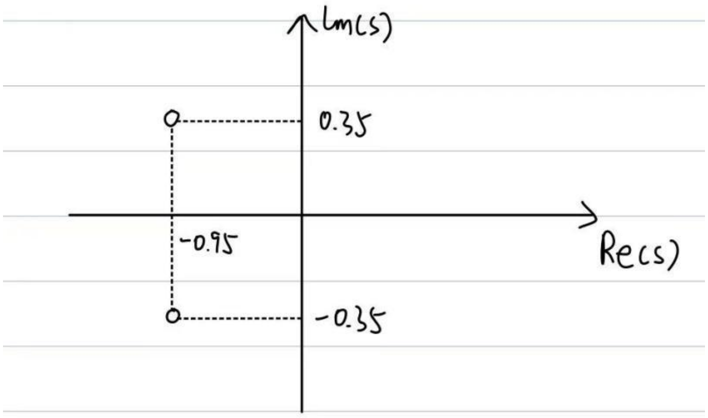

Figure 8: The positions of the open-loop zeros

Therefore, the coordinates of the two zeros are: (-0.95,-0.35) and (-0.95,0.35) respectively

L) State the positions of the open-loop poles (ie the poles of $C(s)H(s)$) for the values of *l* and *m* that you have used. [2E]

According to the Equation (30): $\mathrm{C_sH(s)} = \frac{18*s^2+33.7422*s+18}{3s^3-0.7422s}$, to get the open-loop poles, the following Equation need to be calculated:

$$3s^3 - 0.7422s = 0 \tag{43}$$

After solve the Equation (43), the zeros are:

$$\mathrm{s}_1 = 0,\ \mathrm{s}_2 =- 0.497, \mathrm{s}_3 =- 0.497 \tag{44}$$

Therefore, the positions of all the open-loop poles are on the real axis, and two of the open-loop poles are symmetric about the imaginary axis. The coordinates of the three poles are: (0,0), (-0.497,0) and (0.497,0). The position of the poles in the complex plane is as follows：

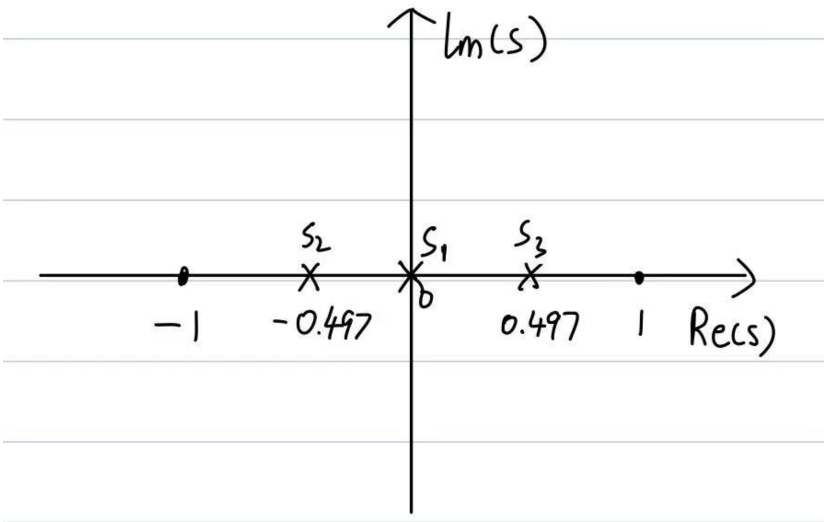

Figure 9: The positions of three open-loop poles

M) Sketch the root locus for $C(s)H(s)$ and identify the points on the root locus that are such that $Re(s) =- 4/t_s$. [3M]

From the above question, the value of t_s is 8.004, therefore,

$$Re(s) = \frac{-4}{8.004} \approx 0.4998\ s^{-1} \tag{45}$$

From the question above, the system contains three poles and two zeros. The coordinates of the three poles are (0,0), (-0.497,0) and (0.497,0). The coordinates of the two zeros are (-0.95,-0.35) and (-0.95,0.35). Therefore, it is first necessary to mark the position of the zeros and poles on the coordinate paper.

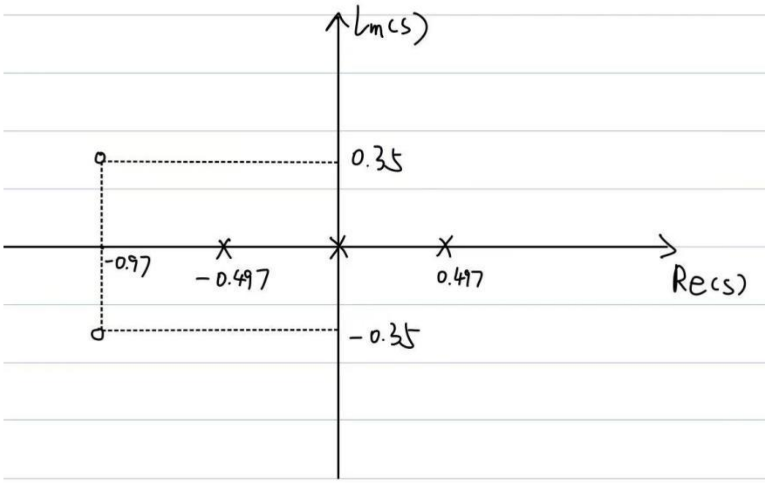

Figure 10: Mark the position of the zeros and poles

Next, we need to know the number of branches. According to the contents of lecture 9 [3], the root locus branches begin from the open-loop poles to the open-loop zeros. Therefore, the number of root locus branches is therefore equal to the number of finite open-loop poles or the number of open-loop zeros, whichever is greater. In this system, the number of poles is three, which is greater than the number of zeros(two). So the system has a total of three branches.

Then, we need to identify and draw the real axis root locus branches. The point is at the root locus when the angle of the open-loop transfer function is odd multiple of 180°. If an odd number of open-loop poles and zeros exists to the left side of one point and are in the real axis , then this point is on the branch of root locus. Thus, the branch of the point satisfying the above is the real axis of the branch of the root locus. Since the system has an imaginary part of the poles, which means there will be a complex conjugate pole. Thus, the root locus will be symmetric about the real axis.

Moreover, the centroid and the angle of the asymptotes also need to be found. Since the number of poles in this system is greater than the number of zeros, 2 (The number of zeros) branches of root locus will start from the open poles to the open loop zeros. 3-2=1 (The number of poles minus the number of zeros) branches of root locus will start from open loop poles to the infinite.

The position of centroid is calculated as follows:

$$\sigma_a = \frac{\sum_{i=1}^{m} p_i - \sum_{i=1}^{n} z_i}{m-n} \tag{46}$$

Where $\sum_{i=1}^{m} p_i$ is the sum of real part of open loop poles and $\sum_{i=1}^{n} z_i$ is the sum of real part of open loop zeros. m is the number of poles and n is the number of zeros.

Therefore, the position of centroid in this system is：

$$\sigma_a = \frac{-0.497+0+0.497+0.97+0.97}{3-2} = 1.94 \tag{47}$$

The coordinates of centroid is (1.94,0).The formula for the angle of asymptotes is:

$$\theta_a = \frac{(2k+1)*\pi}{m-n} \tag{48}$$

Therefore, the angle of asymptotes in this system is:

$$\theta_a = \frac{(2k+1)*\pi}{3-2} = (2k+1)*\pi \tag{49}$$

Next, the intersection points of branches with an imaginary axis also need to be found. The intersection of the root locus with the imaginary axis and the K value corresponding to this point can be calculated through using the Routh array method. If all elements of any row in the Routh array are zero, then the root locus branch intersects the imaginary axis. In this system, the root locus has intersections with the imaginary axis.

More importantly, we need to find Break-away and Break-in points. If there is a real root locus branch between the two open-loop poles, there will be a break away point between two poles. If there is a real root locus branch between the two open-loop zeros, there will be a break in point between two zeros. When calculating the value of K, the following equation will be used:

$$\mathrm{K} = -\frac{1}{\mathrm{C_s H(s)}} \tag{50}$$

Substituting $\mathrm{C_s H(s)}$ into Equation (50) gives:

$$\mathrm{K} = -\frac{3\mathrm{s}^3 - 0.7422\mathrm{s}}{18\mathrm{s}^2 + 33.7422\mathrm{s} + 18} \tag{51}$$

The derivative of K with respect to s is:

$$\frac{\mathrm{dK}}{\mathrm{ds}} = -\frac{(9\mathrm{s}^2 - 0.7422)*(18\mathrm{s}^2 + 33.7422\mathrm{s} + 18) - (3\mathrm{s}^3 - 0.7422\mathrm{s})*(36\mathrm{s} + 33.7422)}{(18\mathrm{s}^2 + 33.7422\mathrm{s} + 18)^2} \tag{52}$$

The value of s will be calculated when $\frac{\mathrm{dK}}{\mathrm{ds}} = 0$:

$$(9\mathrm{s}^2 - 0.7422)*(18\mathrm{s}^2 + 33.7422\mathrm{s} + 18) - (3\mathrm{s}^3 - 0.7422\mathrm{s})*(36\mathrm{s} + 33.7422) = 0 \tag{53}$$

Solve Equation(53) gives:

$$s_1 =-2.43,\ \ s_2 =-1.22, s_3 =-0.35\ ,\ \ s_4 = 0.24 \tag{54}$$

In the previous step, we have drawn the real axis root locus branches. Since the root locus does not have branches at (-0.35, 0), the solution $s_3 =-0.35$ is discarded, which means that the system has (-2.43, 0), (-1.22, 0), (0.24) Break-away and Break-in points above. After finding the above Break-away and Break-in points, we need to find the angle of departure and arrival. The formula to calculate the angels are as follows:

$$\Phi_d = 180^{\circ} - \Phi \tag{55}$$

$$\Phi_a = 180^{\circ} + \Phi \tag{56}$$

Where,

$$\Phi = \sum \Phi_P - \sum \Phi_Z \tag{57}$$

The departure and arrival angles can be calculated at the complex conjugate open-loop poles and complex conjugate open-loop zeros respectively.

After each step of the analysis above, the root locus is sketched as follows：

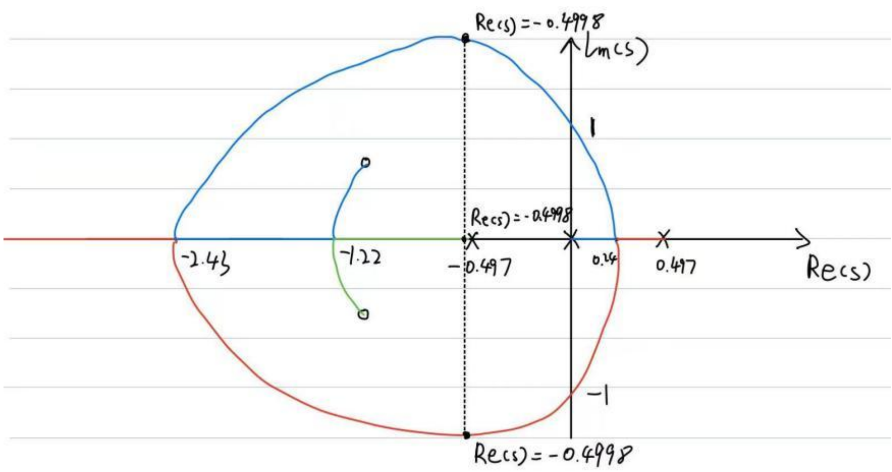

Figure 11: The the root locus for $C(s)H(s)$ and the points on the root locus that are such that $\mathrm{Re}(s) =-4/t_s$

Where the three different coloured lines represent the three different branches of the root locus. Break in and break out points can also be seen on the above figure. At the same time, I have also approximated all three points that $\mathrm{Re}(s) = -4/t_s$ on the above figure. In order to sketch the root locus more accurately, Matlab is also used to sketch the root locus. The root locus simulated by Matlab is as follows:

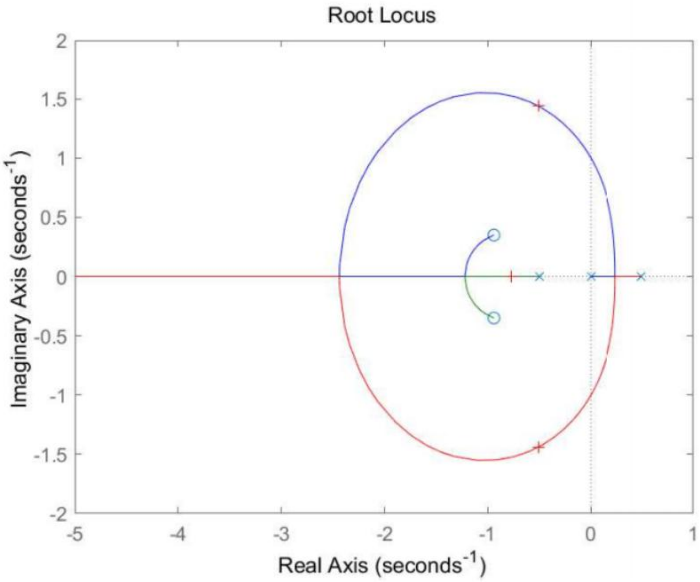

Figure 12: The simulated root locus for $C(s)H(s)$

The three points that $\mathrm{Re}(s) = -4/t_s$ on the root locus are marked in the figure below, as required by the lab script:

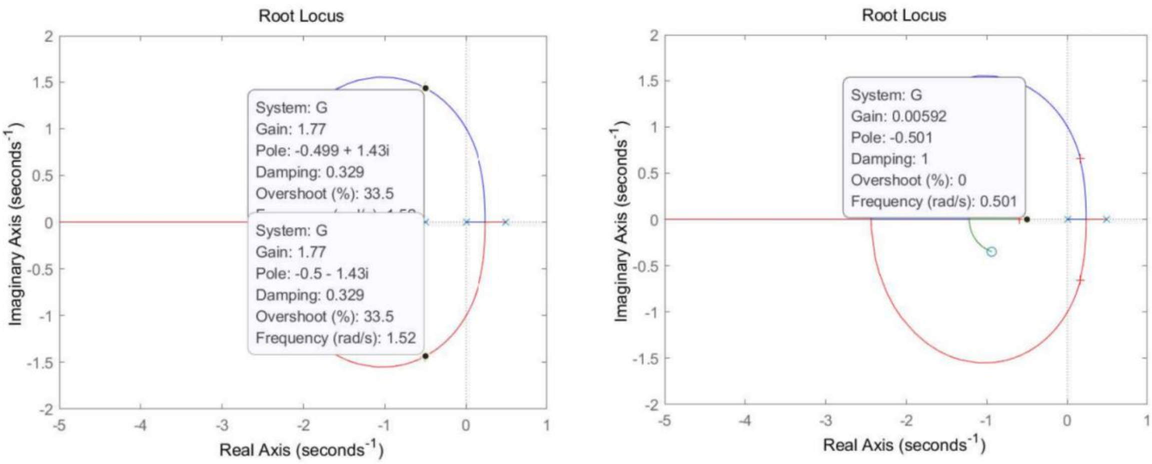

Figure 13: the points on the root locus that are such that $\mathrm{Re}(s) = -4/t_s$

The figure above shows that the three points at root locus are **(-0.5,1.43),(-0.5,-1.43),(-0.5,0)** when $\mathrm{Re}(s) = -4/t_s$. The coordinates of these points will also be calculated in the following question.

The two points at the top and bottom have very close values of gain, damping and overshoot. The value of damping is approximately equal to 1 and the values of gain and overshoot are approximately to 0 in the point on the axis.

N) Write the open-loop transfer function, $C(s)H(s)$, as a ratio of polynomials in s. [1H]

From the above equations, $\mathrm{C_s} = \mathrm{K_p} + \frac{\mathrm{K_i}}{\mathrm{s}} + \mathrm{K_d} * \mathrm{s}$ and $\mathrm{H(s)} = \frac{\theta(s)}{T(s)} = \frac{1}{3s^2 - 0.7422}$, therefore:

$$\mathrm{C_sH(s)} = \frac{\mathrm{K_p} + \frac{\mathrm{K_i}}{\mathrm{s}} + \mathrm{K_d} * \mathrm{s}}{3s^2 - 0.7422} \tag{58}$$

Simplifying Equation (53) gives:

$$\mathrm{C_sH(s)} = \frac{\mathrm{K_p} * \mathrm{s} + \mathrm{K_i} + \mathrm{K_d} * \mathrm{s}^2}{3s^3 - 0.7422s} \tag{59}$$

Substituting the values of $\mathrm{K_p}$, $\mathrm{K_i}$ and $\mathrm{K_d}$ into Equation (54) gives:

$$\mathrm{C_sH(s)} = \frac{18 * \mathrm{s}^2 + 33.7422 * \mathrm{s} + 18}{3s^3 - 0.7422s} \tag{60}$$

O) Write $P(s) + KZ(s) = 0$ as a polynomial in s involving K. [1H]

It is assumed that the numerator of the open-loop transfer function is $\mathrm{Z(s)}$ and the denominator of the open-loop transfer function is $\mathrm{P(s)}$, then the closed-loop poles occur when $P(s) + KZ(s) = 0$

From Equation (60):

$$\mathrm{Z(s)} = 18 * \mathrm{s}^2 + 33.7422 * \mathrm{s} + 18 \tag{61}$$

$$\mathrm{P(s)} = 3s^3 - 0.7422s \tag{62}$$

Therefore, the polynomial in s involving K will be:

$$\mathrm{P(s)} + \mathrm{KZ(s)} = 3s^3 - 0.7422s + K(18 * \mathrm{s}^2 + 33.7422 * \mathrm{s} + 18) = 0 \qquad (63)$$

Simplifying Equation (63) gives:

$$\mathrm{P(s)} + \mathrm{KZ(s)} = 3\mathrm{s}^3 + 18\mathrm{Ks}^2 + (33.7422\mathrm{K} - 0.7422)\mathrm{s} + 18\mathrm{K} = 0 \qquad (64)$$

P) Write $P(\tilde{s}) + KZ(\tilde{s}) = 0$ as a polynomial in $\tilde{s}$ involving K. [1H]

To find the value of K, the above equation (64) need to be re-written by substituting $\mathrm{s} = \tilde{s} - \frac{4}{t_s}$. After substituting the value of $t_s \approx 8.004$:

$$\mathrm{s} \approx \tilde{s} - 0.4997 \qquad (65)$$

For calculation purposes:

$$\mathrm{s} \approx \tilde{s} - 0.5 \qquad (66)$$

Substituting Equation (66) into Equation (64) gives:

$$\mathrm{P}(\tilde{s}) + \mathrm{KZ}(\tilde{s}) = 3(\tilde{s} - 0.5)^3 + 18\mathrm{K}(\tilde{s} - 0.5)^2 + (33.7422\mathrm{K} - 0.7422) * (\tilde{s} - 0.5) + 18\mathrm{K} = 0 \quad (67)$$

Simplifying Equation (67) gives:

$$\mathrm{P}(\tilde{s}) + \mathrm{KZ}(\tilde{s}) = 3\tilde{s}^3 + (18K - 4.5)\tilde{s}^2 + (15.7422K + 1.5078)\tilde{s} + (5.6289\mathrm{K} - 0.0039) = 0 \qquad (68)$$

Q) Complete a Routh table for $P(\tilde{s}) + KZ(\tilde{s})$. Deduce the value of K that is such that $Re(s) = -4/t_s$ [3H]

Because poles in the RHP will cause instability, the Routh-Hurwitz criteria will be used to find K through finding K that forces a row in the Routh-Hurwitz table to be zero. The value of K represents the $\mathrm{Re(s)} = -0.5$ crossings in the root locus. Therefore, the Routh-Hurwitz table is as follows:

First, the first and two rows of Routh-Hurwitz table will be written according to coefficients of the Equation (55):

s^3	3	15.7422k +1.5078
s^2	18k-4.5	5.6289k-0.0039

Figure 14: The first and two rows of Routh-Hurwitz table

According to the method, the example Routh-Hurwitz table is shown below:

Row n	a_n	a_{n-2}	a_{n-4}	$\cdots$
Row $n-1$	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
Row $n-2$	b_{n-1}	b_{n-3}	b_{n-5}	$\cdots$
Row $n-3$	c_{n-1}	c_{n-3}	c_{n-5}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
Row 0	h_{n-1}			

Figure 15:The example of Routh-Hurwitz table

Where,

$$b_{n-1} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \tag{69}$$

Or

$$b_{n-1} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} \tag{70}$$

Therefore, based on the above method, the third row of Routh-Hurwitz table is as follows：

s^1	$-\frac{-283.36K^2+60.5862K+6.7734}{18K-4.5}$	0

Figure 16:The third row of Routh-Hurwitz table

Repeat the above method, the whole Routh-Hurwitz table will be:

s^3	3	$15.7422K+1.5078$
s^2	$18K-4.5$	$5.6289K-0.0039$
s^1	$-\frac{-283.36K^2+60.5862K+6.7734}{18K-4.5}$	0
s^0	$5.6289K-0.0039$	0

Figure 17: The whole Routh-Hurwitz table

Next, we choose to calculate the value of K by making all elements of the third row equal to 0, which means that:

$$-\frac{-283.36K^2+60.5862K+6.7734}{18K-4.5} = 0 \tag{71}$$

The value of K can be known by solving the Equation in (71):

$$\mathrm{K}_1 \approx 0.2949,\ \ \mathrm{K}_2 =-\ 0.0811 \tag{72}$$

When the value of K is equal to -0.0811, the time response figure of the system is:

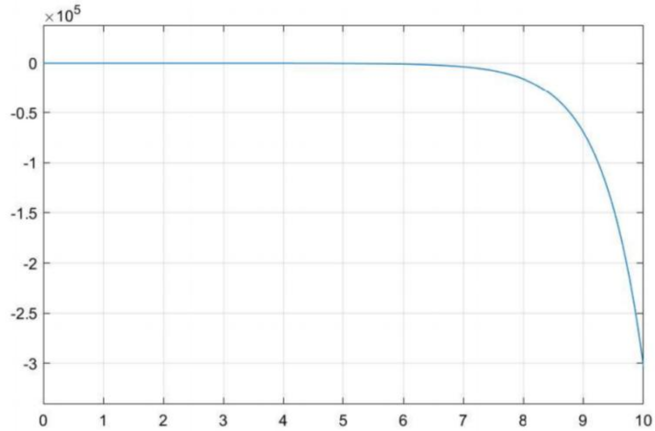

Figure 18: The time-response of the system when K=-0.0811

From the above figure, when the value of K is -0.0811, the system will be unstable. This is because as time increases, the time-response of the system gradually does not converges to a constant value. Therefore, we have to choose a positive value of $K = 0.2911$. The time response figure of the system is in the next question.

After calculating the value of K, we can substituting it into the auxiliary equation, the auxiliary equation for my system is:

$$(18K - 4.5)s^2 + 5.6289K - 0.0039 = 0 \tag{73}$$

Substituting k=0.2949 into Equation (73) gives:

$$0.8082s^2 + 1.6561 = 0 \tag{74}$$

After solve the Equation (74), the solutions are:

$$s_1 \approx -1.43i,\ s_2 = 1.43i \tag{75}$$

Since we used $s \approx \tilde{s} - 0.5$ to replace s earlier, the two solutions of s above should now correspond to the vertical coordinates of the point when $\mathrm{Re}(s) = -0.5$. According to the figure of our simulation experiment.

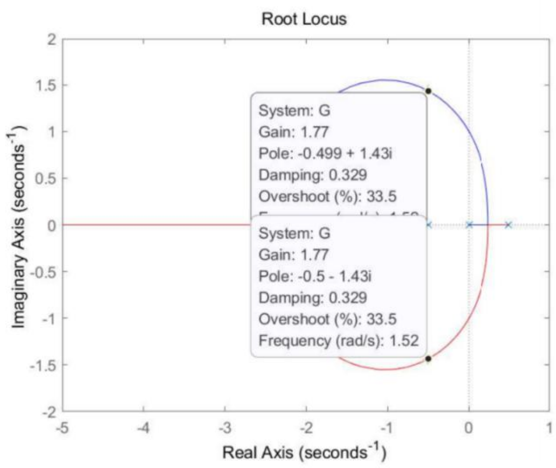

Figure 19: The points on the root locus that are such that $\mathrm{Re}(s) = -4/t_s$

As can be seen from the graph above, the coordinates of the two points are (-0.5,1.43) and (-0.5,-1.43) when$\mathrm{Re}(s) = -0.5$. This is also consistent with the calculation of Equation (75).

6. Validating the Response of Optimised System

R) Insert a picture of the time-response of your improved closed-loop system to the unit-step. [2H]

In the above question, we calculated the value of K : 0.2911. In order to make the setting time of the system to 8s, we need to add a gain to the front of the PID controller we designed. The amplification is the value of K : 0.2911. The improved system is as follows.

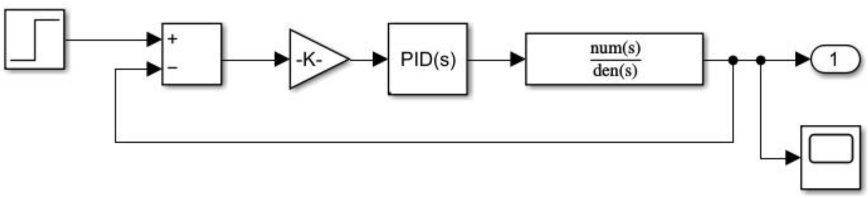

Figure 20: The improved closed-loop system to the unit-step

According to our modified system, the improved time-response of my system is:

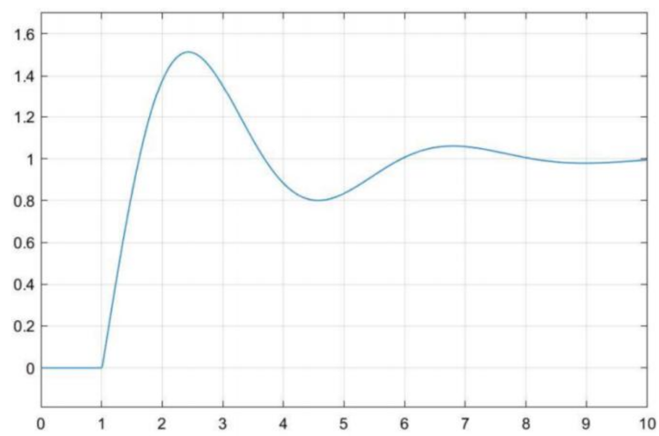

Figure 21: The time-response of my improved closed-loop system

As can be observed from the above figure, the setting time is now about 8s, which means the improvement has been achieved. In addition, this improved system is still stable because as time increases, the time-response of the system gradually does not converges to a constant value of 1. Although the improved time response of my system oscillates a few more times than the original unimproved system, it does not particularly affect the system. Also, according to the data from Matlab, the system now has an overshoot value which is about 53.077%.

As can be seen in the figure of root locus, in the case of some K values before the improvement, the branches of the root locus still have some points appearing in the right half-axis. This means that in some cases the system is still unstable. After the improvement, these points appearing on the right half-axis will be shifted to the left of the axis to make the system stable. More importantly, the system's setting time has also been changed to approximately 8s.The following is the comparation

the new time response (K=0.2911) with the previous response (K=1).

When K=1, the following figure can be seen blow:

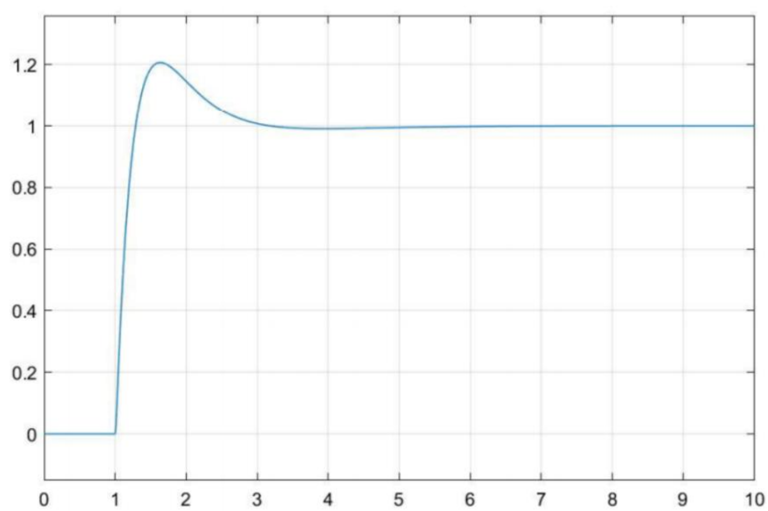

Figure 22: The time-response when K=1

By comparing the figure with K=0.2911, we find that the setting time of the time-response before the improvement is about 2s and the number of oscillations is less than the time-response when K=0.2911. This is because when reducing the value of K, we also change the damping ratio of the system. When damping ratio decreases, the number of oscillations in the time response will increase. However, the system is still a stable system and the improved system achieves the setting time to approximately 8s.

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